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CONVEXIFYING OF POLYNOMIALS BY CONVEX FACTOR

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Abstract. Let $X \subset \mathbb{R}^n$ be a convex closed and semialgebraic set and $b : \mathbb{R}^n \to (0, +\infty)$ be a $\mathscr{C}^2$ class positive strongly convex function. Let f be a polynomial positive on X. If X is compact, we prove that there exists an exponent $N \geq 1$, such that for any $\xi \in X$, the function $\varphi_{N,\xi}(x) = b^N(x-\xi)f(x)$ is strongly convex on X. If $X = \{\xi \in \mathbb{R}^n : f(\xi) \leq r\}$ is bounded we define a mapping $\kappa_N : X \ni \xi \mapsto \operatorname{argmin}_X \varphi_{N,\xi} \in \mathbb{R}^n$, where $\operatorname{argmin}_X \varphi_{N,\xi}$ is the unique point $x \in X$ at which $\varphi_{N,\xi}$ has a global minimum. We prove that κ_N is a mapping of class $\mathscr{C}^1$ of X onto $Y = \kappa_N(X) \subset X$ and that for any $\xi \in X$ the limit of the iterations $\lim_{\nu\to\infty} \kappa_N^\nu(\xi)$ exists and belongs to the set $\sum_f$ of critical points of f. If additionally b is logarithmically strongly convex then κ_N is injective and it is defined on $\mathbb{R}^n$, provided f takes only positive values and the leading form of f is positive except of the origin. In the case $b(x) = \exp|x|^2$ and $f|_X$ has only one critical value we prove that the mapping $X \ni \xi \mapsto \lim_{\nu\to\infty} \kappa_N^\nu(\xi) \in \Sigma_f \cap X$ is continuous. Moreover, assuming that $\lim_{\nu\to\infty} \kappa_N^\nu(\xi) = 0$ we study convergence of the sequence of the spherical parts of $\kappa_N^\nu(\xi)$, $\nu \in \mathbb{N}$.

1. Introduction

The first goal of the paper is to study convexification of polynomial functions by a positive strongly convex function $b : \mathbb{R}^n \to \mathbb{R}$ of class $\mathscr{C}^k$, $k \geq 2$. More precisely, we will prove that (see Corollary 5.1): *If a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ is positive on a compact and convex set $X \subset \mathbb{R}^n$, then there exists an effectively calculable positive integer N_0 such that for any $N \geq N_0$ the function*

$$\varphi_N(x) = b(x)^N f(x)$$

is strongly convex on X. The exponent N_0 depend on $R = \max\{|x| : x \in X\}$, $S = \max\{b(x) : x \in X\}$, the size of coefficients of the polynomial f and $m > 0$

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such that $f(x) \geq m$ for $x \in X$. In case the polynomial f has integer coefficients finding N is fully effective (see Section 7).

A stronger version of the above result we give in Corollary 5.2; *there exists an integer N_0, which can be explicitly estimated, such that for any $N \geq N_0$ the functions*

$$\varphi_{N,\xi}(x) = b(x-\xi)^N f(x), \quad \xi \in X,$$

are strongly convex on X.

The second goal of the paper is to construct a mapping κ_N and investigate its properties. Namely, in the case when $X_{f\leq r} := \{x \in \mathbb{R}^n : f(x) \leq r\} \subset X$, where $r \in \mathbb{R}$ and X is a closed ball, we prove that *the mapping $\kappa_N : X_{f\leq r} \to X_{f\leq r}$ defined by*

$$\kappa_N(\xi) = \operatorname{argmin}_X \varphi_{N,\xi}$$

is of class $\mathscr{C}^{k-1}$ (see Lemma 4.2 and Corollary 5.6). Moreover, *it is a diffeomorphism of class $\mathscr{C}^{k-1}$ provided b is logarithmically strongly convex*, i.e., $\ln b$ is strongly convex (see Lemma 4.3 and Corollary 5.6). For a strongly convex function $g : Y \to \mathbb{R}$ on a closed and convex set Y the unique point $x_0 \in Y$ at which g has a global minimum on Y we denote by $\operatorname{argmin}_Y g$. In Theorem 4.8 we give some properties of the iterations κ_N^ν of the mapping κ_N and prove that: $\kappa_{N,*}(\xi) := \lim_{\nu\to\infty} \kappa_N^\nu(\xi)$ *exists and belongs to the set $X_{f\leq r} \cap \Sigma_f$* of critical points of f in $X_{f\leq r}$. Note that the set of fixed points of κ_N is equal to $X_{f\leq r} \cap \Sigma_f$ (see Lemma 4.5).

Analogous results for unbounded sets we obtain in Section 6 under assumption that b is logarithmically strongly convex and that the *leading form* f_d of f (i.e., a homogeneous polynomial f_d such that $\deg(f - f_d) < \deg f$) satisfy

$$f_d(x) > 0 \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}. \tag{1.1}$$

In Section 8 we give some results on the convergence of the sequence $\kappa_N^\nu(\xi)$, provided $b(x) = \exp|x|^2$. We prove that there is a neighbourhood $U \subset \mathbb{R}^n$ of the set of points, where the function f takes the smallest value such that the mapping assigning to each point $\xi \in U$ the limit point $\kappa_{N,*}(\xi)$ of the proximal algorithm is continuous (see Proposition 8.17). Moreover, we prove that the sequence $\kappa_N^\nu|_U$ uniformly converges to $\kappa_{N,*}|_U$. Without the assumption on U, the assertion of Proposition 8.17 does not hold (see Remark 8.18). We also show that the curve connecting successively the points $\kappa_N^\nu(\xi)$, $\xi \in X$, defined by the formula (8.19), shows a number of properties similar to those of the trajectory of the gradient field $\frac{1}{2N}\nabla(\ln f)$ (see Section 8.2). At the end of the paper we consider the problem of convergence of the sequence of the spherical parts $\kappa_N^\nu(\xi)/|\kappa_N^\nu(\xi)|$ of the sequence $\kappa_N^\nu(\xi)$, provided $\kappa_N^\nu(\xi) \to 0$ as $\nu \to \infty$ (see Fact 8.21).

In the special case when $b(x) = 1 + |x|^2$, a similar results to Corollary 5.1 and Theorem 4.8 are known. In [5, Theorem 5.1] there was proved that: *If a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ is positive on a compact and convex set $X \subset \mathbb{R}^n$, then there exists an effectively calculable positive integer N_0 such that for any integer $N \geq N_0$ the function*

$$\phi_N(x) = (1 + |x|^2)^N f(x)$$

is strongly convex on X. Moreover, a stronger version of [5, Theorem 5.1] was given in [5]; *there exists an effectively calculable positive integer* N_1 *such that for any integer* $N \geq N_1$ *the polynomials* $\phi_{N,\xi}(x) = (1+|x-\xi|^2)^N f(x)$, $\xi \in X$, *are strongly convex on* X. This is a crucial fact for a construction of a proximal algorithm which for a given polynomial f, positive in the convex compact semialgebraic set X, produces a sequence $\xi_\nu \in X$ starting from an arbitrary point $\xi_0 \in X$, defined by induction: $\xi_\nu = \operatorname{argmin}_X \phi_{N,\xi_{\nu-1}}$. The sequence ξ_ν converges to a lower critical point of f on X (see [5, Theorem 7.5]), i.e., a point $a \in X$ for which there exists a neighborhood $\Omega \subset \mathbb{R}^n$ such that $\langle x - a, \nabla f(a)\rangle \geq 0 \quad$ for every $x \in X \cap \Omega$, where ∇f is the gradient of f in the Euclidean norm. In the case of non-compact closed convex set X, under the assumption (1.1) we have that: *if the polynomial* f *is positive on* X *then for any* $R > 0$ *there exists* N_R *such that for any* $\xi \in X$, $|\xi| \leq R$, $N > N_R$ *the polynomial* $\phi_{N,\xi}$ *is strongly convex on* X. Similar results to the above were obtained in [7] for the functions $\psi_{N,\xi}(x) := e^{N|x-\xi|^2} f(x)$ and $\Psi_{N,\xi}(x) := e^{e^{N|x-\xi|^2}} f(x)$.

2. Auxiliary results

2.1. **Convex functions.** Let $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$. The function f is called *convex* if the set X is convex and for any $x, y \in X$ and $0 < t < 1$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

If the above inequality holds with $<$ for $x \neq y$, the function is called *strictly convex.*

Let f be a real function of class $\mathscr{C}^2$ defined on a neighbourhood of a convex set $X \subset \mathbb{R}^n$.

Denote by $\partial_v f(x)$ the directional derivative of the function f in the direction of a vector $v \in \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$, and by $\partial_v^2 f(x)$ the second order derivative of f in the direction v at x. If $v = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 is on the ith place, we write traditionally $\partial_v f = \frac{\partial f}{\partial x_i}$. Then the gradient $\nabla f : X \to \mathbb{R}^n$ of f is of the form

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right).$$

For any $a \in X$ and $v \in \mathbb{R}^n$ we put $I_{a,v} = \{t \in \mathbb{R} : a + tv \in X\}$. Obviously, the set $I_{a,v}$ is an interval or a single point. Recall some known facts (cf. [11]).

Fact 2.1. *The following conditions are equivalent:*

(a) *The function* f *is convex.*

(b) *For any vector* $v \in \mathbb{R}^n$ *and any* $a \in X$ *the function* $I_{a,v} \ni t \mapsto \partial_v f(a+tv) \in \mathbb{R}$ *is increasing.*

(c) *For any vector* $v \in \mathbb{R}^n$ *and any* $a \in X$ *we have* $\partial_v^2 f(a) \geq 0$.

Fact 2.2. *The following conditions are equivalent:*

(a) *The function f is strictly convex.*

(b) *For any vector $v \in \mathbb{R}^n$ of positive length and any $a \in X$ the function $I_{a,v} \ni t \mapsto \partial_v f(a+tv) \in \mathbb{R}$ is strictly increasing.*

(c) *The function f is convex and for any vector $v \in \mathbb{R}^n$ of positive length and any $a \in X$ the set $\{t \in I_{a,v} : \partial_v^2 f(a+tv) = 0\}$ is novhere dense in $I_{a,v}$, provided $I_{a,v}$ is an interval.*

A function $g : X \to \mathbb{R}$ is called *strongly convex* or *μ-strongly convex*, $\mu > 0$, if $X \subset \mathbb{R}^n$ is a convex set and for any $x, y \in X$ and $0 < t < 1$,

$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y) - t(1-t)\frac{\mu}{2}|x-y|^2,$$

If additionally g is of class $\mathscr{C}^1$ then the above condition is equivalent to

$$g(y) \geq g(x) + \langle y - x, \nabla g(x)\rangle + \frac{\mu}{2}|y-x|^2 \quad \text{for } x, y \in X,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$. Obviously, any strongly convex function is strictly convex and consequently, it is also convex.

Denote by S^{n-1} the unit sphere in $\mathbb{R}^n$, i.e., $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

Fact 2.3. *Let $\mu > 0$. The following conditions are equivalent:*

(a) *The function f is μ-strongly convex.*

(b) *For any vector $v \in S^{n-1}$ we have $\partial_v^2 f(x) \geq \mu$ at any point $x \in X$.*

(c) *For any $x \in X$ any eigenvalue of the Hessian matrix of f*

$$H(f) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right]_{1 \leq i,j \leq n}$$

is bounded from below by μ.

Fact 2.4. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a strongly convex function then* $\lim_{|x|\to\infty} f(x) = +\infty$.

If $f(x) > 0$ for $x \in X$, the function f we will call *logarithmically convex*, *logarithmically strictly convex* and *logarithmically μ-strongly convex* if $\ln f$ is convex, strictly convex and μ-strongly convex respectively.

Obviously for any μ-strongly convex function $a : \mathbb{R}^n \to \mathbb{R}$ the function $b = \exp a$ is logarithmically strongly convex, for instance $b(x) = \exp(|x|^2)$, $b(x) = \exp(\exp(|x|^2))$,..., are logarithmically strongly convex functions.

Fact 2.5. *If $b : \mathbb{R}^n \to \mathbb{R}$ is a logarithmically strongly convex function then b is also a strongly convex function.*

Proof. Indeed, for any $\beta \in \S^{n-1}$, we have

$$\partial_\beta^2(\ln b(x)) = \frac{b(x)\partial_\beta^2 b(x) - (\partial_\beta b(x))^2}{b(x)^2} \geq \mu \quad \text{for } x \in \mathbb{R}^n,$$

so,

$$\partial_\beta^2 b(x) \geq \eta b(x) + \frac{(\partial_\beta b(x))^2}{b(x)} \geq \mu b(x_0) > 0 \quad \text{for } x \in \mathbb{R}^n$$

and some $\mu > 0$, where $x_0 = \operatorname{argmin}_{\mathbb{R}^n} b$. □

2.2. **Gradient of convex functions.** Let f be a real function of class $\mathscr{C}^2$ defined in a neighbourhood of a convex set $X \subset \mathbb{R}^n$.

From Fact 2.2 we immediately obtain

Corollary 2.6. *If f is a strictly convex function, then the gradient*

$$\nabla f : X \ni x \mapsto \nabla f(x) \in \mathbb{R}^n$$

is injective.

Proof. Indeed, by Fact 2.2, for any $a, b \in X$, $a \neq b$, the function

$$\varphi : I_{a,b-a} \ni t \mapsto \partial_{b-a} f(a + t(b-a)) \in \mathbb{R}$$

is strictly increasing. Moreover, $0, 1 \in I_{a,b-a}$, so

$$\langle \nabla f(a), b - a \rangle = \varphi(0) < \varphi(1) = \langle \nabla f(b), b - a \rangle.$$

Consequently, $\nabla f(a) \neq \nabla f(b)$. □

From Corollary 2.6 we obtain

Corollary 2.7. *If f is an logarithmically strictly convex function, then the mapping*

$$\frac{1}{f} \nabla f : X \ni x \mapsto \frac{1}{f(x)} \nabla f(x) \in \mathbb{R}^n$$

is injective.

Proof. Indeed, by definition, $\ln f$ is strictly convex and $\nabla(\ln f) = \frac{1}{f}\nabla f$. So, Corollary 2.6 gives the assertion. □

Without assuming logarithmically strict convexity of the function f, the above corollary does not hold. This is demonstrated by the following example.

Example 2.8. *Let $f(x) = 1 + x^2$. Then $\frac{f'}{f}(x) = \frac{2x}{1+x^2}$ and obviously this function is not injective. Moreover, the function f is strongly convex.*

Lemma 2.9. *Let $b : \mathbb{R}^n \to \mathbb{R}$ be a μ-strongly convex function of class $\mathscr{C}^2$, let $x_0 = \operatorname{argmin}_{\mathbb{R}^n} b$ and let $X \subset \mathbb{R}^n$ be a convex and compact set. If $b(x) > 0$ for $x \in X$ and x_0 is an interior point of the set X then there exists $\varepsilon > 0$ such that*

(i) *the function b is an logarithmically strongly convex in the set $X_{x_0,\varepsilon} = \{x \in X : |x - x_0| \leq \varepsilon\}$.*

(ii) *the function $X_{x_0,\varepsilon} \ni x \mapsto \frac{1}{b(x)} \nabla b(x) \in \mathbb{R}^n$ is injective.*

(iii) *there exists $\delta > 0$ such that for any $x \in X$ such that $\frac{|\nabla b(x)|}{b(x)} < \delta$ we have $|x - x_0| < \varepsilon$.*

Proof. Since $b(x) > 0$ for $x \in X$ and b is μ-strongly convex function, for any $x \in X$ and $\beta \in \mathbb{R}^n$, $|\beta| = 1$ we have

$$\partial_\beta^2(\ln b)(x) = \frac{\partial_\beta^2 b(x)}{b(x)} - \left(\frac{\partial_\beta b(x)}{b(x)}\right)^2 \geq \frac{\mu}{b(x)} - \left(\frac{\partial_\beta b(x)}{b(x)}\right)^2.$$

Since b is of class $\mathscr{C}^2$ and $\partial_\beta b(x_0) = 0$ then there exists $\varepsilon > 0$ fulfilling (i). The assertion (ii) immediately follows from (i) and Corollary 2.7. Taking

$$\delta = \min\left\{\varepsilon, \inf\left\{\frac{|\nabla b(x)|}{b(x)} : x \in X,\ |x - x_0| \geq \varepsilon\right\}\right\},$$

where $\inf \emptyset = +\infty$, we see that $\delta > 0$ and deduce the assertion (iii). □

2.3. **Convexifying functions on compact sets.**

Fact 2.10. *If $b : \mathbb{R}^n \to \mathbb{R}$ is a function of class $\mathscr{C}^2$ such that for any compact and convex set $X \subset \mathbb{R}^n$ there exists $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$ the function $x \mapsto b^N(x)$ is strongly convex on X, then b is positive on $\mathbb{R}^n$.*

Proof. Take any compact and convex set $X \subset \mathbb{R}^n$ and let N_0 be such that for any $N \geq N_0$ the function $b^N(x)$ is strongly convex on X. Take $N \geq N_0$. Since b is of class $\mathscr{C}^2$, from Fact 2.3, for any vector $v \in S^{n-1}$ we have

$$\begin{aligned}\partial_v^2 b^N(x) &= N(N-1)b^{N-2}(x)(\partial_v b(x))^2 + N b^{N-1}(x)\partial_v^2 b(x)\\ &= N b^{N-2}(x)\left[(N-1)(\partial_v b(x))^2 + b(x)\partial_v^2 b(x)\right] > 0 \quad \text{for } x \in X.\end{aligned}$$

So, $b(x) \neq 0$ for $x \in \mathbb{R}^n$. Hence, in view of continuity of the functions $x \mapsto b(x)$, $(x, v) \mapsto \partial_v b(x)$, $(x, v) \mapsto \partial_v^2 b(x)$, the Darboux property gives the assertion. □

Example 2.11. *Under assumptions of Fact 2.10 we cannot require that the function b is convex. For example for $b(x) = \sqrt[4]{1 + |x|^2}$, $x \in \mathbb{R}^n$, the assertion of Fact 2.10 holds (see* [5, Theorem 5.1]*) but b is not convex. It can not be expected that $\lim_{|x|\to\infty} b(x) = +\infty$. For example, for the function $b(x) = \exp x$, $x \in \mathbb{R}$, the assertion of Fact 2.10 holds (see Lemma 3.1 in Section 5.1) but $\lim_{x\to-\infty} b(x) = 0$.*

Fact 2.12. *If $b : \mathbb{R}^n \to \mathbb{R}$ is a function of class $\mathscr{C}^2$ such that for any compact and convex set $X \subset \mathbb{R}^n$ there exists $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$ the function $x \mapsto b^N(x)$ is logaritmically strongly convex on X, then b is also logarithmically strongly convex on any compact and convex set $X \subset \mathbb{R}^n$.*

Proof. Sine a logarithmically strongly convex function is also strongly convex, by Fact 2.10, the function b is positive on $\mathbb{R}^n$. Take any compact and convex set $X \subset \mathbb{R}^n$. Let N_0 be such that for any $N \geq N_0$ the function $b^N(x)$ is logarithmically strongly convex on X. Then for $N \geq N_0$ the function $\ln b^N(x) = N \ln b(x)$ is strongly convex on X. Consquently, b is logarithmically strongly convex on X. □

2.4. **Polynomials.** Let $f \in \mathbb{R}[x]$ be a polynomial in $x = (x_1, \ldots, x_n)$ of the form

$$f = \sum_{j=0}^{d} \sum_{|\nu|=j} a_\nu x^\nu, \tag{2.1}$$

where $a_\nu \in \mathbb{R}$, $x^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}$ and $|\nu| = \nu_1 + \cdots + \nu_n$ for $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}^n$ (we assume that $0 \in \mathbb{N}$). Assume that $d = \deg f$. Then $f = f_0 + \cdots + f_d$, where f_j is a homogeneous polynomial of degree j or zero, i.e.,

$$f_j := \sum_{|\nu|=j} a_\nu x^\nu, \quad 0 \le j \le d. \tag{2.2}$$

We will call The polynomial f_d the *leading form* of f. Obviously $\deg(f - f_d) < d$.

We set

$$\|f\| := \sum_{|\nu| \le d} |a_\nu|.$$

Then $||f_0|| = |a_0|$ and

$$||f|| = ||f_0|| + \cdots + ||f_d||.$$

Lemma 2.13. *Take any $\beta \in S^{n-1}$. Then for any $x \in \mathbb{R}^n$ we have*

$$|\partial_\beta f(x)| \le \sum_{j=1}^{d} j||f_j|||x|^{j-1}, \qquad |\partial_\beta^2 f(x)| \le \sum_{j=1}^{d} j(j-1)||f_j|||x|^{j-2}. \tag{2.3}$$

In particular if $|x| \ge 1$ then

$$|\partial_\beta f(x)| \le d||f|| \cdot |x|^{d-1}, \qquad |\partial_\beta^2 f(x)| \le d(d-1)||f|| \cdot |x|^{d-2}. \tag{2.4}$$

Proof. Let $\beta = (\beta_1, \ldots, \beta_n)$. We have

$$\partial_\beta f(x) = \sum_{j=1}^{d} \sum_{|\nu|=j} a_\nu \partial_\beta x^\nu, \qquad \partial_\beta^2 f(x) = \sum_{j=2}^{d} \sum_{|\nu|=j} a_\nu \partial_\beta^2 x^\nu$$

Take any $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}^n$, $|\nu| = \nu_1 + \cdots + \nu_n = j$. Then

$$|\partial_\beta x^\nu| \le \sum_{k=1}^{n} \nu_k |x_1^{\nu_1} \cdots x_k^{\nu_k - 1} \cdots x_n^{\nu_n}| \le j|x|^{j-1}$$

and consequently,

$$|\partial_\beta^2 x^\nu| \le \sum_{k=1}^{n} \nu_k |\partial_\beta x_1^{\nu_1} \cdots x_k^{\nu_k - 1} \cdots x_n^{\nu_n}| \le j(j-1)|x|^{j-2}.$$

This gives (2.3). Consequently, for $|x| \ge 1$ we have

$$|\partial_\beta f(x)| \le \sum_{j=1}^{d} j|x|^{j-1} \sum_{|\nu|=j} |a_\nu| \le d|x|^{d-1}(||f_1|| + \cdots + ||f_d||) \le d|x|^{d-1} \cdot ||f||.$$

and

$$|\partial_\beta^2 f(x)| \le \sum_{j=2}^{d} j(j-1)|x|^{j-2} \sum_{|\nu|=j} |a_\nu| \le d(d-1)|x|^{d-2} \cdot ||f||,$$

which gives (2.4) and ends the proof. □

From Lemma 2.13 we immediately obtain

Corollary 2.14. *If* $\nabla f(0) = 0$ *then*

$$|\nabla f(x)| \le d\sqrt{n}\|f - f_0\| \cdot |x| \quad \textit{for } |x| \le 1.$$

2.5. **Estimation of zeros of a polynomial.** Let $f \in \mathbb{R}[x]$ be a polynomial of form (2.1). Put $f_{d*} = \min_{|x|=1} f_d(x)$. Assume that $f_{d*} > 0$ and set

$$K_f(r) := 2\max\left\{\left(\frac{||f_0|| + r}{f_{d*}}\right)^{1/d}, \max_{1\le j\le d-1}\left|\frac{||f_{d-j}||}{f_{d*}}\right|^{1/j}\right\} \quad \text{for } r > 0.$$

We put $K(f) := K_f(0)$.

Fact 2.15. *For any* $r \ge 0$,

$$\{x \in \mathbb{R}^n : f(x) \le r\} \subset \{x \in \mathbb{R}^n : |x| \le K_f(r)\}.$$

Proof. Under notations of Section 2.4,

$$|f_j(\theta)| \le ||f_j|| \quad \text{for} \quad \theta \in S^{n-1}.$$

Take any $x \in \mathbb{R}^n \setminus \{0\}$ and put $r = |x|$ and $\theta = \frac{1}{|x|}x$. Then $x = r\theta$, $r > 0$, $\theta \in S^{n-1}$ and $f(x)$ can be written in the form

$$f(x) = \sum_{j=0}^{d} f_j(\theta) r^j.$$

Since the number

$$2\max_{1\le j\le d}\left|\frac{f_{d-j}(\theta)}{f_0(\theta)}\right|^{1/j}$$

estimate from above the modul of any zero r of the polynomial $f_d(\theta)r^d + f_{d-1}(\theta)r^{d-1} + \cdots + f_0(\theta)$ in r, where $f_d(\theta) \ge f_{d*} > 0$, then the polynomial $f - r$ have no zeros $x \in \mathbb{R}^n$ such that $|x| > K_f(r)$. Since f have positive values for $x \in \mathbb{R}^n$ such that $|x|$ tends to infinity, then we obtain the assertion. □

3. Convexifying functions on compact sets

3.1. **Strongly convex functions.** Let $b : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathscr{C}^2$ which is μ-strongly convex, $\mu > 0$, and takes only positive values.

Take any convex and compact set $X \subset \mathbb{R}^n$. Let

$$S := \max\{b(x) : x \in X\}.$$

Obviously $S > 0$. Take any function $f : \mathbb{R}^n \to \mathbb{R}$ of class $\mathscr{C}^2$ which is positive on X. Let $m, D \in \mathbb{R}$ be a positive numbers such that

$$f(x) \geq m, \quad |\partial_\beta f(x)| \leq D, \quad |\partial_\beta^2 f(x)| \leq D \quad for\ x \in X \text{ and } \beta \in S^{n-1}.$$

Let

$$N(\mu, S, m, D) := \frac{S}{\mu}\left(\frac{D}{m} + \frac{D^2}{m^2}\right) + 1.$$

The following lemma is a version of Lemma 49 from [13] by Klaudia Rosiak.

Lemma 3.1. *For any $N \geq N(\mu, S, m, D)$ the function $\varphi_N(x) = b^N(x)f(x)$ is strongly convex on the set X.*

Proof. Take any $N \geq N(\mu, S, m, D)$ and $x, \beta \in \mathbb{R}^n$, $|\beta| = 1$. Then

$$\partial_\beta^2 \varphi_N(x) = N(N-1)b^{N-2}(x)f(x)\left(\partial_\beta b(x)\right)^2 + 2Nb^{N-1}(x)\partial_\beta b(x)\partial_\beta f(x) + Nb^{N-1}(x)f(x)\partial_\beta^2 b(x) + b^N(x)\partial_\beta^2 f(x).$$

Since $b(x) > 0$ for $x \in \mathbb{R}^n$, we have

$$\partial_\beta^2 \varphi_N(x) = b^N(x)\Lambda(x),$$

where

$$\Lambda(x) = N(N-1)f(x)\left(\frac{\partial_\beta b(x)}{b(x)}\right)^2 + 2N\frac{\partial_\beta b(x)}{b(x)}\partial_\beta f(x) + \partial_\beta^2 f(x) + Nf(x)\frac{\partial_\beta^2 b(x)}{b(x)}.$$

Since f and b are functions of class $\mathscr{C}^2$, then φ is also class $\mathscr{C}^2$ and it suffices to prove that

$$\Lambda(x) > 0 \quad \text{for } \ x \in X. \tag{3.1}$$

Let now $x \in X$ and put $t = \frac{\partial_\beta b(x)}{b(x)}$. From the assumptions on f and b,

$$\Lambda(x) \geq N(N-1)m|t|^2 - 2ND|t| - D + Nm\frac{\mu}{S}.$$

The discriminant of the quadratic function in $|t|$ on the right hand of the above inequality is of the form

$$\Delta = 4N^2D^2 - 4N(N-1)m\left(-D + Nm\frac{\mu}{S}\right) = -\frac{4Nm^2\mu}{S}\left[N\left(N - 1 - \frac{S}{\mu}\frac{D}{m} - \frac{S}{\mu}\frac{D^2}{m^2}\right) + \frac{S}{\mu}\frac{D}{m}\right]$$

So, for $N \geq N(\mu, S, m, D)$ we have $\Delta < 0$ and consequently

$$N(N-1)m|t|^2 - 2ND|t| - D + Nm\frac{\mu}{S} > 0 \quad \text{for } \ t \in \mathbb{R}.$$

This gives (3.1) and ends the proof. □

Let

$$S' := \max\{b(x - \xi) : x, \xi \in X\}.$$

From Lemma 3.1 we immediately obtain

Corollary 3.2. *For any $N \geq N(\mu, S', m, D)$ and any $\xi \in X$ the function*

$$\varphi_{N,\xi}(x) = b^N(x-\xi)f(x) \tag{3.2}$$

is strongly convex on the set X.

Remark 3.3. *Let $b : \mathbb{R}^n \to \mathbb{R}$ be a μ-strongly convex function, $\mu > 0$, and let $X \subset \mathbb{R}^n$ be a compact and convex set. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathscr{C}^2$ and let $D \in \mathbb{R}$ be a positive number such that*

$$|\partial_\beta^2 f(x)| \leq D \quad for\ x \in X\ and\ \beta \in \mathbb{R}^n,\ |\beta| = 1.$$

Then for any $\xi \in \mathbb{R}^n$ and

$$N > \frac{D}{\mu},$$

the function $\Psi_{N,\xi} : \mathbb{R}^n \to \mathbb{R}$ defined by $\Psi_{N,\xi}(x) = Nb(x-\xi) + f(x)$, $x \in \mathbb{R}^n$, is strongly convex on X (more precisely $(N\mu - D)$-strongly convex).

Indeed, take any $\xi \in \mathbb{R}^n$. Since $N\mu > D$ then for any $\beta \in \mathbb{R}^n$, $|\beta| = 1$ we have

$$\partial_\beta^2 \Psi_{N,\xi}(x) = N\partial_\beta^2 b(x-\xi) + \partial_\beta^2 f(x) \geq N\eta - D > D - D = 0 \quad \text{for } x \in X.$$

This gives the assertion.

3.2. **Logarithmically convex functions.** Let $b : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathscr{C}^2$ which is logarithmically μ-strongly convex, $\mu > 0$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathscr{C}^2$ taking only positive values. Take any convex and compact set $X \subset \mathbb{R}^n$. Let $m, D \in \mathbb{R}$ be a positive numbers such that

$$f(x) \geq m, \quad |\partial_\beta f(x)| \leq D, \quad |\partial_\beta^2 f(x)| \leq D \quad for\ x \in X \text{ and } \beta \in S^{n-1}.$$

Let

$$N_{\exp}(\mu, m, D) := \frac{1}{\mu}\left(\frac{D}{m} + \frac{D^2}{m^2}\right).$$

Lemma 3.4. *For any $N > N_{\exp}(\mu, m, D)$ and any $\xi \in \mathbb{R}^n$ the function $\varphi_{N,\xi}(x) = b^N(x-\xi)f(x)$ is logarithmically strongly convex on the set X.*

Proof. Take any $\xi \in \mathbb{R}^n$. Let $\psi_{N,\xi} = \ln \varphi_{N,\xi}$. Then

$$\psi_{N,\xi}(x) = N \ln b(x-\xi) + \ln f(x), \quad x \in \mathbb{R}^n$$

so for any $\beta \in S^{n-1}$, we have

$$\partial_\beta \psi_{N,\xi}(x) = N\partial_\beta(\ln b(x-\xi)) + \frac{\partial_\beta f(x)}{f(x)}, \quad x \in \mathbb{R}^n,$$

and

$$\partial_\beta^2 \psi_{N,\xi}(x) = N\partial_\beta^2(\ln b(x-\xi)) + \frac{f(x)\partial_\beta^2 f(x) - (\partial_\beta f(x))^2}{f(x)^2}, \quad x \in \mathbb{R}^n.$$

Consequently, for $N > N_{\exp}(\mu, m, D)$ and $x \in X$, we have

$$\partial_\beta^2 \psi_N(x) \geq N\mu - \frac{D}{m} - \frac{D^2}{m^2} > 0, \quad x \in X.$$

Since $\partial_\beta^2\psi_N$ is continuous and X is compact, we obtain the assertion. $\square$

4. Iterations of the mapping $\xi \mapsto \operatorname{argmin} \varphi_{N,\xi}$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathscr{C}^k$, $k \geq 2$. Take any $r > 0$ and assume that the set

$$X_{f\leq r} := \{x \in \mathbb{R}^n : f(x) \leq r\}$$

is bounded and nonempty. Let $R_{f\leq r}$ be the size of $X_{f\leq r}$, i.e.,

$$R_{f\leq r} := \sup\{|x| : x \in X_{f\leq r}\}.$$

Take any $R > R_{f\leq r}$ and put

$$B_R := \{x \in \mathbb{R}^n : |x| \leq R\}.$$

Since $X_{f\leq r} \neq \emptyset$, we have $R_{f\leq r} \geq 0$ and so, $R > 0$.

Let $m_R, D_R \in \mathbb{R}$ be a positive numbers such that

$$f(x) \geq m_R, \;\; |\partial_\beta f(x)| \leq D_R, \;\; |\partial_\beta^2 f(x)| \leq D_R \;\text{ for }\; x \in B_R, \;\; \beta \in S^{n-1}. \tag{4.1}$$

Let $b : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathscr{C}^k$, $k \geq 2$, which is μ-strongly convex, $\mu > 0$, and takes only positive values, let (for simplicity of notations),

$$0 = \operatorname{argmin}_{\mathbb{R}^n} b, \tag{4.2}$$

and let

$$S'_{b,R} := \max\{b(x - \xi) : x, \xi \in B_R\}.$$

Let N be an integer number such that

$$N \geq N(\mu, S'_{b,R}, m_R, D_R). \tag{4.3}$$

By Corollary 3.2 for any $\xi \in B_R$ the function $\varphi_{N,\xi}(x) = b^N(x - \xi)f(x)$ is strongly convex on the set B_R. Let $\kappa_N : B_R \to B_R$ be a mapping defined by

$$\kappa_N(\xi) := \operatorname{argmin}_{B_R} \varphi_{N,\xi} \in B_R \quad \text{for } \xi \in B_R. \tag{4.4}$$

Fact 4.1. $\kappa_N(X_{f\leq r}) \subset X_{f\leq r}$.

Proof. Take any $\xi \in B_{f\leq r}$ and let $x = \kappa_N(\xi)$. Then $\varphi_{N,\xi}(x) \leq \varphi_{N,\xi}(\xi)$ and consequently, $b^N(x-\xi)f(x) \leq b^N(0)f(\xi)$. Since, by (4.2), $b(0) \leq b(x - \xi)$, we have $f(x) \leq f(\xi)$ which gives the assertion. $\square$

Lemma 4.2. *The function $\kappa_N|_{X_{f\leq r}}$ is of class $\mathscr{C}^{k-1}$.*

Proof. Take any $\xi \in X_{f\leq r}$. Observe that $x = \kappa_N(\xi)$ satisfies the following system of equations

$$\nabla\varphi_{N,\xi}(x) = 0. \tag{4.5}$$

Indeed, by the choice of R we have $\min\{f(x) : |x| = R\} > r$, so, $X_{f\leq r} \subset \operatorname{Int} B_R$ and by Fact 4.1, $\kappa_N(\xi) \in \operatorname{Int} B_R$. So, x satisfies (4.5). Since the Jacobian (with respect to x) of the system of equations is equal to the Hessian of $\varphi_{N,\xi}$ then the

Jacobian is nonzero at x, because the Hessian matrix has only positive eigenvalues. Then the Implicit function theorem gives the assertion. □

Lemma 4.3. *Let b be μ-logarithmically strongly convex function of class $\mathscr{C}^k$ and let $N > N_{\exp}(\mu, m_R, D_R)$. Then the mapping*

$$\kappa_N|_{X_{f\leq r}} : X_{f\leq r} \to \kappa_N(X_{f\leq r}) \tag{4.6}$$

is a diffeomorphism of class $\mathscr{C}^{k-1}$.

Proof. Take any $\xi \in X_{f\leq r}$ and let $x = \kappa_N(\xi)$. Since $b(x-\xi) > 0$, under notations of the proof of Lemma 4.2 from (4.5) we have

$$N\nabla b(x-\xi)f(x) + b(x-\xi)\nabla f(x) = 0, \tag{4.7}$$

where $\nabla b(x-\xi)$ is the gradient of $b(x-\xi)$ with respect to x. Then

$$\frac{1}{b(x-\xi)}\nabla b(x-\xi) + \frac{1}{Nf(x)}\nabla f(x) = 0. \tag{4.8}$$

So, by Corollary 2.7, the point ξ is uniquely determined by x. Consequently, the mapping (4.6) is bijective and consequently it is a homeomorphism, because $X_{f,R}$ is compact anf κ_N is continuous. To complete the proof it suffices to show that the mapping $\left(\kappa_N|_{X_{f\leq r}}\right)^{-1} : \kappa_N(X_{f\leq r}) \to X_{f\leq r}$ is of class $\mathscr{C}^{k-1}$. For this it is enough to show that the Jacobian with respect to ξ of the system of equations (4.8) is nonzero for any $(x,\xi) \in X_{f\leq r} \times \kappa_N(X_{f\leq r})$ such that $\xi = \kappa_N(x)$. This is due to the fact that the Jacobian with respect to ξ of the system of equations (4.8) is equal to the Hessian of $\ln(\varphi_{N,\xi})$, so it does not zero anywhere in the set $X_{f\leq r}$. Consequently $\left(\kappa_N|_{X_{f\leq r}}\right)^{-1}$ is a mapping of class $\mathscr{C}^{k-1}$, which completes the proof. □

From Lemma 2.9 we obtain an analogous lemma as Lemma 4.3 for strongly convex functions. Unfortunately, this version is not as effective as Lemma 4.3.

Lemma 4.4. *Let b be strongly convex function. Then there exists N_0 such that for any $N > N_0$, the mapping*

$$\kappa_N|_{X_{f\leq r}} : X_{f\leq r} \to \kappa_N(X_{f\leq r}) \tag{4.9}$$

is a diffeomorphism of class $\mathscr{C}^{k-1}$.

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be as in Lemma 2.9. Then there exists N_1 such that for any $N \geq N_1$ we have

$$\frac{1}{Nf(x)}|\nabla f(x)| < \delta \quad \text{for } x \in X_{f\leq r}$$

Then for $N_0 = \max\left\{N_1, N(\mu, S'_{b,R}, m_R, D_R)\right\}$, analogously as in the proof of Lemma 4.3 (by using Lemma 2.9) we obtain the assertion. □

Let Σ_f be the set of critical points of f, i.e. $\Sigma_f := \{\xi \in \mathbb{R}^n : \nabla f(\xi) = 0\}$.

Lemma 4.5. *The set of fixed points of $\kappa_N|_{X_{f\le r}}$ is equal to $\Sigma_f \cap X_{f\le r}$.*

Proof. Let $\xi \in X_{f\le r}$ be a fixed point of $\kappa_N|_{X_{f\le r}}$. Then, analogously as in the proof of Lemma 4.3, we have $\nabla\varphi_{N,\xi}(\xi)=0$, i.e.,

$$N\nabla b(0)f(\xi)+b(0)\nabla f(\xi)=0.$$

Since b takes the minimal value at zero we have $\nabla b(0)=0$, so $\nabla f(\xi)=0$ and $\xi\in\Sigma_f$. Let now $\xi \in X_{f\le r}$ be a critical point of f and let $x=\kappa_N(\xi)$. Then x is the unique point in $X_{f\le r}$ for which $\nabla\varphi_{N,\xi}(x)=0$. Since $\nabla\varphi_{N,\xi}(\xi)=0$, we have $\xi=x$ and ξ is a fixed point of $\kappa_N|_{X_{f\le r}}$. □

Corollary 4.6. *If $\xi\in X_{f\le r}\setminus\Sigma_f$ and $x=\kappa_N(\xi)$, then*

$$\partial_{x-\xi}f(\xi+t(x-\xi))=\langle\nabla f(\xi+t(x-\xi)),x-\xi\rangle<0 \quad \text{for } t\in[0,1], \tag{4.10}$$

$x\notin\Sigma_f$ and the function

$$f_{\xi,x}:[0,1]\ni t\mapsto f(\xi+t(x-\xi))\in\mathbb{R}$$

is strictly decreasing. In particular, the sequence $f(\kappa_N^\nu(\xi))$, $\nu\in\mathbb{N}$, is strictly decreasing, the sequence $\kappa_N^\nu(\xi)$, $\nu=0,1,\ldots$, is injective and

$$\kappa_N^\nu(\xi)\notin\Sigma_f \quad \text{for } \nu=0,1,\ldots.$$

Proof. Since $\xi\notin\Sigma_f$, by Lemma 4.5 we have $x\ne\xi$. Since x is the unique point of $X_{f\le r}$ at which $\varphi_{N,\xi}$ takes the minimal value in $X_{f\le r}$, then (4.7) holds, i.e., $N\nabla b(x-\xi)f(x)+b(x-\xi)\nabla f(x)=0$. Since $x-\xi\ne 0$, we have $\nabla b(x-\xi)\ne 0$ and, so,

$$\nabla f(x)\ne 0. \tag{4.11}$$

Moreover, the function

$$[0,1]\ni t\mapsto \varphi_{N,\xi}(\xi+t(x-\xi))\in\mathbb{R}$$

is strongly convex with the minimal value at 1, so it is strictly decreasing and its derivative have no zeroes in $(0,1)$. Consequently, for $\beta=\frac{x-\xi}{|x-\xi|}$ we have

$$\partial_\beta\varphi_{N,\xi}(\xi+t(x-\xi))<0 \quad \text{for } t\in(0,1).$$

On the other hand $\partial_\beta b(t(x-\xi))>0$ for $t\in(0,1]$ and

$$\partial_\beta\varphi_{N,\xi}(x)=Nb^{n-1}(x-\xi)\partial_\beta b(x-\xi)f(x)+b^N(x-\xi)\partial_\beta f(x),$$

so, $\partial_\beta f(\xi+t(x-\xi))<0$ and consequently (4.10) holds. In particular $x\notin\Sigma_f$. Moreover, the function $f_{\xi,x}$ is strictly decreasing. The particular part of the assertion is an easy consequence of the above. □

Remark 4.7. *If $\varphi_{N,\xi}$ is μ-strongly convex function then for any $\xi\in X_{f\le r}$,*

$$f(\xi)-f(\kappa_N(\xi))\ge\frac{\mu}{2}|\xi-\kappa_N(\xi)|^2.$$

If additionally $\varphi_{N,\xi}$ is logarithmically μ-strongly convex then for any $\xi\in X_{f\le r}$,

$$\frac{f(\xi)}{f(\kappa_N(\xi))}\ge\exp\left(\frac{\mu}{2}|\xi-\kappa_N(\xi)|^2\right).$$

By using the idea from [5, Section 7] we obtain the following proximity algorithm for semialgebraic functions of class $\mathscr{C}^2$ on convex sets (cf [12]).

Theorem 4.8. *If $f:\mathbb{R}^n\to\mathbb{R}$ is a semialgebraic function of class $\mathscr{C}^2$ satisfying* (4.1) *and N satisfies* (4.3), *then for any $\xi\in X_{f\le r}$*

(a) *the limit point $\lim_{\nu\to\infty}\kappa_N^\nu(\xi)$ exists and belongs to $\Sigma_f\cap X_{f\le r}$.*

(b) *the series $\sum_{\nu=0}^\infty|\kappa_N^{\nu+1}(\xi)-\kappa_N^\nu(\xi)|$ is convergent.*

In particular the curve $\gamma_\xi:[0,+\infty)\to X_{f\le r}$ defined by

$$\gamma_\xi(t)=\kappa_N^\nu(\xi)+(t-k)(\kappa_N^{\nu+1}(\xi)-\kappa_N^\nu(\xi))\quad \textit{for } t\in[k,k+1)$$

has finite length and the function $f\circ\gamma_\xi:[0,+\infty)\to\mathbb{R}$ is decreasing. If additionally $\xi\notin\Sigma_f$ then the function $f\circ\gamma_\xi$ is strictly decreasing.

Proof. Take any $\xi\in X_{f\le r}$. The particular part of the assertion immediately follows from (b) and Corollary 4.6, so it suffices to prove (a) and (b).

Put $\xi_0=\xi$ and $\xi_{\nu+1}=\kappa_N^\nu(\xi_0)$ for $\nu=0,1,\ldots$. Then $\xi_{\nu+1}=\kappa_N(\xi_\nu)$ for $\nu=0,1,\ldots$.

We will quote a sketch of the reasoning used in [5] in the case $X=X_{f\le r}$ and $\xi_0\in X_{f\le r}$. In [5, Theorem 7.5], the assertion was obtained assuming that the function b is of the form $b(x)=1+|x|^2$. Obviously b is strongly convex. In this case we have that (see [5, Lemma 7.1])

$$|\xi_{\nu+1}-\xi_\nu|=\operatorname{dist}(\xi_\nu,f^{-1}(f(\xi_{\nu+1}))).\quad \nu=0,1,\ldots \tag{4.12}$$

and the sequence $f(\xi_\nu)$ is decreasing (see [5, Lemma 7.2] and Corollary 4.6). By using the monotonity of the sequence $f(\xi_\nu)$ and the Comparison principle (see [5, Lemma 7.7]) we obtain that the series

$$\sum_{\nu=0}^{\infty}\operatorname{dist}(\xi_\nu,f^{-1}(f(\xi_{\nu+1}))) \tag{4.13}$$

is convergent. Then, by (4.12), the series

$$\sum_{\nu=0}^{\infty}|\xi_{\nu+1}-\xi_\nu| \tag{4.14}$$

is convergent and consequently the sequence ξ_ν tends to some ξ_*.

To prove that $\xi_*\in\Sigma_f$, observe that by analogously as in the proof of Lemma 4.3 we have (4.7), i.e.,

$$N\nabla b(\xi_{\nu+1}-\xi_\nu)f(\xi_{\nu+1})+b(\xi_{\nu+1}-\xi_\nu)\nabla f(\xi_{\nu+1})=0\quad\text{for }\nu=0,1,\ldots.$$

Since $\nabla b(0)=0$ and ∇b is a Lipschitz mapping on $X_{f\le r}$, there exists $L>0$ such that $|\nabla b(\xi_{\nu+1}-\xi_\nu)-\nabla b(0)|\le L|\xi_{\nu+1}-\xi_\nu|$ for any ν, so,

$$|\nabla f(\xi_{\nu+1})|\le\frac{Nf(\xi_{\nu+1})}{b(\xi_{\nu+1}-\xi_\nu)}L|\xi_{\nu+1}-\xi_\nu|.$$

Hence, by convergence of the series (4.14), we obtain convergence of the series $\sum_{\nu=0}^{\infty} \nabla f(\xi_{\nu+1})$. Moreover, continuity of the gradient ∇f and the necessary condition for series convergence gives $\nabla f(\xi_*) = \lim_{\nu\to\infty} \nabla f(\xi_{\nu+1}) = 0$. This gives the assertion in the case $b(x) = 1 + |x|^2$. Note that the proof of the fact that $\xi_* \in \Sigma_f$ differs from the one in the article [5]. It was carried out without any assumptions about form of the function b, so we proved the assertion (a), provided (b) holds.

Let us return to the proof of the Theorem 4.8. It suffices to prove the part (b) of the assertion.

In the proof of convergence of the series (4.13) the form of the function b was not important, the proof consisted in the use of Comparison principle, semialgebraicity of the function f and monotonity of the sequence $f(\xi_\nu)$. Hence the series (4.13) is convergent. Therefore, taking into account the above considerations, it is enough to prove the convergence of the series (4.14). For this, it is sufficient to show that there is a constant $C > 0$ such that

$$|\xi_{\nu+1} - \xi_\nu| \le C \operatorname{dist}(\xi_\nu, f^{-1}(f(\xi_{\nu+1}))), \quad \nu = 0, 1, \dots \tag{4.15}$$

Let $a_\nu \in f^{-1}(f(\xi_\nu))$, $\nu = 1, 2, \dots$, be such that

$$\operatorname{dist}(\xi_\nu, f^{-1}(f(\xi_{\nu+1})) = |\xi_\nu - a_{\nu+1}|.$$

Then by definition of ξ_ν,

$$b^N(\xi_{\nu+1} - \xi_\nu) f(\xi_{\nu+1}) \le b^N(a_{\nu+1} - \xi_\nu) f(a_{\nu+1}).$$

Since $f(a_{\nu+1}) = f(\xi_{\nu+1}) > 0$, we have

$$b(\xi_{\nu+1} - \xi_\nu) \le b(a_{\nu+1} - \xi_\nu).$$

By convergence of the series (4.13) we have $\lim_{\nu\to\infty}(a_{\nu+1} - \xi_\nu) = 0$, and consequently, $\lim_{\nu\to\infty}(\xi_{\nu+1} - \xi_\nu) = 0$, because the origin is the unique point at which the function b takes minimal value. Take the Taylor expansion of the function b at the origin (recal that $\nabla b(0) = 0$),

$$b(x) = b(0) + \frac{1}{2} x^T H_b(0) x + R_3(x),$$

where $H_b(0)$ is the Hessian matrix of b at 0 and $|R_3(x)| \le M|x|^3$ in a neighbourhood U of the origin for some constant $M > 0$. One can assume that $a_{\nu+1} - \xi_\nu \in U$ and $\xi_{\nu+1} - \xi_\nu \in U$ for $\nu = 0.1. \dots$. Then

$$\begin{aligned}(\xi_{\nu+1} - \xi_\nu)^T H_b(0)(\xi_{\nu+1} - \xi_\nu) - 2M|\xi_{\nu+1} - \xi_\nu|^3& \\ \le (a_{\nu+1} - \xi_\nu)^T H_b(0)(a_{\nu+1} - \xi_\nu) + 2M|a_{\nu+1} - \xi_\nu|^3&.\end{aligned}$$

Since the matrix $H_b(0)$ is symetric and positively defined, we have

$$|\xi_{\nu+1} - \xi_\nu|^2 \le C|a_{\nu+1} - \xi_\nu|^2$$

for some constant $C > 0$. Hence $|\xi_{\nu+1} - \xi_\nu| \le \sqrt{C}|a_{\nu+1} - \xi_\nu|$ which gives (4.15) and ends the proof. □

Remark 4.9. *In the proof of Theorem 4.8 we have shown, inter alia, that if* ∇b *is a Lipschitz mapping in* $X_{f\le r}$ *with a constant* $L>0$*, then the jump* $|\xi_{\nu+1}-\xi_\nu|$ *can be estimated from below as follows*

$$|\xi_{\nu+1}-\xi_\nu| \ge \frac{|\nabla f(\xi_{\nu+1})| b(\xi_{\nu+1}-\xi_\nu|)}{LNf(\xi_{\nu+1})}.$$

5. Convexifying of polynomials

5.1. **Convexifying polynomials on compact sets.** Let $f\in\mathbb{R}[x]$ be a polynomial of form (2.1). Assume that $d=\deg f$. Let $X\subset\mathbb{R}^n$ be a compact and convex set.

For any $R>0$ we put

$$D_n(f,R) := \max\left\{\sum_{j=1}^d j||f_j||R^{j-1}; \sum_{j=1}^d j(j-1)||f_j||R^{j-2}\right\}. \tag{5.1}$$

From Lemma 2.13, for any $\beta,x\in\mathbb{R}^n$ such that $|\beta|=1$ and $|x|\le R$ we have

$$|\partial_\beta f(x)|\le D_n(f,R), \quad |\partial_\beta^2 f(x)|\le D_n(f,R). \tag{5.2}$$

Let $b:\mathbb{R}^n\to\mathbb{R}$ be a function of class $\mathscr{C}^2$ which is μ-strongly convex, $\mu>0$, and takes only positive values, and let

$$S := \max\{b(x) : x\in X\}.$$

Let

$$R := \max\{|x| : x\in X\}.$$

From Lemma 3.1 we obtain

Corollary 5.1. *If*

$$f(x)\ge m \quad \text{for } x\in X \tag{5.3}$$

for some positive constant m*, then for any*

$$N > N(\mu,S,m,D_n(f,R))$$

the function $\varphi_N(x)=b^N(x)f(x)$ *is strongly convex on the set* X*.*

Let

$$S' := \max\{b(x-\xi) : x,\xi\in X\}.$$

From Corollary 3.2 we immediately obtain

Corollary 5.2. *If* f *satisfies* (5.3) *for some positive constant* m*, then for any* $N\ge N(\mu,S',m,D_n(f,R))$ *and any* $\xi\in X$ *the function*

$$\varphi_{N,\xi}(x)=b^N(x-\xi)f(x) \tag{5.4}$$

is strongly convex on the set X*.*

If additionally we assume that b is logarithmically μ-convex function then from Lemma 3.4 we obtain

Corollary 5.3. *If f satisfies* (5.3) *for some positive constant m, then for any $N > N_{\exp}(\mu, m, D_n(f,R))$ and any $\xi \in \mathbb{R}^n$ the function $\varphi_{N,\xi}(x) = b^N(x-\xi)f(x)$ is logarithmically strongly convex on the set X.*

Set

$$||f||_R := \sum_{j=0}^{d} ||f_j|| R^j$$

Then $|f(x)| \le ||f||_R$ and $f(x) + ||f||_R \ge 0$ for $x \in \mathbb{R}^n,\ |x| \le R$. Let

$$\tilde{f} := f + ||f||_R + 1. \tag{5.5}$$

Then $\tilde{f}$ satisfies (5.3) with $m = 1$. So, from Corollaries 5.1 and 5.2 we obtain

Corollary 5.4. *For any*

$$N > N(\mu, S, 1, D_n(f,R) + ||f||_R + 1)$$

the function $\tilde{\varphi}_N(x) = b^N(x)\tilde{f}(x)$ is strongly convex on the set X. For any

$$N \ge N(\mu, S', 1, D_n(f,R) + ||f||_R + 1)$$

and any $\xi \in X$ the function $\tilde{\varphi}_{N,\xi}(x) = b^N(x-\xi)\tilde{f}(x)$ is strongly convex on the set X.

Analogously as in Corollary 5.4, from Corollary 5.3 we obtain

Corollary 5.5. *For any $N > N_{\exp}(\mu, 1, D_n(f,R) + ||f||_R + 1)$ and any $\xi \in \mathbb{R}^n$ the function $\varphi_{N,\xi}(x) = b^N(x-\xi)\tilde{f}(x)$ is logarithmically strongly convex on the set X.*

5.2. **Iteration of the mapping $\xi \mapsto \operatorname{argmin} \varphi_{N,\xi}$ for polynomials.** Let $f \in \mathbb{R}[x]$ be a polynomial of form (2.1). Assume that $f_{d*} > 0$. Take any $r > 0$ and $R > K_f(r)$ and assume that $X_{f \le r} \ne \emptyset$.

Let $b : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathscr{C}^k$, $k \ge 2$, which is μ-strongly convex, $\mu > 0$, and takes only positive values and the minimal value takes at the point $x = 0$, and let

$$S'_{b,R} := \max\{b(x-\xi) : x, \xi \in B_R\},$$

where $B_R = \{x \in \mathbb{R}^n : |x| \le R\}$.

Let N be an integer number such that

$$N \ge N(\mu, S'_{b,R}, 1, D_n(f,R)). \tag{5.6}$$

If $f(x) \ge 1$ for $x \in \mathbb{R}^n$, by Corollary 5.2 for any $\xi \in B_R$ the function $\varphi_{N,\xi}(x) = b^N(x-\xi)f(x)$ is strongly convex on the set B_R. Let $\kappa_N : B_R \to B_R$ be a mapping defined by (4.4). So, from Lemmas 4.2, 4.3, 4.5 and Theorem 4.8 we obtain

Corollary 5.6. *If $f_{d*} > 0$, $f(x) \geq 1$ for $x \in \mathbb{R}^n$ and N meets the inequality* (5.6) *then:*

(a) $\kappa_N(X_{f\leq r}) \subset X_{f\leq r}$.

(b) *the function $\kappa_N|_{X_{f\leq r}}$ is of class $\mathscr{C}^{k-1}$.*

(c) *the set of fixed points of $\kappa_N|_{X_{f\leq r}}$ is equal to $\Sigma_f \cap X_{f\leq r}$.*

(d) *for any $\xi \in X_{f\leq r}$ the limit point $\lim_{\nu\to\infty} \kappa_N^\nu(\xi)$ exists and belongs to Σ_f.*

If additionally b is a logarithmically μ-strongly convex function and

$$N > N_{\exp}(\mu, 1, D_n(f, R))$$

then

(e) *the mapping $\kappa_N|_{X_{f\leq r}} : X_{f\leq r} \to \kappa_N(X_{f\leq r})$ is a diffeomorphism of class $\mathscr{C}^{k-1}$.*

Remark 5.7. *To construct a mapping κ_N satisfying the assertion of Corollary 5.6 we do not have to assume that the polynomial f takes only positive values. It is sufficient to assume that $f_{d*} > 0$. More precisely, let $\tilde{f}$ be of form* (5.5)*, i.e., $\tilde{f} = f + ||f||_R + 1$. Then $\tilde{f}(x) \geq 1$ for $|x| \leq R$ and the polynomials f and $\tilde{f}$ have the same set of critical points. So, for suitable N, the mapping $\tilde{\kappa}_N(\xi) = \operatorname{argmin}_{B_R} b^N(x-\xi)\tilde{f}(x) \in B_R$ for $\xi \in B_R$ satisfy the assertion of Corollary 5.6.*

6. Logarithmically convexification of polynomials on unbounded sets

Let $f \in \mathbb{R}[x]$ be a polynomial of form (2.1), i.e.,

$$f(x) = \sum_{j=0}^{d} \sum_{|\nu|=j} a_\nu x^\nu. \tag{6.1}$$

Assume that $d = \deg f$. Then $f = f_0 + \cdots + f_d$, where f_j is a homogeneous polynomial of degree j or zero. Assume that $f_{d*} > 0$. Recall that $f_{d*} = \min_{|x|=1} f_d(x)$. Then $\|f\| \geq \|f_d\| \geq f_{d*}$. Put

$$\mathbb{K}(f) := \frac{2\|f\|}{f_{d*}}$$

and

$$c(f) := f_{d*} - \sum_{j=0}^{d-1} \mathbb{K}(f)^{j-d} ||f_j||.$$

Obviously, $\mathbb{K}(f) \geq 2$.

We will need the following lemma (see [7, Lemma 3.4]).

Lemma 6.1. *If $d = \deg f > 0$ and $f_{d*} > 0$, then $c(f) > 0$ and $f(x) \geq c(f)|x|^d$ for any $x \in \mathbb{R}^n$ such that $|x| \geq \mathbb{K}(f)$.*

From Lemmas 6.1 and 2.13 we immediately obtain

Corollary 6.2. *Let f be a polynomial of form* (6.1) *such that $f_{d*} > 0$. Take any $\beta \in S^{n-1}$. Then for any $x \in \mathbb{R}^n$, $|x| \geq \mathbb{K}(f)$ we have*

$$\frac{|\partial_\beta f(x)|}{f(x)} \leq \frac{d||f||}{c(f)} \cdot |x|^{-1} \leq \frac{d||f||}{2c(f)} \tag{6.2}$$

and

$$\frac{|\partial_\beta^2 f(x)|}{f(x)} \leq \frac{d(d-1)||f||}{c(f)} \cdot |x|^{-2} \leq \frac{d(d-1)||f||}{4c(f)}. \tag{6.3}$$

For a polynomial f of form (6.1) such that $f_{d*} > 0$ and for any $\mu > 0$ we put

$$N_{\exp,\infty}(\mu, f) := \frac{d(d+1)||f||}{4\mu c(f)}.$$

Obviously, for any $\beta, x \in \mathbb{R}^n$ such that $|\beta| = 1$ and $|x| \leq R$ we have

$$|\partial_\beta f(x)| \leq D_n(f, R), \quad |\partial_\beta^2 f(x)| \leq D_n(f, R), \tag{6.4}$$

where $D_n(f, R)$ is defined by (5.1).

Let $b : \mathbb{R}^n \to \mathbb{R}$ be logarithmically μ-strongly convex function of class $\mathscr{C}^k$, $k \geq 2$. From Lemma 3.4 and Corollaty 6.2 we obtain

Corollary 6.3. *Let $X \subset \mathbb{R}^n$ be a closed and convex set. Let f be a polynomial of form* (6.1) *such that $f_{d*} > 0$ and there exists $m > 0$ such that $f(x) \geq m$ for $x \in X$. For any*

$$N > \max\{N_{\exp}(\mu, m, D_n(f, \mathbb{K}(f))), N_{\exp,\infty}(\mu, f)\}$$

and any $\xi \in \mathbb{R}^n$ the function $\varphi_{N,\xi}(x) = b^N(x - \xi)f(x)$ is logarithmically strongly convex on the set X.

Proof. Take any $\xi \in \mathbb{R}^n$. Let $\psi_{N,\xi}(x) = \ln \varphi_{N,\xi}(x)$. Take any $\beta \in S^{n-1}$. By Lemma 3.4 there exists $\mu_1 > 0$ such that $\partial_\beta^2 \psi_{N,\xi}(x) \geq \mu_1$ for $x \in X$, $|x| \leq \mathbb{K}(f)$. Since

$$\partial_\beta^2 \psi_{N,\xi}(x) = N \partial_\beta^2 (\ln b(x - \xi)) + \frac{\partial_\beta^2 f(x)}{f(x)} - \left(\frac{\partial_\beta f(x)}{f(x)}\right)^2, \quad x \in \mathbb{R}^n,$$

then by Corollary 6.2 there exists $\mu_2 > 0$ such that $\partial_\beta^2 \psi_{N,\xi}(x) \geq \mu_2$ for $x \in X$, $|x| \geq \mathbb{K}(f)$. Consequently, $\partial_\beta^2 \psi_{N,\xi}(x) \geq \min\{\mu_1, \mu_2\} > 0$ for $x \in X$. □

From Corollary 6.3 we obtain

Corollary 6.4. *Let $f \in \mathbb{R}[x]$ be a polynomial of form* (6.1). *If $f_{d*} > 0$ and $f(x) \geq m$ for $x \in \mathbb{R}^n$ and some constant $m > 0$, then for any*

$$N > \max\{N_{\exp}(\mu, m, D_n(f, \mathbb{K}(f))), N_{\exp,\infty}(\mu, f)\}$$

and any $\xi \in \mathbb{R}^n$ the function $\varphi_{N,\xi}(x) = b^N(x - \xi)f(x)$ is logarithymically strongly convex on $\mathbb{R}^n$ and the mapping $\kappa_N : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\kappa_N(\xi) = \operatorname{argmin}_{\mathbb{R}^n} \varphi_{N,\xi} \in \mathbb{R}^n \quad \textit{for } \xi \in \mathbb{R}^n,$$

is a diffeomorphism of class $\mathscr{C}^{k-1}$. *Moreover, for any* $\xi \in \mathbb{R}^n$ *the limit point* $\lim_{\nu\to\infty} \kappa_N^\nu(\xi)$ *exists and belongs to* Σ_f.

Proof. By Corollary 6.3 for any $\xi \in \mathbb{R}^n$ the function $\varphi_{N,\xi}$ is logarithmically strongly convex on $\mathbb{R}^n$. So, $\operatorname{argmin}_{\mathbb{R}^n} \varphi_{N,\xi}$ is a critical point of $\varphi_{N,\xi}$ and consequently by analogous argument as in the proof of Theorem 4.8 we obtain the assertion. □

Remark 6.5. *To determine the diffeomorphism, the successive iterations which converge to the critical points of the polynomial* f, *we do not have to assume that all values of* f *are positive. It is enough to assume that* $f_{d*} > 0$ *and take* $R = K_f$ *and* $\tilde{f} = f + ||f||_R + 1$ *(see Remark 5.7).*

7. Polynomials with integer coefficients

For applications of the above results it is important to estimate the numbers f_{d*}, $m = \min\{f(x) : x \in X\}$ and $R = \max\{|x| : x \in X\}$ for a polynomial f and a compact and convex set $X \subset \mathbb{R}^n$. In the case when f and polynomials describing X have integer coefficients the above numbers can be effectively estimated. More precisely, let $X \subset \mathbb{R}^n$, $n \geq 2$, be a compact semialgebraic set of the form

$$X = \{x \in \mathbb{R}^n : g_1(x) = 0, \dots, g_l(x) = 0, g_{l+1}(x) \geq 0, \dots, g_k(x) \geq 0\}, \tag{7.1}$$

where $g_1, \dots, g_k \in \mathbb{Z}[x]$. Under the above notations G. Jeronimo, D. Perrucci, E. Tsigaridas in [3] proved that

Theorem 7.1. *Let* $f, g_1, \dots, g_k \in \mathbb{Z}[x]$ *be polynomials with degrees bound by an even integer* d *and coefficients of absolute values at most* H, *and let* $\tilde{H} = \max\{H, 2n + 2k\}$. *If* $f(x) > 0$ *for* $x \in X$ *and* X *of form* (7.1) *is compact, then*

$$f(x) \geq \left(2^{4-\frac{n}{2}} \tilde{H} d^n\right)^{-n2^n d^n} \qquad \text{for } x \in X.$$

From Theorem 7.1 we immediately obtain

Corollary 7.2. *Let* $f \in \mathbb{Z}[x]$ *be a homogeneous polynomial with degree bound by an even integer* d *and coefficients of absolute values at most* H, *and let* $\tilde{H} = \max\{H, 2n + 2\}$. *If* $f(x) > 0$ *for* $|x| = 1$. *Then*

$$f(x) \geq \left(2^{4-\frac{n}{2}} \tilde{H} d^n\right)^{-n2^n d^n} \qquad \text{for } |x| = 1.$$

From Theorems 7.1 we immediately obtain (see [7, Theorem 2.7])

Theorem 7.3. *Let* $X \subset \mathbb{R}^n$ *be a compact and convex semialgebraic set of form* (7.1) *and let* $f, g_1, \dots, g_k \in \mathbb{Z}[x]$ *be polynomials with degrees bound by an even integer* d *and coefficients of absolute values at most* H. *Set*

$$\mathfrak{b}(n, d, H, k) = \left(2^{4-\frac{n}{2}} \max\{H, 2n + 2k\} d^n\right)^{-n2^n d^n}$$

and

$$R = \sqrt{\left[\mathfrak{b}(n + 1, \max\{d, 4\}, H, k + 2)\right]^{-1} - 1}, \quad m = \mathfrak{b}(n, d, H, k).$$

Then

$$\max\{|x| : x \in X\} \le R. \tag{7.2}$$

8. The mapping κ_N for $b(x) = \exp|x|^2$

From an IT point of view, it is important to know how fast κ_N^ν converges to its limit. One of the problems that arises here is whether the sequence converges along any direction, that is, whether the spherical part of the sequence (in the polar coordinates) has a limit. It seems to be quite a difficult problem and the methods of solving the gradient conjecture of Rene Thom's used in [4] should be applied. This leads to R. Thom's discrete hypothesis: Does $\kappa_N^\nu/|\kappa_N^\nu|$ have a limit when $\nu \to \infty$. We immediately encounter a difficulty here. While in the case of the gradient field trajectory, the Darboux property holds, it is not the case in the discrete case. We will show in a relatively simple example what are similarities and what are differences in the case of the trajectory and in the case of the sequence.

Let $f \in \mathbb{R}[x]$ be a polynomial of the form

$$f(x) = f_0 + f_k(x) + \cdots + f_d(x), \tag{8.1}$$

where f_j is a homogeneous polynomial of degree j or zero for $j = 0, k, \ldots, d$, $k > 1$, and $f_k \ne 0$, $f_d \ne 0$. Recall that $f_{d*} = \min_{|x|=1} f_d(x)$. Assume that $f_{d*} > 0$ and

$$f(x) \ge 1 \quad \text{for } x \in \mathbb{R}^n. \tag{8.2}$$

Let $g_N : \mathbb{R}^n \to \mathbb{R}$, $N > 0$, be a function defined by

$$g_N(x) := \frac{1}{2N} \ln f(x), \quad x \in \mathbb{R}^n. \tag{8.3}$$

We will assume that

$$b(x) = \exp|x|^2, \qquad x \in \mathbb{R}^n. \tag{8.4}$$

Fact 8.1. *The function b is logarithmically 2-strongly convex in $\mathbb{R}^n$ of class $\mathscr{C}^\infty$. Moreover, $\nabla b^N(x) = 2Nb^N(x) \cdot x$ for $x \in \mathbb{R}^n$.*

Take notations and assumptions from Section 5.2. Let $S'_{b,R} = e^{4R^2}$ and

$$N \ge N(2, S'_{b,R}, 1, D_n(f, R)). \tag{8.5}$$

By Corollary 5.2 the function $\varphi_{N,\xi}(x)$, $\xi \in X_{f \le r}$, is strongly convex on the convex hull of the set $X_{f\le r}$ and the mapping κ_N defined by (4.4) is well defined. By Facts 5.6 and 8.1, analogously as in the proof of Lemma 4.3, from (4.7) we have

Fact 8.2. *The mapping $\kappa_N : X_{f\le r} \to \kappa_N(X_{f\le r})$ is the inverse of*

$$\kappa_N(X_{f\le r}) \ni x \mapsto x + \frac{1}{2Nf(x)} \nabla f(x) \in X_{f\le r}\,, \tag{8.6}$$

so it is an analytic and semialgebraic mapping, i.e., it is a Nash mapping.

Since $\frac{1}{2Nf(x)}\nabla f(x) = \nabla g_N(x)$, so putting $g = g_N$, from Fact 8.2 we have

Fact 8.3. *The Jacobian matrix $J(\kappa_N)$ of κ_N is of the form*

$$J(\kappa_N(\xi)) = (I + H(g)(\kappa_N(\xi)))^{-1},$$

where I is the $n \times n$ unit matrix.

By Fact 8.3 we see that $J(\kappa_N(\xi))$ is a symmetric matrix. So, we have the following corollary suggested by Krzysztof Kurdyka.

Corollary 8.4. *The mapping $\kappa_N : X_{f\le r} \to \kappa_N(X_{f\le r})$ is the gradient of an analytic function $F : X_{f\le r} \to \mathbb{R}$. Moreover, $\xi = \kappa_N(\xi) + \nabla g(\kappa_N(\xi))$ and*

$$\nabla \left(F(\xi) - \frac{|\xi|^2}{2} \right) = -\nabla g(\kappa_N(\xi)).$$

Since we assumed (8.2), from Corollary 6.4 we immediately obtain

Corollary 8.5. *Let $R = K_f$. Assume that $f_{d*} > 0$ and let*

$$N > \max \{N_{\exp}(\mu, 1, D_n(f, \mathbb{K}(f)), N_{\exp,\infty}(\mu, f)\}.$$

Thn the mapping $\kappa_N : \mathbb{R}^m \to \mathbb{R}^m$ is an analytic diffeomorphism. Moreover, for any $\xi \in \mathbb{R}^n$ the limit point $\lim_{\nu\to\infty} \kappa_N^\nu(\xi)$ exists and belongs to $\Sigma_f \cap X_{f\le r}$.

Let $\omega_0 : X_{f\le r} \ni \xi \mapsto \xi \in X_{f\le r}$ be the identity mapping and let $\omega_\nu : X_{f\le r} \to X_{f\le r}$ be mappings defined by

$$\omega_{\nu+1} = \kappa_N(\omega_\nu) \quad \text{for } \nu \ge 0$$

By Fact 5.6 we have that $\omega_\nu(\xi) \in X_{f\le r}$ for any $\xi \in X_{f\le r}$ and $\nu = 1, 2, \ldots$, so the mappings ω_ν are well defined. Obviously $\omega_\nu = \kappa_N^\nu$ for $\nu = 0, 1, \ldots$.

8.1. **Some properties of the sequence $\omega_\nu = \kappa_N^\nu$.** Take any $\xi \in X_{f\le r}$. By [5, Lemma 7.1] (cf., (4.12)),

$$|\omega_{\nu+1}(\xi) - \omega_\nu(\xi)| = \operatorname{dist}(\omega_\nu(\xi), f^{-1}(f(\omega_{\nu+1}(\xi)))), \quad \nu = 0, 1, \ldots, \tag{8.7}$$

and by Theorem 4.8, the sequence

$$\omega_\nu(\xi) \quad \text{has a limit point } \omega_*(\xi) \in \Sigma_f \cap X_{f\le r}, \tag{8.8}$$

the series

$$\sum_{\nu=0}^{\infty} |\omega_{\nu+1}(\xi) - \omega_\nu(\xi)| \quad \text{is convergent} \tag{8.9}$$

and the sequence

$$f(\omega_\nu(\xi)) \quad \text{is decreasing.} \tag{8.10}$$

From Lemma 4.5 and Corollary 4.6 we have

Fact 8.6. *The sequence $\omega_\nu(\xi)$ is constant if and only if $\xi \in X_{f\le r} \cap \Sigma_f$. Moreover, for $\xi \in X_{f\le r} \setminus \Sigma_f$ the sequence $\omega_\nu(\xi)$ is injective and $\omega_\nu(\xi) \ne \omega_*(\xi)$ for any ν.*

By Fact 8.2 (or by Fact 8.1, analogously as in the proof of Lemma 4.3, from (4.7)) we have

$$\omega_{\nu+1}(\xi) - \omega_\nu(\xi) = -\frac{1}{2Nf(\omega_{\nu+1}(\xi))}\nabla f(\omega_{\nu+1}(\xi)), \quad \nu \in \mathbb{N}. \tag{8.11}$$

In particular, by (8.9), the series

$$\sum_{\nu=0}^{\infty} |\nabla f(\omega_\nu(\xi))| \quad \text{is convergent.} \tag{8.12}$$

Remark 8.7. *By the Bochnak-Łojasiewicz inequality (see* [2]*),*

$$|f(x) - f(\omega_*(\xi))| \le C|\nabla f(x)||x - \omega_*(\xi)| \tag{BŁ}$$

in a neighbourhood in $\mathbb{R}^n$ *of the point* $\omega_*(\xi)$ *for some positive constant* C*, so from* (8.12) *we obtain that the series*

$$\sum_{\nu=0}^{\infty} \frac{f(\omega_\nu(\xi)) - f(\omega_*(\xi))}{|\omega_\nu(\xi) - \omega_*(\xi)|} \quad \textit{is convergent,}$$

provided $\xi \notin \Sigma_f$*.*

Remark 8.8. *By the Łojasiewicz gradient inequality (see* [9, 10]*)*

$$|f(x) - f(\omega_*(\xi))|^\varrho \le C|\nabla f(x)| \tag{Ł1}$$

in a neighbourhood in $\mathbb{R}^n$ *of the set* $f^{-1}(f(\omega_*(\xi)))$ *for some constants* $0 < \varrho < 1$ *and* $C > 0$*, we have that the series*

$$\sum_{\nu=0}^{\infty} (f(\omega_\nu(\xi)) - f(\omega_*(\xi)))^\varrho \quad \textit{is convergent.} \tag{8.13}$$

Note that the Łojasiewicz gradient inequality (Ł1) *was proved in a neighbourhood of a point. Since the set* $f^{-1}(f(\omega_*(\xi)))$ *is compact, we easily get this inequality around it.*

By the *global Łojasiewicz inequality*:

$$|f(x) - f(y)| \ge C\left(\frac{\operatorname{dist}(x, f^{-1}(f(y)))}{1+|x|^2}\right)^{d(6d-3)^{n-1}} \quad \text{for } x \in \mathbb{R}^n, \tag{Ł2}$$

under fixed y for some positive constant C and $d = \deg f$ (see [6, Corollary 10]), we have

Fact 8.9. *For any neughbourhood* $U \subset \mathbb{R}^n$ *of the set* $f^{-1}(f(\omega_*(\xi)))$ *there exists* $\varepsilon > 0$ *such that*

$$\{x \in \mathbb{R}^n : |f(x) - f(\omega_*(\xi))| < \varepsilon\} \subset U.$$

Moreover, if $f(\omega_{\nu_0}(\xi)) - f(\omega_*(\xi)) < \varepsilon$ *then* $f(\omega_\nu(\xi)) - f(\omega_*(\xi)) < \varepsilon$ *and* $\omega_\nu(\xi) \in U$ *for any* $\nu \ge \nu_0$*.*

From (8.7), (8.9) and [6, Theorem 1] we obtain

Fact 8.10. *Let C and ϱ be as in* (Ł1). *Then there exists $\delta > 0$ such that for any $\xi \in X_{f\leq r}$ such that $|\omega_\nu(\xi) - \omega_*(\xi)| < \delta$ we have*

$$\tag{8.14} |\omega_{\nu+1}(\xi) - \omega_\nu(\xi)| \leq \frac{1}{C(1-\varrho)}\left[(f(\omega_\nu(\xi)) - f(\omega_*(\xi)))^{1-\varrho} - (f(\omega_{\nu+1}(\xi)) - f(\omega_*(\xi)))^{1-\varrho}\right],$$

in particular, there exists ν_0 such that for any $\nu \geq \nu_0$,

$$\tag{8.15} \operatorname{dist}(\omega_\nu(\xi), f^{-1}(f(\omega_*(\xi)))) \leq \frac{1}{C(1-\varrho)}(f(\omega_\nu(\omega)) - f(\omega_*(\xi)))^{1-\varrho}.$$

Proof. Indeed, for $\omega_\nu(\xi)$ sufficiently close to the origin, from [6, Theorem 1] (more specifically from the proof of this theorem) and (8.7) we obtain (8.14). Since $\lim_{\nu\to\infty}(f(\omega_\nu(\xi)) - f(\omega_*(\xi))) = 0$ and $1 - \varrho > 0$, then

$$\sum_{k=\nu}^{\infty}\left[(f(\omega_k(\xi)) - f(\omega_*(\xi)))^{1-\varrho} - (f(\omega_{k+1}(\xi)) - f(\omega_*(\xi)))^{1-\varrho}\right] = (f(\omega_\nu(\xi)) - f(\omega_*(\xi)))^{1-\varrho}.$$

By (8.8), there exists ν_0 such that foe any $k \geq \nu_0$ the point $\omega_k(\xi)$ is sufficiently close to $\omega_*(\xi)$. So, by (8.7) and (8.9) we have $\operatorname{dist}(\omega_\nu(\xi), f^{-1}(f(\omega_*(\xi)))) \leq \sum_{k=\nu}^{\infty}|\omega_{k+1}(\xi) - \omega_k(\xi)|$. Consequently, the above and (8.14) gives (8.15). □

Remark 8.11. *Let $\xi \in X_{f\leq r}$ Take any $\varepsilon > 0$. If N satisfy* (8.5) *and additionally*

$$\tag{8.16} N \geq \frac{d\sqrt{n}}{2\varepsilon}\|f - f_0\|,$$

then there exists ν_0 such that for any $\nu \geq \nu_0$,

$$|\omega_{\nu+1}(\xi) - \omega_\nu(\xi)| \leq \varepsilon|\omega_{\nu+1}(\xi)|.$$

Indeed, by (8.11) *and Corollary 2.14 there exists ν_0 such that for any $\nu \geq \nu_0$,*

$$|\omega_{\nu+1}(\xi) - \omega_\nu(\xi)| \leq \frac{d\sqrt{n}}{2Nf(\xi_{\nu+1})}\|f - f_0\| \cdot |\omega_{\nu+1}(\xi)| \leq \frac{d\sqrt{n}}{2N}\|f - f_0\| \cdot |\omega_{\nu+1}(\xi)|.$$

So, (8.16) *givs the assertion.*

Remark 8.12. *By Remark 4.7, there exists $\mu > 0$ such that,*

$$\tag{8.17} f(\omega_\nu(\xi)) - f(\omega_{\nu+1}(\xi)) \geq \mu|\omega_\nu(\xi) - \omega_{\nu+1}(\xi)|^2 \quad \textit{for any } \nu.$$

Under additional assumption that $0 \in \mathbb{R}^n$ is an isolated singularity of f, there exist positive constants C, α such that

$$\tag{8.18} |\nabla f(x)| \geq C|x|^\alpha \qquad \text{in a neighbourhood of the origin.}$$

The smallest exponent α is called the *Łojasiewicz exponent of the gradient at the origin* and denoted by $\mathcal{L}_0(\nabla f)$. It is known that $\mathcal{L}_0(\nabla f) \leq (d-1)(6d-9)^{n-1}$, where $d = \deg f$ (see [6, Remark 4]) and (8.18) holds with $\alpha = \mathcal{L}_0(\nabla f)$. Then (8.12) goves that the convergence rate of the sequence $\omega_\nu(\xi)$ is quite fast. Namely, we have the following fact.

Fact 8.13. *Take any $\xi_0 \in X_{f\leq r}\setminus\Sigma_f$ and let $\xi_\nu = \omega_\nu(\xi_0)$ for $\nu = 0, 1, \ldots,$. Assume that $\omega_*(\xi_0) = 0$. If the origin is an isolated singularity of f then the series*

$$\sum_{\nu=0}^{\infty} |\xi_\nu|^\alpha \qquad \text{is convergent,}$$

where $\alpha = (d-1)(6d-9)^{n-1}$ and $d = \deg f$.

8.2. **Some curves with properties similar to trajectories of the gradient field.** Take any $\xi_0 \in X_{f\leq r} \setminus \Sigma_f$ and let $\xi_\nu = \omega_\nu(\xi_0)$ for $\nu = 1, 2, \ldots$.

Take a curve $\gamma_{\xi_0} : [0, +\infty) \to X_{f\leq r}$ defined by

$$\gamma_{\xi_0}(t) = \xi_\nu + (t-k)(\xi_{\nu+1} - \xi_\nu) \quad \text{for } t \in [\nu, \nu+1). \tag{8.19}$$

The curve γ_{ξ_0} has several similarities to the trajectory of a gradient field. Namely, it has the following properties (see Theorem 4.8 and (8.11)):

Fact 8.14. (i) *The curve γ_{ξ_0} has finite length equal to $\sum_{\nu=0}^{\infty} |\xi_{\nu+1} - \xi_\nu|$.*

(ii) *The function $f \circ \gamma_{\xi_0} : [0, +\infty) \to \mathbb{R}$ is strictly decreasing (recall that we assumed that $\xi_0 \notin \Sigma_f$).*

(iii) *For $t \in (\nu, \nu+1)$, $\nu = 0, 1, \ldots$ we have*

$$\gamma'(t) = \xi_{\nu+1} - \xi_\nu = -\frac{1}{2Nf(\xi_{\nu+1})}\nabla f(\xi_{\nu+1}).$$

Condition (iii) does not mean that $\gamma'(t) = -\frac{1}{2Nf(\gamma(t))}\nabla f(\gamma(t))$. This is one of the difficulties in studies of ξ_ν, which does not exist in gradient field trajectory studies.

These curves have another similarity to the trajectories of gradient fields. Namely, we have the following fact.

Proposition 8.15. *Let $0 \in \operatorname{Int} X_{f\leq r}$ and let $f(0)$ be the minimal value of f. Then for any $\varepsilon > 0$ there exists $f(0) < \delta < r$ such that for any $\xi_0 \in X_{f\leq \delta}$ the length of the curve γ_{ξ_0} does not exceed ε.*

Proof. Let $C > 0$ and $0 < \varrho < 1$ be as in (Ł1). Assume that (Ł1) holds in a meighbourhood U of $f^{-1}(f(0))$, i.e.,

$$|f(x) - f(0)|^\varrho \leq C|\nabla f(x)| \quad \text{for } x \in U. \tag{8.20}$$

From Fact 8.9 there exists $c > 0$ such that

$$\{x \in \mathbb{R}^n : (f(x) - f(0))^{1-\varrho} < 2c\} \subset U$$

and $f(0)$ is the unique critical value of $f|_U$.

Take any maximal solution (to the right) $\gamma : [0, \beta) \to U\setminus f^{-1}(f(0))$ of the system of equations

$$x' = -\frac{\nabla f(x)}{|\nabla f(x)|} \qquad \text{in } U \setminus f^{-1}(f(0)). \tag{8.21}$$

From (8.20) we obtain the following *Kurdyka Łojasiewicz inequality* (cf., [6, Proposition 1]):

$$|\nabla(f - f(0))^{1-\varrho}(x)| \geq (1-\varrho)C \quad \text{for } x \in U \setminus f^{-1}(f(0)).$$

Hence it follows that

$$((f - f(0))^{1-\varrho} \circ \gamma)' = -|\nabla(f - f(0))^{1-\varrho}) \circ \gamma| \leq -(1-\varrho)C \quad \text{for } x \in U \setminus f^{-1}(f(0))$$

(cf., the proof of [6, Theorem 1]). Consequently, $(f - f(0))^{1-\varrho} \circ \gamma$ and $f \circ \gamma$ are decreasing functions and for any $0 \leq s_1 < s_2$,

$$\begin{aligned}(f - f(0))^{1-\varrho}(\gamma(s_1)) - (f - f(0))^{1-\varrho}(\gamma(s_2)) &= (s_1 - s_2)((f - f(0))^{1-\varrho} \circ \gamma)'(t) \\ &\geq (s_2 - s_1)(1-\varrho)C.\end{aligned}$$

Since $s_2 - s_1$ is equal to the length of $\gamma|_{[s_1,s_2]}$, we have

$$\text{length}\,\gamma|_{[s_1.s_2]} \leq (f - f(0))^{1-\varrho}(\gamma(s_1)) - (f - f(0))^{1-\varrho}(\gamma(s_2)). \tag{8.22}$$

From the above, for any $s_1 \in [0,\beta)$ we obtain that the length of $\gamma|_{[s_1.\beta)}$ does not exceed $(f - f(0))^{1-\varrho}(\gamma(s_1))$. So, under assumption $(f(\gamma(s)) - f(0))^{1-\varrho} < c$ we obtain that the trajectory $\gamma|_{[s_1.\beta)}$ cannot come out of the set U and, consequently, must have a limit point in the set $f^{-1}(f(0))$. This gives that any maximal solution to the right $\gamma : [0,\beta) \to U \setminus f^{-1}(f(0))$ of the system of equations (8.21) with initial condition $(f(\gamma(0)) - f(0))^{1-\varrho} < c$ runs in the set $U \setminus f^{-1}(f(0))$ and intersects at exactly one point each level $f^{-1}(y)$, $f(0) < y < f(\gamma(0))$.

Take any $\varepsilon > 0$. Without loss of generality we may assume that $\varepsilon < c$. Put

$$\delta = f(0) + c^{1/(1-\varrho)}.$$

Now suppose that $f(\xi_0) < \delta$. Then $(f(\xi_0) - f(0))^{1-\varrho} < c$ and $(f(\xi_\nu) - f(0))^{1-\varrho} < c$ for any ν (see (8.10)). Take the solution $\gamma : [0,\beta) \to U \setminus f^{-1}(f(0))$ of (8.21) such that $\gamma(0) = \xi_\nu$. By the above there exists $s_1 > 0$ such that $f(\gamma(s_1)) = f(\xi_{\nu+1})$ and by (8.22) and (8.7),

$$|\xi_{\nu+1} - \xi_\nu| \leq \text{length}\,\gamma|_{[0,s_1]} \leq (f - f(0))^{1-\varrho}(\xi_\nu) - (f - f(0))^{1-\varrho}(\xi_{\nu+1}).$$

Since $\lim_{\nu\to\infty}(f(\xi_\nu) - f(0)) = 0$ and $1 - \varrho > 0$, then

$$\sum_{k=\nu}^{\infty} \left[(f(\xi_k) - f(0))^{1-\varrho} - (f(\xi_{k+1}) - f(0))^{1-\varrho}\right] = (f(\xi_\nu) - f(0))^{1-\varrho}.$$

From this and Fact 8.14 (i) we obtain that the length of γ_{ξ_0} does not exceed ε. □

From Proposition 8.15 and from the proof of this proposition we immediately obtain

Corollary 8.16. *Let $0 \in \operatorname{Int} X_{f\leq r}$ and let $f(0)$ be the minimal value of f. Then for any $\varepsilon > 0$ there exists $f(0) < \delta < r$ such that for any $\xi \in X_{f\leq\delta}$,*

$$|\omega_\nu(\xi) - \omega_*(\xi)| < \varepsilon \quad \textit{for any } \nu.$$

8.3. **Uniform convergence of the sequence ω_ν.** We will show that the sequence of mappings ω_ν has some property similar to a property of the flow of gradient field (cf., [8, 10], see also subsection 8.2).

Proposition 8.17. *Let $0 \in \operatorname{Int} X_{f\le r}$ and let $f(0)$ be the minimal value of f. Then there exists $f(0) < \delta < r$ such that the sequence ω_ν uniformly convergents to ω_* in the set $U = X_{f\le\delta}$. In particular the mapping $\omega_* : U \to U \cap \Sigma_f$ is continuous and $\omega_*(\xi) = \xi$ for $\xi \in U \cap \Sigma_f$, i.e., ω_* is a deformation retraction and the set $U \cap \Sigma_f$ is a retract of U.*

Proof. Let C, ϱ be as in (Ł1). Assume that (Ł1) is fulfild in the set $U = X_{f\le\delta}$ for some $f(0) < \delta < r$ and that the assertiin of Proposition 8.15 holds for any $\xi \in U$.

By the assumption that $f(0)$ is minimal value of f we have that $f(\omega_*(\xi)) = f(0)$ for $\xi \in U$, so, it is a continuous function. Let $0 < \varrho < 1$ and $C > 0$ be constants fulfilling (Ł1) in Remark 8.8. From Corollary 5.6 (b) we see that

$$(f\circ\omega_\nu - f\circ\omega_*)^{1-\varrho} : U \mapsto \mathbb{R}$$

is a sequence of continuous functions and by (8.10) it is decreasing. Obviously, $\lim_{\nu\to\infty}(f\circ\omega_\nu - f\circ\omega_*) = 0$ is a continuous function. So, by Dini's theorem the sequence

$$(f\circ\omega_\nu - f\circ\omega_*)^{1-\varrho} \quad \text{tends uniformly to 0 on } U. \tag{8.23}$$

By the choice of δ, analogously as in the proof of Proposition 8.15, for any $\xi \in U$ we obtain that (cf., Fact 8.10)

$$\begin{aligned}
|\omega_\nu(\xi) - \omega_*(\xi)| &\le \sum_{k=\nu}^{\infty} |\omega_{\nu+1}(\xi) - \omega_\nu(\xi)| \\
&\le \frac{1}{C(1-\varrho)} \sum_{k=\nu}^{\infty} \left[(f(\omega_k(\xi)) - f(\omega_*(\xi)))^{1-\varrho} - (f(\omega_{k+1}(\xi)) - f(\omega_*(\xi)))^{1-\varrho}\right] \\
&= \frac{1}{C(1-\varrho)} (f(\omega_\nu(\xi)) - f(\omega_*(\xi)))^{1-\varrho}.
\end{aligned}$$

This and (8.23) gives the assertion. □

Remark 8.18. *Without assuming that $f(0)$ is the smallest value of the function, the assertion of Proposition 8.17 does not hold. Namely, if the set $X_{f\le r}$ is connected, and f has at least two critical values in $X_{f\le r}$, we easily get a contradiction.*

8.4. **Gradient of a polynomial in the polar coordinates.** Let $f \in \mathbb{R}[x]$ be a polynomial of form (2.1). Then f can be written as

$$f(x) = \sum_{j=0}^{d} f_j\left(\frac{1}{|x|}x\right)|x|^j, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Denote

$$r = r(x) = |x| \quad \text{and} \quad \theta = \theta(x) = \frac{1}{|x|}x \quad \text{for } x \neq 0.$$

Then $x = r\theta$, $r > 0$, $\theta \in S^{n-1}$ and f can be written in the polar coordinates

$$f(x) = f(r\theta) = \sum_{j=0}^{d} f_j(\theta) r^j, \quad x \neq 0, \tag{8.24}$$

and

$$\nabla f(x) = \partial_r f(r\theta)\theta + \nabla' f(r\theta), \tag{8.25}$$

where

$$\partial_r f(r\theta) = \frac{\langle \nabla f(r\theta), r\theta\rangle}{r} = \partial_\theta f(r\theta) = \frac{\partial f(r\theta)}{\partial r} = \sum_{j=1}^{d} j f_j(\theta) r^{j-1} \tag{8.26}$$

and

$$\nabla' f(r\theta) = \nabla f(r\theta) - \partial_r f(r\theta)\theta.$$

Obviously,

$$\langle \nabla' f(r\theta), \theta\rangle = 0 \quad \text{for } x = r\theta \neq 0$$

and

$$\nabla' f(x) = \nabla f(x) - \frac{\langle \nabla f(x), x\rangle}{|x|^2} x \quad \text{for } x \neq 0.$$

The vector $\partial_r f(r\theta)\theta$ is called *the radial part* of the gradient $\nabla f(x)$ and $\nabla' f(r\theta)$ – *the spherical part* of $\nabla f(x)$.

From the definition of $\nabla' f$ we immediately obtain the following remark.

Remark 8.19. *Let $e_1, \ldots, e_n$ be the standard basis of the linear space $\mathbb{R}^n$, i.e., $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is on the jth place. Take any $x \in \mathbb{R}^n$, $x \neq 0$. Put*

$$\alpha_j = \frac{\langle e_j, x\rangle}{|x|^2} x, \qquad v_j = e_j - \alpha_j \qquad \text{for } j = 1, \ldots, n.$$

Then $|v_j| = 1 - \frac{x_j^2}{|x|^2}$,

$$v_j = \left(-\frac{x_1 x_j}{|x|^2}, \ldots, -\frac{x_{j-1} x_j}{|x|^2}, 1 - \frac{x_j^2}{|x|^2}, -\frac{x_{j+1} x_j}{|x|^2}, \ldots, -\frac{x_n x_j}{|x|^2} \right), \quad j = 1, \ldots, n$$

and

$$\nabla' f(x) = \sum_{j=1}^{n} \langle \nabla f(x), v_j\rangle v_j.$$

8.5. **Spherical part of the sequence $\omega_\nu = \kappa_N^\nu$.** In view of the results of Section 8.2, there is a problem of the convergence of the spherical part of the sequence $\omega_\nu(\xi) = \kappa_N^\nu(\xi)$. We will consider this problem under assumption that $\xi_\nu = \omega_\nu(\xi) \to 0$ as $\nu \to \infty$.

In [4], the key role is played by the sets

$$W_\varepsilon = \{x \in \mathbb{R}^n \setminus \{0\} : \varepsilon|\nabla' f(x)| \le |\partial_r f(x)|\}, \quad \varepsilon > 0,$$

where $\nabla' f(x)$ is the spherical and $\partial_r f(x)\frac{x}{|x|}$ – the radial part of the gradient $\nabla f(x)$. One of the most important properties was the behavior of the gradient field trajectory when crossing the boundary of such a set (and properties of the so called controlling function). More precisely, the trajectory of the gradient field must run through this set from a certain point and must not leave it. In a discrete case, a sequence can jump into or out of that set without crossing its boundary.

In order for the method from [4] to be applied in a discrete case, the following conjecture would have to hold. Take any $\xi_0 \in X_{f\le r} \setminus \Sigma_f$ and let $\xi_\nu = \omega_\nu(\xi_0)$ for $\nu = 0, 1, \ldots,$. Assume that $\omega_*(\xi_0) = 0$.

Conjecture 8.20. *There exists a constant $\varepsilon > 0$ and ν_0 such that for any $\nu \ge \nu_0$*

$$\varepsilon|\xi_{\nu+1} - \xi_\nu| \le ||\xi_{\nu+1}| - |\xi_\nu||,$$

equivalently, $\varepsilon|\nabla f(\xi_\nu)| \le |\partial_r f(\xi_\nu)|$, i.e., $\xi_\nu = \omega_\nu(\xi) \in W_\varepsilon$.

With fairly strong assumptions, we get that the limit of the spherical part of the sequence ξ_ν exists. Namely, the following fact holds.

Fact 8.21. *Assume that $f_k(\theta) > 0$ for $\theta \in S^{n-1}$. Then there is the following limit*

$$\lim_{\nu\to\infty} \frac{1}{|\xi_\nu|}\xi_\nu. \tag{8.27}$$

Moreover, the sequence $|\xi_\nu|$ is strictly decreasing from a certain point.

Proof. Let's write f in a polar coordinates:

$$f(r\theta) = f_0 + r^k f_k(\theta) + \cdots + r^d f_d(\theta),$$

where $r > 0$ and $\theta \in S^{n-1}$. Then

$$\partial_r f(r\theta) = kr^{k-1} f_k(\theta) + \cdots + dr^{d-1} f_d(\theta), \tag{8.28}$$

$$\nabla' f(r\theta) = r^k \nabla' f_k(\theta) + \cdots + r^d \nabla' f_d(\theta).$$

and

$$\nabla f(r\theta) = \partial_r f(r\theta)\theta + \nabla' f(r\theta)$$

So, from the assumption that $f_k(\theta) > 0$ for $\theta \in S^{n-1}$, there exists $r_0 > 0$ such that

$$\frac{|\nabla' f(r\theta)|}{r} \le C_1 r^{k-1} \le C_2 \partial_r f(r\theta) \le C_3 |\nabla f(r\theta)| \quad \text{for } 0 < r < r_0 \tag{8.29}$$

and some positive constants C_1, C_2, C_3.

Take the curve $\gamma = \gamma_{\xi_0}$ defined by (8.19). By Fact 8.14 (ii) the function $f \circ \gamma$ is strictly decreasing, so we have

$$\gamma(t) \neq 0 \quad \text{for } t \in [0, +\infty),$$

and we may write γ in the polar coordinates $\gamma(t) = r_\gamma(t)\theta_\gamma(t)$, $r_\gamma(t) = |\gamma(t)| > 0$ amd $\theta_\gamma(t) \in S^{n-1}$. Then

$$\gamma'(t) = r'_\gamma(t)\theta_\gamma(t) + r_\gamma(t)\theta'_\gamma(t) \quad \text{for } t \in (\nu, \nu+1),\ \ \nu = 0, 1, \ldots,$$

and $\langle \theta_\gamma(t), \theta'_\gamma(t)\rangle = 0$ for $t \in [0, +\infty) \setminus \mathbb{Z}$. On the other hand, by Fact 8.14 (iii),

$$\gamma'(t) = \xi_{\nu+1} - \xi_\nu = -\frac{1}{2Nf(\xi_{\nu+1})}\nabla f(\xi_{\nu+1}) \quad \text{for } t \in (\nu, \nu+1), \quad \nu = 0, 1, \ldots.$$

Since $\nabla f(\xi_{\nu+1}) \neq 0$, we may write $\nabla f(\xi_{\nu+1})$ in the polar coordinates, so

$$r'_\gamma(t) = -\frac{1}{2Nf(\xi_{\nu+1})}\partial_r f(\xi_{\nu+1}) \quad \text{for } t \in (\nu, \nu+1), \quad \nu = 0, 1, \ldots$$

and

$$r_\gamma(t)\theta'_\gamma(t) = -\frac{1}{2Nf(\xi_{\nu+1})}\nabla' f(\xi_{\nu+1}) \quad \text{for } t \in (\nu, \nu+1), \quad \nu = 0, 1, \ldots.$$

So, by (8.28),

$$r'_\gamma(t) = -\frac{1}{2Nf(\xi_{\nu+1})}\left[kr_\gamma^{k-1}(t)f_k(\theta_\gamma(t)) + \cdots + dr_\gamma^{d-1}(t)f_d(\theta_\gamma(t))\right]$$

for $t \in (\nu, \nu+1)$, $\nu = 0, 1, \ldots$. By the assumption that $f_k(\theta) > 0$ for $\theta \in S^{n-1}$ we see that the derivative has a fixed sign $r'_\gamma(t) < 0$ for sufficiently large $t \notin \mathbb{Z}$. Consequently, the sequence $|\xi_\nu|$ is strictly decreasing from a certain point and we proved the moreover part of the assertion. Moreover, $r_\gamma(t)$ tends to 0 as $t \to \infty$ and by (8.29),

$$|\theta'(t)| = \frac{|\nabla' f(\xi_{\nu+1})|}{2Nf(\xi_{\nu+1})r_\gamma(t)} \leq \frac{C_2}{2Nf(\xi_{\nu+1})}\partial_r f(\xi_{\nu+1}) = C_2|r'_\gamma(t)| \leq C_3|\gamma'(t)|$$

for $t \in (\nu, \nu+1)$, and sufficiently large ν. Snce the curve γ has z finite length (see Fact 8.14 (i)), then the above gives that θ_γ also has a finite length. Consequently te curve $\Theta : [0, +\infty) \to \mathbb{R}^n$ defined by

$$\Theta(t) := \theta(\xi_\nu) + (t - k)\left[\theta(\xi_{\nu+1}) - \theta(\xi_\nu)\right] \quad \text{for } t \in [\nu, \nu+1),\ \nu = 0, 1, \ldots$$

has a finite length. This gives that exists a limit $\lim_{\nu\to\infty}\theta(\xi_\nu)$ i.e., the limit (8.27) exists. □

Remark 8.22. *In fact, in the proof of Fact 8.21 we proved that W_ε, for some $\varepsilon > 0$, is equal to some neighbourhood of the origin. Moreover, under the assumption of this fact, we proved that $\varepsilon|\nabla' f(r\theta)| \leq r|\partial_r f(r\theta)$ in a neighbourhood of the origin. This is a stronger condition than the fact that ξ_ν belongs to the set W_ϵ. It seems that it is not enough to prove Conjecture 8.20 to show that $\theta(\xi_\nu)$ converges. The sequence ξ_ν should satisfy $\varepsilon|\nabla' f(\xi_\nu)| \leq |\xi_\nu|^\alpha|\partial_r f(\xi_\nu)$ for some positive constants ε and α.*

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