Analytic and Algebraic Geometry 4
Łódź University Press 2022, 137–146
DOI: https://doi.org/10.18778/8331-092-3.11

SOME NOTES ON THE LÊ NUMBERS IN THE FAMILY OF LINE SINGULARITIES

GRZEGORZ OLEKSIK AND ADAM RÓŻYCKI

Abstract. In this paper we introduce the jumps of the Lê numbers of non-isolated singularity f in the family of line deformations. Moreover, we prove the existence of a deformation of a non-degenerate singularity f such that the first Lê number is constant and the zeroth Lê number jumps down to zero. We also give estimations of the Lê numbers when the critical locus is one-dimensional. These give a version of the celebrated theorem of A. G. Kouchnirenko in this case.

1. Introduction

The most important topological invariant associated with a complex analytic function f with an isolated singularity at 0, is its Milnor number at 0. It is well known that this invariant is upper-semicontinuous in the family of singularities. Therefore it allows to define the jump of the Milnor number as the minimum non-zero difference $\mu(f) - \mu((f_t))$, where (f_t) is a deformation of f. S. Guzein-Zade [6] and A. Bodin [1] began the research devoted to this notion. In the papers [2,7,8,17] authors computed the jump of the Milnor number in the different classes deformations.

If f has a non-isolated singularity at 0, the Milnor number can not be defined. But there exist some numbers called Lê numbers, which play a similar role to the Milnor number in the isolated case. These numbers were defined by D. Massey (see [13–15]). Roughly speaking they describe a handle decomposition of the Milnor fibre (see [15, Theorem 3.3]). We recall that families with constant Lê numbers satisfy remarkable properties. For example, in [14], Massey proved that under appropriate conditions the diffeomorphism type of the Milnor fibrations associated

2010 *Mathematics Subject Classification.* 32S25, 14J17, 14J70.
Key words and phrases. Jump of Lê numbers, Non-isolated hypersurface singularity, Lê numbers, Newton diagram, Modified Newton numbers, Iomdine-Lê-Massey formula.

with the members of such family is constant. In [5], J. Fernández de Bobadilla showed that in the special case of families of 1-dimensional singularities, the constancy of Lê numbers implies the topological triviality of the family at least if $n \geq 5$.

Analogously as the Milnor number, the tuple of the Lê numbers has upper-semicontinuity property in the lexicographical order. Therefore, it is possible to distinguish two types of jumps. The first is the jump up of the tuple of the Lê numbers and the second is the jump up of the Lê number $\lambda^d_{f,z}(0)$, where d is a dimension of the critical locus.

In general, the Lê numbers are not topological invariants. However, it turns out that in the family of aligned singularities they are topological invariants (see [15, Corollary 7.8]). In the paper we focus our attention on the class of line singularities (see definition 5.1). It is the simplest class of aligned singularities. In the paper we consider deformations mainly in this class. Our main theorem (Theorem 5.3) guarantees the existence of a deformation (f_t) of a non-degenerate singularity $f = f_0$ with $\lambda^0_{f_0,z}(0) > 0$, such that $\lambda^0_{(f_t),z}(0) = 0$ and $\lambda^1_{(f_t),z}(0) = \lambda^1_{f,z}(0)$. In terms of a handle decomposition of the Milnor fibre it means that handles of the highest dimension disappear and others remains unchanged (see Remark 5.4).

Using Theorem 5.3 we introduce the minimal jump of the tuple of Lê numbers. In this class we can interpret the jump of the tuple of Lê numbers as a measure of "nearness" of the cycles (see Remark 5.8). Moreover, we show the interesting fact that there exists f such that the minimal jump of $\lambda^1_{f,z}(0)$ is greater then one (see Proposition 5.11). What is surprising, in the class of line singularities $\lambda^0_{f,z}(0) \neq 1$ (see Proposition 5.9). From this fact and Example 5.10 it follows that the "minimal jump" of $\lambda^0_{f,z}(0)$ is greater then one.

In the last section we give estimations of Lê numbers in terms of the Newton diagram when the critical locus is one-dimensional (see Theorem 6.1). This is a generalization of the Kouchnirenko theorem in this case.

2. Preliminary

Lê numbers are intersection multiplicity of certain analytic cycles — so-called *Lê cycles* — with certain affine subspaces. The Lê cycles are defined using the notion of gap sheaf. In this section, we briefly recall these definitions which are essential for the paper. We follow the presentation given by Massey in [13–15].

2.1. **Gap sheaves.** Let $(X, \mathscr{O}_X)$ be a complex analytic space, $W \subseteq X$ be an analytic subset of X, and $\mathscr{I}$ be a coherent sheaf of ideals in $\mathscr{O}_X$. As usual, we denote by $V(\mathscr{I})$ the analytic space defined by the vanishing of $\mathscr{I}$. At each point $x \in V(\mathscr{I})$, we want to consider scheme-theoretically those components of $V(\mathscr{I})$ which are not contained in W. For this purpose, we look at a minimal primary decomposition of the stalk $\mathscr{I}_x$ of $\mathscr{I}$ in the local ring $\mathscr{O}_{X,x}$, and we consider the ideal $\mathscr{I}_x \neg W$ in $\mathscr{O}_{X,x}$ consisting of the intersection of those (possibly embedded) primary components Q of $\mathscr{I}_x$ such that $V(Q) \not\subseteq W$. This definition does not depend on the choice of the minimal primary decomposition of $\mathscr{I}_x$. Now, if we perform the operation described above at the point x simultaneously at all points of $V(\mathscr{I})$, then we obtain a coherent sheaf of ideals called a *gap sheaf* denoted by $\mathscr{I}\neg W$. Hereafter, we shall denote by $V(\mathscr{I})\neg W$ the scheme (i.e., the complex analytic space) $V(\mathscr{I}\neg W)$ defined by the vanishing of the gap sheaf $\mathscr{I}\neg W$.

2.2. **Lê cycles and Lê numbers.** Let $n \geq 2$. Consider an analytic function $f\colon (U,0) \to (\mathbb{C},0)$, where U is an open neighbourhood of 0 in $\mathbb{C}^n$, and fix a system of linear coordinates $z = (z_1,\dots,z_n)$ for $\mathbb{C}^n$. Let Σf be the critical locus of f. For $0 \leq k \leq n-1$, the kth (relative) *polar variety* of f with respect to the coordinates z is the scheme

$$\Gamma^k_{f,z} := V\left(\frac{\partial f}{\partial z_{k+1}},\dots,\frac{\partial f}{\partial z_n}\right)\neg \Sigma f.$$

The analytic cycle

$$[\Lambda^k_{f,z}] := \left[\Gamma^{k+1}_{f,z} \cap V\left(\frac{\partial f}{\partial z_{k+1}}\right)\right] - \left[\Gamma^k_{f,z}\right]$$

is called the kth *Lê cycle* of f with respect to the coordinates z. (We always use brackets $[\cdot]$ to denote analytic cycles.) The kth *Lê number* $\lambda^k_{f,z}(0)$ of f at $0 \in \mathbb{C}^n$ with respect to the coordinates z is defined to be the intersection number

$$\lambda^k_{f,z}(0) := \big([\Lambda^k_{f,z}] \cdot [V(z_1,\dots,z_k)]\big)_0 \tag{2.1}$$

provided that this intersection is 0-dimensional or empty at 0; otherwise, we say that $\lambda^k_{f,z}(0)$ is *undefined.*[1] For $k=0$, the relation (2.1) means

$$\lambda^0_{f,z}(0) = \big([\Lambda^0_{f,z}] \cdot U\big)_0 = \left[\Gamma^1_{f,z} \cap V\left(\frac{\partial f}{\partial z_1}\right)\right]_0.$$

For any $\dim_0 \Sigma f < k \leq n-1$, the Lê number $\lambda^k_{f,z}(0)$ is always defined and equal to zero. For this reason, we usually only consider the Lê numbers

$$\lambda^{\dim_0 \Sigma f}_{f,z}(0),\dots,\lambda^0_{f,z}(0),$$

and we denote this tuple by $\lambda_{f,z}(0)$. Note that if 0 is an *isolated* singularity of f, then $\lambda^0_{f,z}(0)$ (which is the only possible non-zero Lê number) is equal to the Milnor number $\mu_f(0)$ of f at 0.

Now, we introduce the cycle of the critical locus (see [15, Proposition 1.15]). Let $d = \dim_0 \Sigma f$. We define

$$[\Sigma f] = \sum_{i=0}^{d} \lambda^i_{f,z}(0)|[\Lambda^i_{f,z}]|. \tag{2.2}$$

[1] As usual, $[V(z_1,\dots,z_k)]$ denotes the analytic cycle associated to the analytic space defined by $z_1 = \dots = z_k = 0$. The notation $\big([\Lambda^k_{f,z}]\cdot[V(z_1,\dots,z_k)]\big)_0$ stands for the intersection multiplicity at 0 of the analytic cycles $[\Lambda^k_{f,z}]$ and $[V(z_1,\dots,z_k)]$.

3. Lê numbers of a deformation

Let $f\colon (U,0)\to(\mathbb{C},0)$ be an analytic function , where U is an open neighbourhood of 0 in $\mathbb{C}^n$, and fix a system of linear coordinates $z=(z_1,\dots,z_n)$ for $\mathbb{C}^n$.

A deformation of f is an analytic function

$$F\colon (D\times U, D\times\{0\})\to(\mathbb{C},0),$$

where D is an open neighbourhood of the origin in $\mathbb{C}$, such that $F(0,z)=f(z)$ for any $z\in\mathbb{C}^n$. We will shortly write $f_t(z):=F(t,z)$, $(f_t):=F$.

Assume that $d=\dim_0\Sigma f\geq 1$ and the Lê numbers $\lambda^k_{f_t,z}(0)$ are defined for all $k\leq d$ and all t sufficiently small.

Theorem 3.1. *(Uniform Iomdine-Lê-Massey formula, [15, Theorem 4.15]) For sufficiently large integer j and any sufficiently small complex number t, we have the following properties:*

(1) $\Sigma(f_t+z_1^j)=\Sigma f_t\cap V(z_1)$ *in a neighbourhood of the origin;*

(2) $\dim_0\Sigma(f+z_1^j)=d-1$*;*

(3) *the Lê numbers* $\lambda^k_{f_t+z_1^j,\tilde z}(0)$ *exist for all* $0\leq k\leq d-1$ *and*

$$\lambda^0_{f_t+z_1^j,\tilde z}(0)=\lambda^0_{f_t,z}(0)+(j-1)\lambda^1_{f_t,z}(0); \tag{3.1}$$

$$\lambda^k_{f_t+z_1^j,\tilde z}(0)=(j-1)\lambda^{k+1}_{f_t,z}(0)\quad \text{for}\quad 1\leq k\leq d-1; \tag{3.2}$$

where $\lambda^k_{f_t+z_1^j,\tilde z}(0)$ *is the kth Lê number of* $f_t+z_1^j$ *at* 0 *with respect to the rotated coordinates* $\tilde z=(z_2,\dots,z_n,z_1)$.

Now, we define the Lê numbers of a deformation F. For this reason we will prove the following.

Proposition 3.2. *The numbers* $\lambda^k_{f_t,z}(0)$, $k\leq d$ *are independent of small* $t\neq 0$.

Proof. By Uniform Iomdine-Lê-Massey formula inductively we get that for $0\ll j_1\ll\cdots\ll j_d$ and small t,

$$f_{t,d}:=f_t+z_1^{j_1}+\cdots+z_d^{j_d}$$

has an isolated singularity at the origin. By upper-semicontinuity of Milnor number we have the number $\mu(f_{t,d})$ is constant for small $t\neq 0$. By (3.1) we obtain that the number

$$\lambda^1_{f_{t,d-1}}(0)=\mu(f_{t,d+1})-\mu(f_{t,d}) \tag{3.3}$$

is also constant for small $t\neq 0$. Now, by (3.3) and (3.1) we get that

$$\lambda^0_{f_{t,d-1}}(0)=\mu(f_{t,d})-(j_d-1)\lambda^1_{f_{t,d-1}}(0)$$

is also constant for small $t\neq 0$. In similar way, by induction and using (3.1) and (3.2) we finally get the assertion. □

Definition 3.3. By the Lê numbers of a deformation (f_t) we mean

$$\lambda^k_{(f_t),z}(0) := \lambda^k_{f_t,z}(0), \quad k \leq d,$$

for sufficiently small $t \neq 0$.

By Proposition 3.2 this definition is correct.

Like the Milnor number is upper-semicontinuous, the Lê numbers have also this property treated as tuple (see [15]). Precisely, we have the following.

Theorem 3.4. *(Upper-semicontinuity of Lê numbers, [15, Corollary 4.16]) The tuple of Lê numbers*

$$\big(\lambda^d_{f_t,z}(0), \dots, \lambda^0_{f_t,z}(0)\big)$$

is lexicographically upper-semicontinuous in the t variable, i.e. for all sufficiently small $t \neq 0$, either

$$\lambda^d_{f,z}(0) > \lambda^d_{f_t,z}(0)$$

or

$$\lambda^d_{f,z}(0) = \lambda^d_{f_t,z}(0) \quad \textit{and} \quad \lambda^{d-1}_{f,z}(0) > \lambda^{d-1}_{f_t,z}(0)$$

or

$$\vdots$$

or

$$\lambda^d_{f,z}(0) = \lambda^d_{f_t,z}(0), \dots, \lambda^1_{f,z}(0) = \lambda^1_{f_t,z}(0) \quad \textit{and} \quad \lambda^0_{f,z}(0) \geq \lambda^0_{f_t,z}(0).$$

In other words $\lambda_{(f_t),z}(0) \prec \lambda_{f,z}(0)$, where $\prec$ is the lexicographical order.

4. Jump of Lê numbers

Let $F = (f_t)$ be a deformation of f such that $\dim_0 \Sigma f_t = \dim_0 \Sigma f$ for sufficiently small t. By the above semicontinuity, we can consider the jump of Lê numbers of a deformation F in the lexicographical order.

Definition 4.1. By the *jump* $\delta_{F,z}(0)$ of a deformation F we mean

$$\lambda_{f,z}(0) - \lambda_{F,z}(0).$$

By the Theorem 3.4 and the fact that we can always deform f to be smooth, we have

$$\mathbf{0} \prec \delta_{F,z}(0) \prec \lambda_{f,z}(0).$$

Example 4.2. Let $f(x,y,z) = y^2 + z^3$. Then $\Sigma f = \{y = z = 0\}$. It easy to check that $\lambda_{f,z}(0) = (2,0)$. Taking the following sequence of deformations $f^k_t = f + tx^k z^2$, we obtain $\lambda_{f^k_t,z}(0) = (1, 3k-1)$. This shows that $\delta_{f^k_t,z}(0) = (1, 1-3k)$ can be arbitrary small.

5. Main theorem

Let $n \geq 2$ and $f\colon (U,0) \to (\mathbb{C},0)$ be an analytic function , where U is an open neighbourhood of 0 in $\mathbb{C}^n$.

Definition 5.1. We say that f is a *line singularity* if Σf is Oz_1 i.e. $\Sigma f = \{z \in \mathbb{C}^n\colon z_2 = \cdots = z_n = 0\}$ and $f|_{V(z_1)}$ has an isolated singularity at the origin.

Let f be a line singularity and let $F = (f_t)$ be its deformation.

Definition 5.2. We say that (f_t) is a *family of line singularities (F is a line deformation of f)* if Σf_t is z_1-axis and $f_t|_{V(z_1)}$ has an isolated singularity at the origin for each t near $0 \in \mathbb{C}$.

Observe that in the Example 4.2, $\lambda^0_{f,z}(0) = 0$. In the case $\lambda^0_{f,z}(0) > 0$, we give the proof of the following theorem in the class of non-degenerate line singularities (see Appendix A). We believe that it is also true for all line singularities.

Theorem 5.3. *Let $f\colon (U,0) \to (\mathbb{C},0)$ be a non-degenerate line singularity, where U is an open neighbourhood of 0 in $\mathbb{C}^n$. Assume that $z = (z_1,\dots,z_n)$ be prepolar coordinates for f i.e. $f|_{z_1=0}$ has an isolated singularity at 0. If $\lambda^0_{f,z}(0) > 0$, then there exists a line deformation (f_t) such that $\lambda^0_{(f_t),z}(0) = 0$ and $\lambda^1_{(f_t),z}(0) = \lambda^1_{f,z}(0)$.*

Proof. Take

$$f_t(z_1,\dots,z_n) = f(z_1 + t, z_2,\dots,z_n).$$

Since $z = (z_1,\dots,z_n)$ are prepolar coordinates for f, then $f|_{z_1=0}$ has an isolated singularity at 0. Since $(f_t)|_{z_1=0}$ is a deformation of $f|_{z_1=0}$, then $(f_t)|_{z_1=0}$ has an isolated singularity at 0. Therefore by [15, Remark 1.9] $(z_1,\dots,z_n)$ are prepolar coordinates for (f_t) and $\lambda^0_{(f_t),z}(0)$, $\lambda^1_{(f_t),z}(0)$ exist. Since f and (f_t) are the line singularities, by [10, 11, 15] we have

$$\lambda^1_{f,z}(0) = \mu(f|_{z_1=\varepsilon}) = \mu(f|_{z_1=\varepsilon+t}) = \mu((f_t)|_{z_1=\varepsilon}) = \lambda^1_{(f_t),z}(0).$$

We will show $\lambda^0_{(f_t),z}(0) = 0$. Since f is non-degenerate (f_t) is also non-degenerate. Moreover

$$\Gamma((f_t)) = \Gamma((f_t)|_{z_1=0}).$$

To prove it we identify the monomials of (f_t) with associated points of $\operatorname{supp}(f_t)$. The monomials, which are vertices of $\Gamma((f_t))$ do not depend on variable z_1. Indeed, suppose to the contrary that a monomial $z_1^{\alpha_1} z_2^{\beta_2} \dots z_n^{\beta_n}$ is a vertex of $\Gamma((f_t))$. Hence by the form of (f_t) monomial $z_2^{\beta_2} \dots z_n^{\beta_n}$ is a point of $\operatorname{supp}(f_t)$. Take the hyperplane supporting $\Gamma_+((f_t))$ in $z_1^{\alpha_1} z_2^{\beta_2} \dots z_n^{\beta_n}$. Then every point of $\operatorname{supp} f_t$ lies on this hyperplane or above. But the point $z_2^{\beta_2} \dots z_n^{\beta_n}$ lies below it. This gives the contradiction. Therefore by [4] we have

$$\lambda^0_{(f_t),z}(0) = \lambda^0_{(f_t)|_{z_1=0},z}(0) = 0.$$

The last equality follows from the definition of Lê numbers and the fact that $((f_t)|_{z_1=0})'_{z_1} \equiv 0$. This gives the assertion. $\square$

Remark 5.4. Roughly speaking, the deformation in the main theorem "straightens" the line singularity along its critical locus.

Example 5.5. Let $f(x,y,z) = y^2 + z^3 + x^2z^2$. Then $\Sigma f = \{y = z = 0\}$. It is easy to check that $\lambda_{f,z}(0) = (1,5)$. Take the line deformation $f_t = f + tz^2$. We have $\lambda_{(f_t),z}(0) = (1,0)$. Hence $\delta_{(f_t),z}(0) = (0,5)$.

Let f be a line, non-degenerate singularity such that $\lambda^0_{f,z}(0) > 0$. By Theorem 5.3 we can correctly define the minimal jump of f as follows.

Definition 5.6. By the *minimal jump* $\delta_{f,z}(0)$ of a singularity f we mean

$$\min\{\delta_{F,z}(0)\colon F \text{ is a deformation of } f, \delta_{F,z}(0) \succ 0\},$$

where the above minimum is taken in the lexicographical order.

Definition 5.7. By the *minimal jump in the class of line deformation* $\delta^l_{f,z}(0)$ of a singularity f we mean

$$\min\{\delta_{F,z}(0)\colon F \text{ is a line deformation of } f, \delta_{F,z}(0) \succ 0\}.$$

Remark 5.8. By (2.2), when$f$ and (f_t) are line singularities we have

$$[\Sigma f] = \lambda^1_{f,z}(0)[Oz_1] + \lambda^0_{f,z}(0)[0],$$
$$[\Sigma f_t] = \lambda^1_{(f_t),z}(0)[Oz_1] + \lambda^0_{(f_t),z}(0)[0].$$

In this case one can interpret $\delta_{(f_t),z}(0)$ as a "nearness" of the above cycles.

Proposition 5.9. *Let f be a line singularity. Then*

$$\lambda^0_{f,z}(0) \neq 1.$$

Proof. Suppose to the contrary that $\lambda^0_{f,z}(0) = 1$. It means by definition that

$$\left([\Gamma^1_{f,z}] \cdot \left[V\left(\frac{\partial f}{\partial z_1}\right)\right]\right)_0 = 1. \tag{5.1}$$

Let $[\Gamma^1_{f,z}] = \sum_{i=1}^k a_i[\Upsilon^i]$, where Υ^i are irreducible components of $\Gamma^1_{f,z}$. By (5.1) we have

$$\sum_{i=1}^k a_i\left([\Upsilon^i] \cdot \left[V\left(\frac{\partial f}{\partial z_1}\right)\right]\right)_0 = 1.$$

Therefore $k = 1$, $\Gamma^1_{f,z}$ is irreducible. Let $\varphi\colon (\mathbb{C},0) \to (\mathbb{C}^n,0)$ be a parametrization of Υ^1. Hence

$$\operatorname{ord}\left(\frac{\partial f}{\partial z_1} \circ \varphi\right) = 1.$$

This implies that $\operatorname{ord} f'_{z_1} = 1$. Hence, for some i we have

$$f(z_1, \dots, z_n) = az_1z_i + \dots$$

$a \neq 0$. Then $f'_{z_i}(t,0,\dots,0) \neq 0$. This and the assumption Σf is z_1-axis gives the contradiction. ∎

Example 5.10. Let $f(x,y,z) = y^2 + z^3 + xz^2$. Then $\Sigma f = \{y = z = 0\}$. It is easy to check that $\lambda_{f,z}(0) = (1,2)$. Take deformations $f_t = f + tz^2$. Then $\Sigma f_t = \Sigma f$ and $\lambda_{(f_t),z}(0) = (1,0)$. By Proposition 5.9 $\delta^l_{f,z}(0) = (0,2)$.

Proposition 5.11. *There exists a singularity* $f\colon (\mathbb{C}^3,0) \to (\mathbb{C},0)$ *such that*

$$\min\{\lambda^1_{f,z}(0) - \lambda^1_{F,z}(0) > 0\colon F \textit{ is a line deformation of } f\} > 1.$$

Proof. Take

$$f(x,y,z) = y^4 + z^4 + y^2z^2.$$

We check that f is a line singularity and for sufficiently small $\varepsilon \neq 0$ [15, Remark 1.19]

$$\lambda^1_{f,z}(0) = \mu_0(f|_{x=\varepsilon}) = 9.$$

Let $F = (f_t)$ be a line deformation of f. By [15, Remark 1.19] and [2, Theorem 3.1] we have

$$\lambda^1_{(f_t),z}(0) = \mu_0((f_t)|_{x=\varepsilon}) \leq 7.$$

This ends the proof. □

6. Estimation of Lê numbers

Let $f\colon (U,0) \to (\mathbb{C},0)$ be a singularity, where U is an open neighbourhood of 0 in $\mathbb{C}^n$. Suppose that $z = (z_1,\dots,z_n)$ is prepolar coordinates for f and $\dim_0 \Sigma f = 1$.

Theorem 6.1.

$$\lambda_{f,z}(0) \succ (\widetilde{\nu}_1(f_1), (-1)^n + \nu_0(f_1) + \widetilde{\nu}_1(f_1)),$$
$$\lambda^1_{f,z}(0) \geq \widetilde{\nu}_1(f_1),$$

where $f_1 = f + z_1^\alpha$, α *is sufficiently big and* $\nu_0(f_1)$, $\widetilde{\nu}_1(f_1)$ *are modified Newton numbers (see [4]). The equalities hold, if* f *is non-degenerate.*

Proof. If f is non-degenerate, then the assertion follows from [4, Theorem 4.1]. Assume now that f is degenerate. Since the non-degeneracy is open condition (see [16, Appendix]) there exists a non-degenerate deformation (f_t) of f with the same Newton diagram. Since the modified Newton numbers depend only on the Newton diagram, modified Newton numbers of f and (f_t) are the same. Since $z = (z_1,\dots,z_n)$ is prepolar coordinates for f it is also prepolar for (f_t). By [15, Theorem 1.28] the Lê numbers of (f_t) exist. Hence by [4, Theorem 4.1] we have

$$\lambda_{(f_t),z}(0) = (\widetilde{\nu}_1(f_1), (-1)^n + \nu_0(f_1) + \widetilde{\nu}_1(f_1)),$$
$$\lambda^1_{(f_t),z}(0) = \widetilde{\nu}_1(f_1).$$

On the other hand, by the upper-semicontinuity of Lê numbers we get

$$\lambda_{f,z}(0) \succ \lambda_{(f_t),z}(0),$$
$$\lambda^1_{f,z}(0) \geq \lambda^1_{(f_t),z}(0).$$

Summing up, we get the assertion. □

Appendix A. Newton diagram

Here, the reference is Kouchnirenko [9].

Let $z := (z_1, \ldots, z_n)$ be a system of coordinates for $\mathbb{C}^n$, let U be an open neighbourhood of the origin in $\mathbb{C}^n$, and let

$$f \colon (U, 0) \to (\mathbb{C}, 0), \quad z \mapsto f(z) = \sum_{\alpha} c_\alpha z^\alpha,$$

be an analytic function, where $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $c_\alpha \in \mathbb{C}$, and z^α is a notation for the monomial $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

The *Newton polyhedron* $\Gamma_+(f)$ of f (at the origin and with respect to the coordinates $z = (z_1, \ldots, z_n)$) is the convex hull in $\mathbb{R}_+^n$ of the set

$$\bigcup_{c_\alpha \neq 0} (\alpha + \mathbb{R}_+^n).$$

For any $v \in \mathbb{R}_+^n \setminus \{0\}$, put

$$\begin{aligned} \ell(v, \Gamma_+(f)) &:= \min\{\langle v, \alpha\rangle \,;\, \alpha \in \Gamma_+(f)\}, \\ \Delta(v, \Gamma_+(f)) &:= \{\alpha \in \Gamma_+(f) \,;\, \langle v, \alpha\rangle = \ell(v, \Gamma_+(f))\}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$. A subset $\Delta \subseteq \Gamma_+(f)$ is called a *face* of $\Gamma_+(f)$ if there exists $v \in \mathbb{R}_+^n \setminus \{0\}$ such that $\Delta = \Delta(v, \Gamma_+(f))$. The *dimension* of a face Δ of $\Gamma_+(f)$ is the minimum of the dimensions of the affine subspaces of $\mathbb{R}^n$ containing Δ. The *Newton diagram* (also called *Newton boundary*) of f is the union of the compact faces of $\Gamma_+(f)$. It is denoted by $\Gamma(f)$. We say that f is *convenient* if the intersection of $\Gamma(f)$ with each coordinate axis of $\mathbb{R}_+^n$ is non-empty (i.e., for any $1 \leq i \leq n$, the monomial $z_i^{\alpha_i}$, $\alpha_i \geq 1$, appears in the expression $\sum_\alpha c_\alpha z^\alpha$ with a non-zero coefficient).

For any face $\Delta \subseteq \Gamma(f)$, define the *face function* f_Δ by

$$f_\Delta(z) := \sum_{\alpha \in \Delta} c_\alpha z^\alpha.$$

We say that f is *Newton non-degenerate* (in short, non-degenerate) on the face Δ if the equations

$$\frac{\partial f_\Delta}{\partial z_1}(z) = \cdots = \frac{\partial f_\Delta}{\partial z_n}(z) = 0$$

have no common solution on $(\mathbb{C} \setminus \{0\})^n$. We say that f is *(Newton) non-degenerate* if it is non-degenerate on every face Δ of $\Gamma(f)$.

References

[1] A. Bodin, *Jump of Milnor numbers.* Bull. Braz. Math. Soc. (N.S.) 38 (2007), 389–396.
[2] Sz. Brzostowski, T. Krasiński, *The jump of the Milnor number in the X_9 singularity class.* Cent. Eur. J. Math. 12 (2014), 429–435.
[3] C. Eyral, *Uniform stable radius, Lê numbers and topological triviality for line singularities.* Pacific J. Math. 291 (2017), no. 2, 359–367.
[4] C. Eyral, G. Oleksik, A. Różycki, *Lê numbers and Newton diagram.* Adv. Math. 376 (2021), Paper No. 107441, 21 pp.
[5] J. Fernández de Bobadilla, *Topological equisingularity of hypersurfaces with 1-dimensional critical set.* Adv. Math. 248 (2013), 1199–1253.
[6] S. M. Guseĭn-Zade, *On singularities from which an A_1 can be split off.* Funktcional.Anal.iPrilozhen. 27 (1993), 68–71.
[7] T. Krasiński, J. Walewska, *Jumps of Milnor numbers of Brieskorn-Pham singularities in non-degenerate families.* Results Math. 73 (2018), no. 3, Art. 94, 13 pp.
[8] T. Krasiński, J. Walewska, *Non-degenerate jumps of Milnor numbers of quasihomogeneous singularities.* Ann. Polon. Math. 123 (2019), no. 1, 369–386.
[9] A. G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor.* Invent. Math. 32 (1976), no. 1, 1–31.
[10] Lê Dũng Tráng, *Ensembles analytiques complexes avec lieu singulier de dimension un (d'après I. N. Iomdine).* (French) [Complex analytic sets with one-dimensional singular locus (following Y. N. Yomdin)] Seminar on Singularities (Paris, 1976/1977), pp. 87–95, Publ. Math. Univ. Paris VII, 7, Univ. Paris VII, Paris, 1980.
[11] D. B. Massey, *The Lê-Ramanujam problem for hypersurfaces with one-dimensional singular sets.* Math. Ann. 282 (1988), no. 1, 33–49.
[12] D. B. Massey, *A reduction theorem for the Zariski multiplicity conjecture.* Proc. Amer. Math. Soc. 106 (1989), no. 2, 379–383.
[13] D. B. Massey, *The Lê varieties.* I. Invent. Math. 99 (1990), no. 2, 357–376.
[14] D. B. Massey, *The Lê varieties. II.* Invent. Math. 104 (1991), no. 1, 113–148.
[15] D. B. Massey, *Lê cycles and hypersurface singularities.* Lecture Notes Math. 1615, Springer-Verlag, Berlin, 1995.
[16] M. Oka, *On the bifurcation of the multiplicity and topology of the Newton boundary.* J. Math. Soc. Japan 31 (1979), 43–450.
[17] J. Walewska, *The second jump of Milnor numbers.* Demonstr. Math. 43 (2010), 361–374.

(Grzegorz Oleksik) Faculty of Mathematics and Computer Science, University of Łódź, S. Banacha 22, 90-238 Łódź, POLAND

Email address: `grzegorz.oleksik@wmii.uni.lodz.pl`

(Adam Różycki) Faculty of Mathematics and Computer Science, University of Łódź, S. Banacha 22, 90-238 Łódź, POLAND

Email address: `adam.rozycki@wmii.uni.lodz.pl`