Sparse optimal control of a quasilinear elliptic PDE in measure spaces*

Fabian Hoppe

DLR German Aerospace Center
Institute for Software Technology
High Performance Computing
Köln

*work was done at University of Bonn, Institute for Numerical Simulation, partially supported by DFG SFB 1060

13.07.2022

Outline

1. Overview and problem setting
2. Analysis of the state equation
3. First-order optimality conditions
4. Second-order optimality conditions

Literature overview (1/2)

- **Sparse optimal control**
  Different approaches allow to enforce small ("sparse") support of the optimal control of an optimal control problem, e.g.:

Literature overview (1/2)

- **Sparse optimal control**
  Different approaches allow to enforce small ("sparse") support of the optimal control of an optimal control problem, e.g.:
  - *L^1- or mixed L^1/L^2-penalization:* L^1-penalization [Stadler 2009], L^1-L^2-penalization ("directional sparsity") [Herzog, Stadler, Wachsmuth 2012], L^2-L^1-penalization [Casas, Herzog, Wachsmuth 2017]

DLR

Literature overview (1/2)

- **Sparse optimal control**
  Different approaches allow to enforce small ("sparse") support of the optimal control of an optimal control problem, e.g.:
  - *L^1- or mixed L^1/L^2-penalization:* L^1-penalization [Stadler 2009], L^1-L^2-penalization ("directional sparsity") [Herzog, Stadler, Wachsmuth 2012], L^2-L^1-penalization [Casas, Herzog, Wachsmuth 2017]
  - *Controls from measure spaces:* linear elliptic [Casas, Clason, Kunisch 2012] and parabolic [Casas, Kunisch 2016] PDEs, semilinear elliptic PDEs [Casas, Kunisch 2014], [Ponce, Wilmet 2018]

Literature overview (1/2)

- **Sparse optimal control**
  Different approaches allow to enforce small ("sparse") support of the optimal control of an optimal control problem, e.g.:
  - *L^1- or mixed L^1/L^2-penalization:* L^1-penalization [Stadler 2009], L^1-L^2-penalization ("directional sparsity") [Herzog, Stadler, Wachsmuth 2012], L^2-L^1-penalization [Casas, Herzog, Wachsmuth 2017]
  - *Controls from measure spaces:* linear elliptic [Casas, Clason, Kunisch 2012] and parabolic [Casas, Kunisch 2016] PDEs, semilinear elliptic PDEs [Casas, Kunisch 2014], [Ponce, Wilmet 2018]
  - *Combined approaches:* L^2-$\mathcal{M}$ [Casas, Clason, Kunisch 2013], $\mathcal{M}$-L^2 "measure-valued directional sparsity" [Kunisch, Pieper, Vexler 2014],

DLR

Literature overview (1/2)

- **Sparse optimal control**
  Different approaches allow to enforce small ("sparse") support of the optimal control of an optimal control problem, e.g.:
  - *L^1- or mixed L^1/L^2-penalization:* L^1-penalization [Stadler 2009], L^1-L^2-penalization ("directional sparsity") [Herzog, Stadler, Wachsmuth 2012], L^2-L^1-penalization [Casas, Herzog, Wachsmuth 2017]
  - *Controls from measure spaces:* linear elliptic [Casas, Clason, Kunisch 2012] and parabolic [Casas, Kunisch 2016] PDEs, semilinear elliptic PDEs [Casas, Kunisch 2014], [Ponce, Wilmet 2018]
  - *Combined approaches:* L^2-$\mathcal{M}$ [Casas, Clason, Kunisch 2013], $\mathcal{M}$-L^2 "measure-valued directional sparsity" [Kunisch, Pieper, Vexler 2014],
  - *L^p-penalization with $p \in [0, 1)$:* [Ito, Kunisch 2014], [Casas, Wachsmuth 2020]

Literature overview (2/2)

- **Optimal control of quasilinear PDEs**
  Nonlinearity appears in the coefficients of the principal part of the state equation
  here: nonlinearity depends on the solution, *not* on its gradient (= nonmonotone nonlinearity!)

DLR

Literature overview (2/2)

- **Optimal control of quasilinear PDEs**
  Nonlinearity appears in the coefficients of the principal part of the state equation
  here: nonlinearity depends on the solution, *not* on its gradient (= nonmonotone nonlinearity!)
  - *Quasilinear elliptic problems:* smooth nonlinearity [Casas, Tröltzsch 2009; Casas, Tröltzsch 2011; Casas, Tröltzsch 2012], [Casas, Dhamo 2011], nonsmooth nonlinearity [Clason, Nhu, Rösch 2021; Clason, Nhu, Rösch 2021; Clason, Nhu, Rösch 2022]

Literature overview (2/2)

- **Optimal control of quasilinear PDEs**
  Nonlinearity appears in the coefficients of the principal part of the state equation
  here: nonlinearity depends on the solution, *not* on its gradient (= nonmonotone nonlinearity!)
  - *Quasilinear elliptic problems:* smooth nonlinearity [Casas, Tröltzsch 2009; Casas, Tröltzsch 2011; Casas, Tröltzsch 2012], [Casas, Dhamo 2011], nonsmooth nonlinearity [Clason, Nhu, Rösch 2021; Clason, Nhu, Rösch 2021; Clason, Nhu, Rösch 2022]
  - *Quasilinear parabolic problems:* [Meinlschmidt, Rehberg 2016], [Meinlschmidt, Meyer, Rehberg 2017a; Meinlschmidt, Meyer, Rehberg 2017b], [Bonifacius, Neitzel 2018], [Casas, Chrysafinos 2018], [Hoppe, Neitzel 2020; Hoppe, Neitzel 2022a]

Literature overview (2/2)

- **Optimal control of quasilinear PDEs**
  Nonlinearity appears in the coefficients of the principal part of the state equation
  here: nonlinearity depends on the solution, *not* on its gradient (= nonmonotone nonlinearity!)
  - *Quasilinear elliptic problems:* smooth nonlinearity [Casas, Tröltzsch 2009; Casas, Tröltzsch 2011; Casas, Tröltzsch 2012], [Casas, Dhamo 2011], nonsmooth nonlinearity [Clason, Nhu, Rösch 2021; Clason, Nhu, Rösch 2021; Clason, Nhu, Rösch 2022]
  - *Quasilinear parabolic problems:* [Meinlschmidt, Rehberg 2016], [Meinlschmidt, Meyer, Rehberg 2017a; Meinlschmidt, Meyer, Rehberg 2017b], [Bonifacius, Neitzel 2018], [Casas, Chrysafinos 2018], [Hoppe, Neitzel 2020; Hoppe, Neitzel 2022a]
- **Sparse optimal control of quasilinear PDEs**

Literature overview (2/2)

- **Optimal control of quasilinear PDEs**
  Nonlinearity appears in the coefficients of the principal part of the state equation
  here: nonlinearity depends on the solution, *not* on its gradient (= nonmonotone nonlinearity!)
  - *Quasilinear elliptic problems:* smooth nonlinearity [Casas, Tröltzsch 2009; Casas, Tröltzsch 2011; Casas, Tröltzsch 2012], [Casas, Dhamo 2011], nonsmooth nonlinearity [Clason, Nhu, Rösch 2021; Clason, Nhu, Rösch 2021; Clason, Nhu, Rösch 2022]
  - *Quasilinear parabolic problems:* [Meinlschmidt, Rehberg 2016], [Meinlschmidt, Meyer, Rehberg 2017a; Meinlschmidt, Meyer, Rehberg 2017b], [Bonifacius, Neitzel 2018], [Casas, Chrysafinos 2018], [Hoppe, Neitzel 2020; Hoppe, Neitzel 2022a]
- **Sparse optimal control of quasilinear PDEs**
  - *parabolic case:* directional sparsity via L^1-, L^1-L^2-, and L^2-L^1-penalization [Hoppe, Neitzel 2022b]

Literature overview (2/2)

- **Optimal control of quasilinear PDEs**
  Nonlinearity appears in the coefficients of the principal part of the state equation
  here: nonlinearity depends on the solution, *not* on its gradient (= nonmonotone nonlinearity!)
  - *Quasilinear elliptic problems:* smooth nonlinearity [Casas, Tröltzsch 2009; Casas, Tröltzsch 2011; Casas, Tröltzsch 2012], [Casas, Dhamo 2011], nonsmooth nonlinearity [Clason, Nhu, Rösch 2021; Clason, Nhu, Rösch 2021; Clason, Nhu, Rösch 2022]
  - *Quasilinear parabolic problems:* [Meinlschmidt, Rehberg 2016], [Meinlschmidt, Meyer, Rehberg 2017a; Meinlschmidt, Meyer, Rehberg 2017b], [Bonifacius, Neitzel 2018], [Casas, Chrysafinos 2018], [Hoppe, Neitzel 2020; Hoppe, Neitzel 2022a]
- **Sparse optimal control of quasilinear PDEs**
  - *parabolic case:* directional sparsity via L^1-, L^1-L^2-, and L^2-L^1-penalization [Hoppe, Neitzel 2022b]
  - *elliptic case:* **topic of this talk** [Hoppe 2022]

Problem formulation and main difficulties

$$\min_{u \in \mathcal{M}_D(\overline{\Omega})} J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \gamma \|u\|_{\mathcal{M}_D(\overline{\Omega})}, \tag{\textbf{P}}$$

$$\text{s.t.} \quad \left\{ \begin{aligned} -\nabla \cdot \xi(y)\rho\nabla y &= u, && \text{in } \Omega \cup \Gamma_N, \\ y &= 0, && \text{on } \Gamma_D. \end{aligned} \right. \tag{\textbf{Eq}}$$

Problem formulation and main difficulties

$$\min_{u \in \mathcal{M}_D(\overline{\Omega})} J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \gamma \|u\|_{\mathcal{M}_D(\overline{\Omega})}, \qquad \textbf{(P)}$$

$$\text{s.t.} \quad \begin{cases} -\nabla \cdot \xi(y)\rho\nabla y = u, & \text{in } \Omega \cup \Gamma_N, \\ y = 0, & \text{on } \Gamma_D. \end{cases} \qquad \textbf{(Eq)}$$

Challenges:

1. Show existence/uniqueness of solutions to (**Eq**) — in which sense? — highly nonlinear equation with right-hand side of extremly low regularity

Problem formulation and main difficulties

$$\min_{u \in \mathcal{M}_D(\overline{\Omega})} J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \gamma \|u\|_{\mathcal{M}_D(\overline{\Omega})}, \tag{P}$$

$$\text{s.t.} \quad \begin{cases} -\nabla \cdot \xi(y)\rho\nabla y = u, & \text{in } \Omega \cup \Gamma_N, \\ y = 0, & \text{on } \Gamma_D. \end{cases} \tag{Eq}$$

Challenges:

1. Show existence/uniqueness of solutions to (**Eq**) — in which sense? — highly nonlinear equation with right-hand side of extremly low regularity
2. Prove differentiability of the control-to-state mapping — differentiability of superposition operators requires function spaces with high regularity $\leftrightsquigarrow$ solutions of equations with measure right-hand side tend to have low regularity

Problem formulation and main difficulties

$$\min_{u \in \mathcal{M}_D(\overline{\Omega})} J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \gamma\|u\|_{\mathcal{M}_D(\overline{\Omega})}, \qquad \textbf{(P)}$$

$$\text{s.t.} \quad \begin{cases} -\nabla \cdot \xi(y)\rho\nabla y = u, & \text{in } \Omega \cup \Gamma_N, \\ y = 0, & \text{on } \Gamma_D. \end{cases} \qquad \textbf{(Eq)}$$

Challenges:

1. Show existence/uniqueness of solutions to (**Eq**) — in which sense? — highly nonlinear equation with right-hand side of extremly low regularity
2. Prove differentiability of the control-to-state mapping — differentiability of superposition operators requires function spaces with high regularity $\leftrightsquigarrow$ solutions of equations with measure right-hand side tend to have low regularity
3. (**P**) is nonsmooth due to appearance of the total variation norm

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, bounded domain
  $\Gamma_N \subset \partial\Omega$ Neumann and $\Gamma_D = \partial\Omega \setminus \Gamma_N$ Dirichlet boundary,
  $\Omega \cup \Gamma_N$ **regular in the sense of Gröger**

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, bounded domain
  $\Gamma_N \subset \partial\Omega$ Neumann and $\Gamma_D = \partial\Omega \setminus \Gamma_N$ Dirichlet boundary,
  $\Omega \cup \Gamma_N$ **regular in the sense of Gröger**
- Γ_D has nonzero surface measure within $\partial\Omega$
  Subscript "$_D$" $\rightsquigarrow$ homogeneous Dirichlet boundary conditions on Γ_D

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, bounded domain
  $\Gamma_N \subset \partial\Omega$ Neumann and $\Gamma_D = \partial\Omega \setminus \Gamma_N$ Dirichlet boundary,
  $\Omega \cup \Gamma_N$ **regular in the sense of Gröger**
- Γ_D has nonzero surface measure within $\partial\Omega$
  Subscript "$_D$" $\rightsquigarrow$ homogeneous Dirichlet boundary conditions on Γ_D
- $\mathcal{M}_D(\overline{\Omega}) := C_D(\overline{\Omega})^*$, equipped with total variation norm

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, bounded domain
  $\Gamma_N \subset \partial\Omega$ Neumann and $\Gamma_D = \partial\Omega \setminus \Gamma_N$ Dirichlet boundary,
  $\Omega \cup \Gamma_N$ **regular in the sense of Gröger**
- Γ_D has nonzero surface measure within $\partial\Omega$
  Subscript "$_D$" $\rightsquigarrow$ homogeneous Dirichlet boundary conditions on Γ_D
- $\mathcal{M}_D(\overline{\Omega}) := C_D(\overline{\Omega})^*$, equipped with total variation norm
- $\xi \colon \mathbb{R} \to (0, \infty)$ continuous, bounded from below and above

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, bounded domain
  $\Gamma_N \subset \partial\Omega$ Neumann and $\Gamma_D = \partial\Omega \setminus \Gamma_N$ Dirichlet boundary,
  $\Omega \cup \Gamma_N$ **regular in the sense of Gröger**
- Γ_D has nonzero surface measure within $\partial\Omega$
  Subscript "$_D$" $\rightsquigarrow$ homogeneous Dirichlet boundary conditions on Γ_D
- $\mathcal{M}_D(\overline{\Omega}) := C_D(\overline{\Omega})^*$, equipped with total variation norm
- $\xi\colon \mathbb{R} \to (0, \infty)$ continuous, bounded from below and above
- $\rho\colon \Omega \to \mathbb{R}^{d \times d}$ measurable, essentially bounded, and uniformly coercive

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, bounded domain
  $\Gamma_N \subset \partial\Omega$ Neumann and $\Gamma_D = \partial\Omega \setminus \Gamma_N$ Dirichlet boundary,
  $\Omega \cup \Gamma_N$ **regular in the sense of Gröger**
- Γ_D has nonzero surface measure within $\partial\Omega$
  Subscript "$_D$" $\rightsquigarrow$ homogeneous Dirichlet boundary conditions on Γ_D
- $\mathcal{M}_D(\overline{\Omega}) := C_D(\overline{\Omega})^*$, equipped with total variation norm
- $\xi\colon \mathbb{R} \to (0, \infty)$ continuous, bounded from below and above
- $\rho\colon \Omega \to \mathbb{R}^{d \times d}$ measurable, essentially bounded, and uniformly coercive
- **Elliptic regularity for $-\nabla \cdot \rho^T \nabla$**: There is some $\bar{q} > d$ such that

$$-\nabla \cdot \rho^T \nabla\colon \; W_D^{1,\bar{q}} \to W_D^{-1,\bar{q}}$$

  is a topological isomorphism

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, bounded domain
  $\Gamma_N \subset \partial\Omega$ Neumann and $\Gamma_D = \partial\Omega \setminus \Gamma_N$ Dirichlet boundary,
  $\Omega \cup \Gamma_N$ **regular in the sense of Gröger**
- Γ_D has nonzero surface measure within $\partial\Omega$
  Subscript "$_D$" $\rightsquigarrow$ homogeneous Dirichlet boundary conditions on Γ_D
- $\mathcal{M}_D(\overline{\Omega}) := C_D(\overline{\Omega})^*$, equipped with total variation norm
- $\xi\colon \mathbb{R} \to (0, \infty)$ continuous, bounded from below and above
- $\rho\colon \Omega \to \mathbb{R}^{d \times d}$ measurable, essentially bounded, and uniformly coercive
- **Elliptic regularity for** $-\nabla \cdot \rho^T \nabla$: There is some $\bar{q} > d$ such that

$$-\nabla \cdot \rho^T \nabla\colon \; W_D^{1,\bar{q}} \to W_D^{-1,\bar{q}}$$

  is a topological isomorphism
  - no restriction in space dimension $d = 2$: [Gröger 1989]

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, bounded domain
  $\Gamma_N \subset \partial\Omega$ Neumann and $\Gamma_D = \partial\Omega \setminus \Gamma_N$ Dirichlet boundary,
  $\Omega \cup \Gamma_N$ **regular in the sense of Gröger**
- Γ_D has nonzero surface measure within $\partial\Omega$
  Subscript "$_D$" $\rightsquigarrow$ homogeneous Dirichlet boundary conditions on Γ_D
- $\mathcal{M}_D(\overline{\Omega}) := C_D(\overline{\Omega})^*$, equipped with total variation norm
- $\xi\colon \mathbb{R} \to (0, \infty)$ continuous, bounded from below and above
- $\rho\colon \Omega \to \mathbb{R}^{d\times d}$ measurable, essentially bounded, and uniformly coercive
- **Elliptic regularity for $-\nabla \cdot \rho^T \nabla$:** There is some $\bar{q} > d$ such that

$$-\nabla \cdot \rho^T \nabla\colon \; W_D^{1,\bar{q}} \to W_D^{-1,\bar{q}}$$

  is a topological isomorphism
  - no restriction in space dimension $d = 2$: [Gröger 1989]
  - examples in space dimension $d = 3$: [Disser, Kaiser, Rehberg 2015],...

Outline

1. Overview and problem setting
2. Analysis of the state equation
3. First-order optimality conditions
4. Second-order optimality conditions

Solutions of the state equation (1/2)

We call y a solution to (**Eq**) if

$$y \in W_D^{1,\bar{q}'}(\Omega), \quad \text{s.t.} \int_\Omega \xi(y)\rho \nabla y \nabla \varphi \, \mathrm{d}x = \int_{\overline{\Omega}} \varphi \, \mathrm{d}u, \quad \forall \varphi \in C_D^\infty(\Omega).$$

Solutions of the state equation (1/2)

We call y a solution to (**Eq**) if

$$y \in W_D^{1,\bar{q}'}(\Omega), \quad \text{s.t.} \int_\Omega \xi(y)\rho\nabla y \nabla\varphi \,\mathrm{d}x = \int_{\overline{\Omega}} \varphi \,\mathrm{d}u, \ \ \forall \varphi \in C_D^\infty(\Omega).$$

- Define $\Xi\colon \mathbb{R} \to \mathbb{R}$, $s \mapsto \int_0^s \xi(t)\,\mathrm{d}t$

Solutions of the state equation (1/2)

We call y a solution to (**Eq**) if

$$y \in W_D^{1,\bar{q}'}(\Omega), \quad \text{s.t.} \int_\Omega \xi(y)\rho\nabla y \nabla\varphi \,\mathrm{d}x = \int_{\overline{\Omega}} \varphi \,\mathrm{d}u, \ \ \forall \varphi \in C_D^\infty(\Omega).$$

- Define $\Xi\colon \mathbb{R} \to \mathbb{R}$, $s \mapsto \int_0^s \xi(t)\,\mathrm{d}t$
- Ξ^{-1} defines a bijection $W_D^{1,q}(\Omega) \to W_D^{1,q}(\Omega)$ and continuous maps $W_D^{1,q}(\Omega) \to W_D^{1,p}(\Omega)$ for each $q \in (1,\infty)$, $p \in (1,q)$.

DLR

Solutions of the state equation (1/2)

We call y a solution to (**Eq**) if

$$y \in W_D^{1,\bar{q}'}(\Omega), \quad \text{s.t.} \int_\Omega \xi(y)\rho\nabla y \nabla\varphi \,\mathrm{d}x = \int_{\overline{\Omega}} \varphi \,\mathrm{d}u, \;\; \forall \varphi \in C_D^\infty(\Omega).$$

- Define $\Xi\colon \mathbb{R} \to \mathbb{R}$, $s \mapsto \int_0^s \xi(t)\,\mathrm{d}t$
- Ξ^{-1} defines a bijection $W_D^{1,q}(\Omega) \to W_D^{1,q}(\Omega)$ and continuous maps $W_D^{1,q}(\Omega) \to W_D^{1,p}(\Omega)$ for each $q \in (1,\infty)$, $p \in (1,q)$.
- **Kirchhoff transform:** $y \in W_D^{1,\bar{q}'}(\Omega)$ is solution to (**Eq**) if and only if $w = \Xi(y) \in W_D^{1,\bar{q}'}(\Omega)$ satisfies

$$w \in W_D^{1,q}(\Omega) \quad \text{s.t.} \quad \int_\Omega \rho\nabla w \nabla\varphi \,\mathrm{d}x = \langle u, \varphi\rangle_{W_D^{-1,q}, W_D^{1,q'}}, \;\; \forall \varphi \in C_D^\infty(\Omega).$$

DLR

Solutions of the state equation (2/2)

- By assumption: $-\nabla \cdot \rho^T \nabla \colon W_D^{1,q}(\Omega) \to W_D^{-1,q}(\Omega)$ isomorphism for each $q \in [2, \bar{q}]$.

DLR

Solutions of the state equation (2/2)

- By assumption: $-\nabla \cdot \rho^T \nabla \colon W_D^{1,q}(\Omega) \to W_D^{-1,q}(\Omega)$ isomorphism for each $q \in [2, \bar{q}]$.
- Take adjoints [Meyer, Panizzi, Schiela 2011], [Stampacchia 1965]:

$$(-\nabla \cdot \rho \nabla)^{-1} \colon W_D^{-1,q}(\Omega) \to W_D^{1,q}(\Omega)$$

well-defined for each $q \in [\bar{q}', 2]$.

Solutions of the state equation (2/2)

- By assumption: $-\nabla \cdot \rho^T \nabla \colon W_D^{1,q}(\Omega) \to W_D^{-1,q}(\Omega)$ isomorphism for each $q \in [2, \bar{q}]$.
- Take adjoints [Meyer, Panizzi, Schiela 2011], [Stampacchia 1965]:

$$(-\nabla \cdot \rho \nabla)^{-1} \colon W_D^{-1,q}(\Omega) \to W_D^{1,q}(\Omega)$$

  well-defined for each $q \in [\bar{q}', 2]$.
- hence: $(-\nabla \cdot \rho \nabla)^{-1} \colon \mathcal{M}_D(\bar{\Omega}) \to W_D^{1,q}(\Omega)$ well-defined and weak-$\star$-to-strong continuous for each $q \in [\bar{q}', d')$.

Solutions of the state equation (2/2)

- By assumption: $-\nabla \cdot \rho^T \nabla \colon W_D^{1,q}(\Omega) \to W_D^{-1,q}(\Omega)$ isomorphism for each $q \in [2, \bar{q}]$.
- Take adjoints [Meyer, Panizzi, Schiela 2011], [Stampacchia 1965]:

$$(-\nabla \cdot \rho \nabla)^{-1} \colon W_D^{-1,q}(\Omega) \to W_D^{1,q}(\Omega)$$

  well-defined for each $q \in [\bar{q}', 2]$.
- hence: $(-\nabla \cdot \rho \nabla)^{-1} \colon \mathcal{M}_D(\bar{\Omega}) \to W_D^{1,q}(\Omega)$ well-defined and weak-$\star$-to-strong continuous for each $q \in [\bar{q}', d')$.

Solutions of the state equation (2/2)

- By assumption: $-\nabla \cdot \rho^T \nabla \colon W_D^{1,q}(\Omega) \to W_D^{-1,q}(\Omega)$ isomorphism for each $q \in [2, \bar{q}]$.
- Take adjoints [Meyer, Panizzi, Schiela 2011], [Stampacchia 1965]:

$$(-\nabla \cdot \rho \nabla)^{-1} \colon W_D^{-1,q}(\Omega) \to W_D^{1,q}(\Omega)$$

  well-defined for each $q \in [\bar{q}', 2]$.
- hence: $(-\nabla \cdot \rho \nabla)^{-1} \colon \mathcal{M}_D(\overline{\Omega}) \to W_D^{1,q}(\Omega)$ well-defined and weak-$\star$-to-strong continuous for each $q \in [\bar{q}', d')$.

Control-to-state map:

$$S := \Xi^{-1} \circ (-\nabla \cdot \rho \nabla)^{-1} \colon \mathcal{M}_D(\overline{\Omega}) \to W_D^{1,q}(\Omega)$$

well-defined and weak-$\star$-to-strong continuous for each $q \in [\bar{q}', \frac{d}{d-1})$

Solutions of the state equation (2/2)

- By assumption: $-\nabla \cdot \rho^T \nabla\colon W_D^{1,q}(\Omega) \to W_D^{-1,q}(\Omega)$ isomorphism for each $q \in [2, \bar{q}]$.
- Take adjoints [Meyer, Panizzi, Schiela 2011], [Stampacchia 1965]:

$$(-\nabla \cdot \rho \nabla)^{-1}\colon W_D^{-1,q}(\Omega) \to W_D^{1,q}(\Omega)$$

  well-defined for each $q \in [\bar{q}', 2]$.
- hence: $(-\nabla \cdot \rho \nabla)^{-1}\colon \mathcal{M}_D(\overline{\Omega}) \to W_D^{1,q}(\Omega)$ well-defined and weak-$\star$-to-strong continuous for each $q \in [\bar{q}', d')$.

Control-to-state map:

$$S := \Xi^{-1} \circ (-\nabla \cdot \rho \nabla)^{-1}\colon \mathcal{M}_D(\overline{\Omega}) \to W_D^{1,q}(\Omega)$$

well-defined and weak-$\star$-to-strong continuous for each $q \in [\bar{q}', \frac{d}{d-1})$

Solutions of the state equation (2/2)

- By assumption: $-\nabla \cdot \rho^T \nabla \colon W_D^{1,q}(\Omega) \to W_D^{-1,q}(\Omega)$ isomorphism for each $q \in [2, \bar{q}]$.
- Take adjoints [Meyer, Panizzi, Schiela 2011], [Stampacchia 1965]:

$$(-\nabla \cdot \rho \nabla)^{-1} \colon W_D^{-1,q}(\Omega) \to W_D^{1,q}(\Omega)$$

  well-defined for each $q \in [\bar{q}', 2]$.
- hence: $(-\nabla \cdot \rho \nabla)^{-1} \colon \mathcal{M}_D(\overline{\Omega}) \to W_D^{1,q}(\Omega)$ well-defined and weak-$\star$-to-strong continuous for each $q \in [\bar{q}', d')$.

Control-to-state map:

$$S := \Xi^{-1} \circ (-\nabla \cdot \rho \nabla)^{-1} \colon \mathcal{M}_D(\overline{\Omega}) \to W_D^{1,q}(\Omega)$$

well-defined and weak-$\star$-to-strong continuous for each $q \in [\bar{q}', \frac{d}{d-1})$

Corollary (Well-posedness of (P), [Hoppe 2022])

*If $\gamma > 0$, (**P**) admits at least one global solution.*

DLR

Outline

1. Overview and problem setting
2. Analysis of the state equation
3. **First-order optimality conditions**
4. Second-order optimality conditions

First-order differentiability of S

- **Problem:** application of the implicit function theorem requires to show invertibility of

$$-\nabla \cdot \xi(y)\rho\nabla \bullet \; -\nabla \cdot \xi'(y) \; \bullet \; \rho\nabla y$$

with y of low regularity on spaces of low regularity...

First-order differentiability of S

- **Problem:** application of the implicit function theorem requires to show invertibility of

  $$-\nabla \cdot \xi(y)\rho\nabla \bullet \; -\nabla \cdot \xi'(y) \; \bullet \; \rho\nabla y$$

  with y of low regularity on spaces of low regularity...
- **Idea:** exploit structure $S = \Xi^{-1} \circ (-\nabla \cdot \rho\nabla)^{-1}$ due to Kirchhoff transform
  [Recall: $\Xi(s) = \int_0^s \xi(t)\,\mathrm{d}t$, conditions on ξ ensure linear growth of Ξ^{-1}]

DLR

First-order differentiability of S

- **Problem:** application of the implicit function theorem requires to show invertibility of

$$-\nabla \cdot \xi(y)\rho\nabla \bullet \; -\nabla \cdot \xi'(y) \; \bullet \; \rho\nabla y$$

  with y of low regularity on spaces of low regularity...
- **Idea:** exploit structure $S = \Xi^{-1} \circ (-\nabla \cdot \rho\nabla)^{-1}$ due to Kirchhoff transform
  [Recall: $\Xi(s) = \int_0^s \xi(t)\, \mathrm{d}t$, conditions on ξ ensure linear growth of Ξ^{-1}]
- Discuss differentiability of the nonlinear superposition operator Ξ^{-1} between L^p-spaces ($p < \infty$!)

First-order differentiability of S

- **Problem:** application of the implicit function theorem requires to show invertibility of
$$-\nabla \cdot \xi(y)\rho\nabla \bullet \;\; -\nabla \cdot \xi'(y) \;\bullet\;\; \rho\nabla y$$
with y of low regularity on spaces of low regularity...
- **Idea:** exploit structure $S = \Xi^{-1} \circ (-\nabla \cdot \rho\nabla)^{-1}$ due to Kirchhoff transform
[Recall: $\Xi(s) = \int_0^s \xi(t)\,\mathrm{d}t$, conditions on ξ ensure linear growth of Ξ^{-1}]
- Discuss differentiability of the nonlinear superposition operator Ξ^{-1} between L^p-spaces ($p < \infty$!)
- **Result:** S is continuously Fréchet differentiable as map
$$\mathcal{M}_D(\overline{\Omega}) \to L^r(\Omega), \qquad r \in \left[1, \frac{d}{d-2}\right)$$

First-order necessary optimality conditions

Similar arguments as for the semilinear case [Casas, Kunisch 2014] yield

Theorem (FONs, [Hoppe 2022])

Let $\bar{u} \in \mathcal{M}_D(\overline{\Omega})$ be a local solution of (**P**) *with respect to the $W_D^{-1,q}(\Omega)$-topology for some $q \in (1, \frac{d}{d-1})$. Then, there exists a so-called adjoint state $\bar{p} \in W_D^{1,\bar{q}}(\Omega)$ such that*

$$\begin{cases} -\nabla \cdot \xi(\bar{y})\rho\nabla\bar{y} = u, & \text{on } \Omega \cup \Gamma_N, \\ \bar{y} = 0, & \text{on } \Gamma_D, \end{cases} \qquad \begin{cases} -\nabla \cdot \rho^T \nabla\bar{p} = \xi(\bar{y})^{-1}(\bar{y} - y_d) & \text{on } \Omega, \\ \bar{p} = 0, & \text{on } \Gamma_D, \end{cases}$$

$$\gamma\|\bar{u}\|_{\mathcal{M}_D(\overline{\Omega})} + \int_{\overline{\Omega}} \bar{p}\, \mathrm{d}\bar{u} = 0, \qquad \text{and} \qquad \|\bar{p}\|_{C_D(\overline{\Omega})} \begin{cases} = \gamma & \text{if } \bar{u} \neq 0, \\ \leq \gamma & \text{if } \bar{u} = 0. \end{cases}$$

First-order necessary optimality conditions

Similar arguments as for the semilinear case [Casas, Kunisch 2014] yield

Theorem (FONs, [Hoppe 2022])

Let $\bar u \in \mathcal{M}_D(\overline{\Omega})$ *be a local solution of* (**P**) *with respect to the* $W_D^{-1,q}(\Omega)$*-topology for some* $q \in (1, \frac{d}{d-1})$. *Then, there exists a so-called adjoint state* $\bar p \in W_D^{1,\bar q}(\Omega)$ *such that*

$$\left\{\begin{aligned} -\nabla\cdot\xi(\bar y)\rho\nabla\bar y &= u, && \text{on } \Omega\cup\Gamma_N, \\ \bar y &= 0, && \text{on } \Gamma_D, \end{aligned}\right\} \qquad \left\{\begin{aligned} -\nabla\cdot\rho^T\nabla\bar p &= \xi(\bar y)^{-1}(\bar y - y_d) && \text{on } \Omega, \\ \bar p &= 0, && \text{on } \Gamma_D, \end{aligned}\right\}$$

$$\gamma\|\bar u\|_{\mathcal{M}_D(\overline{\Omega})} + \int_{\overline{\Omega}} \bar p \, d\bar u = 0, \qquad \text{and} \qquad \|\bar p\|_{C_D(\overline{\Omega})} \begin{cases} = \gamma & \text{if } \bar u \neq 0, \\ \leq \gamma & \text{if } \bar u = 0. \end{cases}$$

- **"Sparsity" of $\bar u$:** if $\bar u \neq 0$ it holds

$$\operatorname{supp}(\bar u^+) \subset \{x \in \overline{\Omega}\colon \bar p(x) = -\gamma\}, \quad \text{and} \quad \operatorname{supp}(\bar u^-) \subset \{x \in \overline{\Omega}\colon \bar p(x) = +\gamma\}.$$

DLR

First-order necessary optimality conditions

Similar arguments as for the semilinear case [Casas, Kunisch 2014] yield

Theorem (FONs, [Hoppe 2022])

Let $\bar{u} \in \mathcal{M}_D(\overline{\Omega})$ be a local solution of (**P**) *with respect to the $W_D^{-1,q}(\Omega)$-topology for some $q \in (1, \frac{d}{d-1})$. Then, there exists a so-called adjoint state $\bar{p} \in W_D^{1,\bar{q}}(\Omega)$ such that*

$$\begin{cases} -\nabla \cdot \xi(\bar{y})\rho\nabla\bar{y} = u, & \text{on } \Omega \cup \Gamma_N, \\ \bar{y} = 0, & \text{on } \Gamma_D, \end{cases} \qquad \begin{cases} -\nabla \cdot \rho^T \nabla \bar{p} = \xi(\bar{y})^{-1}(\bar{y} - y_d) & \text{on } \Omega, \\ \bar{p} = 0, & \text{on } \Gamma_D, \end{cases}$$

$$\gamma \|\bar{u}\|_{\mathcal{M}_D(\overline{\Omega})} + \int_{\overline{\Omega}} \bar{p} \, d\bar{u} = 0, \qquad \text{and} \qquad \|\bar{p}\|_{C_D(\overline{\Omega})} \begin{cases} = \gamma & \text{if } \bar{u} \neq 0, \\ \leq \gamma & \text{if } \bar{u} = 0. \end{cases}$$

- **"Sparsity" of $\bar{u}$:** if $\bar{u} \neq 0$ it holds

$$\operatorname{supp}(\bar{u}^+) \subset \{x \in \overline{\Omega} : \bar{p}(x) = -\gamma\}, \quad \text{and} \quad \operatorname{supp}(\bar{u}^-) \subset \{x \in \overline{\Omega} : \bar{p}(x) = +\gamma\}.$$

- **Surprisingly: no derivative of ξ required...**, cf., e.g., [Clason, Nhu, Rösch 2022].

DLR

Outline

1. Overview and problem setting
2. Analysis of the state equation
3. First-order optimality conditions
4. Second-order optimality conditions

Second-order differentiability of S

- **Idea:** exploit Kirchhoff transform and discuss second-order differentiablity of Ξ^{-1} between L^p-spaces ($p < \infty$!)

DLR

Second-order differentiability of S

- **Idea:** exploit Kirchhoff transform and discuss second-order differentiablity of Ξ^{-1} between L^p-spaces ($p < \infty$!)
- *Additional assumption:* Let ξ be continuously differentiable and suppose that there are $a, b \in \mathbb{R}$ such that $|\xi'(s)| \leq a + b|s|$, for all $s \in \mathbb{R}$.

Second-order differentiability of S

- **Idea:** exploit Kirchhoff transform and discuss second-order differentiablity of Ξ^{-1} between L^p-spaces ($p < \infty$!)
- *Additional assumption:* Let ξ be continuously differentiable and suppose that there are $a, b \in \mathbb{R}$ such that $|\xi'(s)| \leq a + b|s|$, for all $s \in \mathbb{R}$.
- S is twice continuously Fréchet differentiable as map

$$\mathcal{M}_D(\overline{\Omega}) \to L^r(\Omega), \qquad r \in \left[1, \frac{d}{2(d-2)}\right)$$

Second-order differentiability of S

- **Idea:** exploit Kirchhoff transform and discuss second-order differentiablity of Ξ^{-1} between L^p-spaces $(p < \infty!)$
- *Additional assumption:* Let ξ be continuously differentiable and suppose that there are $a, b \in \mathbb{R}$ such that $|\xi'(s)| \leq a + b|s|$, for all $s \in \mathbb{R}$.
- S is twice continuously Fréchet differentiable as map

$$\mathcal{M}_D(\overline{\Omega}) \to L^r(\Omega), \qquad r \in \left[1, \frac{d}{2(d-2)}\right)$$

- **Consequence:**
  second-order analysis of (P) requires restriction to space dimension $d = 2$
  (however: no second derivative of ξ needed...)

Second-order necessary optimality conditions

Similar arguments as for the semilinear case [Casas, Kunisch 2014] yield

Theorem (SNCs for $d = 2$, [Hoppe 2022])

Assume that $\bar{u} \in \mathcal{M}_D(\overline{\Omega})$ is a local solution to (**P**) *w.r.t. the $W_D^{-1,q}(\Omega)$-topology for some $q \in (1, 2)$. Then, it holds*

$$F''(\bar{u})v^2 = \int_\Omega \left[1 - \frac{\xi'(\bar{y})}{\xi(\bar{y})}(\bar{y} - y_d)\right] z_v^2 \; \mathrm{d}x \geq 0, \qquad \bar{y} = S(\bar{u}),\; z_v = S'(\bar{u})v$$

for all

$$\begin{aligned} v \in C_{\bar{u}} &:= \{v \in \mathcal{M}_D(\overline{\Omega}) \colon F'(\bar{u})v + \gamma\|\cdot\|'_{\mathcal{M}_D(\overline{\Omega})}(\bar{u}, v) = 0\} \\ &= \{v \in \mathcal{M}_D(\overline{\Omega}) \colon \int_{\overline{\Omega}} \bar{p}\, \mathrm{d}v_s + \gamma\|v_s\|_{\mathcal{M}_D(\overline{\Omega})} = 0\} \end{aligned}$$

where $v = v_a + v_s$ with $v_a = g_v\, \mathrm{d}|\bar{u}|$ and $g_v := \frac{\mathrm{d}v}{\mathrm{d}|\bar{u}|} \in L^1(\overline{\Omega}, \mathrm{d}|\bar{u}|)$

Second-order sufficient optimality conditions

- Exended cone of critical directions

$$C_{\bar{u}}^{\tau} := \{v \in \mathcal{M}_D(\overline{\Omega}) \colon F'(\bar{u})v + \gamma j'(\bar{u}, v) \leq \tau \|z_v\|_{L^2(\Omega)}^2\}$$

Second-order sufficient optimality conditions

- Exended cone of critical directions

$$C_{\bar{u}}^{\tau} := \{v \in \mathcal{M}_D(\overline{\Omega})\colon F'(\bar{u})v + \gamma j'(\bar{u}, v) \leq \tau \|z_v\|_{L^2(\Omega)}^2\}$$

- Adaptation of arguments for the semilinear case [Casas, Kunisch 2014] yields

Second-order sufficient optimality conditions

- Exended cone of critical directions

$$C_{\bar{u}}^{\tau} := \{v \in \mathcal{M}_D(\overline{\Omega})\colon F'(\bar{u})v + \gamma j'(\bar{u}, v) \leq \tau \|z_v\|_{L^2(\Omega)}^2\}$$

- Adaptation of arguments for the semilinear case [Casas, Kunisch 2014] yields

Theorem (SSCs in $d = 2$, [Hoppe 2022])

Let $\bar{u} \in \mathcal{M}_D(\overline{\Omega})$ satisfy the first-order necessary optimality conditions and

$$F''(u)v^2 \geq \kappa \|z_v\|_{L^2(\Omega)}^2 \qquad \forall v \in C_{\bar{u}}^{\tau},\ u \in \mathbb{B}_{\rho}^{W_D^{-1,q}(\Omega)}(\bar{u}), \tag{1}$$

with some $\tau, \rho, \kappa > 0$ and $q \in [\max(\bar{q}', \frac{3}{2}), 2)$. Moreover, let $y_d \in L^s(\Omega)$ with $s \geq (q^{-1} - \frac{1}{2})^{-1}$. Then, there are $\varepsilon, \delta > 0$ such that

$$\hat{J}(u) \geq \hat{J}(\bar{u}) + \frac{\delta}{2} \|z_{u-\bar{u}}\|_{L^2(\Omega)}^2 \qquad \forall u \in \mathbb{B}_{\varepsilon}^{W_D^{-1,q}(\Omega)}(\bar{u}). \tag{2}$$

In particular, $\bar{u}$ is a strict $W_D^{-1,q}(\Omega)$-local solution to (**P**).

Second-order sufficient optimality conditions

- Exended cone of critical directions

$$C_{\bar{u}}^{\tau} := \{v \in \mathcal{M}_D(\overline{\Omega})\colon F'(\bar{u})v + \gamma j'(\bar{u}, v) \leq \tau \|z_v\|_{L^2(\Omega)}^2\}$$

- Adaptation of arguments for the semilinear case [Casas, Kunisch 2014] yields

Theorem (SSCs in $d = 2$, [Hoppe 2022])

Let $\bar{u} \in \mathcal{M}_D(\overline{\Omega})$ satisfy the first-order necessary optimality conditions and

$$F''(u)v^2 \geq \kappa \|z_v\|_{L^2(\Omega)}^2 \qquad \forall v \in C_{\bar{u}}^{\tau},\ u \in \mathbb{B}_{\rho}^{W_D^{-1,q}(\Omega)}(\bar{u}), \tag{1}$$

with some $\tau, \rho, \kappa > 0$ and $q \in [\max(\bar{q}', \frac{3}{2}), 2)$. Moreover, let $y_d \in L^s(\Omega)$ with $s \geq (q^{-1} - \frac{1}{2})^{-1}$. Then, there are $\varepsilon, \delta > 0$ such that

$$\hat{J}(u) \geq \hat{J}(\bar{u}) + \frac{\delta}{2} \|z_{u-\bar{u}}\|_{L^2(\Omega)}^2 \qquad \forall u \in \mathbb{B}_{\varepsilon}^{W_D^{-1,q}(\Omega)}(\bar{u}). \tag{2}$$

In particular, $\bar{u}$ is a strict $W_D^{-1,q}(\Omega)$-local solution to (**P**).

- **"exotic" condition** (1) due to lack of continuity of F'' w.r.t. u

An “almost sufficient” second-order condition

- **Idea:** which conclusion can be drawn from coercivity of F'' at $\bar{u}$ only (=“classical” SSC)?

An "almost sufficient" second-order condition

- **Idea:** which conclusion can be drawn from coercivity of F'' at $\bar{u}$ only (="classical" SSC)?
- Exploit structure of S (Kirchhoff transform) and adapt arguments from semilinear problems [Casas, Kunisch 2014] [Casas, Mateos 2020]

An “almost sufficient” second-order condition

- **Idea:** which conclusion can be drawn from coercivity of F'' at $\bar{u}$ only (=“classical” SSC)?
- Exploit structure of S (Kirchhoff transform) and adapt arguments from semilinear problems [Casas, Kunisch 2014] [Casas, Mateos 2020]

Theorem (“Almost sufficient” second-order condition in $d = 2$, [Hoppe 2022])

Let $y_d \in L^\infty(\Omega)$ hold and let $\bar{u} \in \mathcal{M}_D(\overline{\Omega})$ satisfy the first-order necessary conditions with $\bar{y} \in L^\infty(\Omega)$. If

$$F''(\bar{u})v^2 \geq \kappa \|z_v\|^2_{L^2(\Omega)} \qquad \forall v \in C^\tau_{\bar{u}},$$

holds with some $\tau, \kappa > 0$, then there are $\varepsilon, \delta > 0$ such that

$$\hat{J}(u) \geq \hat{J}(\bar{u}) + \frac{\delta}{2}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)} \quad \forall u \in \mathcal{M}_D(\overline{\Omega}) \text{ s.t. } \|S(u) - \bar{y}\|_{L^\infty(\Omega)} < \varepsilon.$$

An "almost sufficient" second-order condition

- **Idea:** which conclusion can be drawn from coercivity of F'' at $\bar{u}$ only (="classical" SSC)?
- Exploit structure of S (Kirchhoff transform) and adapt arguments from semilinear problems [Casas, Kunisch 2014] [Casas, Mateos 2020]

Theorem ("Almost sufficient" second-order condition in $d = 2$, [Hoppe 2022])

Let $y_d \in L^\infty(\Omega)$ hold and let $\bar{u} \in \mathcal{M}_D(\overline{\Omega})$ satisfy the first-order necessary conditions with $\bar{y} \in L^\infty(\Omega)$. If

$$F''(\bar{u})v^2 \geq \kappa \|z_v\|^2_{L^2(\Omega)} \qquad \forall v \in C^\tau_{\bar{u}},$$

holds with some $\tau, \kappa > 0$, then there are $\varepsilon, \delta > 0$ such that

$$\hat{J}(u) \geq \hat{J}(\bar{u}) + \frac{\delta}{2}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)} \quad \forall u \in \mathcal{M}_D(\overline{\Omega}) \text{ s.t. } \|S(u) - \bar{y}\|_{L^\infty(\Omega)} < \varepsilon.$$

- Similarity to the notion of *"strong local minimum"* [Bayen, Bonnans, Silva 2014], but in the present setting actually weaker than "classical" minimum (=*"weak local minimum"*) because $S(u) \in L^\infty(\Omega)$ is not guaranteed for all $u \in \mathcal{M}_D(\overline{\Omega})$

How to remove the restriction to space dimension $d = 2$? (1/2)

- **Idea:** enforce differentiability of S by restriction to a smaller space [Casas, Kunisch 2014]

How to remove the restriction to space dimension $d = 2$? (1/2)

- **Idea:** enforce differentiability of S by restriction to a smaller space [Casas, Kunisch 2014]
- Introduce the Banach space

$$\mathcal{M}_D^\infty(\overline{\Omega}) := \left\{\mu \in \mathcal{M}_D(\overline{\Omega})\colon (-\nabla \cdot \rho \nabla)^{-1}\mu \in L^\infty(\Omega)\right\},$$

equipped with the norm $\|\mu\|_{\mathcal{M}_D^\infty} := \|\mu\|_{\mathcal{M}_D} + \|(-\nabla \cdot \rho\nabla)^{-1}\mu\|_{L^\infty}$

DLR

How to remove the restriction to space dimension $d = 2$? (1/2)

- **Idea:** enforce differentiability of S by restriction to a smaller space [Casas, Kunisch 2014]
- Introduce the Banach space

$$\mathcal{M}_D^\infty(\overline{\Omega}) := \left\{\mu \in \mathcal{M}_D(\overline{\Omega})\colon (-\nabla \cdot \rho\nabla)^{-1}\mu \in L^\infty(\Omega)\right\},$$

  equipped with the norm $\|\mu\|_{\mathcal{M}_D^\infty} := \|\mu\|_{\mathcal{M}_D} + \|(-\nabla \cdot \rho\nabla)^{-1}\mu\|_{L^\infty}$
- Consider the **restricted problem**

DLR

How to remove the restriction to space dimension $d = 2$? (1/2)

- **Idea:** enforce differentiability of S by restriction to a smaller space [Casas, Kunisch 2014]
- Introduce the Banach space

$$\mathcal{M}_D^\infty(\overline{\Omega}) := \left\{\mu \in \mathcal{M}_D(\overline{\Omega})\colon (-\nabla\cdot\rho\nabla)^{-1}\mu \in L^\infty(\Omega)\right\},$$

equipped with the norm $\|\mu\|_{\mathcal{M}_D^\infty} := \|\mu\|_{\mathcal{M}_D} + \|(-\nabla\cdot\rho\nabla)^{-1}\mu\|_{L^\infty}$

- Consider the **restricted problem**

$$\min_{u\in\mathcal{M}_D^\infty(\overline{\Omega})} J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \gamma\|u\|_{\mathcal{M}_D(\overline{\Omega})} \quad \text{s.t.} \quad (\mathbf{Eq}). \qquad (\mathbf{P}^\infty)$$

How to remove the restriction to space dimension $d = 2$? (1/2)

- **Idea:** enforce differentiability of S by restriction to a smaller space [Casas, Kunisch 2014]
- Introduce the Banach space

$$\mathcal{M}_D^\infty(\overline{\Omega}) := \left\{\mu \in \mathcal{M}_D(\overline{\Omega})\colon (-\nabla \cdot \rho \nabla)^{-1}\mu \in L^\infty(\Omega)\right\},$$

equipped with the norm $\|\mu\|_{\mathcal{M}_D^\infty} := \|\mu\|_{\mathcal{M}_D} + \|(-\nabla \cdot \rho\nabla)^{-1}\mu\|_{L^\infty}$

- Consider the **restricted problem**

$$\min_{u \in \mathcal{M}_D^\infty(\overline{\Omega})} J(y,u) := \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \gamma \|u\|_{\mathcal{M}_D(\overline{\Omega})} \qquad (\mathbf{P}^\infty)$$

$$\text{s.t.} \quad (\mathbf{Eq}).$$

- **Drawback:** well-posedness of $(\mathbf{P}^\infty)$ is not clear, in general (under additional assumptions: ✓ if $y_d \in L^\infty(\Omega)$, adapt ideas from [Pieper, Vexler 2013], [Casas, Kunisch 2014])

How to remove the restriction to space dimension $d = 2$? (2/2)

- **FONs:** if solutions to ($\mathbf{P}^\infty$) exist, FONs as for ($\mathbf{P}$) need to be satisfied

How to remove the restriction to space dimension $d = 2$? (2/2)

- **FONs:** if solutions to ($\mathbf{P}^\infty$) exist, FONs as for ($\mathbf{P}$) need to be satisfied
- The "almost sufficient" second-order condition for ($\mathbf{P}$) is a true SSC for ($\mathbf{P}^\infty$):

How to remove the restriction to space dimension $d = 2$? (2/2)

- **FONs:** if solutions to (**P**$^\infty$) exist, FONs as for (**P**) need to be satisfied
- The "almost sufficient" second-order condition for (**P**) is a true SSC for (**P**$^\infty$):

Theorem (SSCs for (P$^\infty$), [Hoppe 2022])

Let $y_d \in L^\infty(\Omega)$ hold, and suppose that ξ is continuously differentiable. If $\bar{u} \in \mathcal{M}_D^\infty(\overline{\Omega})$ satisfies the first-order optimality conditions and

$$F''(\bar{u})v^2 \geq \kappa \|z_v\|^2_{L^2(\Omega)}, \qquad \forall v \in C_{\bar{u}}^\tau \cap \mathcal{M}_D^\infty(\overline{\Omega}),$$

with some $\tau, \kappa > 0$, then there are $\varepsilon, \delta > 0$ such that

$$\hat{J}(u) \geq \hat{J}(\bar{u}) + \frac{\delta}{2}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)}, \qquad \forall u \in \mathbb{B}_\varepsilon^{\mathcal{M}_D^\infty(\overline{\Omega})}(\bar{u}).$$

*In particular, $\bar{u}$ is a strict local solution to (**P**$^\infty$) w.r.t. the $\mathcal{M}_D^\infty(\overline{\Omega})$-topology.*

DLR

Fabian Hoppe (2022). "Sparse optimal control of a quasilinear elliptic PDE in measure spaces". Submitted. Available as INS Preprint No. 2202

Thank you for your attention!

Fabian Hoppe — fabian.hoppe@dlr.de