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Homotopic Distance and Generalized Motion Planning

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Abstract. We prove that the homotopic distance between two maps defined on a manifold is bounded above by the sum of their subspace distances on the critical submanifolds of any Morse–Bott function. This generalizes the Lusternik–Schnirelmann theorem (for Morse functions) and a similar result by Farber for the topological complexity. Analogously, we prove that, for analytic manifolds, the homotopic distance is bounded by the sum of the subspace distances on any submanifold and its cut locus. As an application, we show how navigation functions can be used to solve a generalized motion planning problem.

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Keywords. Morse–Bott function, topological complexity, L–S category, homotopic distance, cut locus.

1. Introduction

Both Lusternik–Schnirelmann category [7] and Farber's topological complexity [11] can be seen as particular cases of the *homotopic distance* between two maps, introduced by the authors in [27].

The importance of those homotopy invariants is well known. On the one hand, the L–S category of a compact differentiable manifold gives a lower bound to the number of critical points of any differentiable function defined on it. On the other hand, the topological complexity is closely related to the problem of designing robot motion planning algorithms for a given configuration space.

In fact, the two approaches above are connected. As noted by Farber [12], following previous ideas by Koditschek and Rimon [21], the negative gradient vector field of a Morse–Bott real valued function gives rise to a flow which moves any given initial condition towards a target critical point. Therefore, so-called *navigation functions* provide motion planning algorithms for moving from an arbitrary source to an arbitrary target.

It has been repeatedly observed that topological complexity shares many properties with L–S category, and that both invariants lead to similar results. Examples are formulas involving products, fibrations and cohomological bounds. As shown in [27], the reason for this phenomenon is that those results can be proven for the homotopic distance between maps.

A paradigmatic result is Lusternik–Schnirelmann's Formula (2.1), that relates, for a given differentiable function $\Phi\colon M \to \mathbb{R}$, the L–S category of the ambient manifold M with the subspace L–S category of the critical levels of Φ. While the original proof of this result for arbitrary differentiable functions involves the subtle mini-max principle which is at the heart of L–S theory, the proof for Morse–Bott functions is much easier and only needs the most basic properties of L–S category. The analogous formula (2.2) for the topological complexity was proved by Farber in [12, Theorem 4.32].

In this paper, we shall prove a similar result for the homotopic distance between two maps (Theorem 4.5). This more general formula can then be applied to other invariants such as the *topological complexity of a work map* [28,37], the *complexity of a fibration* [30] and the *weak category* [39], which are particular cases of the homotopic distance too.

Finally, we adapt our result to the Morse–Bott function given by the (square of) the distance to a submanifold N in a Riemannian manifold M. We obtain that the homotopic distance between two continuous maps on the manifold is bounded above by the sum of the subspace homotopic distances on the submanifold N and its *cut locus* $\operatorname{Cut} N$ (Theorem 5.8). Our proof assumes that the manifold is analytic, in order to guarantee that the cut locus is triangulable.

The idea of using Morse–Bott functions to estimate the L–S category of some homogeneous spaces dates back to Kadzisa and Mimura [18]. However, they did not use Formula (2.1), but instead they constructed cone decompositions of the manifold by the gradient flows. This gives a cone length, which is an upper bound of the L–S category [7, Section 3.5].

In [23], the authors used Formula (2.1) directly, once a convenient function was chosen, to give the upper bound $\operatorname{cat} \operatorname{Sp}(n) \leq (n+1)n/2$ for the L–S category of the symplectic group. In a similar way, an optimal upper bound was given in [24] for the L–S category of the quaternionic Grassmannians $G_{n,k} = \operatorname{Sp}(n)/(\operatorname{Sp}(k) \times \operatorname{Sp}(n-k))$.

On the other hand, Farber [12] and later Costa [8] used Formula (2.2) to study *navigation functions* on the torus T^n, the projective spaces $\mathbb{R}P^n$ and the lens spaces $L(p,q)$, thus bringing new light into known results regarding topological complexity.

We shall define a *general motion planning* problem, meaning that, given two maps $f, g\colon X \to Y$, we need to find, for each $x \in X$, a path $s(x)$ on Y, depending continuously on x and connecting the points $f(x)$ and $g(x)$.

This problem can also be solved with navigation functions. Navigation functions exploit the gradient flow of a Morse–Bott function for constructing motion planning algorithms. Originally, Koditschek and Rimon [21] studied machines that navigate to a fixed goal using a gradient flow technique. Later, Farber [12] considered navigation functions which depend on two variables, the source and the target. We shall adapt his explanation to our generalized setting.

The contents of the paper are as follows. In Sect. 2 we recall the basic definitions of Morse–Bott theory, Lusternik–Schnirelmann category and Farber's topological complexity, as well as the classical theory relating the latter two invariants with the critical submanifolds of a Morse–Bott function.

In Sect. 3, we recall the definition of *homotopic distance* between two continuous maps, introduced by the authors in [27], and we give a *subspace* or *relative* version of it (Definition 3.6). This notion generalizes the subspace L–S-category [7, Definition 1.1] and the relative topological complexity [12, Section 4.3]. We prove its homotopic invariance in Proposition 3.10.

In Sect. 4, we study how to estimate the homotopic distance between two continuous maps defined on a manifold M on which it is also defined a Morse–Bott function Φ, by reducing the computation to the critical levels of Φ. In order to do that, we first show how to modify our definition of subspace homotopic distance to deal with *Euclidean neighbourhood retracts* (ENRs) instead of open subsets. Our main result (Theorem 4.5) states that the homotopic distance on M is bounded above by the sum of the subspace distances on the critical levels. This result generalizes both the classical Lusternik–Schnirelmann theorem [22] and the analogous result for the topological complexity by Farber [12, Theorem 4.32], whose proof we have adapted to our context.

In Sect. 5, we consider a complete Riemannian manifold M, a submanifold N and the function given by the (square of) the distance to N. This function turns to be differentiable on $M\setminus \operatorname{Cut} N$, where $\operatorname{Cut} N$ is the *cut locus* of N. We give a quick survey of the main properties of the cut locus, to prove that the homotopic distance between two maps defined on a compact analytic manifold is bounded by the sum of the subspace homotopic distance on the submanifold N and the subspace homotopic distance on its cut locus $\operatorname{Cut} N$ (Theorem 5.8). An obvious consequence for the L–S category is (Corollary 5.9):

$$\operatorname{cat} M \leq \operatorname{cat}_M(N) + \operatorname{cat}_M(\operatorname{Cut} N) + 1.$$

Finally, we show in Sects. 6 and 7 how to interpret the preceding results in terms of *navigation functions* that solve a *generalized motion planning problem*. In fact, following the original interpretation of the topological complexity, it happens that the homotopic distance $\mathrm{D}(f,g)$ between two continuous maps $f,g\colon X \to Y$ measures the difficulty of finding, for a given $x \in X$, a continuous path in Y that connects the points $f(x)$ and $g(x)$. When $X = M$ is a manifold, and there is a Morse–Bott function Φ defined on it, Theorem 4.5 can be interpreted as follows: first, we solve the subspace motion planning problem on each critical submanifold Σ_i by means of a covering G^i_j by ENRs. Then, if $x \in M$ belongs to the *basin of attraction* V^i_j of G^i_j, we slide along the gradient flow $x(t)$ from $x = x(0)$ to a critical point $\alpha = x(\infty) \in G^i_j$. Since there is a path γ connecting $f(\alpha)$ and $g(\alpha)$, we can concatenate the paths $f(x(t))$,γ and $g(\bar{x}(t))$, where $\bar{x}(t)$ is the reverse path of $x(t)$.

A similar interpretation is valid for Theorem 5.8, because there is a Morse–Bott flow collapsing $M\setminus \mathrm{Cut}\, N$ to N.

We show that these results apply not only to L–S category and topological complexity, but also to other invariants like the *topological complexity* $\mathrm{tc}(f)$ of the work map f, studied by Farber [12, p. 5] and later by Murillo and Wu [28] and by Scott [37, Theorem 3.4]; the *topological complexity* $\mathrm{cx}(f)$ *of a fibration* f, defined by Pavesic [30]; and the *weak category* $c^*_A(f)$ of a continuous map $f\colon X \to X$, reduced to a subspace $A \subset X$, defined by Yokoi [39].

All along this paper we assume that manifolds and topological spaces are path-connected, unless otherwise stated.

2. Basic Definitions

We begin by recalling the basic facts and notations of Morse–Bott theory. Also we recall the definitions of L–S category and topological complexity.

2.1. Morse–Bott Theory

Let M be a compact differentiable manifold. The smooth function $\Phi\colon M \to \mathbb{R}$ is called a *Morse–Bott function* if the critical set $\mathrm{Crit}\,\Phi$ is a disjoint union of connected submanifolds Σ_i and for each critical point $p \in \Sigma_i \subset \mathrm{Crit}\,\Phi$ the Hessian is non-degenerate in the directions transverse to Σ_i. A general reference is Nicolaescu's book [29]. For a complete proof of the Morse–Bott Lemma about the local structure of such a function see Banyaga and Hurtubise's paper [2].

If p is a critical point of Φ, the *index* of p is the number of negative eigenvalues of the Hessian at p. This number is constant along any connected critical submanifold Σ_i. If $\varphi\colon M \times \mathbb{R} \to M$ denotes the negative gradient flow of Φ, the *stable manifold* or *basin of attraction* of Σ_i is the set

$$S(\Sigma_i) = \{p \in M\colon \lim_{t\to+\infty} \varphi(p,t) \in \Sigma_i\}.$$

It is well known that the map $\pi\colon S(\Sigma_i) \to \Sigma_i$, sending each point p to the limit point of its trajectory, is a fibre bundle with fiber $\mathbb{R}^{m-n-k}$, where $m = \dim M$, $n = \dim \Sigma_i$ and k is the index of Σ_i. Moreover, M as a set is the disjoint union of the submanifolds $S(\Sigma_i)$. Notice, however, that the global limit map $M \to \mathrm{Crit}\,\Phi$ is *not* continuous.

2.2. L–S Category

A fundamental reference for L–S category is [7].

Let X be a topological space. A subspace $A \subset X$ is 0-*categorical in* X if it can be contracted to a point, inside X. That is, the inclusion map $A \subset X$ is homotopic to a constant map.

Definition 2.1. The *Lusternik–Schnirelmann* category of A in X, denoted by $\mathrm{cat}_X A$, is the minimum integer $k \geq 0$ such that there is a covering $U_0 \cup \cdots \cup U_k = A$, with the property that each subset U_j is open in A and 0-categorical in X.

If such a covering does not exist, we define $\operatorname{cat}_X A = \infty$. When $A = X$, we simply write $\operatorname{cat}_X X = \operatorname{cat} X$.

There is a well known relationship between the L–S category of a smooth manifold and the number of critical points of any smooth function defined on it [7, Theorem 1.15]. The following more elaborated result already appeared in the Lusternik–Schnirelmann's original work [22]. We heard about it for the first time in Rudyak–Schlenk's paper [34], see also Reeken's work [32, p. 21].

Theorem 2.2. *Let M be a compact smooth manifold. Let $\Phi\colon M \to \mathbb{R}$ be a smooth function with critical values $c_1 < \cdots < c_p$, and let $\Sigma_i = \Phi^{-1}(c_i) \cap \operatorname{Crit}\Phi$ be the set of critical points which lie in the level $\Phi = c_i$. Then*

$$\operatorname{cat} M + 1 \leq \sum_{i=1}^{p} (\operatorname{cat}_M \Sigma_i + 1). \tag{2.1}$$

The following result allows to make more precise the latter formula by observing that different connected critical submanifolds lying on the same critical level can be aggregated.

Proposition 2.3. *Let $C_i^1, \ldots, C_i^{n_i}$ be the connected components of Σ_i. Then*

$$\operatorname{cat}_M \Sigma_i = \max_j \operatorname{cat}_M C_i^j.$$

2.3. Topological Complexity

The fundamental reference for topological complexity is [11].

The following definition is the original one in Farber's paper [12, Definition 4.20], although we have normalized it. Also, we have chosen to say "*subspace* topological complexity" instead of "*relative* topological complexity".

Definition 2.4. Let X be a topological space and let $A \subset X \times X$ be a subspace. The (normalized) *subspace topological complexity* of A, denoted by $\operatorname{TC}_X(A)$, is the smallest integer $k \geq 0$ such that there is a cover $U_0 \cup \cdots \cup U_k = A$ with the property that each $U_j \subset A$ is open in A, and the projections $U_j \rightrightarrows X$ on the first and the second factors are homotopic to each other.

We simply write $\operatorname{TC}_X(X \times X) = \operatorname{TC}(X)$.

The following result is due to Farber [12, Theorem 4.32].

Theorem 2.5. *Let M be a compact smooth manifold (without boundary). Let $\Phi\colon M \times M \to \mathbb{R}$ be a Morse–Bott function such that $\Phi \geq 0$ and $\Phi(x, y) = 0$ if and only if $x = y$. Then*

$$\operatorname{TC}(M) + 1 \leq \sum_{i=1}^{p} (\operatorname{TC}_M(\Sigma_i) + 1), \tag{2.2}$$

where $\Sigma_1, \ldots, \Sigma_p$ are the critical levels of Φ.

Farber also proves that $\operatorname{TC}_M(\Sigma_i)$ equals the maximum of the subspace topological complexities of the connected components.

A function like the Φ of Theorem 2.5 is called by Farber a *navigation function* (cf. Sect. 7).

3. Homotopic Distance

The following notion was introduced by the authors in [27].

Definition 3.1. Let $f, g\colon X \to Y$ be two continuous maps. The *homotopic distance* $\mathrm{D}(f,g)$ between f and g is the least integer $k \geq 0$ such that there exists an open covering $U_0 \cup \cdots \cup U_k = X$ with the property that the restrictions $f_{|U_j}$ and $g_{|U_j}$ are homotopic maps, for all $j = 0, \ldots, k$.

If there is no such covering, we define $\mathrm{D}(f,g) = \infty$.

Example 3.2. Let X be a path-connected topological space. The L–S-category of X equals the homotopic distance between the identity id_X and any constant map, $\operatorname{cat} X = \mathrm{D}(\mathrm{id}_X, x_0)$.

Proposition 3.3. [27, Proposition 2.5] *Given a base point $x_0 \in X$ we define the axis inclusion maps $i_1, i_2\colon X \to X \times X$ as $i_1(x) = (x, x_0)$ and $i_2(x) = (x_0, x)$. The homotopic distance between i_1 and i_2 equals the L–S category of X, that is,* $\mathrm{D}(i_1, i_2) = \operatorname{cat} X$.

Example 3.4. More generally, the L–S category of a map $f\colon X \to Y$ [7, Exercise 1.16, p. 43] is the distance between f and any constant map, $\operatorname{cat} f = \mathrm{D}(f, x_0)$, when Y is path-connected. For instance, the category of the diagonal $\Delta_X\colon X \to X \times X$ equals $\operatorname{cat} X$.

Proposition 3.5. [27, Proposition 2.6] *The topological complexity* $\mathrm{TC}(X)$ *of the topological space X equals the homotopic distance between the two projections $p_1, p_2\colon X \times X \to X$, that is,* $\mathrm{TC}(X) = \mathrm{D}(p_1, p_2)$.

Other examples will be given later (see Sect. 6).

We propose the following definition of *subspace distance*, that is, homotopic distance between two maps with respect to a subspace, as a generalization of Definitions 2.1 and 2.4.

Definition 3.6. Let $f, g\colon X \to Y$ be two continuous maps, and let $A \subset X$ be a subspace. The *subspace distance* between the two maps f, g on A, denoted by $\mathrm{D}_X(A; f, g)$, is defined as the distance between the restrictions of f, g to A, that is,

$$\mathrm{D}_X(A; f, g) := \mathrm{D}(f_{|A}, g_{|A}).$$

Obviously, when $A = X$ we recover the usual homotopic distance. Moreover, observe that $\mathrm{D}_X(A; f, g) = \mathrm{D}(f \circ i_A, g \circ i_A)$, where $i_A\colon A \subset X$ is the inclusion.

Example 3.7. If $i_A\colon A \subset X$ is a subspace, then

$$\operatorname{cat}_X A = \mathrm{D}_X(A; \mathrm{id}_X, x_0) = \mathrm{D}(i_A, x_0).$$

Example 3.8. If $i_A\colon A \subset X \times X$ is a subspace, and $p_1, p_2\colon X \times X \to X$ are the projections, then

$$\mathrm{TC}_X(A) = \mathrm{D}_{X \times X}(A; p_1, p_2).$$

The following result will be used later.

Proposition 3.9. *Let $f, g\colon X \to Y$ be two continuous maps, and let $\{A_i\}_{i=1}^n$ be the connected components of X. Then*

$$\mathrm{D}(f,g) = \mathrm{D}_X(X; f, g) = \max_i \mathrm{D}_X(A_i; f, g).$$

Proof. To simplify the notation, we will do the proof for the case of two connected components A_1 and A_2. The general proof is analogous. Say $\mathrm{D}_X(A_1; f, g) = n_1 \leq \mathrm{D}_X(A_2; f, g) = n_2$ with open coverings $\{U_i\}_{i=0}^{n_1}$ and $\{V_i\}_{i=0}^{n_2}$ of A_1 and A_2, respectively. Then $\{U_i \cup V_i\}_{i=0}^{n_1} \cup \{V_i\}_{i=n_1+1}^{n_2}$ is an open cover of X. Notice that $U_i \cap V_i = \emptyset$ for $i = 0, \ldots, n_1$, thus guarantying that f and g are homotopic on $U_i \cup V_i$. Hence, $\mathrm{D}_X(X; f, g) \leq n_2$.

The inequality $n_2 = \mathrm{D}(A_2; f, g) \leq \mathrm{D}_X(X; f, g)$ simply follows from $A_2 \subset X$. □

The main property of the homotopic distance, and in consequence of cat and TC, is its homotopy invariance [27, Proposition 3.13]. We shall need the following relative version.

Proposition 3.10. *Let $f, g\colon X \to Y$ be two continuous maps. Let $i_A\colon A \hookrightarrow X$ and $i_B\colon B \hookrightarrow X$ be two subspaces, and let $\alpha\colon A \to B$ be a homotopy equivalence, such that $i_B \circ \alpha \simeq i_A$ (we say that A and B are homotopically equivalent* in X)*:*

$$\begin{array}{ccccc} A & \xrightarrow{\ i_A\ } & X & \overset{f}{\underset{g}{\rightrightarrows}} & Y\,. \\ {\scriptstyle\alpha}\big\downarrow & \nearrow_{i_B} & & & \\ B & & & & \end{array}$$

Then $\mathrm{D}_X(A; f, g) = \mathrm{D}_X(B; f, g)$.

Proof. Since α is a homotopy equivalence and $i_B \circ \alpha \simeq i_A$ we have (cf. [27, Propositions 2.2 and 3.12])

$$\begin{aligned} \mathrm{D}_X(B; f, g) &= \mathrm{D}(f_{|B}, g_{|B}) = \mathrm{D}(f \circ i_B, g \circ i_B) = \mathrm{D}(f \circ i_B \circ \alpha, g \circ i_B \circ \alpha) \\ &= \mathrm{D}(f \circ i_A, g \circ i_A) = \mathrm{D}(f_{|A}, g_{|A}) = \mathrm{D}_X(A; f, g). \end{aligned}$$

□

Finally, we have the following sub-additivity property.

Proposition 3.11. *Given two maps $f, g\colon X \to Y$ and a finite open covering $V_1 \cup \cdots \cup V_p = X$, it happens that*

$$\mathrm{D}(f,g) + 1 \leq \sum_{i=1}^{p} \left(\mathrm{D}_X(V_i; f, g) + 1\right).$$

4. Homotopic Distance and Morse–Bott Functions

To generalize Theorems 2.2 and 2.5, we need to adapt Definition 3.1 to a situation that does not demand that the pieces in which we decompose the space are open subsets. In fact, for a Morse–Bott function, the pieces will be the basins of the negative gradient flow, which are submanifolds, but neither open nor closed subspaces, in general.

Hence, we shall restrict ourselves to smooth manifolds and submanifolds, or more generally, to the so-called *Euclidean neighbourhood retracts* (ENR, for short).

Definition 4.1. ([10, p. 81], [38, p. 448]) A topological space E is called a *Euclidean neighbourhood retract* (ENR for short) if it is homeomorphic to a subspace $E' \subset \mathbb{R}^n$ which is a retract of some neighbourhood $E' \subset W \subset \mathbb{R}^n$.

The class of ENRs includes all finite-dimensional cell complexes and all compact topological manifolds [5, Appendix E]. We need the following property.

Proposition 4.2. ([10, Cor. 8.7], [38, Remark 18.4.4]) *Let $A \subset X$ be two ENRs. Then there exists an open neighborhood $A \subset U \subset X$ of A in X and a retraction $r\colon U \to A$ such that the inclusion $i_U\colon U \hookrightarrow X$ is homotopic to $i_A \circ r$, where $i_A\colon A \hookrightarrow X$ denotes the inclusion. That is, we have the diagram, commutative up to homotopy,*

$$\begin{array}{ccc} U & & \\ {\scriptstyle r}\downarrow & \searrow^{i_U} & \\ A & \xrightarrow[i_A]{} & X \end{array} .$$

Corollary 4.3. *Let $A \subset X$ be two ENRs, and let $f, g\colon X \to Y$ be two continuous maps. If f, g are homotopic on A, then there is an open neighbourhood $A \subset U \subset X$ such that f, g are homotopic on U.*

Proof. Let $r\colon U \to A$ as in Proposition 4.2. We have

$$f_{|U} = f \circ i_U \simeq f \circ i_A \circ r = f_{|A} \circ r \simeq g_{|A} \circ r = g \circ i_A \circ r \simeq g \circ i_U = g_{|U}.$$

□

Corollary 4.4. $\mathrm{D}_X(A; f, g) = \mathrm{D}_X(U; f, g)$.

Proof. It is an immediate consequence of Propositions 3.10 and 4.2. □

With the previous ingredients, we are ready to state the following result.

Theorem 4.5. *Let $\Phi\colon M \to \mathbb{R}$ be a Morse–Bott function in the compact smooth manifold M. Let $c_1 < \cdots < c_p$ be its critical values, and let $\Sigma_i = \Phi^{-1}(c_i) \cap \operatorname{Crit}\Phi$ be the set of critical points in the level $\Phi = c_i$. If $f, g\colon M \to Y$ are two continuous maps, then*

$$\mathrm{D}(f, g) + 1 \leq \sum_{i=1}^{p} (\mathrm{D}_M(\Sigma_i; f, g) + 1).$$

Proof. For each Σ_i, let $S(\Sigma_i)$ be the basin $\pi^{-1}(\Sigma_i)$ of points in M whose limit point $\varphi(p, \infty) \in \Sigma_i$. Since $\pi\colon S(\Sigma_i) \to \Sigma_i$ is a homotopy equivalence, by Proposition 3.10 we have

$$\mathrm{D}_M(\Sigma_i; f, g) = \mathrm{D}_M(S(\Sigma_i); f, g).$$

Now, each $S(\Sigma_i)$ is a submanifold of M, so by Proposition 4.2, there is an open subset $\Sigma_i \subset V_i \subset M$ such that

$$\mathrm{D}_M(S(\Sigma_i); f, g) = \mathrm{D}_M(V_i; f, g).$$

Finally, since $\cup_i S(\Sigma_i) = M$, we have $\cup_i V_i = M$, and by Proposition 3.11 we have

$$\mathrm{D}(f, g) + 1 \leq \sum_i (\mathrm{D}_M(V_i; f, g) + 1) = \sum_i (\mathrm{D}_M(\Sigma_i; f, g) + 1).$$

□

5. Cut Locus

In Theorem 4.5 (and in Theorems 2.2 and 2.5, which are particular cases), we gave an upper bound for the homotopic distance between two continuous maps f, g, defined on a manifold M, by considering their restrictions to the critical set of a differentiable Morse–Bott function Φ also defined on M.

In what follows, we shall consider a submanifold N of a complete Riemannian manifold M and the function $\Phi\colon M \to \mathbb{R}$, $\Phi(y) = d^2(y, N)$, given by the square of the distance from the point $y \in M$ to the submanifold N. In general, this function is not differentiable in the *cut locus* of N [36, Proposition 4.8]. Anyway, we shall try to adapt our preceding results to this setting.

5.1. Preliminaries

We begin by recalling some basic facts about the cut locus. See for instance [20, Chapter VIII.7] or [13, Chapter II.C]. Good surveys are Kobayashi's paper [19] or the more recent Angulo's thesis [1].

Let M be a complete $\mathcal{C}^\infty$ Riemannian manifold and let N be an immersed submanifold. The *Riemannian distance* $d(y, N)$, from $y \in M$ to N, is the infimum of the lengths of all piecewise smooth curves joining y to some point $x \in N$.

A geodesic γ between two points $x, y \in M$ is said to be minimizing if its length equals the distance $d(x, y)$. A unit speed geodesic $\gamma\colon [0, t_0] \to M$ emanating from N is an N-*segment* (or N-minimizing geodesic) if its length t equals the distance $d(\gamma(t), N)$, for all $t \in [0, t_0]$ [17]. In this case the geodesic must be orthogonal to the submanifold N.

Definition 5.1. The point $\gamma(t_0)$ is called a *cut point* of N if $\gamma([0, t_0])$ is an N-segment but there is no N-segment properly containing $\gamma([0, t_0])$. The *cut locus* $\mathrm{Cut}\, N$ is the set of all these cut points.

The simplest case is when $N = \{x\}$ is a unique point. On each geodesic curve emanating from the point x, the cut point is the last point to which the geodesic minimizes distance from x.

Example 5.2. If $M = S^n$ is the n-dimensional unit sphere and x is its North Pole, the cut locus of x reduces to the South Pole. If $M = \mathbb{R}P^n$ is the projective space, the Riemannian metric of S^n induces a Riemannian metric on M, so that the projection of S^n onto M is a local isometry. The cut locus of the point x corresponding to the North and South Poles of the sphere S^n is the image of the equator of S^n under the projection, that is a naturally imbedded $(n-1)$-dimensional projective space $\mathbb{R}P^{n-1}$. Analogously, in $\mathbb{C}P^n$ the cut locus of a point x is isometric to $\mathbb{C}P^{n-1}$.

Example 5.3. If $M = T^2 = S^1 \times S^1$ is the torus seen as a quotient of the square $I \times I$, the usual Riemannian metric on $\mathbb{R}^2$ induces a Riemannian metric on M. The cut locus of the point $x = [(1/2, 1/2)]$ can be identified with the image of the boundary of the square, that is,

$$\mathrm{Cut}(x) = (C \times S^1) \cup (S^1 \times C) \cong S^1 \vee S^1, \tag{5.1}$$

where $C = \{[0]\}$ is the cut point of $[1/2]$ in the circle S^1 seen as a quotient of I.

5.2. Structure of the Cut Locus

When N is an arbitrary submanifold, we denote by $\pi\colon (\mathrm{U}\nu)N \to N$ the unit normal bundle of N. Let $\gamma_u(t)$ be the unit speed geodesic emanating from $x \in N$ in the direction of $u \in (\mathrm{U}\nu)_x N$. The transverse exponential map $\mathrm{Exp}_x(tu) = \gamma_u(t)$ is a diffeomorphism from a tubular neighborhood of the zero section of the normal bundle νN of N into a tubular neighborhood of N in M, but singularities can appear for large vectors. For a vector $w = tu \in \nu_x N$ where Exp_x is not regular, the *order of conjugacy* of w is the dimension $k > 0$ of the kernel of the linear map $D_w \mathrm{Exp}_x$. The point $y = \mathrm{Exp}_x(w)$ is called a *conjugate point* of $x \in N$.

It is standard that the points of $\mathrm{Cut}\, N$ are either the first conjugate point $y \in M$ on a length minimizing geodesic starting at N, or a "separating point", that is, a point $y \in M$ where there are at least two length minimizing geodesics from N to y. On a simply connected complete symmetric space, the cut locus of a point coincides with the first conjugate locus [9, Theorem 5].

Proposition 5.4. [3, Theorem 3.26] *If the manifold M is complete, the cut locus* $\mathrm{Cut}\, N$ *of a compact submanifold N is a closed subset of M; in fact, it is the closure of the separating set.*

In practice, the cut locus is very hard to compute, since usually it has a wild structure. For instance, the usual metric on the sphere S^n can be deformed around the equator in such a way that the North Pole has a non-triangulable cut locus [14, Theorem A].

On the other hand, in any compact manifold M there exists some metric such that the cut locus of any point is triangulable [14, p. 348].

Fortunately, the situation is more tractable in Lie groups and homogeneous spaces, which naturally admit analytic structures and metrics.

Theorem 5.5. (Buchner) [3, Theorem 3.9], [6] *Let M be an analytic Riemannian manifold of dimension m, and let N be an analytic submanifold. Then* $\mathrm{Cut}\, N$ *is a simplicial complex of dimension strictly less than m.*

Example 5.6. [3, Theorem 3.23] The cut locus of a point in a real analytic closed orientable surface of genus g is a connected graph, homotopically equivalent to a wedge of $2g$ circles (cf. the torus in Example 5.3).

5.3. The Distance Function

The result we are interested in is the following theorem [3, Theorem 3.28]:

Theorem 5.7. *Let N be a closed embedded submanifold of a complete Riemannian manifold M. Let $d\colon M \to \mathbb{R}$ be the distance function with respect to N. The restriction of the function $\Phi(y) = d^2(y, N)$ to $M \setminus \operatorname{Cut} N$ is a Morse–Bott function, with N as the critical submanifold. As a consequence, the gradient flow of Φ deforms $M \setminus \operatorname{Cut} N$ to N.*

Hence, we have the following formula.

Theorem 5.8. *Let M be a compact analytic manifold and N a closed analytic submanifold. Let $f, g\colon M \to Y$ be two continuous maps. Then*

$$D(f, g) \leq D_M(N; f, g) + D_M(\operatorname{Cut} N; f, g) + 1.$$

Proof. It is known that a finite simplicial complex is an ENR [16, Corollary A.8]. Hence $\operatorname{Cut} N$ is an ENR, by Theorem 5.7. By Corollary 4.3 there is an open neighbourhood U such that $\mathrm{D}_M(\operatorname{Cut} N; f, g) = \mathrm{D}_M(U; f, g)$. Also, we know that $M \setminus \operatorname{Cut} N$ is open in M, by Proposition 5.4. Hence,

$$M = (M \setminus \operatorname{Cut} N) \cup \operatorname{Cut} N = (M \setminus \operatorname{Cut} N) \cup U,$$

so by the subadditivity property (Proposition 3.11) we have

$$\mathrm{D}(f, g) \leq \mathrm{D}_M(N; f, g) + \mathrm{D}_M(\operatorname{Cut} N; f, g) + 1,$$

because $M \setminus \operatorname{Cut} N$ and N are homotopically equivalent in M (see Proposition 3.10). □

Corollary 5.9. *By taking the identity and a constant function, we have*

$$\operatorname{cat} M \leq \operatorname{cat}_M(N) + \operatorname{cat}_M(\operatorname{Cut} N) + 1.$$

Now, to have an analogous result for the topological complexity, we can take the projections $p_1, p_2\colon M \times M \to M$.

Corollary 5.10. $\mathrm{TC}(M) \leq \mathrm{TC}_M(N \times N) + \mathrm{TC}_M(\operatorname{Cut}(N \times N)) + 1.$

This result may be seen as a formalization of the following comment by Blaszczyk and Carrasquel in [4]: "to estimate topological complexity of X, it is enough to understand how to motion plan between points $p, q \in X$ with $q \in \operatorname{Cut}(p)$".

Unfortunately, we do not know any explicit formula for the cut locus of $N \times N$ in $M \times M$. For a point $N = \{x\}$, the following formula is proven by Crittenden in [9, p. 328] (compare with Example 5.3):

$$\operatorname{Cut}_{M \times M}((x, x)) = (\operatorname{Cut}_M(x) \times M) \cup (M \times \operatorname{Cut}_M(x)).$$

Remark. Notice that in general it is *not* true that $\mathrm{TC}(M)$ is bounded by $\mathrm{TC}(N) + \mathrm{TC}(\operatorname{Cut} N) + 1$. For instance, take $M = S^n$ for n even, and as submanifold N the North Pole NP. Since $\operatorname{Cut}(\mathrm{NP}) = \{\mathrm{SP}\}$, the South Pole, we have $\mathrm{TC}(\mathrm{NP}) + \mathrm{TC}(\mathrm{SP}) + 1 = 1$ but $\mathrm{TC}(S^n) = 2$ [11, Theorem 8].

Example 5.11. Currently, the L–S category of the symplectic group $\mathrm{Sp}(n)$ is not known, for $n \geq 4$ [23]. In this Example we show how to compute the cut locus of the identity matrix $N = \{I_n\}$. Unfortunately, we do not know how to compute the subspace L–S category of $\mathrm{Cut}\, N$.

We will follow closely the computation by Sakai [35] for the unitary group $\mathrm{U}(n)$, which is a particular case of his results for symmetric spaces.

Remember that $G = \mathrm{Sp}(n)$ is the compact Lie group of $n \times n$ quaternionic matrices such that $AA^* = I$, where A^* denotes the conjugate transpose of A. Any matrix $A \in \mathrm{Sp}(n)$ can be diagonalized as UDU^*, where $U \in \mathrm{Sp}(n)$ and $D = \mathrm{diag}(z_1, \ldots, z_n)$ is a diagonal matrix with *complex* entries such that $|z_i| = 1$ [33, Theorem 5.3.5]. The Lie algebra $\mathfrak{g}$ of G is formed by the $n \times n$ matrices X which are skew-Hermitian, that is, $X + X^* = 0$. One Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is formed by the diagonal matrices $X = \mathrm{diag}(z_1, \ldots, z_n)$, where the z_i's are complex numbers such that $\Re(z_i) = 0$. It corresponds to the maximal abelian subgroup D_G of diagonal complex matrices D as above. By diagonalization, we have $G = \mathrm{Ad}_G(D_G)$. It is enough to determine the cut points for the vectors $X \in \mathfrak{a}$.

We endow G with the bi-invariant Riemannian metric corresponding to the usual inner product $\langle X, Y \rangle = \Re\, \mathrm{Tr}(X^*Y)$ on $\mathfrak{g}$ (notice that there is an unessential difference of a $-\frac{1}{2}$ factor with the one considered by Sakai). The geodesic emanating from I_n in the direction of $X \in \mathfrak{g}$ is given by the one-parameter subgroup $\exp(tX)$. The matrices

$$A_i = \mathrm{diag}(0, \ldots, \overset{i)}{\mathbf{i}}, \ldots, 0), \quad 0 \leq i \leq n,$$

form a orthonormal basis of $\mathfrak{a}$.

Lemma 5.12. [35, Lemma 3.3] *For $X \in \mathfrak{a}$, the tangent cut point $\tilde{t}_0(X)X$ corresponding to the unit vector $X = \sum x_i A_i$, with $\sum x_i^2 = 1$, is given by*

$$\tilde{t}_0(X) = \min \left\{ \frac{\langle Y, Y \rangle}{|\langle X, Y \rangle|} : Y \in \mathfrak{a}, Y \neq 0, \exp(Y) = I_n \right\} = \frac{\pi}{\max |x_i|}.$$

Hence, the cut locus $\mathrm{Cut}\, N$ of $N = \{I_n\}$ is given by

$$\mathrm{Ad}_G(\exp(\tilde{t}_0(X)X) \colon X \in \mathfrak{a}). \tag{5.2}$$

Proposition 5.13. *The cut locus of $N = \{I_n\}$ in $G = \mathrm{Sp}(n)$ is formed by the matrices $A \in \mathrm{Sp}(n)$ that have -1 as an eigenvalue.*

Proof. According to Formula (5.2), a cut point is given by UDU^*, with $U \in G$ and $D = \exp Y$, with $Y = \tilde{t}_0(X)X$ for some unit vector $X \in \mathfrak{a}$, that is,

$$Y = \sum_i \frac{\pi x_i}{\max |x_i|} A_i, \quad \sum_i x_i^2 = 1.$$

Then, some of the coordinates of $Y = \sum_i y_i A_i$ equals $\pm\pi$, namely $|y_j| = \pi$ if $\max |x_i| = |x_j|$, while the others verify $|y_i| \leq \pi$. By computing the exponential we obtain that

$$D = \mathrm{diag}(e^{y_1 \mathbf{i}}, \ldots, \overset{i)}{-1}, \ldots, e^{y_n \mathbf{i}}),$$

because

$$e^{\pm\pi\mathbf{i}} = -1.$$

On the other hand, any complex number z_i of norm 1 can be written as $z_i = e^{y_i\mathbf{i}}$, with $|y_i| \leq \pi$. Then the result follows. □

Remark. Notice that $\operatorname{Cut} N$ is the orbit, by the adjoint action, of the cut locus in a maximal torus.

Remark. Notice that $\operatorname{Cut} N$ is the complementary of the Cayley open set

$$\Omega_G(I) = \{A \in \mathrm{Sp}(n) \colon A + I \text{ invertible}\}$$

considered in our papers [15,25], which is formed by the matrices which do *not* have -1 as an eigenvalue. Since $\Omega_G(I)$ is known to be contractible, we obtain again that $G \setminus \operatorname{Cut} N$ contracts to N.

6. Motion Planning

Both L–S category and topological complexity measure the difficulty of finding continuous motion planning algorithms for the configuration space X of a mechanical device. We can interpret the homotopic distance between two maps $f, g \colon X \to Y$ in the same way, because it solves the following:

> *generalized planning problem*: given an arbitrary point $x \in X$ find a continuous path $s(x)$, joining $f(x)$ and $g(x)$ in Y, in such a way that the path $s(x)$ depends continuously on x.

More precisely, let $\mathcal{P}(f,g)$ be the space of pairs (x,γ) where $x \in X$ and γ is a continuous path on Y, such that $\gamma(0) = f(x)$ and $\gamma(1) = g(x)$. This space fibers over X, by taking the map $\pi^* \colon \mathcal{P}(f,g) \to X$, where $\pi^*(x,\gamma) = x$. Notice that $\pi^* = (f,g)^*\pi \colon \mathcal{P}(f,g) \to X$ is the pullback fibration of the path fibration $\pi \colon Y^I \to Y \times Y$, where $\pi(\gamma) = (\gamma(0), \gamma(1))$, by the map $(f,g) \colon X \to Y \times Y$:

$$\begin{array}{ccc} \mathcal{P}(f,g) & \longrightarrow & Y^I \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{(f,g)} & Y \times Y. \end{array}$$

Proposition 6.1. [27, Theorem 2.7] *The homotopic distance between the maps* $f, g \colon X \to Y$ *equals the* Svarc genus *of* π^*, *that is, the minimum integer number* $k \geq 0$ *such that there exists an open covering* $U_0 \cup \cdots \cup U_k = X$, *where for each* U_j *there is a continuous section* $s_j \colon U_j \to \mathcal{P}(f,g)$ *of the pullback fibration* π^*.

This situation covers many different scenarios:

6.1. L–S Category

For $f = \mathrm{id}_X$ and $g = x_0$ a constant map, we have $\operatorname{cat} X = \mathrm{D}(\mathrm{id}_X, x_0)$. A homotopy between id_X and x_0 gives a continuous path $H_t(x)$ between each point x and the fixed target x_0.

6.2. Topological Complexity

This was Farbers's original idea for the motion planning problem. For the projections $p_1, p_2\colon X \times X \to X$, one has $\mathrm{D}(p_1, p_2) = \mathrm{TC}(X)$. A homotopy $H_t(x, y)$ between the projections gives a path joining two arbitrary points x, y.

6.3. Topological Complexity of the Work Map

A more elaborated situation, studied by Farber [12, p. 5] and later by Murillo and Wu [28], occurs when X is the configuration space of a multi-link robot arm, and Y is its "workspace", that is, the spatial region that can effectively be attained by the end effector of the arm.

A so-called "work map" $f\colon X \to Y$ (also called a *forward kinematic map*) assigns to each state of the configuration space the corresponding position of the end effector. This map is not assumed to be bijective. When implementing algorithms which control the task performed by the robot, the input of such an algorithm is a pair $(x, y) \in X \times X$ of points of the configuration space, and the output is a curve in the configuration space, whose image by f connects the positions $f(x)$ and $f(y)$. The corresponding invariant is called the *topological complexity of the work map* f, denoted by $\mathrm{tc}(f)$.

Definition 6.2. Let $f\colon X \to Y$ be a continuous map and denote by $\pi\colon X^I \to X \times X$ the path fibration $\pi(\gamma) = \big(\gamma(0), \gamma(1)\big)$, which sends each continuous path $\gamma\colon [0,1] \to X$ to its initial and final points. The *topological complexity of* f, denoted $\mathrm{tc}(f)$, is the least integer k such that $X \times X$ can be covered by $k+1$ open subsets $\mathrm{U}_0, \ldots, U_k$, such that for each U_i there exists a continuous map $\sigma_i\colon U_i \to X^I$ satisfying

$$(f \times f) \circ \pi \circ \sigma_i \simeq (f \times f)_{|U_i}.$$

Later, Scott proved [37, Theorem 3.4] that the previous definition can be equivalently stated as follows:

Proposition 6.3. *Let $f\colon X \to Y$ be a continuous map. The topological complexity of the work map $\mathrm{tc}(f)$ equals the least integer k such that $X \times X$ can be covered by $k+1$ open subsets $\{U_i\}_{i=0}^k$ such that for each U_i there exists a continuous map $f_i\colon U_i \to Y^I$ satisfying $f_i(x_0, x_1)(0) = f(x_0)$ and $f_i(x_0, x_1)(1) = f(x_1)$.*

Theorem 6.4. *Let $f\colon X \to Y$ be a continuous map. Then, the (normalized) topological complexity of f equals the distance of the projections composed with the map f. That is, $\mathrm{tc}(f) = \mathrm{D}(f \circ p_1, f \circ p_2)$.*

Proof. Let us denote by $\pi\colon Y^I \to Y \times Y$ the path fibration. Then, Scott proved [37, Theorem 3.4] that there is an open subset $U_i \subset X \times X$ such that that there exists a continuous map $f_i\colon U_i \to Y^I$ satisfying $f_i(x_0, x_1)(0) = f(x_0)$ and $f_i(x_0, x_1)(1) = f(x_1)$ if and only if there is a section $s\colon U_i \to (f \times f)^* Y^I$ of the pullback fibration of π by the map $f \times f$:

$$\begin{array}{ccc} (f\times f)^*Y^I & \longrightarrow & Y^I \\ {\scriptstyle (f\times f)^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X\times X & \xrightarrow{f\times f} & Y\times Y. \end{array}$$

Observe that $f\times f\colon X\times X\to Y\times Y$ is just the map $(f\circ p_1, f\circ p_2)\colon X\times X\to Y\times Y$. Therefore, by Proposition 6.1, $\mathrm{tc}(f)=\mathrm{D}(f\circ p_1, f\circ p_2)$. □

6.4. Topological Complexity of a Map

The latter invariant $\mathrm{tc}(f)$ is different from $\mathrm{cx}(f)$, the *topological complexity of a map* $f\colon X\to Y$, defined by Pavesic [30] as the sectional complexity (that is, the number of partial solutions to the motion planning problem) of the map

$$\pi\colon X^I\to X\times Y,\quad \pi(\gamma)=(\gamma(0), f(\gamma(1)))\,,$$

which assigns, to each path in the configuration space, the initial state and the end effector position at the final state. Later [31], he modified this definition, but in a way that does not alter the original one when f is a fibration. We can interpret it as a distance:

Theorem 6.5. *Let $f\colon X\to Y$ be a fibration and consider the map $\pi\colon X^I\to X\times Y$, $\pi(\gamma)=(\gamma(0), f(\gamma(1)))$ and the projections $\pi_X\colon X\times Y\to X$ and $\pi_Y\colon X\times Y\to Y$. Then,* $\mathrm{cx}(f)=\mathrm{D}(f\circ\pi_X,\pi_Y)$.

Proof. First, suppose that there exists an open subset U of $X\times Y$ and a continuous section $s\colon U\to X^I$ for π. Then we define a homotopy between $f\circ\pi_X$ and π_Y, $H\colon U\times I\to Y$ given by $H(x,y,t)=f(s(x,y)(t))$. Conversely, suppose that there exists an open subset U of $X\times Y$ and a homotopy $H\colon U\times I\to Y$ between $f\circ\pi_X$ and π_Y. Consider the following commutative diagram:

$$\begin{array}{ccc} U\times\{0\} & \xrightarrow{\pi_X} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ U\times I & \xrightarrow{H} & Y. \end{array}$$

Since f is a fibration, there exists a lifting $\widetilde{H}\colon U\times I\to X$ of H, which makes the diagram commutative. Now, we define a section $s\colon U\to X^I$ of π, given by $s(x,y)(t)=\widetilde{H}(x,y,t)$. □

6.5. Weak Category

Yokoi [39] defined the following *weak category* of a continuous map $f\colon X\to X$ from a topological space X into itself, with respect to a subspace $A\subset X$. This invariant is of interest when studying a notion of discrete Conley index.

Definition 6.6. The *weak category of f reduced to A*, denoted by $c_A^*(f)$, is the smallest integer k such that $A=U_0\cup\cdots\cup U_k$, where the U_i are open in A and each restriction ${f^{n_i}}_{|U_i}\colon U_i\to X$ is null-homotopic for some n_i, where $f^{n_i}=f\circ\overset{n_i)}{\cdots}\circ f$.

As always, we have normalized it. We found the following interpretation in terms of the homotopic distance.

Proposition 6.7. *If X is path connected. and $x_0 \in X$, then the weak category of f reduced to A equals*

$$c^*_A(f) = \lim_{n\to\infty} \mathrm{D}_X(A; f^n, x_0).$$

Proof. First, observe that

$$\mathrm{D}_X(A; f^{n+1}, x_0) = \mathrm{D}_X(A; f^{n+1}, f(x_0)) \leq \mathrm{D}_X(A; f^n, x_0),$$

because $f^n \simeq x_0$ on U_i implies $f^{n+1} \simeq f(x_0)$ on U_i, and the constant maps x_0 and $f(x_0)$ are homotopic.

If $c^*_A(f) \leq k$, we take an open covering $U_0 \cup \cdots \cup U_k = A$ such that ${f^{n_i}}_{|U_i} \simeq x_0$ for some n_i. By taking n as the maximum of the n_i, we can assume that n is the same for all U_i. Then $\mathrm{D}_X(A; f^n, x_0) \leq k$, hence $\lim_{n\to\infty} \mathrm{D}(f^n, x_0) \leq k$.

On the other hand, if $\lim_{n\to\infty} \mathrm{D}_X(A; f^n, x_0) \leq k$, then $\mathrm{D}_X(A; f^n, x_0) \leq k$ for some integer n. By the definition of subspace distance, there is an open covering $U_0 \cup \cdots \cup U_k = A$ such that ${f^n}_{|U_i} \simeq x_0$ for all i. Then $c^*_A(f) \leq k$. □

7. Navigation Functions

Navigation functions exploit the gradient flow of a Morse–Bott function for constructing motion planning algorithms.

Originally, Koditschek and Rimon [21] studied machines that navigate to a fixed goal using a gradient flow technique. Later, Farber [12] considered navigation functions, as in Theorem 2.5, which depend on two variables, the source and the target. He gave an explicit description of motion planning algorithms based in a navigation function. We shall adapt his explanation to our generalized setting (see Fig. 1), described in Sect. 6, as a direct application of Theorem 5.8.

Assume that we have two continuous maps $f, g\colon M \to Y$, defined on the manifold M, and a Morse–Bott function $\Phi\colon M \to \mathbb{R}$, with critical values $c_1, \ldots, c_p$.

- Call r_i the subspace distance $\mathrm{D}_M(\Sigma_i; f, g)$, for each critical level Σ_i of Φ. Find a decomposition $G^i_1 \cup \cdots G^i_{r_i} = \Sigma_i$ of Σ_i into ENRs, which solves the generalized motion planning in Σ_i for the restrictions of f and g.
- Consider the basins of attraction $V^i_j \subset M$ of each G^i_j, $i = 1, \ldots, p$, $j = 1, \ldots, r_i$. If $x \in V^i_j$, we can move x following the trajectory $x(t)$ of the negative gradient flow (may be in an infinite time), arriving to some point $\alpha \in G^i_j \subset \Sigma_i$. The path $f(x(t))$ connects $f(x)$ to $f(\alpha)$. We then consider the path γ on Y from $f(\alpha)$ to $g(\alpha)$ which solves the motion problem for α in G^i_j.
- Finally, we consider the image $g(\bar{x}(t))$ by g of the inverse path $\bar{x}(t)$.

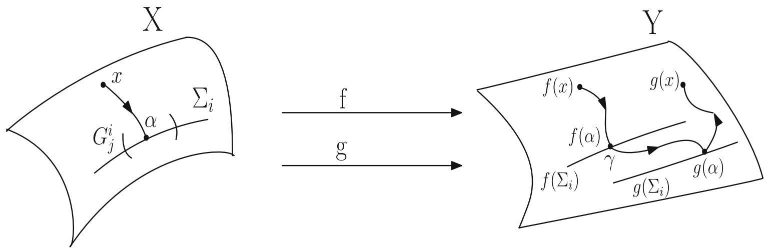

Figure 1. Navigation algorithm

Remark. Since $x'(t) = (\operatorname{grad}\Phi)_{x(t)}$ for all t, we can reparametrize the flow by considering the change of variable

$$s = \int_0^t |(\operatorname{grad}\Phi)_{x(t)}|\mathrm{d}t.$$

In this way, the trajectory $x(s)$ reaches the critical submanifold Σ_i in a finite time.

The same ideas can be applied to the cut locus of a submanifold, in order to apply Theorem 5.8.

See [15, Section 3.2], [26, Section 5] or [3, Remark 2.12, Lemma 3.18 and Theorem 3.28] for explicit computations of gradient flows.

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