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ZARISKI MULTIPLICITY CONJECTURE IN FAMILIES OF NON-DEGENERATE SINGULARITIES

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Abstract. We give a new, elementary proof of the Zariski multiplicity conjecture in μ-constant families of non-degenerate singularities.

1. Introduction

One of the most longstanding conjectures in singularity theory is the Zariski multiplicity conjecture [Zar71] that if two hypersurface singularities are embedded topologically equivalent, then their multiplicities (= the orders of reduced functions defining them), are the same. By definition, two hypersurface singularities, not necessarily isolated, $(V,0) = (V(f),0)$ and $(W,0) = (V(\widetilde{f}),0)$ in $\mathbb{C}^n$ are *embedded topologically equivalent* iff there exists a homeomorphism $\Phi : (U,0) \to (U',0)$ of small neighbourhoods of the origin in $\mathbb{C}^n$ which transforms $V \cap U$ onto $W \cap U'$. Fifty years have passed, but the conjecture has been solved only in a few special cases. Information on these particular results one can find in the survey by Ch. Eyral [Eyr07] (up to 2007) and in the monograph by the same author [Eyr16]. One of the general results is that the conjecture is true for plane curve singularities (because in this case, we have complete, discrete characteristics of embedded topological types, for instance so-called Puiseux pairs, and one member of this characteristic is the multiplicity). It seems to be a simpler problem to prove the conjecture for pairs $f, \widetilde{f}$ that are members of a holomorphic family (f_t) of pairwise embedded topologically equivalent singularities. But this last assumption is implied, in the case of isolated singularities, by the fact that (f_t) is μ-constant, i.e., the Milnor number $\mu(f_t)$ at 0 in this family is constant. This follows from the Lê and Ramanujam theorem [LR76]. Because of this, B. Teissier [Tei77] posed the following conjecture

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Conjecture 1 (B. Teissier)**.** *Let (f_t) be a holomorphic family of isolated singularities. If $\mu(f_t)$ is constant, then (f_t) is equimultiple.*

Until recently, this has been a wide-open problem, except for several special cases that have been settled. In [dBP22], J. F. de Bobadilla and T. Pełka announced a positive solution to Teissier's conjecture. Since, however, this paper counts 80 pages and has not yet been published in a recognized journal, the result still requires independent confirmation. Somewhat earlier, Y. O. M. Abderrahmane [Abd16] proved this conjecture in the case the family is additionally non-degenerate, i.e., all f_t are non-degenerate in the Kushnirenko sense. He proved even more – that the family (f_t) is also topologically trivial. He used advanced results of the singularity theory (characterizations of (c)-regularity and μ-constancy). In the paper, we give a simpler, elementary proof of the Teissier conjecture in the Abderrahmane case, based on the recent result by M. Leyton-Álvarez, H. Mourtada and M. Spivakovsky [LÁMS21] concerning a characterization of the difference of the Newton polyhedra of singularities with the same Newton number.

2. Preliminaries

Let $0 \neq f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function defined by a convergent power series $\sum_{\nu\in\mathbb{N}^n} a_\nu z^\nu$, $z = (z_1, \dots, z_n)$, $\nu = (\nu_1, \dots, \nu_n)$. Let $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geqslant 0,\ i = 1, \dots, n\}$. We define $\operatorname{supp} f := \{\nu \in \mathbb{N}^n : a_\nu \neq 0\} \subset \mathbb{R}^n_+$ and the *Newton polyhedron* $\Gamma_+(f) \subset \mathbb{R}^n_+$ *of* f as the convex hull of the set $\{\nu + \mathbb{R}^n_+ : \nu \in \operatorname{supp} f\}$. It is a non-compact polyhedron with a finite number of vertices $\operatorname{Vert}(f)$. We say f is *convenient* if $\Gamma_+(f)$ has non-empty intersection with each coordinate x_i-axis, $i = 1, \dots, n$. Let $\Gamma(f)$ be the set of compact boundary faces of any dimension of $\Gamma_+(f)$ – the *Newton boundary* of f. Denote by $\Gamma^k(f)$ the subset of $\Gamma(f)$ of all k-dimensional faces, $k = 0, \dots, n-1$. Then $\Gamma(f) = \bigcup_k \Gamma^k(f)$ and $\Gamma^0(f) = \operatorname{Vert}(f)$. Elements of $\Gamma^1(f)$ we will call *edges*. For each $(n-1)$-dimensional face (compact) $S \in \Gamma^{n-1}(f)$ we denote by $v_S = (v_1, \dots, v_n)$ the unique vector, perpendicular to S with positive, integer coordinates satisfying $\operatorname{GCD}(v_1, \dots, v_n) = 1$. From this we get that the projection of any $S \in \Gamma^{n-1}(f)$ on any coordinate hyperplane $H_i := \{x \in \mathbb{R}^n : x_i = 0\}$ is a linear homeomorphism. For each face $S \in \Gamma(f)$ of any dimension, we define the quasihomogeneous polynomial $f_S := \sum_{\nu\in S} a_\nu z^\nu$. We say f is *non-degenerate on* S if the system of polynomial equations $\partial f_S/\partial z_i = 0$, $i = 1, \dots, n$, has no solution in $(\mathbb{C}^*)^n$; f is *non-degenerate (in the Kushnirenko sense)* if f is non-degenerate on each face $S \in \Gamma(f)$.

For convenient f we define $\Gamma_-(f)$ as $\overline{\mathbb{R}^n_+ \setminus \Gamma_+(f)}$. It is a compact polyhedron (not necessarily convex) which is the union of cones over faces from $\Gamma^{n-1}(f)$ with vertex at 0. We define the Newton number $\nu(f)$ of f as

$$\nu(f) := n!\, V_n - (n-1)!\, V_{n-1} + \cdots + (-1)^{n-1} V_1 + (-1)^n,$$

where V_n is the n-dimensional volume of $\Gamma_-(f)$ and V_i is the sum of the i-dimensional volumes of the intersections of $\Gamma_-(f)$ with all the coordinate hyperplanes of dimension i, $1 \leqslant i \leqslant (n-1)$. The Newton number may also be defined in the non-convenient case, but we will not use this notion. Now we recall two important results. The first one is the formula for the Milnor number of an isolated singularity f in the generic case, expressed in terms of the Newton polyhedron of f.

Theorem 1 (Kushnirenko [Kou76])**.** *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic, convenient function with an isolated critical point at 0 (= an isolated singularity) and $\mu(f)$ be the Milnor number of f at 0. Then*

$$\mu(f) \geqslant \nu(f)$$

and the equality holds if f is non-degenerate. Moreover, non-degeneracy is a generic property in the space of coefficients corresponding to integer points of $\Gamma(f)$.

The second result is a recent one, by M. Leyton-Álvarez, H. Mourtada and M. Spivakovsky [LÁMS21, Thm. 2.25], giving a characterization of the difference of the Newton polyhedra of isolated singularities with the same Newton number. The same result in the particular case of isolated surface singularities ($n = 3$) was proved in [BKW19]. We will formulate this theorem in a form convenient for us.

Theorem 2. *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two holomorphic, convenient functions such that $\Gamma_+(f) \subsetneq \Gamma_+(g)$ (equivalently $\Gamma_-(g) \subsetneq \Gamma_-(f)$). Then $\nu(f) = \nu(g)$ if and only if for each vertex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathrm{Vert}(g) \setminus \mathrm{Vert}(f)$:*

1. α lies in one of the coordinate hyperplanes H_i, i.e., there exists $i \in \{1, \dots, n\}$ such that $\alpha_i = 0$. Denote the set of such i by I.

2. There exists $i_0 \in I$ for which there exists a unique edge $\overline{\alpha\beta'}$ of $\Gamma_+(g)$, $\beta' \in \mathrm{Vert}(g)$, which does not lie in H_{i_0}. Moreover, there exists $\beta = (\beta_1, \dots, \beta_n) \in \overline{\alpha\beta'} \cap \mathrm{Vert}(f)$ with coordinates $\beta_{i_0} = 1$ and $\beta_i = 0$ for $i \in I \setminus \{i_0\}$.

Remark 3. *The possible configurations for $n = 3$ are illustrated in Fig. 1 (the case $\beta' = \beta$) and Fig. 2 (the case $\beta' \neq \beta$). Notice that in the case $\beta' \neq \beta$ the segment $\overline{\alpha\beta'}$ is an extension of the segment $\overline{\alpha\beta}$.*

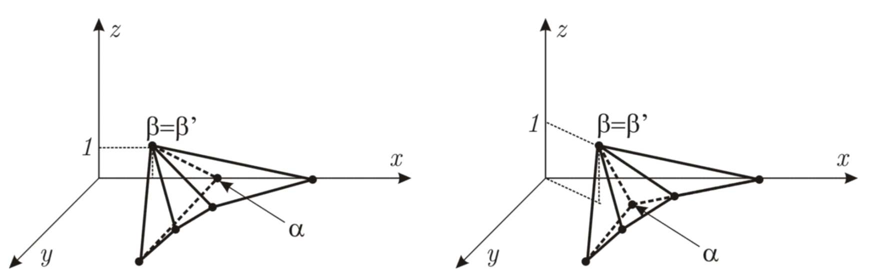

FIGURE 1.

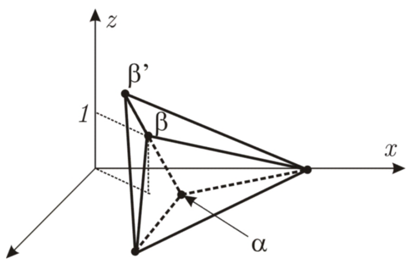

FIGURE 2.

Remark 4. *Geometrically, if* $\operatorname{Vert}(g) \setminus \operatorname{Vert}(f)$ *consists of only one vertex* $\alpha \in H_{i_0}$ *(as in Fig. 1 and Fig. 2), then conditions 1 and 2 in the theorem mean that the difference* $\overline{\Gamma_-(f) \setminus \Gamma_-(g)}$ *is an* n*-dimensional pyramid of height* 1 *with the apex in* β *and the base in* H_{i_0}*, and, moreover,* β *has the same zero coordinates as* α *except for one equal to* 1*.*

We also recall the following monotonicity property (see e.g. [Gwo08]).

Proposition 5. *If* Γ_1*,* Γ_2 *are two convenient Newton polyhedra of holomorphic functions such that* $\Gamma_1 \subset \Gamma_2$*, then*

$$\nu(\Gamma_2) \leqslant \nu(\Gamma_1) < \infty.$$

3. The main theorem

Let $0 \neq f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function defined by a convergent power series $\sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu$. By $\operatorname{ord} f$ we denote the order of f. If f is *reduced* in $\mathbb{C}\{z_1, \dots z_n\}$, i.e., has no multiple factors in the factorization into irreducible elements in $\mathbb{C}\{z_1, \dots z_n\}$, then the multiplicity $\operatorname{mult} V(f)$ of $V(f)$ is equal to $\operatorname{ord} f$. Before the main theorem, we give a geometric lemma that easily follows from properties of the Newton polyhedron of a holomorphic function gathered in Preliminaries. By $\operatorname{Pr}_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$ we denote the projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}$: $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$ and, accordingly, Pr_i $(1 \leqslant i \leqslant n-1)$.

Lemma 6. *If* f *is convenient and we put* $\delta := \bigcup \Gamma^{n-1}(f)$*, the union of compact* $(n-1)$*-dimensional faces of* $\Gamma_+(f)$*, then, for any* $i \in \{1, \dots, n\}$*, we have* $\operatorname{Pr}_i(\delta) = \Gamma_-(f) \cap H_i$ *and the restriction* $\operatorname{Pr}_i|_\delta$ *is a homeomorphism (piecewise linear). In particular, if* $\Gamma^{n-1}(f) = \{S_1, \dots, S_k\}$*, then* $\operatorname{Pr}_i(S_1), \dots, \operatorname{Pr}_i(S_k)$ *are* $(n-1)$*-dimensional convex polyhedra which give a partition of* $\Gamma_-(f) \cap H_i$ *preserving the boundary relation. Moreover, from any point* $\tilde{\alpha} \in \operatorname{Pr}_i(S_j)$ *we "see" all the vertices of* S_j*, i.e., the segments joining* $\tilde{\alpha}$ *with the vertices of* S_j *lie in* $\Gamma_-(f)$*.*

Now we may pass to the main aim of our paper – a new proof of the Abderrahmane theorem.

Theorem 7. *Let (f_t) be a holomorphic family of isolated, non-degenerate singularities, where t is a parameter in a neighbourhood of* 0 *in* $\mathbb{C}$. *If $\mu(f_t)$ is constant, then* $\operatorname{ord} f_t$ *is also constant for small t.*

The same assertion holds for any holomorphic family of functions (f_t) if f_0 is convenient and $\nu(f_t) = \text{const}$.

Proof. Notice that the first part of the theorem follows from the second one. Indeed, if (f_t) are non-degenerate, isolated singularities, then if we add to f_t the sum of specific monomials $a_1 z_1^N + \cdots + a_n z_n^N$ with sufficiently large N and generic $a_1, \ldots, a_n$, we get a new holomorphic family of *convenient*, isolated singularities, which are also non-degenerate with the same Milnor numbers and the same orders as f_t. By the Kushnirenko theorem, we now have constant ν for this new family; thus, we may assume from the beginning that f_0 is some convenient function and $\nu(f_t) = \text{const}$.

Let us pass to the proof of the second part of the theorem. Because both the Newton number and the multiplicity depend only on the Newton diagram, we may change f_t at will, demanding that $\operatorname{Supp} f_t = \operatorname{Vert} f_t$, for all $|t| \ll 1$; in particular, $\operatorname{Supp} f_0$ is finite. Clearly, $\Gamma_+(f_0) \subset \Gamma_+(f_t)$ and we may assume the containment is strict for $t \neq 0$. Hence, Proposition 5 allows us to reduce the problem further, to the case where $\Gamma_+(f_0)$ and $\Gamma_+(f_t)$ "differ by one point only", i.e., $\Gamma_+(f_t) = \operatorname{conv}(\Gamma_+(f_0), \alpha)$, where $\{\alpha\} = \operatorname{Vert} f_t \setminus \operatorname{Vert} f_0$ $(t \neq 0)$. Accordingly, we may put $f_t := f_0 + t \cdot z^\alpha$. Now, let us note the following

Claim. *We may additionally assume that in f_0 there are no surplus vertices (not on any axis), in the sense that removing any vertex monomial from f_0 changes its Newton number.*

Proof of Claim. Indeed, let ι be a vertex of $\Gamma_+(f_0)$ not lying on any axis and let $c_\iota \cdot z^\iota$ be the corresponding monomial with the property that for $\tilde{f}_0 := f_0 - c_\iota \cdot z^\iota$ we have $\Gamma_+(\tilde{f}_0) \subsetneq \Gamma_+(f_0)$ and $\nu(\tilde{f}_0) = \nu(f_0)$. Set $\tilde{f}_t := f_t - c_\iota \cdot z^\iota$. We have

$$\nu(\tilde{f}_0) \geqslant \nu(\tilde{f}_t) \geqslant \nu(f_t) = \nu(f_0) = \nu(\tilde{f}_0),$$

where the inequalities follow from the monotonicity of the Newton number (Proposition 5). Thus, $\nu(\tilde{f}_t) = \nu(\tilde{f}_0)$. Moreover, we obviously have $\operatorname{ord} \tilde{f}_0 \geqslant \operatorname{ord} f_0$, with strict inequality if, and only if, $|\iota| = \operatorname{ord} f_0$ and $c_\iota \cdot z^\iota$ is the only monomial appearing in f_0 and having the degree equal to $\operatorname{ord} f_0$. It follows that if we prove $\operatorname{ord} \tilde{f}_t = \operatorname{ord} \tilde{f}_0$, then $\operatorname{ord} f_t = \min(\operatorname{ord} \tilde{f}_t, |\iota|) = \min(\operatorname{ord} \tilde{f}_0, |\iota|) = \operatorname{ord} f_0$. Note also that we still have $\tilde{f}_t = \tilde{f}_0 + t \cdot z^\alpha$ and $\Gamma_+(\tilde{f}_t) = \operatorname{conv}(\Gamma_+(\tilde{f}_0), \alpha)$, where $\{\alpha\} = \operatorname{Vert} \tilde{f}_t \setminus \operatorname{Vert} \tilde{f}_0$, $t \neq 0$. Hence, we may replace the pair (f_0, f_t) by $(\tilde{f}_0, \tilde{f}_t)$ in our reasoning. Repeating this reduction finitely many times (bounded by the number of elements of $\operatorname{Supp} f_0$), we reach the conclusion of the claim. $\diamondsuit$

Continuing the main reasoning, we have $\operatorname{ord} f_t = \text{const}$ for $t \neq 0$, and we need to prove $\operatorname{ord} f_t = \operatorname{ord} f_0$. By upper semicontinuity of the order, we have

$$\operatorname{ord} f_t \leqslant \operatorname{ord} f_0.$$

Assume to the contrary that

$$\operatorname{ord} f_t < \operatorname{ord} f_0. \tag{3.1}$$

If $\alpha = (\alpha_1, \dots, \alpha_n)$, then from (3.1)

$$\alpha_1 + \cdots + \alpha_n < \operatorname{ord} f_0. \tag{3.2}$$

By assumption, the family is ν-constant, i.e, $\nu(f_t) = \nu(f_0)$. Since of course $\Gamma_-(f_t) \subsetneq \Gamma_-(f_0)$ for $t \neq 0$, Theorem 2 implies that the vertex α lies in one of the coordinate hyperplanes, say H_n, i.e., $\alpha = (\alpha_1, \dots, \alpha_{n-1}, 0)$, and α is a vertex of the unique edge $\overline{\alpha\beta'}$ of $\Gamma_+(f_t)$ not lying in H_n, which joins α with $\beta' \in \operatorname{Vert}(f_t)$ and for which there exists $\beta = (\beta_1, \dots, \beta_{n-1}, 1) \in \overline{\alpha\beta'} \cap \operatorname{Vert}(f_0)$ satisfying $\beta_i = 0$ if $\alpha_i = 0$ $(i \neq n)$. Since $\alpha \in \Gamma_-(f_t) \cap H_n \subset \Gamma_-(f_0) \cap H_n$, by Lemma 6 we have that $\alpha \in \operatorname{Pr}_n(S)$, for some $S \in \Gamma^{n-1}(f_0)$.

We shall show that the face S has only one vertex, exactly β, not lying in H_n. To this end, we will first exclude vertices outside the set $\{\beta, \beta'\}$. Indeed, suppose there is a vertex $\gamma \notin \{\beta, \beta'\}$ of S not lying in H_n. Since, by Lemma 6, γ is visible from α and the edge $\overline{\alpha\beta'}$ of $\Gamma_+(f_t)$ is the unique one containing α and lying outside H_n, it follows that $\gamma \notin \operatorname{Vert}(f_t)$ for $t \neq 0$. Consider $g_0 := f_0 - c_\gamma z^\gamma$, i.e., f_0 without the monomial corresponding to γ. Note that $\gamma \notin H_n$ cannot lie on any axis; otherwise, γ would still be a vertex of $\Gamma_+(f_t)$ for $t \neq 0$. Hence, g_0 is convenient. By the Claim, we have that $\nu(g_0) > \nu(f_0)$. Putting $g_t := f_t - c_\gamma z^\gamma$, we get $\Gamma_+(g_t) = \Gamma_+(f_t)$ $(t \neq 0)$ because $\gamma \notin \operatorname{Vert}(f_t)$ for $t \neq 0$. Thus, $\infty > \nu(g_0) > \nu(f_0) = \nu(f_t) = \nu(g_t)$. This contradicts Theorem 2 because $\{\alpha\} = \operatorname{Vert}(g_t) \setminus \operatorname{Vert}(g_0) = \operatorname{Vert}(f_t) \setminus \operatorname{Vert}(f_0)$ and $\Gamma_+(g_t) = \Gamma_+(f_t)$ $(t \neq 0)$ so we should have $\nu(g_0) = \nu(g_t)$.

Now, note that for $\beta \neq \beta'$ we must also have $\beta' \notin S$; for, in the opposite case, $\beta \in \overline{\alpha\beta'}$ and we cannot "see" the point β' from α, contrary to Lemma 6.

Summing up, the only vertex of S outside H_n is β, i.e., S is a pyramid with the apex β and the base $T \in \Gamma^{n-2}(f_0)$, where T is an $(n-2)$-dimensional convex polyhedron lying in H_n (see Fig. 3).

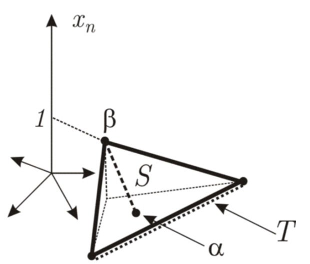

FIGURE 3.

Of course, $\alpha \notin T$ as T is a face of $\Gamma(f_0)$. From (3.2)

$$\alpha_1 + \cdots + \alpha_{n-1} < \operatorname{ord} f_0 \leqslant \beta_1 + \cdots + \beta_{n-1} + 1$$

and, hence,

$$\alpha_1 + \cdots + \alpha_{n-1} \leqslant \beta_1 + \cdots + \beta_{n-1}. \tag{3.3}$$

Consider the hyperplane $\Pi : x_1 + \cdots + x_{n-1} = \beta_1 + \cdots + \beta_{n-1}$ in H_n, treated as $\mathbb{R}^{n-1}$, which passes through $\mathrm{Pr}_n(\beta)$. Then from (3.3), α lies beneath or on Π. Since S is a pyramid with the base T lying in H_n and the apex β, $\mathrm{Pr}_n(S)$ is also a pyramid with the base T and the apex $\mathrm{Pr}_n(\beta)$. Notice $\mathrm{Pr}_n(\beta) \neq \alpha$ because otherwise the edge $\overline{\alpha\beta}$ would be vertical (perpendicular to H_n). Hence the unique line passing through $\mathrm{Pr}_n(\beta)$ and $\alpha \in \mathrm{Pr}_n(S)$ intersects the base T in a point, say $\kappa = (\kappa_1, \ldots, \kappa_{n-1})$. Of course

$$\kappa_1 + \cdots + \kappa_{n-1} \leqslant \alpha_1 + \cdots + \alpha_{n-1} \tag{3.4}$$

as α lies between $\mathrm{Pr}_n(\beta)$ and κ on this line, and by (3.3). Since T is a convex, compact polyhedron and has points lying beneath the hyperplane $\widetilde{\Pi} : x_1 + \cdots + x_{n-1} = \alpha_1 + \cdots + \alpha_{n-1}$ (by (3.4)), it also has a vertex lying beneath $\widetilde{\Pi}$. But such a vertex is in $\mathrm{supp}(f_0)$ and, hence, we obtain a contradiction with (3.1). $\square$

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