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Critical Transitions in Piecewise Uniformly Continuous Concave Quadratic Ordinary Differential Equations

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Abstract

A critical transition for a system modelled by a concave quadratic scalar ordinary differential equation occurs when a small variation of the coefficients changes dramatically the dynamics, from the existence of an attractor–repeller pair of hyperbolic solutions to the lack of bounded solutions. In this paper, a tool to analyze this phenomenon for asymptotically nonautonomous ODEs with bounded uniformly continuous or bounded piecewise uniformly continuous coefficients is described, and used to determine the occurrence of critical transitions for certain parametric equations. Some numerical experiments contribute to clarify the applicability of this tool.

Keywords Critical transition · Rate-induced tipping · Nonautonomous bifurcation

Mathematics Subject Classification 37B55 · 37G35 · 37M22

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1 Introduction

Substantial and irreversible changes in the output of a system upon a negligible change in the input are referred to as *critical transitions* or *tipping points*. Motivated by current exceptional challenges in nature and society [18, 38], the study of the several mechanisms leading to a critical transition has experienced a renewed scientific thrust. In recent years, for example, it has been observed that a time-dependent transition connecting a *past* dynamical system to a *future* one can give rise to critical transitions when the transition dynamics "fails to connect the limit ones" [5]. This type of phenomenon has been identified in several real scenarios including ecology [39, 42], climate [2, 5, 27, 43], biology [21], and quantum mechanics [23], among others.

Frequently in the literature (see for example [5, 24, 32, 34]), the evolution of the system from the past to the future is modeled by an asymptotically autonomous differential equation. An asymptotically nonautonomous version of this theory has been considered recently for the first time in [29], where also the past and future systems are time-dependent: this reference deals with scalar quadratic differential equations of the type

$$y' = -\big(y - \Gamma(t)\big)^2 + p(t) \tag{1.1}$$

with $\Gamma(t) := (2/\pi)\arctan(ct)$ for $c > 0$ and $p\colon \mathbb{R} \to \mathbb{R}$ bounded and uniformly continuous. There are two main reasons for this choice. First, the global dynamics induced by a quadratic differential equation is basically described by the presence or the absence of a (classical) attractor–repeller pair of (bounded) hyperbolic solutions. In consequence, these equations offer a solid structure to formulate and study the possible occurrence of critical transitions: small changes in the coefficients may cause an attractor–repeller pair to disappear. In fact, quadratic differential equations have been identified as prototype models for the so-called *rated-induced tipping* (which we will describe below) since the very beginning [6], and have been further studied in this context [5, 20, 35]. Second, quadratic differential equations appear as mathematical models in many different areas of applied sciences, which makes this formulation interesting by itself. For instance: several model in mathematical finance respond to this type of Eq. [7, 8]; the relation (1.1) is also the Riccati equation of a two-dimensional linear hamiltonian system and the possible presence of the attractor–repeller pair is related with the existence of an exponential dichotomy of this linear equation, which in turn determines the existence of a local attractor or the lack of bounded solutions in some associated nonlinear models [15, 22]; and Eq. (1.1) are simple models of concave differential equations, which appear often in applications and share a common dynamical description given by the presence or absence of an attractor–repeller pair [12, 31].

In this paper, with the aim to contribute to a more robust mathematical theory of critical transitions, we go deeper in the theoretical and numerical analysis initiated in [29], which is now extended to Eq. (1.1) with much more general coefficients, as well as to more general types of critical transitions. When Γ and p are arbitrary measurable functions belonging to the Banach space $L^\infty(\mathbb{R}, \mathbb{R})$, (1.1) fits in the class of Carathéodory differential equations, which have well-known regularity properties. We analyze the case where these coefficients are bounded and piecewise uniformly continuous functions with an at most countable set of discontinuity points (BPUC, for short). A highly technical and far from trivial extension of the methods used in [29] allows us to show that the description of the dynamical possibilities there given remains valid in this extended framework. In particular, the bifurcation analysis for $y' = -(y - \Gamma(t))^2 + p(t) + \lambda$ associates a certain real value $\lambda^*(\Gamma, p)$ to (1.1), in such a way that (1.1) admits an attractor–repeller pair of hyperbolic solutions if $\lambda^*(\Gamma, p) < 0$

(CASE A), it admits bounded but no hyperbolic solutions if $\lambda^*(\Gamma, p) = 0$ (CASE B), and bounded solutions do not exist if $\lambda^*(\Gamma, p) > 0$ (CASE C). This description shows that the map $(\Gamma, p) \mapsto \lambda^*(\Gamma, p)$ is a strong tool to analyze the occurrence of critical transitions as Γ and p vary. In fact, CASE A is equivalent to the existence of at least one bounded hyperbolic solution, and a key point in this assertion is the choice of BPUC coefficients: we prove that a BPUC function has a compact hull Ω for the L^1_{loc}-topology and induces a continuous skew-product flow on $\Omega \times \mathbb{R}$; and these properties are required in the proof of the mentioned equivalence. For reasons which will become clear in the next paragraphs, we need to deal with bounded piecewise constant functions (which also appear in some applications), and hence the set of BPUC maps provides an optimal framework to formulate our results.

We add two more hypotheses to our BPUC coefficients Γ and p: the asymptotic limits $\Gamma(\pm\infty)$ exist and are finite; and the equation $y' = -y^2 + p(t)$ has an attractor–repeller pair, which implies this same dynamical structure for the past and future systems $y' = -(y - \Gamma(\pm\infty))^2 + p(t)$. These conditions will be in force in the next paragraphs. They ensure the existence of: a local pullback attractor for (1.1) which "connects with the attractor for the past" as time decreases, meaning that the distance between both maps goes to 0 as $t \to -\infty$; and of a local pullback repeller for (1.1) which "connects with the repeller for the future" as time increases. When this local pullback attractor and repeller are globally defined and different, they form an attractor–repeller pair which, in addition, connects those of the past and the future, and we are in CASE A: this is the situation usually called (*end-point*) *tracking*. If the local pullback attractor is globally defined and coincides with the local pullback repeller, then they provide a unique bounded solution, and we are in CASE B. And the only remaining possibility is that none of them is globally defined, which corresponds to CASE C, and is sometimes called *tipping*. When a small variation of Γ and p changes the dynamics from CASE A to CASE C (from tracking to tipping), we have a critical transition.

In this paper, we analyze the occurrence of critical transitions as a parameter c varies for two different types of one-parametric equations of (1.1) type, which now we write as $y' = -(y - \Gamma_c(t))^2 + p(t)$. For both models, the function $\lambda^*(\Gamma_c, p)$ varies continuously with the parameter c, and the most basic type of critical transition (which we call *transversal*) occurs when its graph crosses the vertical axis: this means a change from CASE A to CASE C at a particular *tipping value* c_0 of the parameter. In particular, as expected, the dynamics fits in CASE B for $c = c_0$. The previous description of these cases shows the link between this type of tipping points and a simple nonautonomous saddle-node bifurcation pattern [30]: a transversal critical transition occurs when the attractor–repeller pair collides in just one bounded solution. Such a collision has been explored analytically and numerically in several contexts: in one-dimensional systems [5, 26]; in higher-dimensional systems [1, 36, 44, 45]; in set-valued dynamical systems [11]; in random dynamical systems [20]; in regards to early-warning signals [35, 36]; and in the nonautonomous formulation [29]. There are other points of connection between the two considered cases. For instance, a large enough transition $\Gamma_c(+\infty) - \Gamma_c(-\infty)$ guarantees the occurrence of critical transitions, while a decreasing function Γ_c makes this occurrence impossible. The role played by the size of the coefficients of the model in the occurrence of tipping points is a key question, which appears implicit in several works, as [3, 32, 34].

For our first model, $\Gamma_c(t) := c\,\Gamma(t)$ for a C^1 function Γ (always with finite asymptotic limits), and p is a BPUC function. An in-depth analysis of the map $c \mapsto \widehat{\lambda}(c) := \lambda^*(c\,\Gamma, p)$ shows its continuity as well as some fundamental monotonicity properties. This allows us to prove that, if Γ has a local increasing point, then $\widehat{\lambda}(c) > 0$ if c is large enough. Since, by hypothesis, $\widehat{\lambda}(0) < 0$, at least a critical transition occurs. In addition, there is a unique zero of $\widehat{\lambda}$ (a unique critical transition) if Γ is nondecreasing.

Our second model fits in a rate-induced tipping pattern, as in almost all the afore-mentioned references. In this case, we take $\Gamma_c(t) := \Gamma(c\,t)$ for a fixed Γ, so that c determines the speed of the transition from the past system to the future system, which are common for all $c > 0$. As before, p is assumed to be BPUC; and now we include the analysis of bounded piecewise constant transition functions Γ. These models seem to be physically reasonable. When the rate c tends to infinite, the transition function tends to a new piecewise constant function, and hence the limit equation is included in the theoretical formulation. The function $\lambda_*(c) := \lambda^*(\Gamma_c, p)$ varies continuously with respect to c on $\mathbb{R}^+ \cup \{\infty\}$. From this continuity, it is posible to deduce the tracking when the rate c is small and also the occurrence of tracking or tipping when it is large enough, based on the analysis equation corresponding to $c = \infty$. In addition, if the piecewise constant function Γ^h is defined by coinciding with an initially fixed continuous Γ at the discrete set $\{jh \mid j \in \mathbb{Z}\}$, and $\Gamma_c^h(t) := \Gamma^h(c\,t)$ (so that Γ_c^h is BPUC), then the function $(\Gamma_c^h, p) \to \lambda^*(\Gamma_c^h, p)$ varies continuously with respect to the L^1_{loc}-topology on the subset $\{(\Gamma_c^h, p) \mid c \in \mathbb{R} \cup \{\pm\infty\},\ h \in [0, h_0]\} \subset \text{BPUC} \times \text{BPUC}$ for any $h_0 \geq 0$. Getting this continuity is one of the most challenging problems in this paper. (In fact, the map $(\Gamma, p) \mapsto \lambda^*(\Gamma, p)$ is locally Lipschitz for the L^∞-norm on BPUC $\times$ BPUC, but λ^* is not a continuous function for the L^1_{loc}-topology.) As a consequence of the continuity, the properties of the continuous case can be understood by taking limits as h tends to 0. These facts, combined with a simple numerical analysis and with an easy characterization of $\lambda_*(\infty, h)$, allow us to show interesting tipping phenomena for a quite simple example (as its possibly revertible character) and to explain the concept of *partial tipping* in our setting. The occurrence of tipping points in piecewise constant transition functions is also analyzed in [3, 27].

The paper is organized as follows. Section 2 extends to the most general situation considered in the paper some dynamical properties previously known for quadratic differential equation with continuous coefficients. An important part of the (highly technical) proofs is postponed to Appendix A. Section 3 starts an in-depth study of the bifurcation function $\lambda^*(\Gamma, p)$ and includes the analysis of the first model above mentioned. The last two sections of the paper concern the occurrence of rate-induced tipping for the second model. Section 4 deals with the case where the functions Γ is continuous, whereas in Sect. 5 the transition function is taken piecewise constant. The phenomenon of partial tipping is described in Sect. 4. Appendix B, which completes the paper, justifies the accuracy of the numerical examples included in the previous sections.

2 General Results for Concave Quadratic Scalar ODEs

Throughout the paper, $L^\infty(\mathbb{R}, \mathbb{R})$ is the Banach space of essentially bounded functions $q\colon \mathbb{R} \to \mathbb{R}$ endowed with the norm $\|q\|$ given by the inferior of the set of real numbers $k \geq 0$ such that the Lebesgue measure of $\{\, t \in \mathbb{R} \mid |q(t)| > k \,\}$ is zero.

Let us consider the nonautonomous concave quadratic scalar equation

$$x' = -x^2 + q(t)\,x + p(t)\,, \tag{2.1}$$

where q, p belong to $L^\infty(\mathbb{R}, \mathbb{R})$. Later on, we will have to be more restrictive in the choice of q and p, but we will first establish some general properties. Throughout this section, $t \mapsto x(t, s, x_0)$ represents the unique maximal solution of (2.1) satisfying $x(s, s, x_0) = x_0$, defined for $t \in \mathcal{I}_{s,x_0} = (\alpha_{s,x_0}, \beta_{s,x_0})$ with $-\infty \leq \alpha_{s,x_0} < s < \beta_{s,x_0} \leq \infty$. Recall that, in this setting, a solution is an absolutely continuous function on each compact interval of $\mathcal{I}_{s,x_0}$

which satisfies (2.1) at Lebesgue almost every $t \in \mathcal{I}_{s,x_0}$; and that $\mathcal{I}_{s,x_0} = \mathbb{R}$ if $x(t, s, x_0)$ is bounded. The results establishing the existence and properties of this unique maximal solution can be found in [13, Chapter 2]. Recall also that the real map x, defined on an open subset of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ containing $\{(s, s, x_0) \mid s, x_0 \in \mathbb{R}\}$, satisfies $x(s, s, x_0) = x_0$ and $x(t, l, x(l, s, x_0)) = x(t, s, x_0)$ whenever all the involved terms are defined. In fact, these results hold for Carathéodory differential equations of more general type. For instance, those of the form (2.1) with $q, p \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$, where $L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ is the space of Borel functions $b \colon \mathbb{R} \to \mathbb{R}$ which are integrable on compact intervals (which, as explained in Appendix A, is a complete metric space).

2.1 Hyperbolic Solutions and Their Persistence

Let q, p belong to $L^\infty(\mathbb{R}, \mathbb{R})$. A bounded solution $\widetilde{b} \colon \mathbb{R} \to \mathbb{R}$ of (2.1) is said to be *hyperbolic* if the corresponding variational equation $z' = (-2\,\widetilde{b}(t) + q(t))\,z$ has an exponential dichotomy on $\mathbb{R}$. That is (see [14]), if there exist $k_b \geq 1$ and $\beta_b > 0$ such that either

$$\exp \int_s^t (-2\,\widetilde{b}(l) + q(l))\,dl \leq k_b\,e^{-\beta_b(t-s)} \quad \text{whenever} \quad t \geq s \tag{2.2}$$

or

$$\exp \int_s^t (-2\,\widetilde{b}(l) + q(l))\,dl \leq k_b\,e^{\beta_b(t-s)} \quad \text{whenever} \quad t \leq s \tag{2.3}$$

holds. If (2.2) holds, the hyperbolic solution $\widetilde{b}$ is *(locally) attractive*, and if (2.3) holds, $\widetilde{b}$ is *(locally) repulsive*. In both cases, we call (k_b, β_b) a (non-unique) *dichotomy constant pair* for the solution $\widetilde{b}$ (or for the equation $z' = (-2\,\widetilde{b}(t) + q(t))\,z$).

Proposition 2.1 *Assume that* (2.1) *has an attractive (resp. repulsive) hyperbolic solution* $\widetilde{b}_{q,p}$. *Then, this hyperbolic solution is persistent in the following sense: for any* $\varepsilon > 0$ *there exists* $\delta_\varepsilon > 0$ *such that, if* $\bar{q}, \bar{p} \in L^\infty(\mathbb{R}, \mathbb{R})$ *satisfy* $\|\bar{q} - q\| < \delta_\varepsilon$ *and* $\|\bar{p} - p\| < \delta_\varepsilon$, *then also the perturbed differential equation*

$$x' = -x^2 + \bar{q}(t)\,x + \bar{p}(t)$$

has an attractive (resp. repulsive) hyperbolic solution $\widetilde{b}_{\bar{q},\bar{p}}$ *which satisfies* $\|\widetilde{b}_{q,p} - \widetilde{b}_{\bar{q},\bar{p}}\| < \varepsilon$. *In addition, there exists a common dichotomy constant pair for the variational equations* $z' = (-2\,\widetilde{b}_{\bar{q},\bar{p}}(t) + \bar{q}(t))\,z$ *corresponding to all the functions* $\bar{q}$ *and* $\bar{p}$ *which satisfy* $\|\bar{q} - q\| < \delta_\varepsilon$ *and* $\|\bar{p} - p\| < \delta_\varepsilon$.

Proof The proof follows step by step that of [29, Proposition 3.2]. Note that given $s \in L^\infty(\mathbb{R}, \mathbb{R})$, the equation

$$y' = (-2\,\widetilde{b}_{q,p}(t) + q(t))\,y + s(t) \tag{2.4}$$

has a (unique) bounded solution, given by $t \mapsto \int_{-\infty}^t u(t)\,u^{-1}(l)\,s(l)\,dl$ for $u(t) := \exp \int_0^t (-2\widetilde{b}(l) + q(l))\,dl$. This allows us to define the operator T on the Banach space of real bounded continuous functions on $\mathbb{R}$ as in [29], and repeat the whole argument used there. □

The next result shows the persistence also of those solutions for which the variational equation has exponential dichotomy not in the whole of $\mathbb{R}$, but in a half-line. We represent by "$\sup\mathrm{ess}_{t\in\mathcal{I}}$" the restriction of the L^∞-norm to an interval $\mathcal{I}$, and by $L^\infty(\mathcal{I}, \mathbb{R})$ the corresponding Banach space.

Proposition 2.2 *Let $\widehat{q}, \widehat{p}\colon (-\infty, t_*] \to \mathbb{R}$ belong to $L^\infty((-\infty, t_*], \mathbb{R})$, where $t_* \in \mathbb{R}$. Assume that the equation*

$$x' = -x^2 + \widehat{q}(t)\,x + \widehat{p}(t) \tag{2.5}$$

has a bounded solution $\widetilde{b}_{\widehat{p},\widehat{q}}\colon (-\infty, t_*] \to \mathbb{R}$ satisfying

$$\exp \int_s^t (-2\,\widetilde{b}_{\widehat{q},\widehat{p}}(l) + \widehat{q}(l))\,dl \le k\,e^{-\beta(t-s)} \quad \text{whenever} \quad t_* \ge t \ge s$$

for some constants $k \ge 1$ and $\beta > 0$. Given $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that, if $\bar{q}, \bar{p}\colon (-\infty, t_*] \to \mathbb{R}$ belong to $L^\infty((-\infty, t_*], \mathbb{R})$ and satisfy $\sup\mathrm{ess}_{t\le t_*}|\bar{q}(t) - \widehat{q}(t)| < \delta_\varepsilon$ and $\sup\mathrm{ess}_{t\le t_*}|\bar{p}(t) - \widehat{p}(t)| < \delta_\varepsilon$, then the equation

$$x' = -x^2 + \bar{q}(t)\,x + \bar{p}(t)$$

has a solution $\widetilde{b}_{\bar{q},\bar{p}}\colon (-\infty, t_*] \to \mathbb{R}$ such that $\sup_{t\le 0}|\widetilde{b}_{\widehat{q},\widehat{p}}(t) - \widetilde{b}_{\bar{q},\bar{p}}(t)| < \varepsilon$ and

$$\exp \int_s^t (-2\,\widetilde{b}_{\bar{q},\bar{p}}(l) + \bar{q}(l))\,dl \le \bar{k}\,e^{-\bar{\beta}(t-s)} \quad \text{whenever} \quad t_* \ge t \ge s \tag{2.6}$$

for some constants $\bar{k} \ge 1$ and $\bar{\beta} > 0$.

Let now $\widehat{q}, \widehat{p}\colon [t_*, \infty) \to \mathbb{R}$ belong to $L^\infty([t_*, \infty), \mathbb{R})$, and assume that the Eq. (2.5) has a bounded solution $\widetilde{b}_{\widehat{p},\widehat{q}}\colon [t_*, \infty)) \to \mathbb{R}$ satisfying

$$\exp \int_s^t (-2\,\widetilde{b}_{\widehat{q},\widehat{p}}(l) + \widehat{q}(l))\,dl \ge k\,e^{\beta(t-s)} \quad \text{whenever} \quad t_* \le t \le s$$

for some constants $k \ge 1$ and $\beta > 0$. Then, the conclusions are analogous to those of the first case.

Proof The proof is almost identical to that of Proposition 2.1. The differences, in the first case, are that now we work just on $(-\infty, t_*]$, and that (2.4) may have solutions different from $\widetilde{b}_{\widehat{p},\widehat{q}}$ which are bounded in this interval. Nevertheless, we can define the operator T by the same expression, acting now on the Banach space of the bounded continuous real functions on $(-\infty, t_*]$; and the argument of [29] works. The proof of the second case is analogous. □

2.2 Concavity, and the Sets of Half-bounded and Bounded Solutions

Let q, p belong to $L^\infty(\mathbb{R}, \mathbb{R})$. The concavity on x of the function giving rise to (2.1) ensures the concavity with respect to the state of the corresponding solutions:

Proposition 2.3 *As long as the involved terms are defined, we have*

$$x(t, s, \rho\,x_1 + (1-\rho)\,x_2) > \rho\,x(t, s, x_1) + (1-\rho)\,x(t, s, x_2) \quad \text{if} \quad \rho \in (0,1) \quad \text{and} \quad t > s\,,$$
$$x(t, s, \rho\,x_1 + (1-\rho)\,x_2) < \rho\,x(t, s, x_1) + (1-\rho)\,x(t, s, x_2) \quad \text{if} \quad \rho \in (0,1) \quad \text{and} \quad t < s\,.$$

Proof We rewrite the equation as $x' = f(t, x)$. Then, since f is strictly concave in its second argument, $f(t, \rho\,x_1 + (1-\rho)\,x_2) > \rho\,f(t, x_1) + (1-\rho)\,f(t, x_2)$ if $\rho \in (0, 1)$. This inequality and the comparison result for Carathéodory equations given in [33, Theorem 2] (based on the previous results of [10]) prove the assertions. □

The concavity has also fundamental consequences on the properties of the sets

$$\mathcal{B}^- := \left\{ (s, x_0) \in \mathbb{R}^2 \,\middle|\, \sup_{t\in(\alpha_{s,x_0},s]} x(t,s,x_0) < \infty \right\},$$

$$\mathcal{B}^+ := \left\{ (s, x_0) \in \mathbb{R}^2 \,\middle|\, \inf_{t\in[s,\beta_{s,x_0})} x(t,s,x_0) > -\infty \right\},$$

which may be empty. We fix $\varepsilon > 0$ and $m > 0$ large enough to satisfy $-m^2 + |q(t)|\,m + |p(t)| \le -\varepsilon$ for all $t \in \mathbb{R}$, which yields $-x^2 + p(t)x + q(t) \le -\varepsilon$ for all $t \in \mathbb{R}$ and $|x| \ge m$. Then, for all $(s, x_0) \in \mathbb{R}^2$, $\liminf_{t\to(\alpha_{s,x_0})^+} x(t,s,x_0) > -m$ and $\limsup_{t\to(\beta_{s,x_0})^-} x(t,s,x_0) < m$. In other words, **any solution remains upper bounded as time increases and lower bounded as time decreases**. We will use this property repeatedly in the paper without further reference. In particular, $\alpha_{s,x_0} = -\infty$ for all $(s, x_0) \in \mathcal{B}^-$ and $\beta_{s,x_0} = \infty$ for all $(s, x_0) \in \mathcal{B}^+$; and $\mathcal{B} := \mathcal{B}^- \cap \mathcal{B}^+$ is the (possibly empty) set of pairs (s, x_0) giving rise to (globally defined) bounded solutions of (2.1).

Remark 2.4 Recall that, given a continuous function $f : [a, b] \to \mathbb{R}$ of bounded variation (as is the case with any monotonic continuous function), there exists a finite Borel measure μ such that $f(x) - f(a) = \mu([a, x))$. The Radon-Nikodym decomposition of μ with respect to the Lebesgue measure l, $\mu = \mu_{ac} + \mu_s$, provides the *singular part* of f, $f_s(x) := \mu_s([a, x))$. In addition, f is differentiable at l-a.e. $t \in [a, b]$ and f' is L^1 with respect to l. Moreover, if f is nondecreasing, then $f'(t) \ge 0$ whenever it exists, and $f(x) - f(a) = \int_a^x f'(t)\,dt + f_s(x)$, with f_s nondecreasing and with $f_s'(t) = 0$ for l-a.e. $t \in [a.b]$. Finally, f is absolutely continuous on $[a, b]$ if and only if $f_s \equiv 0$. (See e.g. [37, Exercises 1.13 and 1.12, and Theorem 6.10].) In particular, any bounded solution of a Carathéodory equation satisfies the initial conditions of Theorem 2.5(v).

Theorem 2.5 *Let $\mathcal{B}^\pm$, $\mathcal{B}$ and m be the sets and constant above defined.*

(i) *If $\mathcal{B}^-$ is nonempty, then there exist a set $\mathcal{R}^-$ coinciding with $\mathbb{R}$ or with a negative open half-line and a maximal solution $a : \mathcal{R}^- \to (-\infty, m)$ of (2.1) such that, if $s \in \mathcal{R}^-$, then $x(t, s, x_0)$ remains bounded as $t \to -\infty$ if and only if $x_0 \le a(s)$; and if $\sup \mathcal{R}^- < \infty$, then $\lim_{t\to(\sup \mathcal{R}^-)^-} a(t) = -\infty$.*

(ii) *If $\mathcal{B}^+$ is nonempty, then there exist a set $\mathcal{R}^+$ coinciding with $\mathbb{R}$ or with a positive open half-line and a maximal solution $r : \mathcal{R}^+ \to (-m, \infty)$ of (2.1) such that, if $s \in \mathcal{R}^+$, then $x(t, s, x_0)$ remains bounded as $t \to +\infty$ if and only if $x_0 \ge r(s)$; and if $\inf \mathcal{R}^+ > -\infty$, then $\lim_{t\to(\inf \mathcal{R}^+)^+} r(t) = \infty$.*

(iii) *Let x be a solution defined on a maximal interval (α, β). If it satisfies $\liminf_{t\to\beta^-} x(t) = -\infty$, then $\beta < \infty$; and if $\limsup_{t\to\alpha^+} x(t) = \infty$, then $\alpha > -\infty$. In particular, any globally defined solution is bounded.*

(iv) *The set $\mathcal{B}$ is nonempty if and only if $\mathcal{R}^- = \mathbb{R}$ or $\mathcal{R}^+ = \mathbb{R}$, in which case both equalities hold, a and r are globally defined and bounded solutions of (2.1), and $\mathcal{B} = \{(s, x_0) \in \mathbb{R}^2 \mid r(s) \le x_0 \le a(s)\} \subset \mathbb{R} \times [-m, m]$.*

(v) *Let the function $b : \mathbb{R} \to \mathbb{R}$ be bounded, continuous, of bounded variation and with nonincreasing singular part on every compact interval of $\mathbb{R}$. Assume that $b'(t) \le -b^2(t) + q(t)\,b(t) + p(t)$ for almost all $t \in \mathbb{R}$. Then, $\mathcal{B}$ is nonempty, and $r \le b \le a$. If, in addition, there exists $t_0 \in \mathbb{R}$ such that $b'(t_0) < -b^2(t_0) + q(t_0)\,b(t_0) + p(t_0)$, then $r < a$. And, if $b'(t) < -b^2(t) + q(t)\,b(t) + p(t)$ for almost all $t \in \mathbb{R}$, then $r < b < a$.*

Proof The proofs of (i)-(iv) repeat step by step those of [29, Theorem 3.1]. The unique required change is in (iii), where we substitute "for all $t \geq s_0$" by "for Lebesgue a.a. $t \geq s_0$". Let us prove (v). The comparison theorem for Carathéodory equations (see [33, Theorem 2]) yields $x(t, s, b(s)) \geq b(t)$ for all $s \in \mathbb{R}$ and $t \geq s$, so that $(s, b(s)) \in \mathcal{B}^+$; and $x(t, s, b(s)) \leq b(t)$ for all $t \leq s$, so that $(s, b(s)) \in \mathcal{B}^-$. Consequently, $(s, b(s)) \in \mathcal{B}$ for all $s \in \mathbb{R}$: $\mathcal{B}$ is nonempty, and $r \leq b \leq a$. If, in addition, there is $t_0 \in \mathbb{R}$ with $b'(t_0) < -b^2(t_0)+q(t_0)\,b(t_0)+p(t_0) = (d/dt)x(t, t_0, b(t_0))|_{t=t_0}$, then an easy contradiction argument shows that there exists $t_1 > t_0$ such that $b(t_1) < x(t_1, t_0, b(t_0))$. Hence, $x(t, t_0, b(t_0))$ and $x(t, t_1, b(t_1))$ are different bounded solutions of (2.1). Hence, $(t_1, b(t_1)), (t_1, x(t_1, t_0, b(t_0)) \in \mathcal{B}$, which ensures that $r < a$. Finally, under the last assumption in (v), we can adapt the argument in [33] to prove that $x(t, s, b(s)) > b(t)$ whenever $t > s$ and $x(t, s, b(s)) < b(t)$ whenever $t < s$. Hence, $a(t) = a(t, t-1, a(t-1)) \geq x(t, t-1, b(t-1)) > b(t)$ and $r(t) \leq x(t, t+1, b(t+1)) < b(t)$ for any $t \in \mathbb{R}$, which completes the proof. □

Remark 2.6 Note that (2.1) has a bounded solution if and only if there exist times $t_1 \leq t_2$ (which can be equal) such that the solutions a and r defined in Theorem 2.5 are respectively defined at least on $(-\infty, t_2]$ and $[t_1, \infty)$, and $a(t) \geq r(t)$ for $t \in [t_1, t_2]$. The "only if" follows from Theorem 2.5(iv). To check the "if", we assume that, despite the described situation, a is unbounded. Then, it is not globally defined and, since it is upper bounded, its graph goes to $-\infty$ (that is, it has a vertical asymptote) at a certain time to the right of t_2; but, if so, this graph intersects that of r, impossible. Note also that the inequality $a(t) > r(t)$ for $t \in [t_1, t_2]$ is equivalent to the existence of at least two bounded solutions.

2.3 Occurrence of an Attractor–Repeller Pair

As said before, the main results in this paper require us to be more exigent with the properties assumed on the coefficients of the quadratic equation (2.1). Let $\Delta \subset \mathbb{R}$ be a *disperse* set, i.e., $\Delta = \{a_j \in \mathbb{R} \mid j \in \mathbb{Z}\}$ with $\inf_{j\in\mathbb{Z}}(a_{j+1} - a_j) > 0$. We denote by $BPUC_\Delta(\mathbb{R}, \mathbb{R})$ the set of bounded real functions which are defined and uniformly continuous on $\mathbb{R}-\Delta$. More precisely, $q\colon \mathbb{R}-\Delta \to \mathbb{R}$ belongs to $BPUC_\Delta(\mathbb{R}, \mathbb{R})$ if and only if

c1 there is $c > 0$ such that $|q(t)| < c$ for all $t \in \mathbb{R}-\Delta$;
c2 for all $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that, if $t_1, t_2 \in (a_j, a_{j+1})$ for some $j \in \mathbb{Z}$ and $t_2 - t_1 < \delta$, then $|q(t_2) - q(t_1)| < \varepsilon$.

The "P" in the notation makes reference to the piecewise continuity of q: it is clear that, if $q \in BPUC_\Delta(\mathbb{R}, \mathbb{R})$, then the lateral limits $q(a_j^+) := \lim_{t\to a_j^+} q(t)$ and $q(a_j^-) := \lim_{t\to a_j^-} q(t)$ exist for all $j \in \mathbb{Z}$, although possibly $q(a_j^+) \neq q(a_j^-)$. We will assume that any function of $BPUC_\Delta(\mathbb{R}, \mathbb{R})$ is defined and right-continuous on the whole real line. This assumption causes no difference in our results, but slightly simplifies the description of some of their proofs.

Definition 2.7 A *bounded* function $q\colon \mathbb{R} \to \mathbb{R}$ is *piecewise uniformly continuous* (*BPUC* for short) if there exists a finite number of disperse sets $\Delta_1, \dots, \Delta_n$ and functions $q_i \in BPUC_{\Delta_i}(\mathbb{R}, \mathbb{R})$ for $i = 1, \dots, n$ such that $q = q_1 + \cdots + q_n$.

Note that a finite union of disperse sets may be non disperse, which justifies this last definition. Note also that the vector space $BPUC(\mathbb{R}, \mathbb{R})$ of BPUC functions is a subset of $L^\infty(\mathbb{R}, \mathbb{R})$, and that the L^∞-norm of a BPUC function coincides with $\|q\| := \sup_{t\in\mathbb{R}} |q(t)|$. Clearly, any bounded and uniformly continuous function is BPUC.

Many of our results referring (2.1) consider BPUC coefficients q and p. Let us explain the reason for this restriction. Theorems 2.9 and 2.11 provide fundamental insight in the dynamics of (2.1): they extend several properties proved in [29] for bounded and uniformly continuous functions q, p to the BPUC case. As in that paper, the construction of the hull Ω_r in $L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^2)$ for $r := (q, p)$ (i.e., the closure in $L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^2)$ of the set of shifts $r_t(s) := r(s+t)$), as well as of continuous flows on Ω_r and on $\Omega_r \times \mathbb{R}$, are crucial tools: these constructions, standard for nonautonomous differential equations, allow us to use techniques from topological dynamics. The definitions of hull and flows, and the proofs of their properties, are more technical in the present setting of BPUC coefficients than in that of [29]. The point is that taking $r: \mathbb{R} \to \mathbb{R}^d$ with any number d of BPUC component functions guarantees the compactness of Ω_r and the continuity of the flows. In order to avoid drawing focus away from the objective of this work, we prefer to postpone a more detailed description of these quite technical concepts and results, as well as their proofs, to Appendix A. We point out here that, if $r: \mathbb{R} \to \mathbb{R}^d$ is almost-periodic, the topology used to define Ω_r on $L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^d)$ coincides with that of the uniform convergence on $\mathbb{R}$: see, e.g., [16, Chapter 1].

Theorem 2.9 shows that, if q, p are BPUC functions, then the solutions a and r associated to (2.1) by Theorem 2.5 are globally defined and uniformly separated if and only if they are hyperbolic. Its proof is given in Appendix A.

Definition 2.8 Two globally defined solutions $x_1(t)$ and $x_2(t)$ of (2.1) with $x_1 \leq x_2$ are *uniformly separated* if $\inf_{t\in\mathbb{R}}(x_2(t) - x_1(t)) > 0$.

Theorem 2.9 *Let $q, p: \mathbb{R} \to \mathbb{R}$ be BPUC functions, assume that the Eq.* (2.1) *has bounded solutions, and let a and r be the (globally defined) functions provided by Theorem* 2.5. *Then, the following assertions are equivalent:*

(a) *The solutions a and r are uniformly separated.*
(b) *The solutions a and r are hyperbolic, with a attractive and r repulsive.*
(c) *The Eq.* (2.1) *has two different hyperbolic solutions.*

In this case,

(i) *let (k_a, β_a) and (k_r, β_r) be dichotomy constant pairs for the hyperbolic solutions a and r, respectively, and let us choose any $\bar\beta_a \in (0, \beta_a)$ and any $\bar\beta_r \in (0, \beta_r)$. Then, given $\varepsilon > 0$, there exist $k_{a,\varepsilon} \geq 1$ and $k_{r,\varepsilon} \geq 1$ (depending also on the choice of $\bar\beta_a$ and of $\bar\beta_r$, respectively) such that*

$$|a(t) - x(t, s, x_0)| \leq k_{a,\varepsilon}\, e^{-\bar\beta_a(t-s)}|a(s) - x_0| \quad \textit{if} \quad x_0 \geq r(s) + \varepsilon \quad \textit{and} \quad t \geq s\,,$$
$$|r(t) - x(t, s, x_0)| \leq k_{r,\varepsilon}\, e^{\bar\beta_r(t-s)}|r(s) - x_0| \quad \textit{if} \quad x_0 \leq a(s) - \varepsilon \quad \textit{and} \quad t \leq s\,.$$

In addition,

$$|a(t) - x(t, s, x_0)| \leq k_a\, e^{-\beta_a(t-s)}|a(s) - x_0| \quad \textit{if} \quad x_0 \geq a(s) \quad \textit{and} \quad t \geq s\,,$$
$$|r(t) - x(t, s, x_0)| \leq k_r\, e^{\beta_r(t-s)}|r(s) - x_0| \quad \textit{if} \quad x_0 \leq r(s) \quad \textit{and} \quad t \leq s\,.$$

(ii) *The Eq.* (2.1) *does not have more hyperbolic solutions, and a and r are the only bounded solutions of* (2.1) *which are uniformly separated.*

Definition 2.10 In the situation described by Theorem 2.9, (a, r) is a (*classical*) *attractor–repeller pair* for (2.1).

Note that the global dynamics in the case of existence of an attractor–repeller pair is described by Theorems 2.5 and 2.9.

We include in this subsection the definitions of local pullback attractors and repellers, which are to some extent related to the classical ones, and which play a fundamental role in the dynamical description of the next sections: see e.g. Remark 3.5. These definitions adapt those given in Section 3.8 of [25] to the case of a (possibly) locally defined solution. A solution $\bar{a}\colon(-\infty,\beta)\to\mathbb{R}$ (with $\beta\le\infty$) of (2.1) is *locally pullback attractive* if there exist $s_0<\beta$ and $\delta>0$ such that, if $s\le s_0$ and $|x_0-\bar{a}(s)|<\delta$, then $x(t,s,x_0)$ is defined on $[s,s_0]$ and, in addition,

$$\lim_{s\to-\infty}\max_{x_0\in[\bar{a}(s)-\delta,\bar{a}(s)+\delta]}|\bar{a}(t)-x(t,s,x_0)|=0\quad\text{for all}\quad t\le s_0\,.$$

Note that, in our scalar case, this is equivalent to say that, if $s\le s_0$, then the solutions $x(t,s,a(s)\pm\delta)$ are defined on $[s,s_0]$ and, in addition,

$$\lim_{s\to-\infty}|\bar{a}(t)-x(t,s,\bar{a}(s)\pm\delta)|=0\quad\text{for all}\quad t\le s_0\,.$$

A solution $\bar{r}\colon(\alpha,\infty)\to\mathbb{R}$ (with $\alpha\ge-\infty$) of (2.1) is *locally pullback repulsive* if the solution $\bar{r}^*\colon(-\infty,-\alpha)\to\mathbb{R}$ of $y'=-h(-t,y)$ given by $\bar{r}^*(t)=\bar{r}(-t)$ is locally pullback attractive. In other words, it there exist $s_0>\alpha$ and $\delta>0$ such that, if $s\ge s_0$, then the solutions $x(t,s,\bar{r}(s)\pm\delta)$ are defined on $[s_0,s]$ and, in addition,

$$\lim_{s\to\infty}|\bar{r}(t)-x(t,s,\bar{r}(s)\pm\delta)|=0\quad\text{for all}\quad t\ge s_0\,.$$

2.4 One-Parametric Variation of the Global Dynamics

Let us now consider the parametric family of equations

$$x'=-x^2+q(t)\,x+p(t)+\lambda\,,\tag{2.7}$$

where q and p are BPUC functions and λ varies in $\mathbb{R}$. Let $\mathcal{B}_\lambda$ be the (possibly empty) set of bounded solutions, and a_λ and r_λ the corresponding bounded solutions provided by Theorem 2.5 when $\mathcal{B}_\lambda$ is nonempty. The next result, proved in Appendix A, shows the existence of a bifurcation value λ^*: for smaller values of the parameter, there are no bounded solutions, while for greater ones two hyperbolic solutions exist. We will talk hence about nonautonomous saddle-node bifurcation.

Theorem 2.11 *There exists a unique* $\lambda^*=\lambda^*(q,p)\in[-\|q^2/4+p\|,\|p\|\,]$ *such that*

(i) $\mathcal{B}_\lambda$ *is empty if and only if* $\lambda<\lambda^*$.
(ii) *If* $\lambda^*\le\lambda_1<\lambda_2$, *then* $\mathcal{B}_{\lambda_1}\subsetneq\mathcal{B}_{\lambda_2}$. *More precisely,*

$$r_{\lambda_2}<r_{\lambda_1}\le a_{\lambda_1}<a_{\lambda_2}\,.$$

In addition, $\lim_{\lambda\to\infty}a_\lambda(t)=\infty$ *and* $\lim_{\lambda\to\infty}r_\lambda(t)=-\infty$ *uniformly on* $\mathbb{R}$.
(iii) $\inf_{t\in\mathbb{R}}(a_{\lambda^*}(t)-r_{\lambda^*}(t))=0$, *and* $(2.7)_{\lambda^*}$ *has no hyperbolic solution.*
(iv) *If* $\lambda>\lambda^*$, *then* a_λ *and* r_λ *are uniformly separated and the unique hyperbolic solutions of* $(2.7)_\lambda$.
(v) $\lambda^*(q,p+\lambda)=\lambda^*(q,p)-\lambda$ *for any* $\lambda\in\mathbb{R}$.

Theorem 2.12 *Let $q, \bar{q}, p, \bar{p} \colon \mathbb{R} \to \mathbb{R}$ be BPUC functions which are norm-bounded by a constant κ, and let $\lambda^*(q, p)$ and $\lambda^*(\bar{q}, \bar{p})$ be the constants provided by Theorem 2.11. Then, there exists a constant m_κ such that*

$$|\lambda^*(\bar{q}, \bar{p}) - \lambda^*(q, p)| \le m_\kappa \big(\|\bar{q} - q\| + \|\bar{p} - p\| \big) \,.$$

In particular, the map $\lambda^* \colon \mathrm{BPUC} \times \mathrm{BPUC} \to \mathbb{R}$ is continuous for the L^∞-topology.

Proof Theorem 2.11 ensures that $\lambda^*(q, p)$ is bounded by $\kappa + \kappa^2/4$. Let $m_\kappa \ge 1$ satisfy $-m_\kappa^2 + \kappa\, m_\kappa + \kappa + \kappa^2/4 < 0$. Then, $\|b\| \le m_\kappa$ for any bounded solution b of $x' = -x^2 + q(t)\, x + p(t) + \lambda^*(q, p)$: see Theorem 2.5. Consequently, at almost all $t \in \mathbb{R}$, this bounded solution b satisfies

$$\begin{aligned} b'(t) &= -b^2(t) + \bar{q}(t)\, b(t) + \bar{p}(t) + (q(t) - \bar{q}(t))\, b(t) + (p(t) - \bar{p}(t)) + \lambda^*(q, p) \\ &\le -b^2(t) + \bar{q}(t)\, b(t) + \bar{p}(t) + m_\kappa \big(\|\bar{q} - q\| + \|\bar{p} - p\| \big) + \lambda^*(q, p) \,. \end{aligned}$$

Theorem 2.5(v) (see also Remark 2.4) ensures that $x' = -x^2 + \bar{q}(t)\, x + \bar{p}(t) + m_\kappa \big(\|\bar{q} - q\| + \|\bar{p} - p\| \big) + \lambda^*(q, p)$ has a bounded solution, and hence Theorem 2.11(i) ensures that $\lambda^*(\bar{q}, \bar{p}) \le m_\kappa \big(\|\bar{q} - q\| + \|\bar{p} - p\| \big) + \lambda^*(q, p)$. The same argument shows that $\lambda^*(q, p) \le m_\kappa \big(\|\bar{q} - q\| + \|\bar{p} - p\| \big) + \lambda^*(\bar{q}, \bar{p})$, and both inequalities prove the first assertion. The second one is clear. $\square$

Remarks 2.13 1. Theorem 2.11 shows that the variation in λ of the family (2.7) determines a nonautonomous bifurcation pattern of saddle-node type: the absence of bounded solutions for $\lambda < \lambda^*(q, p)$ gives rise to the existence of an attractor–repeller pair for $\lambda > \lambda^*(q, p)$. See, e.g., [4, 17, 30]. Note also that the equation corresponding to the bifurcation value $\lambda^*(q, p)$ has either a unique bounded solution or infinitely many ones, none of them hyperbolic. The first situation is simpler and more common, but there are well-known examples of the second case: we refer the interested reader to [30] for the details. Corollary 3.8(ii) provides a simple way to get examples of this nontrivial bifurcation pattern.

2. The function $\lambda^* \colon \mathrm{BPUC} \times \mathrm{BPUC} \to \mathbb{R}$, which according to Theorem 2.12 is continuous for the L^∞-topology, is not continuous when the topology of convergence in L^∞-norm on compact sets (or a weaker one, as L^1_{loc}) is considered, as the next simple example shows. We fix $\Gamma \equiv 0$, define p_n as the continuous piecewise linear map taking the values -1 on $(-\infty, -n-1] \cup [n+1, \infty)$ and 1 on $[-n, n]$, and observe that the sequence (p_n) converges to $p \equiv 1$ for the compact-open topology of $C(\mathbb{R}, \mathbb{R}) \subset BPUC(\mathbb{R}, \mathbb{R})$. However, since the equation $x' = -x^2 + p_n(t)$ does not have bounded solutions, $\lambda^*(0, p_n) > 0$, while $\lambda^*(0, p) = -1$.

3 A Particular Case of Concave Quadratic Equations

Let us fix BPUC functions (see Definition 2.7) $\Gamma, p \colon \mathbb{R} \to \mathbb{R}$ such that the asymptotic limits of Γ, $\gamma_\pm := \lim_{t \to \pm\infty} \Gamma(t)$, exist and are finite. These conditions will be in force in this initial part of Sect. 3, whereas in some of the subsections we will impose more or less restrictive conditions on Γ and p which we will describe in due time. Observe that $-2\,\Gamma$ and $p - \Gamma^2$ are also BPUC functions. In what follows, we will analyze some general facts concerning the dynamical possibilities for

$$y' = -\big(y - \Gamma(t)\big)^2 + p(t) \,, \tag{3.1}$$

whose solution with value y_0 at $t = s$ is represented by $y(t, s, y_0)$. We understand Γ as a *transition* from γ_- (in the past) to γ_+ (in the future). In this way,

$$y' = -(y - \gamma_+)^2 + p(t)\,, \tag{3.2}$$

and

$$y' = -(y - \gamma_-)^2 + p(t) \tag{3.3}$$

play the role of “limit" equations for (3.1) as $t \to \infty$ and as $t \to -\infty$, respectively. We will refer to them also as *future equation* and *past equation*. Note also that the global dynamics of these two equations is “identical" to that of

$$x' = -x^2 + p(t) \tag{3.4}$$

since they are obtained from this one by the trivial changes of variables.

Definition 3.1 – The Eq. (3.1) is in CASE A if it has two different hyperbolic solutions.
– The Eq. (3.1) is in CASE B if it has at least one bounded solution but no hyperbolic ones.
– The Eq. (3.1) is in CASE C if it has no bounded solutions.

It follows from Theorem 2.11 that these three cases exhaust the possibilities. Theorem 2.9 proves that CASE A is equivalent to the existence of an attractor–repeller pair, which determines the global dynamics of (3.1). We will see below that much more can be said in any of the three situations if the next condition (assumed when indicated) holds:

Hypothesis 3.2 The Eq. (3.4) has an attractor–repeller pair $(\widetilde{a}, \widetilde{r})$.

Remark 3.3 Hypothesis 3.2 is equivalent to any of these assertions: $(\widetilde{a} + \gamma_+, \widetilde{r} + \gamma_+)$ is an attractor–repeller pair for (3.2); $(\widetilde{a} + \gamma_-, \widetilde{r} + \gamma_-)$ is an attractor–repeller pair for (3.3); $(\widetilde{a} + \Gamma(0), \widetilde{r} + \Gamma(0))$ is an attractor–repeller pair for $y' = -(y - \Gamma(0))^2 + p(t)$.

The next result and Remark 3.5 below are fundamental to understand the dynamics of (3.1) in CASES A, B and C under Hypothesis 3.2.

Theorem 3.4 *Assume Hypothesis 3.2, and let* $(\widetilde{\mathfrak{a}}_\pm, \widetilde{\mathfrak{r}}_\pm) := (\widetilde{a} + \gamma_\pm, \widetilde{r} + \gamma_\pm)$ *be the attractor–repeller pairs for the future and past equations (3.2) and (3.3). Then,*

(i) *there exist the functions* $\mathfrak{a}$ *and* $\mathfrak{r}$ *associated to (3.1) by Theorem 2.5.*
(ii) $\lim_{t\to-\infty} |\mathfrak{a}(t) - \widetilde{\mathfrak{a}}_-(t)| = 0$, $\lim_{t\to-\infty} |y(t, s, y_0) - \widetilde{\mathfrak{r}}_-(t)| = 0$ *whenever* $\mathfrak{a}(s)$ *exists and* $y_0 < \mathfrak{a}(s)$, $\lim_{t\to+\infty} |\mathfrak{r}(t) - \widetilde{\mathfrak{r}}_+(t)| = 0$, *and* $\lim_{t\to+\infty} |y(t, s, y_0) - \widetilde{\mathfrak{a}}_+(t)| = 0$ *whenever* $\mathfrak{r}(s)$ *exists and* $y_0 > \mathfrak{r}(s)$.
(iii) *The solutions* $\mathfrak{a}$ *and* $\mathfrak{r}$ *are respectively locally pullback attractive and locally pullback repulsive.*
(iv) *If* $\mathfrak{a}$ *and* $\mathfrak{r}$ *are globally defined and different, then they are uniformly separated, and hence* $(\widetilde{\mathfrak{a}}, \widetilde{\mathfrak{r}}) := (\mathfrak{a}, \mathfrak{r})$ *is an attractor–repeller pair for (3.1).*
(v) *If the Eq. (3.1) does not have hyperbolic solutions, then it has at most one bounded solution* $\mathfrak{a} = \mathfrak{r}$.

Proof (i) Proposition 2.1 applied to the attractor–repeller pair $(\widetilde{\mathfrak{a}}_-, \widetilde{\mathfrak{r}}_-)$ of (3.3) states that, given $\varepsilon > 0$, there exists $\delta_- = \delta_-(\varepsilon) > 0$ such that if $\left\| \Sigma - \gamma^- \right\| \le \delta_-$ then the equation $y' = -(y - \Sigma(t))^2 + p(t)$ also has an attractor–repeller pair $(\widetilde{\mathfrak{a}}_\Sigma, \widetilde{\mathfrak{r}}_\Sigma)$ with $\|\widetilde{\mathfrak{a}}_\Sigma - \widetilde{\mathfrak{a}}_-\| \le \varepsilon$ and $\|\widetilde{\mathfrak{r}}_\Sigma - \widetilde{\mathfrak{r}}_-\| \le \varepsilon$. It also states that there exists a common dichotomy pair $(k_\varepsilon, \beta_\varepsilon)$ for all these functions Σ which can be assumed to be valid for both hyperbolic solutions.

We choose $t^- = t^-(\varepsilon) < 0$ such that $|\Gamma(t) - \gamma^-| \le \delta_-$ if $t \le t^-$, and define $\Sigma^-(t)$ as $\Gamma(t)$ on $(-\infty, t^-)$ and as $\Gamma(t^-)$ on $[t^-, \infty)$. Then, $\left\|\Sigma^- - \gamma^-\right\| \le \delta$, and hence $y' = -(y - \Sigma^-(t))^2 + p(t)$ has an attractor–repeller pair $(\widetilde{\mathfrak{a}}_{\Sigma^-}, \widetilde{\mathfrak{r}}_{\Sigma^-})$, with $\left\|\widetilde{\mathfrak{a}}_{\Sigma^-} - \widetilde{\mathfrak{a}}_-\right\| \le \varepsilon$ and $\left\|\widetilde{\mathfrak{r}}_{\Sigma^-} - \widetilde{\mathfrak{r}}_-\right\| \le \varepsilon$. In particular,

$$\exp \int_s^t (-2\,\widetilde{\mathfrak{a}}_{\Sigma^-}(l) + 2\,\Sigma^-(l))\,dl \le k_\varepsilon\, e^{-\beta_\varepsilon (t-s)} \quad \text{whenever} \quad t \ge s\,. \tag{3.5}$$

Let us now define $\widehat{\mathfrak{a}}_{\Sigma^-}$ as the solution of (3.1) with value $\widehat{\mathfrak{a}}_{\Sigma^-}(t^-) = \widetilde{\mathfrak{a}}_{\Sigma^-}(t^-)$. Our goal is to check that $\widehat{\mathfrak{a}}_{\Sigma^-}$ coincides with the function $\mathfrak{a}$ of the statement. Since $\widehat{\mathfrak{a}}_{\Sigma^-}(t) = \widetilde{\mathfrak{a}}_{\Sigma^-}(t)$ for $t \le t^-$, it remains bounded as t decreases, which proves that $\mathfrak{a}$ exists and that $\widehat{\mathfrak{a}}_{\Sigma^-} \le \mathfrak{a}$. To prove the converse inequality, we take $y_0 > \widehat{\mathfrak{a}}_{\Sigma^-}(t^-)$ and check that $y(t, t^-, y_0)$ is unbounded as t decreases. This property follows from

$$\frac{1}{k_\varepsilon}\, e^{\beta_\varepsilon (t^- - t)} \le \exp \int_{t^-}^t (-2\,\widehat{\mathfrak{a}}_{\Sigma^-}(l) + 2\,\Sigma^-(l))\,dl \le \frac{y(t, t^-, y_0) - \widehat{\mathfrak{a}}_{\Sigma^-}(t)}{y_0 - \widehat{\mathfrak{a}}_{\Sigma^-}(t^-)}$$

if $t \le t^-$: the first inequality comes from (3.5), and the second one can be obtained, for instance, as (3.15) in [29].

To complete the proof of (i), we work with $(\widetilde{\mathfrak{a}}_+, \widetilde{\mathfrak{r}}_+)$ and use an analogous argument in order to obtain t^+ such that $\mathfrak{r}$ is defined al least on $[t^+, \infty)$.

(ii) We keep the notation established in the proof of (i). There, we have checked that, given $\varepsilon > 0$, there exists t^- such that $|\mathfrak{a}(t) - \widetilde{\mathfrak{a}}_-(t)| = |\widetilde{\mathfrak{a}}_{\Sigma^-}(t) - \widetilde{\mathfrak{a}}_-(t)| \le \varepsilon$ if $t \le t^-$, which proves the first assertion for $\mathfrak{a}$ in this case. On the other hand, if $y_0 < \mathfrak{a}(s)$, then there exists $t_0 < t^-$ such that $y(t_0, s, y_0) < \mathfrak{a}(t_0) = \widetilde{\mathfrak{a}}_{\Sigma^-}(t_0)$. Since $y(t, s, y_0) = y(t, t_0, y(t_0, s, y_0))$ solves $y' = -(y - \Sigma^-(t))^2 + p(t)$ for $t \le t_0$, we conclude from Theorem 2.9(i) that $\lim_{t\to-\infty} |y(t, s, y_0) - \widetilde{\mathfrak{r}}_-(t)| = 0$. The proofs of the two remaining assertions are similar.

(iii) Let us take $\varepsilon \in \big(0, \inf_{t\in\mathbb{R}}(\widetilde{a}(t) - \widetilde{r}(t))\big)$. We have obtained in (i) the time t^- and the functions $\widetilde{\mathfrak{a}}_{\Sigma^-}$ and $\widetilde{\mathfrak{r}}_{\Sigma^-}$ satisfying $\inf_{s\in(-\infty,t^-]}(\mathfrak{a}(s) - \widetilde{\mathfrak{r}}_{\Sigma^-}(s)) = \inf_{s\in(-\infty,t^-]}(\widetilde{\mathfrak{a}}_{\Sigma^-}(s) - \widetilde{\mathfrak{r}}_{\Sigma^-}(s)) > \varepsilon$. Hence, Theorem 2.5(ii) applied to $y' = -(y - \Sigma^-(t))^2 + p(t)$ ensures that its solutions $y^-(t, s, \mathfrak{a}(s) \pm \varepsilon)$ are defined for any $t \ge s$ if $s \le t^-$. Now we fix $t \le t^-$ and take $s \le t$. If $l \in [s, t]$, then $\mathfrak{a}(l) = \widetilde{\mathfrak{a}}_{\Sigma^-}(l)$ and $y(l, s, \mathfrak{a}(s) \pm \varepsilon)$ coincide with the solutions $y^-(l, s, \widetilde{\mathfrak{a}}_{\Sigma^-}(s) \pm \varepsilon)$ of $y' = -(y - \Sigma^-(t))^2 + p(t)$. Therefore, Theorem 2.9(i) applied to this last equation and ε provides, for any $\beta_0 \in (0, \beta_\varepsilon)$, a constant $k_0 = k_0(\beta_0, \varepsilon) \ge 1$ (independent of s) with

$$|\mathfrak{a}(t) - y(t, s, \mathfrak{a}(s) \pm \varepsilon)| = |\widetilde{\mathfrak{a}}_{\Sigma^-}(t) - y^-(t, s, \widetilde{\mathfrak{a}}_{\Sigma^-}(s) \pm \varepsilon)| \le k_0\, e^{-\beta_0 (t-s)}\varepsilon\,, \tag{3.6}$$

which is as small as desired if $-s$ is large enough. This proves (iii) in the case of $\mathfrak{a}$. The proof for $\mathfrak{r}$ is analogous.

(iv) Assume the global existence of $\mathfrak{a}$ and $\mathfrak{r}$, with $\mathfrak{r} < \mathfrak{a}$. According to (ii), $\lim_{t\to-\infty} |\mathfrak{a}(t) - \widetilde{\mathfrak{a}}_-(t)| = 0$ and $\lim_{t\to-\infty} |\mathfrak{r}(t) - \widetilde{\mathfrak{r}}_-(t)| = 0$, so that their distance is bounded from below on $(-\infty, 0]$. Point (ii) also ensures $\lim_{t\to+\infty} |\mathfrak{a}(t) - \widetilde{\mathfrak{a}}_+(t)| = 0$ and $\lim_{t\to+\infty} |\mathfrak{r}(t) - \widetilde{\mathfrak{r}}_+(t)| = 0$. Therefore, $\mathfrak{a}$ and $\mathfrak{r}$ are uniformly separated. Theorem 2.9 proves that they form an attractor–repeller pair.

(v) It follows from (iv) that the unique possibility for the existence of bounded solutions but not of hyperbolic ones is that $\mathfrak{a} = \mathfrak{r}$, which proves (v). □

Remark 3.5 Assume the conditions on Γ and p described at the beginning of the section, and Hypothesis 3.2, and let $(\widetilde{\mathfrak{a}}_\pm, \widetilde{\mathfrak{r}}_\pm) := (\widetilde{a} + \gamma_\pm, \widetilde{r} + \gamma_\pm)$ be the attractor–repeller pairs for the future and past equations (3.2) and (3.3). Under the assumed conditions on Γ and p, Theorem

3.4, combined with Theorems 2.11 and 2.9, proves the next statements (among many other properties).

- Case A holds for (3.1) if and only if the equation has an attractor–repeller pair $(\widetilde{\mathfrak{a}}, \widetilde{\mathfrak{r}})$ (see Definition 2.10); or, equivalently, if it has two different bounded solutions. In this case, this attractor–repeller pair connects $(\widetilde{\mathfrak{a}}_-, \widetilde{\mathfrak{r}}_-)$ to $(\widetilde{\mathfrak{a}}_+, \widetilde{\mathfrak{r}}_+)$: $\lim_{t\to\pm\infty} |\widetilde{\mathfrak{a}}(t) - \widetilde{\mathfrak{a}}_\pm(t)| = 0$ and $\lim_{t\to\pm\infty} |\widetilde{\mathfrak{r}}(t) - \widetilde{\mathfrak{r}}_\pm(t)| = 0$. This situation is often referred to as *end-point tracking*. In addition, $\widetilde{\mathfrak{a}}(t)$ is the unique solution approaching $\widetilde{\mathfrak{a}}_-$ as time decreases, and $\widetilde{\mathfrak{r}}(t)$ is the unique solution approaching $\widetilde{\mathfrak{r}}_+$ as time increases.

- Case B holds for (3.1) if and only if the equation has a unique bounded solution $\mathfrak{b}$. In this case, this solution is locally pullback attractive and repulsive (see Sect. 2.3), and it connects $\widetilde{\mathfrak{a}}_-$ to $\widetilde{\mathfrak{r}}_+$: $\lim_{t\to-\infty} |\mathfrak{b}(t) - \widetilde{\mathfrak{a}}_-(t)| = 0$ and $\lim_{t\to+\infty} |\mathfrak{b}(t) - \widetilde{\mathfrak{r}}_+(t)| = 0$. In addition, no other solution of (3.1) satisfies any of these two properties.

- Case C holds if and only if the equation has no bounded solutions. In this case, there exists a locally pullback attractive solution $\mathfrak{a}$ which is the unique solution bounded at $-\infty$ approaching $\widetilde{\mathfrak{a}}_-$ as time decreases (i.e., with $\lim_{t\to-\infty} |\mathfrak{a}(t) - \widetilde{\mathfrak{a}}_-(t)| = 0$); and it exists a locally pullback repulsive solution $\mathfrak{r}$ which is the unique solution bounded at $+\infty$ approaching $\widetilde{\mathfrak{r}}_+$ as time increases (i.e., with $\lim_{t\to+\infty} |\mathfrak{r}(t) - \widetilde{\mathfrak{r}}_+(t)| = 0$). This situation of loss of connection is sometimes referred to as *tipping*.

The interested reader can in find [29, Figures 1-6] some drawings showing the dynamical behavior in each one of these three cases. (There is a last-version typo there: the graphs of cases A and C are interchanged).

We also point out that, in the three dynamical cases, the constants β_0 and k_0 appearing in (3.6) can be chosen to get

$$|\mathfrak{a}(t) - y(t,s,y_0)| \le k_0\, e^{-\beta_0(t-s)} |\mathfrak{a}(s) - y_0| \quad \text{for} \quad y_0 \ge \widetilde{\mathfrak{r}}_-(s) + \varepsilon \quad \text{and if} \quad s \le t \le t^- .$$

That is, $\mathfrak{a}(t)$ forwardly attracts exponentially fast all the solutions $y(t,s,y_0)$ starting above $\widetilde{\mathfrak{r}}_-(s) + \varepsilon$ for $s < t^-$ while $t \le t^-$. Similar bounds can be found for $\mathfrak{r}$.

3.1 Some Fundamental Inequalities for $\lambda^*(2\,\Gamma,\ p - \Gamma^2)$

Recall that Theorem 2.11 associates the value $\lambda^*(2\,\Gamma,\ p - \Gamma^2)$ to (3.1): $\lambda^*(2\,\Gamma,\ p - \Gamma^2)$ is the bifurcation point in λ of $x' = -(x - \Gamma(t))^2 + p(t) + \lambda$. We will establish some interesting facts concerning this value under different assumptions on Γ and p which will be clarified in the statement of each result. Hypothesis 3.2 is not in force in this subsection.

Our first "comparison" result relates $\lambda^*(0, q)$ to $\lambda^*(2\,\Gamma,\ p - \Gamma^2)$ for certain functions q. Recall that the construction of the hull Ω_p of a BPUC function p, referred to in Sect. 2.3, is detailed in Appendix A. The function p is *recurrent* when every orbit of the flow on its hull is dense. It is well-known that every almost periodic function is recurrent. In addition, the hull of any BPUC function contains recurrent functions. We say that a function $q \in \Omega_p$ *belongs to the alpha limit* (resp. to the omega limit) of p if there exists a sequence $(t_n)_{n\ge1}$ with limit $-\infty$ (resp. $+\infty$) such that $q = \lim_{n\to\infty} p_{t_n}$ on Ω_p (where $p_t(s) := p(s+t)$ for $t, s \in \mathbb{R}$).

Proposition 3.6 *Let $\Gamma, p\colon \mathbb{R} \to \mathbb{R}$ be BPUC functions and let Γ have finite asymptotic limits. Assume that $q\colon \mathbb{R} \to \mathbb{R}$ belongs to the alpha limit or to the omega limit of p. Then, $\lambda^*(0, q) \le \lambda^*(2\,\Gamma,\ p - \Gamma^2)$. In particular, if p is recurrent, then $\lambda^*(0, p) \le \lambda^*(2\,\Gamma,\ p - \Gamma^2)$.*

Proof We fix q and Γ as in the statement, and denote $\lambda^* := \lambda^*(2\,\Gamma, p - \Gamma^2)$. Theorem 2.11 ensures the existence of a globally bounded solution $\mathfrak{b}$ of $y' = -(y - \Gamma(t))^2 + p(t) + \lambda^*$.

Our goal is to check the existence of a bounded solution of $x' = -x^2 + q(t) + \lambda^*$: this and Theorem 2.11 prove that $\lambda^*(0, q) \le \lambda^*$.

Let us work in the case of existence of $(t_n) \uparrow \infty$ such that $q = \lim_{n\to\infty} p_{t_n}$ in Ω_p. Then $\mathfrak{b}_{t_n}(t) := \mathfrak{b}(t + t_n)$ solves $y' = -(y - \Gamma_{t_n}(t))^2 + p_{t_n}(t) + \lambda^*$, where $\Gamma_{t_n}(t) := \Gamma(t + t_n)$. We can assume without restriction the existence of $\lim_{n\to\infty} \mathfrak{b}_{t_n}(0) =: \mathfrak{b}_0$. Clearly, $\lim_{n\to\infty}(-2\,\Gamma_{t_n}, p_{t_n} - \Gamma_{t_n}^2 + \lambda^*) = (-2\gamma_+, q - \gamma_+^2 + \lambda^*)$ in the common hull $\Omega_{-2\Gamma,\, p-\Gamma^2+\lambda^*}$. Therefore, Theorem A.2 guarantees that the sequence of functions $(\mathfrak{b}_{t_n})_{n\ge 1}$ converges uniformly on compact sets as $n \to \infty$ to the solution $\mathfrak{b}_{\gamma_+}$ of $y' = -(y - \gamma_+)^2 + q(t) + \lambda^*$ with $\mathfrak{b}_{\gamma_+}(0) = \mathfrak{b}_0$. In particular, $\mathfrak{b}_{\gamma_+}$ is defined on the whole $\mathbb{R}$ and bounded. Hence, $b := \mathfrak{b}_{\gamma_+} - \gamma_+$ is a bounded solution of $x' = -x^2 + q(t) + \lambda^*$, and the assertion is proved.

The proof is analogous if $q = \lim_{n\to\infty} p_{t_n}$ in Ω_p for $(t_n) \downarrow -\infty$, working now with γ_- instead of γ_+. The last assertion is a trivial consequence of the first one. □

The next result compares the values of $\lambda^*(2\,\Gamma, p - \Gamma^2)$ for two different functions Γ under some conditions including the nondecreasing character of their difference.

Theorem 3.7 *Let $\Gamma_1, \Gamma_2, p\colon \mathbb{R} \to \mathbb{R}$ be BPUC functions with $\Gamma_2 - \Gamma_1$ nondecreasing, and let $\lambda_i := \lambda^*(2\,\Gamma_i,\ p - (\Gamma_i)^2)$ be the values provided by Theorem 2.11.*

(i) *If $\Gamma_2 - \Gamma_1$ is continuous, then $\lambda_1 \le \lambda_2$. If, in addition, $\Gamma_2 - \Gamma_1$ is absolutely continuous and nonconstant on a nondegenerate interval, and $\lambda_1 = \lambda_2$, then $y' = -(y - \Gamma_1(t))^2 + p(t) + \lambda_1$ has infinitely many bounded solutions (but no hyperbolic ones), and the same happens for all the equations $y' = -(y - \Gamma_\mu(t))^2 + p(t) + \lambda_\mu$ for $\mu \in (0, 1)$, where $\Gamma_\mu := \mu\,\Gamma_1 + (1 - \mu)\,\Gamma_2$ and $\lambda_\mu := \lambda^*(2\,\Gamma_\mu,\ p - (\Gamma_\mu)^2)$.*

(ii) *Assume that Γ_1 and Γ_2 have finite asymptotic limits. Then, $\lambda_1 \le \lambda_2$.*

Proof (i) As recalled in Remark 2.4, the continuous nondecreasing function $\Gamma_2 - \Gamma_1$ is of bounded variation, and hence there exists $(\Gamma_2 - \Gamma_1)'(t) \ge 0$ for Lebesgue-a.a. $t \in \mathbb{R}$. Let $\mathfrak{b}_2$ be a bounded solution of $y' = -(y - \Gamma_2(t))^2 + p(t) + \lambda_2$. Then, the bounded continuous function $b_2 := \mathfrak{b}_2 - (\Gamma_2 - \Gamma_1)$, which is of bounded variation and has nonincreasing singular part on every compact interval of $\mathbb{R}$ (see Remark 2.4), satisfies $b_2'(t) = -(b_2(t) - \Gamma_1(t))^2 + p(t) + \lambda_2 - (\Gamma_2 - \Gamma_1)'(t) \le -(b_2(t) - \Gamma_1(t))^2 + p(t) + \lambda_2$ for almost all $t \in \mathbb{R}$. Theorem 2.5(v) guarantees the existence of at least one bounded solution of $x' = -(x - \Gamma_1(t))^2 + p(t) + \lambda_2$. Therefore, Theorem 2.11 ensures that $\lambda_1 \le \lambda_2$, which is the first assertion in (i).

If, in addition, $\Gamma_2 - \Gamma_1$ is absolutely continuous and nonnonconstant on an interval $[s, t]$, with $s < t$, it follows from $(\Gamma_2 - \Gamma_1)(t) - (\Gamma_2 - \Gamma_1)(s) = \int_s^t (\Gamma_2 - \Gamma_1)'(l)\, dl$ (see Remark 2.4) that there exists $t_0 \in \mathbb{R}$ such that $(\Gamma_2 - \Gamma_1)'(t_0) > 0$. Therefore, Theorem 2.5(v) ensures that $x' = -(x - \Gamma_1(t))^2 + p(t) + \lambda_2$ has more than one bounded solution. The fact that $\lambda_1 = \lambda_2$ implies infinitely many bounded nonhyperbolic solutions for $x' = -(x - \Gamma_1(t))^2 + p(t) + \lambda_1$ follows hence from Theorem 2.11, as explained in Remark 2.13. Finally, if we define Γ_μ and λ_μ as in the statement, the initial assertion of (i) shows that $\lambda_1 \le \lambda_\mu \le \lambda_2$ for any $\mu \in [0, 1]$. If $\mu \in (0, 1]$ and $\Gamma_2 - \Gamma_1$ is nonconstant, so is $\Gamma_2 - \Gamma_\mu$, which is also absolutely continuous on compact intervals of $\mathbb{R}$. Therefore, the argument used for Γ_1 allows us to show the last assertion for all these functions Γ_μ.

(ii) Let us fix $\varepsilon > 0$. Our goal is to prove that $\lambda_1 \le \lambda_2 + \varepsilon$, which ensures (ii). Let κ be a common bound for $\|\Gamma_1\|$ and $\|\Gamma_2\|$. Theorem 2.12 provides a constant $\delta_\varepsilon = \delta_\varepsilon(\varepsilon, \kappa) > 0$ such that if $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ are BPUC functions norm-bounded by κ such that $\|\widetilde{\Gamma}_1 - \widetilde{\Gamma}_2\| \le \delta_\varepsilon$, then $|\lambda^*\big(2\,\widetilde{\Gamma}_1,\ p - (\widetilde{\Gamma}_1)^2\big) - \lambda^*\big(2\,\widetilde{\Gamma}_2,\ p - (\widetilde{\Gamma}_2)^2\big)| < \varepsilon/2$. We call $\gamma_i^\pm := \lim_{n\to\pm\infty} \Gamma_i(t)$, and look for a common $t_\varepsilon > 0$ such that $|\Gamma_i(t) - \gamma_i^\pm| \le \delta_\varepsilon/2$ if $\pm t \ge t_\varepsilon$ for $i = 1, 2$, assuming without restriction that $\Gamma_i(t)$ is continuous at $\pm t_\varepsilon$ for $i = 1, 2$. Let us define the

BUPC functions $\Gamma_{i,\varepsilon}^{\infty}$ for $i = 1, 2$ by

$$\Gamma_{i,\varepsilon}^{\infty}(t) := \begin{cases} \Gamma_i(-t_\varepsilon) & \text{if} \quad t < -t_\varepsilon\,, \\ \Gamma_i(t) & \text{if} \quad -t_\varepsilon \le t < t_\varepsilon\,, \\ \Gamma_i(t_\varepsilon) & \text{if} \quad t \ge t_\varepsilon\,, \end{cases}$$

so that

$$|\lambda^*(2\,\Gamma_i,\ p - \Gamma_i^2) - \lambda^*(2\,\Gamma_{i,\varepsilon}^{\infty},\ p - (\Gamma_{i,\varepsilon}^{\infty})^2)| < \varepsilon/2 \quad \text{for} \quad i = 1, 2\,. \tag{3.7}$$

Now we take the smallest (finite) ordered set $\{a_0, \ldots, a_m\}$ composed by the points of $(-t_\varepsilon, t_\varepsilon)$ at which either Γ_1 or Γ_2 are not continuous and by $a_0 := -t_\varepsilon$ and $a_m := t_\varepsilon$. Recall that Γ_i is right-continuous on a_j for all $j = 0, \ldots, m$ and $i = 1, 2$. We call $h := \inf_{j\in\{0,\ldots,m-1\}}(a_{j+1} - a_j) > 0$. For all $n \in \mathbb{N}$ and for $i = 1, 2$, we define $\Lambda_{i,\varepsilon}^n : [-t_\varepsilon, t_\varepsilon] \to \mathbb{R}$ as follows: if $t \in [a_j, a_{j+1} - h/n)$, then $\Lambda_{i,\varepsilon}(t) := \Gamma_i(t)$, whereas if $t \in [a_{j+1} - h/n, a_{j+1})$, then $\Lambda_{i,\varepsilon}(t) := \Gamma_i(a_{j+1}) + (a_{j+1} - t)(n/h)\big(\Gamma_i(a_{j+1} - h/n) - \Gamma_i(a_{j+1})\big)$. We complete the definition to the whole line as follows:

$$\Gamma_{i,\varepsilon}^{n}(t) := \begin{cases} \Gamma_i(-t_\varepsilon) & \text{if} \quad t < -t_\varepsilon\,, \\ \Lambda_{i,\varepsilon}^n(t) & \text{if} \quad -t_\varepsilon \le t < t_\varepsilon\,, \\ \Gamma_i(t_\varepsilon) & \text{if} \quad t \ge t_\varepsilon\,, \end{cases}$$

Clearly, each function $\Gamma_{i,\varepsilon}^n$ is continuous on $\mathbb{R}$ but it is BPUC, and $\lim_{n\to\infty} \Gamma_{i,\varepsilon}^n(t) = \Gamma_{i,\varepsilon}^{\infty}(t)$ for all $t \in \mathbb{R}$. In particular, Lebesgue's dominated convergence theorem ensures that the sequence $(\Gamma_{i,\varepsilon}^n)_{n\ge1}$ converges to $\Gamma_{i,\varepsilon}^{\infty}$ in $L^1_{\text{loc}}(\mathbb{R}, \mathbb{R})$; i.e., $\lim_{n\to\infty} \int_a^b |\Gamma_{i,\varepsilon}^n(t) - \Gamma_{i,\varepsilon}^{\infty}(t)|\, dt = 0$ whenever $a < b$. In addition, $\Gamma_{1,\varepsilon}^n - \Gamma_{2,\varepsilon}^n$ is nondecreasing for all $n \in \mathbb{N}$: it coincides with the function $(\Gamma_1 - \Gamma_2)_\varepsilon^n$ which we obtain by the same procedure starting with $\Gamma_1 - \Gamma_2$, and this procedure provides a nondecreasing function. Hence, according to (i), $\lambda^*(2\,\Gamma_{1,\varepsilon}^n,\ p - (\Gamma_{1,\varepsilon}^n)^2) \le \lambda^*(2\,\Gamma_{2,\varepsilon}^n,\ p - (\Gamma_{2,\varepsilon}^n)^2)$. Our next purpose is showing that $\lim_{n\to\infty} \lambda^*(2\,\Gamma_{i,\varepsilon}^n,\ p - (\Gamma_{i,\varepsilon}^n)^2) = \lambda^*(2\,\Gamma_{i,\varepsilon}^{\infty},\ p - (\Gamma_{i,\varepsilon}^{\infty})^2)$ for $i = 1, 2$, which yields

$$\lambda^*(2\,\Gamma_{1,\varepsilon}^{\infty},\ p - (\Gamma_{1,\varepsilon}^{\infty})^2) \le \lambda^*(2\,\Gamma_{2,\varepsilon}^{\infty},\ p - (\Gamma_{2,\varepsilon}^{\infty})^2)\,.$$

In turn, this inequality and (3.7) prove $\lambda_1 \le \lambda_2 + \varepsilon$ and complete the proof.

Since the proof is the same for both values of i, we fix one and omit the subindex. Let us call $\lambda_\varepsilon(n) := \lambda^*(2\,\Gamma_\varepsilon^n,\ p - (\Gamma_\varepsilon^n)^2)$ for $n \in \mathbb{N} \cup \{\infty\}$, i.e., the index associated to $y' = -\big(y - \Gamma_\varepsilon^n(t)\big)^2 + p(t)$ by Theorem 2.11 for $n \in \mathbb{N} \cup \{\infty\}$. We must prove:

1 given $\lambda < \lambda_\varepsilon(\infty)$, there exists n_1 such that $\lambda \le \lambda_\varepsilon(n)$ for all $n \ge n_1$,
2 given $\lambda > \lambda_\varepsilon(\infty)$, there exists n_2 such that $\lambda \ge \lambda_\varepsilon(n)$ for all $n \ge n_2$.

Let us check **1**. Reasoning by contradiction, we assume the existence of $\bar\lambda < \lambda_\varepsilon(\infty)$ and a subsequence $(\Gamma_\varepsilon^k)_{k\ge1}$ of $(\Gamma_\varepsilon^n)_{n\ge1}$ such that $\bar\lambda > \lambda_\varepsilon(k)$ for all $k \ge 1$. Theorem 2.11(i) ensures the existence of a bounded solution $\mathfrak{b}_\varepsilon^k$ of $y' = -\big(y - \Gamma_\varepsilon^k(t)\big)^2 + p(t) + \bar\lambda$ for $k \ge 1$. The existence of a common bound for $\big\|\Gamma_\varepsilon^k\big\|$ for all $k \ge 1$ ensures the existence of $m > 0$ and $\rho > 0$ such that $-m^2 + 2\,|\Gamma_\varepsilon^k(t)|\,m + |p(t) - (\Gamma_\varepsilon^k)^2(t) + \bar\lambda| < -\rho$ for all $t \in \mathbb{R}$ and $k \ge 1$. Hence, $\big\|\mathfrak{b}_\varepsilon^k\big\| \le m$ for any $k \ge 1$: see Theorem 2.5(iv). Now we take a new subsequence $(\Gamma_\varepsilon^j)_{j\ge1}$ of $(\Gamma_\varepsilon^k)_{k\ge1}$ such that there exists $y_0 := \lim_{j\to\infty} \mathfrak{b}_\varepsilon^j(0)$. Theorem A.3 ensures that the solution $y_\varepsilon^{\infty}(t, 0, y_0)$ of $y' = -\big(y - \Gamma_\varepsilon^{\infty}(t)\big)^2 + p(t) + \bar\lambda$ coincides with $\lim_{j\to\infty} \mathfrak{b}_\varepsilon^j(t)$ for any t in its maximal interval of definition; therefore, it is bounded by m (and hence globally defined). This and Theorem 2.11(i) contradict $\bar\lambda < \lambda_\varepsilon(\infty)$. Thus, **1** is proved.

Let us now sketch the idea to prove **2**. We fix $\bar\lambda > \lambda_\varepsilon(\infty)$, so that the equation

$$y' = -\big(y - \Gamma_\varepsilon^n(t)\big)^2 + p(t) + \bar\lambda \tag{3.8}$$

corresponding to $n = \infty$ has an attractor–repeller pair $(\widetilde{\mathfrak{a}}_\varepsilon^\infty, \widetilde{\mathfrak{r}}_\varepsilon^\infty)$. We will check that, if n is large enough, then there exist the functions $\mathfrak{a}_\varepsilon^n$ and $\mathfrak{r}_\varepsilon^n$ associated to $(3.8)_\varepsilon^n$ by Theorem 2.5, they are respectively defined at least on the intervals $(-\infty, t_\varepsilon]$ and $[t_\varepsilon, \infty)$, and they satisfy $\mathfrak{a}_\varepsilon^n(t_\varepsilon) \geq \mathfrak{r}_\varepsilon^n(t_\varepsilon)$. As explained in Remark 2.6, this proves the existence of a bounded solution, and hence that $\bar\lambda \geq \lambda_\varepsilon(n)$, as **2** asserts.

Observe that, outside the interval $[-t_\varepsilon, t_\varepsilon]$, the coefficients of the Eq. $(3.8)_\varepsilon^n$ are common for any $n \geq 1$. We can repeat the proof of Theorem 3.4(i), working with the attractor–repeller pair $(\widetilde{\mathfrak{a}}_\varepsilon^\infty, \widetilde{\mathfrak{r}}_\varepsilon^\infty)$ of $(3.8)_\varepsilon^\infty$ instead of $(\widetilde{\mathfrak{a}}_-, \widetilde{\mathfrak{r}}_-)$, and with time $-t_\varepsilon$ instead of t^-. In this way we prove that, for any $n \geq 1$, $\mathfrak{a}_\varepsilon^n$ is defined at least on $(-\infty, -t_\varepsilon]$, where it coincides with $\widetilde{\mathfrak{a}}_\varepsilon^\infty$. Analogously, for any $n \geq 1$, $\mathfrak{r}_\varepsilon^n$ is defined at least on $[t_\varepsilon, \infty)$, where it coincides with $\widetilde{\mathfrak{r}}_\varepsilon^\infty$. We call $\rho := \min_{t\in[-t_\varepsilon,t_\varepsilon]} \big(\widetilde{\mathfrak{a}}_\varepsilon^\infty(t) - \widetilde{\mathfrak{r}}_\varepsilon^\infty(t)\big) > 0$. Theorem A.3 provides n_2 such that, for $n \geq n_2$,

$$\max_{t\in[-t_\varepsilon,t_\varepsilon]} \big| y_\varepsilon^n(t, -t_\varepsilon, \widetilde{\mathfrak{a}}_\varepsilon^\infty(-t_\varepsilon)) - y_\varepsilon^\infty(t, -t_\varepsilon, \widetilde{\mathfrak{a}}_\varepsilon^\infty(-t_\varepsilon))\big| \leq \rho\,,$$

where $y_\varepsilon^n(t, s, y_0)$ is the solution of $(3.8)_\varepsilon^n$ with value y_0 at $t = s$. Hence,

$$\min_{t\in[-t_\varepsilon,t_\varepsilon]} \big(y_\varepsilon^n(t, -t_\varepsilon, \widetilde{\mathfrak{a}}_\varepsilon^\infty(-t_\varepsilon)) - \widetilde{\mathfrak{r}}_\varepsilon^\infty(t)\big) \geq 0\,.$$

Altogether, we conclude that, if $t \in [-t_\varepsilon, t_\varepsilon]$ and $n \geq n_2$, then

$$\mathfrak{a}_\varepsilon^n(t) = y_\varepsilon^n(t, -t_\varepsilon, \mathfrak{a}_\varepsilon^n(-t_\varepsilon)) = y_\varepsilon^n(t, -t_\varepsilon, \widetilde{\mathfrak{a}}_\varepsilon^\infty(-t_\varepsilon)) \geq \widetilde{\mathfrak{r}}_\varepsilon^\infty(t)\,:$$

the lower bound ensures that $\mathfrak{a}_\varepsilon^n$ is also defined on $[-t_\varepsilon, t_\varepsilon]$. Taking $t = t_\varepsilon$ in the previous formula provides the sought-for inequality and ensures **2**. □

Corollary 3.8 *Let $p\colon \mathbb{R} \to \mathbb{R}$ be a BPUC function.*

(i) *Let $\Gamma^+, \Gamma^-\colon \mathbb{R} \to \mathbb{R}$ be bounded, uniformly continuous, and nondecreasing, and define $\Gamma := \Gamma^+ - \Gamma^-$. Then, $\lambda^*(-2\,\Gamma^-,\ p-(\Gamma^-)^2) \leq \lambda^*(2\,\Gamma,\ p-\Gamma^2) \leq \lambda^*(2\,\Gamma^+,\ p-(\Gamma^+)^2)$.*

(ii) *Let $\Gamma\colon \mathbb{R} \to \mathbb{R}$ be nondecreasing, and either be a BPUC function and have finite asymptotic limits or be bounded and uniformly continuous. Then, $\lambda^*(-2\,\Gamma,\ p-\Gamma^2) \leq \lambda^*(0, p) \leq \lambda^*(2\,\Gamma,\ p-\Gamma^2)$. Moreover,*

- *$\lambda^*(-2\,\Gamma,\ p-\Gamma^2) = \lambda^*(0, p)$ if p is recurrent and Γ has finite asymptotic limits.*
- *Assume also that Γ is continuous, and absolutely continuous and nonconstant on a nongenenerate compact interval of $\mathbb{R}$. If $\lambda^*(0, p) = \lambda^*(-2\,\Gamma,\ p-\Gamma^2)$, then $y' = -(y+\Gamma(t))^2 + p(t) + \lambda^*(-2\,\Gamma,\ p-\Gamma^2)$ has infinitely many bounded solutions; and $\lambda^*(0, p) < \lambda^*(2\,\Gamma,\ p-\Gamma^2)$ if $x' = -x^2 + p(t) + \lambda^*(0, p)$ has just one bounded solution.*

Proof Theorem 3.7(i) ensures (i). The first (or second) inequality in (ii) follows from Theorem 3.7 applied to $\Gamma_1 := 0$ and $\Gamma_2 := \Gamma$ (or $\Gamma_1 := -\Gamma$ and $\Gamma_2 := 0$). The assertion in (ii) concerning a recurrent p follows from Proposition 3.6, and the last assertions follow also from Theorem 3.7(i). □

Corollary 3.9 *Let $p\colon \mathbb{R} \to \mathbb{R}$ be a BPUC function, and assume that $x' = -x^2 + p(t)$ does not have bounded solutions. Then, the equation $y' = -(y - \Gamma(t))^2 + p(t)$ has no bounded solutions in the following cases:*

(a) *if p is recurrent and the function $\Gamma\colon \mathbb{R} \to \mathbb{R}$ is BPUC and has finite asymptotic limits;*
(b) *or if the function $\Gamma\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing and either is BPUC and has finite asymptotic limits or is bounded and uniformly continuous.*

Assume now that $x' = -x^2 + p(t)$ has an attractor–repeller pair and the conditions of (b). Then, the equation $y' = -(y + \Gamma(t))^2 + p(t)$ has an attractor–repeller pair.

Proof Assume the lack of bounded solutions. In case (a), the result is an easy consequence of Proposition 3.6 and Theorem 2.11: the lack of bounded solutions for $x' = -x^2 + p(t)$ means $\lambda^*(0, p) > 0$, so that $\lambda^*(2\,\Gamma, p - \Gamma^2) > 0$, and hence $y' = -(y - \Gamma(t))^2 + p(t)$ has no bounded solutions. The same arguments and Corollary 3.8(ii) prove case (b), as well as the last assertion. $\square$

3.2 Tipping Induced by a Local Increment of the Transition Function

The results already proved allow us to analyze the existence of *tipping values of c* (see Definition 3.10 below) for the parametric family of equations

$$y' = -\big(y - c\,\Gamma(t)\big)^2 + p(t) \tag{3.9}$$

for $c \in \mathbb{R}$ under more restrictive conditions on Γ and p which we will describe in due time. We will represent by $(3.9)_c$ the equation corresponding to a fixed c. Observe that the corresponding future and past equations also depend on the value of the multiplicative parameter c.

Our tipping analysis studies the change of the global dynamics as c varies under some assumptions involving the existence of an strictly increasing point for Γ. This dynamics corresponds to CASES A, B or C of Definition 3.1. Recall that Theorem 2.9 shows that CASE A is equivalent to the existence of an attractor–repeller pair. With the aim of talking about *occurrence of tipping* when an attractor–repeller pair "persists for a while and then disappears", we define:

Definition 3.10 The point $c_0 \in \mathbb{R}$ is a *tipping value for the family* $(3.9)_c$ if the equation $(3.9)_c$ is in CASE A for c in an open interval of endpoint c_0, but not at c_0.

Theorem 3.4 and Remark 3.5 provide more details concerning the three dynamical situations under Hypothesis 3.2 and the conditions assumed on Γ and p at the beginning of Sect. 3. But these conditions will not in force unless otherwise indicated. Theorem 2.11 establishes a one-to-one relation between the dynamical case of $(3.9)_c$ and the sign at c of the map

$$\widehat{\lambda}\colon \mathbb{R} \cup \{\pm\infty\} \to \mathbb{R}\,, \quad c \mapsto \widehat{\lambda}(c) := \lambda^*\big(2\,c\,\Gamma,\ p - c^2\Gamma^2\big)\,, \tag{3.10}$$

given by the value associated to $(3.9)_c$ by this theorem; that is, the bifurcation point in λ of $x' = -(x - c\,\Gamma(t))^2 + p(t) + \lambda$. More precisely, CASE A (resp. CASE B, resp. CASE C) occurs if and only if $\widehat{\lambda}(c)$ is strictly negative (resp. null, resp. strictly positive). The next result implies that, as one might expect, if $(3.9)_c$ undergoes a tipping at c_0 then $(3.9)_{c_0}$ is in CASE B.

Proposition 3.11 *Let $\Gamma, p\colon \mathbb{R} \to \mathbb{R}$ be BPUC functions, and let $\widehat{\lambda}$ be the map defined by* (3.10). *Then,*

(i) *for every $\kappa > 0$ there exists $m_\kappa > 0$ such that, if $c_1, c_2 \in [-\kappa, \kappa]$, then $|\widehat{\lambda}(c_1) - \widehat{\lambda}(c_2)| \le m_\kappa |c_1 - c_2|$. In particular, $\widehat{\lambda}$ is continuous and locally Lipschitz on $\mathbb{R}$.*

(ii) *If, in addition, Γ is C^1 and $\|\Gamma'\| := \sup_{t\in\mathbb{R}} |\Gamma'(t)| < \infty$, then $|\widehat{\lambda}(c_1) - \widehat{\lambda}(c_2)| \le \|\Gamma'\|\; |c_1 - c_2|$ for all $c_1, c_2 \in \mathbb{R}$. That is, under these conditions, $\widehat{\lambda}$ is Lipschitz on $\mathbb{R}$.*

Proof Assertions (i) follow easily from Theorem 2.12. Under the hypothesis of (ii), for each $c \in \mathbb{R}$, the (bounded) change of variable $x = y - c\,\Gamma(t)$ takes $(3.9)_c$ to

$$x' = -x^2 + p(t) - c\,\Gamma'(t)\,, \tag{3.11}$$

without changing its dynamics: CASES A, B or C are preserved. From this point, we check (ii) by repeating the argument of the proof of [29, Theorem 4.13(ii)]. □

Proposition 3.12 *Let $p\colon \mathbb{R} \to \mathbb{R}$ be a BPUC function.*

(i) *Assume that $\Gamma\colon \mathbb{R} \to \mathbb{R}$ is C^1 and that there exists a point t_0 at which it is strictly increasing. Then, there exists a value $c_0 > 0$ such that $(3.9)_c$ is in* CASE C *for all $c \ge c_0$. Moreover, $\lim_{c\to\infty} \widehat{\lambda}(c) = \infty$.*
(ii) *Assume that Hypothesis 3.2 holds, that $\Gamma\colon \mathbb{R} \to \mathbb{R}$ is nonincreasing, and that either is BPUC and has finite asymptotic limits or is bounded and uniformly continuous. Then, $(3.9)_c$ is in* CASE A *for all $c \ge 0$.*

Proof (i) To avoid extra technical difficulties in the proof, we assume that $\Gamma'(t) \ge \delta > 0$ for all $t \in [0, 1]$. The general case can be proved by adapting the argument we will follow. For each $c \in \mathbb{R}$, the (bounded) change of variable $x = y - c\,\Gamma(t)$ takes $(3.9)_c$ to $(3.11)_c$, preserving its global dynamics. We look for $c_0 > 0$ such that $c_0\Gamma'(t) \ge \pi^2 + p(t)$ for all $t \in [0, 1]$, and observe that the same inequality holds for all $c \ge c_0$. Then, if $c \ge c_0$, the solution $x_c(t, 0, x_0)$ of $(3.11)_c$ with value x_0 at $t = 0$ satisfies $x_c(t, 0, x_0) \le \pi \tan(-\pi t + \arctan(x_0/\pi))$ (which is the solution of $x' = -x^2 - \pi^2$ with value x_0 at $t = 0$) for all the values of $t \in [0, 1]$ for which they are defined. (As usual, we take $\arctan(x_0/\pi) \in (-\pi/2, \pi/2)$.) Since $-\pi + \arctan(x_0/\pi) < -\pi/2$, there exists $t_0 \in [0, 1]$ such that $\lim_{t\to t_0^-} \tan(-\pi t + \arctan(x_0/\pi)) = -\infty$. Consequently, $x_c(t, 0, x_0)$ is unbounded for any $x_0 \in \mathbb{R}$ and $c \ge c_0$, which proves the first assertion in (i).

Let us now take $k > 0$. The previous property provides $c_k > 0$ such that $\lambda^*\big(2\,c\,\Gamma,\; p + k - c^2\Gamma^2\big) > 0$ for all $c \ge c_k$, and hence Theorem 2.11(v) ensures that $\lambda^*\big(2\,c\,\Gamma,\; p - c^2\Gamma^2\big) > k$ for all $c \ge c_k$. This proves the last assertion in (i).

(ii) It follows from Corollary 3.8(ii) and Hypothesis 3.2 that $\lambda^*(2\,c\,\Gamma, p - c^2\Gamma^2) \le \lambda^*(0, p) < 0$ whenever $c \ge 0$, which proves (ii). □

Proposition 3.13 *Let $p\colon \mathbb{R} \to \mathbb{R}$ be a BPUC function. Assume that Hypothesis 3.2 holds, and that $\Gamma\colon \mathbb{R} \to \mathbb{R}$ has finite asymptotic limits and is C^1, nondecreasing, and nonconstant. Then there exists exactly a tipping value $\widehat{c}$ for the family $(3.9)_c$, which is strictly positive.*

Proof Hypothesis 3.2 ensures $\widehat{\lambda}(0) < 0$, and Proposition 3.12 provides at least a value of $c > 0$ with $\widehat{\lambda}(c) > 0$. The continuity of $\widetilde{\lambda}$ established by Proposition 3.11(i) shows the existence of a minimum $c_1 > 0$ with $\widehat{\lambda}(c_1) = 0$. Let us assume for contradiction the existence of $c_2 > c_1$ with $\widehat{\lambda}(c_2) = 0$. By applying Theorem 3.7(i) to $\Gamma_1 = c_1\,\Gamma$ and $\Gamma_2 = c_2\,\Gamma$, we deduce that $y' = -(y - c_1\,\Gamma(t))^2 + p(t)$ has infinitely many bounded solutions but no hyperbolic ones. But this contradicts the information provided by Remark 3.5 in CASE B. □

To close this section we point out that the tipping analysis just performed can also be understood as a bifurcation analysis depending on c: Proposition 3.13(ii) establishes conditions under which the c-parametric family (3.9) follows a global saddle-node nonautonomous bifurcation pattern (see also Remark 2.13).

4 Rate-Induced Tipping in the Continuous Case

In the rest of the paper, $\Gamma\colon \mathbb{R}\to\mathbb{R}$ represents a continuous map with finite asymptotic limits $\gamma_\pm := \lim_{t\to\pm\infty}\Gamma(t)$, and $p\colon \mathbb{R}\to\mathbb{R}$ is a BPUC function. One of the main goals of the paper is to analyze the possibility of occurrence of *rate-induced* tipping for the one-parametric family of equations

$$y' = -\big(y-\Gamma_c(t)\big)^2 + p(t)\,,\quad \text{with}\quad \Gamma_c(t) := \Gamma(c\,t) \tag{4.1}$$

for $c\in\mathbb{R}$ (which will be referred to as $(4.1)_c$ if c is fixed). The parameter c is the *rate*. For $c>0$, Γ_c is often understood as a *transition* from γ_- to γ_+ as time increases, and c determines the velocity of this transition. Note that the function $\Gamma^-(t) := \Gamma(-t)$ for $t\in\mathbb{R}$ maintains the same properties required to Γ, and $\Gamma^-(ct)=\Gamma(-ct)$ for every $c\in\mathbb{R}$. Therefore, the analysis of $(4.1)_c$ for $c<0$ is implicitly contained in the analysis of $(4.1)_c$ for any Γ and $c>0$. However, we will formulate several properties also for $c<0$, to provide a better understanding of the global picture.

In our rate-induced tipping analysis for (4.1), a fundamental role is played by the Carathéodory equations

$$y' = -(y-\Gamma_\infty(t))^2 + p(t)\,,\quad \text{where } \Gamma_\infty(t) := \begin{cases}\gamma_- & \text{if } \; t<0\,,\\ \gamma_+ & \text{if } \; t\ge 0\,,\end{cases} \tag{4.2}$$

and

$$y' = -(y-\Gamma_{-\infty}(t))^2 + p(t)\,,\quad \text{where } \Gamma_{-\infty}(t) := \begin{cases}\gamma_+ & \text{if } \; t<0\,,\\ \gamma_- & \text{if } \; t\ge 0\,.\end{cases} \tag{4.3}$$

Note that (4.2) and (4.3) can be respectively understood as the limiting systems of $(4.1)_c$ as $c\to\infty$ and $c\to-\infty$. (We will describe this limiting behaviour more precisely in Sect. 5.) From now on, $(4.1)_\infty$ and $(4.1)_{-\infty}$ represent (4.2) and (4.3). Note also that $\Gamma_{\pm\infty}\in BPUC_{\mathbb{R}-\{0\}}(\mathbb{R},\mathbb{R})$ (see Sect. 2.3).

Following the ideas explained in Sect. 3.2, our tipping analysis studies the change of the global dynamics, determined by CASES A, B or C:

Definition 4.1 The point $c_0\in\mathbb{R}$ is a *tipping rate for the family* $(4.1)_c$ if the equation $(4.1)_c$ is in CASE A for c in an open interval of finite endpoint c_0, but not at c_0. A tipping rate c_0 is *transversal* if there is an open interval containing c_0 such that, for values of c at one side of c_0, the equation $(4.1)_c$ is in CASE A whereas at the other side is in CASE C. In the case of existence of a (transversal) tipping point c_0, we have a (*transversal*) *rate-induced tipping at* c_0.

Observe that a transversal tipping can be understood as a local saddle-node bifurcation phenomenon occurring as the parameter c varies. In Sect. 4.1, we will explain why we use the word *transversal* in Definition 4.1. We now anticipate that it is related to the properties of the map $c\mapsto\lambda_*(c)=\lambda^*(2\,\Gamma_c,\,p-\Gamma_c^2)$ for fixed Γ and p, where $\lambda^*(2\,\Gamma_c,\,p-\Gamma_c^2)$ is the value associated to $(4.1)_c$ by Theorem 2.11 for $c\in\mathbb{R}\cup\{\pm\infty\}$; that is, the bifurcation point in λ of $x'=-(x-\Gamma_c(t))^2+p(t)+\lambda$. In particular, the sign of $\lambda_*(c)$ determines the dynamics of $(4.1)_c$. Observe that, unlike the situation in Sect. 3.2, Theorem 2.12 does not imply immediately the continuity of the map λ_* on the extended real line. But we will prove this continuity in Sect. 5. Therefore, as in Sect. 3.2, if (4.1) undergoes a rate-induced tipping at c_0, then $(4.1)_{c_0}$ is in CASE B.

Observe that the future and past equations are given for any $c>0$ by

$$y' = -(y-\gamma_+)^2 + p(t) \tag{4.4}$$

and

$$y' = -(y - \gamma_-)^2 + p(t)\,, \tag{4.5}$$

while the roles of (4.4) and (4.5) are interchanged for $c < 0$. We consider also the Eq. $(4.1)_0$, namely

$$y' = -(y - \gamma_0)^2 + p(t) \tag{4.6}$$

for $\gamma_0 := \Gamma(0)$. Note that the global dynamics of these three equations is "identical", since all of them are obtained from

$$x' = -x^2 + p(t) \tag{4.7}$$

by trivial changes of variables: see Remark 3.3.

For the reader's convenience, we complete this initial part of the section by repeating the fundamental Hypothesis 3.2:

Hypothesis 4.2 The equation (4.7) has an attractor–repeller pair $(\widetilde{a}, \widetilde{r})$.

Remarks 4.3 1. Under this condition, all the information provided by Theorem 3.4 and Remark 3.5 applies to $(4.1)_c$ for any $c \in \mathbb{R} \cup \{\pm\infty\}$. But one must have in mind that the future and past equations, and hence the corresponding attractor–repeller pairs, depend on the sign of c: $(\widetilde{a} + \gamma_+, \widetilde{r} + \gamma_+)$ is the future (resp. past) pair for $c > 0$ (resp. $c < 0$), $(\widetilde{a} + \gamma_-, \widetilde{r} + \gamma_-)$ is the past (resp. future) pair for $c > 0$ (resp. $c < 0$), and $(\widetilde{a} + \gamma_0, \widetilde{r} + \gamma_0)$ is the future and past pair for $c = 0$.

2. Proposition 5.2, proved in the next section, shows that part of the dynamical properties described in Theorem 3.4 and Remark 3.5 also hold when Hypothesis 4.2 is substituted by the existence of attractor–repeller pair for (4.2) or (4.3).

4.1 The Bifurcation Curve $\lambda_*(c)$

Let us define

$$\lambda_* \colon \mathbb{R} \cup \{\pm\infty\} \to \mathbb{R}\,, \quad c \mapsto \lambda_*(c) := \lambda^*\big(2\,\Gamma_c,\ p - (\Gamma_c)^2\big) \tag{4.8}$$

and recall the relation between the dynamical situation of $(4.1)_c$ (CASE A, B, or C) and the sign of $\lambda_*(c)$ (negative, null, or positive). In particular, Hypothesis 4.2 can be reformulated as $\lambda_*(0) < 0$: see Remark 3.3. This hypothesis is not in force for the next result, already mentioned, and proved by Theorem 5.3 in the next section.

Theorem 4.4 *Let* $\lambda_* \colon \mathbb{R} \cup \{\pm\infty\} \to \mathbb{R}$ *be defined by* (4.8). *Then,*

(i) *the map* λ_* *is bounded: it takes values in* $[-||p||, ||p|| + ||\Gamma||^2]$.
(ii) *the map* λ_* *is continuous on the extended real line.*

So, according to Definition 4.1, $c_0 \in \mathbb{R}$ is a tipping rate if $\lambda_*(c_0) = 0$ and there is $\delta_0 > 0$ such that $\lambda_*(c) < 0$ either for $c \in (c_0 - \delta_0, c_0)$ or for $c \in (c_0, c_0 + \delta_0)$; and a transversal tipping rate if, in addition, $\lambda_*(c) > 0$ either for $c \in (c_0 - \delta_0, c_0)$ or for $c \in (c_0, c_0 + \delta_0)$. Hence, the graph of the continuous map λ_* crosses the horizontal axis transversally at a transversal tipping-rate c_0. Although one might expect this situation to be the most frequent one, other types of tipping are also possible.

Assuming Hypothesis 4.2, the continuity of λ_* established in Theorem 4.4 allows us to determine the dynamical case of $(4.1)_c$ for small values of the rate, and also for large ones under additional conditions. This is what the next theorem states. Its scope will be clearer in Sect. 4.2.

Theorem 4.5 *Assume Hypothesis 4.2, and let $(\widetilde{a}_\lambda, \widetilde{r}_\lambda)$ be the attractor–repeller pair for $x' = -x^2 + p(t) + \lambda$ for $\lambda > \lambda_*(0)$ (with $(\widetilde{a}_0, \widetilde{r}_0) = (\widetilde{a}, \widetilde{r})$). Then,*

- (i) *there exists $c_0 > 0$ such that $(4.1)_c$ is in* CASE A *for $c \in (-c_0, c_0)$.*
- (ii) *If $\widetilde{a}(0) - \widetilde{r}(0) > \gamma_+ - \gamma_-$, then the Eq. (4.2) has an attractor–repeller pair, and hence there exists a minimum $c_M \geq 0$ such that $(4.1)_c$ is in* CASE A *for $c > c_M$.*
- (iii) *If $\widetilde{a}(0) - \widetilde{r}(0) < \gamma_+ - \gamma_-$, then the Eq. (4.2) has no bounded solutions, and hence there is a minimum $c_M^* > 0$ such that $(4.1)_c$ is in* CASE C *for $c > c_M^*$. In this case, $\lambda_*(\infty) = \lambda_\infty$, where $\lambda_\infty > 0$ is the unique value of the parameter such that $\widetilde{a}_{\lambda_\infty}(0) - \widetilde{r}_{\lambda_\infty}(0) = \gamma_+ - \gamma_-$.*
- (iv) *If $\widetilde{a}(0) - \widetilde{r}(0) > \gamma_- - \gamma_+$, then the Eq. (4.3) has an attractor–repeller pair, and hence there exists a maximum $c_m \leq 0$ such that $(4.1)_c$ is in* CASE A *for $c < c_m$.*
- (v) *If $\widetilde{a}(0) - \widetilde{r}(0) < \gamma_- - \gamma_+$, then the Eq. (4.3) has no bounded solutions, and hence there is a maximum $c_m^* < 0$ such that $(4.1)_c$ is in* CASE C *for $c < c_m^*$. In this case, $\lambda_*(-\infty) = \lambda_{-\infty} > 0$, where $\lambda_{-\infty}$ is the unique value of the parameter such that $\widetilde{a}_{\lambda_{-\infty}}(0) - \widetilde{r}_{\lambda_{-\infty}}(0) = \gamma_- - \gamma_+$.*

Proof This result is included in Theorem 5.5, proved in the next section. □

Remark 4.6 It follows from Theorems 4.5 and 3.4 that the Eq. (4.2) (or (4.3)) has only a bounded solution if and only if $\widetilde{a}(0) - \widetilde{r}(0) = \gamma_+ - \gamma_-$ (or $\widetilde{a}(0) - \widetilde{r}(0) = \gamma_- - \gamma_+$). But $\lambda_*(\infty) = 0$ (or $\lambda_*(-\infty) = 0$) does not lead us to any conclusion for large values of c (or $-c$). Observe also that, since $\widetilde{a}$ and $\widetilde{r}$ depend just on p while γ^+ and γ^- depend just on Γ, a suitable choice of Γ once p is fixed (or the converse) determines the dynamical situation of (4.2): tipping occurs when $\widetilde{a}(0) - \widetilde{r}(0) < \gamma_+ - \gamma_-$ (which is not possible if $\gamma_+ < \gamma_-$), and there is tracking whenever $\gamma^+ - \gamma^-$ is large enough. Analogous conclusions hold for (4.3).

Remark 4.7 Proposition 5.7 applied to the case $h = 0$ establishes some inequalities regarding $\lambda_*(c)$ which provide valuable information about the corresponding dynamical case for $(4.1)_c$ depending on that for $(4.1)_0 = (4.7)$.

4.2 Partial and Total Tipping on the Hull

Partial tipping and total tipping are phenomena introduced in [1] in the context of two-dimensional asymptotically autonomous systems. In particular, the attractors of the future and past systems in [1] are compact sets, each given by the trajectory of an orbitally asymptotically stable solution. The associated pullback attractor for the nonautonomous system is hence a compact nonautonomous set which is not a singleton. Upon the variation of the rate, it is shown that the associated pullback attractor can break up in the sense that some of the trajectories limiting at the limit cycle of the past limit system also limit at the limit cycle of the future system, but others fail to do so, and *partial tipping* occurs. If all the trajectories which determine the pullback attractor do not limit to the limit cycle of the future system, then a *total tipping* happens.

While the current state of the art does not allow us to pose the same question in the context of two-dimensional asymptotically nonautonomous systems, the results of the previous section do allow us to address a different phenomenon which can still be regarded as an instance of partial and total tipping.

The key point is that, in the nonautonomous case, our Hypothesis 4.2 of existence of an attractor–repeller pair for the equation $x' = -x^2 + p(t)$, and hence for the future and past equations $y' = -(y - \gamma_\pm)^2 + p(t)$, means the existence of two hyperbolic copies of the base for the corresponding skew-product flow defined on the hull Ω_p of p (described in Sect. 3.1). The proof of this assertion is the fundamental point in the proof of [29, Theorem 3.5], where the interested reader can find a more detailed explanation of the meaning of hyperbolic copy of the base. What is interesting for us, now, is that this property ensures that each equation $x' = -x^2 + q(t)$ given by $q \in \Omega_p$, as well as the corresponding future and past equations, has an attractor–repeller pair (given by the corresponding sections of the hyperbolic copies of the base). So, a natural question arises: for a given value of $c > 0$ (or $c = \infty$), are all the equations $y' = -(y - \Gamma_c(t))^2 + q(t)$, where Γ is fixed and q varies in the hull Ω_p, in the same dynamical case? We will talk about *partial tipping on the hull* when CASES A and C coexist for different functions in the hull for a given value of c, and about *total tipping on the hull* when the dynamics is always in CASE C. The global occurrence of CASE A is *total tracking on the hull*.

The next example has a double purpose: to illustrate a simple way to determine the dynamical situation of (4.2) = (4.2)$_\infty$, and hence that of (4.1)$_c$ for large enough c; and to show a situation of partial tipping on the hull. Let us define

$$\Gamma(t) := \frac{2}{\pi}\arctan(t) \quad \text{and} \quad p(t) := 0.962 - \sin(t/2) - \sin(\sqrt{5}\,t). \tag{4.9}$$

The choice of Γ and p is not coincidental: it permits a direct connection to the numerical analysis carried out in [29], which features the problem given by the same Γ and $p(t) := 0.895 - \sin(t/2) - \sin(\sqrt{5}\,t)$. Theorem 2.11(v) guarantees that the bifurcation curve λ_* of the Eq. (4.1), defined by (4.8) for the chosen Γ and p in (4.9), is a vertical translation (of -0.067) of that depicted in Figure 8 of [29]. As justified in [29], we can assume that Hypothesis 4.2 holds. We can also assume that, for all $c \in (0, 50] \cup \{\infty\}$, we are able to approximate beyond machine precision the (possibly locally defined) solutions $\mathfrak{a}_c$ and $\mathfrak{r}_c$ associated to (4.1) by Theorem 2.5. A detailed clarification supporting this last assumption is given in Appendix B.

As said in Sect. 2.3, since p is a quasiperiodic function, the corresponding hull Ω_p is constructed as the closure of the set of the shifts $p_s(t) := p(t + s)$ in the uniform topology. For this reason (as we will explain later), instead of working with the whole hull, it suffices to our purposes working with the shifts of p. Therefore, we consider the equations

$$y' = -\big(y - \Gamma_c(t)\big)^2 + p_s(t) \tag{4.10}$$

and their limits as $c \to \infty$,

$$y' = -(y - \Gamma_\infty(t))^2 + p_s(t)\,, \tag{4.11}$$

for $s \in \mathbb{R}$. Let us call $\widetilde{a}_s(t) := \widetilde{a}(t + s)$ and $\widetilde{r}_s(t) := \widetilde{r}(t + s)$. It is easy to check that $(\widetilde{a}_s, \widetilde{r}_s)$ is the attractor–repeller pair of $x' = -x^2 + p_s(t)$. Theorem 4.5(ii) and (iii) (see also their proofs) reveal that, for a given value of s, (4.11) is in CASE A (resp. CASE C), and hence the same happens with (4.1)$_c$ for large enough c, if and only if

$$d^\infty(s) := \gamma_+ - \gamma_- - \widetilde{a}_s(0) + \widetilde{r}_s(0) = 2 - \widetilde{a}(s) + \widetilde{r}(s)$$

is strictly negative (resp. positive). That is, if the distance from $\widetilde{r}(s)$ to $\widetilde{a}(s)$ is large enough, then (4.11) and all the Eq. (4.10)$_c$ for large enough c (depending on s) have an attractor–repeller pair which, according to Remark 3.5, connects that of the past equation $y' = -(y + 1)^2 + p_s(t)$ to that of the future equation $y' = -(y - 1)^2 + p_s(t)$; i.e, $(\widetilde{a}_s - 1, \widetilde{r}_s - 1)$ to $(\widetilde{a}_s + 1, \widetilde{r}_s + 1)$. And if the distance is small, then the tracking is lost and tipping occurs: there are no longer bounded solutions. This guarantees the existence of at least a tipping rate $c_0 > 0$ for the c-parametric family (4.10), as Theorem 4.5(iii) ensures.

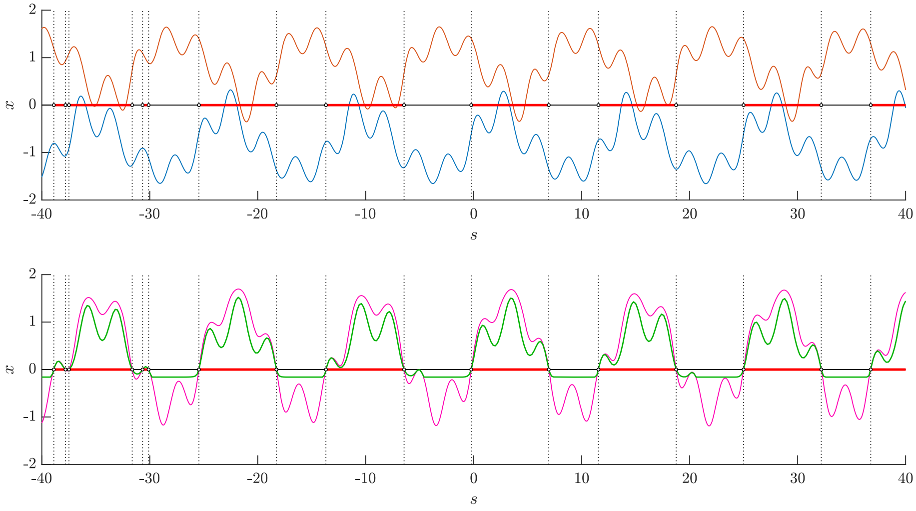

Fig. 1 Characterization of the dynamics for the differential equation (4.11) for $s \in [-40, 40]$. In the upper panel the attractor $\widetilde{a}$ (in red) and repeller $\widetilde{r}$ (in blue) of $x' = -x^2 + p(t)$. In the lower panel, the curves $\lambda^\infty(s)$ (solid green curve in the lower panel) and $d^\infty(s) = 2 - \widetilde{a}(s) + \widetilde{r}(s)$ (magenta in the lower panel). The (common) points s on which they are strictly positive (i.e., the points s for which $\widetilde{a}(s)$ and $\widetilde{r}(s)$ are close enough) are highlighted in thick red on the axis $y = 0$ in both panels. These are the points for which (4.11) has no bounded solutions, The complementary of the closure of this set, given by the points for which $\lambda^\infty(s)$ and $d^\infty(s)$ are strictly negative, is composed by the points for which (4.11) has an attractor–repeller pair (Color figure online)

We point out that the argument of the proofs of Theorem 4.5(ii) and (iii) relies on showing that $d^\infty(s)$ is strictly positive or negative whenever $\lambda^\infty(s)$ is strictly positive or negative, where $\lambda^\infty(s) := \lambda^*(2\,\Gamma_\infty,\ p_s - (\Gamma_\infty)^2)$ is the bifurcation value associated to (4.11) by Theorem 2.11. This fact provides two methods to identify the values of $s \in \mathbb{R}$ at which (4.11) is in CASES A or C (with tracking or tipping: see Remark 3.5): one can numerically calculate $\lambda^\infty(s)$, which is rather computationally expensive; or calculate $d^\infty(s)$, which is a considerably more economic alternative. In the upper panel of Fig. 1, the attractor–repeller pair $(\widetilde{a}, \widetilde{r})$ of $x' = -x^2 + p(t)$ is depicted on the plane (s, y) for $s \in [-40, 40]$, and the values of $s \in \mathbb{R}$ for which $d(s) > 0$ are highlighted in thick red on the axis $y = 0$. The lower panel shows the graphs of λ^∞ (in green) and $d^\infty(s)$ (in magenta). Of course, the two curves have the same signs. We recall once more that, when this sign is positive (resp. negative), we can assure the tipping (resp. the tracking) for $(4.10)_c$ if $c > 0$ is large enough.

In Fig. 2, we show how the bifurcation curve $\lambda^\infty(s)$ (green curve) of $y' = -(y - \Gamma_\infty(t))^2 + p_s(t)$ depending on the variation of $s \in [-20, 20]$ seems to be rapidly approached by the bifurcation curve $\lambda_*(c, s)$ of $(4.10)_c$ as c increases: $\lambda^\infty(s)$ is very similar to $\lambda_*(50, s)$.

Coming back to our notion of partial tipping, what we have in this example is the following. For a small value of $c_0 > 0$, we have tracking for all the Eq. $(4.10)_{c_0}$ for $s \in \mathbb{R}$. More precisely, if c_0 is small enough, we have $\lambda_*(c_0, s) < -\varepsilon$ for all $s \in \mathbb{R}$ and an $\varepsilon > 0$. From the hull definition and the continuity of $s \mapsto \lambda_*(c_0, s)$ guaranteed by Theorem 2.12, it can be easily deduced that this means tracking for all the equations corresponding to c_0 and any q in the hull of p: we have total tracking on the hull. This means that the whole hyperbolic copiess2 of the base existing for the future and past families of equations on the hull are connected by the hyperbolic families existing for c_0. But at a certain value $c_1 > c_0$, $\lambda(c, s)$ is no longer

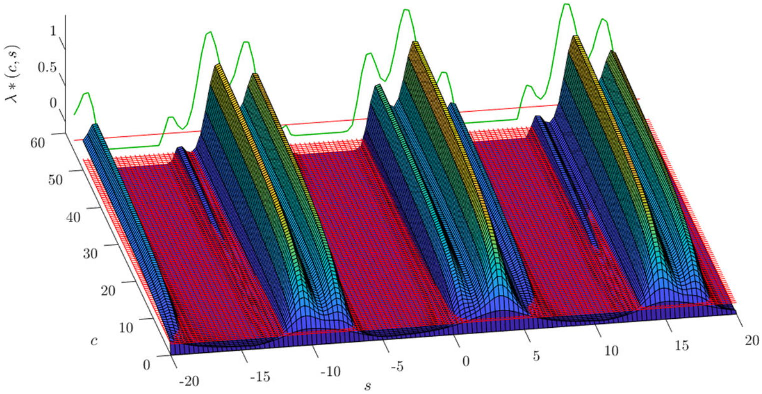

Fig. 2 Numerical simulation of the bifurcation map $\lambda_*(c, s)$ of $(4.10)_c$ (surface with gradient color). The red grid identifies the plane $\lambda = 0$. Consequently, the points of the surface below it correspond to CASE A, the points above to CASE C, and the points of intersection to CASE B. The curve in green is the graph of the bifurcation curve $\lambda^\infty(s)$ of (4.11) (see also Fig. 1). Theorem 5.3 guarantees the convergence of $\lambda_*(c, s)$ to $\lambda^\infty(s)$ as c increases. The figure indicates how fast this convergence is

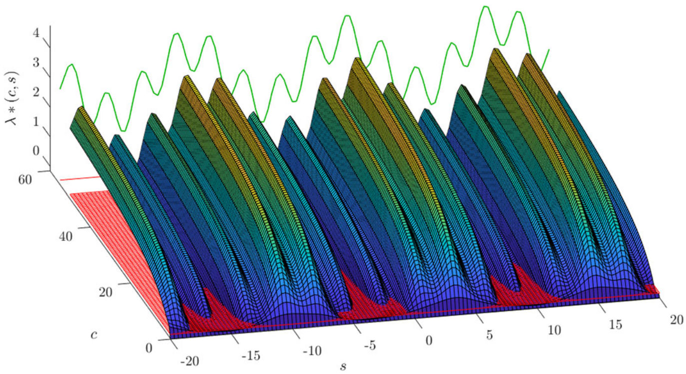

Fig. 3 Total tipping on the hull for $y' = -\big(y - 2\,\Gamma_c(t)\big)^2 + p_s(t)$ for large enough c

negative for all $s \in \mathbb{R}$, and for $c_2 > c_1$, it takes positive and negative values at non degenerate intervals. The functions $\widetilde{a}_s$ and $\widetilde{r}_s$ for which $\lambda_*(c_2, s)$ is negative, which are contained in the hyperbolic copies of the base for the past family, approach the hyperbolic families of the future as time increases (by the action of $(4.10)_{c_2}$). But the function $\widetilde{a}_s$ for which $\lambda_*(c_2, s)$ is positive gives rise to unbounded solutions (always under the action of $(4.10)_{c_2}$). Therefore, the global connection is lost. This is the phenomenon which we have called partial tipping on the hull. Observe that the continuity of the function $s \to \lambda_*(c_2, s)$ guaranteed by Theorem 2.12 ensures that CASE B also occurs for some $s \in \mathbb{R}$ in this situation. (Incidentally, observe also that Fig. 2 shows the existence of many values of $s \in \mathbb{R}$ such that tracking occurs for the all the systems in c-parametric family (4.10), since $\lambda(c, s) < 0$ for all $c \geq 0$.)

Figure 2 also indicates that partial tipping persists for $c \in (c_1, \infty) \cup \{\infty\}$. The simple modification of changing Γ by $2\,\Gamma$ gives rise to an always positive d^∞ (see the next paragraph to understand this phenomenon), and hence to an always positive λ^∞. This fact combined with the previous arguments of continuity of $s \mapsto \lambda_*(c, s)$ for a fixed c means the existence of a large enough value of c_3 (for 2Γ) such that the tipping is total on the hull for $c > c_3$. See Fig. 3.

We want to insist in the fact that Theorem 4.5 is the key to talk about partial and total tipping (or total tracking) on the hull for the family (4.11), which corresponds to $c = \infty$ (and hence also for the family $(4.10)_c$ if c is large enough). As explained in Remark 4.6, a suitable choice of Γ once p is fixed can determine this dynamical situation: partial tipping on the hull occurs when $s \mapsto \widetilde{a}(s) - \widetilde{r}(s)$ takes values greater and smaller than $\gamma^+ - \gamma^-$; when $\gamma_+ - \gamma_- < 0$, there is total tracking on the hull for (4.11); and, a large enough value of $\gamma^+ - \gamma^-$ guarantees the occurrence of total tipping.

The phenomenon that we have described admits also a different interpretation: the change of variable $s + t = l$ transforms (4.10) and (4.11) into

$$y' = -(y - \Gamma_c(l - s))^2 + p(l), \tag{4.12}$$

and

$$y' = -(y - \Gamma_\infty(l - s))^2 + p(l)\,, \tag{4.13}$$

respectively. Observe that (4.13) is obtained from (4.12) by taking limits as $c \to \infty$. Therefore, one can read Figs. 1 and 2 as a characterization of the dynamical scenario for $(4.12)_c$ depending on s for c sufficiently large. In particular, a time shift of the connecting function Γ can change the scenario, from the occurrence to the absence of rate-induced tipping.

We close this section by recalling that, recently, other notion of partial tipping has been described for some switched predator-prey models, in [3]: tipping as the climate varies occurs or not depending on the initial point of the phase space. The model is given by a Carathéodory equation, which can be understood as a limit of equations with bounded and uniformly continuous coefficients, as (4.11) is the limit of $(4.10)_c$ as $c \to \infty$. In this way, our analysis of partial tipping on the hull for (4.11) is, to some extent, related to that of [3].

5 Approaching Γ by Piecewise Uniformly Continuous Functions

Throughout this whole section, and unless otherwise indicated, $\Gamma\colon \mathbb{R} \to \mathbb{R}$ will be a (bounded and uniformly) continuous function such that there exist the real limits $\gamma_\pm := \lim_{t\to\pm\infty} \Gamma(t)$. From this map, we define $\Gamma_{\pm\infty}$ as in Eqs. (4.2) and (4.3), and Γ_c as in Eq. (4.1). We also define, for $c \in \mathbb{R}$ and $h > 0$,

$$\Gamma_c^0(t) := \Gamma_c(t)\,, \qquad \Gamma_{\pm\infty}^0(t) := \Gamma_{\pm\infty}(t)\,,$$
$$\Gamma_c^h(t) := \Gamma(cjh) \quad \text{if } \ t \in [\,jh\,, (j+1)h\,) \text{ for } j \in \mathbb{Z}\,,$$
$$\Gamma_\infty^h(t) := \begin{cases} \gamma_- & \text{if } \ t < 0\,, \\ \gamma_0 & \text{if } \ 0 \le t < h\,, \\ \gamma_+ & \text{if } \ t \ge h\,, \end{cases} \qquad \Gamma_{-\infty}^h(t) := \begin{cases} \gamma_+ & \text{if } \ t < 0\,, \\ \gamma_0 & \text{if } \ 0 \le t < h\,, \\ \gamma_- & \text{if } \ t \ge h\,. \end{cases}$$

Note that, if $h > 0$, then $\Gamma_c^h \in BPUC_{\mathbb{R}-\Delta_h}(\mathbb{R}, \mathbb{R})$ if $c \in \mathbb{R}$ for $\Delta_h := \{jh \mid j \in \mathbb{Z}\}$, and that $\Gamma_{\pm\infty}^h \in BPUC_{\mathbb{R}-\{0,h\}}(\mathbb{R}, \mathbb{R})$. We fix (also for the whole section) a BPUC function $p\colon \mathbb{R} \to \mathbb{R}$ and consider the equations

$$y' = -(y - \Gamma_c^h(t))^2 + p(t)\,. \tag{5.1}$$

We will represent by $(5.1)_{c,h}$ the equation (5.1) for fixed $c \in \mathbb{R} \cup \{\pm\infty\}$ and $h \ge 0$, and by $y_{c,h}(t, s, y_0)$ its maximal solution with value y_0 at $t = s$. Note that $(5.1)_{0,h}$, $(5.1)_{c,0}$, $(5.1)_{\infty,0}$ and $(5.1)_{-\infty,0}$ respectively coincide with (4.6), $(4.1)_c$, (4.2) and (4.3), that $(5.1)_{\pm\infty,h}$ play the role of limit equations for $(5.1)_{c,h}$ when $c \to \pm\infty$ for any fixed $h \geq 0$, that $(5.1)_{c,0}$ plays the role of limit equation for $(5.1)_{c,h}$ when $h \to 0$ for any fixed $c \in \mathbb{R} \cup \{\pm\infty\}$, and that (4.4) and (4.5) are the future and past equations of $(5.1)_{c,h}$ for all $c \in \mathbb{R} - \{0\}$ and $h \geq 0$. In this formulation, c is again the *rate* of the *transition function* Γ_c. The notions of tipping rate and transversal tipping rate of Definition 4.1 are extended without changes to the newly presented context of families $(5.1)_{c,h_0}$ (given by piecewise uniformly continuous transition functions Γ_c^h) for a fixed $h_0 \geq 0$.

Remark 5.1 Theorem 3.4 establishes a fundamental consequence of the existence of an attractor–repeller pair for (4.7) for the dynamics induced by $(5.1)_{c,h}$: the existence of the special functions $\mathfrak{a}_{c,h}$ and $\mathfrak{r}_{c,h}$ provided by Theorem 2.5 for any $c \in \mathbb{R} \cup \{\pm\infty\}$ and for any $h \geq 0$. In particular, under Hypothesis 4.2, the description made in Remarks 3.5 and 4.3.1 of the dynamics in CASE A (or tracking), B and C (or tipping) for $h = 0$ and $c \in (\mathbb{R}-\{0\}) \cup \{\pm\infty\}$ is also valid for any $h > 0$.

As indicated in Remark 4.3.2 for the continuous case, the information provided in the previous remark can be partially extended to some situations in which Hypothesis 4.2 does not hold, but instead an attractor–repeller pair for one of the Eq. $(5.1)_{\pm\infty,0}$ exists. It establishes the existence of a local pullback attractor and a local pullback repeller of $(5.1)_{c,h}$ for $\pm c > 0$ and $h \geq 0$, respectively connecting with the attractor and repeller of $(5.1)_{\pm\infty,0}$ as times decreases and increases, as well as the behavior of the rest of (at least) half-bounded solutions. If both equations have an attractor–repeller pair, the comments in Remarks 5.1 also apply.

Proposition 5.2 (i) *Assume that* $(4.2) = (5.1)_{\infty,0}$ *has an attractor–repeller pair,* $(\widetilde{\mathfrak{a}}_\infty, \widetilde{\mathfrak{r}}_\infty)$. *Then, there exist the functions* $\mathfrak{a}_{c,h}$ *and* $\mathfrak{r}_{c,h}$ *associated to* $(5.1)_{c,h}$ *by Theorem 2.5 for* $c \in (0,\infty) \cup \{\infty\}$ *and* $h \geq 0$, *and they satisfy:* $\lim_{t\to-\infty} |\mathfrak{a}_{c,h}(t) - \widetilde{\mathfrak{a}}_\infty(t))| = 0$, $\lim_{t\to-\infty} |y_{c,h}(t,s,y_0) - \widetilde{\mathfrak{r}}_\infty(t)| = 0$ *whenever* $y_0 < \mathfrak{a}_{c,h}(s)$, $\lim_{t\to+\infty} |\mathfrak{r}_{c,h}(t) - \widetilde{\mathfrak{r}}_\infty(t)| = 0$, *and* $\lim_{t\to+\infty} |y_{c,h}(t,s,y_0) - \widetilde{\mathfrak{a}}_\infty(t)| = 0$ *whenever* $y_0 > \mathfrak{r}_{c,h}(s)$.

(ii) *Assume that* $(4.3) = (5.1)_{-\infty,0}$ *has an attractor–repeller pair,* $(\widetilde{\mathfrak{a}}_{-\infty}, \widetilde{\mathfrak{r}}_{-\infty})$. *Then, there exist the functions* $\mathfrak{a}_{c,h}$ *and* $\mathfrak{r}_{c,h}$ *associated to* $(5.1)_{c,h}$ *by Theorem 2.5 for* $c \in \{-\infty\} \cup (-\infty, 0)$ *and* $h \geq 0$, *with* $\lim_{t\to-\infty} |\mathfrak{a}_{c,h}(t) - \widetilde{\mathfrak{a}}_{-\infty}(t)| = 0$, $\lim_{t\to-\infty} |y_{c,h}(t,s,y_0) - \widetilde{\mathfrak{r}}_{-\infty}(t)| = 0$ *whenever* $y_0 < \mathfrak{a}_{c,h}(s)$, $\lim_{t\to+\infty} |\mathfrak{r}_{c,h}(t) - \widetilde{\mathfrak{r}}_{-\infty}(t)| = 0$, *and* $\lim_{t\to+\infty} |y_{c,h}(t,s,y_0) - \widetilde{\mathfrak{a}}_{-\infty}(t)| = 0$ *whenever* $y_0 > \mathfrak{r}_{c,h}(s)$.

(iii) *In both cases, the solutions* $\mathfrak{a}_{c,h}$ *and* $\mathfrak{r}_{c,h}$ *are respectively locally pullback attractive and locally pullback repulsive.*

(iv) *Assume that* (4.3) *and* (4.2) *have attractor–repeller pairs. If* $\mathfrak{a}_{c,h}$ *and* $\mathfrak{r}_{c,h}$ *are globally defined and different, then they are uniformly separated, and hence* $(\widetilde{\mathfrak{a}}_{c,h}, \widetilde{\mathfrak{r}}_{c,h}) := (\mathfrak{a}_{c,h}, \mathfrak{r}_{c,h})$ *is an attractor–repeller pair for* $(5.1)_{c,h}$. *Consequently, if the Eq.* $(5.1)_{c,h}$ *does not have hyperbolic solutions, it has at most one bounded solution.*

Proof We assume the existence of an attractor–repeller pair $(\widetilde{\mathfrak{a}}_\infty, \widetilde{\mathfrak{r}}_\infty)$ for (4.2), and take $c \in (0,\infty) \cup \{+\infty\}$ and $h \geq 0$. We repeat the arguments of the proof of Theorem 3.4(i) and (ii) taking $(\widetilde{\mathfrak{a}}_\infty, \widetilde{\mathfrak{r}}_\infty)$ instead of $(\widetilde{\mathfrak{a}}_-, \widetilde{\mathfrak{r}}_-)$ as starting point, and working just on $(-\infty, 0]$. This requires to make now use of Proposition 2.2 instead of Proposition 2.1, which provides (3.5) just for $0 \geq t \geq s$. In this way, we prove the existence of $\mathfrak{a}_{c,h}$, as well as the limit behavior as time decreases of all the functions which are bounded at $-\infty$. To check the rest of the assertions in (i), we work on $[0,\infty)$. The proof of (ii) is analogous, and the proofs of (iii) and (iv) repeat those of (iii) and (iv) in Theorem 3.4. □

5.1 The Bifurcation Map $\lambda_*(c, h)$

For each $c \in \mathbb{R} \cup \{\pm\infty\}$ and $h \geq 0$, we represent by $\lambda_*(c, h)$ the value of the parameter associated to (5.1)$_{c,h}$ by Theorem 2.11; that is, the bifurcation value of $y' = -(y - \Gamma_c^h(t))^2 + p(t) + \lambda$,

$$\lambda_*(c, h) := \lambda^* \left(2\,\Gamma_c^h,\, p - (\Gamma_c^h)^2\right).$$

Theorem 5.3 *Let* $\lambda_* \colon (\mathbb{R} \cup \{\pm\infty\}) \times [0, \infty) \to \mathbb{R}$ *be defined as above. Then,*

(i) *the map* λ_* *is bounded: it takes values in* $[-||p||, ||p|| + ||\Gamma||^2]$.
(ii) *The map* λ_* *is jointly continuous.*

Proof (i) This assertion is a consequence of the first statement of Theorem 2.11, since $\|\Gamma\| \geq \left\|\Gamma_c^h\right\| \geq \left\|\Gamma_{\pm\infty}^h\right\|$ if $c \in \mathbb{R} - \{0\}$ and $h \geq 0$, and $\|\Gamma\| \geq \left\|\Gamma_0^h\right\|$ if $h \geq 0$.

(ii) We will prove (ii) in three steps.

Step 1. First we will check the continuity at (c_0, h_0) with $c_0 \in \mathbb{R} \cup \{\pm\infty\} - \{0\}$ and $h_0 \geq 0$. We begin by assuming $c_0 > 0$. Let us take a sequence $((c_k, h_k))_{k\geq 1}$ with limit (c_0, h_0), and assume without restriction that $c_k \geq c_0/2$ for all $k \geq 1$. What we do in the following paragraphs reproduces the ideas of the proof of Theorem 3.7(ii), where we also prove a continuity property for the bifurcation function. The reader is referred there for the details which we omit here.

We will associate below a suitable time $t_\varepsilon > 0$ to any $\varepsilon > 0$. Once fixed, we represent the index associated by Theorem 2.11 to

$$y' = -\big(y - (\Gamma_c^h)^\varepsilon(t)\big)^2 + p(t)\,, \quad \text{where} \quad (\Gamma_c^h)^\varepsilon(t) := \begin{cases} \gamma_- & \text{if} \quad t < -t_\varepsilon\,, \\ \Gamma_c^h(t) & \text{if} \quad -t_\varepsilon \leq t < t_\varepsilon\,, \\ \gamma_+ & \text{if} \quad t \geq t_\varepsilon \end{cases}$$

as $\lambda_\varepsilon(c, h)$. To check that $\lim_{k\to\infty} \lambda_*(c_k, h_k) = \lambda_*(c_0, h_0)$ we will prove that, given $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

1 $|\lambda_*(c_k, h_k) - \lambda_\varepsilon(c_k, h_k)| < \varepsilon$ for any k, including $k = 0$; and,
2 $\lim_{k\to\infty} \lambda_\varepsilon(c_k, h_k) = \lambda_\varepsilon(c_0, h_0)$.

Let us fix $\varepsilon > 0$ and prove **1**. Theorem 2.12 ensures that, for each $k \geq 1$, there exists $\delta_\varepsilon > 0$ such that, if $\left\|\Gamma_{c_k}^{h_k} - (\Gamma_{c_k}^{h_k})^\varepsilon\right\| \leq \delta_\varepsilon$, then $|\lambda_*(c_k, h_k) - \lambda_\varepsilon(c_k, h_k)| < \varepsilon$. The goal is finding t_ε and hence δ_ε such that this bound works for all $k \geq 0$. We look for $\kappa > 0$ and $\eta > 0$ such that $c_k \geq \kappa$ and $h_k \leq \eta$ for any $k \geq 0$, and for $t_\varepsilon = t_\varepsilon(\varepsilon, \kappa, \eta) > \eta$ such that $|\Gamma(t) - \gamma_-| \leq \delta_\varepsilon$ if $t \leq -\kappa\, t_\varepsilon$ and $|\Gamma(t) - \gamma_+| \leq \delta_\varepsilon$ if $t \geq \kappa\, t_\varepsilon$. If $t \leq -t_\varepsilon$, then $c_k t \leq -\kappa\, t_\varepsilon$, so that $|\Gamma_{c_k}^0(t) - \gamma_-| \leq \delta_\varepsilon$; and if, in addition, $h_k > 0$ and $t \in [jh_k, (j+1)h_k)$, then $jh_kc_k \leq c_kt \leq -\kappa\, t_\varepsilon$, so that $|\Gamma_{c_k}^{h_k}(t) - \gamma_-| \leq \delta_\varepsilon$. If $t \geq t_\varepsilon$, then $c_k t \geq \kappa\, t_\varepsilon$, so that $|\Gamma_{c_k}^0(t) - \gamma_+| \leq \delta_\varepsilon$; and if, in addition, $h_k > 0$ and $t \in [jh_k, (j+1)h_k)$, then $jh_kc_k \geq c_kt \leq \kappa\, t_\varepsilon$, so that $|\Gamma_{c_k}^{h_k}(t) - \gamma_+| \leq \delta_\varepsilon$. Hence, the time t_ε is fixed once $\varepsilon > 0$ is fixed, and **1** is proved.

Let us fix $\varepsilon > 0$, which determines $t_\varepsilon > 0$ and hence λ_ε. To prove **2**, it suffices to check that

2.1 given $\lambda < \lambda_\varepsilon(c_0, h_0)$, there exists k_0 such that if $k \geq k_0$ then $\lambda \leq \lambda_\varepsilon(c_k, h_k)$,
2.2 given $\lambda > \lambda_\varepsilon(c_0, h_0)$, there exists k_0 such that if $k \geq k_0$ then $\lambda \geq \lambda_\varepsilon(c_k, h_k)$.

The proof of **2.1** reproduces without changes that of point **1** of Theorem 3.7(ii). The same happens with the idea to prove **2.2** and point **2** of that theorem. We take $\bar{\lambda} > \lambda_\varepsilon(c_0, h_0)$, so that the equation

$$y' = -\big(y - (\Gamma_c^h)^\varepsilon(t)\big)^2 + p(t) + \bar\lambda\,, \tag{5.2}$$

corresponding to $(c, h) = (c_0, h_0)$ has an attractor–repeller pair $(\widetilde{\mathfrak{a}}_0, \widetilde{\mathfrak{r}}_0)$. The goal is to check the existence of at least one bounded solution for $(5.2)^\varepsilon_{c_k,h_k}$ if k is large enough, what we achieve by checking the existence corresponding functions $\mathfrak{a}^k_\varepsilon$ and $\mathfrak{r}^k_\varepsilon$ given by Theorem 2.5 at least on $(-\infty, t_\varepsilon]$ and $[t_\varepsilon, \infty)$, with $\mathfrak{a}^k_\varepsilon(t_\varepsilon) \ge \mathfrak{r}^k_\varepsilon(t_\varepsilon)$.

Observe that the coefficients of the Eq. $(5.2)^\varepsilon_{c_k,h_k}$ are common for any $k \ge 0$ outside the interval $[-t_\varepsilon, t_\varepsilon]$. This fact allows us to repeat the procedure followed to prove **2** in Theorem 3.7(ii) in order to check the previous assertion. This completes STEP 1 for $c_0 > 0$, and the proof is analogous if $c_0 < 0$.

STEP 2. We will prove that $\lim_{c\to 0} \lambda_*(c, 0) = \lambda_*(0, 0)$. Recall that $\Gamma^0_c(t) = \Gamma(c\,t)$. Let us assume first that Γ is C^1 with $\Gamma' \colon \mathbb{R} \to \mathbb{R}$ bounded. For each $c \in \mathbb{R}$, the change of variables $x = y - \Gamma(c\,t)$ takes equation $y' = -(y - \Gamma(c\,t))^2 + p(t) + \lambda$ to $x' = -x^2 - c\,\Gamma'(c\,t) + p(t) + \lambda$, and transforms bounded solutions in bounded solutions. Therefore, the role of $\lambda_*(c, 0)$ does not change. Let us take $c \ne 0$, and let b_c be a bounded solution for $x' = -x^2 - c\,\Gamma'(c\,t) + p(t) + \lambda_*(c, 0)$. Then $b'_c(t) \le -b^2_c(t) + p(t) + |c|\,\|\Gamma'\| + \lambda_*(c, 0)$, so that Theorem 2.5(v) and Theorem 2.11(i) ensure that $\lambda_*(0, 0) \le |c|\,\|\Gamma'\| + \lambda_*(c, 0)$; that is, $\lambda_*(0, 0) - \lambda_*(c, 0) \le |c|\,\|\Gamma'\|$. Now, let $b_0(t)$ be a bounded solution of $x' = -x^2 + p(t) + \lambda_*(0, 0)$, so that $b'_0(t) \le -x^2 + p(t) + |c|\,\|\Gamma'\| + \lambda_*(0, 0)$. Reasoning as before, we get $\lambda_*(c, 0) - \lambda_*(0, 0) \le |c|\,\|\Gamma'\|$. Consequently, $|\lambda_*(c, 0) - \lambda_*(0, 0)| \le |c|\,\|\Gamma'\|$, which proves the assertion in this case.

Still in STEP 2, we look for a sequence $(\Gamma_n)_{n\ge 1}$ of bounded C^1 functions with bounded derivatives and such that $\lim_{n\to\infty} \Gamma_n = \Gamma$ uniformly on $\mathbb{R}$. This can be easily done since Γ is asymptotically constant and $C^1(\mathcal{I}, \mathbb{R})$ is dense in $C(\mathcal{I}, \mathbb{R})$ for any compact interval $\mathcal{I}$. We represent by $\lambda^*_n(c, 0)$ the parameter associated to the equation $y' = -(y - \Gamma_n(c\,t))^2 + p(t)$ by Theorem 2.11. Then $|\lambda_*(c, 0) - \lambda_*(0, 0)| \le |\lambda_*(c, 0) - \lambda^*_n(c, 0)| + |\lambda^*_n(c, 0) - \lambda^*_n(0, 0)| + |\lambda^*_n(0, 0) - \lambda_*(0, 0)|$. Let us take $\varepsilon > 0$. Note that $\sup_{t\in\mathbb{R}} |\Gamma_n(c\,t) - \Gamma(c\,t)| \le \|\Gamma_n - \Gamma\|$ for any $c \in \mathbb{R}$ (in fact they are equal if $c \ne 0$). Theorem 2.12 provides $n_0 \in \mathbb{N}$ such that $\|\Gamma_{n_0} - \Gamma\|$ is small enough as to guarantee that $|\lambda^*_{n_0}(c, 0) - \lambda_*(c, 0)| \le \varepsilon/3$ for any $c \in \mathbb{R}$. Besides, we have proved in the previous paragraph that $|\lambda^*_{n_0}(c, 0) - \lambda^*_{n_0}(0, 0)| \le |c|\,\|\Gamma'_{n_0}\|$. Let us take $c_0 > 0$ such that if $|c| \le c_0$ then $|c|\,\|\Gamma'_{n_0}\| \le \varepsilon/3$. Altogether, we have $|\lambda_*(c, 0) - \lambda_*(0, 0)| \le \varepsilon$ if $|c| \le c_0$, and this completes the second step.

STEP 3. Note now that $\lambda_*(0, h_0) = \lambda_*(0, 0)$, since $\Gamma^{h_0}_0 = \Gamma^0_0 \equiv \Gamma(0)$. Therefore, in the third and last step we will prove that, if the sequence $((c_k, h_k))_{k\ge 1}$ tends to $(0, h_0)$, with $h_0 \ge 0$, then $\lim_{k\to\infty} \lambda_*(c_k, h_k) = \lambda_*(0, 0)$. We write $|\lambda_*(c_k, h_k) - \lambda_*(0, 0)| \le |\lambda_*(c_k, h_k) - \lambda_*(c_k, 0)| + |\lambda_*(c_k, 0) - \lambda_*(0, 0)|$. We have proved in the second step that $\lim_{k\to\infty} |\lambda_*(c_k, 0) - \lambda_*(0, 0)| = 0$. In addition, $\lim_{k\to\infty} \left\|\Gamma^{h_k}_{c_k} - \Gamma^0_{c_k}\right\| = 0$, since $|c_k h_k j - c_k t| \le |c_k h_k| \to 0$ if $t \in [j h_k, (j + 1)h_k)$ for a $j \in \mathbb{Z}$ and Γ is uniformly continuous. Hence, Theorem 2.12 ensures that $\lim_{k\to\infty} |\lambda_*(c_k, h_k) - \lambda_*(c_k, 0)| = 0$, and this completes the proof of (ii). □

Remarks 5.4 1. Observe that Theorem 2.11 and the definition of tipping rate given at the beginning of Sect. 5 (which repeats Definition 4.1) ensure that, for a fixed value of $h_0 \ge 0$, c_0 is a tipping rate for the c-parametric family of Eq. $(5.1)_{c,h_0}$ if $\lambda_*(c_0, h_0) = 0$ and there is $\delta > 0$ such that $\lambda_*(c, h_0) < 0$ either for $c \in (c_0 - \delta, c_0)$ or for $c \in (c_0, c_0 + \delta)$; and that the tipping rate is transversal if, in addition, $\lambda_*(c, h_0) > 0$ either for $c \in (c_0 - \delta, c_0)$ or for $c \in (c_0, c_0 + \delta)$. This characterization combined with the just proved joint continuity of λ_* shows that, at a tipping rate c_0, the graph of the continuous map $c \mapsto \lambda_*(c, h_0)$ reaches the horizontal axis coming from negative values to the left side or to the right side of c_0; and it crosses the horizontal axis transversally at c_0 if the tipping rate is transversal.

2. Assume that the family $(4.1)_c = (5.1)_{c,0}$ has a transversal tipping rate at c_0, passing from CASE A to C as c increases. This means the existence of $\delta > 0$ such that $\lambda_*(c, 0) < 0$ for $c \in (c_0 - \delta, c_0)$ and $\lambda_*(c, 0) > 0$ for $c \in (c_0, c_0 + \delta)$. In particular, $\lambda_*(c_0, 0) = 0$. The continuity of $\lambda_*(c, h)$ ensures the existence of $h_0 > 0$ such that $\lambda_*(c_0 - \delta, h) > 0$ and $\lambda_*(c_0 + \delta, h) < 0$ for every $h \in [0, h_0]$. Therefore, $c(h) := \min\{c \in (c_0 - \delta, c_0 + \delta) \mid \lambda_*(c, h) = 0\}$ is a (non necessarily transversal) tipping rate of the c-parametric family $(5.1)_{c,h}$, and in addition $\lim_{h\to 0^+} c(h) = c_0$. In consequence, every transversal tipping rate of $(4.1)_c$ can be approximated by tipping rates of the piecewise continuous transition equations $(5.1)_{c,h}$ as $h \to 0^+$. The other type of rate-induced transversal tipping leads to the same conclusion.

The next theorem, which includes and extends Theorem 4.5, combines Hypothesis 4.2 with the continuity of λ_* in order to analyze the dynamical case of $(5.1)_{c,h}$ for small and large values of $|c|$ and a fixed $h \ge 0$. We represent by $x(t, s, x_0)$ the solution of $x' = -x^2 + p(t)$ with value x_0 at $t = s$.

Theorem 5.5 *Assume Hypothesis 4.2, and let $(\widetilde{a}_\lambda, \widetilde{r}_\lambda)$ be the attractor–repeller pair for $x' = -x^2 + p(t) + \lambda$ for $\lambda > \lambda_*(0)$. Let us fix $h \ge 0$. Then,*

(i) *there exists $c_{0,h} > 0$ such that $(5.1)_{c,h}$ is in CASE A for $c \in (-c_{0,h}, c_{0,h})$.*
(ii) *If there exists $x(h, 0, \widetilde{a}(0) + \gamma_- - \gamma_0) > \widetilde{r}(h) + \gamma_+ - \gamma_0$, then the Eq. $(5.1)_{\infty,h}$ has an attractor–repeller pair $(\widetilde{\mathfrak{a}}_{\infty,h}, \widetilde{\mathfrak{r}}_{\infty,h})$. In this case, there exists a minimum $c_{M,h} \ge 0$ such that $(5.1)_{c,h}$ is in CASE A for $c > c_{M,h}$.*
(iii) *If $x(h, 0, \widetilde{a}(0)+\gamma_- -\gamma_0)$ does not exist, or if $x(h, 0, \widetilde{a}(0)+\gamma_- -\gamma_0) < \widetilde{r}(h)+\gamma_+ -\gamma_0$, then the Eq. $(5.1)_{\infty,h}$ has no bounded solutions. In this case, there is a minimum $c^*_{M,h} > 0$ such that $(5.1)_{c,h}$ is in CASE C for $c > c^*_{M,h}$. In addition, if $\widetilde{a}(0) + \gamma_- < \widetilde{r}(0) + \gamma_+$, then $\lambda_*(\infty, 0) = \lambda_\infty$, where $\lambda_\infty > 0$ is the unique value of the parameter such that $\widetilde{a}_{\lambda_\infty}(0) - \widetilde{r}_{\lambda_\infty}(0) = \gamma_+ - \gamma_-$.*
(iv) *If there exists $x(h, 0, \widetilde{a}(0)+\gamma_+ -\gamma_0) > \widetilde{r}(h)+\gamma_- -\gamma_0$, then the Eq. $(5.1)_{-\infty,h}$ has an attractor–repeller pair $(\widetilde{\mathfrak{a}}_{-\infty,h}, \widetilde{\mathfrak{r}}_{-\infty,h})$. In this case, there exists a maximum $c_{m,h} \le 0$ such that $(5.1)_{c,h}$ is in CASE A for $c < c_{m,h}$.*
(v) *If $x(h, 0, \widetilde{a}(0)+\gamma_+ -\gamma_0)$ does not exits, or if $x(h, 0, \widetilde{a}(0)+\gamma_+ -\gamma_0) > \widetilde{r}(h)+\gamma_- -\gamma_0$ then the Eq. $(5.1)_{-\infty,h}$ has no bounded solutions. In this case, there is a maximum $c^*_{m,h} > 0$ such that $(5.1)_{c,h}$ is in CASE C for $c < c^*_{m,h}$. In addition, if $\widetilde{a}(0) + \gamma_+ < \widetilde{r}(h) + \gamma_-$, then $\lambda_*(-\infty, 0) = \lambda_{-\infty} > 0$, where $\lambda_{-\infty}$ is the unique value of the parameter such that $\widetilde{a}_{\lambda_{-\infty}}(0) - \widetilde{r}_{\lambda_{-\infty}}(0) = \gamma_- - \gamma_+$.*

Proof (i) Hypothesis 4.2 ensures that $\lambda_*(0, h) < 0$ for any $h \ge 0$, and hence (i) is a trivial consequence of the continuity of λ_*.

(ii) The goal is proving that the functions $\mathfrak{a}_{\infty,h}$ and $\mathfrak{r}_{\infty,h}$ associated to $(5.1)_{\infty,h}$ by Theorem 2.5 form an attractor–repeller pair if $x(h, 0, \widetilde{a}(0) + \gamma_- - \gamma_0) > \widetilde{r}(h) + \gamma_+ - \gamma_0$. In these conditions, Theorem 2.11 ensures $\lambda_*(\infty, h) < 0$, and hence the continuity established in Theorem 4.4 provides the value $c_{M,h}$ of statement (ii).

The existence of $(\widetilde{a}, \widetilde{r})$ ensures that of the attractor-repeler pairs $(\widetilde{\mathfrak{a}}_\pm, \widetilde{\mathfrak{r}}_\pm) := (\widetilde{a}+\gamma_\pm, \widetilde{r}+\gamma_\pm)$ for the future and past equations $y' = -(y - \gamma_\pm)^2 + p(t)$: see Remark 3.3. Let us prove that $\mathfrak{a}_{\infty,h}(t) = \widetilde{\mathfrak{a}}_-(t)$ for all $t \le 0$. First, we observe that the existence of $\mathfrak{a}_{\infty,h}$ on $(-\infty, 0]$ is guaranteed by the existence of a solution of $(5.1)_{\infty,h}$ bounded on $(-\infty, 0]$, which is the case of $y_{\infty,h}(t, 0, \widetilde{\mathfrak{a}}_-(0))$: it coincides with $\widetilde{\mathfrak{a}}_-(t)$ for $t \le 0$, since $\Gamma^h_\infty(t) = \gamma_-$ for $t < 0$. This fact also proves that $\mathfrak{a}_{\infty,h}(t) \ge \widetilde{\mathfrak{a}}_-(t)$ for $t \le 0$. To prove the converse inequality, we take

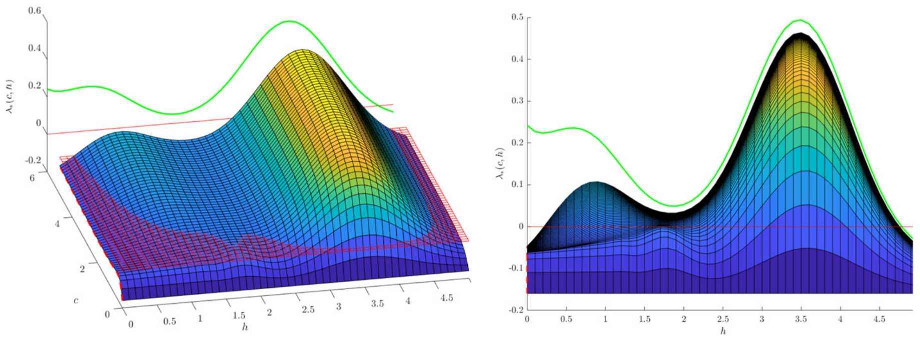

Fig. 4 Numerical simulation of the bifurcation map $\lambda_*(c, h)$ of $(5.1)_{c,h}$ for Γ and p as in (4.9) and $c, h \in [0, 5]$. On the left: the gradient surface represents the graph of $\lambda_*(c, h)$; the red grid identifies the plane $\lambda_* = 0$: the points of the surface below this plane correspond to Case A, the points above to Case C, and the points of intersection to Case B. The red dashed line is the graph of the bifurcation curve $\lambda_*(c, 0) = \lambda_*(c)$ of the family (4.1), whereas the solid green line is the graph of the bifurcation curve $\lambda_*(\infty, h)$ of $(5.1)_{\infty,h}$, represented at $c = 6$ for convenience. On the right: a projection of the same picture on the plane $c = 0$

$s \le 0$ and $y_0 > \mathfrak{a}_{\infty,h}(s)$. Then, the solution $y_-(t, s, y_0)$ of $y' = -(y - \gamma_-)^2 + p(t)$, which coincides with $y_{\infty,h}(t, s, y_0)$ for $t \le 0$, is unbounded as t decreases, so that $y_0 > \widetilde{\mathfrak{a}}_-(s)$ and hence $\mathfrak{a}_{\infty,h}(s) \ge \widetilde{\mathfrak{a}}_-(s)$.

The same argument allows us check that $\mathfrak{r}_{\infty,h}(t) = \widetilde{\mathfrak{r}}_+(t)$ for $t \ge h$. Note also that for those values of $t \in [0, h]$ for which $\mathfrak{a}_{\infty,h}(t)$ exists, it coincides with $x(t, 0, \widetilde{\mathfrak{a}}_-(0) - \gamma_0) + \gamma_0$. These previous properties and the existence and inequality assumed in (ii) ensure that $\mathfrak{a}_{\infty,h}(h) = x(h, 0, \widetilde{\mathfrak{a}}_-(0) - \gamma_0) + \gamma_0 > \widetilde{\mathfrak{r}}_+(h) = \mathfrak{r}_{\infty,h}(h)$. According to Remark 2.6, (4.2) has at least two bounded solutions, and hence the information provided by Remark 3.5 ensures that it has an attractor–repeller pair. This completes the proof of (ii).

(iii) Under the assumptions on (iii), and according to the proof of (ii), either $\mathfrak{a}_{\infty,h}(h)$ does not exist or we have $\mathfrak{a}_{\infty,h}(h) < \mathfrak{r}_{\infty,h}(h)$. This precludes the existence of globally bounded solutions for $(5.1)_{\infty,h}$, and hence Theorem 2.11 ensures that $\lambda_*(\infty) > 0$. This fact combined with the hypothesis $\lambda_*(0) < 0$ and the continuity of λ_* ensures the existence of a maximum $c^*_M > 0$ with $\lambda_*(c^*_M) = 0$, which proves the first assertion in (iii).

Assume now that $h = 0$ and that $\widetilde{a}(0) + \gamma_- < \widetilde{r}(0) + \gamma_+$. Theorem 2.11(ii) ensures the existence of a unique value $\lambda_\infty > 0$ of the parameter with $\widetilde{a}_{\lambda_\infty}(0) - \widetilde{r}_{\lambda_\infty}(0) = \gamma_+ - \gamma_-$. We repeat the arguments of the proof of (ii) taking as starting point the attractor–repeller pair $(\widetilde{a}_{\lambda_\infty}, \widetilde{r}_{\lambda_\infty})$ of $x' = -x^2 + p(t) + \lambda_\infty$ in order to conclude that the functions $\bar{a}_{\infty,0}$ and $\bar{r}_{\infty,0}$ associated to the equation $y' = -(y - \Gamma^0_\infty(t))^2 + p(t) + \lambda_\infty$ by Theorem 2.5 satisfy $\bar{a}_{\infty,0}(0) = \widetilde{a}_{\lambda_\infty}(0) + \gamma_- = \widetilde{r}_{\lambda_\infty}(0) + \gamma_+ = \bar{r}_{\infty,0}(0)$. This ensures that $y' = -(y - \Gamma^0_\infty(t))^2 + p(t) + \lambda_\infty$ has a unique bounded solution, which in turn yields $\lambda_*(\infty) = \lambda_\infty$, as asserted.

(iv) and (v) The arguments to prove these properties are the analogues of those previously used. □

Remark 5.6 Theorems 5.5 and 3.4 show that the Eq. $(5.1)_{\infty,h}$ (or $(5.1)_{-\infty,h}$) has only a bounded solution if and only if there exists $x(h, 0, \widetilde{a}(0) + \gamma_- - \gamma_0) > \widetilde{r}(h) + \gamma_+ - \gamma_0$ (or $x(h, 0, \widetilde{a}(0) + \gamma_+ - \gamma_0) > \widetilde{r}(h) + \gamma_- - \gamma_0$). But $\lambda_*(\infty, h) = 0$ (or $\lambda_*(-\infty, h) = 0$) does not take us to any conclusion for large values of c (or $-c$).

We complete this subsection by adapting to Eq. (5.1) part of the information obtained in Sect. 3.1. Observe that the value $\lambda_*(0)$ appearing in the next statement coincides with $\lambda_*(0, h)$

for any $h \geq 0$, since $\Gamma_0^h(t) \equiv \gamma_0 := \Gamma(0)$ and dynamics of $x' = -x^2 + p(t)$ and $y' = -(y - \gamma_0)^2 + p(t)$ are identical.

Proposition 5.7 *Let $\lambda_*(0) := \lambda^*(0, p)$ be the value associated to $x' = -x^2 + p(t)$ by Theorem 2.11.*

(i) *If p is recurrent, then $\lambda_*(0) \leq \lambda_*(c, h)$ for all $c \in \mathbb{R}$ and $h \geq 0$.*
(ii) *If Γ is nondecreasing (resp. nonincreasing), then*

$$\lambda_*(-c, h) \leq \lambda_*(0) \leq \lambda_*(c, h), \quad (\textit{resp. } \lambda_*(c, h) \leq \lambda_*(0) \leq \lambda_*(-c, h))$$

for all $c > 0$ and $h \geq 0$. If, in addition, p is recurrent, then: $\lambda_*(0) = \lambda_*(-c, h)$ (resp. $\lambda_*(0) = \lambda_*(c, h)$) for all $c > 0$ and $h \geq 0$; and $\lambda_*(0) < \lambda_*(c, 0)$ (resp. $\lambda_*(c, 0) < \lambda_*(0)$) if the equation $x' = -x^2 + p(t) + \lambda^*(0, p)$ has just one bounded solution.

Proof Statement (i) is a direct consequence of Proposition 3.6. The properties of (ii) follow from Corollary 3.8(ii), having in mind that Γ_c^h and $-\Gamma_{-c}^k$ are nondecreasing for $c > 0$ if Γ is nondecreasing, and that Γ_{-c}^h and $-\Gamma_c^k$ are nondecreasing for $c > 0$ if Γ is nonincreasing. □

The bifurcation map $\lambda_*(c, h)$ of $(5.1)_{c,h}$ when Γ and p are given by (4.9) and $c, h \in [0, 5]$ is depicted in Fig. 4. Besides the joint continuity of λ_*, observe that the section map $h \mapsto \lambda^*(c, h)$ is not increasing for a fixed $c > 0$, unlike the map $h \mapsto \Gamma_c^h$.

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Declarations

Conflict of interest All authors declare that they have no conflicts of interest.

Appendix A. Compactness of the Hull and Continuity of the Flow

Hereby, we recall some facts on nonautonomous equations of the type (2.1),

$$x' = -x^2 + q(t)\,x + p(t)\,, \tag{A.1}$$

where p and q are BPUC functions: see Definition 2.7. The first objective is proving Theorems 2.9 and 2.11, which rely on Theorem A.2 (in turn based on Theorem A.1). The second one is to prove Theorem A.3, a result on continuous variation of the solutions with respect to the coefficients which we have used several times.

Let $\Delta = \{a_j \in \mathbb{R} \mid j \in \mathbb{Z}\} \subset \mathbb{R}$ be a disperse set (see Sect. 2.3). Recall that $q : \mathbb{R} \to \mathbb{R}$ belongs to $BPUC_\Delta(\mathbb{R}, \mathbb{R})$ if and only it is right-continuous and

c1 there is $c > 0$ such that $|q(t)| < c$ for all $t \in \mathbb{R}$;
c2 for all $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that, if $t_1, t_2 \in (a_j, a_{j+1})$ for some $j \in \mathbb{Z}$ and $t_2 - t_1 < \delta$, then $|q(t_2) - q(t_1)| < \varepsilon$.

Recall also a function $q \in BPUC(\mathbb{R},\mathbb{R})$ is a finite sum of a finite number of functions $q_i \in BPUC_{\Delta_i}(\mathbb{R},\mathbb{R})$, for possibly different disperse sets Δ_i. It is clear that $BPUC(\mathbb{R},\mathbb{R}) \subset L^\infty(\mathbb{R},\mathbb{R}) \subset L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$. Recall that $L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ is a complete metric space for the distance defined by

$$d(q_1,q_2) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\int_{-k}^{k} |q_1(t)-q_2(t)|\,dt}{1+\int_{-k}^{k} |q_1(t)-q_2(t)|\,dt} \,.$$

In addition, for every $q \in BPUC(\mathbb{R},\mathbb{R})$ and $s \in \mathbb{R}$, the shift $q_s\colon \mathbb{R} \to \mathbb{R}$ defined by $q_s(t) := q(t+s)$, belongs to $L^\infty(\mathbb{R},\mathbb{R})$ and has norm $\|q\|$. We define

$$\Omega_q := \mathrm{closure}_{L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R})}\{q_t \mid t \in \mathbb{R}\} \subset L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}) \cap L^\infty(\mathbb{R},\mathbb{R})\,.$$

The set Ω_q is the *hull* of q (in $L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$). Theorem A.1 shows that Ω_q is a compact metric space, and that the time-translation map

$$\sigma\colon \mathbb{R}\times\Omega_q \to \Omega_q\,,\ \ (t,\omega)\mapsto \omega_t\,, \qquad \text{with}\ \ \omega_t(s) := \omega(t+s)$$

defines a (real) continuous flow on Ω_q. Recall that being a flow means that $\sigma_0 = \mathrm{Id}$ and $\sigma_{s+t} = \sigma_t \circ \sigma_s$ for each $s,t \in \mathbb{R}$, where $\sigma_t(\omega) := \sigma(t,\omega)$.

Theorem A.1 *Let $q\colon \mathbb{R}\to\mathbb{R}$ belong to $BPUC(\mathbb{R},\mathbb{R})$. Then its hull Ω_q is a compact subset of $L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$, and σ defines a continuous flow.*

Proof Let $\Delta \subset \mathbb{R}$ be a disperse set, and let us take $q \in BPUC_\Delta(\mathbb{R},\mathbb{R})$. We will first prove the compactness of Ω_q in this case. Note that, since q is bounded, there exists a common bound for all the shifts q_s. Therefore, and according to [40, Theorem 1], to prove the compactness of Ω_δ it suffices to show that given $\varepsilon > 0$ and a compact interval $\mathcal{I} \subset \mathbb{R}$ there exists $\delta = \delta(\varepsilon,q,\mathcal{I}) > 0$ such that, for any $s \in \mathbb{R}$,

$$\int_{\mathcal{I}} |q_s(t+\tau)-q_s(t)|\,dt < \varepsilon \quad \text{whenever}\ |\tau| < \delta\,. \tag{A.2}$$

Let us write $\Delta = \{a_j \in \mathbb{R} \mid j \in \mathbb{Z}\}$ and define $h := \inf_{j\in\mathbb{Z}}(a_{j+1}-a_j) > 0$. We fix $\varepsilon > 0$, a non-degenerate interval $\mathcal{I} = [r_1,r_2]$, and $s \in \mathbb{R}$, and look for $a_{k+1}, a_{k+m-1} \in \Delta$ (depending on s), if they exist, such that $[a_{k+1},a_{k+m-1}] \subseteq (r_1+s, r_2+s) \subset [a_k, a_{k+m}]$. We set $\widetilde{a}_k := r_1+s$, $\widetilde{a}_j := a_j$ for $j = k+1,\ldots,k+m-1$ and $\widetilde{a}_{k+m} := r_2+s$, so that $\widetilde{a}_{j+1}-\widetilde{a}_j \le r_2-r_1$ for $j = k,\ldots,k+m-1$. Then,

$$\int_{\mathcal{I}} |q_s(t+\tau)-q_s(t)|\,dt = \int_{\mathcal{I}+s} |q(t+\tau)-q(t)|\,dt \le \sum_{j=k}^{k+m-1} \int_{\widetilde{a}_j}^{\widetilde{a}_{j+1}} |q(t+\tau)-q(t)|\,dt\,.$$

Condition **c1** gives $\delta_1 = \delta_1(\varepsilon,q,\mathcal{I}) > 0$ such that, if $0 \le \tau < \delta_1$ and $d \in \mathbb{R}$, then

$$\int_{d-\tau}^{d} |q(t+\tau)-q(t)|\,dt < \frac{\varepsilon}{2m}\,.$$

In addition, **c2** provides $\delta_2 = \delta_2(\varepsilon,q,\mathcal{I}) \in (0,h]$ such that, if $0 < \tau < \delta_2$, then

$$|q(t+\tau)-q(t)| < \frac{\varepsilon}{2m(r_2-r_1)} \quad \text{for all} \quad j \in \mathbb{Z} \quad \text{and all} \quad t \in (a_j, a_{j+1}-\tau)\,.$$

Let us call $\delta := \min(\delta_1, \delta_2)$ and fix $\tau \in [0, \delta)$. For $j \in \{k+1, \dots, k+m-2\}$ (if there is any),

$$\int_{\widetilde{a}_j}^{\widetilde{a}_{j+1}} |q(t+\tau) - q(t)|\, dt = \int_{a_j}^{a_{j+1}-\tau} |q(t+\tau) - q(t)|\, dt + \int_{a_{j+1}-\tau}^{a_{j+1}} |q(t+\tau) - q(t)|\, dt$$
$$< \frac{\varepsilon}{2m(r_2 - r_1)}\,(a_{j+1} - \tau - a_j) + \frac{\varepsilon}{2m} \le \frac{\varepsilon}{m}\,.$$

In the case or cases $j = k$ and $j = k+m-1$, the length of the interval $[\widetilde{a}_j, \widetilde{a}_{j+1}]$ can be less than h, and hence greater than τ. We proceed in the same way as before if $\widetilde{a}_j \le \widetilde{a}_{j+1} - \tau$, getting the bound ε/m; if this is not the case, we forget about $\int_{\widetilde{a}_j}^{\widetilde{a}_{j+1}-\tau}$ (which is negative), getting $\varepsilon/(2m)$ as a bound. It follows that (A.2) holds for $0 \le \tau < \delta$. To work with $\tau < 0$, we write $\int_{\widetilde{a}_j}^{\widetilde{a}_{j+1}} = \int_{\widetilde{a}_j}^{\widetilde{a}_j-\tau} + \int_{\widetilde{a}_j-\tau}^{\widetilde{a}_{j+1}}$ and use the same arguments. This proves the compactness of Ω_q in the case $q \in BPUC_\Delta(\mathbb{R}, \mathbb{R})$.

To extend the result to the general BPUC case, it is enough to observe that (A.2) holds for $q = q_1 + \cdots + q_n$ if it holds for every q_i. It is also easy to check that σ defines a flow on Ω_q, and its continuity follows from [41, Theorem III.11]. □

The function $f(x,t) := -x^2 + q(t)\,x + p(t)$ giving rise to (A.1) is Lipschitz Carathéodory whenever $q, p \in BPUC(\mathbb{R}, \mathbb{R})$. Recall that a function $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is said to be *Lipschitz Carathéodory*, which we represent by $f \in \mathfrak{LC}(\mathbb{R}^2, \mathbb{R})$, if

- f is Borel measurable,
- for every compact interval $\mathcal{I} \subset \mathbb{R}$ there exists a function $m^{\mathcal{I}} \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ such that $|f(t,x)| \le m^{\mathcal{I}}(t)$ for any $x \in \mathcal{I}$ and almost every $t \in \mathbb{R}$,
- for every compact interval $\mathcal{I} \subset \mathbb{R}$ there exists a function $l^{\mathcal{I}} \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ such that $|f(t,x_1) - f(t,x_2)| \le l^{\mathcal{I}}(t)|x_1 - x_2|$ for any $x_1, x_2 \in \mathcal{I}$ and almost every $t \in \mathbb{R}$.

We endow the set $\mathfrak{LC}(\mathbb{R}^2, \mathbb{R})$ with the $\mathcal{T}_{\mathbb{Q}}$ topology, which is generated by the countable family of seminorms

$$n_{[r_1,r_2],s}(f) = \int_{r_1}^{r_2} |f(t,s)|\, dt \quad \text{for} \quad r_1,\, r_2,\, s \in \mathbb{Q} \quad \text{with} \quad r_1 < r_2\,,$$

and for which $\mathfrak{LC}(\mathbb{R}^2, \mathbb{R})$ is a locally convex metric space: see e.g. [40].

The results on existence, uniqueness, and basic properties of the solutions of the initial value problems of equations $x' = f(t,x)$ for $f \in \mathfrak{LC}(\mathbb{R}^2, \mathbb{R})$ are classical: see e.g. [13, Chapter 2]. But working with $f \in \mathfrak{LC}(\mathbb{R}^2, \mathbb{R})$ with the topology $\mathcal{T}_{\mathbb{Q}}$ does not allow to define a *continuous* flow from the solutions of the equation via the hull procedure, which is required to apply techniques from topological dynamics. One needs to restrict to a suitable subset of functions f. Theorem A.2 deals with this question when $f(x,t)$ takes the shape $-x^2 + q(t)\,x + p(t)$ with $q, p \in BPUC_\Delta(\mathbb{R}, \mathbb{R})$. An in-depth analysis of different topologies and additional conditions in different spaces of Carathéodory functions giving rise to continuous flows appears in [28].

Before stating Theorem A.2, we fix some notation. Given $q, p \in BPUC(\mathbb{R}, \mathbb{R})$, we can consider the joint hull of (q, p),

$$\Omega_{q,p} := \mathrm{closure}_{L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2)}\{(q_t, p_t) \mid t \in \mathbb{R}\} \subset L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)\,.$$

For every $(\bar{q}, \bar{p}) \in \Omega_{q,p}$ and $x_0 \in \mathbb{R}$, the map $t \mapsto x(t, \bar{q}, \bar{p}, x_0)$ is the unique maximal solution of $x' = -x^2 + \bar{q}(t)\,x + \bar{p}(t)$ with $x(0) = x_0$; and $\mathcal{U} \subseteq \mathbb{R} \times \Omega_{q,p} \times \mathbb{R}$ is the domain of the function x.

Theorem A.2 *Let $q, p\colon \mathbb{R} \to \mathbb{R}$ belong to $BPUC(\mathbb{R}, \mathbb{R})$. Then, $\mathcal{U}$ is an open set, and*

$$\Phi\colon \mathcal{U} \subseteq \mathbb{R} \times \Omega_{q,p} \times \mathbb{R} \to \Omega_{q,p} \times \mathbb{R}, \quad \big(t, (\bar{q}, \bar{p}), x_0\big) \mapsto \big((\bar{q}_t, \bar{p}_t), x(t, \bar{q}, \bar{p}, x_0)\big)$$

defines a continuous local flow on $\Omega_{q,p} \times \mathbb{R}$, which is C^1 in x_0.

Proof The flow properties follow easily from the uniqueness of the solutions. Since $\Omega_{q,p} \subset \Omega_q \times \Omega_p$, it is compact, and the continuity of the flow translation follows from those of Ω_q and Ω_p. Hence, it remains to check the continuity of the second component of Φ.

Set $f(\omega, x) := -x^2 + q(t)\,x + p(t)$, and let $\mathrm{Hull}_{\mathcal{T}_{\mathbb{Q}}}(f)$ be the closure in $\mathfrak{LC}(\mathbb{R}^2, \mathbb{R})$ endowed with $\mathcal{T}_{\mathbb{Q}}$ of the set $\{f_t \mid t \in \mathbb{R}\}$ of time-translations of f. There exists a one-to-one correspondence between $\mathrm{Hull}_{\mathcal{T}_{\mathbb{Q}}}(f)$ and $\Omega_{q,p}$: for $(\widetilde{q}, \widetilde{p}) \in \Omega_{q,p}$ we define $\widetilde{f}(t, x) := -x^2 + \widetilde{q}(t)\,x + \widetilde{p}(t)$ and check that it belongs to $\mathrm{Hull}_{\mathcal{T}_{\mathbb{Q}}}(f)$; and given $\widetilde{f} \in \mathrm{Hull}_{\mathcal{T}_{\mathbb{Q}}}(f)$, we define $\widetilde{q}(t) := \widetilde{f}(t, 0)$ and $\widetilde{p}(t) := \widetilde{f}(t, 1) + 1 - \widetilde{q}(t)$, and check that $(\widetilde{q}, \widetilde{p}) \in \Omega_{q,p}$. In addition, it is not hard to check that the bijection $(\widetilde{q}, \widetilde{p}) \mapsto \widetilde{f}$ is continuous, and hence a homeomorphism between both compact spaces.

On the other hand, the expression of f makes it easy to check that, for every $j \in \mathbb{N}$, there is $\kappa_j > 0$ such that, for a.e. $t \in \mathbb{R}$,

$$|f(t, x_1) - f(t, x_2)| < \kappa_j |x_1 - x_2| \quad \text{whenever } x_1, x_2 \in [-j, j]\,.$$

In this situation, [28, Theorem 5.9(i)] (which is formulated for $\mathrm{Hull}_{\mathcal{T}_{\mathbb{Q}}}(f)$) shows that $\mathcal{U}$ is open as well as the continuity of the second component of Φ. The C^1 character in x_0 follows in a standard way from the fact that the derivative solves the corresponding variational equation (see [9, Theorem 2.3.1]) combined with the just established continuity. □

Proofs of Theorems 2.9 and 2.11 Once established the continuity of the flow induced by (2.1) on $\Omega_{q,p} \times \mathbb{R}$, in Theorem A.2, we can repeat the arguments leading to the (long and complex) proof of [29, Theorem 3.5], which deals with the analogous properties in the case of bounded and uniformly continuous functions q, p. These arguments allow us to prove (a)⇒(b), as well as points (i) and (ii) of Theorem 2.9 when (a) (and hence (b)) holds. The proof of the analogue of [29, Theorem 3.5] requires to apply the first approximation theorem to a scalar equation of the type $z' = (-2\,\widetilde{b}(t) + q(t))\,z - z^2$, where $\widetilde{b}(t)$ is a bounded continuous function, but with $q \in L^\infty(\mathbb{R}, \mathbb{R})$ instead of bounded continuous. This is not a problem: the proof of Theorem III.2.4 in [19] works without changes for this situation.

The assertion (b)⇒(c) of Theorem 2.9 is trivial, and (c)⇒(a) can be deduced, for instance, of Theorem 2.11(iv), whose proof is independent of Theorem 2.9. The whole proof of Theorem 2.11 repeats that of [29, Theorem 3.6]. The results required there for bounded and uniformly continuous coefficients q, p have been established in this paper for $q, p \in BPUC(\mathbb{R}, \mathbb{R})$. □

Now we consider, as in the previous sections, a BPUC function $p\colon \mathbb{R} \to \mathbb{R}$ and a continuous function $\Gamma\colon \mathbb{R} \to \mathbb{R}$ with finite asymptotic limits $\gamma_\pm := \lim_{t\to\pm\infty} \Gamma(t)$. And we define Γ_c^h for all $c \in \mathbb{R} \cup \{\pm\infty\}$ and $h \geq 0$ as at the beginning of Sect. 5.

Theorem A.3 *Let us define $f_c^h(t, y) := -(y - \Gamma_c^h(t))^2 + p(t)$ for $c \in \mathbb{R} \cup \{\pm\infty\}$ and $h \geq 0$. Then, $f_c^h \in \mathfrak{LC}(\mathbb{R}^2, \mathbb{R})$. In addition, let us denote by $\tilde{y}_{c,h}(\cdot, 0, y_0)$ the solution of $y' = f_c^h(t, y)$ with $y(0) = y_0$. If the sequence $((c_k, h_k))$ in $\mathbb{R} \times [0, \infty)$ converges to (c_0, h_0), with $c_0 \in \mathbb{R} \cup \{\pm\infty\}$ and $h_0 \in [0, \infty)$, and the sequence (y_k) in $\mathbb{R}$ converges to $y_0 \in \mathbb{R}$, then*

$$\lim_{k\to\infty} \tilde{y}_{c_k,h_k}(t, 0, y_k) = \tilde{y}_{c_0,h_0}(t, 0, y_0)$$

uniformly in t varying in any compact interval contained in the maximal interval of definition of $\tilde{y}_{c_0,h_0}(\cdot, 0, y_0)$.

Proof Note first that, if $j \in \mathbb{N}$ and $y_1, y_2 \in [-j, j]$, then

$$|f_c^h(t, y_1) - f_c^h(t, y_2)| \le (2j + 2\,\|\Gamma\|)|y_1 - y_2|$$

for all $c \in \mathbb{R} \cup \{\pm\infty\}$ and $h \ge 0$. This ensures the third of the conditions ensuring that $f_c^h \in \mathfrak{LC}(\mathbb{R}^2, \mathbb{R})$, and the two first ones are easier to check. Now, let us take a sequence $((c_k, h_k))$ in $\mathbb{R} \times [0, \infty)$ with limit (c_0, h_0), as in the statement. According to Theorem 5.8(i) in [28], to prove the last assertion it suffices to check that

$$\lim_{k\to\infty} \int_{r_1}^{r_2} |f_{c_k}^{h_k}(t, y) - f_{c_0}^{h_0}(t, y))|\, ds = 0$$

for all $r_1, r_2, y \in \mathbb{Q}$ with $r_1 < r_2$. Clearly, this limiting behavior is guaranteed by

$$\lim_{k\to\infty} \int_{r_1}^{r_2} \left|\Gamma_{c_k}^{h_k}(t) - \Gamma_{c_0}^{h_0}(t)\right| dt = 0 \quad \text{for} \quad r_1, r_2 \in \mathbb{Q} \quad \text{with} \quad r_1 < r_2\,,$$

which is hence the property to be proved. In turn, this property follows from the dominated convergence theorem, since Γ is bounded and $\lim_{k\to\infty} \Gamma_{c_k}^{h_k}(t) = \Gamma_{c_0}^{h_0}(t)$ for almost every $t \in \mathbb{R}$. The detailed proof of this last assertion is a nice exercise, for which we give some hints. Given $h > 0$ and $t \in \mathbb{R}$, if j^t is the unique integer number with $t \in [j^t h, (j^t + 1)\,h)$, we have $j^t h \in (t - h, t\,]$. This is the key point to prove that in the case $c_0 = 0$ and $h_0 \ge 0$, as well as in the case $c_0 \in \mathbb{R} - \{0\}$ and $h_0 = 0$, the convergence holds for every $t \in \mathbb{R}$. If $c_0 \in \mathbb{R} - \{0\}$ and $h_0 > 0$, the convergence holds when $t \ne j\,h_0$ for every $j \in \mathbb{Z}$. For $c = \pm\infty$ and $h_0 = 0$, it holds for $t \ne 0$. And finally, for $c = \pm\infty$ and $h_0 > 0$, it holds for every $t \ne h_0$. □

Appendix B. Clarification on the Numerical Analysis

Hereby, we clarify the way in which we obtain the figures in Sects. 4.2 and 5.1, corresponding to the differential equation $(5.1)_{c,h}$ (equal to $(4.1)_c$ for $h = 0$) for

$$\Gamma(t) := \frac{2}{\pi}\arctan(t)\,, \quad p(t) := 0.962 - \sin(t/2) - \sin(\sqrt{5}\,t), \quad c \ge 0, \quad \text{and} \quad h \ge 0\,.$$

All the involved equations have been numerically integrated using the MATLAB function `ode45` with double precision and the options on the relative and absolute tolerance respectively set to `RelTol=1e-9` and `AbsTol=1e-9`. The numerical method used to compute λ_* is based on the bisection idea outlined in [29], to which we refer the reader for further details. In that example, Γ is the same, and $p(t) := 0.892 - \sin(t/2) - \sin(\sqrt{5}\,t)$. We point out here that the section $\lambda_*(c, 0)$ of the bifurcation map λ_* (see Sect. 5.1) corresponding to our present example coincides with $\widetilde{\lambda}(c) - (0.962 - 0.895) = \widetilde{\lambda}(c) - 0.067$, where $\widetilde{\lambda}(c)$ is the function corresponding to the example in [29]: see Theorem 2.11(v). In particular, the detailed justification given in [29], taken for valid also in what follows, allows us to ensure that $\lambda_*(c, 0) < 0$ for $c \in [0, 0.25]$ (at least). Hence, Hypothesis 4.2 is fulfilled for our coefficients Γ and p.

We work under the next fundamental assumption, which is based on a consistent numerical evidence and which we will explain later: if $\widetilde{\Gamma}$ belongs to $BPUC_\Delta(\mathbb{R}, \mathbb{R})$ for a disperse set

Δ and satisfies $\|\widetilde{\Gamma}\| \le 0.1$, then the equation $x' = -(x - \widetilde{\Gamma}(t))^2 + p(t)$ has an attractor–repeller pair. In these conditions, Hypothesis 3.2 and Theorem 3.4 guarantee the existence of the (possibly locally defined) solutions $\mathfrak{a}_{c,h}$ and $\mathfrak{r}_{c,h}$ of $(5.1)_{c,h}$ for $c \in \mathbb{R} \cup \{\pm\infty\}$ and $h \ge 0$ described in Theorem 2.5. In addition, since the constant $m = 3.4$ satisfies the condition required in Theorem 2.5 for all the differential equations $(5.1)_{c,h}$ (as deduced from $\|\Gamma_c^h\| \le 1$ and $\|p\| \le 3$), we know that $\mathfrak{a}_{c,h}(t) < 3.5$ and $\mathfrak{r}_{c,h}(t) > -3.5$ on their respective domains, and that any bounded solution, if it exists, takes values in $(-3.5, 3.5)$.

We already know that $\lambda_*(c, 0) < 0$ for $c \in [0, 0.25]$. Therefore, $\mathfrak{a}_{c,0}$ and $\mathfrak{r}_{c,0}$ are globally defined hyperbolic solutions for $c \in [0, 0.25]$. We will now check that $\mathfrak{a}_{c,h}$ and $\mathfrak{r}_{c,h}$ are respectively defined on $(-\infty, -35]$ and $[35, \infty)$ whenever $c \ge 0.25$ (including $c = \infty$) and $h \in (0, 6]$. Let us define

$$(\Gamma_c^h)^-(t) := \begin{cases} \Gamma_c^h(t) & \text{if } \ t < -35\,, \\ \Gamma_c^h(-35) & \text{if } \ t \ge -35\,, \end{cases} \qquad (\Gamma_c^h)^+(t) := \begin{cases} \Gamma_c^h(35) & \text{if } \ t < 35\,, \\ \Gamma_c^h(t) & \text{if } \ t \ge 35\,, \end{cases}$$

and observe that $(\Gamma_c^h)^-(t) \in (-1, -0.9)$ and $(\Gamma_c^h)^+(t) \in (0.9, 1)$. In the case of $c = \infty$ and $h \in (0, 6]$, these assertions are trivial. In the remaining cases, they follow from these facts: given $h > 0$ and $t \in \mathbb{R}$, if j^t is the unique integer number with $t \in [j^t h, (j^t + 1)\,h)$, then $j^t h \in (t - h, t\,] \subseteq (t - 6, t]$; $-1 < (2/\pi)\arctan(c\, j^t h) < (2/\pi)\arctan(c\,t) \le (2/\pi)\arctan(0.25\cdot(-35)) < -0.9$ if $t \le -35$ and $c \ge 0.25$; and $1 > (2/\pi)\arctan(c\, j^t h) > (2/\pi)\arctan(c\,(t-6)) \ge (2/\pi)\arctan(0.25\cdot(35-6)) > 0.9$ if $t \ge 35$ and $c \ge 0.25$. Then, $y' = -\big(y - (\Gamma_c^h)^-(t)\big)^2 + p(t)$ has an attractor–repeller pair $(\widetilde{a}^-_{c,h}, \widetilde{r}^-_{c,h})$, since the trivial change of variables $x = y + 1$ provides the equation $x' = -\big(x - (1 + (\Gamma_c^h)^-(t))\big)^2 + p(t)$, with $\|1 + (\Gamma_c^h)^-\| < 0.1$. As before, we have $-3.5 < \widetilde{r}^-_{c,h}(t) < \widetilde{a}^-_{c,h} < 3.5$ for all $t \in \mathbb{R}$. In addition, by reviewing the proof of Theorem 3.4, we observe that $\mathfrak{a}_{c,h}(t) = \widetilde{a}^-_{c,h}(t)$ whenever $t \le -35$. This proves our assertion concerning $\mathfrak{a}_{c,h}$. To prove it for $\mathfrak{r}_{c,h}$, we work with $(\Gamma_c^h)^+$ and with the change of variables $x = y - 1$, using now that $\|1 - (\Gamma_c^h)^+\| < 0.1$.

Our goal now is finding suitable pairs (initial time, initial value) to reliably approximate $\mathfrak{a}_{c,h}$ and $\mathfrak{r}_{c,h}$ in the range of values $c \in (0, 50] \cup \{\infty\}$ and $h \in [0, 6]$ by finite integration. Recall that $\mathfrak{a}_{c,h}$ behaves like $\widetilde{a}^-_{c,h}$ on $(-\infty, -35]$, and hence it attracts exponentially fast solutions starting above $\widetilde{r}^-_{c,h}$ as time increases: see Theorem 2.9. Having in mind this fact, and trusting the simplicity of the numerical integration that we are performing, we can say that the computer does not distinguish $\mathfrak{a}_{c,h}(-35)$ from $y_{c,h}(-35, -500, 3.5)$. In fact, independently of the value of $(c, h) \in (0, 50]\times \in [0, 6]$, we observe that the graph of any solution $y_{c,h}(t, s, 3.5)$ with $s \le -85$ "collides" after less that 20 units of time with the graph of $y_{c,h}(-35, -500, 3.5)$: see Fig. 5. And the same happens with $\mathfrak{r}_{c,h}(35)$ and $y_{c,h}(35, 500, -3.5)$, so that the data we are taking are very precautionary.

The way to proceed is clear now. If we can continue the solution $y_{c,h}(t, -500, 3.5)$ at least until $t = 35$, and observe that $y_{c,h}(35, -500, 3.5) > y_{c,h}(35, 500, -3.5)$, then we are in CASE A (see Remark 2.6). If this is not the case, we will find $t_a > -35$ with $y_{c,h}(t_a, -500, 3.5) < -3.5$, which means that the graph of $y_{c,h}(t_a, -500, 3.5)$ intersects that of any function taking values on $[-3.5, 3.5]$, as is the case of any possible bounded solution; therefore, there are no bounded solutions, and hence the dynamics is given by CASE C.

Let us justify our initial assumption. First, we check that the equation $x' = -x^2 - 0.2\,|x| - 0.011 + p(t)$ has a bounded solution. In fact, using the same MATLAB routine to represent a large number of solutions of this equation, we observe that, independently of the initial time, the numerical approximation of every solution starting at an initial value

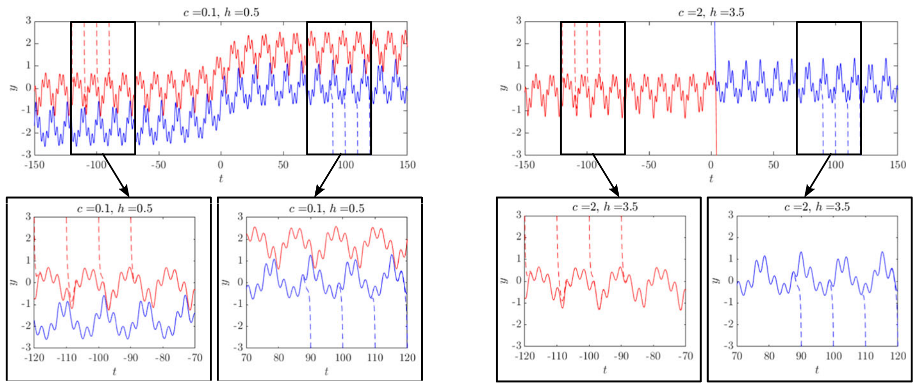

Fig. 5 Phase planes for two values of (c, h). In the left figure, an attractor–repeller pair exists (Case A), while in the right one there are no bounded solutions (Case C). The observed behavior is similar for any value of $(c, h) \in (0, 50] \times [0, 6]$, being the collision time always less than 20

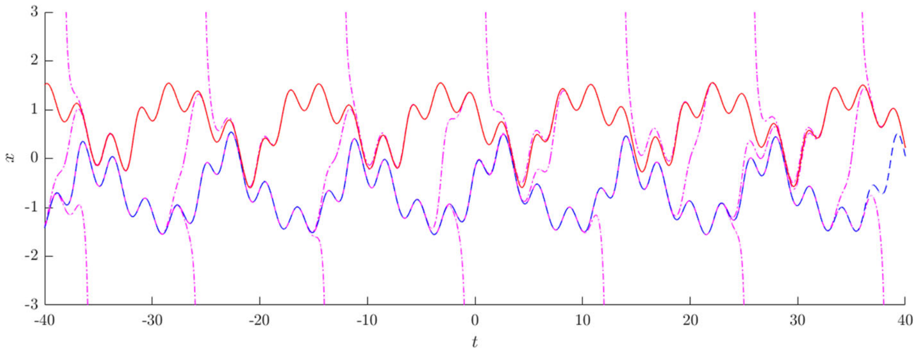

Fig. 6 Global dynamics of $x' = -x^2 - 0.2\,|x| - 0.011 + p(t)$. The behavior is analogous at any interval of integration

greater than 3 eventually falls onto the graph of the function represented in solid red in Fig. 6. The analogous behavior is observed backwards in time when computing solutions with initial value less than -3, which are eventually mapped on the graph of the function represented in dashed blue in Fig. 6. In addition, the solution corresponding to any initial pair (initial time, initial value) between the graphs of both functions falls onto the red curve as time increases and onto the blue curve as time decreases. In other words, we observe numerically that the dynamics for the (concave) equation $x' = -x^2 - 0.2\,|x| - 0.011 + p(t)$ is that of existence of an attractor–repeller pair, which is more than required. Let b be a bounded solution, and take $\widetilde{\Gamma} \in BPUC_\Delta(\mathbb{R}, \mathbb{R})$ for a disperse set Δ with $\|\widetilde{\Gamma}\| \le 0.1$. Then $b'(t) = -b^2(t) - 0.2\,|b(t)| - 0.011 + p(t) < -(b(t) - \widetilde{\Gamma}(t))^2 + p(t)$ for all $t \in \mathbb{R} - \Delta$, and hence Theorem 2.5(v) (see also Remark 2.4) ensures that $x' = -(x - \widetilde{\Gamma}(t))^2 + p(t)$ has at least two different bounded solutions. According to Remark 3.5, this ensures the existence of the attractor–repeller pair, which was our initial assumption. This completes our explanation, and the appendix.

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