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Global rigidity of (quasi-)injective frameworks on the line

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ABSTRACT

A realization of a graph G is a pair (G, p) where p maps the vertices of G into Euclidean space $\mathbb{R}^d$. The realization is injective if p is injective and quasi-injective if for each edge of G, p maps the endpoints of the edge to different points in space. The realization is globally rigid if any realization (G, q) in $\mathbb{R}^d$ with the same edge lengths is congruent to (G, p).
In this paper we characterize graphs that have an injective (quasi-injective, respectively) non-globally rigid realization in $\mathbb{R}^1$, and we show that the problem of recognizing these graphs is NP-complete in both the injective and the quasi-injective cases. Our characterizations are based on the notion of NAC-colorings, which have been used previously to investigate similar problems in the plane. We also give an overview of related results and open problems in rigidity theory.

1. Introduction

A *(bar-and-joint) framework* in $\mathbb{R}^d$ is a pair (G, p) where $G = (V, E)$ is a (simple) graph and $p : V \to \mathbb{R}^d$ maps the vertices of G into Euclidean space. We also say that (G, p) is a *realization* of G. As the naming suggests, the vertices of the framework may be thought of as universal joints and the edges as rigid bars. A framework is *rigid* if it cannot be deformed continuously while keeping the bar lengths fixed. If this is true for non-continuous deformations as well, so that the bar lengths uniquely determine the configuration of the vertices in $\mathbb{R}^d$, then the framework is *globally rigid*. We can formalize these notions as follows.

Two frameworks (G, p) and (G, q) in $\mathbb{R}^d$ are *equivalent* if for every edge $uv \in E$ we have $\|p(u) - p(v)\| = \|q(u) - q(v)\|$, i.e. the corresponding edge lengths coincide in the two frameworks. If $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ holds for all pairs of vertices $u, v \in V$, then we say that the frameworks are *congruent*. A *motion* of a framework (G, p) in $\mathbb{R}^d$ is a family of maps $p_t : V \to \mathbb{R}^d, 0 \leq t \leq 1$, continuous with respect to t, with $p_0 = p$ and such that (G, p_t) is equivalent to (G, p) for all t. The motion is *non-trivial* if (G, p_t) and (G, p) are non-congruent for all $t > 0$. If there exists a non-trivial motion of (G, p) then we say that it is *flexible*; otherwise it is *rigid*. We say that (G, p) is *globally rigid* if every equivalent framework (G, q) in $\mathbb{R}^d$ is congruent to (G, p).

In general, it is NP-hard to decide the rigidity (global rigidity, respectively) of a framework in dimension $\mathbb{R}^d$ for any fixed $d \geq 2$ ($d \geq 1$, respectively) ([1,17]). On the other hand, for *generic* frameworks, in which the set of coordinates of $p(v), v \in V$ is algebraically independent over $\mathbb{Q}$, rigidity and global rigidity is completely determined by the structure of the underlying graph, so that either all generic realizations are rigid in $\mathbb{R}^d$ (globally rigid in $\mathbb{R}^d$, respectively) or none of them are (see

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[3,4,6]). A combinatorial characterization of graphs with generic (globally) rigid realizations is known in $d = 1, 2$ dimensions but is a major open problem in $d \geq 3$ dimensions.

It is natural to also consider the family of graphs whose realizations in a given dimension are all (globally) rigid. However, if we do not make any non-degeneracy assumptions on the realizations in question, then these families turn out to be rather uninteresting. In particular, if a graph G has a pair of non-neighboring vertices u, v, then a realization mapping u and v to distinct points and the rest of the vertices to a third point is flexible in $\mathbb{R}^d, d \geq 2$ and non-globally rigid in $\mathbb{R}^d, d \geq 1$. This suggests only considering *injective* frameworks, in which different vertices are mapped to different points in space. More leniently, we may only require that neighboring vertices are mapped to different points (in other words, that the edge lengths are non-zero in the framework). We shall call such frameworks *quasi-injective*.

Thus, we are led to the question of characterizing graphs whose (quasi-)injective realizations in $\mathbb{R}^d$ are (globally) rigid. In the case of rigidity, this problem can be easily settled in the $d = 1$ and $d \geq 3$ cases,[1] while the $d = 2$ case is much more difficult. In [7], a combinatorial characterization was given for graphs that have a flexible quasi-injective realization in the plane in terms of the existence of a so-called *NAC-coloring* of its edges. In the case of injective realizations, only partial results are known. Graphs with no flexible injective realizations in the plane have been called absolutely 2-rigid graphs [15], or non-movable graphs [8]. In the latter paper, the authors give a necessary, but not sufficient, condition for movability that is also based on the notion of NAC-colorings.

Our aim in this paper is to examine the analogous questions regarding global rigidity. Here, the only non-trivial case is when $d = 1$: for a non-complete graph G with distinct non-neighboring vertices u, v, we can construct a non-globally rigid injective realization in $\mathbb{R}^d, d \geq 2$ by mapping the vertices other than u and v onto a line injectively, and mapping u and v to distinct points not on this line; then reflecting u through a hyperplane containing the line but not containing u and v gives an equivalent, non-congruent realization.

The $d = 1$ case turns out to be related to the planar rigidity case discussed above. In particular, a graph has a non-globally rigid quasi-injective realization in $\mathbb{R}^1$ if and only if it has a flexible quasi-injective realization in $\mathbb{R}^2$ (Theorems 2.1 and 2.2). As an analogue to Theorem 2.2, we also give a characterization of graphs that have a non-globally rigid injective realization in $\mathbb{R}^1$ (Theorem 2.4).[2] This graph family, however, is not the same as the family of movable graphs; in fact, the former is a strict subset of the latter.

We also prove that in both the injective and quasi-injective case, the problem of recognizing such graphs is NP-complete. In the case of injective realizations this is simply done by reformulating the characterization given in Theorem 2.4 as a known NP-complete problem (Corollary 3.1), while in the case of quasi-injective frameworks, we prove that the 3-SAT problem can be polynomially reduced to the problem of deciding whether a graph has a NAC-coloring (Theorem 3.5).

The rest of the paper is laid out as follows. In Section 2 we recall the definition of NAC-colorings, as well as introduce *grid-like* frameworks and characterize graphs that have a non-globally rigid (quasi-)injective realization in $\mathbb{R}^1$ in terms of these notions. In Section 3 we give the aforementioned hardness proofs. Finally, in Section 4 we consider more generally the type of questions studied in this paper. These questions have the form "which graphs have a non-degenerate realization with a given rigidity property?" A survey of these problems in the case of various non-degeneracy and rigidity notions is given in Tables 1 to 3, with the purpose of highlighting open problems.

2. NAC-colorings and grid-like frameworks

Let G be a graph. Following [7], we say that a 2-coloring of the edges of G is a *NAC-coloring* if both colors are used and no cycle has exactly one edge of a given color. We shall always refer to the two colors as red and blue. The name stands for "no almost (unicolored) cycles". As we shall see, NAC-colorings are related to the following, special kind of frameworks.

Let us say that a framework (G, p) in $\mathbb{R}^2$ is *grid-like* if each edge in (G, p) is either vertical or horizontal. We say that a grid-like framework is *non-trivial* if there is at least one horizontal and at least one vertical edge. More generally, we could consider realizations of G in $\mathbb{R}^d$ that use exactly d linearly independent directions, for some $d \geq 2$. However, a graph that has such a realization also has a non-trivial grid-like realization. This follows from the observation that a non-trivial grid-like realization is always flexible: one can "fold" it onto the line (or, in the case of a realization that uses d directions in $\mathbb{R}^d$, into a realization in $\mathbb{R}^{d-1}$ that uses $d - 1$ directions). We shall give a concrete description of this folding motion in the next proof.

The following is the main result in [7]. Since we shall use it, we recall the proof of the first two implications, but omit the third (which is the most difficult one).

Theorem 2.1. *[7, Theorem 3.1] For a graph $G = (V, E)$, the following are equivalent:*

a) *G has a NAC-coloring,*

[1] See Tables 1 to 3 at the end of this paper for an overview of these different graph families, among others.

[2] Jim Geelen gave essentially the same characterization for these graphs, although in a different formulation, see [11, Theorem B.1]. Thus, our main contribution in the injective case is in clarifying its relationship with the quasi-injective case, as well as with the notions of NAC-colorings and grid-like frameworks.

b) *G has a quasi-injective, non-trivial grid-like realization in* $\mathbb{R}^2$,
c) *G has a quasi-injective realization in* $\mathbb{R}^2$ *that is not rigid.*

Proof. $a) \Rightarrow b)$: Fix a NAC-coloring of G and let $R_1, \ldots, R_k$ and $B_1, \ldots, B_l$ be the vertex sets of the connected components of the subgraph of red and blue edges, respectively. We claim that the framework (G, p) in $\mathbb{R}^2$ defined by $p(v) = (i, j)$ if $v \in R_i \cap B_j$ is a quasi-injective, non-trivial grid-like realization of G. Indeed, consider an edge $uv \in E$ in G. Without loss of generality we may suppose that it is colored red. Then there cannot be a path of blue edges between u and v, since together with the edge uv this would give a cycle with precisely one red edge. This shows that (G, p) is quasi-injective. It is also clear that each red edge is vertical and each blue edge is horizontal, so that (G, p) is grid-like. Finally, it is non-trivial, since there exists at least one edge of both color classes.

$b) \Rightarrow c)$: As we have noted, non-trivial grid-like frameworks are always flexible; for completeness, we give a concrete example of their motion in $\mathbb{R}^2$. Let (G, p) be a quasi-injective, non-trivial grid-like realization in $\mathbb{R}^2$ and let $p(v) = (x_v, y_v)$ denote the coordinates of each vertex v. Identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ so that $p(v) = x_v + iy_v$. Now the intuitive "folding" motion of (G, p) is given by $p_t(v) = x_v + ie^{it} y_v, 0 \leq t \leq \frac{\pi}{2}$. This preserves the edge lengths of (G, p): horizontal edges are only translated, while vertical edges are translated and rotated throughout the motion. This observation also shows that the angle between horizontal and vertical edges changes during the motion, so it is non-trivial. □

The framework $(G, p_{\pi/2})$ in the previous proof lies on a line in $\mathbb{R}^2$. We can also fold the framework into $\mathbb{R}^1$ in the other direction via the motion $p_t, 0 \geq t \geq \frac{-\pi}{2}$. It is not difficult to see that if the grid-like framework was non-trivial, then these one-dimensional realizations are non-congruent. Thus we have that graphs with a NAC-coloring have quasi-injective non-globally rigid realizations in $\mathbb{R}^1$. It turns out that the reverse implication is true as well.

Theorem 2.2. *A graph $G = (V, E)$ has a quasi-injective realization in $\mathbb{R}^1$ that is not globally rigid if and only if it has a quasi-injective non-trivial grid-like realization in $\mathbb{R}^2$.*

Proof. $\Rightarrow$: If G is not connected, then the statement is trivial since we can just draw one connected component of G on a vertical line in $\mathbb{R}^2$ and the rest of the graph on a horizontal line to obtain a non-trivial grid-like realization. Thus, let us suppose that G is connected. Let (G, p) be a quasi-injective framework in $\mathbb{R}^1$ and (G, q) equivalent, but not congruent to (G, p).

Consider the framework (G, p') in $\mathbb{R}^2$ defined by

$$p'(v) = \left(\frac{p(v) + q(v)}{2}, \frac{p(v) - q(v)}{2} \right), \qquad v \in V.$$

Let $uv \in E$ be an edge. Since (G, p) and (G, q) are equivalent, we have that $|p(u) - p(v)| = |q(u) - q(v)|$. It is immediate from the definition of p' that if $p(u) - p(v) = q(u) - q(v)$ then uv is horizontal in (G, p'), while if $p(u) - p(v) = q(v) - q(u)$, then it is vertical, so (G, p') is indeed grid-like. Moreover, it is non-trivial, since if (for example) every edge was horizontal, then $p(u) - p(v) = q(u) - q(v)$ for all edges uv; but since G is connected, this would uniquely determine q up to translations, so (G, p) and (G, q) would be congruent, contradicting the choice of (G, q). The same reasoning applies in the case when every edge is vertical. Finally, note that (G, p) and (G, p') are equivalent, so (G, p') is quasi-injective as well.

$\Leftarrow$: Let (G, p) be a quasi-injective non-trivial grid-like realization in $\mathbb{R}^2$ and let us consider the mappings $f, g : \mathbb{R}^2 \to \mathbb{R}^1$ defined by $f(x, y) = x + y$ and $g(x, y) = x - y$. We claim that $(G, f \circ p)$ and $(G, g \circ p)$ are equivalent but non-congruent quasi-injective frameworks. Note that, using the notation in the proof of Theorem 2.1, $f \circ p = p_{\pi/2}$ and $g \circ p = p_{-\pi/2}$. Thus, both frameworks are equivalent to (G, p). The fact that they are non-congruent follows from the fact that (G, p) is non-trivial: indeed, this implies the existence of vertices u, v of G with $p(u) = (x_u, y_u), p(v) = (x_v, y_v)$ and such that $x_u \neq x_v$ and $y_u \neq y_v$. Then it is easy to see that

$$|f(p(u)) - f(p(v))| = |x_u - x_v + y_u - y_v| \neq |x_u - x_v - (y_u - y_v)| = |g(p(u)) - g(p(v))|,$$

so $(G, f \circ p)$ and $(G, g \circ p)$ are non-congurent, as desired. □

Corollary 2.3. *A graph has a quasi-injective realization in $\mathbb{R}^1$ that is not globally rigid if and only if it has a NAC-coloring.* □

We can also use NAC-colorings and grid-like frameworks to give an analogue of Theorem 2.2 for injective frameworks. The first implication in the following proof can be found in [8, Lemma 4.2]. The equivalence of the first two conditions has been independently observed multiple times, see e.g. [9, Appendix A] and references therein.

Theorem 2.4. *For a graph $G = (V, E)$, the following are equivalent:*

a) *G has a NAC-coloring for which $|R_i \cap B_j| \leq 1$ for all $i \leq i \leq k$, $1 \leq j \leq l$, where $R_1, \ldots, R_k$ and $B_1, \ldots, B_l$ are the vertex sets of the connected components of the subgraph of red and blue edges, respectively,*

b) *G has an injective non-trivial grid-like realization in* $\mathbb{R}^2$,
c) *G has an injective realization in* $\mathbb{R}^1$ *that is not globally rigid.*

Proof. $a) \Rightarrow b)$: The proof is the same as in Theorem 2.1; the condition on the coloring ensures that the grid-like realization constructed there is injective.

$b) \Rightarrow a)$: Given an injective non-trivial grid-like realization (G, p), color the edges of G that are vertical in the realization red, and the horizontal edges blue. It is easy to see that this is a NAC-coloring with the desired property.

$b) \Rightarrow c)$: Let (G, p) be an injective non-trivial grid-like realization in $\mathbb{R}^2$. As in the proof of Theorem 2.2, the frameworks $(G, f \circ p)$ and $(G, g \circ p)$ are equivalent and non-congruent; we only need to make sure that $(G, f \circ p)$ is injective. In other words, we need that $x_u + y_u \neq x_v + y_v$ for any pair of vertices $u, v \in V$ with $p(u) = (x_u, y_u)$ and $p(v) = (x_v, y_v)$. We can ensure this by stretching (G, p) vertically, i.e. by considering the framework (G, q) defined by $q(v) = (x_v, ty_v)$, $v \in V$, for some $t > 0$, instead of (G, p). Suppose that $x_u + ty_u = x_v + ty_v$ for some pair of vertices $u, v \in V$. If t is sufficiently large, then this implies $ty_u = ty_v$ and thus $x_u = x_v$ as well, contradicting the assumption that (G, p) is injective.

$c) \Rightarrow b)$: The proof is the same as in Theorem 2.2, noting that if (G, p) is injective in $\mathbb{R}^1$, then the grid-like framework (G, p') constructed in that proof is injective as well. □

Note that in contrast to the quasi-injective case, Theorem 2.4 does not give a characterization of movable graphs, i.e. ones with an injective flexible realization in $\mathbb{R}^2$. Indeed, it is well-known that the complete bipartite graph $K_{3,3}$ is movable, while it is not difficult to see that it does not have an injective non-trivial grid-like realization in $\mathbb{R}^2$.

3. Hardness results

The characterizations given by Corollary 2.3 and Theorem 2.4 leave open the question whether the recognition of graphs with these properties is algorithmically tractable. To be more precise, we may consider the following decision problems.

Problem. HAS NAC-COLORING.
Input: a graph G.
Output: YES *if G has a NAC-coloring,* NO *otherwise.*

Problem. HAS GRID-LIKE REALIZATION.
Input: a graph G.
Output: YES *if G has an injective non-trivial grid-like realization in* $\mathbb{R}^2$, NO *otherwise.*

It turns out that both of these problems are NP-complete. In the case of HAS GRID-LIKE REALIZATION, this can be shown by reformulating the problem in the following way. Given two graphs G, H, their *Cartesian product*, denoted by $G \square H$, is the graph on vertex set $V(G) \times V(H)$ in which there is an edge between the vertices (u_1, v_1) and (u_2, v_2) precisely if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. This graph can be visualized as a $V(G) \times V(H)$ grid in which each row is a copy of H and each column is a copy of G. The product $K_n \square K_m$ is also called the $m \times n$ *rook's graph*. It is immediate from the definitions that for any graphs G and H, $G \square H$ is a subgraph of the $|V(G)| \times |V(H)|$ rook's graph. A subgraph of $G \square H$ is *non-trivial* if it is not contained in any row or column. A graph is said to be *S-composite* if it can be written as a non-trivial subgraph of $G \square H$ for some graphs G and H; by the previous remark, we can equivalently consider non-trivial subgraphs of $K_n \square K_m$ for some $n, m \geq 1$. A graph is *S-prime* if it is not S-composite. The “S” in the name stands for “subgraph”.

Now a graph is S-composite if and only if it has an injective non-trivial grid-like realization in $\mathbb{R}^2$. Indeed, by adding all horizontal and vertical edges, any grid-like realization can be augmented to $K_n \square K_m$ for some n and m, while the definition of $K_n \square K_m$ immediately suggests a grid-like realization on the $n \times m$ grid. This gives the following corollary to Theorem 2.4.

Corollary 3.1. *A graph has an injective realization in* $\mathbb{R}^1$ *that is not globally rigid if and only if it is S-composite.* □

Thus, HAS GRID-LIKE REALIZATION is equivalent to the following problem, shown to be NP-complete in [10].

Problem. IS S-COMPOSITE.
Input: a graph G.
Output: YES *if G is S-composite,* NO *otherwise.*

Theorem 3.2. *[10, Theorem 2.12]* IS S-COMPOSITE *is NP-complete.*

Corollary 3.3. *The recognition of graphs that have a non-globally rigid injective realization in* $\mathbb{R}^1$ *is NP-complete.*

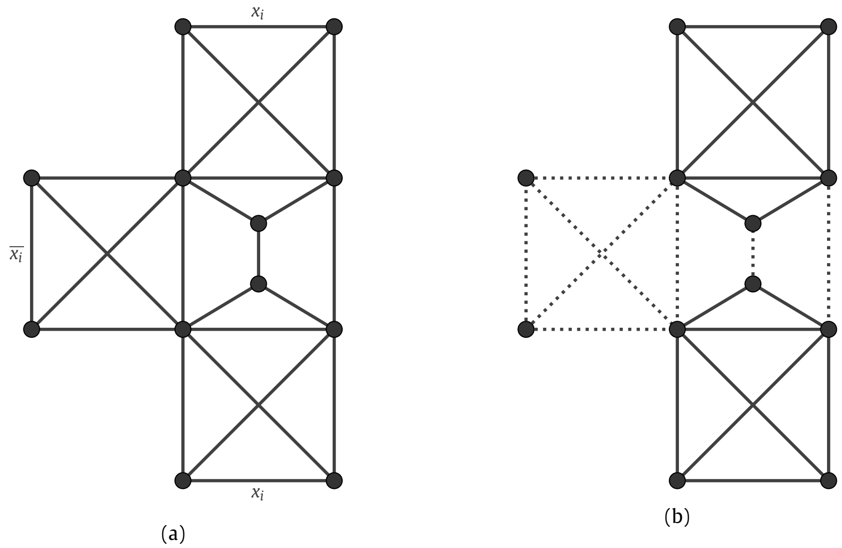

Fig. 1. (a) A connecting element, made up of a "prism graph" with three copies of K_4 attached to it. We shall refer to the labeled edges as the *ends* of the connecting element. (b) In a coloring of a connecting element with no almost unicolored cycles, each of the solid edges and each of the dotted edges must have the same color. In particular, either all of the edges have the same color, or the top and bottom ends have the same color and the left end has a different color.

The rest of this section is devoted to showing that Has NAC-coloring is NP-complete. First, the following observation implies that this problem is in NP.

Lemma 3.4. *[7, Lemma 2.4] A partition of the edges of a graph G into two non-empty sets E_b, E_r is a NAC-coloring if and only if each connected component of $G[E_b]$ and $G[E_r]$ is an induced subgraph of G.*

We shall show that the well-known NP-complete 3-SAT problem can be reduced to Has NAC-coloring:

Problem. 3-SAT.
Input: A boolean formula φ in conjunctive normal form in which each clause contains at most three literals.
Output: YES *if φ is satisfiable,* NO *otherwise.*

Essentially the same reduction works for any boolean formula in conjunctive normal form, regardless of the number of literals in each clause.

Theorem 3.5. Has NAC-coloring *is NP-complete.*

Proof. Given a 3-SAT instance φ with variables $x_1, \dots, x_n$ and clauses $L_1, \dots, L_k$, we shall construct a graph G_φ of size $O(n+k)$ such that φ is satisfiable if and only if G_φ has a NAC-coloring. During the construction we shall label the edges of G_φ with the literals x_i and $\overline{x_i}$, as well as the true literal t and false literal f, in such a way that if two edges have the same label, then in any NAC-coloring of G_φ they must have the same color. First, we shall create a number of disjoint edges and cycles, and then we connect some triplets of edges by gluing onto them a *connecting element* in a particular way. In the following, we will use the notation $x_i^1 = x_i$ and $x_i^{-1} = \overline{x_i}$, as well as $\bar{t} = f$ and $\overline{f} = t$.

The construction goes as follows. First, take $2n+2$ disjoint edges with the labels $t, f, x_i, \overline{x_i}, i = 1, \dots, n$; we shall refer to the edge with label $x_i^{\varepsilon_i}, \varepsilon_i \in \{-1, 1\}$ as the *terminal of the literal* $x_i^{\varepsilon_i}$; similarly, the edges with labels t and f will be referred to as the *true terminal* and *false terminal*, respectively. Then for each variable x_i we create two cycles A_i and B_i of lengths 5 and 4, respectively. We label the edges of A_i with $t, x_i, x_i, \overline{x_i}, \overline{x_i}$ in order and the edges of B_i with $t, f, x_i, \overline{x_i}$. Also, for each clause $L_i = \{x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, x_3^{\varepsilon_3}\}$ we create a cycle C_i of length 7 with edge labels $t, x_1^{\varepsilon_1}, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, x_2^{\varepsilon_2}, x_3^{\varepsilon_3}, x_3^{\varepsilon_3}$. Let G_0 denote the union of the cycles $A_i, B_i, i = 1, \dots, n$ and $C_j, j = 1, \dots, k$.

Finally, we connect each edge in G_0 with the corresponding terminal via *connecting elements*, graphs isomorphic to the one depicted in Fig. 1. Let $e \in E(G_0)$ be an edge and let l denote the literal with which e is labeled. We attach a connecting element so that its bottom edge (as drawn in the figure) is e, its top edge is the terminal corresponding to l, and its left-most edge is the terminal corresponding to $\bar{l}$ (the negation of the literal). We shall refer to these three edges as the *ends* of the connecting element. We do this for each edge separately in G_0. Denote the graph obtained in this way by G_φ.

We would like to show that G_φ has a NAC-coloring if and only if φ is satisfiable. Suppose first that there is a NAC-coloring $\delta : E(G_\varphi) \to \{blue, red\}$ of G_φ. We may assume that the true terminal is colored blue. Note that the construction of the connecting elements ensures that each edge labeled with the literal l must have the same color as the terminal corresponding to l (see Fig. 1b). Now from the cycles A_i, B_j, C_l we have the following properties of δ.

Claim.

a) *If there is a literal $x_i^{\varepsilon_i}$ such that its terminal is colored red, then the terminal of $x_i^{-\varepsilon_i}$ is colored blue. Indeed, otherwise the cycle A_i would have precisely one blue edge.*
b) *If there is a variable x_i such that the terminals of x_i and $\overline{x_i}$ have different colors, then the true and false terminals must have different colors as well (i.e. the false terminal must be colored red); otherwise the cycle B_i would have only one red edge.*
c) *Similarly, if the false terminal is colored red, then for each variable x_i, the terminals of x_i and $\overline{x_i}$ must have different colors.*
d) *Finally, in each clause L_j there must be a literal $x_i^{\varepsilon_i}$ such that its terminal is colored blue, since otherwise C_j would contain only one blue edge.* □

By definition, there must be at least one red edge in G_φ. Note that each edge is contained in some connecting element, so, as shown in Fig. 1b, one of the terminals must be red as well. Then *a*) and *b*) in the previous claim imply that the false terminal must be colored red. Now from *c*) it follows that the terminals of x_i and $\overline{x_i}$ have different colors, for every $i = 1, \dots, n$. Consider the truth assignment in which x_i is true if and only if its terminal has color blue under δ; claim *d*) implies that this truth assignment satisfies φ, as needed.

Now we work in the other direction. Given a truth assignment satisfying φ, we construct a coloring δ of G_φ by coloring the terminals labeled with true literals (including t) blue and the rest of the terminals red, and then coloring the edges in each connecting element according to the colors of its end terminals as in Fig. 1b. We claim that this is a NAC-coloring of G_φ. To the contrary, suppose that there is an almost unicolored cycle C in G_φ. For convenience, orient the edges of C cyclically.

It is immediate from the construction that δ is a NAC-coloring of each cycle A_i, B_j, C_k and each connecting element, so C cannot be contained in either of these subgraphs. It follows that C must enter some connecting element K in the sense that there is an end uv of K, a vertex w not in K and a vertex w' in K such that wu and uw' are oriented edges of C. Now C must exit K through some vertex of an end of K. If it exits through v, then we can shortcut C through uv without destroying the almost unicolored property. Thus, we may assume that C enters and exits K at different ends. Then C must either enter and exit K twice, or enter and exit another connecting element as well. But a path that enters and exits a connecting element through different ends must contain edges of both colors, so the existence of two such (disjoint) paths contradicts the assumption that C is almost unicolored. □

Corollary 3.6. *The recognition of graphs that have a non-globally rigid quasi-injective realization in $\mathbb{R}^1$ is NP-complete.*

4. Concluding remarks

In this section we survey more generally the type of questions considered in this paper. These questions have the following general form: which graphs are such that, in a given dimensions, all realizations subject to a given non-degeneracy condition have a given rigidity property? Similarly, which graphs have, in a given dimension, a non-degenerate realization with a given rigidity property? In this paper we have encountered three such non-degeneracy conditions (genericity, injectivity and quasi-injectivity), but there are other natural choices as well. One that has been considered in the literature before is *general position*: a framework in $\mathbb{R}^d$ is in general position if no $k+1$ points in it lie on an affine $(k-1)$-dimensional subspace for all $1 \le k \le d$. Note that in $\mathbb{R}^1$ this notion coincides with injectivity.

Table 1
Graph properties relating to rigid realizations.

	$d=1$	$d=2$	$d \ge 3$
∃ rigid realization	connected		
∃ rigid (quasi-)injective realization	connected	2-connected	
∃ rigid general position realization	connected	open	
∃ rigid generic realization ⟺ ∀ generic realization is rigid	connected	∃ spanning Laman subgraph [16,14]	open
∀ general position realization is rigid	connected	open	
∀ injective realization is rigid	connected	open	complete graph
∀ quasi-injective realization is rigid	connected	∄ NAC-coloring [7] (co-NP-complete)	complete graph
∀ realization is rigid	connected	complete graph	

Table 2
Graph properties relating to globally rigid realizations.

	$d=1$	$d=2$	$d\geq 3$
∃ globally rigid realization	connected		
∃ globally rigid (quasi-)injective realization	2-connected		
∃ globally rigid general position realization	$(d+1)$-connected [2]		
∃ globally rigid generic realization $\Longleftrightarrow$ ∀ generic realization is globally rigid	2-connected	3-connected and redundantly rigid [12]	open
∀ general position realization is globally rigid	S-prime (co-NP-complete)	open	
∀ injective realization is globally rigid	S-prime (co-NP-complete)	complete graph	
∀ quasi-injective realization is globally rigid	∄ NAC-coloring (co-NP-complete)	complete graph	
∀ realization is globally rigid	complete graph		

Table 3
Graph properties relating to universally rigid realizations.

	$d=1$	$d=2$	$d\geq 3$
∃ universally rigid realization	connected		
∃ universally rigid (quasi-)injective realization	2-connected		
∃ universally rigid general position realization	$(d+1)$-connected [2]		
∃ universally rigid generic realization $\Longleftrightarrow$ ∃ globally rigid generic realization	2-connected	3-connected and redundantly rigid [12]	open
∀ generic realization is universally rigid	open		
∀ general position realization is universally rigid	open		
∀ injective realization is universally rigid	open	complete graph	
∀ quasi-injective realization is universally rigid	open	complete graph	
∀ realization is universally rigid	complete graph		

These questions and their answers, where known, are catalogued in Table 1 in the case of rigidity and Table 2 in the case of global rigidity. The results without citations are either simple constructions, considered folkore, or can be found in this paper.

There is also a third rigidity notion, universal rigidity, which has received considerable attention in recent years. A framework in $\mathbb{R}^d$ is *universally rigid* if it is globally rigid when viewed as a framework in $\mathbb{R}^D$ for all $D \geq d$. This form of rigidity is less well-behaved than those considered in this paper in that the existence of a generic universally rigid realization of a graph G does not guarantee that every generic realization of G in the same dimension is universally rigid. Indeed, the former of these conditions is shown to be equivalent to G being globally rigid in [5]; on the other hand, no characterization is known for graphs that are "generically universally rigid", even in $\mathbb{R}^1$. See [13] for a discussion of this problem and several related conjectures. Questions related to universal rigidity are summarized in Table 3.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] T.G. Abbott, Generalizations of Kempe's Universality Theorem, PhD thesis, MIT, 2008.
[2] A.Y. Alfakih, Graph connectivity and universal rigidity of bar frameworks, Discrete Appl. Math. 217 (Jan. 2017) 707–710.
[3] L. Asimow, B. Roth, The rigidity of graphs, Trans. Am. Math. Soc. 245 (1978) 279–289.
[4] R. Connelly, Generic global rigidity, Discrete Comput. Geom. 33 (4) (Apr. 2005) 549–563.
[5] R. Connelly, S.J. Gortler, L. Theran, Generically globally rigid graphs have generic universally rigid frameworks, Combinatorica 40 (2020) 1–37.
[6] S.J. Gortler, A.D. Healy, D.P. Thurston, Characterizing generic global rigidity, Am. J. Math. 132 (4) (Jul. 2010) 897–939.
[7] G. Grasegger, J. Legerský, J. Schicho, Graphs with flexible labelings, Discrete Comput. Geom. 62 (2) (Sep. 2019) 461–480.
[8] G. Grasegger, J. Legerský, J. Schicho, Graphs with flexible labelings allowing injective realizations, Discrete Math. 343 (6) (Jun. 2020).
[9] G. Grasegger, J. Legersky, Bracing frameworks consisting of parallelograms, Art of Discrete Appl. Math. (2021), https://adam-journal.eu/index.php/ADAM/article/view/1379.
[10] M. Hellmuth, On the complexity of recognizing S-composite and S-prime graphs, Discrete Appl. Math. 161 (7–8) (May 2013) 1006–1013.
[11] B. Jackson, Notes on the rigidity of graphs, in: Levico Conference Notes, 2007.
[12] B. Jackson, T. Jordán, Connected rigidity matroids and unique realizations of graphs, J. Comb. Theory, Ser. B 94 (1) (May 2005) 1–29.
[13] T. Jordán, V.-H. Nguyen, On universally rigid frameworks on the line, Contrib. Discrete Math. 10 (2) (2015) 10–21.
[14] G. Laman, On graphs and rigidity of plane skeletal structures, J. Eng. Math. 4 (4) (Oct. 1970) 331–340.
[15] H. Maehara, Geometry of frameworks, Yokohama Math. J. 47 (1999) 41–65.
[16] H. Pollaczek-Geiringer, Uber die gliederung ebener fachwerke, Z. Angew. Math. Mech. 7 (1) (1927) 58–72.
[17] J.B. Saxe, Embeddability of Weighted Graphs in k-Space is Strongly NP-Hard, Technical Report, Computer Science Department, Carnegie-Mellon University, Pittsburgh, PA, 1979.