Nonminimal non-Abelian quantum vector fields in curved spacetime

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The quantum effective action of nonminimal vector fields with Abelian or non-Abelian gauge degrees of freedom in curved spacetime is studied. The Proca or Yang-Mills fields are coupled to a local masslike term acting in both coordinate and gauge spaces. Pathologies due to gauge invariance in the ultraviolet are avoided through the introduction of a non-Abelian version of the Stueckelberg field. It is found that the breaking of gauge invariance induced by the mass term affects only the tree-level part of the effective action. The ultraviolet divergent part of the effective action to one loop is obtained using the method of covariant symbols and dimensional regularization. Formulas are given valid for any spacetime dimension and explicit results are shown for the two-dimensional case. As already happened for a single vector field, the ultraviolet divergences are local but not of polynomial type.

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I. INTRODUCTION

Vector fields play a prominent role in the Standard Model of particles, as mediators of gauge interactions. In turn there is currently a growing interest in the role that various types of vector fields could play in relativistic gravity and cosmology [1–9]. As noted in [9], "Imposing the conditions of Lorentz symmetry, unitarity, locality and a (pseudo-)Riemannian spacetime, any attempt of modifying gravity inevitably introduces new dynamical degrees of freedom. They could be additional scalar, vector or tensor fields." The subject has received a further boost with the discovery of ghost-free consistent nonlinear theories of Proca interactions [10–15]. The crucial issue of the quantum stability of these theories has been analyzed in [16,17].

In this work, we consider a set of N vector fields in curved spacetime endowed with Abelian (Proca) and/or non-Abelian (Yang-Mills) internal degrees of freedom. No self-interactions are included beyond those implied by the Yang-Mills structure, but the vector fields are coupled to an external masslike x-dependent tensor field which is allowed to arbitrarily mix them [see Eq. (2.1)]. Our focus is on the proper quantization of such a theory and on the structure of the quantum fluctuations.

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Early work studying the subject of quantum fluctuations for vectors fields was carried out in [18–21] (see [22–29] for recent related work). Particular nonminimal couplings (the minimal case being a standard mass term) were considered in [30] at the classical level and in [31] at the quantum level. The quantized theory for general nonminimal couplings was first studied in [32] for particular spacetime backgrounds. There it was found that pathologies arise in the quantization of the theory since the mass term couples effectively as a metric field. Technically the problem is that the masslike field breaks gauge invariance but does not suppress the fluctuations in the longitudinal polarization at large wave numbers. In other words, the principal symbol of the fluctuation operator is singular. General backgrounds were considered in [33] solving the above-mentioned pathology by means of a Stueckelberg field. In this way, a proper gauge symmetry is present in the theory and one can proceed through a standard gauge fixing procedure. However, approximations were introduced in the analysis of [33] giving rise to a nonlocal result. A full solution to the problem of computing the ultraviolet (UV) divergent part of the effective action, within dimensional regularization, was obtained in [34] using the Schwinger-DeWitt technique and later in [35] using the method of covariant symbols, finding perfect concordance in both calculations. The pathologies identified in [32] translate to the fact that the UV divergences are local but not polynomial in the masslike external field.

The results just noted refer to a single vector field. Here we address the case of several vector fields. This allows us to consider the non-Abelian scenario. In fact, we consider sets of vector fields organized in Abelian and non-Abelian

multiplets. We treat the Abelian and non-Abelian versions simultaneously since the formalism is identical in both cases. In the absence of a mass term, there would be a gauge symmetry present in the Lagrangian. This symmetry is explicitly broken by the mass term. Nevertheless we obtain the remarkable result that the breaking only affects the effective action at tree level, while the contributions from one or more loops are fully gauge invariant. This results in an important simplification of the calculation. Another insight comes from the introduction of the Stueckelberg field in our present non-Abelian setting (see [36] for a review on this subject). In the Abelian case, the Stueckelberg field appears through $A_\mu = B_\mu + \partial_\mu \phi$. In this way, a $U(1)$ gauge symmetry arises from $B_\mu \to B_\mu + \partial_\mu \Lambda$, $\phi \to \phi - \Lambda$. In the non-Abelian case, a literal translation of this prescription would take the form $A^a_\mu = (B^\Omega)^a_\mu$ where Ω refers to a non-Abelian gauge transformation with parameters ϕ^a (in Lie algebra of the gauge group). The resulting theory enjoys a non-Abelian gauge invariance and one can then proceed to fix the gauge through the Fadeev-Popov method. Such approach is, in principle, correct but exceedingly complicated, as the dependence on ϕ^a is nonlinear. In particular, this would imply a reorganization of the loop expansion from the original theory (field A^a_μ) to that with B^a_μ and ϕ^a. We develop a completely different approach where the Stueckelberg field is introduced linearly also in the non-Abelian setting.

Once the Stueckelberg field is introduced, the (UV regulated) quantum theory is no longer pathological and it is possible to proceed to a systematic computation of its effective action. Our focus is on the UV part of the effective action to one-loop order. As already known from the study of the Abelian case [34,35], the mass term acts as an effective second metric tensor. In the present case, this is in fact a non-Abelian effective metric. Presumably, the generalized Schwinger-DeWitt technique [18] can be adapted to this situation, but such an approach is not presently available. Instead here we apply the method of covariant symbols [37]. This method is simple to use and allows one to formulate the loop momentum integration while preserving manifest covariance under diffeomorphism and gauge transformations. Details of the method are provided below.

The problem studied here is an extension of that already solved for $N = 1$ (just one vector field), so some specific features of that case are inherited in the more general setting analyzed in this work. In particular, the loop momentum integrals cannot be written in closed form. For $N > 1$ this problem is even worse, as the propagators are now matrices with respect to the gauge indices. Also for this reason some contributions to the effective action (see $\Gamma_{L,0}$ below) cannot be expressed in a standard form involving just integration over x, p, and traces in internal space, and it is necessary to resort to a parametric form, with integration over one more parameter. Unfortunately, while the problem is well posed and the method fully appropriate to solve it, we have found an unexpected impediment, namely, the number of terms obtained for the physically relevant case of four spacetime dimensions is prohibitively large (at least hundreds of terms are generated). In view of this, we develop the formulas for the general case but only present detailed results for two spacetime dimensions (note that there are no UV divergences for odd dimensions within dimensional regularization).

In Sec. II we expose the theory to be analyzed. In Sec. III the background field approach is introduced for the effective action. It is shown that its quantum part admits a gauge-covariant treatment. The effective action to one loop is constructed, showing its limitations in the UV sector. Those obstacles are overcome in Sec. IV by introducing the Stueckelberg field. The nonpathological one-loop effective action is constructed and then decomposed into various contributions to be computed subsequently. Section V introduces some notational conventions. Section VI introduces general considerations to undertake the calculation and presents explicit results for $d = 2$. Some nontrivial symmetries related to metric deformations are also verified. The actual calculations are worked out in Sec. VII. To this end, the method of covariant symbols is reviewed first, and its application to the various contributions is discussed, including the extraction of the coefficients of the UV divergence. The conclusions are summarized in Sec. VIII. The proof of some formulas is provided in Appendix A. Properties of the operator $Z_{\mu\nu}$ and its relation to $\hat{\mathcal{R}}_{\mu\nu}$ of [18] are discussed in Appendix B. The canonical form of Γ_S using a basis of standard operators is displayed in Appendix C. Explicit results for perturbative mass expansions are presented in Appendix D. Details of the method of noncovariant symbols, used as a check of the calculations, are given in Appendix E. Finally, the method of covariant symbols is illustrated through a sample computation in Appendix F.

II. FORMULATION OF THE PROBLEM

We consider N real vector fields $\hat{\mathcal{A}}^a_\mu(x)$, $a = 1, \ldots, N$ in an Euclidean d-dimensional spacetime with metric $g_{\mu\nu}(x)$ and action[1]

$$S[\hat{\mathcal{A}}; \mathcal{M}, g] = \int d^d x \sqrt{g} \left(\frac{1}{4} \hat{\mathcal{F}}^{\mu\nu}_a \hat{\mathcal{F}}^a_{\mu\nu} + \frac{1}{2} \mathcal{M}^{\mu\nu}_{ab} \hat{\mathcal{A}}^a_\mu \hat{\mathcal{A}}^b_\nu \right), \quad (2.1)$$

where $\mathcal{M}^{\mu\nu}_{ab}(x)$ is a positive definite[2] local mass term fulfilling the symmetry condition

$$\mathcal{M}^{\mu\nu}_{ab}(x) = \mathcal{M}^{\nu\mu}_{ba}(x). \quad (2.2)$$

[1]The ugly notation $\hat{\mathcal{A}}^a_\mu$ and $\hat{\mathcal{F}}^{\mu\nu}_a$ will soon be traded by A^a_μ and $F^{\mu\nu}$.

[2]As a matrix with indices (μa) and (νb).

Coordinate indices are raised, lowered, and contracted with the metric $g_{\mu\nu}$.

The N fields are organized in n gauge sectors. Each sector has gauge symmetry of either SU(n_i) or U(1) type and the full gauge group is the direct product of these.[3] The fields fall in the Lie algebra of the group, i.e., the adjoint representation, with gauge coupling g_i in the gauge sector i. Without loss of generality, one can choose $g_i = 1$ for the Abelian factors. Hence, $N = \sum_{i=1}^{n} N_i$ where $N_i = n_i^2 - 1$ for an SU(n_i) sector and $N_i = 1$ for a U(1) one. With a standard normalization of the fields, the field strength tensor is

$$\hat{\mathcal{F}}^a_{\mu\nu} = \partial_\mu \hat{\mathcal{A}}^a_\nu - \partial_\nu \hat{\mathcal{A}}^a_\mu + g_a f_{abc} \hat{\mathcal{A}}^b_\mu \hat{\mathcal{A}}^c_\nu, \tag{2.3}$$

where f_{abc} are the structure constants of the gauge group. Since the gauge group is a direct product, the structure constants are block diagonal, one block for each gauge sector, and $g_a = g_i$ is the coupling of the ith sector (the g_a take a common value within each block). Of course, in a U(1) sector the structure constant vanishes. The field strength tensor $\hat{\mathcal{F}}^a_{\mu\nu}(x)$ is covariant under gauge and coordinate transformations. The kinetic part of the action is block diagonal, while the mass term may mix different gauge sectors.

When all the gauge sectors are of the U(1) type, the theory is Abelian and reduces to a generalized Proca field with N flavors. Nevertheless, since all the cases can be treated within the same scheme, we will refer to the internal space as gauge space.

Regarding the symmetries, the kinetic term is fully local gauge invariant but such symmetry is reduced to a global one by the mass term: the action is invariant under $\mathcal{M} \to \mathcal{M}^\Omega = \Omega \mathcal{M} \Omega^{-1}$ for a global transformation Ω in the gauge group. If some gauge sectors are equivalent, namely, with equal gauge group SU(n_i) or U(1) and same g_i, there is an additional global symmetry under rotations among those equivalent sectors, with a corresponding rotation of $\mathcal{M}$. A further symmetry, special for the case of $d = 4$ spacetime dimensions, is that of local Weyl-like transformations, namely, the action is unchanged under the simultaneous replacements $g_{\mu\nu}(x) \to \xi(x) g_{\mu\nu}(x)$ and $\mathcal{M}^{\mu\nu}_{ab}(x) \to \xi^{-2}(x) \mathcal{M}^{\mu\nu}_{ab}(x)$.

III. THE EFFECTIVE ACTION

A. The background gauge field

Within the background field approach [38], the field is split as a background plus a fluctuation, $\hat{\mathcal{A}}^a_\mu(x) = A^a_\mu(x) + \mathcal{A}^a_\mu(x)$.

In this approach, the effective action $\Gamma[A; \mathcal{M}, g]$ follows from

$$Z = e^{-\Gamma[A;\mathcal{M},g]} = \int \mathcal{D}\mathcal{A} e^{-S[\mathcal{A}+A] + \int d^d x \sqrt{g} J \mathcal{A}}, \tag{3.1}$$

where $A^a_\mu(x)$ is the background field and the current $J^\mu_a(x)$ is adjusted so that $\langle \mathcal{A}^a_\mu(x) \rangle = 0$. As usual,

$$J^\mu_a(x) = \frac{1}{\sqrt{g(x)}} \frac{\delta \Gamma[A; \mathcal{M}, g]}{\delta A^a_\mu(x)}. \tag{3.2}$$

In the background gauge field approach, the field $\mathcal{A}^a_\mu(x)$ transforms homogeneously under local gauge transformations, the inhomogeneity being saturated by the transformation of $A^a_\mu(x)$. Correspondingly, the gauge-covariant derivative relies on $A^a_\mu(x)$ as a gauge connection.

We will use a single covariant derivative ∇_μ containing coordinate and gauge connections [39]. The coordinate connection is that of Levi-Civita for the metric and the gauge connection is that of the background field A^a_μ. So, for instance, for coordinate-scalar and coordinate-vector fields ϕ^a and $\mathcal{B}^a_\mu$, respectively, both in the adjoint gauge representation,

$$\begin{aligned} \nabla_\mu \phi^a &= \partial_\mu \phi^a + g_a f_{abc} A^b_\mu \phi^c, \\ \nabla_\mu \mathcal{B}^a_\nu &= \partial_\mu \mathcal{B}^a_\nu - \Gamma^\lambda_{\mu\nu} \mathcal{B}^a_\lambda + g_a f_{abc} A^b_\mu \mathcal{B}^c_\nu. \end{aligned} \tag{3.3}$$

Throughout, coordinate indices are contracted with the metric $g_{\mu\nu}$ and gauge-vector indices with δ_{ab}.

The effective action can be split into the classical or tree-level component $S[A]$ and the quantum correction $\Gamma_Q[A]$ which contains graphs with one or more loops,

$$\Gamma[A] = S[A] + \Gamma_Q[A]. \tag{3.4}$$

Here we find a fundamental result given by the following

Theorem.—$\Gamma_Q[A]$ is invariant under *local* gauge transformations and all the gauge breaking in the effective action is saturated by the mass term at tree level. That is,

$$\Gamma_Q[A; \mathcal{M}, g] = \Gamma_Q[A^\Omega; \mathcal{M}^\Omega, g], \tag{3.5}$$

where $\Omega(x)$ is any local gauge transformation, and A^Ω and $\mathcal{M}^\Omega$ are the gauge-transformed fields.

Proof.—The reason is fairly simple. The semiclassical expansion follows from a Taylor expansion of the action $S[\mathcal{A} + A] - \int d^d x \sqrt{g} J \mathcal{A}$ in powers of the fluctuation $\mathcal{A}$. The zeroth order gives the classical action, and the first order in $\mathcal{A}$ cancels due to the equations of motion, i.e., the choice of J^μ. The quantum component Γ_Q depends only on terms that are quadratic or higher order in $\mathcal{A}$. The breaking of gauge invariance would come solely from the mass term $\frac{1}{2} \mathcal{M}^{\mu\nu}_{ab} \mathcal{A}^a_\mu \mathcal{A}^b_\nu$ but this is covariant since the field $\mathcal{A}^a_\mu$ transforms homogeneously under gauge transformations.

The property (3.5) is important because it allows us to use a gauge-covariant formalism for Γ_Q.

[3] More generally, one could take a Lie subgroup of SO(N) and the results and formulas derived in this work hold equally well in that case.

B. One-loop effective action

The one-loop effective action follows from the quadratic part of the action:

$$\Gamma_1[A] = \frac{1}{2}\log \mathrm{Det}_1\left(\frac{\delta^2 S[\mathcal{A}+A]}{\delta \mathcal{A}^2}\right)\Big|_{\mathcal{A}=0}. \tag{3.6}$$

The subindex 1 in Det_1 indicates that the determinant is to be evaluated in the space $\mathcal{A}_\mu(x)$, i.e., of coordinate vectors. Also, for the gauge degrees of freedom, the determinant is taken in the adjoint gauge representation space. This is not explicitly indicated but will be implicit in all formulas as no other gauge representations will be present.

Therefore, we need to isolate the terms quadratic in $\mathcal{A}$ from the action $S[\mathcal{A}+A]$. After the shift $\hat{\mathcal{A}} = \mathcal{A} + A$, the field strength tensor in Eq. (2.3) becomes

$$\hat{\mathcal{F}}^a_{\mu\nu} = \mathcal{F}^a_{\mu\nu} + \nabla_\mu \mathcal{A}^a_\nu - \nabla_\nu \mathcal{A}^a_\mu + g_a f_{abc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu \tag{3.7}$$

with

$$\mathcal{F}^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_a f_{abc} A^b_\mu A^c_\nu. \tag{3.8}$$

In the shifted variables, the quadratic part of the kinetic term of the action takes the form[4]

$$S^{(2)}_{\mathrm{kin}}[\mathcal{A}] = \int d^dx\sqrt{g}\left(\frac{1}{4}(\nabla_\mu \mathcal{A}^a_\nu - \nabla_\nu \mathcal{A}^a_\mu)^2 - \frac{1}{2}F^{ab}_{\mu\nu}\mathcal{A}^a_\mu \mathcal{A}^b_\nu\right), \tag{3.9}$$

where we have the introduced field strength tensor $F^{ab}_{\mu\nu}$ as an antisymmetric matrix in gauge space (as well as in coordinate space)

$$F^{ab}_{\mu\nu} \equiv g_c f_{acb}\mathcal{F}^c_{\mu\nu}. \tag{3.10}$$

Note that $F^{ab}_{\mu\nu}$ vanishes in the Abelian sectors.

In what follows, we adopt the convention that covariant derivatives are indicated by adding new indices *to the left*, hence $\phi^a_\mu \equiv \nabla_\mu \phi^a$, $\mathcal{B}^a_{\mu\nu} \equiv \nabla_\mu \mathcal{B}^a_\nu$, etc. The only exceptions to this rule are the operators $Z_{\mu_1\mu_2\ldots}$ and $Z^R_{\mu_1\mu_2\ldots}$.[5] With this convention,

$$S^{(2)}_{\mathrm{kin}}[\mathcal{A}] = \int d^dx\sqrt{g}\left(\frac{1}{4}(\mathcal{A}^a_{\mu\nu} - \mathcal{A}^a_{\nu\mu})^2 - \frac{1}{2}F^{ab}_{\mu\nu}\mathcal{A}^a_\mu \mathcal{A}^b_\nu\right). \tag{3.11}$$

Using integration by parts and Bianchi identities, the quadratic part of the kinetic term can be written as (see Appendix A)

$$S^{(2)}_{\mathrm{kin}}[\mathcal{A}] = \int d^dx\sqrt{g}\left(\frac{1}{2}(\mathcal{A}^a_{\mu\nu})^2 - \frac{1}{2}(\mathcal{A}^a_{\mu\mu})^2 - F^{ab}_{\mu\nu}\mathcal{A}^a_\mu \mathcal{A}^b_\nu + \frac{1}{2}\mathcal{R}_{\mu\nu}\mathcal{A}^a_\mu \mathcal{A}^a_\nu\right), \tag{3.12}$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor. Thus, adding the mass term,

$$S^{(2)}_{\mathrm{mass}}[\mathcal{A}] = \int d^dx\sqrt{g}\left(\frac{1}{2}\mathcal{M}^{\mu\nu}_{ab}\mathcal{A}^a_\mu \mathcal{A}^b_\nu\right), \tag{3.13}$$

the full quadratic Lagrangian controlling the one-loop fluctuations is

$$\mathcal{L}^{(2)}(x) = \frac{1}{2}\mathcal{A}_\mu K^{\mu\nu}_0 \mathcal{A}_\nu, \tag{3.14}$$

where

$$K^{\mu\nu}_0 = -g^{\mu\nu}\nabla^2 + \nabla^\mu\nabla^\nu - 2F^{\mu\nu} + \mathcal{R}^{\mu\nu} + \mathcal{M}^{\mu\nu}. \tag{3.15}$$

Here, and also in what follows, we use a matrix notation for the gauge indices, which will be implicit. As advertised, the Lagrangian $\mathcal{L}^{(2)}(x)$ is manifestly gauge invariant.

The term $+\nabla^\mu\nabla^\nu$ in $K^{\mu\nu}_0$ is a direct consequence of gauge invariance of the kinetic energy part of the action (2.1) and is needed to retain just three polarizations in the Proca field. While the differential operator K_0 needs not be singular in the presence of a positive definite mass term $\mathcal{M}^{\mu\nu}$, its principal symbol, i.e., the $O(\nabla^2)$ leading UV divergent component is singular, since the longitudinal polarizations are not penalized at large wave numbers. The fact that the principal symbol is singular introduces pathologies in the effective action which prevent one from carrying out an extraction of the UV divergent terms.

In the special case of a standard Proca field, with a constant scalar mass, the UV divergent part of the effective action is a polynomial in the mass [18], but this is no longer so for a nonconstant mass term even in the Abelian case [34]. This confirms that K_0 cannot be directly used as the fluctuation operator.

IV. THE STUECKELBERG FIELD

A. The non-Abelian Stueckelberg field

In order to bypass the above-mentioned pathologies in the UV, we will adapt the Stueckelberg approach introduced in [33] (and also applied in [34,35]) for the nonminimal Proca field to the non-Abelian case. To this end, we rewrite the partition function as

[4]We will occasionally place all the coordinate indices as lower indices when no ambiguity arises. Repeated coordinate indices are always contracted with the metric $g_{\mu\nu}$.

[5]Some conventions used in this work are summarized in Sec. V.

$$Z = \int \mathcal{D}\mathcal{A} e^{-S[\mathcal{A}+A]+\int d^dx\sqrt{g}J\mathcal{A}} \int \mathcal{D}\chi e^{-S_{\rm gf}[\chi]}, \tag{4.1}$$

where $S_{\rm gf}[\chi]$ can be any action. The partition function is unchanged as the new factor is just a constant.[6] We take a standard choice

$$S_{\rm gf}[\chi] = \int d^dx\sqrt{g}\frac{1}{2}\chi^a\chi_a, \tag{4.2}$$

where $\chi^a(x)$ is a coordinate scalar and a gauge vector (i.e., in the adjoint representation).

Subsequently, a change of variables $(\mathcal{A},\chi) \to (\mathcal{B},\phi)$ is applied in (4.1), where $\mathcal{B}^a_\mu(x)$ is a real coordinate-vector and gauge-vector field and $\phi^a(x)$ is real coordinate-scalar and gauge-vector field,

$$Z = \int \mathcal{D}\mathcal{B}\mathcal{D}\phi J[\mathcal{B},\phi] e^{-S[\mathcal{A}+A]-S_{\rm gf}[\chi]+\int d^dx\sqrt{g}J\mathcal{A}},$$
$$J[\mathcal{B},\phi] = \mathrm{Det}\left(\frac{\partial(\mathcal{A},\chi)}{\partial(\mathcal{B},\phi)}\right). \tag{4.3}$$

By construction, the effective action does not depend on the detailed choice of gauge-fixing function(al) $\chi[\mathcal{B},\phi]$, moreover, the expectation value of any functional written in the form $F[\mathcal{A},\chi]$ is independent of this choice (unless the very functional F depends on it). This property provides identities for the gauge-fixing dependence of the expectation values [40].

We choose a linear change of variables. Besides simplicity, the virtue of such a choice is that the loop expansion in the new variables coincides with that in the old ones. Specifically, we take

$$\mathcal{A}^a_\mu = \mathcal{B}^a_\mu + \nabla_\mu\phi^a, \qquad \chi^a = \nabla_\mu\mathcal{B}^a_\mu, \tag{4.4}$$

or, using our notational convention for the covariant derivatives,

$$\mathcal{A}^a_\mu = \mathcal{B}^a_\mu + \phi^a_\mu, \qquad \chi^a = \mathcal{B}^a_{\mu\mu}. \tag{4.5}$$

The corresponding Fadeev-Popov determinant is easily obtained as (see Appendix A)

$$J[\mathcal{B},\phi] = \mathrm{Det}_0(\delta_{ab}\nabla^\alpha\nabla_\alpha). \tag{4.6}$$

The subindex 0 in Det_0 indicates that the determinant is to be evaluated in the ϕ^a space, i.e., the coordinate-scalar space. As already noted, the reference to the adjoint gauge representation is not explicitly displayed, as its presence is ubiquitous and no other gauge representation will be needed. With our choice of a linear change of variables, the determinant J does not depend on the quantum fields $\mathcal{B}$, ϕ. It depends on A^a_μ and $g_{\mu\nu}$. As usual, the determinant can be implemented through a complex ghost field with quadratic action.

It is worth noticing that one could have introduced the Stueckelberg field in a different manner, to wit, through the change of variable $\mathcal{A} = \mathcal{B}^\Omega$ in Eq. (3.1), where $\Omega(x)$ is an arbitrary gauge transformation, and $\phi^a(x)$ enters through $\Omega = e^{i\phi}$. In addition, the measure $\mathcal{D}\mathcal{A}$ is replaced by $\mathcal{D}\mathcal{B}\mathcal{D}\Omega$.[7] In this way, the full theory $S[\mathcal{B}^\Omega]$ becomes gauge invariant even in the presence of the mass term. Then one fixes the gauge as usual with the Fadeev-Popov method. A more involved question is how to introduce the background gauge machinery. In such alternative approach, the change of variables from $(\mathcal{A},\chi)$ to $(\mathcal{B},\phi)$ is not linear and so it should be considerably more complicated than the method adopted above. In the Abelian case, the two approaches are equivalent.

B. The one-loop effective action revisited

The introduction of the gauge-fixing action $S_{\rm gf}[\chi]$ adds an irrelevant constant to the effective action. Hence, in variables $(\mathcal{B},\phi)$ the effective action is just

$$\Gamma_1[A] = \frac{1}{2}\log\mathrm{Det}_{1+0}\left(\frac{\delta^2(S[\mathcal{A}+A]+S_{\rm gf}[\chi])}{\delta(\mathcal{B},\phi)^2}\right)\Big|_{\substack{\mathcal{B}=0\\ \phi=0}} - \log\mathrm{Det}_0(\delta_{ab}\nabla^2), \tag{4.7}$$

where the last term comes from the Fadeev-Popov determinant. The subindex $1+0$ indicates the direct sum of coordinate-vector and -scalar spaces.

The kinetic energy term (3.12) in variables $(\mathcal{B},\phi)$ becomes (see Appendix A)

$$S^{(2)}_{\rm kin}[\mathcal{A}] = \int d^dx\sqrt{g}\left(\frac{1}{2}(\mathcal{B}^a_{\mu\nu})^2 - \frac{1}{2}(\mathcal{B}^a_{\mu\mu})^2 - F^{ab}_{\mu\nu}\mathcal{B}^a_\mu\mathcal{B}^b_\nu + \frac{1}{2}\mathcal{R}_{\mu\nu}\mathcal{B}^a_\mu\mathcal{B}^a_\nu + F^{ab}_{\mu\mu\nu}\phi^a\mathcal{B}^b_\nu + \frac{1}{2}F^{ab}_{\mu\mu\nu}\phi^a\phi^b_\nu\right), \tag{4.8}$$

while $S_{\rm gf}[\chi]$ is already quadratic, namely,

$$\mathrm{S}_{\rm gf}[\chi] = \int d^dx\sqrt{g}\frac{1}{2}(\mathcal{B}^a_{\mu\mu})^2. \tag{4.9}$$

This contribution removes the problematic longitudinal term in the kinetic energy. The price to pay is the introduction of a kinetic term for ϕ which has a metriclike coupling to the mass tensor, namely, the last term in

[6] It does not depend on $\mathcal{A}^\mu_a(x)$, $J^\mu_a(x)$, nor $\mathcal{M}^{\mu\nu}_{ab}(x)$. It is also independent of the metric if no derivatives are involved.

[7] Or just $\mathcal{D}\mathcal{B}\mathcal{D}\phi$. The two measures $\mathcal{D}\Omega$ and $\mathcal{D}\phi$ are equivalent. As is well known, the Jacobian of an ultralocal change of variables such as $\partial\Omega/\partial\phi$ has no effect in dimensional regularization.

$$S^{(2)}_{\rm mass}[\mathcal{A}] = \int d^dx\sqrt{g}\left(\frac{1}{2}\mathcal{M}^{\mu\nu}_{ab}\mathcal{B}^a_\mu\mathcal{B}^b_\nu + \mathcal{M}^{\mu\nu}_{ab}\phi^a_\mu\mathcal{B}^b_\nu + \frac{1}{2}\mathcal{M}^{\mu\nu}_{ab}\phi^a_\mu\phi^b_\nu\right). \tag{4.10}$$

In summary, the new full quadratic Lagrangian controlling the one-loop fluctuations is

$$\mathcal{L}^{(2)} = \frac{1}{2}(\mathcal{B},\phi)K(\mathcal{B},\phi)^T + \omega^*_a\nabla^2\omega^a. \tag{4.11}$$

The ghost field ω_a is a complex fermionic coordinate-scalar and gauge-vector field. On the other hand K is a second order differential operator acting on the space $(\mathcal{B},\phi)$,

$$K = \begin{pmatrix} -\nabla^2 g^{\mu\nu} - 2F^{\mu\nu} + \mathcal{R}^{\mu\nu} + \mathcal{M}^{\mu\nu} & -F_{\alpha\alpha}{}^{\mu} + \mathcal{M}^{\mu\alpha}\nabla_\alpha \\ F_{\alpha\alpha}{}^{\nu} - \nabla_\alpha\mathcal{M}^{\alpha\nu} & \frac{1}{2}\{F_{\alpha\alpha}{}^{\beta},\nabla_\beta\} - \nabla_\alpha\mathcal{M}^{\alpha\beta}\nabla_\beta \end{pmatrix}. \tag{4.12}$$

The matrix gauge indices are implicit. $\{,\}$ denotes the anticommutator.

Note that several differential operators can be read from the last term in Eq. (4.8), namely, $F_{\alpha\alpha}{}^{\beta}\nabla_\beta$, $\nabla_\beta F_{\alpha\alpha}{}^{\beta}$ or $\frac{1}{2}\{F_{\alpha\alpha}{}^{\beta},\nabla_\beta\}$. All of them are equivalent in the ϕ-ϕ sector, since the matrix $F_{\mu\nu}$ is antisymmetric and the Hilbert space spanned by ϕ^a is real. However, the functional integral over $\mathcal{B}$ and ϕ is only related to the determinant of the *symmetric* version of K, the one presented in Eq. (4.12).

As expected, after the introduction of the Stueckelberg field, the principal symbol of the operator K is no longer singular. Nevertheless, even though the technical problems have been sorted, the pathologies still will reflect on the effective action; in particular, one finds that the UV divergences do not depend polynomially on $\mathcal{M}$, as already happened in the case $N = 1$ studied in [34,35].

The leading UV divergent terms, with two derivatives, are at the diagonal of the matrix K. Particularly problematic will be the term $-\nabla_\alpha\mathcal{M}^{\alpha\beta}\nabla_\beta$ in the ϕ-ϕ sector. Because the leading divergence in the covariant derivatives is Abelian, only the symmetric component of $\mathcal{M}^{\mu\nu}$ is truly of second order. Hence, we will introduce the separation of the mass tensor into symmetric and antisymmetric components,

$$\mathcal{M}^{\mu\nu}_{ab} = M^{\mu\nu}_{ab} + Q^{\mu\nu}_{ab}, \qquad M^{\mu\nu}_{ab} = M^{\nu\mu}_{ab} = M^{\mu\nu}_{ba}, \qquad Q^{\mu\nu}_{ab} = -Q^{\nu\mu}_{ab} = -Q^{\mu\nu}_{ba}. \tag{4.13}$$

It can be noted that $M^{\mu\nu}$ must be positive definite and should dominate $Q^{\mu\nu}$, which is not. The mass term from Q is subdivergent since it is of first order in the derivatives. Indeed, after integration by parts,

$$\int d^dx\sqrt{g}\frac{1}{2}Q^{\mu\nu}_{ab}\phi^a_\mu\phi^b_\nu = \int d^dx\sqrt{g}\left(-\frac{1}{2}Q^{\mu\mu\nu}_{ab}\phi^a\phi^b_\nu - \frac{1}{4}Q^{\mu\nu}_{ac}F^{cb}_{\mu\nu}\phi^a\phi^b\right). \tag{4.14}$$

Then the (symmetric) fluctuation operator K takes the final form

$$K = \begin{pmatrix} -\nabla^2 g^{\mu\nu} + Y^{\mu\nu} & -\Phi^\mu + \mathcal{M}^{\mu\alpha}\nabla_\alpha \\ \Phi^\nu - \nabla_\alpha\mathcal{M}^{\alpha\nu} & -\nabla^2_M + \frac{1}{2}\{P^\beta,\nabla_\beta\} + W \end{pmatrix}, \tag{4.15}$$

where we have introduced the following shorthand notation:

$$Y_{\mu\nu} = \mathcal{M}_{\mu\nu} - 2F_{\mu\nu} + \mathcal{R}_{\mu\nu}, \qquad \Phi_\mu = F_{\alpha\alpha\mu}, \qquad P_\mu = F_{\alpha\alpha\mu} - Q_{\alpha\alpha\mu}, \qquad W = -\frac{1}{4}\{Q_{\mu\nu},F_{\mu\nu}\}, \tag{4.16}$$

as well as

$$\nabla^2_M \equiv \nabla_\alpha M^{\alpha\beta}\nabla_\beta. \tag{4.17}$$

As already noted, the field $M^{\mu\nu}(x)$, which was seemingly UV subdominant in the original action, is in fact UV dominant in the sector of the field ϕ in K and acts effectively as a second (inverse) metric. In the Abelian case ($N = 1$) such metric is an ordinary one, and even so it introduced a considerable amount of complication in the calculation of the effective action in [34,35]. In the setting discussed in this work, the "effective metric" $M^{\mu\nu}(x)$ is a non-Abelian one in gauge space, so we can certainly expect a higher degree of difficulty in the resources needed to attack this problem.

From the Lagrangian in Eq. (4.11), the effective action to one loop is thus

$$\Gamma_1[A;\mathcal{M},g] = \Gamma_K[A;\mathcal{M},g] + \Gamma_{\rm gh}[A;g], \tag{4.18}$$

with

$$\Gamma_K[A;\mathcal{M},g] = \frac{1}{2}{\rm Tr}_{1+0}\log(K), \qquad \Gamma_{\rm gh}[A;g] = -{\rm Tr}_0\log(\nabla^2). \tag{4.19}$$

C. Contributions to the effective action

As just said, the effective action can be split into

$$\Gamma_1 = \Gamma_K + \Gamma_{\rm gh}. \tag{4.20}$$

The operator K can be split into UV leading $O(\nabla^2)$ and subdivergent $O(\nabla)$ components

$$\begin{aligned} K &= K_L + K_S, \\ K_L &\equiv \mathrm{diag}(-\nabla^2 g^{\mu\nu}, -\nabla_M^2). \end{aligned} \tag{4.21}$$

Correspondingly, we also separate the effective action as

$$\Gamma_K = \frac{1}{2}\mathrm{Tr}_{1+0}(K_L(1 + K_L^{-1}K_S)) = \Gamma_L + \Gamma_S, \tag{4.22}$$

with

$$\begin{aligned} \Gamma_L &= \frac{1}{2}\mathrm{Tr}_{1+0}\log(K_L), \\ \Gamma_S &= \frac{1}{2}\mathrm{Tr}_{1+0}\log(1 + K_L^{-1}K_S). \end{aligned} \tag{4.23}$$

The relation $\mathrm{Tr}(\log(AB)) = \mathrm{Tr}(\log(A)) + \mathrm{Tr}(\log(B))$ is only guaranteed for sufficiently convergent pseudodifferential operators A and B. Nevertheless, it is expected to correctly reproduce the UV divergent terms within dimensional regularization when A and B are both coordinate-scalar operators.

Γ_L can be split further as

$$\Gamma_L = \Gamma_{L,1} + \Gamma_{L,0}, \tag{4.24}$$

with

$$\begin{aligned} \Gamma_{L,1} &= \frac{1}{2}\mathrm{Tr}_1\log(-\nabla^2 g^{\mu\nu}), \\ \Gamma_{L,0} &= \frac{1}{2}\mathrm{Tr}_0\log(-\nabla_M^2). \end{aligned} \tag{4.25}$$

For the purpose of obtaining the UV divergences of the effective action, the subdivergent terms can be treated perturbatively

$$\Gamma_S = \frac{1}{2}\mathrm{Tr}_{1+0}\log(1 + K_L^{-1}K_S) = \sum_{n=1}^{\infty}\Gamma_{S,n}, \tag{4.26}$$

where

$$\Gamma_{S,n} = -\frac{1}{2n}\mathrm{Tr}_{1+0}((-K_L^{-1}K_S)^n). \tag{4.27}$$

Since $K_L^{-1}K_S = O(\nabla^{-1})$, terms with $n > d$ are UV finite in d spacetime dimensions. Thus, collecting the various contributions,

$$\Gamma_1^{\rm div} = \Gamma_{\rm gh}^{\rm div} + \Gamma_{L,0}^{\rm div} + \Gamma_{L,1}^{\rm div} + \sum_{n=1}^{d}\Gamma_{S,n}^{\rm div}. \tag{4.28}$$

The contributions to $\Gamma_1^{\rm div}[A; \mathcal{M}, g]$ are analyzed in the following sections, after introducing some notation.

V. SOME NOTATIONAL CONVENTIONS

A. Covariant derivatives

Let us first recall that the covariant derivative operator ∇_μ contains all connections (and not only the Christoffel symbols), and also our convention that covariant derivatives are indicated by adding coordinate indices to the left, e.g.,

$$R_{\rho\mu\nu\alpha\beta} \equiv [\nabla_\rho, R_{\mu\nu\alpha\beta}], \qquad \mathcal{R}_{\lambda\lambda\mu} = \frac{1}{2}\mathbf{R}_\mu. \tag{5.1}$$

Here $R_{\mu\nu\alpha\beta}$, $\mathcal{R}_{\mu\nu}$, and $\mathbf{R}$ denote the Riemann tensor, the Ricci tensor, and the scalar curvature, respectively.

All quantities in the fluctuation operator K are to be regarded as operators acting on the vector space spanned by the fields $\mathcal{B}_\mu$ and ϕ. Hence ∇_μ acts on such quantities through the commutator. In particular $g_{\mu\nu}$, $\mathcal{M}^{\mu\nu}$, and $F_{\mu\nu}$ are purely multiplicative operators, which means that they are ordinary functions (possibly matrices in gauge space).[8]

B. Operators $Z_{\mu_1\cdots\mu_n}$

We will make use of the operator $Z_{\mu\nu}$, which is defined as

$$Z_{\mu\nu} \equiv [\nabla_\mu, \nabla_\nu]. \tag{5.2}$$

This operator is multiplicative because its action on a quantity does not involve derivatives of that quantity; $Z_{\mu\nu}$ is diagonal in x space. However, it is not purely multiplicative because it acts (is not diagonal) on coordinate indices. For instance, for a purely multiplicative tensor field $V_{\mu\nu}$,

$$[Z_{\mu\nu}, V_{\alpha\beta}] = R_{\mu\nu\alpha\lambda}V_{\lambda\beta} + R_{\mu\nu\beta\lambda}V_{\alpha\lambda} + [F_{\mu\nu}, V_{\alpha\beta}]. \tag{5.3}$$

The operator $Z_{\mu\nu}$ admits a natural separation between coordinate and gauge actions

$$Z_{\mu\nu} = Z_{\mu\nu}^R + F_{\mu\nu}, \tag{5.4}$$

where $Z_{\mu\nu}^R$ acts only on coordinate indices. As illustrated in (5.3), the operator $Z_{\mu\nu}^R$ acts on every coordinate index in turn.

Higher order operators $Z_{\mu_1\cdots\mu_n}$, with n covariant derivatives, are defined recursively (see Appendix B) so that they are also multiplicative. Letting $I = \mu_1\cdots\mu_n$ denote a

[8]Of class $\mathcal{C}(\underline{\nabla}, \underline{Z})$ in the notation of [37].

string of coordinate indices, the operators Z_I have again a clean separation between coordinate and gauge

$$Z_I = Z_I^R + F_I, \tag{5.5}$$

and also fulfill

$$[Z_I^R, V_{\mu_1\mu_2}] = R_{I\mu_1\lambda}V_{\lambda\mu_2} + R_{I\mu_2\lambda}V_{\mu_1\lambda}. \tag{5.6}$$

In fact, these operators are an anti-Hermitian version of the derivatives of the operator $\hat{\mathcal{R}}_{\mu\nu}$ of [18]. Specifically,

$$Z^R_{\alpha_1\cdots\alpha_n\mu\nu} = \nabla_{\alpha_1}\cdots\nabla_{\alpha_n}\hat{\mathcal{R}}_{\mu\nu} + C_{\alpha_1\cdots\alpha_n\mu\nu}, \tag{5.7}$$

where the C_I are purely multiplicative operators constructed with the Riemann tensor,

$$\begin{aligned} C_{\mu\nu} &= 0, \\ C_{\alpha_1\cdots\alpha_n\mu\nu} &= \frac{1}{2}R_{\lambda\alpha_2\alpha_3\cdots\alpha_n\mu\nu\alpha_1\lambda} + \frac{1}{2}R_{\alpha_1\lambda\alpha_3\cdots\alpha_n\mu\nu\alpha_2\lambda} + \cdots \\ &\quad + \frac{1}{2}R_{\alpha_1\cdots\alpha_{n-1}\lambda\mu\nu\alpha_n\lambda}. \end{aligned} \tag{5.8}$$

Eventually we will need to take traces of operators with a factor Z_I^R on the left. The formulas for the coordinate-scalar and -vector spaces are, respectively,

$$\begin{aligned} \mathrm{tr}_0(Z_I^R\mathcal{O}) &= -\mathrm{tr}_0(C_I\mathcal{O}), \\ \mathrm{tr}_1(Z_I^R\mathcal{O}^\mu{}_\nu) &= \mathrm{tr}_1((R_I{}^\mu{}_\lambda - g^\mu{}_\lambda C_I)\mathcal{O}^\lambda{}_\nu). \end{aligned} \tag{5.9}$$

Further details and proofs are given in Appendix B.

C. Integrals and traces

We will use the shorthand notation

$$\langle X\rangle_x \equiv \int d^dx\sqrt{g}X, \tag{5.10}$$

as well as

$$\langle X\rangle_{x,g} \equiv \int d^dx\sqrt{g}\,\mathrm{tr}_g(X) = \langle \mathrm{tr}_g(X)\rangle_x, \tag{5.11}$$

where $\mathrm{tr}_g()$ refers to trace over gauge space. In particular $\mathrm{tr}_g(1) = N$, the dimension of the gauge space is the number of dynamical real vector fields in the theory.

In addition, for integrals over a momentum variable in Sec. VII,

$$\langle X\rangle_p \equiv \frac{1}{\sqrt{g}}\int\frac{d^{d+2\varepsilon}p}{(2\pi)^d}X, \tag{5.12}$$

where $d+2\varepsilon$ refers to dimensional regularization. We also use combinations such as $\langle X\rangle_{x,p}$ for $\langle\langle X\rangle_p\rangle_x$, etc.

VI. RESULTS FOR Γ_1^{DIV}

A. Results for Γ_{gh} and $\Gamma_{L,1}$

The contributions Γ_{gh} and $\Gamma_{L,1}$ to the one-loop effective action are given by Eqs. (4.19) and (4.25), respectively. The computation of their UV divergent part is straightforward in dimension regularization, in $d+2\varepsilon$ dimensions, using the identity

$$\mathrm{Tr}\log(-\nabla^2)|_{\mathrm{div}} = \frac{1}{(4\pi)^{d/2}}\frac{1}{\varepsilon}\int d^dx\sqrt{g}\,\mathrm{tr}(b_{d/2}(x)), \tag{6.1}$$

where the trace is taken in the corresponding space and b_n is the nth heat-kernel coefficient of the Laplacian [39]. For $d=2$ and $d=4$, the required coefficients are

$$\begin{aligned} b_1 &= \frac{1}{6}\mathbf{R}, \\ b_2 &= \frac{1}{12}Z_{\mu\nu}^2 + \frac{1}{180}R_{\mu\nu\alpha\beta}^2 - \frac{1}{180}\mathcal{R}_{\mu\nu}^2 + \frac{1}{72}\mathbf{R}^2. \end{aligned} \tag{6.2}$$

As they stand, these formulas hold for an arbitrary space since $Z_{\mu\nu}$ takes care of all required curvatures (coordinate, gauge, or other in more general cases). For the space of coordinate tensors of rank r in d dimensions (and adjoint gauge representation), one easily finds

$$\begin{aligned} \mathrm{tr}_r(Z_{\mu\nu}^2) &= \mathrm{tr}_r(F_{\mu\nu}^2) + \mathrm{tr}_r((Z_{\mu\nu}^R)^2) \\ &= d^r\mathrm{tr}_g(F_{\mu\nu}^2) - rd^{r-1}NR_{\mu\nu\alpha\beta}^2, \end{aligned} \tag{6.3}$$

where as already said $\mathrm{tr}_g()$ denotes the trace over gauge space. Of course, this result is fully consistent with Eq. (5.9).

Therefore, for $d=2$,

$$\begin{aligned} \Gamma_{\mathrm{gh}}^{\mathrm{div}} &= -\frac{1}{4\pi\varepsilon}\left\langle\frac{1}{6}\mathbf{R}\right\rangle_{x,g}, \\ \Gamma_{L,1}^{\mathrm{div}} &= \frac{1}{4\pi\varepsilon}\left\langle\frac{1}{6}\mathbf{R}\right\rangle_{x,g}. \end{aligned} \tag{6.4}$$

These two contributions cancel each other, as they should in $d=2$.

For $d=4$,

$$\begin{aligned} \Gamma_{\mathrm{gh}}^{\mathrm{div}} &= -\frac{1}{(4\pi)^2\varepsilon}\left\langle\frac{1}{12}F_{\mu\nu}^2 + \frac{1}{180}R_{\mu\nu\alpha\beta}^2 - \frac{1}{180}\mathcal{R}_{\mu\nu}^2 + \frac{1}{72}\mathbf{R}^2\right\rangle_{x,g}, \\ \Gamma_{L,1}^{\mathrm{div}} &= \frac{1}{(4\pi)^2\varepsilon}\left\langle\frac{1}{6}F_{\mu\nu}^2 - \frac{11}{360}R_{\mu\nu\alpha\beta}^2 - \frac{1}{90}\mathcal{R}_{\mu\nu}^2 + \frac{1}{36}\mathbf{R}^2\right\rangle_{x,g}. \end{aligned} \tag{6.5}$$

B. Results for Γ_S

1. Contributions to Γ_S

The UV divergent part of Γ_S in d dimensions is contained in $\sum_{n=1}^{d}\Gamma_{S,n}$, where $\Gamma_{S,n}$ is given in Eq. (4.27). The quantity K_L, defined in Eq. (4.21), is homogeneous in ∇ of degree $+2$, while $K_S \equiv K - K_L$ contains terms of degrees 0 and 1. We will expand $\Gamma_{S,n}$ in powers of ∇ keeping terms up to $O(\nabla^{-4})$, which is sufficient for $\Gamma_1^{\rm div}$ in spacetime dimensions $d \le 4$. The trace cyclic property can be used to collect equivalent terms and also we choose whenever possible to bring the trace to the scalar-coordinate space.

Introducing the notation

$$\Delta \equiv \frac{1}{\nabla^2}, \qquad \Delta_M \equiv \frac{1}{\nabla_M^2}, \tag{6.6}$$

this procedure yields the following expressions:

$$\begin{aligned}
\Gamma_{S,1} &= \mathrm{Tr}_1\left(-\frac{1}{2}\Delta Y_{\mu\nu}\right) + \mathrm{Tr}_0\left(-\frac{1}{2}\Delta_M W - \frac{1}{4}\Delta_M\{P_\mu,\nabla_\mu\}\right),\\
\Gamma_{S,2} &= \mathrm{Tr}_1\left(-\frac{1}{4}\Delta Y_{\mu\alpha}\Delta Y_{\alpha\nu}\right) + \mathrm{Tr}_0\left(\frac{1}{2}\Delta_M\Phi_\mu\Delta\Phi_\mu + \frac{1}{2}\Delta_M\nabla_\mu\mathcal{M}_{\mu\nu}\Delta\mathcal{M}_{\nu\alpha}\nabla_\alpha\right.\\
&\quad -\frac{1}{2}\Delta_M\Phi_\mu\Delta\mathcal{M}_{\mu\nu}\nabla_\nu - \frac{1}{2}\Delta_M\nabla_\mu\mathcal{M}_{\mu\nu}\Delta\Phi_\nu\\
&\quad \left.-\frac{1}{4}\Delta_M W\Delta_M W - \frac{1}{16}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\{P_\nu,\nabla_\nu\} - \frac{1}{4}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M W\right),\\
\Gamma_{S,3} &= \mathrm{Tr}_0\left(\frac{1}{2}\Delta_M\nabla_\mu\mathcal{M}_{\mu\nu}\Delta Y_{\nu\alpha}\Delta\mathcal{M}_{\alpha\beta}\nabla_\beta + \frac{1}{2}\Delta_M W\Delta_M\nabla_\mu\mathcal{M}_{\mu\nu}\Delta\mathcal{M}_{\nu\alpha}\nabla_\alpha + \frac{1}{4}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\nabla_\nu\mathcal{M}_{\nu\alpha}\Delta\mathcal{M}_{\alpha\beta}\nabla_\beta\right.\\
&\quad -\frac{1}{4}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\nabla_\nu\mathcal{M}_{\nu\alpha}\Delta\Phi_\alpha - \frac{1}{4}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\Phi_\nu\Delta\mathcal{M}_{\nu\alpha}\nabla_\alpha\\
&\quad \left.-\frac{1}{8}\Delta_M W\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\{P_\nu,\nabla_\nu\} - \frac{1}{48}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\{P_\nu,\nabla_\nu\}\Delta_M\{P_\alpha,\nabla_\alpha\} + O(\nabla^{-5})\right),\\
\Gamma_{S,4} &= \mathrm{Tr}_0\left(-\frac{1}{4}\Delta_M\nabla_\mu\mathcal{M}_{\mu\nu}\Delta\mathcal{M}_{\nu\alpha}\nabla_\alpha\Delta_M\nabla_\beta\mathcal{M}_{\beta\rho}\Delta\mathcal{M}_{\rho\sigma}\nabla_\sigma + \frac{1}{8}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\{P_\nu,\nabla_\nu\}\Delta_M\nabla_\alpha\mathcal{M}_{\alpha\beta}\Delta\mathcal{M}_{\beta\rho}\nabla_\rho\right.\\
&\quad \left.-\frac{1}{128}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\{P_\nu,\nabla_\nu\}\Delta_M\{P_\alpha,\nabla_\alpha\}\Delta_M\{P_\beta,\nabla_\beta\} + O(\nabla^{-5})\right).
\end{aligned} \tag{6.7}$$

The functional traces are of the form

$$\mathrm{Tr}_r(A) = \int d^dx\sqrt{g}\,\mathrm{tr}_r(\langle x|A|x\rangle), \quad r = 0, 1, \tag{6.8}$$

where d is the spacetime dimension and tr_0 or tr_1 refer to the trace over coordinate labels, scalar or vector, respectively, and also include trace over gauge labels. In detail, the diagonal (in x space) matrix element $\langle x|A|x\rangle$ is of the form $\langle x, I', a|A|x, I, b\rangle$, where a, b are gauge labels in the adjoint representation and I, I' are coordinate labels. For $\mathrm{Tr}_0()$ these coordinate labels are absent, while for $\mathrm{Tr}_1()$ they are of the type $I = \mu$, $I' = \nu$.

At this point, one could already attempt the computation of the functional traces to obtain $\Gamma_S^{\rm div}$. Nevertheless, it is convenient to first simplify the expressions by bringing the operators to a canonical form. This is in the same spirit as the universal functional traces of [18]. The goal is to put together terms involving powers of ∇ (to wit, ∇_μ, Δ, and Δ_M) on one side and the purely multiplicative terms on another. That is, bring the various operators in (6.7) to the form

$$A = \sum_n \mathcal{O}_n A_n, \tag{6.9}$$

where $\mathcal{O}_n$ form a basis of pseudodifferential operators and A_n are purely multiplicative operators. We have chosen to put the latter on the right-hand side. Thus, for the diagonal matrix elements,

$$\langle x,I',a|A|x,I,b\rangle = \sum_{n,c}\langle x,I',a|\mathcal{O}_n|x,I,c\rangle(A_n(x))_{cb}, \tag{6.10}$$

or just $\langle x|A|x\rangle = \sum_n\langle x|\mathcal{O}_n|x\rangle A_n(x)$. The coefficients A_n do not modify the UV degree of divergence of the term, which is controlled by $\mathcal{O}_n$, so the hardest work is computing $\langle x|\mathcal{O}_n|x\rangle$.

To obtain the expansion in Eq. (6.9), we apply the following commutation identities:

$$\begin{aligned}
[\nabla_\mu, X_I] &= X_{\mu I},\\
[\Delta, X_I] &= \Delta(X_{\mu\mu I} - 2\nabla_\mu X_{\mu I})\Delta,\\
[\Delta, \nabla_\mu] &= \Delta\left(2\nabla_\nu Z_{\mu\nu} - \nabla_\nu \mathcal{R}_{\mu\nu} + Z_{\nu\nu\mu} + \frac{1}{4}\mathbf{R}_\mu\right)\Delta,\\
[\Delta_M, \nabla_\mu] &= \Delta_M(\nabla_\alpha\nabla_\beta M_{\mu\alpha\beta} - \nabla_\alpha M_{\beta\mu\beta\alpha} + \nabla_\alpha M_{\alpha\beta} Z_{\mu\beta} + Z_{\mu\alpha}\nabla_\beta M_{\alpha\beta} - Z_{\mu\alpha} M_{\beta\beta\alpha})\Delta_M,\\
[\Delta, Z_I^R] &= \Delta(\nabla_\mu R_{\nu I\mu\nu} - 2\nabla_\mu Z^R_{\mu I} + Z^R_{\mu\mu I})\Delta,\\
[\Delta_M, Z_I^R] &= \Delta_M\left(\nabla_\mu M_{\mu\nu}(R_{\alpha I\nu\alpha} - 2Z^R_{\nu I}) + M_{\mu\nu}\left(Z^R_{\mu\nu I} - \frac{1}{2}R_{\mu\alpha I\nu\alpha} - \frac{1}{2}R_{\alpha\mu I\nu\alpha}\right) + M_{\mu\mu\nu}\left(Z^R_{\nu I} - \frac{1}{2}R_{\alpha I\nu\alpha}\right)\right)\Delta_M.
\end{aligned} \tag{6.11}$$

In these formulas $I = \mu_1 \cdots \mu_n$ represents any (possibly empty) string of coordinate indices (e.g., $\alpha\beta$) and μI the new string adding μ to the left (e.g., $\mu\alpha\beta$). X_I represents a purely multiplicative operator, that is, any coordinate tensor (in particular, may be a coordinate scalar) and a matrix in gauge space, not involving ∇_μ nor Z_I^R.

The usefulness of the commutation relations (6.11) is that the true degree of UV divergence of the operator on the left-hand side is actually smaller than the nominal one. So, for instance, $[\Delta, X_I]$ would be nominally of $O(\nabla^{-2})$, while this commutator is actually of order $O(\nabla^{-3})$. Note that Z_I and Z_I^R count as degree $O(\nabla^0)$ as they do not add to the UV degree of divergence of a term, as follows from the property in Eq. (B4).

A very conspicuous and relevant absence in the list of commutators is the combination $[\Delta_M, X_I]$. This is not in the list because that commutator is still of $O(\nabla^{-2})$ unless $[X_I, M_{\mu\nu}] = 0$. This absence prevents one from putting together all terms involving ∇ and is a consequence of the non-Abelian character of the theory.

For Γ_S, the operators required in the basis are

$$\begin{aligned}
(\mathcal{O}_1)_\mu &= \Delta_M\nabla_\mu, & \mathcal{O}_2 &= \Delta_M,\\
(\mathcal{O}_3)_{\mu\nu} &= \Delta_M\Delta\nabla_\mu\nabla_\nu, & (\mathcal{O}_4[X])_{\mu\nu} &= \Delta_M X\Delta_M\nabla_\mu\nabla_\nu,\\
(\mathcal{O}_5)_\mu &= \Delta_M\Delta\nabla_\mu, & (\mathcal{O}_6)_{\mu\nu\alpha} &= \Delta_M\Delta^2\nabla_\mu\nabla_\nu\nabla_\alpha,\\
(\mathcal{O}_7[X])_\mu &= \Delta_M X\Delta_M\nabla_\mu, & (\mathcal{O}_8[X])_{\mu\nu\alpha} &= \Delta_M X\Delta_M\Delta\nabla_\mu\nabla_\nu\nabla_\alpha,\\
(\mathcal{O}_9[X,X'])_{\mu\nu\alpha} &= \Delta_M X\Delta_M X'\Delta_M\nabla_\mu\nabla_\nu\nabla_\alpha, & \mathcal{O}_{10} &= \Delta_M\Delta,\\
(\mathcal{O}_{11})_{\mu\nu} &= \Delta_M\Delta^2\nabla_\mu\nabla_\nu, & (\mathcal{O}_{12})_{\mu\nu\alpha\beta} &= \Delta_M\Delta^3\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta,\\
\mathcal{O}_{13}[X] &= \Delta_M X\Delta_M, & (\mathcal{O}_{14}[X])_{\mu\nu} &= \Delta_M X\Delta_M\Delta\nabla_\mu\nabla_\nu,\\
(\mathcal{O}_{15}[X])_{\mu\nu\alpha\beta} &= \Delta_M X\Delta_M\Delta^2\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta, & (\mathcal{O}_{16}[X,X'])_{\mu\nu} &= \Delta_M X\Delta_M X'\Delta_M\nabla_\mu\nabla_\nu,\\
(\mathcal{O}_{17}[X,X'])_{\mu\nu\alpha\beta} &= \Delta_M X\Delta_M X'\Delta_M\Delta\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta, & (\mathcal{O}_{18}[X,X',X''])_{\mu\nu\alpha\beta} &= \Delta_M X\Delta_M X'\Delta_M X''\Delta_M\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta,\\
\mathcal{O}_{19} &= \Delta, & \mathcal{O}_{20} &= \Delta^2.
\end{aligned} \tag{6.12}$$

Here X, X', and X'' are arbitrary purely multiplicative operators, possibly with coordinate indices. The presence of operators in the basis with such insertions of purely multiplicative operators is a direct consequence of $[\Delta_M, X]$ being of $O(\nabla^{-2})$ in the non-Abelian case. Nevertheless, the computation of the diagonal matrix elements of the operators $\mathcal{O}_n$ can be done for generic X, X', and X''.

The operators $\mathcal{O}_1$–$\mathcal{O}_{18}$ are required for the terms $\mathrm{Tr}_0()$ of Γ_S in $d = 4$ dimensions, $\mathcal{O}_{19}$ and $\mathcal{O}_{20}$ appear in the terms $\mathrm{Tr}_1()$. The explicit expansion of the operators in Γ_S, Eq. (6.7), in the basis (6.12) is presented in Appendix C.

The traces indicated in Eq. (6.7) can then be obtained from

$$\mathrm{Tr}_r(A) = \left\langle \sum_n \mathrm{tr}_r(\langle x|\mathcal{O}_n|x\rangle A_n(x)) \right\rangle_x, \quad r = 0, 1. \tag{6.13}$$

The diagonal matrix elements of the operators $\mathcal{O}_n$ can be regarded as a generalization of the well-known "universal functional traces" introduced in [18] and computed there using a Schwinger-DeWitt technique. The same technique

was adapted in [34] for the Abelian case, where $M^{\mu\nu}$ acts effectively as a second metric. Our problem involves considerably more complicated operators since now such effective metric is gauge non-Abelian. We will use the method of covariant symbols, already applied in [35] in the Abelian setting. Of course, it would be interesting to adapt the Schwinger-DeWitt approach to the present non-Abelian case.

2. $\Gamma_S^{\rm div}$ in $d=2$

Details of the calculation of the divergent part of the diagonal matrix elements will be given in Sec. VII. However, from the data displayed in Table I, we already anticipate that the number of terms contributing to $\Gamma_S^{\rm div}$ is very large for $d=4$ (namely, 265 terms from $\mathcal{O}_1$, 123 from $\mathcal{O}_4$, etc). In view of this, we will only display explicit results for $d=2$.

In $d=2$, Eq. (6.7) reduces to

$$\Gamma_{S,1}=\mathrm{Tr}_1\left(-\frac{1}{2}\Delta Y_{\mu\nu}\right)+\mathrm{Tr}_0\left(-\frac{1}{2}\Delta_M W-\frac{1}{4}\Delta_M\{P_\mu,\nabla_\mu\}\right),$$
$$\Gamma_{S,2}=\mathrm{Tr}_0\left(\frac{1}{2}\Delta_M\nabla_\mu\mathcal{M}_{\mu\nu}\Delta\mathcal{M}_{\nu\alpha}\nabla_\alpha -\frac{1}{16}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\{P_\nu,\nabla_\nu\}+O(\nabla^{-3})\right). \quad (6.14)$$

This can be rewritten using the basis of operators $\mathcal{O}_n$ in Eq. (6.12) (see Appendix C),

$$\Gamma_{S,1}=\mathrm{Tr}_1\left(-\frac{1}{2}\mathcal{O}_{19}Y_{\mu\nu}\right)+\mathrm{Tr}_0\left(-\frac{1}{2}\mathcal{O}_2 W-\frac{1}{2}(\mathcal{O}_1)_\mu P_\mu +\frac{1}{4}\mathcal{O}_2 P_{\mu\mu}+O(\nabla^{-3})\right),$$
$$\Gamma_{S,2}=\mathrm{Tr}_0\left(\frac{1}{2}(\mathcal{O}_3)_{\mu\nu}\mathcal{M}_{\mu\alpha}\mathcal{M}_{\alpha\nu}-\frac{1}{4}(\mathcal{O}_4[P_\mu])_{\mu\nu}P_\nu+O(\nabla^{-3})\right). \quad (6.15)$$

Details of the calculation are given in Sec. VII B. The result is[9]

$$\Gamma_{S,1}^{\rm div}=\frac{1}{4\pi\varepsilon}\left\langle -\frac{1}{2}Y_{\mu\mu}-\frac{1}{2}\hat{N}_M W+\frac{1}{4}\hat{N}_M P_{\mu\mu} -\frac{1}{2}\hat{N}_M M_{\mu\nu}\hat{N}_M M_{\mu\alpha\beta}\hat{N}_M P_\lambda\hat{k}_\nu\hat{k}_\alpha\hat{k}_\beta\hat{k}_\lambda +\frac{1}{2}\hat{N}_M M_{\mu\alpha\beta}\hat{N}_M M_{\mu\nu}\hat{N}_M P_\lambda\hat{k}_\nu\hat{k}_\alpha\hat{k}_\beta\hat{k}_\lambda\right\rangle_{x,g,\mathrm{ang}},$$
$$\Gamma_{S,2}^{\rm div}=\frac{1}{4\pi\varepsilon}\left\langle\frac{1}{2}\hat{N}_M\mathcal{M}_{\mu\alpha}\mathcal{M}_{\alpha\nu}\hat{k}_\mu\hat{k}_\nu-\frac{1}{4}\hat{N}_M P_\mu\hat{N}_M P_\nu\hat{k}_\mu\hat{k}_\nu\right\rangle_{x,g,\mathrm{ang}}. \quad (6.16)$$

[9]Note that $Y_{\mu\mu}=M_{\mu\mu}+\mathbf{R}$, $P_{\mu\mu}=\frac{1}{2}[F_{\mu\nu},Q_{\mu\nu}]$.

TABLE I. For the various operators in Eq. (6.12), the second column displays the UV divergence degree, while the columns labeled with $-2,-4,-6$ show the number of terms of each degree in the expansion of the diagonal matrix element of the operator.

Operator	Degree	-2	-4	-6
$\mathcal{O}_1$	-1	2	265	
$\mathcal{O}_2$	-2	1	11	3303
$\mathcal{O}_3$	-2	1	44	
$\mathcal{O}_4$	-2	1	123	
$\mathcal{O}_{5,6}$	-3		2	
$\mathcal{O}_{7,8}$	-3		6	
$\mathcal{O}_9$	-3		12	
$\mathcal{O}_{10\text{-}18}$	-4		1	
$\mathcal{O}_{19}$	-2	1	1	
$\mathcal{O}_{20}$	-4		1	

Here $\hat{k}_\mu$ is a normalized momentum variable, $g^{\mu\nu}\hat{k}_\mu\hat{k}_\nu=1$, whereas

$$\hat{N}_M\equiv(\hat{k}_\mu\hat{k}_\nu M^{\mu\nu})^{-1}. \quad (6.17)$$

The symbol $\langle\rangle_{\rm ang}$ denotes angular average over $\hat{k}_\mu$,

$$\langle X\rangle_{\rm ang}\equiv\frac{\Gamma(d/2)}{2\pi^{d/2}}\int d^{d-1}\Omega_{\hat{k}}X. \quad (6.18)$$

This average is to be applied together with $\langle\rangle_{x,g}$, already introduced in Sec. V C.

Equation (6.16) is our final result for $\Gamma_S^{\rm div}$ in $d=2$. The expression for $d=4$ is qualitatively similar, but considerably longer. Further perturbative results are given in Appendix D.

While the angular averages in Eq. (6.16) cannot be evaluated in closed form in general, they are perfectly convergent and well defined. In any case, these integrals introduce a wild local but nonpolynomial dependence on the field $M_{ab}^{\mu\nu}(x)$ in the divergent part of the effective action, implying that the UV divergences cannot be removed by polynomial counterterms, rendering the theory not renormalizable in a standard sense. This was true already when $N=1$, the Abelian case studied in [32–35].

C. Results for $\Gamma_{L,0}$

1. Preliminaries

The remaining contribution to the effective action is $\Gamma_{L,0}$,

$$\Gamma_{L,0}=\frac{1}{2}\mathrm{Tr}_0\log(-\nabla_M^2)=\frac{1}{2}\mathrm{Tr}_0\log(-\nabla_\alpha M^{\alpha\beta}\nabla_\beta). \quad (6.19)$$

This is the effective action of a scalar field with action

$$S_{L,0}=\int d^dx\sqrt{g}\frac{1}{2}\phi_\mu^a M_{ab}^{\mu\nu}\phi_\nu^b. \quad (6.20)$$

In the special case of $N = 1$, that is, when there is no gauge sector (the case considered in [32–35]), one can combine $g_{\mu\nu}$ and $M^{\alpha\beta}$ into a new metric $\tilde{g}_{\mu\nu}$, namely, $\sqrt{g}M^{\mu\nu} = \sqrt{\tilde{g}}\tilde{g}^{\mu\nu}$ [34,41], in such a way that

$$S_{L,0} = \int d^dx\sqrt{\tilde{g}}\frac{1}{2}\phi_\mu\tilde{g}^{\mu\nu}\phi_\nu \quad (N = 1). \tag{6.21}$$

Hence, $\Gamma_{L,0} = \frac{1}{2}\mathrm{Tr}_0\log(-\tilde{\nabla}^2)$ and the heat-kernel result (6.1) immediately applies. Note that this method works in any spacetime dimension except $d = 2$, since in that case $\det(\sqrt{\tilde{g}}\tilde{g}_{\mu\nu}) \equiv 1$.

Unfortunately, no such simplification takes place for $N > 1$. In fact the situation is even worse, namely, $\Gamma^{\mathrm{div}}_{L,0}$ does not admit a standard form, like that displayed in Eq. (7.18). By definition, we say that the terms in an expansion of a diagonal matrix element $\langle x|\mathcal{O}|x\rangle$ adopt a "standard form" when all the pieces are covariant with no "free" ∇_μ operators (all ∇_μ are in the form $[\nabla_\mu,]$) and only remains to carry out a momentum integration. Also, labeled operators [42] are not permitted in a standard form. The momentum integration can be traded by another parameter, e.g., a proper time as in a Schwinger parametrization.

While a standard form can always be achieved for matrix elements of the type $\langle x|f(\nabla, X)|x\rangle$ (X being purely multiplicative operators) when the dependence on ∇_μ is of rational type, this is not guaranteed for more general functions f. However, it is often the case that a standard form exists for a pseudodifferential operator of the type $\log(\hat{f})$ with $\hat{f}$ a differential operator. A well-known instance of standard form for a Trlog is that found by Chan (for flat spacetime to fourth order in a derivative expansion) [43]. To second order,

$$\begin{aligned}&\mathrm{Tr}\log(-\nabla_\mu^2 + X)\\ &= \left\langle \int \frac{d^dk}{(2\pi)^d}\left(-\log(N_C) + \frac{k^2}{d}[\nabla_\mu, N_C]^2 + O(\nabla^4)\right)\right\rangle_{x,g},\\ &N_C \equiv (k^2 + X)^{-1}.\end{aligned} \tag{6.22}$$

Extensions of this formula exist to sixth order [44], also for curved spacetime [45] and for finite temperature [46]. Yet the techniques applied in those cases cannot be translated to evaluating $\Gamma_{L,0}$ because in the present case what plays the role of a metric, $M^{\mu\nu}$, is actually a nontrivial matrix in gauge space and appears already in the leading term $\log(k_\mu k_\nu M^{\mu\nu})$. The obstruction to a standard form does not depend on the method used to compute the effective action; it is intrinsic to $\Gamma_{L,0}$ in the non-Abelian case.

2. Special case of separable mass term

Here we mention the special case of a mass term separable in coordinate and gauge spaces, that is,

$$M^{\mu\nu}_{ab}(x) = T_{ab}(x)G^{\mu\nu}(x). \tag{6.23}$$

The fluctuation operator can then be written as

$$\begin{aligned}\nabla_\alpha M^{\alpha\beta}\nabla_\beta &= \nabla_\alpha T G^{\alpha\beta}\nabla_\beta = T\nabla_\alpha G^{\alpha\beta}\nabla_\beta + T_\alpha G^{\alpha\beta}\nabla_\beta\\ &= T(\nabla_G^2 + L^\alpha\nabla_\alpha),\end{aligned} \tag{6.24}$$

where we have defined

$$\nabla_G^2 \equiv \nabla_\alpha G^{\alpha\beta}\nabla_\beta, \qquad L^\mu \equiv T^{-1}T_\nu G^{\nu\mu}. \tag{6.25}$$

Now, the factor $T(x)$ can be dropped from $\mathrm{Tr}(\log(-\nabla_M^2))$, being ultralocal (carries no derivative operators), hence

$$\begin{aligned}\Gamma_{L,0} &= \frac{1}{2}\mathrm{Tr}_0\log(-\nabla_G^2 - L^\alpha\nabla_\alpha)\\ &= \frac{1}{2}\mathrm{Tr}_0\log(-\nabla_G^2) + \frac{1}{2}\mathrm{Tr}_0\log(1 + \Delta_G L^\alpha\nabla_\alpha),\end{aligned} \tag{6.26}$$

with $\Delta_G = 1/\nabla_G^2$. The first term is just like $\Gamma_{L,0}$ of the Abelian case $N = 1$ (with $G^{\mu\nu}$ instead of $M^{\mu\nu}$). The second term can be expanded in powers of $\Delta_G L^\alpha\nabla_\alpha$, similarly as done for Γ_S in (4.26). So this contribution does admit a standard form.

It is noteworthy that there is an ambiguity in the separation (6.23) of the Weyl-transformation type, namely,

$$G^{\mu\nu}(x) \to \lambda(x)G^{\mu\nu}(x), \qquad T_{ab}(x) \to \lambda^{-1}(x)T_{ab}(x), \tag{6.27}$$

where $\lambda(x)$ is local but both scalar coordinate and gauge singlet. Such ambiguity can be used to fix some gauge condition on $G^{\mu\nu}(x)$ or $T_{ab}(x)$, or as a check of the calculation, since the sum of the two terms in (6.26) should be λ independent.

We do not pursue the subject of the separable mass case any further in this work.

3. Method of contour integration

Coming back to the case of a general mass term $M^{\mu\nu}_{ab}(x)$, to deal with the logarithm we follow here the approach of introducing a parametric integral. This is based on the observation that a standard form would easily follow for a rational function instead of the log.

The contour integration method is based on the identity

$$\Gamma_{L,0} = \int_\gamma \frac{dz}{2\pi i}\log(z)\frac{1}{2}\mathrm{Tr}_0\left(\frac{1}{z + \nabla_M^2}\right), \tag{6.28}$$

where the path γ is meant to enclose counterclockwise the spectrum of $-\nabla_M^2$. More precisely, the path γ on the z complex plane starts at $-\infty$ toward the origin just above the negative real semiaxis, encircles $z = 0$ clockwise, and goes

back to $-\infty$ just below the negative real semiaxis.[10] Hence, introducing a convenient notation,

$$\langle X\rangle_z \equiv \int_\gamma \frac{dz}{2\pi i}\log(z)X, \qquad \mathcal{O}'_2 \equiv \frac{1}{z+\nabla_M^2}, \tag{6.29}$$

we can express Eq. (6.28) as

$$\Gamma_{L,0} = \left\langle \frac{1}{2}\mathrm{Tr}_0(\mathcal{O}'_2)\right\rangle_z. \tag{6.30}$$

The calculation of the diagonal matrix element $\langle x|\mathcal{O}'_2|x\rangle$ is formally identical to that of $\langle x|\mathcal{O}_2|x\rangle$; unfortunately, the presence of the z integration implies that the relevant quantity $\langle\mathcal{O}'_2\rangle_z = \log(-\nabla_M^2)$ is actually of degree $O(\nabla^0)$ instead of $O(\nabla^{-2})$. Hence, the amount of work required to evaluate $\langle x|\langle\mathcal{O}'_2\rangle_z|x\rangle$ is similar to that of $\langle x|\mathcal{O}_2|x\rangle$ for $d+2$ dimensions to achieve the same degree of UV divergence (see Sec. VII C).

The convergence problem would not improve if instead of a z integration the sought-for rational dependence on ∇_μ was obtained through the variation of $\Gamma_{L,0}$ under a deformation of $M^{\mu\nu}$,

$$\delta\Gamma_{L,0} = \frac{1}{2}\mathrm{Tr}_0(\Delta_M\nabla_\mu\delta M^{\mu\nu}\nabla_\nu), \tag{6.31}$$

as this requires one to evaluate $\langle x|\Delta_M\nabla_\mu\nabla_\nu|x\rangle$, which is still of $O(\nabla^0)$.

4. $\Gamma_{L,0}$ in $d=2$

Once again, the number of terms is prohibitively large in $d=4$ [it requires $\mathcal{O}_2$ to $O(p^{-6})$ in Table I] and we present results for $d=2$. The calculation gives

$$\begin{aligned}\Gamma^{\mathrm{div}}_{L,0} = \frac{1}{4\pi\varepsilon}\Big\langle & -\hat N_M M_{\mu\nu}\hat N_M M_{\mu\alpha\beta}\hat N_M M_{\rho\sigma}\hat N_M M_{\rho\lambda\eta}\hat N_M \hat k_\nu\hat k_\alpha\hat k_\beta\hat k_\sigma\hat k_\lambda\hat k_\eta - \hat N_M M_{\mu\nu\alpha}\hat N_M M_{\mu\beta}\hat N_M M_{\rho\sigma\lambda}\hat N_M M_{\rho\eta}\hat N_M\hat k_\nu\hat k_\alpha\hat k_\beta\hat k_\sigma\hat k_\lambda\hat k_\eta\\
&+\frac{1}{2}\hat N_M M_{\mu\nu}\hat N_M M_{\mu\alpha\beta}\hat N_M M_{\alpha\rho\sigma}\hat N_M\hat k_\nu\hat k_\beta\hat k_\rho\hat k_\sigma - \frac{1}{2}\hat N_M M_{\mu\nu}\hat N_M M_{\alpha\alpha\beta}\hat N_M M_{\mu\rho\sigma}\hat N_M\hat k_\nu\hat k_\beta\hat k_\rho\hat k_\sigma\\
&+\frac{1}{2}\hat N_M M_{\mu\nu\alpha}\hat N_M M_{\beta\mu\rho}\hat N_M M_{\beta\sigma}\hat N_M\hat k_\nu\hat k_\alpha\hat k_\rho\hat k_\sigma - \frac{1}{2}\hat N_M M_{\mu\nu\alpha}\hat N_M M_{\beta\beta\rho}\hat N_M M_{\mu\sigma}\hat N_M\hat k_\nu\hat k_\alpha\hat k_\rho\hat k_\sigma\\
&-\frac{1}{2}\hat N_M M_{\mu\nu}\hat N_M M_{\alpha\beta}\hat N_M M_{\mu\rho}\hat N_M\mathbf{R}\hat k_\nu\hat k_\alpha\hat k_\beta\hat k_\rho + \frac{1}{2}\hat N_M M_{\mu\mu\nu}\hat N_M M_{\alpha\alpha\beta}\hat N_M\hat k_\nu\hat k_\beta - \frac{1}{2}\hat N_M M_{\mu\nu}F_{\mu\alpha}\hat N_M M_{\alpha\beta}\hat N_M\hat k_\nu\hat k_\beta\\
&-\frac{1}{2}\hat N_M M_{\mu\nu}\hat N_M F_{\mu\alpha}M_{\alpha\beta}\hat N_M\hat k_\nu\hat k_\beta + \frac{1}{3}\hat N_M M_{\mu\nu}\hat N_M M_{\mu\alpha}\hat N_M\mathbf{R}\hat k_\nu\hat k_\alpha + \frac{1}{24}\hat N_M M_{\mu\mu}\hat N_M\mathbf{R}\Big\rangle_{x,g,\mathrm{ang},z}.\end{aligned} \tag{6.32}$$

In this formula $\hat N_M \equiv (-z+\hat k_\mu\hat k_\nu M^{\mu\nu})^{-1}$. Besides integration over x, trace over gauge space, and angular average, an integral over z, as defined in Eq. (6.29), is applied.

Integration by parts with respect to x has been used to have at most one covariant derivative on $M_{\mu\nu}$. Also the following two-dimensional identities have been used:

$$\begin{aligned}R_{\mu\nu\alpha\beta} &= \frac{1}{2}(g_{\mu\alpha}g_{\nu\beta}-g_{\mu\beta}g_{\nu\alpha})\mathbf{R},\\ \mathcal{R}_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}\mathbf{R}.\end{aligned} \tag{6.33}$$

In addition, integration by parts identities in momentum space have been applied to bring the expression to a manifestly Hermitian form.

Perturbative expansions are presented in Appendix D.

[10]The ζ function result would follow from understanding $\log(z)$ as $-\frac{dz^{-s}}{ds}$ at $s=0$ in the sense of analytical continuation in the s variable, taking the derivative with respect to s only after the Tr() has been computed [47].

5. Verification of metric-related symmetries in $\Gamma_{L,0}$

The Weyl-like transformation noted at the end of Sec. II, $g_{\mu\nu}(x)\to\xi(x)g_{\mu\nu}(x)$, $\mathcal{M}^{\mu\nu}_{ab}(x)\to\xi^{-2}(x)\mathcal{M}^{\mu\nu}_{ab}(x)$, is a symmetry of the full action only in $d=4$. However, in any spacetime dimension, $\Gamma_{L,0}$ has a large symmetry as this term only depends on the pair $(g_{\mu\nu},M^{\mu\nu})$ through the combination $\sqrt{g}M^{\mu\nu}_{ab}$. This follows from (6.20) since ϕ^a are coordinate scalars and so ϕ^a_μ does not depend on the metric tensor.

It is convenient to distinguish two types of symmetry transformations leaving invariant $\Gamma_{L,0}$, which will be called "transverse" and "longitudinal," respectively,

(i) A transverse transformation corresponds to leaving $M^{\mu\nu}$ intact, while $g_{\mu\nu}$ changes arbitrarily but keeping $\sqrt{g}$ invariant, thus preserving the volume element.

(ii) A longitudinal transformation corresponds to the simultaneous change

$$\begin{aligned}g_{\mu\nu}(x) &\to \xi(x)g_{\mu\nu}(x),\\ M^{\mu\nu}_{ab}(x) &\to \xi^{-d/2}(x)M^{\mu\nu}_{ab}(x),\end{aligned} \tag{6.34}$$

in d spacetime dimensions.

We have checked that both symmetries are fulfilled by the expression in Eq. (6.32). It is sufficient to consider the infinitesimal case.[11] For *transverse* transformations, the first order variations are

$$\begin{aligned}
\delta g_{\mu\nu} &= \omega_{\mu\nu} \quad \text{with} \quad \omega^{\lambda}{}_{\lambda} = 0,\\
\delta M^{\mu\nu} &= \delta p_{\mu} = \delta F_{\mu\nu} = \delta N_M = 0,\\
\delta M_{\alpha}{}^{\mu\nu} &= \frac{1}{2}(\omega_{\alpha}{}^{\mu}{}_{\lambda} + \omega_{\lambda}{}^{\mu}{}_{\alpha} - \omega^{\mu}{}_{\alpha\lambda})M^{\lambda\nu}\\
&\quad + \frac{1}{2}(\omega_{\alpha}{}^{\nu}{}_{\lambda} + \omega_{\lambda}{}^{\nu}{}_{\alpha} - \omega^{\nu}{}_{\alpha\lambda})M^{\mu\lambda},\\
\delta(g_{\mu\nu}\mathbf{R}) &= \omega^{\lambda}{}_{\mu\lambda\nu} + \omega^{\lambda}{}_{\nu\lambda\mu} - \omega^{\lambda}{}_{\lambda\mu\nu} \quad (d=2).
\end{aligned} \tag{6.35}$$

It should be noted that the metric does not appear explicitly anywhere in Eq. (7.21), and the metric is only contained in $R_{\mu\nu}{}^{\alpha}{}_{\beta}$, $\mathcal{R}_{\mu\nu}$, and the covariant derivatives. This is because the same is true for the covariant symbols of ∇_μ and the multiplicative operators, and no explicit metric appears in the original expression of $\mathcal{O}'_2$. After using the two-dimensional identities (6.33) the metric does enter in Eq. (7.22) in the combination $g_{\mu\nu}\mathbf{R}$. In $d=2$, the variation of this quantity depends only on *derivatives* of $\omega_{\mu\nu}$. Also note that the integral $\langle\rangle_{x,p}$ does not contain a net $\sqrt{g}$. When the above first order variations are applied to (7.22), and after integration by parts in x space to remove terms with $\omega_{\mu\nu\alpha\beta}$ (two derivatives), one obtains an expression which vanishes upon integration by parts in p space. We have checked this to third order in an expansion of the type $M^{\mu\nu} = m^2 g^{\mu\nu} + H^{\mu\nu}$ in powers of $H^{\mu\nu}$. The integration over z plays no role in the cancellation of terms.

For first order *longitudinal* variations, one has instead (in $d=2$),

$$\begin{aligned}
&\delta g_{\mu\nu} = w g_{\mu\nu}, \qquad \delta p_{\mu} = \delta F_{\mu\nu} = 0, \qquad \delta M^{\mu\nu} = -w M^{\mu\nu},\\
&\delta M_{\alpha}{}^{\mu\nu} = -w M_{\alpha}{}^{\mu\nu} + \frac{1}{2} w_{\lambda} g^{\mu}{}_{\alpha} M^{\lambda\nu} + \frac{1}{2} w_{\lambda} g^{\nu}{}_{\alpha} M^{\mu\lambda}\\
&\qquad - \frac{1}{2} w^{\mu} M_{\alpha}{}^{\nu} - \frac{1}{2} w^{\nu} M^{\mu}{}_{\alpha},\\
&\delta(g_{\mu\nu}\mathbf{R}) = -w^{\lambda}{}_{\lambda}.
\end{aligned} \tag{6.36}$$

The terms with w without a covariant derivative correspond to global transformations. It is easy to verify that (7.22) remains invariant under global longitudinal transformations by applying the simultaneous rescaling $z \to \xi^{-1} z$ (which in turn implies $N_M \to \xi N_M$ or $\delta N_M = w N_M$). Therefore, it is only necessary to keep terms with w_μ and $w_{\mu\nu}$ in the variation of (7.22). After variation and integration by parts in x to remove terms with $w_{\mu\nu}$ (two derivatives), one finds that the result cancels upon integration over p. Once again we have checked this through third order in an expansion in powers of $H^{\mu\nu}$.

The cancellation of transverse and longitudinal variations of the effective action is a highly nontrivial check of Eq. (6.32).

VII. COMPUTATION OF THE DIAGONAL MATRIX ELEMENTS

A. Method of covariant symbols

The diagonal matrix elements can be computed using the method of covariant symbols. This method was introduced in [48] for flat spacetime and extended to curved spacetime in [37] where it is described in great detail. It has also been extended to finite temperature in [46,49]. A summary can be found in [35] (Sec. III. 4 and Appendix B). Nevertheless, as the method is not widely known, and to have a more self-contained work, we give some details here.

For an operator $\hat{\mathcal{O}}$, constructed with ∇_μ plus some purely multiplicative fields, such as those in the basis (6.12), its covariant symbol will be denoted $\bar{\mathcal{O}}$. This quantity is obtained by applying two successive similarity transformations[12]

$$\bar{\mathcal{O}} := e^{-\frac{1}{2}\{\nabla_\mu, \partial^\mu\}} e^{-\xi^\alpha p_\alpha} \hat{\mathcal{O}} e^{\xi^\beta p_\beta} e^{\frac{1}{2}\{\nabla_\nu, \partial^\nu\}}\big|_{\xi^\mu = 0}. \tag{7.1}$$

Here $\{,\}$ denotes the anticommutator, ξ^μ are the Riemann coordinates corresponding to the affine connection in ∇_μ located at the point x where the diagonal matrix element $\langle x|\hat{\mathcal{O}}|x\rangle$ will be evaluated, p_μ is a momentum variable, and $\partial^\mu \equiv \partial/\partial p_\mu$. For convenience, we use a purely imaginary variable, $p_\mu = i k_\mu$, $k \in \mathbb{R}^d$ but $d^d p \equiv d^d k$. This definition of covariant symbols holds for ∇_μ having any affine connections (e.g., with torsion); here we assume the Levi-Civita connection for the coordinate indices, plus the gauge connection.

The original operator $\hat{\mathcal{O}}$ acts only in x space (and possibly in some internal space), while its covariant symbol $\bar{\mathcal{O}}$ acts both on x and p spaces. The first remarkable property of $\bar{\mathcal{O}}$ is that this operator is multiplicative with respect to x (although not with respect to p). This means that it commutes with functions of x which are coordinate scalar and gauge singlet.

The second important property of the map $\hat{\mathcal{O}} \mapsto \bar{\mathcal{O}}$ is that, being a similarity transformation, it is a faithful algebra homomorphism (which also preserves Hermiticity). Therefore, the covariant symbol of a pseudodifferential operator of the type $f(\nabla_\mu, X)$ is simply $f(\bar{\nabla}_\mu, \bar{X})$. This implies that it is sufficient to obtain (once and for all) the covariant symbol of a few basic blocks and, once this is done, there is no need to go back to the original definition in Eq. (7.1). For instance,

[11] Here we need to refer to results to be established in Sec. VII C.

[12] This formula is schematic, see full construction in [37].

$$\bar{\nabla}_\mu = p_\mu - \frac{1}{4}\{Z_{\alpha\mu}, \partial^\alpha\} + \frac{1}{12}\{[Z_{\alpha\mu}, p_\beta], \partial^\alpha\partial^\beta\} + O(p^{-2}),$$
$$\bar{X} = X - X_\alpha\partial^\alpha + \frac{1}{2!}X_{\alpha\beta}\partial^\alpha\partial^\beta + O(p^{-3}),$$
$$\bar{Z}_{\mu\nu} = Z_{\mu\nu} - \frac{1}{2}\{Z_{\alpha\mu\nu}, \partial^\alpha\} + O(p^{-2}). \quad (7.2)$$

Extensive formulas can be found in [35,37]. As advertised, explicit ∇_μ are no longer present and the covariant symbol is a multiplicative operator. The expressions take the form of an expansion in powers of ∇/p, i.e., with terms ordered by the number of covariant derivatives or equivalently by powers of p_μ [counting ∂^μ as $O(p^{-1})$]. Here X is any purely multiplicative operator (i.e., not containing ∇_μ nor Z_I^R), so in particular,

$$\bar{M}_{\mu\nu} = M_{\mu\nu} - M_{\alpha\mu\nu}\partial^\alpha + \frac{1}{2!}M_{\alpha\beta\mu\nu}\partial^\alpha\partial^\beta + O(p^{-3}). \quad (7.3)$$

And, of course, $\bar{Z}_{\mu\nu} = [\bar{\nabla}_\mu, \bar{\nabla}_\nu]$ is verified.

It might seem that $Z_{\mu\nu}$ or $Z_{\alpha\mu\nu}$ would commute with p_λ or ∂^λ in Eq. (7.2), but in fact this is not so. Since p_μ and ∂^μ carry coordinate indices, Eq. (B4) applies and one has instead

$$[Z_{\mu\nu}, p_\alpha] = R_{\mu\nu\alpha\beta}p_\beta, \qquad [Z_{\mu\nu}, \partial^\alpha] = R_{\mu\nu\alpha\beta}\partial^\beta. \quad (7.4)$$

Equations (7.2) are written so that the Hermiticity properties are manifest. Note that the metric does not appear in the covariant symbols of ∇_μ or X; in fact, those formulas hold for a completely arbitrary torsionless connection.[13] The concrete connection will show up through the action of $Z_{\mu\nu}$, as in Eq. (5.9). In our case, we have also $\bar{g}_{\mu\nu} = g_{\mu\nu}$ since ∇_μ is metric preserving.

The last important property of the covariant symbols to be noted is their relation with diagonal matrix elements,

$$\langle x|\hat{\mathcal{O}}|x\rangle = \frac{1}{\sqrt{g}}\int \frac{d^d p}{(2\pi)^d}\bar{\mathcal{O}}\mathbf{1}. \quad (7.5)$$

Here the object $\mathbf{1}$ (which is usually not written explicitly) is the function identically equal to 1 in momentum space. In practice, this means that the operator ∂^μ in $\bar{\mathcal{O}}$ vanishes when it is placed on the right. On the other hand, ∂^μ is also zero when placed on the left due to the integral over p_μ.

The quantity $\bar{\mathcal{O}}\mathbf{1}$ (with all the ∂^μ already canceled by moving them to the right) is just a function of x and p and is closer to the standard definition of the symbol of a pseudodifferential operator. Hence, the function $\bar{\mathcal{O}}\mathbf{1}$ (multiplicative with respect to p_μ) is what is needed to obtain the diagonal matrix elements, but the full covariant symbol $\bar{\mathcal{O}}$ is the object carrying a faithful algebra representation.

[13]In this sense $\bar{\nabla}_\mu$ or $\bar{X}$ are truly universal.

If instead of Eq. (7.1) only the first similarity transformation is applied, $\hat{\mathcal{O}} \mapsto e^{-\xi^\alpha p_\alpha}\hat{\mathcal{O}}e^{\xi^\beta p_\beta}$, one obtains $f(\nabla, X) \mapsto f(\nabla + p, X)$, and the result is the method of noncovariant symbols,

$$\langle x|f(\nabla, X)|x\rangle = \frac{1}{\sqrt{g}}\int \frac{d^d p}{(2\pi)^d} f(\nabla + p, X). \quad (7.6)$$

The integrand is not a multiplicative operator (nor manifestly covariant) but it becomes so after integration over p_μ. What the additional similarity transformation achieves in Eq. (7.1) is precisely to have an integrand which is manifestly covariant by systematically applying integration by parts in p space. Further details on the method of noncovariant symbols are given in Appendix E.

The manifest covariance of $\bar{\mathcal{O}}$ (and hence $\bar{\mathcal{O}}\mathbf{1}$) follows from using Riemann coordinates at x (rather than the Synge function as in the Schwinger-DeWitt approach), so the method of covariant symbols is suited to computing diagonal matrix elements (within a derivative expansion approach), but not for nondiagonal matrix elements. On the other hand, the method makes no assumptions on $\hat{\mathcal{O}}$ so it works equally well even for a pseudo-Laplacian like ∇_M^2 with a "metric" which is non-Abelian.

We will illustrate the use of the method of covariant symbols below, but its application is straightforward. $\bar{\mathcal{O}}$ is obtained from $f(\nabla, X) \mapsto f(\bar{\nabla}, \bar{X})$, and $\bar{\nabla}_\mu$, $\bar{X}$ are taken from the already compiled tables to the required order. The quantity $\bar{\mathcal{O}}$ so obtained contains only purely multiplicative operators (covariant derivatives of X), plus $Z_{\mu_1\cdots\mu_n}$, p_μ and ∂^μ.

The natural next step is to remove all ∂^μ by moving them to the right where they vanish, acting on all dependence on p_μ. In doing so the commutator between $Z_{\mu_1\cdots\mu_n}$ and ∂^μ has to be applied. This generates some Riemann tensors. One can also choose to move some of the ∂^μ to the left (where they also vanish) if this produces a smaller number of terms. The two choices are related through integration by parts in momentum space.

Since p_μ does not commute with $Z_{\mu_1\cdots\mu_n}$, in order to carry out the momentum integration it will be convenient to move all the $Z_{\mu_1\cdots\mu_n}$ together, to the right (or to the left) using their commutation relations. This produces more coordinate curvatures and also gauge curvatures $F_{\mu\nu}$. All these manipulations produce a diagonal matrix element which does not assume a particular vector space for the action of the operator $\hat{\mathcal{O}}$. The concrete space is used when the operators $Z_{\mu_1\cdots\mu_n}$ are removed after its action is evaluated, as in Eq. (5.9).

Once the ∂^μ have been eliminated and all the $Z_{\mu_1\cdots\mu_n}$ are together (or eliminated) it only remains to carry out the momentum integration, if possible. When the nonpolynomial dependence on p_μ comes from a single (and Abelian) metric, the integrals can often be obtained explicitly.

If there are two metrics $g_{\mu\nu}$ and $M^{\mu\nu}$ (and even more so when the latter is non-Abelian), the integrals cannot be evaluated in closed form in general. In this case, one should be aware of ambiguities in the final expression due to integration by parts in momentum space. Complicated expressions can occasionally reach a simpler form through integration by parts in p_μ. For this reason, the numbers quoted in Table I are upper bounds.

An explicit application of the method of the covariant symbols to illustrate the procedure just described is displayed in Appendix F, by computing one of the universal functional traces of [18].

In what follows, we proceed to give details of the calculation of $\Gamma_S^{\rm div}$ and $\Gamma_{L,0}^{\rm div}$.

B. Calculation of $\Gamma_S^{\rm div}$ in $d=2$

The starting point is Eq. (6.15). Applying the covariant symbols formula (7.5),

$$\langle x|\hat{\mathcal{O}}|x\rangle = \langle\bar{\mathcal{O}}\rangle_p \tag{7.7}$$

[with $\langle\rangle_p$ defined in Eq. (5.12) and the **1** is implicit)], one has[14]

$$\begin{aligned}\Gamma_{S,1} &= \left\langle \mathrm{tr}_1\left(-\frac{1}{2}\bar{\mathcal{O}}_{19}Y_{\mu\nu}\right)\right.\\ &\quad + \mathrm{tr}_0\left(-\frac{1}{2}\bar{\mathcal{O}}_2 W - \frac{1}{2}(\bar{\mathcal{O}}_1)_\mu P_\mu\right.\\ &\quad \left.\left.+\frac{1}{4}\bar{\mathcal{O}}_2 P_{\mu\mu} + O(p^{-3})\right)\right\rangle_{x,p},\\ \Gamma_{S,2} &= \left\langle \mathrm{tr}_0\left(\frac{1}{2}(\bar{\mathcal{O}}_3)_{\mu\nu}\mathcal{M}_{\mu\alpha}\mathcal{M}_{\alpha\nu}\right.\right.\\ &\quad \left.\left.-\frac{1}{4}(\overline{\mathcal{O}_4[P_\mu]})_{\mu\nu}P_\nu + O(p^{-3})\right)\right\rangle_{x,p}.\end{aligned} \tag{7.8}$$

Since $\bar{\nabla}_\mu = O(p_\mu)$, while for multiplicative operators $\bar{X} = O(1)$ for large p_μ, the terms $O(\nabla^{-n})$ in $\hat{\mathcal{O}}$ become $O(p^{-n})$ in $\bar{\mathcal{O}}$.

Within dimensional regularization $\langle p^{-n}\rangle_p$ vanishes for all n with the exception $n=d$. Hence, we need to isolate terms $1/p^2$ in $d=2$, and in particular more UV convergent terms can be neglected.

Of the basic operators present in the formula, $\mathcal{O}_{2,3,4,19}$, are of $O(\nabla^{-2})$. Therefore, the covariant symbols of the latter only require the leading terms of the building blocks,

$$\begin{aligned}\bar{\nabla}_\mu &= p_\mu + O(p^{-1}), \qquad \bar{\Delta} = -N_g + O(p^{-3}),\\ \bar{\Delta}_M &= -N_M + O(p^{-3}), \qquad \bar{X} = X + O(p^{-1}),\end{aligned} \tag{7.9}$$

where X is any purely multiplicative operator. Here we have introduced the definitions

$$N_g \equiv (-g^{\mu\nu}p_\mu p_\nu)^{-1}, \qquad N_M \equiv (-M^{\mu\nu}p_\mu p_\nu)^{-1}. \tag{7.10}$$

Note that N_M is a matrix in gauge space. This produces

$$\begin{aligned}\bar{\mathcal{O}}_2 &= -N_M + O(p^{-3}),\\ \bar{\mathcal{O}}_3 &= N_M N_g p_\mu p_\nu + O(p^{-3}),\\ \overline{\mathcal{O}_4[X]}_{\mu\nu} &= N_M X N_M p_\mu p_\nu + O(p^{-3}),\\ \overline{\mathcal{O}}_{19} &= -N_g + O(p^{-3}).\end{aligned} \tag{7.11}$$

The terms shown explicitly are homogeneous of degree p^{-2}.

The remaining operator $\mathcal{O}_1$ is $O(\nabla^{-1})$ and its covariant symbol $O(p^{-1})$. To isolate the $1/p^2$ term we have to take one more term in the expansion. The expansion in (7.2) is effectively in powers of ∇/p, so terms with one more covariant derivative are needed. Note that in this counting $Z_{\mu_1,\ldots,\mu_n}$ counts as $O(\nabla^n)$. From its definition $(\mathcal{O}_1)_\lambda \equiv \Delta_M\nabla_\lambda$, one obtains

$$(\bar{\mathcal{O}}_1)_\lambda = \bar{\Delta}_M\bar{\nabla}_\lambda, \tag{7.12}$$

with

$$\bar{\Delta}_M = (\bar{\nabla}_\mu \bar{M}_{\mu\nu}\bar{\nabla}_\nu)^{-1}. \tag{7.13}$$

The expansion of $\bar{\nabla}_\lambda$ in Eq. (7.9) is already sufficient, but $\bar{\Delta}_M$ needs to be expanded to order $1/p^3$ which in turn requires $\bar{M}_{\mu\nu}$ to order $1/p$,

$$\begin{aligned}\bar{\Delta}_M &= (p_\mu(M_{\mu\nu} - M_{\alpha\mu\nu}\partial^\alpha)p_\nu + O(1))^{-1}\\ &= -N_M + N_M p_\mu M_{\nu\mu\alpha}\partial^\nu p_\alpha N_M + O(p^{-4}).\end{aligned} \tag{7.14}$$

Therefore,

$$(\bar{\mathcal{O}}_1)_\lambda = -N_M p_\lambda + N_M p_\mu M_{\nu\mu\alpha}\partial^\nu p_\alpha N_M p_\lambda + O(p^{-3}). \tag{7.15}$$

To the order needed, the operators $\bar{\mathcal{O}}_{2,3,4,19}$ do not have any ∂^μ, hence they already coincide with $\bar{\mathcal{O}}_n\mathbf{1}$. For $\bar{\mathcal{O}}_1$ the momentum derivatives can be moved to the right using

$$[\partial^\mu, p_\nu] = \delta^\mu_\nu, \qquad [\partial^\mu, N_M] = 2p_\nu N_M M^{\mu\nu} N_M. \tag{7.16}$$

[14]For a multiplicative X, $\langle x|\mathcal{O}X|x\rangle = \langle x|\mathcal{O}|x\rangle X(x)$, consistently $\langle\bar{\mathcal{O}}\bar{X}\rangle_p = \langle\bar{\mathcal{O}}\rangle_p X(x)$ since $\bar{X}-X$ contains ∂^μ but not p_μ [see Eq. (7.2)], and such terms vanish inside $\langle\rangle_p$.

In this way,[15]

$$(\bar{\mathcal{O}}_1)_\lambda \mathbf{1} = -N_M p_\lambda - N_M M_{\mu\mu\nu} N_M p_\lambda p_\nu - 2N_M M_{\mu\nu} N_M M_{\mu\alpha\beta} N_M p_\lambda p_\nu p_\alpha p_\beta + O(p^{-3}). \tag{7.17}$$

The first term is homogeneous of degree $1/p$ and the two other explicit terms are homogeneous of degree $1/p^2$. The first term vanishes within momentum integration due to parity. For the same reason, all odd order terms have been omitted in Table I. In any case, only the contributions $1/p^d$ are relevant in d dimensions for $\Gamma_1^{\rm div}$.

No operators Z_I appear in the expansions of $\bar{\mathcal{O}}_{1,2,3,4,19}$ to the order required in Eq. (7.8). The most UV divergent operator is $\mathcal{O}_1 = O(\nabla^{-1})$, so its leading term is $1/p$ and the term relevant in $d = 2$, $1/p^2$, comes from contributions of the type ∇/p^2, while Z_I needs at least two covariant derivatives.

Operators Z_I do appear in $d = 4$. Just as ∂^μ, the operators Z_I^R have to be resolved before having a useful expression for $\langle x|\mathcal{O}_n|x\rangle$. To this end, the operators Z_I^R can be moved to the left or to the right, using their commutation relations noted in (B4), including (7.4). In accordance with the choice in Eq. (6.9), where the multiplicative operators A_n have been placed at the right, Z_I^R should be moved to the left. This allows one to apply the rules in Eq. (5.9) (see Appendix B).

It remains to insert in Eq. (7.8) the various expressions for $\bar{\mathcal{O}}_n\mathbf{1}$ just obtained. In the resulting expression, the terms involving P_μ in $\Gamma_{S,1}$ are not manifestly Hermitian. This can be fixed by applying integration by parts in p and x spaces[16] as well as the trace cyclic property. This gives,

$$\Gamma_{S,1}^{\rm div} = \left\langle \frac{1}{2}N_g Y_{\mu\mu} + \frac{1}{2}N_M W - \frac{1}{4}N_M P_{\mu\mu} + \frac{1}{2}N_M M_{\mu\nu} N_M M_{\mu\alpha\beta} N_M P_\lambda p_\nu p_\alpha p_\beta p_\lambda - \frac{1}{2}N_M M_{\mu\alpha\beta} N_M M_{\mu\nu} N_M P_\lambda p_\nu p_\alpha p_\beta p_\lambda \right\rangle_{x,p,g},$$
$$\Gamma_{S,2}^{\rm div} = \left\langle \frac{1}{2}N_g N_M \mathcal{M}_{\mu\alpha}\mathcal{M}_{\alpha\nu} p_\mu p_\nu - \frac{1}{4}N_M P_\mu N_M P_\nu p_\mu p_\nu \right\rangle_{x,p,g}. \tag{7.18}$$

The integrand is a homogeneous function of p of degree -2. It only remains to extract the $1/\varepsilon$ coefficient to isolate the UV divergent contributions. The details are given in Sec. VII D. An application of the rules provided there immediately produces the result quoted in Eq. (6.16).

In Eq. (7.17) there is one term of degree p^{-1} and two terms of degree p^{-2}. Table I shows the number of terms of each degree for the diagonal matrix elements of the operators $\mathcal{O}_n$. Only even orders are displayed since odd orders vanish upon integration over p_μ in any parity preserving regularization, such as dimensional regularization. For a given operator, the number of terms increases rapidly with (minus) the degree. Nevertheless, the number of terms displayed in the table is an upper bound; this number is subject to variations due to various identities which allow one to write a given expression in different forms. Such identities include integration by parts in momentum space and reordering of the covariant derivatives acting on a tensor due to the Jacobi identity,

$$[\nabla_\mu, [\nabla_\nu, X]] = [\nabla_\nu, [\nabla_\mu, X]] + [Z_{\mu\nu}, X]. \tag{7.19}$$

Furthermore, integration by parts in x space and trace cyclic property is allowed within the functional trace operations $\mathrm{Tr}_{0,1}$ in Eq. (6.13). We have not attempted a systematic minimization of the number of terms as there is no practical procedure to do this, and in any case we do not expect a significant reduction in the length of the expressions. An exception is the operator $\mathcal{O}_2$ at p^{-4} which is used below, Eq. (7.21), in the computation of $\Gamma_{L,0}$, in Sec. VI C 4.

Nevertheless, it should be noted that, recently, important progress has been achieved in the counting and classification of allowed independent terms in effective field theories, through the construction of Hilbert series of the operator basis [50]. In this technique, the basic blocks (fields or composite operators) plus their symmetrized derivatives are identified with representations of the d-dimensional conformal group.[17] Computation of the Clebsch-Gordan series then allows one to obtain generating functions for basis operators and count them. The key point is that both equation of motion as well as integration by part identities are automatically accounted for, in addition to spacetime and internal group symmetries. The method has been successfully applied to pure Einstein relativity and also to general relativity combined with the Standard Model of particle physics [51]. The adaptation of the Hilbert series technique to obtain basis of operators in diagonal matrix elements and the effective action contributions as those displayed in Eqs. (7.18) or (7.22) would be extremely interesting, and more so in $d = 4$ where the number of terms becomes huge. Serious complications arise due to the presence of an additional momentum variable, with its own integration by parts identities, and also the existence of constraints relating some of the building blocks, such as $M^{\mu\nu}$ and N_M. No attempt will be made here to adapt the promising technique of Hilbert

[15] Actually, for convenience here ∂^μ has been moved to the left (and then removed) so the rhs differs from $\bar{\mathcal{O}}\mathbf{1}$ by terms which vanish upon momentum integration.

[16] As shown in Appendix C of [35], p_μ can be treated as a constant when integrating by parts in x space, when ∇_μ and Z_I^R are no longer present.

[17] Alternatively, cohomological techniques can be applied [50].

series to working with covariant symbols; we defer such a study to future work.

C. Calculation of $\Gamma^{\rm div}_{L,0}$ in $d=2$

Because of the similarity between the operators $\mathcal{O}'_2$ and $\mathcal{O}_2$, the expression of their diagonal matrix elements are identical when written in terms of N_M, with the only proviso of using the new definition

$$N_M = (-z - M^{\mu\nu}p_\mu p_\nu)^{-1} \tag{7.20}$$

for $\mathcal{O}'_2$, instead of that in (7.10).

The UV divergent terms in $\Gamma_{L,0}$ must have exactly d covariant derivatives. The operator $\mathcal{O}'_2$ is of UV degree $O(\nabla^{-2})$, so the leading term in $\overline{\mathcal{O}'}_2$ is $O(p^{-2})$ and no derivatives. Since one still needs to expand its covariant symbol to d covariant derivatives, the order $p^{-d-2}\nabla^d$ is needed. The proper divergence $O(p^{-d})$ is only recovered after integration over z, a parameter of dimension squared mass.

We present results only for the case $d=2$. For the terms of order exactly p^{-4}, the calculation gives the following result:

$$\begin{aligned}\langle x|\mathcal{O}'_2|x\rangle^{(-4)} = \Big\langle & 4N_M M_{\mu\nu}N_M M_{\mu\alpha\beta}N_M M_{\rho\sigma\lambda}N_M M_{\rho\eta}N_M p_\nu p_\alpha p_\beta p_\sigma p_\lambda p_\eta + 2N_M M_{\mu\nu}N_M M_{\mu\alpha\beta}N_M M_{\rho\rho\sigma}N_M p_\nu p_\alpha p_\beta p_\sigma \\ & + 2N_M M_{\mu\mu\nu}N_M M_{\alpha\beta\rho}N_M M_{\alpha\sigma}N_M p_\nu p_\beta p_\rho p_\sigma + N_M M_{\mu\nu}N_M M_{\mu\alpha\beta\rho}N_M M_{\alpha\sigma}N_M p_\nu p_\beta p_\rho p_\sigma \\ & + N_M M_{\mu\nu}N_M M_{\alpha\mu\beta\rho}N_M M_{\alpha\sigma}N_M p_\nu p_\beta p_\rho p_\sigma + \frac{2}{3}N_M M_{\mu\nu}N_M M_{\alpha\beta}N_M M_{\rho\sigma}N_M R_{\mu\alpha\rho\lambda}p_\nu p_\beta p_\sigma p_\lambda \\ & - \frac{2}{3}N_M M_{\mu\nu}N_M M_{\alpha\beta}N_M M_{\rho\sigma}N_M R_{\mu\lambda\alpha\rho}p_\nu p_\beta p_\sigma p_\lambda + N_M M_{\mu\mu\nu}N_M M_{\alpha\alpha\beta}N_M p_\nu p_\beta \\ & - N_M M_{\mu\nu}F_{\mu\alpha}N_M M_{\alpha\beta}N_M p_\nu p_\beta - N_M M_{\mu\nu}N_M F_{\mu\alpha}M_{\alpha\beta}N_M p_\nu p_\beta - \frac{1}{6}N_M M_{\mu\nu}N_M \mathcal{R}_{\mu\nu}\Big\rangle_p . \end{aligned} \tag{7.21}$$

The expression obtained after applying the method of covariant symbols has been simplified by using integration by parts in momentum space, also achieving a manifest Hermitian form (since $\mathcal{O}_2$ is Hermitian).

It can be noted that the expression in (7.21) holds for arbitrary d. For $d=2$, one can use the identities in (6.33). Also we integrate by parts with respect to x to have at most one covariant derivative on $M_{\mu\nu}$. This gives for the effective action

$$\begin{aligned}\Gamma^{\rm div}_{L,0} = \frac{1}{2}\Big\langle & -2N_M M_{\mu\nu}N_M M_{\mu\alpha\beta}N_M M_{\rho\sigma}N_M M_{\rho\lambda\eta}N_M p_\nu p_\alpha p_\beta p_\sigma p_\lambda p_\eta - 2N_M M_{\mu\nu\alpha}N_M M_{\mu\beta}N_M M_{\rho\sigma\lambda}N_M M_{\rho\eta}N_M p_\nu p_\alpha p_\beta p_\sigma p_\lambda p_\eta \\ & - N_M M_{\mu\nu}N_M M_{\mu\alpha\beta}N_M M_{\alpha\rho\sigma}N_M p_\nu p_\beta p_\rho p_\sigma + N_M M_{\mu\nu}N_M M_{\alpha\alpha\beta}N_M M_{\mu\rho\sigma}N_M p_\nu p_\beta p_\rho p_\sigma \\ & - N_M M_{\mu\nu\alpha}N_M M_{\beta\mu\rho}N_M M_{\beta\sigma}N_M p_\nu p_\alpha p_\rho p_\sigma + N_M M_{\mu\nu\alpha}N_M M_{\beta\beta\rho}N_M M_{\mu\sigma}N_M p_\nu p_\alpha p_\rho p_\sigma \\ & + N_M M_{\mu\nu}N_M M_{\alpha\beta}N_M M_{\mu\rho}N_M \mathbf{R}\, p_\nu p_\alpha p_\beta p_\rho + N_M M_{\mu\mu\nu}N_M M_{\alpha\alpha\beta}N_M p_\nu p_\beta - N_M M_{\mu\nu}F_{\mu\alpha}N_M M_{\alpha\beta}N_M p_\nu p_\beta \\ & - N_M M_{\mu\nu}N_M F_{\mu\alpha}M_{\alpha\beta}N_M p_\nu p_\beta + \frac{2}{3}N_M M_{\mu\nu}N_M M_{\mu\alpha}N_M \mathbf{R}\, p_\nu p_\alpha - \frac{1}{12}N_M M_{\mu\mu}N_M \mathbf{R}\Big\rangle_{x,p,g,z} . \end{aligned} \tag{7.22}$$

One could apply further the trace cyclic property and also the identity $N_M M_{\mu\nu}N_M p_\mu p_\nu = -N_M - zN_M^2$ in one of the terms, but no simplification would be achieved. The final step is to extract the $1/\varepsilon$ coefficient from the radial part of the momentum integral, as described in the next subsection. This procedure yields the expression quoted in Eq. (6.32).

D. Extraction of the UV divergent component

Let us consider first the case when the parameter z is not present, as in Eq. (7.18). Once the operators ∂_μ and Z_I^R have been removed, the structure of a general term $\bar{\mathcal{O}}_n$ to be integrated over p_μ is a sum of products with factors N_M, N_g, and p_μ, as well as p_μ-independent multiplicative operators M_I, R_I, F_I, etc. Hence, the momentum integral affects only terms of the form

$$N_g^n (N_M)_{a_1 b_1}\cdots(N_M)_{a_m b_m} p_{\mu_1}\cdots p_{\mu_{2j}} \equiv N_g^n N_M^{\otimes m} p^{\otimes 2j}. \tag{7.23}$$

This monomial is homogeneous in p_μ with degree $2(j-n-m)$.

We are only interested in the UV divergent part and to extract it we will use dimensional regularization, namely, in $d+2\varepsilon$ dimensions with $\varepsilon \to 0^-$. For the UV divergent part, d can be used instead of $d+2\varepsilon$ in the UV-finite contributions. Within dimensional regularization, the integral

$$I_d^{m,j} = \frac{1}{\sqrt{g}} \int \frac{d^{d+2\varepsilon}p}{(2\pi)^d} N_g^n N_M^{\otimes m} p^{\otimes 2j} \tag{7.24}$$

vanishes unless $d = 2(n+m-j)$, which is assumed in what follows.

Recalling that $p_\mu = ik_\mu$ and $d^d p_\mu \equiv d^d k_\mu$,

$$I_d^{m,j} = (-1)^j \frac{1}{\sqrt{g}} \int \frac{d^{d+2\varepsilon}k_\mu}{(2\pi)^d} N_g^n N_M^{\otimes m} k^{\otimes 2j}. \tag{7.25}$$

Let us introduce a standard tetrad field $e_\mu^A(x)$,

$$g_{\mu\nu} = e_\mu^A e_\nu^B \delta_{AB}, \qquad e_\mu^A e_B^\mu = \delta_B^A, \tag{7.26}$$

in such a way that

$$\begin{aligned} k_\mu &= e_\mu^A k_A, \qquad d^d k_\mu = \sqrt{g} d^d k_A, \\ N_g &= \frac{1}{k_A^2}, \qquad N_M = \frac{1}{k_A k_B M^{AB}}, \end{aligned} \tag{7.27}$$

thus

$$\begin{aligned} (I_d^{m,j})_{\mu_1\cdots\mu_{2j}} &= (I_d^{m,j})_{A_1\cdots A_{2j}} e_{\mu_1}^{A_1} \cdots e_{\mu_{2j}}^{A_{2j}}, \\ (I_d^{m,j})_{A_1\cdots A_{2j}} &= (-1)^j \int \frac{d^{d+2\varepsilon}k_A}{(2\pi)^d} N_g^n N_M^{\otimes m} k_A^{\otimes 2j}. \end{aligned} \tag{7.28}$$

The UV divergence comes from the radial part of the integral, hence we introduce spherical coordinates,

$$k_A = k\hat{k}_A, \qquad k = \sqrt{k_A^2} = N_g^{-1/2}, \qquad \hat{k}_A^2 = 1. \tag{7.29}$$

This allows one to separate the integral into radial and angular average factors,

$$I_d^{m,j} = \frac{(-1)^j}{(4\pi)^{d/2}} \frac{2}{\Gamma(d/2)} I_\varepsilon \hat{I}_d^{m,j}, \tag{7.30}$$

where the angular average is

$$\begin{aligned} \hat{I}_d^{m,j} &= \frac{\Gamma(d/2)}{2\pi^{d/2}} \int d^{d-1}\Omega_{\hat{k}} \hat{N}_M^{\otimes m} \hat{k}_A^{\otimes 2j} \equiv \langle \hat{N}_M^{\otimes m} \hat{k}_A^{\otimes 2j} \rangle_{\text{ang}}, \\ \hat{N}_M &\equiv \frac{1}{\hat{k}_A \hat{k}_B M^{AB}} = N_M / N_g, \end{aligned} \tag{7.31}$$

and I_ε is the radial part (introducing a cutoff mass m_0 to avoid a trivial infrared divergence for negative ε), and using the condition $2(n+m-j) = d$,

$$I_\varepsilon = \int_{m_0}^\infty dk k^{2\varepsilon-1} = -\frac{m_0^{2\varepsilon}}{2\varepsilon} = -\frac{1}{2\varepsilon} + O(1). \tag{7.32}$$

In summary, for the UV divergent part, one obtains

$$I_d^{m,j,\text{div}} = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \frac{1}{\varepsilon} (-1)^{j+1} \hat{I}_d^{m,j}. \tag{7.33}$$

As noted, the angular averages $\hat{I}_d^{m,j}$ are perfectly UV convergent and well defined, but they cannot be written in closed form in general.

The analysis is similar for $\Gamma_{L,0}$, which involves an additional integration over z. As mentioned, in this case and for $d = 2$, the relevant terms are of order p^{-4}, and to extract the UV divergent part one must integrate over z and p_μ. The point can be elucidated following the steps shown above (for $d = 2$), noting that now N_M contains z:

$$\begin{aligned} \langle p^{\otimes 2j} N_M^{\otimes(2+j)} \rangle_{z,p} &= \frac{1}{\sqrt{g}} \int \frac{d^{2+2\varepsilon}p}{(2\pi)^2} \int_\gamma \frac{dz}{2\pi i} \\ &\quad \times \log(z) p^{\otimes 2j} \left(\frac{1}{-z - M^{\mu\nu} p_\mu p_\nu} \right)^{\otimes(2+j)} \\ &= (-1)^j \int_{m_0}^\infty dk k^{2j+1+2\varepsilon} \int \frac{d\Omega}{(2\pi)^2} \int_\gamma \frac{dz}{2\pi i} \\ &\quad \times \log(z) \hat{k}^{\otimes 2j} \left(\frac{1}{-z + k^2 M^{\mu\nu} \hat{k}_\mu \hat{k}_\nu} \right)^{\otimes(2+j)}. \end{aligned} \tag{7.34}$$

Applying the rescaling $z \to zk^2$ and noting that the induced term with $\log(k^2)$ vanishes since no singularities are enclosed by the path γ in the z complex plane,

$$\begin{aligned} \langle p^{\otimes 2j} N_M^{\otimes(2+j)} \rangle_{z,p} &= (-1)^{j+1} \frac{m_0^{2\varepsilon}}{2\varepsilon} \int \frac{d\Omega}{(2\pi)^2} \int_\gamma \frac{dz}{2\pi i} \\ &\quad \times \log(z) \hat{k}^{\otimes 2j} \left(\frac{1}{-z + M^{\mu\nu} \hat{k}_\mu \hat{k}_\nu} \right)^{\otimes(2+j)} \\ &= \frac{(-1)^{j+1}}{4\pi\varepsilon} \langle \hat{k}^{\otimes 2j} \hat{N}_M^{\otimes(2+j)} \rangle_{\text{ang},z} + O(1), \end{aligned} \tag{7.35}$$

where $\hat{N}_M = (-z + M^{\mu\nu} \hat{k}_\mu \hat{k}_\nu)^{-1}$. Applying this angular average in Eq. (7.22) yields Eq. (6.32).

VIII. SUMMARY AND CONCLUSIONS

In this work we have addressed the problem of quantizing a system of, in general, non-Abelian vector fields with a completely general local nonminimal mass term coupling all of them. The case of N Abelian fields is a particular instance in our formulation. We make use of a background field approach. A remarkable result is that, although the

mass term breaks gauge invariance, the effective action is fully gauge (as well as coordinate) invariant beyond tree level, Eq. (3.5). The technical problems present in the original theory (namely, the UV region is blind to the mass term and so requires some type of gauge fixing) are satisfactorily removed by introducing a non-Abelian Stueckelberg field, Eqs. (4.1) and (4.4). This is done after background and fluctuation fields have been separated and the Stueckelberg field only affects the latter. This auxiliary field is introduced linearly so that the loop structure of the original theory is preserved. As a consequence, the computation of the UV divergent part of the effective action to one loop can be carried out systematically preserving coordinate and gauge symmetries during the calculation. To this end, we apply dimensional regularization and the method of covariant symbols. This produces terms which are local, i.e., they contain a finite number derivatives of the external fields (no more than d in d spacetime dimensions), however, they are not polynomial with respect to the mass term, a fact already observed in the simpler case of a single vector field. The one-loop effective action is expressed in Eq. (4.28) as a sum of four terms, $\Gamma_{\rm gh}$ defined in (4.19), $\Gamma_{L,1}$ and $\Gamma_{L,0}$ in (4.25), and Γ_S in (4.26) [and expanded in Eq. (6.7)]. The formalism is developed for arbitrary spacetime dimensions and we present explicit results for the two-dimensional case. Regrettably in the four-dimensional case too many terms are produced (even selecting particular settings, such as purely Abelian, flat spacetime, or perturbative expansions) so their explicit expression would be of little practical use. The explicit two-dimensional results for the UV divergent component are displayed in (6.4) for $\Gamma_{\rm gh}$ and $\Gamma_{L,1}$, in (6.16) for Γ_S, and in (6.32) for $\Gamma_{L,0}$. Checks have been applied to the results of the calculation. Particularly stringent are the tests related to invariance with respect to metric transformations in Sec. VI C 5. Perturbative results are also presented in Appendix D. Although not explicitly discussed in the text, we have repeated most of the calculations of diagonal matrix elements using the method of noncovariant symbols [37,46] to find an identical result, modulo integration by parts in momentum space. In some cases, the latter method has produced shorter expressions. Details of the noncovariant method are discussed in Appendix E.

As already pointed out, the nonminimal masslike coupling discussed in this work (or also its Abelian version) respects locality but introduces terms in the effective action which are nonpolynomial in the field $M^{\mu\nu}(x)$. All current efforts for an effective field theory description of general relativity or the Standard Model of particles, or both (e.g., [51]), naturally assume an expansion in local and polynomial operators over some power of the cutoff (a new-physics scale), $\mathcal{O}_\alpha(x)/\Lambda^n$, consistent with the renormalization group analysis of Wilson [52]. In this light, the analyses presented in [34,35] and in this work should indicate that a nonminimal coupling of the type Eq. (2.1) can be ruled out in vector field theories, also in the non-Abelian setting. If the presence of such nonminimal coupling could not be prevented through some mechanism (such as the requirement of strict gauge invariance), one would be impelled to assume a much larger class of effective field theories, including local but nonpolynomial operators. On the other hand, even in that case, reexpansions as that in Appendix D would bring the expression again to the standard form, requiring only local and polynomial composite operators, provided that the scale m can be interpreted as a proper cutoff of the theory and a separation of the type $M^{\mu\nu} = m^2 g^{\mu\nu} + H^{\mu\nu}$ is somehow natural.

The fluctuation operator in Eq. (4.15) is a rather involved one, due to the presence of a non-Abelian field $M^{\mu\nu}$ coupling like a metric in the ϕ sector. Chan's method or even the Schwinger-DeWitt technique are not readily available to deal with such term. Yet the formalism of covariant symbols could be applied also to $\Gamma_{L,0}$, upon introduction of a parametric form to remove the logarithm. In fact, the method of covariant symbols is a practical and easy-to-use tool to obtain diagonal matrix elements of local operators $f(\nabla, X)$, provided the dependence on ∇ is of rational type. This latter requirement follows from the fact that, in practice, the covariant symbols are only obtained as an expansion in powers of ∇/p. It can also be noted that the covariant symbols depend only on the connection; the presence of a metric is not required, as the Riemann coordinates can be defined directly from the connection [53]. The method can be used to obtain not only the counterterms but also covariant derivative expansions of the effective action itself [45,54,55]. In particular, the treatment of fermionic modes in curved spacetime poses no special problems once the spin connection is included in the covariant derivative [56]. The extension of the method for finite temperature also exists [46,49] (but not yet for temperature and curvature at the same time). The method of covariant symbols should apply whenever the generalized Schwinger-DeWitt technique applies, so it can be used as an alternative approach in the analysis of quantum field theories in curved spacetime, effective field theories involving gravity, or in the study of the newly developed Proca theories noted in the Introduction.

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APPENDIX A: DERIVATION OF SOME FORMULAS

1. Derivation of Eq. (3.12)

Applying integration by parts in the first term in (3.11) [using the notation of (5.10)],

$$\begin{aligned}
&\frac{1}{4}\langle(\mathcal{A}^a_{\mu\nu}-\mathcal{A}^a_{\nu\mu})^2\rangle_x \\
&=\frac{1}{2}\langle(\mathcal{A}^a_{\mu\nu})^2-\mathcal{A}^a_{\mu\nu}\mathcal{A}^a_{\nu\mu}\rangle_x=\frac{1}{2}\langle(\mathcal{A}^a_{\mu\nu})^2+\mathcal{A}^a_{\nu}\mathcal{A}^a_{\nu\mu\mu}\rangle_x \\
&=\frac{1}{2}\langle(\mathcal{A}^a_{\mu\nu})^2+\mathcal{A}^a_{\nu}\mathcal{A}^a_{\nu\mu\mu}+\mathcal{A}^a_{\nu}Z^{ab}_{\mu\nu}\mathcal{A}^b_{\mu}\rangle_x \\
&=\frac{1}{2}\langle(\mathcal{A}^a_{\mu\nu})^2-\mathcal{A}^a_{\nu\nu}\mathcal{A}^a_{\mu\mu}+\mathcal{A}^a_{\nu}(F^{ab}_{\mu\nu}\mathcal{A}^b_{\mu}+R_{\mu\nu\mu\alpha}\mathcal{A}^a_{\alpha})\rangle_x \\
&=\frac{1}{2}\langle(\mathcal{A}^a_{\mu\nu})^2-(\mathcal{A}^a_{\mu\mu})^2-F^{ab}_{\mu\nu}\mathcal{A}^a_{\mu}\mathcal{A}^b_{\nu}+\mathcal{R}_{\mu\alpha}\mathcal{A}^a_{\alpha})\rangle_x. \qquad \text{(A1)}
\end{aligned}$$

Added to the other term in (3.11), $\langle-\frac{1}{2}F^{ab}_{\mu\nu}\mathcal{A}^a_{\mu}\mathcal{A}^b_{\nu}\rangle$, produces (3.12).

2. Derivation of Eq. (4.6)

From the change of variables

$$\mathcal{A}^a_{\mu}=\mathcal{B}^a_{\mu}+\phi^a_{\mu}, \qquad \chi^a=\mathcal{B}^a_{\mu\mu}, \qquad \text{(A2)}$$

one obtains

$$\frac{\partial(\mathcal{A}^a_{\mu},\chi^a)}{\partial(\mathcal{B}^b_{\nu},\phi^b)}=\begin{pmatrix}\delta_{ab}g_{\mu}{}^{\nu} & \delta_{ab}\nabla_{\mu}\\ \delta_{ab}\nabla^{\nu} & 0\end{pmatrix} \qquad \text{(A3)}$$

(with rows for the numerator and columns for the denominator). More conveniently, doing the change of variables in two steps $(\mathcal{B},\phi)\to(\mathcal{A},\phi)\to(\mathcal{A},\chi)$,

$$\frac{\partial(\mathcal{A}^a_{\mu},\chi^a)}{\partial(\mathcal{B}^b_{\nu},\phi^b)}=\frac{\partial(\mathcal{A}^a_{\mu},\chi^a)}{\partial(\mathcal{A}^c_{\lambda},\phi^c)}\frac{\partial(\mathcal{A}^c_{\lambda},\phi^c)}{\partial(\mathcal{B}^b_{\nu},\phi^b)}, \qquad \text{(A4)}$$

using $\chi^a=\mathcal{A}^a_{\mu\mu}-\phi^a_{\mu\mu}$ in the first factor, produces

$$\begin{aligned}
&\begin{pmatrix}\delta_{ab}g_{\mu}{}^{\nu} & \delta_{ab}\nabla_{\mu}\\ \delta_{ab}\nabla^{\nu} & 0\end{pmatrix} \\
&=\begin{pmatrix}\delta_{ac}g_{\mu}{}^{\lambda} & 0\\ \delta_{ac}\nabla^{\lambda} & -\delta_{ac}\nabla^2\end{pmatrix}\begin{pmatrix}\delta_{cb}g_{\lambda}{}^{\nu} & \delta_{cb}\nabla_{\lambda}\\ 0 & \delta_{cb}\end{pmatrix}. \qquad \text{(A5)}
\end{aligned}$$

The second matrix has a unit determinant and likewise for the upper-left block in the first matrix. This produces Eq. (4.6).

3. Derivation of Eq. (4.8)

Starting from (3.11), and using $\mathcal{A}^a_{\mu}=\mathcal{B}^a_{\mu}+\phi^a_{\mu}$, one obtains terms of the types $\mathcal{BB}$, $\mathcal{B}\phi$, and $\phi\phi$. The terms $\mathcal{BB}$ are just those in (3.12) with $\mathcal{A}^a_{\mu}\to\mathcal{B}^a_{\mu}$.

The terms $\mathcal{B}\phi$ are given by twice (3.11) replacing one of the $\mathcal{A}$ with $\mathcal{A}^a_{\mu}\to\mathcal{B}^a_{\mu}$ and the other one with $\mathcal{A}^a_{\mu}\to\phi^a_{\mu}$. This produces

$$(S^{(2)}_{\rm kin})_{\mathcal{B}\phi}=\left\langle\frac{1}{2}(\phi^a_{\mu\nu}-\phi^a_{\nu\mu})(\mathcal{B}^a_{\mu\nu}-\mathcal{B}^a_{\nu\mu})-F^{ab}_{\mu\nu}\phi^a_{\mu}\mathcal{B}^b_{\nu}\right\rangle_x. \qquad \text{(A6)}$$

Using $\phi^a_{\mu\nu}-\phi^a_{\nu\mu}=F^{ab}_{\mu\nu}\phi^b$ and integration by parts in the other term,

$$\begin{aligned}
(S^{(2)}_{\rm kin})_{\mathcal{B}\phi}&=\langle-\phi^aF^{ab}_{\mu\nu}\mathcal{B}^b_{\mu\nu}+F^{ab}_{\mu\mu\nu}\phi^a\mathcal{B}^b_{\nu}+F^{ab}_{\mu\nu}\phi^a\mathcal{B}^b_{\mu\nu}\rangle_x \\
&=\langle F^{ab}_{\mu\mu\nu}\phi^a\mathcal{B}^b_{\nu}\rangle_x. \qquad \text{(A7)}
\end{aligned}$$

Finally, the term $\phi\phi$ is half the previous one after the replacement $\mathcal{B}^a_{\mu}\to\phi^a_{\mu}$. This produces (4.8).

APPENDIX B: THE OPERATORS $Z_{\mu_1\cdots\mu_n}$

1. Definition and properties of $Z^R_{\mu_1\cdots\mu_n}$

The operator $Z_{\mu\nu}$ is defined as

$$Z_{\mu\nu}=[\nabla_{\mu},\nabla_{\nu}]=Z^R_{\mu\nu}+F_{\mu\nu}. \qquad \text{(B1)}$$

$Z^R_{\mu\nu}$ acts on coordinate indices and $F_{\mu\nu}$ on gauge indices. $Z^R_{\mu\nu}$ is multiplicative but not "purely multiplicative" (by definition) as it is not diagonal in the coordinate indices. The higher rank tensors are defined recursively, namely (recall that $I=\mu_1\cdots\mu_n$ stands for a string of coordinate indices),

$$Z_{\alpha I}=[\nabla_{\alpha},Z_I]+\frac{1}{2}\{\nabla_{\lambda},R_{I\alpha\lambda}\}. \qquad \text{(B2)}$$

The extra term ensures that Z_I is a multiplicative operator. The same formula applies to Z^R_I.

The clean separation between coordinate and gauge sectors,

$$Z_I=Z^R_I+F_I, \qquad \text{(B3)}$$

holds too for higher rank tensors. The operators Z_I, Z^R_I, and F_I are all anti-Hermitian. From its definition, Z^R_I has the property

$$[Z^R_I,V_{\mu_1\mu_2\cdots}]=R_{I\mu_1\lambda}V_{\lambda\mu_2\cdots}+R_{I\mu_2\lambda}V_{\mu_1\lambda\cdots}+\cdots, \qquad \text{(B4)}$$

where V is a coordinate tensor. In particular, for a scalar field $\phi(x)$,

$$[Z_I^R, \phi] = 0. \tag{B5}$$

Now, let $|0\rangle$ be the scalar function that takes the value 1 for all x. This is a coordinate scalar. The relation $\nabla_\mu|0\rangle = 0$ implies

$$Z^R_{\mu\nu}|0\rangle = 0 = \langle 0|Z^R_{\mu\nu}. \tag{B6}$$

More generally [using Eq. (B2)],

$$Z_I^R|0\rangle = C_I|0\rangle, \qquad \langle 0|Z_I^R = -\langle 0|C_I, \tag{B7}$$

with

$$\begin{aligned} C_{\mu\nu} &= 0, \\ C_{\alpha\mu\nu} &= \frac{1}{2}R_{\lambda\mu\nu\alpha\lambda}, \\ C_{\alpha_1\cdots\alpha_n\mu\nu} &= \frac{1}{2}R_{\lambda\alpha_2\alpha_3\cdots\alpha_n\mu\nu\alpha_1\lambda} + \frac{1}{2}R_{\alpha_1\lambda\alpha_3\cdots\alpha_n\mu\nu\alpha_2\lambda} + \cdots \\ &\quad + \frac{1}{2}R_{\alpha_1\cdots\alpha_{n-1}\lambda\mu\nu\alpha_n\lambda}. \end{aligned} \tag{B8}$$

2. Relation with the operator $\hat{\mathcal{R}}_{\mu\nu}$ of [18]

The operator $Z^R_{\mu\nu}$ coincides with $\hat{\mathcal{R}}_{\mu\nu}$ in [18]. The higher rank operators $(\nabla_{\alpha_1}\cdots\nabla_{\alpha_n}\hat{\mathcal{R}}_{\mu\nu})^\rho{}_\sigma$ are introduced in [18], with the convention that the covariant derivative connections do not act on the matrix indices ρ, σ. Using the notation

$$\hat{\mathcal{R}}_{\alpha_1\cdots\alpha_n\mu\nu} \equiv \nabla_{\alpha_1}\cdots\nabla_{\alpha_n}\hat{\mathcal{R}}_{\mu\nu}, \tag{B9}$$

these operators fulfill the recursion

$$\hat{\mathcal{R}}_{\alpha I} = [\nabla_\alpha, \hat{\mathcal{R}}_I] + R_{I\alpha\lambda}\nabla_\lambda, \tag{B10}$$

as well as

$$\begin{aligned} [\hat{\mathcal{R}}_I, V_{\mu_1\mu_2\cdots}] &= [Z_I^R, V_{\mu_1\mu_2\cdots}] \\ &= R_{I\mu_1\lambda}V_{\lambda\mu_2\cdots} + R_{I\mu_2\lambda}V_{\mu_1\lambda\cdots} + \cdots. \end{aligned} \tag{B11}$$

The operators Z_I^R and $\hat{\mathcal{R}}_I$ are related through

$$Z_I^R = \hat{\mathcal{R}}_I + C_I. \tag{B12}$$

Therefore,

$$\hat{\mathcal{R}}_I|0\rangle = 0, \qquad \langle 0|\hat{\mathcal{R}}_I = -2\langle 0|C_I. \tag{B13}$$

The operators $\hat{\mathcal{R}}_I$ and Z_I^R have identical commutation properties on purely multiplicative fields. The $\hat{\mathcal{R}}_I$ are simpler than Z_I^R when acting on states on the right (since they vanish on coordinate scalars), while the Z_I^R have the virtue of being anti-Hermitian.

3. Derivation of Eqs. (5.9)

The first relation in (5.9), the trace in the coordinate-scalar space, follows from the fact that Z_I^R coincides with $-C_I$ when acting on scalars on the left,

$$\langle\phi|Z_I^R = \langle 0|\phi Z_I^R = \langle 0|(Z_I^R\phi - [Z_I^R, \phi]) = -\langle\phi|C_I. \tag{B14}$$

For the second relation, the trace in the coordinate-vector space, consider an operator $\mathcal{O}^\mu{}_\nu$ acting on the coordinate-vector space, $(\mathcal{O}V)^\mu = \mathcal{O}^\mu{}_\nu V^\nu$. Disregarding the gauge-space sector for simplicity, the trace can be written as

$$\mathrm{tr}_1(\mathcal{O}^\mu{}_\nu) = \sum_A u_\mu^A \mathcal{O}^\mu{}_\nu u_A^\nu, \tag{B15}$$

where $u_A^\mu(x)$ is any local basis of vectors at x and $u_\mu^A(x)$ is its dual basis, $u_\mu^A u_B^\mu = \delta_B^A$. When $\mathcal{O}^\mu{}_\nu$ is purely multiplicative (i.e., it does not contain ∇_μ nor Z_I^R), the trace is simply

$$\mathrm{tr}_1(\mathcal{O}^\mu{}_\nu) = \mathcal{O}^\mu{}_\nu u_\mu^A u_A^\nu = \mathcal{O}^\mu{}_\nu g^\nu{}_\mu = \mathcal{O}^\mu{}_\mu. \tag{B16}$$

If the operator has a factor Z_I^R on the left, where I is any string of coordinate indices, without loss of generality we can assume that it has the form $Z_I^R\mathcal{H}^{I\mu}{}_\nu$. That is, the row-column indices $\mu\nu$ are not in Z_I^R, and all the indices in I are different. For instance, $Z_\alpha^{R\alpha\mu}X_\nu$ can be rewritten as $Z^R_{\alpha\beta\lambda}g^{\alpha\beta}g^{\mu\lambda}X_\nu$. Using now

$$\begin{aligned} \langle u_\mu^A|Z_I^R &= \langle 0|u_\mu^A Z_I^R = \langle 0|(Z_I^R u_\mu^A - [Z_I^R, u_\mu^A]) \\ &= \langle 0|(-C_I u_\mu^A + R_I{}^\lambda{}_\mu u_\lambda^A) = \langle u_\lambda^A|(R_I{}^\lambda{}_\mu - g^\lambda{}_\mu C_I), \end{aligned} \tag{B17}$$

it follows that

$$\begin{aligned} \mathrm{tr}_1(Z_I^R\mathcal{O}^\mu{}_\nu) &= \sum_A u_\mu^A Z_I^R \mathcal{O}^\mu{}_\nu u_A^\nu \\ &= \sum_A u_\mu^A (R_I{}^\mu{}_\lambda - g^\mu{}_\lambda C_I)\mathcal{O}^\lambda{}_\nu u_A^\nu \\ &= \mathrm{tr}_1((R_I{}^\mu{}_\lambda - g^\mu{}_\lambda C_I)\mathcal{O}^\lambda{}_\nu). \end{aligned} \tag{B18}$$

This proves Eqs. (5.9). If the operator contains more than one factor Z^R on the left, the procedure is applied recursively,

$$\begin{aligned} \mathrm{tr}_1(Z_I^R Z_J^R \mathcal{O}^\mu{}_\nu) &= \mathrm{tr}_1((R_I{}^\mu{}_\lambda - g^\mu{}_\lambda C_I)Z_J^R\mathcal{O}^\lambda{}_\nu) \\ &= \mathrm{tr}_1(Z_J^R(R_I{}^\mu{}_\lambda - g^\mu{}_\lambda C_I)\mathcal{O}^\lambda{}_\nu \\ &\quad - [Z_J^R, R_I{}^\mu{}_\lambda - g^\mu{}_\lambda C_I]\mathcal{O}^\lambda{}_\nu). \end{aligned} \tag{B19}$$

APPENDIX C: CANONICAL FORM OF Γ_S

The following formulas display the expression of the operators in $\Gamma_{S,n}$, $n = 1, 2, 3, 4$ in Eq. (6.7) using the basis of operators in (6.12) to bring them to the canonical form in (6.9), by means of the commutation identities in (6.11). These expressions hold in any spacetime dimension. Equations (C1)–(C4) correspond to $\Gamma_{L,1}$, $\Gamma_{L,2}$, $\Gamma_{L,3}$, and $\Gamma_{L,4}$, respectively,

$$\begin{aligned}
\Delta Y_{\mu\nu} &= \mathcal{O}_{19} Y_{\mu\nu},\\
\Delta_M W &= \mathcal{O}_2 W,\\
\frac{1}{2}\Delta_M\{P_\mu,\nabla_\mu\} &= (\mathcal{O}_1)_\mu P_\mu - \frac{1}{2}\mathcal{O}_2 P_{\mu\mu},
\end{aligned} \tag{C1}$$

$$\begin{aligned}
\Delta Y_{\alpha\mu}\Delta Y_{\mu\beta} &= \mathcal{O}_{20} Y_{\alpha\mu} Y_{\mu\beta} + O(\nabla^{-5}),\\
\Delta_M\Phi_\mu\Delta\Phi_\mu &= \mathcal{O}_{10}\Phi_\mu^2 + O(\nabla^{-5}),\\
\Delta_M\nabla_\mu\mathcal{M}_{\mu\nu}\Delta\mathcal{M}_{\nu\alpha}\nabla_\alpha &= (\mathcal{O}_3)_{\mu\nu}\mathcal{M}_{\mu\alpha}\mathcal{M}_{\alpha\nu} + 2(\mathcal{O}_6)_{\mu\nu\alpha}\mathcal{M}_{\nu\mu\beta}\mathcal{M}_{\beta\alpha} + 4(\mathcal{O}_{12})_{\mu\nu\alpha\beta}\mathcal{M}_{\alpha\nu\mu\rho}\mathcal{M}_{\rho\beta}\\
&\quad - (\mathcal{O}_{11})_{\mu\nu}(2\mathcal{M}_{\nu\mu\alpha}\mathcal{M}_{\beta\alpha\beta} + \mathcal{M}_{\alpha\alpha\mu\beta}\mathcal{M}_{\beta\nu} + 2\mathcal{M}_{\alpha\nu\mu\beta}\mathcal{M}_{\beta\alpha} + \mathcal{M}_{\alpha\beta}\mathcal{M}_{\beta\nu}\mathcal{R}_{\mu\alpha}\\
&\quad - 2F_{\mu\alpha}\mathcal{M}_{\alpha\beta}\mathcal{M}_{\beta\nu}) - (\mathcal{O}_5)_\mu(\mathcal{M}_{\mu\nu}\mathcal{M}_{\alpha\nu\alpha} + \mathcal{M}_{\nu\mu\alpha}\mathcal{M}_{\alpha\nu}) + O(\nabla^{-5}),\\
\Delta_M\Phi_\mu\Delta\mathcal{M}_{\mu\nu}\nabla_\nu &= (\mathcal{O}_5)_\mu\Phi_\nu\mathcal{M}_{\nu\mu} + \mathcal{O}_{10}(-\Phi_\mu\mathcal{M}_{\nu\mu\nu} - \Phi_{\mu\nu}\mathcal{M}_{\nu\mu}) + 2(\mathcal{O}_{11})_{\mu\nu}\Phi_{\mu\alpha}\mathcal{M}_{\alpha\nu} + O(\nabla^{-5}),\\
\Delta_M\nabla_\mu\mathcal{M}_{\mu\nu}\Delta\Phi_\nu &= (\mathcal{O}_5)_\mu\mathcal{M}_{\mu\nu}\Phi_\nu + 2(\mathcal{O}_{11})_{\mu\nu}\mathcal{M}_{\nu\mu\alpha}\Phi_\alpha + O(\nabla^{-5}),\\
\Delta_M W\Delta_M W &= (\mathcal{O}_{13}[W])_{\mu\nu} W,\\
\frac{1}{4}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\{P_\nu,\nabla_\nu\} &= (\mathcal{O}_4[P_\mu])_{\mu\nu}P_\nu + \frac{1}{2}(\mathcal{O}_7[P_{\mu\mu}])_\nu P_\nu - (\mathcal{O}_{16}[P_\mu, M_{\mu\nu\nu\alpha}])_{\alpha\beta}P_\beta - (\mathcal{O}_9[P_\mu, M_{\mu\nu\alpha}])_{\nu\alpha\beta}P_\beta\\
&\quad - (\mathcal{O}_{16}[P_\mu, M_{\nu\alpha}R_{\mu\nu\alpha\beta}])_{\beta\rho}P_\rho\\
&\quad + (\mathcal{O}_{18}[P_\mu, M_{\mu\nu\alpha}, M_{\nu\beta\rho}])_{\beta\rho\alpha\sigma}P_\sigma + (\mathcal{O}_{18}[P_\mu, M_{\mu\nu\alpha}, M_{\nu\beta\rho}])_{\alpha\beta\rho\sigma}P_\sigma\\
&\quad - \frac{1}{4}\mathcal{O}_{13}[P_{\mu\mu}]P_{\nu\nu} - \frac{1}{2}(\mathcal{O}_7[P_\mu])_\mu P_{\nu\nu} + \frac{1}{2}(\mathcal{O}_{16}[P_\mu, M_{\mu\nu\alpha}])_{\nu\alpha}P_{\beta\beta} + O(\nabla^{-5}),\\
\frac{1}{2}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M W &= (\mathcal{O}_7[P_\mu])_\mu W + \frac{1}{2}\mathcal{O}_{13}[P_{\mu\mu}]W - (\mathcal{O}_{16}[P_\mu, M_{\mu\nu\alpha}])_{\nu\alpha}W + O(\nabla^{-5}),
\end{aligned} \tag{C2}$$

$$\begin{aligned}
\Delta_M\nabla_\mu\mathcal{M}_{\mu\nu}\Delta Y_{\nu\alpha}\Delta\mathcal{M}_{\alpha\beta}\nabla_\beta &= (\mathcal{O}_{11})_{\mu\nu}\mathcal{M}_{\mu\alpha}Y_{\alpha\beta}\mathcal{M}_{\beta\nu} + O(\nabla^{-5}),\\
\Delta_M W\Delta_M\nabla_\mu\mathcal{M}_{\mu\nu}\Delta\mathcal{M}_{\nu\alpha}\nabla_\alpha &= (\mathcal{O}_{14}[W])_{\mu\nu}\mathcal{M}_{\mu\alpha}\mathcal{M}_{\alpha\nu} + O(\nabla^{-5}),\\
\frac{1}{2}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\nabla_\nu\mathcal{M}_{\nu\alpha}\Delta\mathcal{M}_{\alpha\beta}\nabla_\beta &= \frac{1}{2}(\mathcal{O}_{14}[P_{\mu\mu}])_{\nu\alpha}\mathcal{M}_{\nu\beta}\mathcal{M}_{\beta\alpha} + (\mathcal{O}_8[P_\mu])_{\mu\nu\alpha}\mathcal{M}_{\nu\beta}\mathcal{M}_{\beta\alpha} + 2(\mathcal{O}_{15}[P_\mu])_{\mu\nu\alpha\beta}\mathcal{M}_{\alpha\nu\rho}\mathcal{M}_{\rho\beta}\\
&\quad - (\mathcal{O}_{17}[P_\mu, M_{\mu\nu\alpha}])_{\nu\alpha\beta\rho}\mathcal{M}_{\beta\sigma}\mathcal{M}_{\sigma\rho}\\
&\quad - (\mathcal{O}_{14}[P_\mu])_{\mu\nu}(\mathcal{M}_{\nu\alpha}\mathcal{M}_{\beta\alpha\beta} + \mathcal{M}_{\alpha\nu\beta}\mathcal{M}_{\beta\alpha}) + O(\nabla^{-5}),\\
\frac{1}{2}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\nabla_\nu\mathcal{M}_{\nu\alpha}\Delta\Phi_\alpha &= (\mathcal{O}_{14}[P_\mu])_{\mu\nu}\mathcal{M}_{\nu\alpha}\Phi_\alpha + O(\nabla^{-5}),\\
\frac{1}{2}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\Phi_\nu\Delta\mathcal{M}_{\nu\alpha}\nabla_\alpha &= (\mathcal{O}_{14}[P_\mu])_{\mu\nu}\Phi_\alpha\mathcal{M}_{\alpha\nu} + O(\nabla^{-5}),\\
\frac{1}{4}\Delta_M W\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\{P_\nu,\nabla_\nu\} &= (\mathcal{O}_{16}[W, P_\mu])_{\mu\nu}P_\nu + O(\nabla^{-5}),\\
\frac{1}{8}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\{P_\nu,\nabla_\nu\}\Delta_M\{P_\alpha,\nabla_\alpha\} &= (\mathcal{O}_{16}[P_\mu, P_{\mu\nu}])_{\nu\alpha}P_\alpha + \frac{1}{2}(\mathcal{O}_{16}[P_\mu, P_{\nu\nu}])_{\mu\alpha}P_\alpha\\
&\quad + \frac{1}{2}(\mathcal{O}_{16}[P_{\mu\mu}, P_\nu])_{\nu\alpha}P_\alpha + (\mathcal{O}_9[P_\mu, P_\nu])_{\mu\nu\alpha}P_\alpha - (\mathcal{O}_{18}[P_\mu, M_{\mu\nu\alpha}, P_\beta])_{\nu\alpha\beta\rho}P_\rho\\
&\quad - (\mathcal{O}_{18}[P_\mu, P_\nu, M_{\nu\alpha\beta}])_{\alpha\beta\nu\rho}P_\rho - (\mathcal{O}_{18}[P_\mu, P_\nu, M_{\nu\alpha\beta}])_{\mu\alpha\beta\rho}P_\rho\\
&\quad - \frac{1}{2}(\mathcal{O}_{16}[P_\mu, P_\nu])_{\mu\nu}P_{\alpha\alpha} + O(\nabla^{-5}),
\end{aligned} \tag{C3}$$

$$\Delta_M\nabla_\mu\mathcal{M}_{\mu\nu}\Delta\mathcal{M}_{\nu\alpha}\nabla_\alpha\Delta_M\nabla_\beta\mathcal{M}_{\beta\rho}\Delta\mathcal{M}_{\rho\sigma}\nabla_\sigma = (\mathcal{O}_{15}[\mathcal{M}_{\mu\nu}\mathcal{M}_{\nu\alpha}])_{\mu\alpha\beta\rho}\mathcal{M}_{\beta\sigma}\mathcal{M}_{\sigma\rho} + O(\nabla^{-5}),$$
$$\frac{1}{4}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\{P_\nu,\nabla_\nu\}\Delta_M\nabla_\alpha\mathcal{M}_{\alpha\beta}\Delta\mathcal{M}_{\beta\rho}\nabla_\rho = (\mathcal{O}_{17}[P_\mu,P_\nu])_{\mu\nu\alpha\beta}\mathcal{M}_{\alpha\rho}\mathcal{M}_{\rho\beta} + O(\nabla^{-5}),$$
$$\frac{1}{16}\Delta_M\{P_\mu,\nabla_\mu\}\Delta_M\{P_\nu,\nabla_\nu\}\Delta_M\{P_\alpha,\nabla_\alpha\}\Delta_M\{P_\beta,\nabla_\beta\} = (\mathcal{O}_{18}[P_\mu,P_\nu,P_\alpha])_{\mu\nu\alpha\beta}P_\beta + O(\nabla^{-5}). \tag{C4}$$

APPENDIX D: PERTURBATIVE EXPANSIONS

For completeness, we give here the expressions for $\Gamma^{\rm div}_{L,0}$ and $\Gamma^{\rm div}_{S}$ in $d=2$ [Eqs. (6.32) and (6.16), respectively] to second order in an expansion in powers of $H^{\mu\nu}$, where

$$M^{\mu\nu} = m^2 g^{\mu\nu} + H^{\mu\nu} \tag{D1}$$

and to all orders in the other fields. m^2 is a constant c number. In $d=2$, the combination $\Gamma^{\rm div}_{L,1}+\Gamma^{\rm div}_{\rm gh}$ cancels, so there are no further contributions to $\Gamma^{\rm div}_1$.

The result is obtained by expanding Eqs. (6.32) and (6.16) and carrying out the angular average, and integration over z in the case of $\Gamma^{\rm div}_{L,0}$, as such evaluations can be done explicitly when $M^{\mu\nu}=m^2g^{\mu\nu}$,

$$\Gamma^{\rm div}_{L,0} = \frac{1}{4\pi\varepsilon}\left\langle \frac{1}{12}\mathbf{R} + \frac{1}{m^4}\left(-\frac{1}{12}F_{\mu\nu}H_{\mu\alpha}H_{\nu\alpha} - \frac{1}{96}H_{\mu\mu}H_{\nu\nu}\mathbf{R} + \frac{1}{48}H_{\mu\nu}H_{\mu\nu}\mathbf{R} \right.\right.$$
$$\left.\left. - \frac{1}{24}H_{\mu\mu\nu}H_{\nu\alpha\alpha} - \frac{1}{96}H_{\mu\nu\nu}H_{\mu\alpha\alpha} - \frac{1}{48}H_{\mu\nu\alpha}H_{\mu\nu\alpha} + \frac{1}{24}H_{\mu\nu\alpha}H_{\nu\mu\alpha}\right) + O(H^3)\right\rangle_{x,g}. \tag{D2}$$

$$\begin{aligned}
\Gamma^{\rm div}_{S} = \frac{1}{4\pi\varepsilon}\Bigg\langle & -\frac{1}{2}m^2 - \frac{1}{2}\mathbf{R} - \frac{1}{4}H_{\mu\mu} + \frac{1}{m^2}\left(-\frac{1}{2}\tilde{W} - \frac{1}{4}Q_{\mu\nu}Q_{\mu\nu} - \frac{1}{16}H_{\mu\mu}H_{\nu\nu} + \frac{1}{8}H_{\mu\nu}H_{\mu\nu}\right) \\
& + \frac{1}{m^4}\left(-\frac{1}{8}P_\mu P_\mu + \frac{1}{4}\tilde{W}H_{\mu\mu} + \frac{1}{16}Q_{\mu\nu}Q_{\mu\nu}H_{\alpha\alpha} + \frac{1}{8}Q_{\mu\nu}Q_{\mu\alpha}H_{\nu\alpha} + \frac{1}{4}Q_{\mu\nu}H_{\mu\alpha}H_{\nu\alpha}\right) \\
& + \frac{1}{m^6}\left(\frac{1}{16}P_\mu P_\mu H_{\nu\nu} + \frac{1}{8}P_\mu P_\nu H_{\mu\nu} - \frac{1}{24}P_\mu H_{\mu\nu}H_{\nu\alpha\alpha} + \frac{1}{24}P_\mu H_{\mu\nu}H_{\alpha\nu\alpha} + \frac{1}{96}P_\mu H_{\nu\nu}H_{\mu\alpha\alpha}\right. \\
& + \frac{1}{48}P_\mu H_{\nu\nu}H_{\alpha\mu\alpha} + \frac{1}{48}P_\mu H_{\nu\alpha}H_{\mu\nu\alpha} - \frac{1}{12}P_\mu H_{\nu\alpha}H_{\nu\mu\alpha} - \frac{1}{96}P_\mu H_{\mu\nu\nu}H_{\alpha\alpha} - \frac{1}{48}P_\mu H_{\mu\nu\alpha}H_{\nu\alpha} \\
& - \frac{1}{48}P_\mu H_{\nu\mu\nu}H_{\alpha\alpha} + \frac{1}{12}P_\mu H_{\nu\mu\alpha}H_{\nu\alpha} - \frac{1}{24}P_\mu H_{\nu\nu\alpha}H_{\mu\alpha} + \frac{1}{24}P_\mu H_{\nu\alpha\alpha}H_{\mu\nu} - \frac{1}{16}\tilde{W}H_{\mu\mu}H_{\nu\nu} \\
& - \frac{1}{8}\tilde{W}H_{\mu\nu}H_{\mu\nu} - \frac{1}{96}Q_{\mu\nu}Q_{\mu\nu}H_{\alpha\alpha}H_{\beta\beta} - \frac{1}{48}Q_{\mu\nu}Q_{\mu\nu}H_{\alpha\beta}H_{\alpha\beta} - \frac{1}{48}Q_{\mu\nu}Q_{\mu\alpha}H_{\nu\alpha}H_{\beta\beta} \\
& \left. - \frac{1}{24}Q_{\mu\nu}Q_{\mu\alpha}H_{\nu\beta}H_{\alpha\beta} - \frac{1}{24}Q_{\mu\nu}Q_{\mu\alpha}H_{\alpha\beta}H_{\nu\beta} - \frac{1}{48}Q_{\mu\nu}Q_{\mu\alpha}H_{\beta\beta}H_{\nu\alpha}\right) \\
& + \frac{1}{m^8}\left(-\frac{1}{96}P_\mu P_\mu H_{\nu\nu}H_{\alpha\alpha} - \frac{1}{48}P_\mu P_\mu H_{\nu\alpha}H_{\nu\alpha} - \frac{1}{48}P_\mu P_\nu H_{\mu\nu}H_{\alpha\alpha} - \frac{1}{24}P_\mu P_\nu H_{\mu\alpha}H_{\nu\alpha} - \frac{1}{24}P_\mu P_\nu H_{\nu\alpha}H_{\mu\alpha}\right. \\
& - \frac{1}{48}P_\mu P_\nu H_{\alpha\alpha}H_{\mu\nu} - \frac{1}{48}P_\mu H_{\mu\nu}P_\nu H_{\alpha\alpha} - \frac{1}{48}P_\mu H_{\mu\nu}P_\alpha H_{\nu\alpha} - \frac{1}{192}P_\mu H_{\nu\nu}P_\mu H_{\alpha\alpha} \\
& \left. - \frac{1}{96}P_\mu H_{\nu\alpha}P_\mu H_{\nu\alpha} - \frac{1}{48}P_\mu H_{\nu\alpha}P_\nu H_{\mu\alpha}\right) + O(H^3)\Bigg\rangle_{x,g}.
\end{aligned} \tag{D3}$$

In this formula $\tilde{W} \equiv W - \frac{1}{2}P_{\mu\mu}$.

The term $\mathbf{R}$ in $\Gamma^{\rm div}_{L,0}$ is the Gauss-Bonnet invariant in two dimensions, complying by itself with the transverse and longitudinal symmetry invariance discussed in Sec. VI C 5. In the two-dimensional case, these symmetries do not allow terms with one $H^{\mu\nu}$ nor of order $1/m^2$ in $\Gamma^{\rm div}_{L,0}$.

APPENDIX E: METHOD OF NONCOVARIANT SYMBOLS

1. Covariant vs noncovariant method of symbols

The method of noncovariant symbols [57,58] allows one to obtain diagonal matrix elements of pseudodifferential operators and it can be extended to curved spacetime [37].

The difference between the covariant and noncovariant versions can be elucidated already in the case of flat spacetime. Let $\hat{f} = f(D, X)$ be a pseudodifferential operator constructed out of the gauge-covariant derivative D_μ and one (or more) non-Abelian fields $X(x)$ (a purely multiplicative operator). Then,

$$\begin{aligned}\langle x|\hat{f}|x\rangle &= \int \frac{d^d p}{(2\pi)^d}\langle x|\hat{f}|p\rangle\langle p|x\rangle \\ &= \int \frac{d^d p}{(2\pi)^d} e^{-xp}\langle x|\hat{f}|p\rangle \\ &= \int \frac{d^d p}{(2\pi)^d}\langle x|e^{-xp}\hat{f}e^{xp}|0\rangle \\ &= \int \frac{d^d p}{(2\pi)^d}\langle x|f(D+p,X)|0\rangle, \end{aligned} \tag{E1}$$

where p_μ is imaginary and $|0\rangle$ is the state $\langle x|0\rangle = 1$. The quantity $\langle x|f(D+p,X)|0\rangle$ is the (noncovariant) symbol of $\hat{f}$.

After momentum integration, the result is gauge covariant in the sense that D_μ will only appear in the form of a commutator $[D_\mu,]$. This follows from the fact that under the shift $D_\mu \to D_\mu + a_\mu$, where a_μ is an arbitrary constant imaginary c number, the dependence on a_μ cancels upon momentum integration (since a_μ can be compensated by a corresponding shift in p_μ). The virtue of the covariant symbol,

$$\bar{f} = e^{-D\partial} f(D+p,X) e^{D\partial}, \tag{E2}$$

is that it is multiplicative (with respect to x) and is already covariant without momentum integration [37].

To illustrate the point, let us apply the method of noncovariant symbols to $\hat{f} = (D_\mu^2 - X)^{-1}$,

$$\langle x|\hat{f}|x\rangle = \left\langle \frac{1}{(p_\mu + D_\mu)^2 - X}\right\rangle_p. \tag{E3}$$

Defining $N = 1/(p_\mu^2 - X)$ and expanding in powers of D_μ,

$$\begin{aligned}\langle x|\hat{f}|x\rangle = \langle & N - N(2p_\mu D_\mu + D_\mu^2)N \\ & + N(2p_\mu D_\mu)N(p_\nu D_\nu)N + O(D^3)\rangle_p. \end{aligned} \tag{E4}$$

Instead of doing the momentum integration, one can add terms which are identically zero by integration by parts in momentum space to bring the expression to a covariant form. For instance, the first order term[18]

$$\langle x|\hat{f}|x\rangle^{(1)} = \langle -2p_\mu N D_\mu N\rangle_p \tag{E5}$$

can be supplemented with

$$0 = \langle \partial^\mu(-D_\mu N)\rangle_p = \langle 2p_\mu D_\mu N^2\rangle_p, \tag{E6}$$

yielding a manifestly gauge-covariant result

$$\langle x|\hat{f}|x\rangle^{(1)} = \langle 2p_\mu [D_\mu, N]N\rangle_p. \tag{E7}$$

This procedure can be carried out systematically. Rewriting Eq. (E4) as

$$\begin{aligned}&\langle x|\hat{f}|x\rangle = \langle T_0 + T_1 + T_2 + \cdots\rangle_p, \\ &T_0 = N, \qquad T_1 = -2NpDN, \\ &T_2 = -NDDN + 4NpDNpDN, \end{aligned} \tag{E8}$$

the systematic integration by parts suggested by the method of covariant symbols can be implemented as[19]

$$\begin{aligned}e^{-D\partial}\sum_n T_n|0\rangle &= \sum_n \tilde{T}_n|0\rangle, \\ \tilde{T}_n &= \sum_{j=0}^{n}\frac{(-1)^j}{j!}(D\partial)^j T_{n-j}, \end{aligned} \tag{E9}$$

and now

$$\langle x|\hat{f}|x\rangle = \langle \tilde{T}_0 + \tilde{T}_1 + \tilde{T}_2 + \cdots\rangle_p. \tag{E10}$$

In this way,

$$\tilde{T}_0 = T_0 = N \tag{E11}$$

is already covariant, and

$$\tilde{T}_1 = T_1 - D\partial T_0 = -2NpDN - D\partial N = 2[pD, N]N \tag{E12}$$

is the term obtained previously in Eq. (E7). For the second order,

[18] Actually this term vanishes by parity; nevertheless, it serves to illustrate the point.

[19] Equivalently, one can put instead a factor e^{pD} at the right and move ∂^μ to left.

$$\tilde{T}_2 = T_2 - D\partial T_1 + \frac{1}{2}(D\partial)^2 T_0 = -NDDN + 4NpDNpDN - D\partial(-2NpDN) + \frac{1}{2}(D\partial)^2 N. \tag{E13}$$

Carrying out the momentum derivatives and moving the D_μ to the left yields a manifestly covariant form

$$\tilde{T}_2 = -N_{\mu\mu}N + 4p_\mu p_\nu (N_\mu N_\nu + N_{\mu\nu}N)N, \tag{E14}$$

where

$$N_\mu = [\nabla_\mu, N], \qquad N_{\mu\nu} = [\nabla_\mu, N_\nu]. \tag{E15}$$

2. Method of noncovariant symbols in curved spacetime

The results presented in this work, obtained using the method of covariant symbols, have been reproduced also using the method of noncovariant symbols. For the latter, the technique in Eqs. (E9) and (E10) is applied, which produces covariant expressions without requiring further integration by parts in momentum space. This holds too in curved spacetime using the covariant derivative ∇_μ which includes all connections, and $\frac{1}{2}\{\nabla_\mu, \partial^\mu\}$ instead of $D_\mu\partial^\mu$, in the exponential.

In this approach, typically the ∂^μ [generated in Eq. (E9)] are removed by moving them to the right. Then the free ∇_μ are moved together, say to the right, to form covariant combinations $Z_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$, etc. Therefore, one has to specify how ∇_μ commutes with p_ν and ∂^ν, as these commutators do not vanish (consistently with $[Z_{\mu\nu}, p_\alpha] = R_{\mu\nu\alpha\lambda}p_\lambda$).

To this end, let us introduce a set of coordinates $\xi^A(x)$ (technically d coordinate scalars). Eventually, after all free covariant derivative operators have been removed, these will be the Riemann coordinates at x_0 (the point at which the diagonal matrix element is being computed). And let the vector fields $t^A_\mu(x)$ and $t^\mu_A(x)$ be defined through the relations

$$t^A_\mu = [\nabla_\mu, \xi^A], \qquad t^A_\mu t^\mu_B = \delta^A_B. \tag{E16}$$

Note that the d vector fields $t^A_\mu(x)$ do not define a tetrad. In terms of these,

$$p_\mu = t^A_\mu p_A, \qquad \partial^\mu = t^\mu_A \partial^A, \qquad \partial^A \equiv \partial/\partial p_A, \tag{E17}$$

where p_A are (imaginary) constant c-number parameters, hence $[\nabla_\mu, p_A] = [\nabla_\mu, \partial^A] = 0$. This allows one to write

$$[\nabla_\mu, p_\nu] = t^A_{\mu\nu}p_A = t^\lambda_A t^A_{\mu\nu}p_\lambda \equiv -G_\mu{}^\lambda{}_\nu p_\lambda,$$
$$[\nabla_\mu, \partial^\nu] = G_\mu{}^\nu{}_\lambda \partial^\lambda, \tag{E18}$$

using $[\nabla_\mu, t^\nu_A] = -t^\nu_B t^B_{\mu\alpha} t^\alpha_A$ in the second relation. Note that $t^A_{\mu\nu} = t^A_{\nu\mu}$, hence $G_{\mu\lambda\nu} = G_{\nu\lambda\mu}$. Successive derivatives require derivatives of $G_{\mu\nu\alpha}(x)$ which in turn follow from those of $t^A_\mu(x)$. Here it enters the Riemann coordinate condition at x_0, which requires [53]

(i) $t^A_\mu = \delta^A_\mu$ at x_0, and
(ii) the vanishing of the completely symmetry component of $t^A_{\mu_1\cdots\mu_n}$ at x_0 for $n \geq 2$.

From these conditions, using the Jacobi identity (7.19), it follows

$$t^A_{\mu\nu}|_{x_0} = 0,$$
$$t^A_{\alpha\mu\nu}|_{x_0} = \frac{1}{3}(R_{\alpha\mu\nu}{}^\lambda + R_{\alpha\nu\mu}{}^\lambda)t^A_\lambda,$$
$$t^A_{\alpha\beta\mu\nu}|_{x_0} = \left(\frac{1}{4}R_{\alpha\beta\mu\nu}{}^\lambda + \frac{1}{4}R_{\alpha\beta\nu\mu}{}^\lambda + \frac{1}{6}R_{\beta\alpha\mu\nu}{}^\lambda + \frac{1}{6}R_{\beta\alpha\nu\mu}{}^\lambda + \frac{1}{12}R_{\mu\alpha\nu\beta}{}^\lambda + \frac{1}{12}R_{\nu\alpha\mu\beta}{}^\lambda\right)t^A_\lambda. \tag{E19}$$

Hence,

$$G_{\mu\lambda\nu}(x_0) = 0,$$
$$G_{\alpha\mu\lambda\nu}(x_0) = -\delta^\lambda_A t^A_{\alpha\mu\nu}(x_0),$$
$$G_{\alpha\beta\mu\lambda\nu}(x_0) = -\delta^\lambda_A t^A_{\alpha\beta\mu\nu}(x_0). \tag{E20}$$

It is important to remark that the conditions at $x = x_0$ can only be imposed after all free ∇_μ have been removed, and also that the p_μ's in N_g and N_M have to be differentiated too; p^2_μ cannot be treated as a constant during this calculation. On the other hand, as proven in Appendix C of [35], in the final expression (i.e., after x is set to x_0) one can freely integrate by parts neglecting derivatives of p_μ.

APPENDIX F: SAMPLE CALCULATION USING THE METHOD OF COVARIANT SYMBOLS

As the method of covariant symbols is not well known, we will present here the calculation of one of the universal functional traces of [18] using this method. Concretely, we consider

$$\langle x|\nabla_\mu \Delta|x\rangle^{\rm div}_{d=4}. \tag{F1}$$

We want the UV divergent part in $4 + 2\varepsilon$ dimensions. Therefore, we need the terms of order $1/p^4$ of the covariant symbol of $\nabla_\mu\Delta$. Since this operator is of order $1/p$ and the expansion of the covariant symbols is in powers of ∇/p, it will necessary to go to order $(\nabla/p)^3$. That is,

$$(\overline{\nabla_\mu \Delta})_{-4} = (\bar{\nabla}_\mu)_1(\bar{\Delta})_{-5} + (\bar{\nabla}_\mu)_{-2}(\bar{\Delta})_{-2}, \tag{F2}$$

where the subindex indicates the number of p_μ minus the number of ∂^μ in the term. Note that $(\bar{\nabla}_\mu)_0 = 0$, as can be seen in Eq. (7.2), and also $(\bar{\Delta})_{-3} = 0$ due to $(\overline{\nabla^2})_1 = 0$.

Equations (B1) of [35] show the expressions of $(\bar{\nabla}_\mu)_n$ for $n = 1, \ldots, -3$ and of $(\bar{\nabla}^2)_n$ for $n = 2, \ldots, -2$ (i.e., to four covariant derivatives). The expressions of $(\bar{\Delta})_n$ are not tabulated there but they need to be computed only once, using

$$\bar{\Delta} = (\overline{\nabla^2})^{-1} = ((\overline{\nabla^2})_2 + (\overline{\nabla^2})_0 + (\overline{\nabla^2})_{-1} + O(p^{-2}))^{-1}. \quad \text{(F3)}$$

This gives

$$(\bar{\Delta})_{-2} = (p_\mu p_\nu g^{\mu\nu})^{-1} = -N_g \quad \text{(F4)}$$

[with N_g already defined in (7.10)] and

$$(\bar{\Delta})_{-5} = -N_g (\overline{\nabla^2})_{-1} N_g. \quad \text{(F5)}$$

The formulas $(\bar{\nabla}_\mu)_1 = p_\mu$ and $(\bar{\Delta})_{-2} = -N_g$, as well as the tabulated values of $(\bar{\nabla}_\mu)_{-2}$ and $(\overline{\nabla^2})_{-1}$, are inserted in Eq. (F2) and this results in an expression containing ∂_μ. These derivatives are removed by using

$$[\partial^\mu, N_g] = 2p^\mu N_g^2 \quad \text{(F6)}$$

and (7.4). Also this very equation is used to move all $Z_{\mu_1 \cdots \mu_n}$ to the right. This procedure yields

$$\langle x|\nabla_\mu \Delta|x\rangle^{\rm div}_{d=4} = \Big\langle -\frac{2}{3}N_g^2 Z_{\nu\nu\mu} - \frac{2}{3}N_g^3 p_\mu p_\nu Z_{\alpha\alpha\nu} - \frac{8}{3}N_g^3 p_\nu p_\alpha Z_{\nu\alpha\mu} + \frac{1}{2}N_g^3 p_\mu p_\nu \mathbf{R}_\nu + N_g^3 p_\nu p_\alpha \mathcal{R}_{\mu\nu\alpha} - 2N_g^3 p_\nu p_\alpha \mathcal{R}_{\nu\alpha\mu} \Big\rangle_p. \quad \text{(F7)}$$

The integral over p_μ is immediate and gives

$$\langle x|\nabla_\mu \Delta|x\rangle^{\rm div}_{d=4} = -\frac{1}{(4\pi)^2\varepsilon}\left(\frac{1}{6}Z_{\alpha\alpha\mu} - \frac{1}{8}\mathbf{R}_\mu\right), \quad \text{(F8)}$$

using $\mathcal{R}_{\nu\nu\mu} = \frac{1}{2}\mathbf{R}_\mu$.

This result can be expressed in terms of the operator $\hat{\mathcal{R}}_I$. Using the relations $Z_I = F_I + Z_I^R$ and $Z_I^R = \hat{\mathcal{R}}_I + C_I$, and also $C_{\alpha\alpha\mu} = \frac{1}{4}\mathbf{R}_\mu$, from Eq. (5.8), one obtains

$$\langle x|\nabla_\mu \Delta|x\rangle^{\rm div}_{d=4} = -\frac{1}{(4\pi)^2\varepsilon}\left(\frac{1}{6}F_{\alpha\alpha\mu} + \frac{1}{6}\hat{\mathcal{R}}_{\alpha\alpha\mu} - \frac{1}{12}\mathbf{R}_\mu\right), \quad \text{(F9)}$$

a result in agreement with Eq. (4.54) of [18]. Other universal functional traces are also reproduced with the method of covariant symbols.

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