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AN ESTIMATION OF THE JUMP OF THE MILNOR NUMBER OF PLANE CURVE SINGULARITIES

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Abstract. The jump of the Milnor number of an isolated singularity f_0 is the minimal non-zero difference between the Milnor numbers of f_0 and one of its deformations f_s. We estimate the jump using the Enriques diagram of f_0.

1. Introduction

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an **isolated singularity**, i.e. a function germ for which there exists a representative $\widehat{f_0} : U \to \mathbb{C}$ of f_0, holomorphic in an open neighbourhood U of the point $0 \in \mathbb{C}^n$ such that $\widehat{f_0}(0) = 0$, $\nabla\widehat{f_0}(0) = 0$, $\nabla\widehat{f_0}(z) \neq 0$ for $z \in U \setminus \{0\}$. We put $\nabla f := \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$. In the sequel a singularity means an isolated singularity.

A **deformation of a singularity** f_0 is the germ of a holomorphic function $f = f(s, z) : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that

(1) $f(0, z) = f_0(z)$,
(2) $f(s, 0) = 0$.

The deformation $f(s, z)$ of the singularity f_0 will also be treated as a family (f_s) of function germs, taking $f_s(z) := f(s, z)$. Since f_0 is an isolated singularity, f_s for sufficiently small s also has isolated singularities near 0 ([GLS06] Theorem 2.6 I). Hence, for sufficiently small s one can define the **Milnor number of** f_s

$$\mu_s := \mu(f_s) = \dim_{\mathbb{C}} \mathcal{O}_n/(\nabla f_s),$$

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where $\mathcal{O}_n$ is the ring of holomorphic function germs at 0, and (∇f_s) is the ideal in $\mathcal{O}_n$ generated by $\frac{\partial f_s}{\partial z_1}, \ldots, \frac{\partial f_s}{\partial z_n}$.

Since the Milnor number is upper semi-continuous in the Zariski topology in families of singularities ([GLS06], Theorem 2.6 I and Proposition 2.57 II), there exists an open neighbourhood S of the point 0 such that

(1) $\mu_s = \text{const.}$ for $s \in S \setminus \{0\}$,
(2) $\mu_0 \geq \mu_s$ for $s \in S$.

The constant difference $\mu_0 - \mu_s$ (for $s \neq 0$) will be called **the jump of the deformation** (f_s) and denoted by $\lambda((f_s))$. The smallest non-zero value among all the jumps of deformations of the singularity f_0 will be called **the jump of the Milnor number of the singularity** f_0 and denoted by $\lambda(f_0)$.

From now on, we will consider only plane curve singularities $f_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$.

The first general result concerning the jump of the Milnor number was obtained by Sabir Gusein-Zade([GZ93]), who proved that there exist singularities f_0 for which $\lambda(f_0) > 1$ and gave some sufficient conditions for which $\lambda(f_0) = 1$. These conditions are in terms of branches and the resolution process of plane surve singularities. In particular from his result follows $\lambda(f_0) = 1$ for irreducible plane singularities.

S. Brzostowski, T. Krasiński and J. Walewska in [BKW21] proved that for the special reducible singularities $f_0^n(x, y) = x^n + y^n$, $n \geq 2$, we have $\lambda(f_0) = \left[\frac{n}{2}\right]$. Determining the jump of a singularity is a difficult task because it is not a topological invariant ([BK14], [dPW95] Section 7.3). For specific classes of deformations i.e. for non-degenerated deformations (it means each element of the family f_s is a non-degenerated singularity in the Kouchnirenko sense [Kou76]) the jump problem was considered in [Bod07], [Wal13], [BKW21], [KW19].

One of the results of this article is an extension of the Gusein-Zade result ([GZ93]) by giving a next sufficient condition for plane curve singularities f_0 under which $\lambda(f_0) = 1$ (Theorem 4.1). Our methods give also the Gusein-Zade conditions.

The second result of the article (Theorem 3.1) is an estimation (from above) of $\lambda(f_0)$ in terms of branches and the resolution process of plane curve singularities using previous result concerning the jump in the case f_0 is a homogenous (quasi-homogenous) singularities ([Zak17],[Zak]).

We obtain both above results in the framework of narrower class of deformations - linear deformations of the form $f_0 + sg$, where g is a holomorphic function in the neighbourhood of 0 such that $g(0) = 0$. We will denote the jump of f_0 for this class of deformations by $\lambda^{lin}(f_0)$. Of course $\lambda(f_0) \leq \lambda^{lin}(f_0)$ and so any estimation of $\lambda^{lin}(f_0)$ from above is automatically an estimation of $\lambda(f_0)$,

To get this formula the **Enriques diagrams** will be used. To any singularity we can assign a weighted Enriques diagram (D, ν) which represents the whole resolution process of this singularity ([CA00] Chapter 3.9). It is a tree with two types of edges and a weight function $\nu : D \to \mathbb{Z}$ on vertices of the diagram. M. Alberich-Carraminñana and J. Roé ([ACR05] Theorem 1.3, Remark 1.4) gave a necessary and sufficient condition for two Enriques diagrams of singularities to be linear adjacent. It means that one singularity is a linear deformation of another. They used a wider class of Enriques diagrams, so-called abstract Enriques diagrams, which are described in Section 2.

In Section 3 we estimate the jump $\lambda(f_0)$ in terms of its Enriques diagram and in Section 4 we give sufficient conditions under which $\lambda^{lin}(f_0) = 1$ and consequently $\lambda(f_0) = 1$.

2. Enriques diagrams

Information about abstract Enriques diagrams can be found in [ACR05] and [KP99]. Moreover in my previous paper [Zak17], in which I gave the estimation of $\lambda^{lin}(f_0)$ for homogeneous singularities, abstract Enriques diagrams are described in more details with examples.

Definition 2.1. *An **abstract Enriques diagram** (in short an Enriques diagram) is a rooted tree D with a binary relation between vertices, called proximity, which satisfies:*

(1) *The root is proximate to no vertex.*
(2) *Every vertex that is not the root is proximate to its immediate predecessor.*
(3) *No vertex is proximate to more than two vertices.*
(4) *If a vertex Q is proximate to two vertices, then one of them is the immediate predecessor of Q and this is proximate to the other.*
(5) *Given two vertices P, Q with Q proximate to P, there is at most one vertex proximate to both of them.*

The fact that Q is proximate to P we will denote by $Q \to P$. The vertices which are proximate to two points are called **satellite**, the other vertices (except the root) are called **free**. The vertex is **final** if has no successor. To show graphically the proximity relation, Enriques diagrams are drawn according to the following rules:

(1) If Q is a free successor of P, then the edge going from P to Q is smooth and curved and, if P is not the root, it has at P the same tangent as the edge joining P to its predecessor.
(2) The sequence of edges connecting a maximal succession of vertices proximate to the same vertex P are shaped into a line segment, orthogonal to the edge joining P to the first vertex of the sequence.

The example of an abstract Enriques diagram is shown in Figure 1.

We will now introduce few basic notations that are needed in the sequel.

A **weight function** of an abstract Enriques diagram D is any function $\nu : D \to \mathbb{Z}$ defined on vertices of D. A pair (D, ν), where D is an abstract Enriques diagram

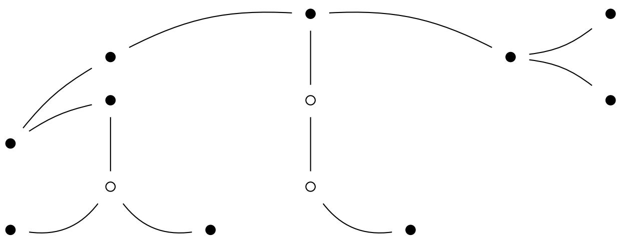

FIGURE 1. The abstract Enriques diagram. Satellite vertices are marked in white

and ν its weight function, is called a **weighted Enriques diagram**. A **consistent Enriques diagram** is a weighted Enriques diagram such that for all $P \in D$

$$\nu(P) \geq \sum_{Q \to P} \nu(Q). \tag{1}$$

A **complete Enriques diagram** is a weighted Enriques diagram such that for all non-final $P \in D$ the equality in (1) holds and for all final $P \in D$ it is a free vertex with weight 1 not proximate to another free vertex with weight 1. To the weight function ν of a weighted diagram D we associate a **system of values**, which is another map $\mathrm{ord}_\nu : D \to \mathbb{Z}$, defined recursively as

$$\mathrm{ord}_\nu(P) := \begin{cases} \nu(P), & \text{if } P \text{ is the root,} \\ \nu(P) + \sum\limits_{P \to Q} \mathrm{ord}_\nu(Q), & \text{otherwise.} \end{cases}$$

For any consistent (D, ν) we define **the Milnor number of** (D, ν) by

$$\mu((D,\nu)) := \sum_{P \in D} \nu(P)(\nu(P) - 1) + 1 - r_D,$$

where $r_D := \sum_{P \in D} r_D(P)$, $r_D(P) := \Big(\nu(P) - \sum_{Q \to P} \nu(Q)\Big)$ for every $P \in D$.

A **subdiagram** of an abstract Enriques diagram D is a subtree $D_0 \subset D$ with the same proximity relation such that if $Q \in D_0$ then its predecessor belongs to D_0.

In the class of weighted Enriques diagrams, we introduce equivalence relation. We say that weighted diagrams (D, ν) and (D', ν') are **equivalent** if they differ at most in free vertices of weight 1. The equivalence class of (D, ν) is denoted by $[(D, \nu)]$ and called the **type** of (D, ν). Of course, the Milnor number is constant in the class $[(D, \nu)]$.

A **minimal Enriques diagram** is a consistent Enriques diagram (D, ν) with:

(1) no free vertices of weight 0,
(2) no free vertices of weight 1 except for these such $P \in D$ for which there exists a satellite vertex $Q \in D$ satisfying $Q \to P$.

It is easy to see ([Zak17], Theorem 2.12) that

Theorem 2.2. *Let (D,ν) be a consistent weighted diagram. There exists exactly one minimal diagram which belongs to $[(D,\nu)]$.*

The theory of Enriques diagrams has its roots in the theory of plane curve singularities. The embedded resolution of a plane curve singularity using blow-ups can be explicitly presented as a complete Enriques diagram. A precise description can be found in [CA00] Chapter 3.8 and Chapter 3.9. Two plane curve singularities are topologically equivalent if and only if their Enriques diagrams are isomorphic (as graphs). For the Enriques diagram (D,ν) of a plane curve singularity f_0, the weight function represents the orders of the consecutive proper transforms of f_0 while the system of values – the orders of the total transforms. The number $r_D(P)$ equals to the number of branches at P of a proper transform of f_0 for which next blow-up at P "resolve" these branches. Hence, r_D is the number of branches of f_0. Moreover (D,ν) is complete. We need only the next fact which easily follows from these results.

Theorem 2.3 ([CA00] Theorem 3.8.6)**.** *There exists a bijection between minimal Enriques diagrams and topological types of singularities.*

In the paper [ACR05], M. Alberich-Carramiñana and J. Roé gave a necessary and sufficient condition for two Enriques diagrams of singularities to be linear adjacent. This is the key result we will use in the sequel. First we give definitions.

Definition 2.4. *Let (D,ν) and (D',ν') be weighted Enriques diagrams, with (D',ν') consistent. We will write $(D',\nu')\geq(D,\nu)$ when there exist isomorphic subdiagrams $D_0\subset D$, $D_0'\subset D'$ with an isomorphism (that preserves proximity relations)*

$$i: D_0\to D_0'$$

such that the new weight function $\kappa: D\to\mathbb{Z}$ for D, defined by

$$\kappa(P):=\begin{cases}\nu'(i(P)), & P\in D_0\\ 0, & P\notin D_0\end{cases}$$

satisfies

$$\mathrm{ord}_\nu(P)\leq\mathrm{ord}_\kappa(P)$$

for any $P\in D$.

Definition 2.5. *Let $[(D,\nu)]$ and $[(\widetilde{D},\widetilde{\nu})]$ be types of Enriques diagrams. $[(\widetilde{D},\widetilde{\nu})]$ is* ***linear adjacent*** *to $[(D,\nu)]$ if there exists a consistent Enriques diagram $(D',\nu')\in[(\widetilde{D},\widetilde{\nu})]$ such that $(D',\nu')\geq(D_{min},\nu_{min})$, where (D_{min},ν_{min}) is the minimal diagram of type $[(D,\nu)]$.*

Theorem 2.6 ([ACR05] Theorem 1.3 and Remark 1.4)**.** *Let $[(D,\nu)]$ and $[(\widetilde{D},\widetilde{\nu})]$ be types of consistent Enriques diagrams. The following conditions are equivalent:*

(1) $[(\widetilde{D},\widetilde{\nu})]$ *is linear adjacent to* $[(D,\nu)]$.

(2) *For every singularity f_0 whose Enriques diagram belongs to $[(\widetilde{D}, \widetilde{\nu})]$, there exists a linear deformation (f_s) of f_0 such that the Enriques diagram of a generic element f_s belongs to $[(D, \nu)]$.*

(3) *There exists a singularity f_0 whose Enriques diagram belongs to $[(\widetilde{D}, \widetilde{\nu})]$ and a linear deformation (f_s) of f_0 such that the Enriques diagram of a generic element f_s belongs to $[(D, \nu)]$.*

This theorem was also formulated using prime divisors by J. Fernández de Bobadilla, M. Pe Pereira and P. Popescu-Pampu in Theorem 3.25 ([dBPPP17]).

Theorems 2.3 and 2.6 imply the following corollary:

Corollary 2.7. *$\lambda^{lin}(f_0)$ is a topological invariant.*

3. Estimation of the jump of the Milnor number for linear deformation

Let $f_0 : \mathbb{C}^2 \to \mathbb{C}$ be a singularity and (D, ν) its minimal Enriques diagram. The jump of the Milnor number for linear deformation can be estimated as follows.

Theorem 3.1.

$$\lambda^{lin}(f_0) \leq \min\{l(P) : P \text{ - } a \text{ } leaf \text{ } in \text{ } D\},$$

where l(P) can be read from the table

vertex P \ $\nu(P)$	*root*	*free*	*satellite*
1	-	-	*1*
2	*1*	*1*	*2*
≥ 3	$\nu(P) - 2$	$\nu(P) - 1$	$\nu(P)$

Proof. Let $D_L = \{L_1, \dots, L_m\}$ be a set of leaves of (D, ν). For each $i = 1, \dots, m$ we will define the diagram (E_i, λ_i) by a modification of (D, ν), for which the difference of the Milnor number of (E_i, λ_i) and (D, ν) is equal to $l(L_i)$. If $\nu(L_i) = 1$ we remove only the L_i from (D, ν) and this will be (E_i, λ_i). If $\nu(L_i) = 2$ and L_i is the root, then E_i will have only one vertex with weight 1. If $\nu(L_i) = 2$ and L_i is not a root we change the weight of L_i to 1 and add one additional satellite vertex W with weight 1, so that $W \to L_i$ (Figure 2(a)) and this will be (E_i, λ_i).

If $\nu(L_i) \geq 3$ we change the weight of L_i to $\nu(L_i) - 1$ and add new vertices free U and satellite $W_1, \dots, W_{\nu(L_i)-3}$ (if $\nu(L_i) = 3$ there is no W_j vertices), all proximate to L_i. The weight of new vertices are: $\lambda_i(U) = 2$, $\lambda_i(W_j) = 1$ (for

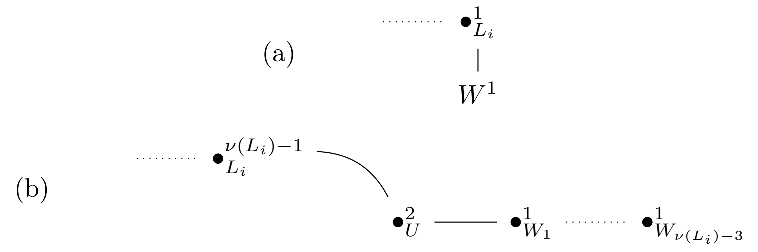

FIGURE 2. The Enriques diagram (E, λ)

$j = 1, \dots, \nu(L_i) - 3)$. The proximity relation between new vertices is

$$\begin{aligned} &W_{\nu(L_i)-3} \to W_{\nu(L_i)-4}, L_i \\ &\dots \\ &W_2 \to W_1, L_i \\ &W_1 \to U, L_i \\ &U \to L_i, \end{aligned}$$

see Figure 2(b).

It is easy to check that (E_i, λ_i) is a minimal (and hence consistent) diagram and that $(E_i, \lambda_i) \notin [(D, \nu)]$. From the above detailed description of (E_i, λ_i) we easily show that $[(D, \nu)]$ is linear adjacent to $[(E_i, \lambda_i)]$.

Now we may compute the Milnor number of (E_i, λ_i). It is easy to notice that

$$r_{E_i} = \begin{cases} r_D + 1, & \text{if } \nu(L_i) = 1 \\ r_D - 1, & \text{if } \nu(L_i) = 2, L_i \text{ is a root} \\ r_D - 2 + w_{L_i}, & \text{if } \nu(L_i) = 2, L_i \text{ is not a root} \\ r_D - d + 2 + w_{L_i}, & \text{if } \nu(L_i) \geq 3 \end{cases},$$

where w_{L_i} is a number of vertices to which L_i is proximate to. Then we get $\mu\left((E_i, \lambda_i)\right) = \mu\left((D, \nu)\right) - l(L_i)$. Since this formula is true for every $i = 1, \dots, m$ and from Theorem 2.6 we get $\lambda^{lin}(f_0) \leq \min_{i=1,\dots,m} l(L_i)$. □

Hence, we get a corollary for the general jump $\lambda(f_0)$.

Corollary 3.2.

$$\lambda(f_0) \leq \min\{l(P) : P \text{ - a leaf in } D\}.$$

Remark 3.3. *In Theorem 3.1 the estimation cannot be replace by an equality. Let consider the singularity $f_0(x, y) = x^8 + y^5$, its minimal Enriques diagram (D, ν) is shown in Figure 3. It is easy to check that $[(D, \nu)]$ is linear adjacent to $[(E, \lambda)]$ shown in Figure 4. Since $\mu((D, \nu)) - \mu((E, \lambda)) = 22 - 21 = 1$, we have $\lambda^{lin}(f_0) = 1$. On the other hand from Theorem 3.1 we get only such an estimation $\lambda^{lin}(f_0) \leq 3 - 1 = 2$.*

Figure 3. Minimal Enriques diagram of $f_0(x,y) = x^8 + y^5$

Figure 4. Enrqiues diagram (E, λ)

4. Singularities with the Milnor number 1

In this Section we gave next sufficient conditions for plane curve singularities f_0 under which $\lambda(f_0) = 1$. In [GZ93] Gusain-Zade proved that if for a singularity f_0 there exists a maximal exceptional divisor which intersects no more than three other components of the total preimage of the curve $f_0 = 0$, then $\lambda(f_0) = 1$. In terms of Enriques diagram this condition is equivalent to the first three conditions of the next theorem. We add the next one condition.

Theorem 4.1. *Let $f_0 : \mathbb{C}^2 \to \mathbb{C}$ be a singularity and (D, ν) its minimal diagram. If one of below conditions is true:*

(1) *there exists a leaf $P \in D$ such that P is satellite with weight 1,*
(2) *the diagram D contains only root with weight 2,*
(3) *there exists a leaf $P \in D$ such that P is free with weight 2,*
(4) $\nu(R_D) \geq 2 + \sum_{P \to R_D} \nu(P)$ *and there exists $P \in D$ such that* $\nu(P) = \nu(R_D) - 2$,

then $\lambda(f_0) = \lambda^{lin}(f_0) = 1$.

Proof. If (D, ν) satisfies one of first three conditions then from Theorem 3.1 we get immediately that $\lambda(f_0) = \lambda^{lin}(f_0) = 1$.

If (D, ν) satisfies the fourth condition we will construct (E, λ) such that, $[(D, \nu)]$ is linear adjacent to $[(E, \lambda)]$ and $\mu((E, \lambda)) = \mu((D, \nu)) - 1$. Let $\{P_1, \ldots, P_m\}$ will be the set of vertices of the diagram D, where P_1 is a root, and $\nu(P_2) = \nu(P_1) - 2$. We can assume that $\nu(P_2) < \nu(\tilde{P_2})$ where $\tilde{P_2}$ is a predecessor of P_2. Indeed, otherwise (if their weights are the same) we take $\tilde{P_2}$ instead of P_2. We put $E = D$ with changed weights λ,

$$\lambda(P_i) = \begin{cases} \nu(P_1) - 1, & \text{if } i = 1 \\ \nu(P_2) + 1, & \text{if } i = 2 \\ \nu(P_i), & \text{if } i \geq 3 \end{cases}.$$

The diagram E is consistent and it is easy to check that $[(D, \nu)]$ is linear adjacent to $[(E, \lambda)]$. Since $r_E = r_D - 1$, $\mu((E, \lambda)) = \mu((D, \nu)) - 1$. □

Remark 4.2. *The singularity f_0 from Remark 3.3 is an example of a singularity that does not meet the first three conditions and meets the last one.*

Remark 4.3. *The above theorem seems to describe all Enriques diagrams of singularity such that* $\lambda^{lin}(f_0) = 1$.

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