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Non-commutative propositional logic with short-circuited biconditional and NAND

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Abstract

Short-circuit evaluation denotes the semantics of propositional connectives in which the second argument is evaluated only if the first argument does not suffice to determine the value of the expression. In programming, short-circuit evaluation is widely used, with left-sequential conjunction and disjunction as primitive connectives.

We consider left-sequential, non-commutative propositional logic, also known as MSCL (memorising short-circuit logic), and start from a previously published, equational axiomatisation. First, we extend this logic with a left-sequential version of the biconditional connective, which allows for an elegant axiomatisation of MSCL. Next, we consider a left-sequential version of the NAND operator (the Sheffer stroke) and again give a complete, equational axiomatisation of the corresponding variant of MSCL. Finally, we consider these logical systems in a three-valued setting with a constant for 'undefined', and again provide completeness results.

Keywords: Non-commutative propositional logic, conditional connective, sequential connective, NAND, short-circuit evaluation, proposition algebra

Contents

1 Introduction

This paper is about non-commutative propositional logic, also known as Memorising Short-Circuit Logic (MSCL), enriched with some alternative connectives. In [3], MSCL is defined as a logic for equational reasoning about sequential propositions with the property that atomic side effects do not occur, so that in the evaluation of a compound statement, the first evaluation result of each atom is memorised. Furthermore, the prescribed evaluation strategy is *short-circuit evaluation*: the second argument in a conjunction or a disjunction is evaluated only if the first argument does not suffice to determine its evaluation result. In MSCL, the binary connectives are left-sequential and written as

$$\mathbin{{}_{\circ}\!\wedge} \quad \text{and} \quad \mathbin{{}^{\circ}\!\vee},$$

where the little circle prescribes that the left-argument must be evaluated first. We define these connectives using Hoare's *conditional*, a ternary connective that naturally prescribes short-circuit evaluation (in an if-then-else manner), and which has an elegant duality property.

Left-sequential conjunction $\mathbin{{}_{\circ}\!\wedge}$ and disjunction $\mathbin{{}^{\circ}\!\vee}$ are not commutative, but many common equational laws hold, such as the double negation shift, and idempotence and left-distributivity of the binary connectives.

We first extend MSCL with the connective $\mathbin{\circ\!\!\leftrightarrow}$, a left-sequential version of the biconditional connective $\leftrightarrow$, which in the MSCL-setting can be defined in different, equivalent ways:

$$x \mathbin{\circ\!\!\leftrightarrow} y = (\neg x \mathbin{{}^{\circ}\!\vee} y) \mathbin{{}_{\circ}\!\wedge} (x \mathbin{{}^{\circ}\!\vee} \neg y) = (x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\vee} (\neg x \mathbin{{}_{\circ}\!\wedge} \neg y).$$

We provide a complete, equational axiomatisation for this extension. The combination of $\mathbin{\circ\!\!\leftrightarrow}$ with $\mathbin{{}_{\circ}\!\wedge}$, $\mathbin{{}^{\circ}\!\vee}$ and negation allows for an equational axiomatisation that is simple and perhaps more natural than the one for MSCL.

Then, we consider an alternative for MSCL, based on a left-sequential version of the NAND-operator (the Sheffer stroke) and again provide a complete, equational axiomatisation.

Finally, following [4], we consider extensions of these logical systems with U, a constant for the truth value "undefined". We discuss an advantage of the resulting NAND-system.

Structure of the paper. In Section 2, Hoare's ternary *conditional* and 'basic forms' for this connective are discussed, and the left-sequential connectives are defined with this connective.
- In Section 3 memorising valuation congruence is discussed, the congruence that characterises MSCL, and a normalisation function for so-called 'mem-basic forms'.
- In Section 4, we discuss in detail the equational axiomatisation for MSCL that is the starting point for this paper.
- In Section 5 we introduce a left-sequential variant of the biconditional connective and give an equational axiomatisation for the extension of MSCL with this connective. We prove a correspondence result, and briefly discuss duality.
- In Section 6 we discuss a left-sequential variant of the NAND connective, provide axioms for the memorising variant of this extension, and prove a correspondence result.
- In Section 7 we consider the extension of the new logical systems with U.
- In Section 8 we present some conclusions and discuss related work.

All derivations from equational axiomatisations were found by the theorem prover *Prover9*, and all independence results were found by the tool *Mace4*, see [10] for both tools.

$$x \triangleleft \mathsf{T} \triangleright y = x \tag{CP1}$$
$$x \triangleleft \mathsf{F} \triangleright y = y \tag{CP2}$$
$$\mathsf{T} \triangleleft x \triangleright \mathsf{F} = x \tag{CP3}$$
$$x \triangleleft (y \triangleleft z \triangleright u) \triangleright v = (x \triangleleft y \triangleright v) \triangleleft z \triangleright (x \triangleleft u \triangleright v) \tag{CP4}$$

Table 1: CP (Conditional Propositions), a set of equational axioms for free valuation congruence

2 The conditional, basic forms, and propositional connectives

In this section we recall Hoare's ternary *conditional* connective and 'basic forms' for the conditional. Next, we define sequential versions of the common propositional connectives using this connective.

In 1985, Hoare introduced in [9] the ternary *conditional* connective $p \triangleleft q \triangleright r$ in order to express "`if` q `then` p `else` r" [1] and provided eleven equational axioms to show that the conditional and the two constants T and F for truth and falsehood characterise the propositional calculus.[2]

We are interested in both duality and equational axioms, and moreover in the short-circuit evaluation strategy suggested by the if-then-else reading that prescribes that in $p \triangleleft q \triangleright r$, *first* q is evaluated, and only *then* either p or r to determine the result of the evaluation.

Throughout this paper let A be a set of atoms (propositional variables). The signature we consider is $\Sigma_{\mathrm{CP}}(A) = \{\_ \triangleleft \_ \triangleright \_, \mathsf{T}, \mathsf{F}, a \mid a \in A\}$ and we write

$$C_A$$

for the set of closed terms over this signature. Table 1 displays of a set of equational axioms for terms over this signature, and we will refer to these axioms as CP (for Conditional Propositions).

The dual of a closed term $P \in C_A$, notation P^{dl}, is defined as follows (for $a \in A$):

$$\begin{aligned} \mathsf{T}^{dl} &= \mathsf{F}, & a^{dl} &= a, \\ \mathsf{F}^{dl} &= \mathsf{T}, & (P \triangleleft Q \triangleright R)^{dl} &= R^{dl} \triangleleft Q^{dl} \triangleright P^{dl}. \end{aligned}$$

The duality mapping is an involution, $(P^{dl})^{dl} = P$. Setting $x^{dl} = x$ for each variable x, the duality principle extends to equations and it is easy to see that CP is a self-dual axiomatisation: (CP1) and (CP2) are each other's dual, and (CP3) and (CP4) are self-dual (i.e., identical to their own duals). Hence,

$$\text{For all terms } s, t \text{ over } \Sigma_{\mathrm{CP}}(A),\ \mathrm{CP} \vdash s = t \iff \mathrm{CP} \vdash s^{dl} = t^{dl}.$$

We define the subset of 'basic forms' of C_A that can be used to prove that CP is a complete set of axioms.

[1] However, in 1948, Church introduced in [5] the *conditioned disjunction* connective $[p, q, r]$, which, following the author, may be read "p or r according as q or not q." and which expresses exactly the same connective as Hoare's conditional. We further discuss this in Section 8 (Related work).

[2] In 1963, Dicker provided in [8] a set of five independent and elegant axioms for the conditioned disjunction. Unaware of this, we provided in [4] a set of four simple, independent axioms that is also complete.

Definition 2.1. ***Basic forms over*** A *are defined by the following grammar*

$$t ::= \mathsf{T} \mid \mathsf{F} \mid t \triangleleft a \triangleright t \quad \textit{for } a \in A.$$

We write BF_A *for the set of basic forms over* A*.*

Basic forms[3] can be seen as *evaluation trees*, which are binary, rooted trees with internal nodes labelled from A and leaves in $\{\mathsf{T}, \mathsf{F}\}$: a branch from the root to a leaf represents the process of evaluation and all internal nodes represent the evaluation of the atom it is labelled with, while the leaf represents the final evaluation result. Also, T and F are seen as evaluation trees that represent the evaluation of the constants T and F, respectively. Typically, the evaluation tree associated with $\mathsf{T} \triangleleft a \triangleright \mathsf{F}$ is

$$\begin{array}{ccc} & a & \\ \diagup & & \diagdown \\ \mathsf{T} & & \mathsf{F} \end{array}$$

Evaluation trees were introduced in [14], and a formal relation between their equality and that of the associated basic forms is established in [2]. However, basic forms themselves can be represented as evaluation trees, as explained in the following example.

Example 2.2. The basic form $\mathsf{F} \triangleleft b \triangleright (\mathsf{T} \triangleleft a \triangleright \mathsf{F})$ can be represented as follows, where $\triangleleft$ yields a left branch (the *true*-case), and $\triangleright$ a right branch (the *false*-case):

$$\begin{array}{ccccc} & b & & & \\ \diagup & & \diagdown & & \\ \mathsf{F} & & & a & \\ & & \diagup & & \diagdown \\ & & \mathsf{T} & & \mathsf{F} \end{array}$$

and expresses that if b evaluates to *true* (left branch), the expression evaluates to *false*, while if b evaluates to *false* (right branch), the evaluation of a determines the overall evaluation result.

We now recall some definitions and results from [2]. In order to support the intuitions, we spell out the proofs of Lemma 2.8 and Theorem 2.9 (in [2, La.2.17 and Thm.2.18]).

Lemma 2.3. *For each* $P \in C_A$ *there exists* $Q \in BF_A$ *such that* $\mathrm{CP} \vdash P = Q$*.*

Definition 2.4. *Given* $Q, R \in BF_A$*, the auxiliary function* $[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R] : BF_A \to BF_A$ *for which postfix notation* $P[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R]$ *is used, is defined as follows:*

$$\begin{aligned}
\mathsf{T}[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R] &= Q, \\
\mathsf{F}[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R] &= R, \\
(P_1 \triangleleft a \triangleright P_2)[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R] &= P_1[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R] \triangleleft a \triangleright P_2[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R].
\end{aligned}$$

The ***basic form function*** $bf : C_A \to BF_A$ *is defined as follows:*

$$\begin{aligned}
bf(\mathsf{T}) &= \mathsf{T}, \\
bf(\mathsf{F}) &= \mathsf{F}, \\
bf(a) &= \mathsf{T} \triangleleft a \triangleright \mathsf{F} \qquad \textit{for all } a \in A, \\
bf(P \triangleleft Q \triangleright R) &= bf(Q)[\mathsf{T} \mapsto bf(P), \mathsf{F} \mapsto bf(R)].
\end{aligned}$$

[3] We speak of 'basic form' rather than 'normal form' because the basic form associated with atom a is $\mathsf{T} \triangleleft a \triangleright \mathsf{F}$, while for normal forms, one could have expected the reverse.

The following lemma implies that $bf()$ is a normalisation function; both statements easily follow by structural induction.

Lemma 2.5. *For all $P \in C_A$, $bf(P)$ is a basic form, and for each basic form P, $bf(P) = P$.*

Definition 2.6. *The binary relation $=_{bf}$ on C_A is defined as follows:*

$$P =_{bf} Q \iff bf(P) = bf(Q).$$

Lemma 2.7. *The relation $=_{bf}$ is a congruence relation.*

Before proving that CP is an axiomatization of the relation $=_{bf}$, we show that each closed instance of axiom (CP4) satisfies $=_{bf}$.

Lemma 2.8. *For all $P, P_1, P_2, Q_1, Q_2 \in C_A$,*

$$bf(Q_1 \triangleleft (P_1 \triangleleft P \triangleright P_2) \triangleright Q_2) = bf((Q_1 \triangleleft P_1 \triangleright Q_2) \triangleleft P \triangleright (Q_1 \triangleleft P_2 \triangleright Q_2)).$$

Proof. By definition, the lemma's statement is equivalent with

$$\begin{aligned}&\big(bf(P)[\mathsf{T} \mapsto bf(P_1), \mathsf{F} \mapsto bf(P_2)]\big)\,[\mathsf{T} \mapsto bf(Q_1), \mathsf{F} \mapsto bf(Q_2)]\\ &\qquad = bf(P)[\mathsf{T} \mapsto bf(Q_1 \triangleleft P_1 \triangleright Q_2), \mathsf{F} \mapsto bf(Q_1 \triangleleft P_2 \triangleright Q_2)].\end{aligned}$$

We prove this by structural induction on the form that $bf(P)$ can have. If $bf(P) = \mathsf{T}$, then

$$\begin{aligned}&\big(\mathsf{T}[\mathsf{T} \mapsto bf(P_1), \mathsf{F} \mapsto bf(P_2)]\big)\,[\mathsf{T} \mapsto bf(Q_1), \mathsf{F} \mapsto bf(Q_2)]\\ &\quad = bf(P_1)[\mathsf{T} \mapsto bf(Q_1), \mathsf{F} \mapsto bf(Q_2)]\\ &\quad = bf(Q_1 \triangleleft P_1 \triangleright Q_2)\\ &\quad = \mathsf{T}[\mathsf{T} \mapsto bf(Q_1 \triangleleft P_1 \triangleright Q_2), \mathsf{F} \mapsto bf(Q_1 \triangleleft P_2 \triangleright Q_2)].\end{aligned}$$

The case $bf(P) = \mathsf{F}$ follows in a similar way.

The inductive case $bf(P) = R_1 \triangleleft a \triangleright R_2$ is trivial (by Definition 2.4). □

Theorem 2.9. *For all $P, Q \in C_A$,* $\mathrm{CP} \vdash P = Q \iff P =_{bf} Q$.

Proof. (⇒) By Lemma 2.7, $=_{bf}$ is a congruence relation and it easily follows that closed instances of the CP-axioms (CP1) – (CP3) satisfy $=_{bf}$. By Lemma 2.8, closed instances of axiom (CP4) also satisfy $=_{bf}$.

(⇐) Assume $P =_{bf} Q$. According to Lemma 2.3, there exist basic forms P' and Q' such that $\mathrm{CP} \vdash P = P'$ and $\mathrm{CP} \vdash Q = Q'$, so by (⇒), $P' =_{bf} Q'$ and thus $P' = Q'$. Hence, $\mathrm{CP} \vdash P = P' = Q' = Q$. □

The relation $=_{bf}$ coincides with *free valuation congruence*, which is in [1] defined in terms of valuation algebras, and in [2] in terms of evaluation trees. Basic forms have a 1-1-relation with evaluation trees, as shown by their pictorial representation in Example 2.2. Evaluation trees were introduced in [14] by a function CE() that assigns these trees to closed terms: the function CE() is very comparable with the basic form function $bf()$.

We now present definitions of the left-sequential variants of the common propositional connectives. The connective $\mathbin{_{\circ}\!\!\wedge}$ is called *left-sequential conjunction*, and the little circle in its symbol prescribes that the left-argument is evaluated first and that evaluation stops if it yields *false*. This evaluation strategy is called *short-circuit evaluation*: evaluation stops as soon as the evaluation result is known.

We define the signature $\Sigma_{\mathrm{SCL}}(A) = \{\mathbin{_{\circ}\!\!\wedge}, \mathbin{^{\circ}\!\!\vee}, \neg, \mathsf{T}, \mathsf{F}, a \mid a \in A\}$ where SCL stands for short-circuit logic. Negation and sequential conjunction are defined in terms of the conditional connective:

$$\neg x = \mathsf{F} \triangleleft x \triangleright \mathsf{T}, \tag{1}$$

$$x \mathbin{_{\circ}\!\!\wedge} y = y \triangleleft x \triangleright \mathsf{F}. \tag{2}$$

Left-sequential disjunction $\mathbin{^{\circ}\!\!\vee}$ is defined by the following axiom:

$$x \mathbin{^{\circ}\!\!\vee} y = \neg(\neg x \mathbin{_{\circ}\!\!\wedge} \neg y). \tag{3}$$

Note that $\mathbin{_{\circ}\!\!\wedge}$ and $\mathbin{^{\circ}\!\!\vee}$ are each others duals, thus $(P \mathbin{_{\circ}\!\!\wedge} Q)^{dl} = P^{dl} \mathbin{^{\circ}\!\!\vee} Q^{dl}$, and $(\neg P)^{dl} = \neg(P^{dl})$.

Next, we extend CP with the equations (1), (2), and (3), notation

$$\mathrm{CP}(\neg, \mathbin{_{\circ}\!\!\wedge}, \mathbin{^{\circ}\!\!\vee}).$$

The following equations are easily proved in $\mathrm{CP}(\neg, \mathbin{_{\circ}\!\!\wedge}, \mathbin{^{\circ}\!\!\vee})$:

$$\neg \mathsf{T} = \mathsf{F}, \tag{4}$$

$$x \mathbin{^{\circ}\!\!\vee} y = \mathsf{T} \triangleleft x \triangleright y, \tag{5}$$

$$\neg\neg x = x, \tag{6}$$

for example,

$$\begin{aligned}
x \mathbin{^{\circ}\!\!\vee} y &= \mathsf{F} \triangleleft ((\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleleft (\mathsf{F} \triangleleft x \triangleright \mathsf{T}) \triangleright \mathsf{F}) \triangleright \mathsf{T} && \text{by } (1) - (3)\\
&= \mathsf{F} \triangleleft (\mathsf{F} \triangleleft x \triangleright (\mathsf{F} \triangleleft y \triangleright \mathsf{T})) \triangleright \mathsf{T} && \text{by (CP4)}\\
&= (\mathsf{F} \triangleleft \mathsf{F} \triangleright \mathsf{T}) \triangleleft x \triangleright (\mathsf{F} \triangleleft (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleright \mathsf{T}) && \text{by (CP4)}\\
&= \mathsf{T} \triangleleft x \triangleright y. && \text{by (CP1)} - \text{(CP4)}
\end{aligned}$$

For the signature $\Sigma(A) = \Sigma_{\mathrm{CP}}(A) \cup \Sigma_{\mathrm{SCL}}(A)$, let T_A be the set of all its closed terms.

Definition 2.10. *The domain of the function bf() (Def.2.4) is extended to T_A as follows:*

$$bf(\neg P) = bf(\mathsf{F} \triangleleft P \triangleright \mathsf{T}), \quad bf(P \mathbin{_{\circ}\!\!\wedge} Q) = bf(Q \triangleleft P \triangleright \mathsf{F}), \quad bf(P \mathbin{^{\circ}\!\!\vee} Q) = bf(\mathsf{T} \triangleleft P \triangleright Q).$$

The relation $=_{bf}$ (Def.2.6) is extended to T_A.

It follows easily that $=_{bf}$ is a congruence and that Theorem 2.9 can be generalised.

Theorem 2.11. *For all $P, Q \in T_A$,* $\mathrm{CP}(\neg, \mathbin{_{\circ}\!\!\wedge}, \mathbin{^{\circ}\!\!\vee}) \vdash P = Q \iff P =_{bf} Q$.

3 Memorising valuation congruence and mem-basic forms

In this section we discuss memorising valuation congruence, a congruence obtained by extending CP with one axiom. Then we recall a normalisation function for so-called mem-basic forms.

We extend CP as defined in Table 1 with the axiom

$$x \triangleleft y \triangleright (z \triangleleft u \triangleright (v \triangleleft y \triangleright w)) = x \triangleleft y \triangleright (z \triangleleft u \triangleright w). \tag{CPmem}$$

This axiom expresses that the first evaluation value of y is *memorised*. We write

$$\mathrm{CP}_{mem}$$

for this extension of CP. The dual of the axiom (CPmem) is easily derived in CP_{mem}:

$$\begin{aligned}((w \triangleleft y \triangleright v) \triangleleft u \triangleright z) \triangleleft y \triangleright x &= x \triangleleft (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleright (z \triangleleft (\mathsf{F} \triangleleft u \triangleright \mathsf{T}) \triangleright (v \triangleleft (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleright w))\\ &= x \triangleleft (\mathsf{F} \triangleleft y \triangleright \mathsf{T}) \triangleright (z \triangleleft (\mathsf{F} \triangleleft u \triangleright \mathsf{T}) \triangleright w) \qquad \text{by (CPmem)}\\ &= (w \triangleleft u \triangleright z) \triangleleft y \triangleright x,\end{aligned}$$

so CP_{mem} also satisfies the duality principle:

$$\text{For all terms } s, t \text{ over } \Sigma_{\mathrm{CP}}(A),\ \mathrm{CP}_{mem} \vdash s = t \iff \mathrm{CP}_{mem} \vdash s^{dl} = t^{dl}.$$

We note that in CP_{mem} other variants of the axiom (CPmem) are derivable, such as

$$(x \triangleleft y \triangleright (z \triangleleft u \triangleright v)) \triangleleft u \triangleright w = (x \triangleleft y \triangleright z) \triangleleft u \triangleright w, \tag{7}$$

and that *contraction* is also derivable:

$$(x \triangleleft y \triangleright z) \triangleleft y \triangleright u = x \triangleleft y \triangleright u, \tag{8}$$

$$x \triangleleft y \triangleright (z \triangleleft y \triangleright u) = x \triangleleft y \triangleright u. \tag{9}$$

We define a proper subset of basic forms with the property that each propositional statement can be proved equal to such a basic form.

Definition 3.1. ***Mem-basic forms over*** A *are inductively defined:*

- T *and* F *are mem-basic forms, and*
- *For* $a \in A$, $P \triangleleft a \triangleright Q$ *is a mem-basic form if* P *and* Q *are mem-basic forms in which* a *does not occur.*

We write MBF_A *for the set of mem-basic forms over* A.

Note that if A is finite, the number of mem-basic forms is also finite. Mem-basic forms characterise evaluation trees with the property that in each path from the root to a leaf, the internal nodes have distinct labels.

The following normalisation function transforms closed terms to basic forms and then strips off repeated occurrences of atoms by auxiliary functions $\ell_a()$ and $r_a()$ that can be called 'left-a-reduction' and 'right-a-reduction', respectively.

Definition 3.2. *The* ***mem-basic form function*** $mbf : C_A \to MBF_A$ *is defined by*

$$mbf(P) = mf(bf(P)).$$

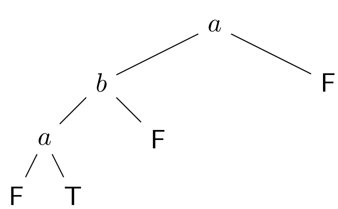
The auxiliary function $mf : BF_A \to BF_A$ is defined inductively:

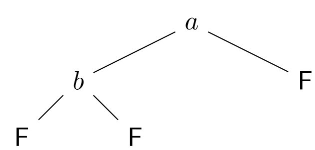
$$\begin{aligned} mf(\mathsf{T}) &= \mathsf{T},\\ mf(\mathsf{F}) &= \mathsf{F},\\ mf(P \triangleleft a \triangleright Q) &= mf(\ell_a(P)) \triangleleft a \triangleright mf(r_a(Q)). \end{aligned}$$

For $a \in A$, the auxiliary functions $\ell_a : BF_A \to BF_A$ and $r_a : BF_A \to BF_A$ are defined by

$$\ell_a(B) = r_a(B) = B \text{ if } B \in \{\mathsf{T},\mathsf{F}\}, \text{ and } \begin{cases} \ell_a(P \triangleleft b \triangleright Q) = \begin{cases} \ell_a(P) & \text{if } b = a,\\ \ell_a(P) \triangleleft b \triangleright \ell_a(Q) & \text{otherwise,} \end{cases} \\ \\ r_a(P \triangleleft b \triangleright Q) = \begin{cases} r_a(Q) & \text{if } b = a,\\ r_a(P) \triangleleft b \triangleright r_a(Q) & \text{otherwise.} \end{cases} \end{cases}$$

It is not hard to see that $mbf(P) \in MBF_A$ for each $P \in C_A$. As an example we depict the basic form $((\mathsf{F} \triangleleft a \triangleright \mathsf{T}) \triangleleft b \triangleright \mathsf{F}) \triangleleft a \triangleright \mathsf{F}$ and its mbf-image $(\mathsf{F} \triangleleft b \triangleright \mathsf{F}) \triangleleft a \triangleright \mathsf{F}$:





Definition 3.3. *The binary relation $=_{mbf}$ on C_A is defined as follows:*

$$P =_{mbf} Q \iff mbf(P) = mbf(Q).$$

In [2, Thm.5.9] it is proved that $mbf()$ is a normalisation function with the following property:

Theorem 3.4. *For all $P, Q \in C_A$, $\mathrm{CP}_{mem} \vdash P = Q \iff P =_{mbf} Q$.*

From a more general point of view, a mem-basic form represents a *decision tree*, that is a labelled, rooted, binary tree with internal nodes labelled from A and leaves labelled from $\{\mathsf{T},\mathsf{F}\}$ such that for any path from the root to a leaf, the internal nodes receive distinct labels [11].

The relation $=_{mbf}$ coincides with *memorising valuation congruence*, which is in [1] defined in terms of so-called memorising valuation algebras, and in [2] in terms of so-called memorising evaluation trees. The rightmost tree displayed above exactly represents the memorising evaluation tree of the conditional statement

$$((\mathsf{F} \triangleleft a \triangleright \mathsf{T}) \triangleleft b \triangleright \mathsf{F}) \triangleleft a \triangleright \mathsf{F},$$

and of course also the memorising evaluation tree of $(\mathsf{F} \triangleleft b \triangleright \mathsf{F}) \triangleleft a \triangleright \mathsf{F}$.

Using Definition 2.10, we extend both the domain of the function $mbf()$ (Def.3.2) and the relation $=_{mbf}$ to T_A. It follows that $=_{mbf}$ is a congruence and that Theorem 3.4 can be generalised.

Theorem 3.5. *For all $P, Q \in T_A$, $\mathrm{CP}_{mem}(\neg, \leftand, \leftor) \vdash P = Q \iff P =_{mbf} Q$.*

$$\mathsf{F} = \neg\mathsf{T} \tag{Neg}$$
$$x \mathbin{{}^{\circ}\!\vee} y = \neg(\neg x \mathbin{{}_{\circ}\!\wedge} \neg y) \tag{Or}$$
$$\mathsf{T} \mathbin{{}_{\circ}\!\wedge} x = x \tag{Tand}$$
$$x \mathbin{{}_{\circ}\!\wedge} (x \mathbin{{}^{\circ}\!\vee} y) = x \tag{Abs}$$
$$(x \mathbin{{}^{\circ}\!\vee} y) \mathbin{{}_{\circ}\!\wedge} z = (\neg x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}_{\circ}\!\wedge} z)) \mathbin{{}^{\circ}\!\vee} (x \mathbin{{}_{\circ}\!\wedge} z) \tag{Mem}$$

Table 2: EqMSCL, a complete, independent set of axioms for MSCL

4 MSCL, Memorising Short-Circuit Logic

In this section we consider the set EqMSCL of equational axioms in Table 2 and recall the fact that EqMSCL axiomatises MSCL, that is, memorising short-circuit logic.

Axioms (Neg) and (Or) are explained in Section 2, and axiom (Tand) needs no explanation. Axiom (Abs) is a left-sequential variant of the absorption law and captures a first aspect of memorising valuation congruence: if x evaluates to *false*, then this axiom holds, and if x evaluates to *true*, then its second evaluation does so as well, and prevents evaluation of y.

Axiom (Mem) captures another, less obvious aspect of memorising valuation congruence: if x evaluates to *true*, then z determines the evaluation result of both expressions because the evaluation result of x is memorised; if x evaluates to *false*, the evaluation result of both expressions is determined by $y \mathbin{{}_{\circ}\!\wedge} z$ because the right disjunct $(x \mathbin{{}_{\circ}\!\wedge} z)$ also evaluates to *false* (because the evaluation result of x is memorised).

We define $\mathrm{CP}_{mem}(\neg, \mathbin{{}_{\circ}\!\wedge}, \mathbin{{}^{\circ}\!\vee})$ as CP_{mem} extended with the equations (1), (2), and (3). We note that in $\mathrm{CP}_{mem}(\neg, \mathbin{{}_{\circ}\!\wedge}, \mathbin{{}^{\circ}\!\vee})$, Hoare's conditional connective can be defined:

$$\begin{aligned}
(x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\vee} (\neg x \mathbin{{}_{\circ}\!\wedge} z) &= \mathsf{T} \triangleleft (y \triangleleft x \triangleright \mathsf{F}) \triangleright (z \triangleleft (\mathsf{F} \triangleleft x \triangleright \mathsf{T}) \triangleright \mathsf{F}) && \text{by (1)-(3)}\\
&= \mathsf{T} \triangleleft (y \triangleleft x \triangleright \mathsf{F}) \triangleright (\mathsf{F} \triangleleft x \triangleright z) && \text{by (CP4), (CP2), (CP1)}\\
&= (\mathsf{T} \triangleleft y \triangleright (\mathsf{F} \triangleleft x \triangleright z)) \triangleleft x \triangleright (\mathsf{F} \triangleleft x \triangleright z) && \text{by (CP4), (CP2)}\\
&= y \triangleleft x \triangleright z. && \text{by (7), (CP3), (9)}
\end{aligned} \tag{10}$$

Furthermore, as a simple example of contraction (equation (8)), axiom (Abs) easily follows from $\mathrm{CP}_{mem}(\neg, \mathbin{{}_{\circ}\!\wedge}, \mathbin{{}^{\circ}\!\vee})$: $x \mathbin{{}_{\circ}\!\wedge} (x \mathbin{{}^{\circ}\!\vee} y) = (\mathsf{T} \triangleleft x \triangleright y) \triangleleft x \triangleright \mathsf{F} = \mathsf{T} \triangleleft x \triangleright \mathsf{F} = x$.

In [3], *memorising short-circuit logic*, notation MSCL, is defined as the equational logic that implies that part of the equational theory of $\mathrm{CP}_{mem}(\neg, \mathbin{{}_{\circ}\!\wedge}, \mathbin{{}^{\circ}\!\vee})$ that is expressed in $\Sigma_{\mathrm{SCL}}(A)$. In [4] it is proved that MSCL is axiomatised by EqMSCL as defined in Table 2, thus

$$\text{For all (open) terms } s, t \text{ over } \Sigma_{\mathrm{SCL}}(A),\ \mathrm{EqMSCL} \vdash s = t \iff \mathrm{MSCL} \vdash s = t. \tag{11}$$

From the correspondence result (11) and Theorem 3.5 it follows that $\mathbin{{}_{\circ}\!\wedge}$ is not commutative:

$$\mathit{mbf}(a \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) = \mathit{mbf}(\mathsf{F} \triangleleft a \triangleright \mathsf{F}) = \mathsf{F} \triangleleft a \triangleright \mathsf{F} \neq \mathsf{F} = \mathit{mbf}(\mathsf{F} \mathbin{{}_{\circ}\!\wedge} a),$$

and that for closed terms over $\Sigma_{\mathrm{SCL}}(A)$, EqMSCL axiomatises $=_{mbf}$.

$$\neg\neg x = x \tag{F3}$$
$$x \mathbin{{}_{\circ}\!\wedge} \mathsf{T} = x \tag{F5}$$
$$\mathsf{F} \mathbin{{}_{\circ}\!\wedge} x = \mathsf{F} \tag{F6}$$
$$(x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}_{\circ}\!\wedge} z = x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}_{\circ}\!\wedge} z) \tag{F7}$$
$$\neg x \mathbin{{}_{\circ}\!\wedge} \mathsf{F} = x \mathbin{{}_{\circ}\!\wedge} \mathsf{F} \tag{F8}$$
$$(x \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \mathbin{{}^{\circ}\!\vee} y = (x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} y \tag{F9}$$
$$(x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\vee} (z \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) = (x \mathbin{{}^{\circ}\!\vee} (z \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}^{\circ}\!\vee} (z \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) \tag{F10}$$

$$x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}_{\circ}\!\wedge} x) = x \mathbin{{}_{\circ}\!\wedge} y \tag{C1}$$
$$x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}_{\circ}\!\wedge} \neg x) = x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \tag{C2}$$

$$(x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\vee} (\neg x \mathbin{{}_{\circ}\!\wedge} z) = (\neg x \mathbin{{}^{\circ}\!\vee} y) \mathbin{{}_{\circ}\!\wedge} (x \mathbin{{}^{\circ}\!\vee} z) \tag{M1}$$
$$(x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\vee} (\neg x \mathbin{{}_{\circ}\!\wedge} z) = (\neg x \mathbin{{}_{\circ}\!\wedge} z) \mathbin{{}^{\circ}\!\vee} (x \mathbin{{}_{\circ}\!\wedge} y) \tag{M2}$$
$$((x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\vee} (\neg x \mathbin{{}_{\circ}\!\wedge} z)) \mathbin{{}_{\circ}\!\wedge} u = (x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}_{\circ}\!\wedge} u)) \mathbin{{}^{\circ}\!\vee} (\neg x \mathbin{{}_{\circ}\!\wedge} (z \mathbin{{}_{\circ}\!\wedge} u)) \tag{M3}$$

$$x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}^{\circ}\!\vee} z) = (x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\vee} (x \mathbin{{}_{\circ}\!\wedge} z) \tag{Dis}$$

Table 3: Some consequences of EqMSCL, where (Dis) stands for left-distributivity

Some nice and natural consequences of EqMSCL are collected in Table 3 (more of them in [4]). Note that with (F3), (M1) and (M2) it follows that

$$(x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\vee} (\neg x \mathbin{{}_{\circ}\!\wedge} z) = (x \mathbin{{}^{\circ}\!\vee} z) \mathbin{{}_{\circ}\!\wedge} (\neg x \mathbin{{}^{\circ}\!\vee} y), \tag{12}$$

which, in addition to equation (10), provides another, simple definition of the conditional connective in terms of the Boolean connectives. Next, with (12), (M1) and (M3)^{dl} it follows that

$$((x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\vee} (\neg x \mathbin{{}_{\circ}\!\wedge} z)) \mathbin{{}^{\circ}\!\vee} u = (x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}^{\circ}\!\vee} u)) \mathbin{{}^{\circ}\!\vee} (\neg x \mathbin{{}_{\circ}\!\wedge} (z \mathbin{{}^{\circ}\!\vee} u)).$$

As was proved in [12], the axioms (Neg), (Or) and (Tand) from Table 2 together with (F3) and (F5) – (F10) from Table 3 axiomatise FSCL for closed terms (where FSCL is the abbreviation of *Free short-circuit logic*, see further Section 8 – Related work). In [12], the names (F1), (F2) and (F4) are used for the axioms (Neg), (Or) and (Tand), respectively.

5 The left-sequential biconditional connective

As is well-known, the biconditional is the logical connective that requires both of its arguments to evaluate to the same truth value to return true, so that

$$x \leftrightarrow y = (x \wedge y) \vee (\neg x \wedge \neg y).$$

In this section we discuss a left-sequential variant of the biconditional connective $\leftrightarrow$. We write ℓIFF ("left-iff") for this connective and use the notation

$$\mathbin{\circ\!\!\leftrightarrow}$$

$$
\begin{aligned}
x \mathbin{^{\circ}\!\vee} y &= \neg(\neg x \mathbin{_{\circ}\!\wedge} \neg y) && \text{(Or)}\\
x \mathbin{_{\circ}\!\wedge} (x \mathbin{^{\circ}\!\vee} y) &= x && \text{(Abs)}\\
(x \mathbin{_{\circ}\!\wedge} y) \mathbin{_{\circ}\!\wedge} z &= x \mathbin{_{\circ}\!\wedge} (y \mathbin{_{\circ}\!\wedge} z) && \text{(Assoc)}\\
\mathsf{T} \mathbin{\circ\!\!\leftrightarrow} x &= x && \text{(Tx)}\\
x \mathbin{\circ\!\!\leftrightarrow} \mathsf{F} &= \neg x && \text{(xF)}\\
(x \mathbin{_{\circ}\!\wedge} y) \mathbin{\circ\!\!\leftrightarrow} z &= (x \mathbin{_{\circ}\!\wedge} (y \mathbin{\circ\!\!\leftrightarrow} z)) \mathbin{^{\circ}\!\vee} (\neg x \mathbin{_{\circ}\!\wedge} \neg z) && \text{(AndIff)}
\end{aligned}
$$

Table 4: $\mathrm{EqMSCL}_{\ell\mathrm{I}}$, a complete, independent set of axioms for $\mathrm{EqMSCL}_{\ell\mathrm{I}}$

to mark that short-circuit evaluation is prescribed.[4] We provide axioms for this extension, prove a correspondence result, and briefly discuss duality.

In the setting of Hoare's ternary conditional connective, ℓIFF is easy to define.

Definition 5.1. *The connective* $\circ\!\!\leftrightarrow$ *is in* CP *defined by the axiom* $x \mathbin{\circ\!\!\leftrightarrow} y = y \triangleleft x \triangleright (\mathsf{F} \triangleleft y \triangleright \mathsf{T})$.

So, $x \mathbin{\circ\!\!\leftrightarrow} y$ only evaluates to *true* if x and y evaluate to the same truth value. In $\mathrm{CP}_{mem}(\neg, \mathbin{_{\circ}\!\wedge}, \mathbin{^{\circ}\!\vee})$ it follows with equation (10), i.e., $x \triangleleft y \triangleright z = (y \mathbin{_{\circ}\!\wedge} x) \mathbin{^{\circ}\!\vee} (\neg y \mathbin{_{\circ}\!\wedge} z)$, that

$$x \mathbin{\circ\!\!\leftrightarrow} y = (x \mathbin{_{\circ}\!\wedge} y) \mathbin{^{\circ}\!\vee} (\neg x \mathbin{_{\circ}\!\wedge} \neg y).$$

Memorising valuation congruence is axiomatised by $\mathrm{EqMSCL}_{\ell\mathrm{I}}$, the set of axioms in Table 4 over the signature $\Sigma_{\mathrm{SCL}\ell\mathrm{I}}(A) = \Sigma_{\mathrm{SCL}}(A) \cup \{\circ\!\!\leftrightarrow\}$. The axiomatisation result for MSCL mentioned in equation (11) gives rise to the following result.

Theorem 5.2.

1. $\mathrm{EqMSCL}_{\ell\mathrm{I}} \vdash \mathrm{EqMSCL}$*,*
2. $\mathrm{EqMSCL} \cup \{x \mathbin{\circ\!\!\leftrightarrow} y = (x \mathbin{_{\circ}\!\wedge} y) \mathbin{^{\circ}\!\vee} (\neg x \mathbin{_{\circ}\!\wedge} \neg y)\} \vdash \mathrm{EqMSCL}_{\ell\mathrm{I}}$*,*
3. *The axioms of* $\mathrm{EqMSCL}_{\ell\mathrm{I}}$ *in Table 4 are independent.*

Proof.

1. Axiom (Neg), thus $\mathsf{F} = \neg\mathsf{T}$, follows immediately from axioms (Tx) and (xF).

   An auxiliary result is $\neg\neg x = x$ (DNS, the double negation shift), which follows with Prover9.

   Axiom (Tand), that is $\mathsf{T} \mathbin{_{\circ}\!\wedge} x = x$, follows with Prover9 (faster if DNS is added to $\mathrm{EqMSCL}_{\ell\mathrm{I}}$).

   Axiom (Mem), that is $(x \mathbin{^{\circ}\!\vee} y) \mathbin{_{\circ}\!\wedge} z = (\neg x \mathbin{_{\circ}\!\wedge} (y \mathbin{_{\circ}\!\wedge} z)) \mathbin{^{\circ}\!\vee} (x \mathbin{_{\circ}\!\wedge} z)$, follows with Prover9: if DNS and (Tand) are added to $\mathrm{EqMSCL}_{\ell\mathrm{I}}$, a proof with option kbo is relatively fast.

2. With Prover9 (immediate).

[4] As mentioned before, the circle in the $\circ\!\!\leftrightarrow$ symbol indicates that the left argument must be evaluated first and prescribes short-circuit evaluation. Observe that to determine the value of the expression $x \mathbin{\circ\!\!\leftrightarrow} y$, the second argument y must always be evaluated. For left-sequential conjunction this is different, the variant of left-sequential conjunction that prescribes so-called *full* sequential evaluation (as opposed to short-circuit evaluation) has notation $\mathbin{_{\bullet}\!\wedge}$ and always evaluates both conjuncts from left to right (indicated by the black circle), and can be defined by $x \mathbin{_{\bullet}\!\wedge} y = (x \mathbin{^{\circ}\!\vee} (y \mathbin{_{\circ}\!\wedge} \mathsf{F})) \mathbin{_{\circ}\!\wedge} y$, or, by $x \mathbin{_{\bullet}\!\wedge} y = y \triangleleft x \triangleright (\mathsf{F} \triangleleft y \triangleright \mathsf{F})$ (see [14]).

3. Independence of $\mathrm{EqMSCL}_{\ell\mathrm{I}}$ follows immediately with Mace4. The only case to be mentioned is the independence of axiom (Assoc): a counter model (with domain size 8) is quickly generated by Mace4 if (DNS) is added to $\mathrm{EqMSCL}_{\ell\mathrm{I}} \setminus \{(\mathrm{Assoc})\}$.

$\square$

Some nice consequences of $\mathrm{EqMSCL}_{\ell\mathrm{I}}$, all checked with Prover9: [5]

$$\begin{aligned}
\mathsf{T} \mathbin{\circ\!\!\!\leftrightarrow} x &= x \mathbin{\circ\!\!\!\leftrightarrow} \mathsf{T}\\
\mathsf{F} \mathbin{\circ\!\!\!\leftrightarrow} x &= x \mathbin{\circ\!\!\!\leftrightarrow} \mathsf{F}\\
\neg(x \mathbin{\circ\!\!\!\leftrightarrow} y) &= x \mathbin{\circ\!\!\!\leftrightarrow} \neg y\\
\neg x \mathbin{\circ\!\!\!\leftrightarrow} \neg y &= x \mathbin{\circ\!\!\!\leftrightarrow} y\\
(x \mathbin{{}^{\circ}\!\!\vee} \mathsf{T}) \mathbin{\circ\!\!\!\leftrightarrow} y &= (x \mathbin{{}^{\circ}\!\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\!\wedge} y.\\
(x \mathbin{\circ\!\!\!\leftrightarrow} y) \mathbin{\circ\!\!\!\leftrightarrow} z &= x \mathbin{\circ\!\!\!\leftrightarrow} (y \mathbin{\circ\!\!\!\leftrightarrow} z)
\end{aligned}$$

With reference to correspondence result (11), we note that these consequences can also be easily proved in $\mathrm{CP}_{mem}(\neg, \mathbin{{}_{\circ}\!\!\wedge}, \mathbin{{}^{\circ}\!\!\vee})$ extended with the equation $x \mathbin{\circ\!\!\!\leftrightarrow} y = y \triangleleft x \triangleright \neg y$. A consequence of $\mathrm{EqMSCL}_{\ell\mathrm{I}}$ that is more difficult to prove with Prover9,[6] is recorded in [7, dual of axiom (AX4)]:

$$(x \mathbin{\circ\!\!\!\leftrightarrow} y) \mathbin{{}_{\circ}\!\!\wedge} (z \mathbin{{}_{\circ}\!\!\wedge} \mathsf{F}) = (x \mathbin{{}^{\circ}\!\!\vee} (\neg y \mathbin{{}_{\circ}\!\!\wedge} (z \mathbin{{}_{\circ}\!\!\wedge} \mathsf{F}))) \mathbin{{}_{\circ}\!\!\wedge} (y \mathbin{{}_{\circ}\!\!\wedge} (z \mathbin{{}_{\circ}\!\!\wedge} \mathsf{F})).$$

However, there is a simple proof in $\mathrm{CP}(\neg, \mathbin{{}_{\circ}\!\!\wedge}, \mathbin{{}^{\circ}\!\!\vee})$ extended with $x \mathbin{\circ\!\!\!\leftrightarrow} y = y \triangleleft x \triangleright \neg y$.

Duality and ℓIFF. The duality principle can be extended to terms over $\Sigma_{\mathrm{SCL}\ell\mathrm{I}}(A)$ if the left-sequential version of the connective *exclusive or* (XOR, notation $\oplus$) is added as the dual of ℓIFF. We write ℓXOR ("left-xor") for this connective and use the symbol $\mathbin{\circ\!\!\oplus}$.

Define $\Sigma_{\mathrm{SCL}\ell\mathrm{I}\ell\mathrm{X}}(A) = \Sigma_{\mathrm{SCL}\ell\mathrm{I}}(A) \cup \{\mathbin{\circ\!\!\oplus}\}$. So, for all $P, Q \in \Sigma_{\mathrm{SCL}\ell\mathrm{I}\ell\mathrm{X}}(A)$,

$$P \mathbin{\circ\!\!\oplus} Q = (P^{dl} \mathbin{\circ\!\!\!\leftrightarrow} Q^{dl})^{dl}.$$

With the definining axiom for $\mathbin{\circ\!\!\!\leftrightarrow}$, that is, $x \mathbin{\circ\!\!\!\leftrightarrow} y = y \triangleleft x \triangleright (\mathsf{F} \triangleleft y \triangleright \mathsf{T})$ we find

$$P \mathbin{\circ\!\!\oplus} Q = (Q^{dl} \triangleleft P^{dl} \triangleright (\mathsf{F} \triangleleft Q \triangleright \mathsf{T})^{dl})^{dl} = (\mathsf{F} \triangleleft Q \triangleright \mathsf{T}) \triangleleft P \triangleright Q,$$

and thus $P \mathbin{\circ\!\!\oplus} Q = P \mathbin{\circ\!\!\!\leftrightarrow} \neg Q$. With respect to the signature $\Sigma_{\mathrm{SCL}\ell\mathrm{I}\ell\mathrm{X}}(A)$, the axiom

$$x \mathbin{\circ\!\!\oplus} y = x \mathbin{\circ\!\!\!\leftrightarrow} \neg y$$

defines $\mathbin{\circ\!\!\oplus}$ and we write $\mathrm{EqMSCL}_{\ell\mathrm{I}\ell\mathrm{X}}$ for the addition of this axiom to $\mathrm{EqMSCL}_{\ell\mathrm{I}}$ (see Table 4). Hence, in $\mathrm{EqMSCL}_{\ell\mathrm{I}\ell\mathrm{X}}$ it follows that

$$\neg(x \mathbin{\circ\!\!\oplus} y) = \neg(x \mathbin{\circ\!\!\!\leftrightarrow} \neg y) = x \mathbin{\circ\!\!\!\leftrightarrow} y,$$

so $\mathbin{\circ\!\!\!\leftrightarrow}$ is also "the negation of $\mathbin{\circ\!\!\oplus}$". From $\mathrm{EqMSCL}_{\ell\mathrm{I}\ell\mathrm{X}}$ and the EqMSCL-identities discussed in Section 4 it easily follows that

$$x \mathbin{\circ\!\!\oplus} y = (x \mathbin{{}_{\circ}\!\!\wedge} \neg y) \mathbin{{}^{\circ}\!\!\vee} (\neg x \mathbin{{}_{\circ}\!\!\wedge} y).$$

[5] For the last consequence, a run in Prover9 with the extra assumption $\neg\neg x = x$ and options kbo and fold is relatively fast.

[6] With all other EqMSCL-axioms added to $\mathrm{EqMSCL}_{\ell\mathrm{I}}$, a run in Prover9 with options kbo and fold required 112 CPU seconds.

Theorem 5.3 (Duality)**.** *For all terms s,t over* $\Sigma_{\mathrm{SCL}\ell\mathrm{I}\ell\mathrm{X}}(A)$*,*

$$\mathrm{EqMSCL}_{\ell\mathrm{I}\ell\mathrm{X}} \vdash s = t \iff \mathrm{EqMSCL}_{\ell\mathrm{I}\ell\mathrm{X}} \vdash s^{dl} = t^{dl}.$$

Of course, we could have started with adding ℓXOR to $\Sigma_{\mathrm{SCL}}(A)$ instead of ℓIFF. It appears that taking all duals of the axioms (Tx) – (AndIff) in Table 4, that is,

$$\mathsf{F} \mathbin{\circ\!\!\oplus} x = x \qquad \text{(Tx)}^{dl}$$

$$x \mathbin{\circ\!\!\oplus} \mathsf{T} = \neg x \qquad \text{(xF)}^{dl}$$

$$(x \mathbin{^{\circ}\!\vee} y) \mathbin{\circ\!\!\oplus} z = (x \mathbin{^{\circ}\!\vee} (y \mathbin{\circ\!\!\oplus} z)) \mathbin{_{\circ}\!\wedge} (\neg x \mathbin{^{\circ}\!\vee} \neg z) \qquad \text{(AndIff)}^{dl}$$

yields an axiomatisation that is also independent and has the same equational theory (modulo $x \mathbin{\circ\!\!\oplus} y = x \mathbin{\circ\!\!\leftrightarrow} \neg y$).

6 The left-sequential NAND connective

The Sheffer stroke $|$, also known as the NAND (not and) connective, requires at least one of its arguments to be false so that it returns true:

$$x \mid y = \neg(x \wedge y).$$

In this section we discuss a left-sequential variant of the NAND connective. We write ℓNAND ("left-NAND") for this connective and use the notation

$$\mathbin{\circ\!\!\mid}$$

to mark that a left-sequential evaluation strategy is prescribed. We provide axioms for this extension and prove a correspondence result.

In [7], the connective ℓNAND is defined in CP by

$$x \mathbin{\circ\!\!\mid} y = (\mathsf{F} \lhd y \rhd \mathsf{T}) \lhd x \rhd \mathsf{T},$$

and hence satisfies $x \mathbin{\circ\!\!\mid} y = \neg(x \mathbin{_{\circ}\!\wedge} y) = \mathsf{F} \lhd (y \lhd x \rhd \mathsf{F}) \rhd \mathsf{T} = (\mathsf{F} \lhd y \rhd \mathsf{T}) \lhd x \rhd \mathsf{T}$. Conversely, negation and the sequential connectives can be defined in terms of ℓNAND by

$$\neg x = x \mathbin{\circ\!\!\mid} \mathsf{T}, \tag{1ℓ}$$

$$x \mathbin{_{\circ}\!\wedge} y = (x \mathbin{\circ\!\!\mid} y) \mathbin{\circ\!\!\mid} \mathsf{T}, \tag{2ℓ}$$

$$x \mathbin{^{\circ}\!\vee} y = (x \mathbin{\circ\!\!\mid} \mathsf{T}) \mathbin{\circ\!\!\mid} (y \mathbin{\circ\!\!\mid} \mathsf{T}). \tag{3ℓ}$$

To obtain the ℓNAND-translation of the EqMSCL-axioms, we apply these definitions to each occurrence of the $\{\neg, \mathbin{_{\circ}\!\wedge}, \mathbin{^{\circ}\!\vee}\}$-connectives for every axiom of EqMSCL:

$$\mathsf{F} = \mathsf{T} \mathbin{\circ\!\!\mid} \mathsf{T}, \tag{13}$$

$$(x \mathbin{\circ\!\!\mid} \mathsf{T}) \mathbin{\circ\!\!\mid} (y \mathbin{\circ\!\!\mid} \mathsf{T}) = (((x \mathbin{\circ\!\!\mid} \mathsf{T}) \mathbin{\circ\!\!\mid} (y \mathbin{\circ\!\!\mid} \mathsf{T})) \mathbin{\circ\!\!\mid} \mathsf{T}) \mathbin{\circ\!\!\mid} \mathsf{T}, \tag{14}$$

$$(\mathsf{T} \mathbin{\circ\!\!\mid} x) \mathbin{\circ\!\!\mid} \mathsf{T} = x, \tag{15}$$

$$(x \mathbin{\circ\!\!\mid} ((x \mathbin{\circ\!\!\mid} \mathsf{T}) \mathbin{\circ\!\!\mid} (y \mathbin{\circ\!\!\mid} \mathsf{T}))) \mathbin{\circ\!\!\mid} \mathsf{T} = x, \tag{16}$$

$$\begin{aligned} &(((x \mathbin{\circ\!\!\mid} \mathsf{T}) \mathbin{\circ\!\!\mid} (y \mathbin{\circ\!\!\mid} \mathsf{T})) \mathbin{\circ\!\!\mid} z) \mathbin{\circ\!\!\mid} \mathsf{T} = \\ &\qquad ((((x \mathbin{\circ\!\!\mid} \mathsf{T}) \mathbin{\circ\!\!\mid} ((y \mathbin{\circ\!\!\mid} z) \mathbin{\circ\!\!\mid} \mathsf{T})) \mathbin{\circ\!\!\mid} \mathsf{T}) \mathbin{\circ\!\!\mid} \mathsf{T}) \mathbin{\circ\!\!\mid} (((x \mathbin{\circ\!\!\mid} z) \mathbin{\circ\!\!\mid} \mathsf{T}) \mathbin{\circ\!\!\mid} \mathsf{T}). \end{aligned} \tag{17}$$

$$\mathsf{F} = \mathsf{T} \mathbin{^{\circ}\!\!\mid} \mathsf{T} \tag{N1}$$

$$(\mathsf{T} \mathbin{^{\circ}\!\!\mid} x) \mathbin{^{\circ}\!\!\mid} (x \mathbin{^{\circ}\!\!\mid} y) = x \tag{N2}$$

$$(x \mathbin{^{\circ}\!\!\mid} (y \mathbin{^{\circ}\!\!\mid} \mathsf{T})) \mathbin{^{\circ}\!\!\mid} z = ((x \mathbin{^{\circ}\!\!\mid} ((y \mathbin{^{\circ}\!\!\mid} z) \mathbin{^{\circ}\!\!\mid} \mathsf{T})) \mathbin{^{\circ}\!\!\mid} ((x \mathbin{^{\circ}\!\!\mid} \mathsf{T}) \mathbin{^{\circ}\!\!\mid} z)) \mathbin{^{\circ}\!\!\mid} \mathsf{T} \tag{N3}$$

Table 5: $\mathrm{EqMSCL}_{\ell\mathrm{N}}$, a set of equational axioms for ℓNAND, left-sequential NAND

Next, we simplify the translation equations (14) – (17) as follows:

$$(x \mathbin{^{\circ}\!\!\mid} \mathsf{T}) \mathbin{^{\circ}\!\!\mid} \mathsf{T} = x \qquad \text{[in (14), replace } (x \mathbin{^{\circ}\!\!\mid} \mathsf{T}) \mathbin{^{\circ}\!\!\mid} (y \mathbin{^{\circ}\!\!\mid} \mathsf{T}) \text{ by } x\text{],} \tag{18}$$

$$\mathsf{T} \mathbin{^{\circ}\!\!\mid} x = x \mathbin{^{\circ}\!\!\mid} \mathsf{T} \qquad \text{[in (15), add } (\_) \mathbin{^{\circ}\!\!\mid} \mathsf{T} \text{ and apply (18)],} \tag{19}$$

$$x \mathbin{^{\circ}\!\!\mid} ((x \mathbin{^{\circ}\!\!\mid} \mathsf{T}) \mathbin{^{\circ}\!\!\mid} y) = x \mathbin{^{\circ}\!\!\mid} \mathsf{T} \tag{20}$$

[in (16), add $(\_) \mathbin{^{\circ}\!\!\mid} \mathsf{T}$, replace $(y \mathbin{^{\circ}\!\!\mid} \mathsf{T})$ by y, and apply (18)],

$$(x \mathbin{^{\circ}\!\!\mid} (y \mathbin{^{\circ}\!\!\mid} \mathsf{T})) \mathbin{^{\circ}\!\!\mid} z = ((x \mathbin{^{\circ}\!\!\mid} ((y \mathbin{^{\circ}\!\!\mid} z) \mathbin{^{\circ}\!\!\mid} \mathsf{T})) \mathbin{^{\circ}\!\!\mid} ((x \mathbin{^{\circ}\!\!\mid} \mathsf{T}) \mathbin{^{\circ}\!\!\mid} z)) \mathbin{^{\circ}\!\!\mid} \mathsf{T} \tag{21}$$

[in (17), add $(\_) \mathbin{^{\circ}\!\!\mid} \mathsf{T}$, replace$(x \mathbin{^{\circ}\!\!\mid} \mathsf{T})$ by x and the rightmost x by $(x \mathbin{^{\circ}\!\!\mid} \mathsf{T})$, and apply (18)].

Finally, if in equation (20) we replace the leftmost x by $(\mathsf{T} \mathbin{^{\circ}\!\!\mid} x)$ and the two occurrences $(x \mathbin{^{\circ}\!\!\mid} \mathsf{T})$ by x, then equations (18) and (19) are consequences of the three remaining axioms, this follows easily with Prover9. These three axioms are listed in Table 5 and we call this set of axioms $\mathrm{EqMSCL}_{\ell\mathrm{N}}$.

Theorem 6.1.

1. $\mathrm{EqMSCL}_{\ell\mathrm{N}} \cup \{(1\ell), (2\ell), (3\ell)\} \vdash \mathrm{EqMSCL}$,
2. $\mathrm{EqMSCL} \cup \{x \mathbin{^{\circ}\!\!\mid} y = \neg(x \mathbin{_{\circ}\!\!\wedge} y)\} \vdash \mathrm{EqMSCL}_{\ell\mathrm{N}}$,
3. *The axioms of* $\mathrm{EqMSCL}_{\ell\mathrm{N}}$ *in Table 5 are independent.*

Proof.

1. This follows easily with Prover9. Recall equation (3): $x \mathbin{^{\circ}\!\!\vee} y = \neg(\neg x \mathbin{_{\circ}\!\!\wedge} \neg y)$. We note that

   $$\mathrm{EqMSCL}_{\ell\mathrm{N}} \cup \{(1\ell), (2\ell), (3)\} \vdash \mathrm{EqMSCL}$$

   also follows easily with Prover9.

2. This follows easily with Prover9.

3. By Theorem 7.1 (which states that a superset of $\mathrm{EqMSCL}_{\ell\mathrm{N}}$ is independent).

□

A first consequence of $\mathrm{EqMSCL}_{\ell\mathrm{N}}$ stems from the characterisation of $y \triangleleft x \triangleright z$ in $\mathrm{EqMSCL}_{\ell\mathrm{I}}$, i.e., $y \triangleleft x \triangleright z = (x \mathbin{_{\circ}\!\!\wedge} y) \mathbin{^{\circ}\!\!\vee} (\neg x \mathbin{_{\circ}\!\!\wedge} z) = (\neg x \mathbin{_{\circ}\!\!\wedge} z) \mathbin{^{\circ}\!\!\vee} (x \mathbin{_{\circ}\!\!\wedge} y)$ (equations (10) and (M1) from Table 3). With the ℓNAND definitions of the Boolean connectives we find

$$(x \mathbin{^{\circ}\!\!\mid} y) \mathbin{^{\circ}\!\!\mid} ((x \mathbin{^{\circ}\!\!\mid} \mathsf{T}) \mathbin{^{\circ}\!\!\mid} z) = ((x \mathbin{^{\circ}\!\!\mid} \mathsf{T}) \mathbin{^{\circ}\!\!\mid} z) \mathbin{^{\circ}\!\!\mid} (x \mathbin{^{\circ}\!\!\mid} y).$$

Furthermore, with (M1), (M2) and $(x \mathbin{^{\circ}\!\!\mid} \mathsf{T}) \mathbin{^{\circ}\!\!\mid} \mathsf{T} = x$ it follows that

$$(x \mathbin{^{\circ}\!\!\mid} y) \mathbin{^{\circ}\!\!\mid} ((x \mathbin{^{\circ}\!\!\mid} \mathsf{T}) \mathbin{^{\circ}\!\!\mid} z) = (((x \mathbin{^{\circ}\!\!\mid} \mathsf{T}) \mathbin{^{\circ}\!\!\mid} (z \mathbin{^{\circ}\!\!\mid} \mathsf{T})) \mathbin{^{\circ}\!\!\mid} (x \mathbin{^{\circ}\!\!\mid} (y \mathbin{^{\circ}\!\!\mid} \mathsf{T}))) \mathbin{^{\circ}\!\!\mid} \mathsf{T}.$$

Abbreviation. If we write x' for $x \mathbin{{}^{\circ}\!\!\uparrow} \mathsf{T}$, terms become more readable. In the remainder of our discussion about $\mathbin{{}^{\circ}\!\!\uparrow}$ we will mostly use this abbreviation, e.g., $x'' = x$. The characterisations of the conditional then look like this:

$$\begin{aligned} y \triangleleft x \triangleright z &= (x \mathbin{{}^{\circ}\!\!\uparrow} y) \mathbin{{}^{\circ}\!\!\uparrow} (x' \mathbin{{}^{\circ}\!\!\uparrow} z) \\ &= (x' \mathbin{{}^{\circ}\!\!\uparrow} z) \mathbin{{}^{\circ}\!\!\uparrow} (x \mathbin{{}^{\circ}\!\!\uparrow} y) \\ &= ((x' \mathbin{{}^{\circ}\!\!\uparrow} z') \mathbin{{}^{\circ}\!\!\uparrow} (x \mathbin{{}^{\circ}\!\!\uparrow} y'))' \\ &= ((x \mathbin{{}^{\circ}\!\!\uparrow} y') \mathbin{{}^{\circ}\!\!\uparrow} (x' \mathbin{{}^{\circ}\!\!\uparrow} z'))'. \end{aligned} \tag{22}$$

Some more consequences of $\mathrm{EqMSCL}_{\ell\mathrm{N}}$, all of which can be easily verified with Prover9: [7]

$$((x \mathbin{{}^{\circ}\!\!\uparrow} y) \mathbin{{}^{\circ}\!\!\uparrow} (x' \mathbin{{}^{\circ}\!\!\uparrow} z)) \mathbin{{}^{\circ}\!\!\uparrow} v = (x \mathbin{{}^{\circ}\!\!\uparrow} (y \mathbin{{}^{\circ}\!\!\uparrow} v)) \mathbin{{}^{\circ}\!\!\uparrow} (x' \mathbin{{}^{\circ}\!\!\uparrow} (z \mathbin{{}^{\circ}\!\!\uparrow} v)), \tag{23}$$

$$\mathsf{F} \mathbin{{}^{\circ}\!\!\uparrow} x = \mathsf{T}, \tag{24}$$

$$x' \mathbin{{}^{\circ}\!\!\uparrow} (x' \mathbin{{}^{\circ}\!\!\uparrow} \mathsf{F}) = x, \tag{25}$$

$$x \mathbin{{}^{\circ}\!\!\uparrow} ((x \mathbin{{}^{\circ}\!\!\uparrow} y) \mathbin{{}^{\circ}\!\!\uparrow} (x' \mathbin{{}^{\circ}\!\!\uparrow} z)) = x \mathbin{{}^{\circ}\!\!\uparrow} y, \tag{26}$$

$$x \mathbin{{}^{\circ}\!\!\uparrow} ((y \mathbin{{}^{\circ}\!\!\uparrow} z) \mathbin{{}^{\circ}\!\!\uparrow} (y' \mathbin{{}^{\circ}\!\!\uparrow} w)) = x \mathbin{{}^{\circ}\!\!\uparrow} ((y \mathbin{{}^{\circ}\!\!\uparrow} (x \mathbin{{}^{\circ}\!\!\uparrow} z')) \mathbin{{}^{\circ}\!\!\uparrow} (y' \mathbin{{}^{\circ}\!\!\uparrow} (x \mathbin{{}^{\circ}\!\!\uparrow} w'))). \tag{27}$$

An advantage of the signature with ℓNAND is that inductive properties are easier to prove than in the case of $\Sigma_{\mathrm{SCL}}(A)$. We return to this point in the next section.

Duality and ℓNAND. We write ℓNOR for the dual connective of ℓNAND. In [7], the left-sequential version of the NOR connective, notation $\mathbin{{}^{\circ}\!\!\downarrow}$ (with notation $\downarrow$ for NOR), is defined by

$$x \mathbin{{}^{\circ}\!\!\downarrow} y = \mathsf{F} \triangleleft x \triangleright (\mathsf{F} \triangleleft y \triangleright \mathsf{T}).$$

Using CP, it easily follows that $\mathbin{{}^{\circ}\!\!\downarrow}$ is expressible in terms of $\mathbin{{}^{\circ}\!\!\uparrow}$, T and F:

$$x \mathbin{{}^{\circ}\!\!\downarrow} y = ((x \mathbin{{}^{\circ}\!\!\uparrow} \mathsf{T}) \mathbin{{}^{\circ}\!\!\uparrow} (y \mathbin{{}^{\circ}\!\!\uparrow} \mathsf{T})) \mathbin{{}^{\circ}\!\!\uparrow} \mathsf{T} \quad (= (x' \mathbin{{}^{\circ}\!\!\uparrow} y')', \text{ compare the symmetry with axiom(Or)}).$$

However, we see no reason why adding the dual connective $\mathbin{{}^{\circ}\!\!\downarrow}$ could be attractive, nor to prefer $\mathbin{{}^{\circ}\!\!\downarrow}$ to $\mathbin{{}^{\circ}\!\!\uparrow}$.

7 The three-valued case

In this section we discuss the addition of a third constant to CP_{mem}, and also to $\mathrm{EqMSCL}_{\ell\mathrm{I}}$ and $\mathrm{EqMSCL}_{\ell\mathrm{N}}$, which represents the truth value "undefined". This is a follow-up to the results in [4] on $\mathrm{MSCL}^{\mathsf{U}}$, that is, MSCL with undefinedness.

The constant U is used to represent the third truth value *undefined*. In the setting with the conditional connective, this constant is defined by the axiom

$$x \triangleleft \mathsf{U} \triangleright y = \mathsf{U}, \tag{CP-U}$$

which should be added to the axiom system under consideration. We write $\mathrm{CP}^{\mathsf{U}}_{mem}$ for the extension of CP_{mem} with axiom (CP-U). With $\mathsf{U}^{dl} = \mathsf{U}$, it follows that $\mathrm{CP}^{\mathsf{U}}_{mem}$ also satisfies the duality principle. Let C^{U}_A be the set of closed terms belonging to $\mathrm{CP}^{\mathsf{U}}_{mem}$.

[7] Verification with Prover9 is fastest if an auxiliary function $f(x) = x \mathbin{{}^{\circ}\!\!\uparrow} \mathsf{T}$ is added to the axioms of $\mathrm{EqMSCL}_{\ell\mathrm{N}}$, the options rpo and fold are used, and $f()$ and $()'$ are *not* used in these consequences, i.e., the goal formulas use $x \mathbin{{}^{\circ}\!\!\uparrow} \mathsf{T}$. Without the addition of $f(x) = x \mathbin{{}^{\circ}\!\!\uparrow} \mathsf{T}$ (or $x' = x \mathbin{{}^{\circ}\!\!\uparrow} \mathsf{T}$), proofs of (23), (26) and (27) seem not feasible.

$\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{I}}$:	$\neg\mathsf{U} = \mathsf{U}$	(Und)
$\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{N}}$:	$\mathsf{U} \mathbin{\lhd\!\!\!\mid} x = \mathsf{U}$	(NU)

Table 6: The axiom for U in $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{I}}$ and in $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{N}}$

For each of the equational logics defined by $\mathrm{EqMSCL}_{\ell \mathrm{I}}$ and $\mathrm{EqMSCL}_{\ell \mathrm{N}}$, the additional axiom for U is given in Table 6, and we write $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{I}}$ and $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{N}}$ for the resulting axiom systems. It follows that

$$\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{I}} \vdash \mathsf{U} \mathbin{{}_{\circ}\!\wedge} x = \mathsf{U} \mathbin{{}^{\circ}\!\vee} x = \mathsf{U} \mathbin{{}_{\circ}\!\!\leftrightarrow} x = \mathsf{U}, \quad \mathsf{F} \mathbin{{}_{\circ}\!\wedge} \mathsf{U} = \mathsf{F},$$
$$\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{N}} \vdash \mathsf{F} \mathbin{\lhd\!\!\!\mid} \mathsf{U} = \mathsf{T}.$$

In [4], *memorising short-circuit logic with undefinedness*, notation $\mathrm{MSCL}^{\mathsf{U}}$, is defined as the equational logic that implies the part of the equational theory of $\mathrm{CP}^{\mathsf{U}}_{mem}(\neg, \mathbin{{}_{\circ}\!\wedge}, \mathbin{{}^{\circ}\!\vee})$ that is expressed in $\Sigma_{\mathrm{SCL}^{\mathsf{U}}}(A) = \Sigma_{\mathrm{SCL}}(A) \cup \{\mathsf{U}\}$, and it is proved that $\mathrm{MSCL}^{\mathsf{U}}$ is axiomatised by $\mathrm{EqMSCL}^{\mathsf{U}} = \mathrm{EqMSCL} \cup \{(\mathrm{Und})\}$ (see [4, Thm.7.16]):

$$\text{For all terms } s, t \text{ over } \Sigma_{\mathrm{SCL}^{\mathsf{U}}}(A),\ \mathrm{EqMSCL}^{\mathsf{U}} \vdash s = t \iff \mathrm{MSCL}^{\mathsf{U}} \vdash s = t.$$

It immediately follows that when $\mathrm{EqMSCL}^{\mathsf{U}}$ is replaced by $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{I}}$, this result is preserved. Moreover, the axioms of $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{I}}$ are independent: this follows quickly with Mace4.

In the remainder of this section, we will further discuss $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{N}}$.

Theorem 7.1.
1. $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{N}} \cup \{(1\ell), (2\ell), (3\ell)\} \vdash \mathrm{EqMSCL}^{\mathsf{U}}$,
2. $\mathrm{EqMSCL}^{\mathsf{U}} \cup \{x \mathbin{\lhd\!\!\!\mid} y = \neg(x \mathbin{{}_{\circ}\!\wedge} y)\} \vdash \mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{N}}$,
3. *The axioms for* $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{N}}$ *in Tables 5 and 6 are independent.*

Proof. Statements 1 and 2 follow easily with Prover9, and 3 follows quickly with Mace4. ☐

To illustrate an advantage of $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{N}}$ over $\mathrm{EqMSCL}^{\mathsf{U}}$ (or, $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell \mathrm{I}}$), we prove a representation result for basic forms. This still requires a lot of detail, but is simpler and more straightforward than the proof of the corresponding representation result for $\mathrm{EqMSCL}^{\mathsf{U}}$ in [4, La.7.9 and La.7.10]).

Definition 7.2. ***Memorising*** U***-Nand Basic Forms*** **(*mUNBFs*)** ***over*** A *are inductively defined:*

- $\mathsf{T}, \mathsf{F}, \mathsf{U}$ *are mUNBFs, and*
- *For* $a \in A$, $(a \mathbin{\lhd\!\!\!\mid} P) \mathbin{\lhd\!\!\!\mid} (a' \mathbin{\lhd\!\!\!\mid} Q)$ *is a mUNBF if* P *and* Q *are mUNBFs that do not contain* a.

We write $MUNBF_A$ *for the set of all mUNBFs over* A.

The following functions on $MUNBF_A$ are used to compose mUNBFs.

Definition 7.3. *For $a \in A$, the function $\mathsf{T}^{\mathsf{U}}_a : MUNBF_A \to MUNBF_A$ is defined by*

$$\mathsf{T}^{\mathsf{U}}_a(\mathsf{T}) = \mathsf{T}, \quad \mathsf{T}^{\mathsf{U}}_a(\mathsf{F}) = \mathsf{F}, \quad \mathsf{T}^{\mathsf{U}}_a(\mathsf{U}) = \mathsf{U},$$

$$\mathsf{T}^{\mathsf{U}}_a((b \triangleleft P_1) \triangleleft (b' \triangleleft P_2)) = \begin{cases} P_1 & \text{if } b = a, \\ (b \triangleleft \mathsf{T}^{\mathsf{U}}_a(P_1)) \triangleleft (b' \triangleleft \mathsf{T}^{\mathsf{U}}_a(P_2)) & \text{otherwise.} \end{cases}$$

For $a \in A$ the function $\mathsf{F}^{\mathsf{U}}_a : MUNBF_A \to MUNBF_A$ is defined by

$$\mathsf{F}^{\mathsf{U}}_a(\mathsf{T}) = \mathsf{T}, \quad \mathsf{F}^{\mathsf{U}}_a(\mathsf{F}) = \mathsf{F}, \quad \mathsf{F}^{\mathsf{U}}_a(\mathsf{U}) = \mathsf{U},$$

$$\mathsf{F}^{\mathsf{U}}_a((b \triangleleft P_1) \triangleleft (b' \triangleleft P_2)) = \begin{cases} P_2 & \text{if } b = a, \\ (b \triangleleft \mathsf{F}^{\mathsf{U}}_a(P_1)) \triangleleft (b' \triangleleft \mathsf{F}^{\mathsf{U}}_a(P_2)) & \text{otherwise.} \end{cases}$$

So, $\mathsf{T}^{\mathsf{U}}_a$ removes the a-occurrences in a mUNBF under the assumption that a evaluates to *true*, and $\mathsf{F}^{\mathsf{U}}_a$ does this under the assumption that a evaluates to *false*. Note that for each $P \in MUNBF_A$, both $\mathsf{T}^{\mathsf{U}}_a(P)$ and $\mathsf{F}^{\mathsf{U}}_a(P)$ are also mUNBFs. To compose mUNBFs we use the following lemma.

Lemma 7.4. *For all $a \in A$ and $P \in MUNBF_A$,*
1. $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell\mathrm{N}} \vdash a \triangleleft P = a \triangleleft \mathsf{T}^{\mathsf{U}}_a(P)$ *and* $a \triangleleft P' = a \triangleleft \mathsf{T}^{\mathsf{U}}_a(P)'$,
2. $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell\mathrm{N}} \vdash a' \triangleleft P = a' \triangleleft \mathsf{F}^{\mathsf{U}}_a(P)$ *and* $a' \triangleleft P' = a' \triangleleft \mathsf{F}^{\mathsf{U}}_a(P)'$.

Proof. Statement 1 follows by induction on the structure of P. If $P \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ this is trivial because $\mathsf{T}^{\mathsf{U}}_a(P) = P$. For the induction step, four cases have to be dealt with:
- If $P = (a \triangleleft P_1) \triangleleft (a' \triangleleft P_2)$ then

$$a \triangleleft P \stackrel{(26)}{=} a \triangleleft P_1 = a \triangleleft \mathsf{T}^{\mathsf{U}}_a(P),$$

and

$$\begin{aligned} a \triangleleft P' &= a \triangleleft ((a \triangleleft P_1') \triangleleft (a' \triangleleft P_2')) && \text{by (22)} \\ &= a \triangleleft P_1' && \text{by (26)} \\ &= a \triangleleft \mathsf{T}^{\mathsf{U}}_a(P_1)' && \text{by IH} \\ &= a \triangleleft \mathsf{T}^{\mathsf{U}}_a(P)'. \end{aligned}$$

- If $P = (b \triangleleft P_1) \triangleleft (b' \triangleleft P_2)$ for $b \neq a$, then

$$\begin{aligned} a \triangleleft P &= a \triangleleft ((b \triangleleft (a \triangleleft P_1')) \triangleleft (b' \triangleleft (a \triangleleft P_2'))) && \text{by (27)} \\ &= a \triangleleft ((b \triangleleft (a \triangleleft \mathsf{T}^{\mathsf{U}}_a(P_1)')) \triangleleft (b' \triangleleft (a \triangleleft \mathsf{T}^{\mathsf{U}}_a(P_2)'))) && \text{by IH} \\ &= a \triangleleft ((b \triangleleft \mathsf{T}^{\mathsf{U}}_a(P_1)) \triangleleft (b' \triangleleft \mathsf{T}^{\mathsf{U}}_a(P_2))) && \text{by (27)} \\ &= a \triangleleft \mathsf{T}^{\mathsf{U}}_a((b \triangleleft P_1) \triangleleft (b' \triangleleft P_2)), \end{aligned}$$

and

$$\begin{aligned}
a \mathbin{\triangleleft\!\!\!|} P' &= a \mathbin{\triangleleft\!\!\!|} ((b \mathbin{\triangleleft\!\!\!|} P_1') \mathbin{\triangleleft\!\!\!|} (b' \mathbin{\triangleleft\!\!\!|} P_2')) && \text{by (22)}\\
&= a \mathbin{\triangleleft\!\!\!|} ((b \mathbin{\triangleleft\!\!\!|} (a \mathbin{\triangleleft\!\!\!|} P_1)) \mathbin{\triangleleft\!\!\!|} (b' \mathbin{\triangleleft\!\!\!|} (a \mathbin{\triangleleft\!\!\!|} P_2))) && \text{by (27)}\\
&= a \mathbin{\triangleleft\!\!\!|} ((b \mathbin{\triangleleft\!\!\!|} (a \mathbin{\triangleleft\!\!\!|} \mathsf{T}^{\mathsf{U}}_a(P_1))) \mathbin{\triangleleft\!\!\!|} (b' \mathbin{\triangleleft\!\!\!|} (a \mathbin{\triangleleft\!\!\!|} \mathsf{T}^{\mathsf{U}}_a(P_2)))) && \text{by IH}\\
&= a \mathbin{\triangleleft\!\!\!|} ((b \mathbin{\triangleleft\!\!\!|} \mathsf{T}^{\mathsf{U}}_a(P_1)') \mathbin{\triangleleft\!\!\!|} (b' \mathbin{\triangleleft\!\!\!|} \mathsf{T}^{\mathsf{U}}_a(P_2)')) && \text{by (27)}\\
&= a \mathbin{\triangleleft\!\!\!|} ((b \mathbin{\triangleleft\!\!\!|} \mathsf{T}^{\mathsf{U}}_a(P_1)) \mathbin{\triangleleft\!\!\!|} (b' \mathbin{\triangleleft\!\!\!|} \mathsf{T}^{\mathsf{U}}_a(P_2)))' && \text{by (22)}\\
&= a \mathbin{\triangleleft\!\!\!|} \mathsf{T}^{\mathsf{U}}_a((b \mathbin{\triangleleft\!\!\!|} P_1) \mathbin{\triangleleft\!\!\!|} (b' \mathbin{\triangleleft\!\!\!|} P_2))'.
\end{aligned}$$

Statement 2 follows in a similar way. □

Theorem 7.5. *For each term P constructed from $\{\mathbin{\triangleleft\!\!\!|}, \mathsf{T}, \mathsf{F}, \mathsf{U}, a \mid a \in A\}$ there is $Q \in \mathit{MUNBF}_A$ such that* $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell\mathrm{N}} \vdash P = Q$.

Proof. We first prove an auxiliary result: if P and Q are mUNBFs, there is a mUNBF R such that $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell\mathrm{N}} \vdash P \mathbin{\triangleleft\!\!\!|} Q = R$. We prove this by induction on the structure of P.

- If $P = \mathsf{T}$, make a case distinction on Q: if $Q \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$, this is trivial, and if $Q = (a \mathbin{\triangleleft\!\!\!|} Q_1) \mathbin{\triangleleft\!\!\!|} (a' \mathbin{\triangleleft\!\!\!|} Q_2)$, then by induction there are mUNBFs R_i such that $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell\mathrm{N}} \vdash R_i = \mathsf{T} \mathbin{\triangleleft\!\!\!|} Q_i$, and since a does not occur in the Q_i, it does not occur in the R_i either. Derive

$$\begin{aligned}
\mathsf{T} \mathbin{\triangleleft\!\!\!|} ((a \mathbin{\triangleleft\!\!\!|} Q_1) \mathbin{\triangleleft\!\!\!|} (a' \mathbin{\triangleleft\!\!\!|} Q_2)) &= ((a \mathbin{\triangleleft\!\!\!|} Q_1) \mathbin{\triangleleft\!\!\!|} (a' \mathbin{\triangleleft\!\!\!|} Q_2))' && \text{by (19)}\\
&= (a \mathbin{\triangleleft\!\!\!|} Q_1') \mathbin{\triangleleft\!\!\!|} (a' \mathbin{\triangleleft\!\!\!|} Q_2') && \text{by (23)}\\
&= (a \mathbin{\triangleleft\!\!\!|} (\mathsf{T} \mathbin{\triangleleft\!\!\!|} Q_1)) \mathbin{\triangleleft\!\!\!|} (a' \mathbin{\triangleleft\!\!\!|} (\mathsf{T} \mathbin{\triangleleft\!\!\!|} Q_2)) && \text{by (19)}\\
&= (a \mathbin{\triangleleft\!\!\!|} R_1) \mathbin{\triangleleft\!\!\!|} (a' \mathbin{\triangleleft\!\!\!|} R_2).
\end{aligned}$$

- If $P = \mathsf{F}$, then $\mathsf{F} \mathbin{\triangleleft\!\!\!|} Q = \mathsf{T}$ by (24).
- If $P = \mathsf{U}$, then $\mathsf{U} \mathbin{\triangleleft\!\!\!|} Q = \mathsf{U}$ by axiom (Und).
- If $P = (a \mathbin{\triangleleft\!\!\!|} P_1) \mathbin{\triangleleft\!\!\!|} (a' \mathbin{\triangleleft\!\!\!|} P_2)$, then by induction there are mUNBFs R_i such that $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell\mathrm{N}} \vdash R_i = P_i \mathbin{\triangleleft\!\!\!|} Q$. Note that Q, and therefore each of R_i, can contain the atom a. Derive

$$\begin{aligned}
((a \mathbin{\triangleleft\!\!\!|} P_1) \mathbin{\triangleleft\!\!\!|} (a' \mathbin{\triangleleft\!\!\!|} P_2)) \mathbin{\triangleleft\!\!\!|} Q &= (a \mathbin{\triangleleft\!\!\!|} (P_1 \mathbin{\triangleleft\!\!\!|} Q)) \mathbin{\triangleleft\!\!\!|} (a' \mathbin{\triangleleft\!\!\!|} (P_2 \mathbin{\triangleleft\!\!\!|} Q)) && \text{by (23)}\\
&= (a \mathbin{\triangleleft\!\!\!|} R_1) \mathbin{\triangleleft\!\!\!|} (a' \mathbin{\triangleleft\!\!\!|} R_2)\\
&= (a \mathbin{\triangleleft\!\!\!|} \mathsf{T}^{\mathsf{U}}_a(R_1)) \mathbin{\triangleleft\!\!\!|} (a' \mathbin{\triangleleft\!\!\!|} \mathsf{F}^{\mathsf{U}}_a(R_2)). && \text{by Lemma 7.4}
\end{aligned}$$

This concludes the proof of the auxiliary result.

The theorem follows easily by structural induction on P. The base cases T, F, U are trivial and by (25), $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell\mathrm{N}} \vdash a = (a \mathbin{\triangleleft\!\!\!|} \mathsf{T}) \mathbin{\triangleleft\!\!\!|} (a' \mathbin{\triangleleft\!\!\!|} \mathsf{F})$. The inductive case follows from the auxiliary result. □

Finally, we note there is a one-to-one correspondence between $\mathit{MBF}^{\mathsf{U}}_A$ (the mem-basic forms that can also contain the constant U) and MUNBF_A: define f and g by $f(n) = g(n) = n$ for $n \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$, $f(P \triangleleft a \triangleright Q) = (a \mathbin{\triangleleft\!\!\!|} f(P)) \mathbin{\triangleleft\!\!\!|} (a' \mathbin{\triangleleft\!\!\!|} f(Q))$, and $g((a \mathbin{\triangleleft\!\!\!|} P) \mathbin{\triangleleft\!\!\!|} (a' \mathbin{\triangleleft\!\!\!|} Q)) = g(P) \triangleleft a \triangleright g(Q)$. It easily follows that $g \circ f$ and $f \circ g$ are the identity on MBF_A and MUNBF_A, respectively.

8 Conclusions

We begin with a remark about the incorporation of the constants T and F in short-circuit logics, the logics that in programming model the use of conditions and prescribe short-circuit evaluation (more information and motivation can be found in [12, 4]). Non-commutative propositional logic, or MSCL, deals with the case where atomic side effects do not occur and requires incorporation of (at least one) of these constants because they are not definable: for no atom a does it hold that $a \mathbin{{}^{\circ}\!\vee} \neg a = \mathsf{T}$ or $a \mathbin{{}_{\circ}\!\wedge} \neg a = \mathsf{F}$ (compare this with CP and CP_{mem}, which require both constants). As for the "non-commutativity" from the title of this paper, the difference between $a \mathbin{{}_{\circ}\!\wedge} \mathsf{F}$ and $\mathsf{F} \mathbin{{}_{\circ}\!\wedge} a$ is that the first expression requires evaluation of a and the second does not, so these two expressions are not identified in MSCL.

This paper can be seen as a continuation of [4] because it introduces variants of MSCL with additional or alternative connectives:

1. The addition of the connective ℓIFF to MSCL (and/or its dual ℓXOR) can be motivated as a matter of convenience. The connectives ℓIFF and ℓXOR are definable in EqMSCL, so their addition is not essential and can only contribute to more comprehensible axioms and derivations.

   Conversely, negation is definable by the addition of any of these two by $\neg x = \mathsf{F} \mathbin{\circ\!\!\leftrightarrow} x$ (or $\neg x = x \mathbin{\circ\!\!\leftrightarrow} \mathsf{F}$) and $\neg x = \mathsf{T} \mathbin{\circ\!\!\oplus} x$ (or $\neg x = x \mathbin{\circ\!\!\oplus} \mathsf{T}$), and thereby also the other connective is definable: $x \mathbin{\circ\!\!\oplus} y = x \mathbin{\circ\!\!\leftrightarrow} \neg y = \neg(x \mathbin{\circ\!\!\leftrightarrow} y)$. However, it does not seem an attractive idea to omit negation.

   Finally, omitting ${}_{\circ}\!\wedge$ and ${}^{\circ}\!\vee$ from $\mathrm{EqMSCL}_{\ell\mathrm{I}}$, either one or both of $\circ\!\!\leftrightarrow$ and $\circ\!\!\oplus$ is modulo memorising valuation congruence not sufficiently expressive, which can be easily seen by considering $a \mathbin{{}_{\circ}\!\wedge} b$.

2. A preference for $\mathrm{EqMSCL}_{\ell\mathrm{N}}$ over EqMSCL can be motivated as a technical improvement. As suggested in Section 6 (and illustrated in Section 7), the completeness result for EqMSCL proved in [4] is easier to prove with help of $\mathrm{EqMSCL}_{\ell\mathrm{N}}$, which has only three axioms and allows smaller and simpler inductive proofs of the supporting lemmas. Then, Theorem 6.1 implies the completeness of EqMSCL. Similar remarks can be made for preferring $\mathrm{EqMSCL}^{\mathsf{U}}_{\ell\mathrm{N}}$ to $\mathrm{EqMSCL}^{\mathsf{U}}$.

A second goal of this paper is to emphasise that the mem-basic forms introduced in [1] themselves provide a semantics for closed terms: this is based on the results in [2] on the correspondence between mem-basic forms and evaluation trees, and on the coinciding congruences defined by each (see [2, Prop.5.13]). It is not difficult to prove that this is also true for the extension to T_A, the set of closed terms belonging to $\mathrm{CP}(\neg, {}_{\circ}\!\wedge, {}^{\circ}\!\vee)$ and $\mathrm{CP}_{mem}(\neg, {}_{\circ}\!\wedge, {}^{\circ}\!\vee)$, compare Theorems 2.11 and 3.5.

We conclude with a reflection on the definition of MSCL and the advantages and disadvantages of using Hoare's conditional as an auxiliary operator to define this short-circuit logic. For the question of which laws axiomatise short-circuit evaluation, the appeal to an auxiliary operator does not seem appropriate, and axioms for the Boolean short-circuit connectives are more interesting, so this can be considered a disadvantage. On the other hand, the usefulness of the conditional connective as a means of proving properties of short-circuit logics[8] can be easily demonstrated. A first advantage of $\mathrm{CP}_{mem}(\neg, {}_{\circ}\!\wedge, {}^{\circ}\!\vee)$ (the underlying set of axioms of MSCL) over $\mathrm{EqMSCL}_{\ell\mathrm{I}}$ and

[8] Note that all short-circuit logics introduced in [3] are defined with help of the conditional connective.

$\text{EqMSCL}_{\ell\text{N}}$ is that in its mem-basic forms (MBF_A) each atom occurs at most once, which is also true of their U-variants. The analogue of Theorem 7.5 for C_A^{U}-terms, viz,

$$\text{For each } P \in C_A^{\mathsf{U}} \text{ there is } Q \in MBF_A^{\mathsf{U}} \text{ such that } \text{CP}_{mem}^{\mathsf{U}} \vdash P = Q,$$

is therefore much easier to prove (for the case without U this is shown in [1, La.8.2], and U does not provide a worrying extension here). As a second example, equation (F9) in Table 3 has a simple proof in $\text{CP}_{mem}(\neg, \mathbin{{}_{\circ}\!\wedge}, \mathbin{{}^{\circ}\!\vee})$ (in which the axiom (CPmem) is not needed):

$$(x \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \mathbin{{}^{\circ}\!\vee} y = \mathsf{T} \lhd (\mathsf{F} \lhd x \rhd \mathsf{F}) \rhd y = y \lhd x \rhd y = y \lhd (\mathsf{T} \lhd x \rhd \mathsf{T}) \rhd \mathsf{F} = (x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} y,$$

but a proof in EqMSCL is not so simple. As a last example, one can compare a proof of the associativity of $\mathbin{\circ\!\!\leftrightarrow}$ in $\text{CP}_{mem} \cup \{x \mathbin{\circ\!\!\leftrightarrow} y = y \lhd x \rhd (\mathsf{F} \lhd y \rhd \mathsf{T})\}$ with one in $\text{EqMSCL}_{\ell\text{I}}$. Elaborating on this, there is a strong case for introducing the abbreviation x' for $\mathsf{F} \lhd x \rhd \mathsf{T}$ in CP, which yields equations like

$$(x \lhd y \rhd x')' = x \lhd y' \rhd x'$$

(which follow easily in CP). This allows a very simple proof of the associativity of $\mathbin{\circ\!\!\leftrightarrow}$ in $\text{CP}_{mem} \cup \{x \mathbin{\circ\!\!\leftrightarrow} y = y \lhd x \rhd y'\}$ (again, the axiom (CPmem) is not needed):

$$\begin{aligned}(x \mathbin{\circ\!\!\leftrightarrow} y) \mathbin{\circ\!\!\leftrightarrow} z = z \lhd (y \lhd x \rhd y') \rhd z' &= (z \lhd y \rhd z') \lhd x \rhd (z \lhd y' \rhd z')\\ &= (z \lhd y \rhd z') \lhd x \rhd (z \lhd y \rhd z')' = x \mathbin{\circ\!\!\leftrightarrow} (y \mathbin{\circ\!\!\leftrightarrow} z).\end{aligned}$$

Related work. In 1948, Church introduced in [5] the *conditioned disjunction* $[p, q, r]$, which, following the author, may be read "p or r according as q or not q" and which expresses exactly the same connective as Hoare's conditional (introduced in 1985). Church showed that this connective together with constants T and F form a complete set of independent primitive connectives for the propositional calculus. Church also noted that for propositional variables a, b, c, the dual of $[a, b, c]$ is simply $[c, b, a]$, so that to dualize an expression of the propositional calculus in which the only connectives occurring are conditioned disjunction, T, and F, it is sufficient to write the expression backwards and at the same time to interchange T and F. For the conditioned disjunction, reference [6] is often used, and also the name *conditional disjunction.* Although $[x, y, z]$ has explicit scoping, more complex expressions such as $[[a, b, [b, a, c]], [a, b, [c, a, b], [a, b, c]]$ are difficult to read and for this reason we prefer Hoare's conditional connective $x \lhd y \rhd z$.

In 2013, we defined *free short-circuit logic* (notation FSCL) in [3] as the equational logic that implies the part of the equational theory of $\text{CP}(\neg, \mathbin{{}_{\circ}\!\wedge}, \mathbin{{}^{\circ}\!\vee})$ that is expressed in $\Sigma_{\text{SCL}}(A)$. In [12], the set of axioms EqFSCL is defined as (F1) − (F10), where (F1) = (Neg), (F2) = (Or), (F4) = (Tand), and the remaining axioms are listed in Table 3, and it is proved that EqFSCL axiomatises FSCL for closed terms. We note that in FSCL, not all basic forms can be expressed, for example not those of $a \lhd b \rhd c$ and $a \lhd a \rhd (\mathsf{F} \lhd a \rhd \mathsf{T})$ (see [1, Prop.12.1 and Thm.12.2]).

In 2020, Cornets de Groot defined in [7] the two left-sequential connectives ℓXOR and ℓNAND and studied their relation with FSCL. In that paper, complete equational axiomatisations of the resulting logical systems for closed terms are provided, while also attention is paid to the dual connectives ℓIFF and ℓNOR, respectively, and to expressiveness issues under free valuation congruence.

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