Reciprocal Frames
The Flat Beam Grillage

ARCE 453 | INTERDISCIPLINARY SENIOR PROJECT

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Definitions

False Floor: floor system that contains access of a void underneath it.

False Formwork: method used to construct flat beam grillage models; ability to lay out grillage patterns flat on formwork and attach to perimeter walls, once dowels are released, formwork is dropped leaving behind a planar grillage with self-supporting members

Flat Beam Grillage: synonym for planar grillage, beams are flat and span along a plane

Grillage: framework of crossing beams that form a mesh like repetitive pattern

Pinned Support: support that restrains a structure both horizontally and vertically

Planar Grillage: grillage framework that spans two-dimensionally along a plane

Reciprocity: mutual exchange of load between members at a connection

Reciprocal Frame Structures: two or three-dimensional grillage structures with short members that mutually support each other to span a relatively large distance

Roller Support: support that restrains a structure vertically

Shoring: temporary support that holds up an unstable structure

I. Introduction

This report follows the pursuit towards attaining information and researching academic resources regarding the elusive reciprocal frame structures throughout history, in particular the flat beam grillage. In the following pages, the reader should expect to learn about reciprocal frames in a historical context throughout the globe, as well as, gain insight on how to potentially analyze these frames when they span two-dimensionally. As seen in Figure 1 (Pugnale 2011) and Figure 2 (Godthelp 2019), reciprocal frame structures consist of multiple groups of three or more members that are mutually supported. Along the perimeter of the structure, the members are supported by walls, columns, or the ground; where members meet to a certain extent from the ends of an adjacent member, they are supported by such subsequent member. In structural engineering, it is an intuitive instinct to attempt to follow the load path of a structure until the load is safely distributed into the ground. Only considering gravity, when looking at a planar reciprocal frame layout, it is difficult to visualize exactly how the loading is being transferred within the structure. Furthermore, how does one go about doing statics on a problem that is undergoing a perpetual cycle of load transfer. Hopefully, with the data that has been gathered within this research paper, a path can begin to be paved in regards to the design and analysis of two-dimensional reciprocal frames.

Figure 1. Two-Dimensional Reciprocal Frame (Pugnale 2011)

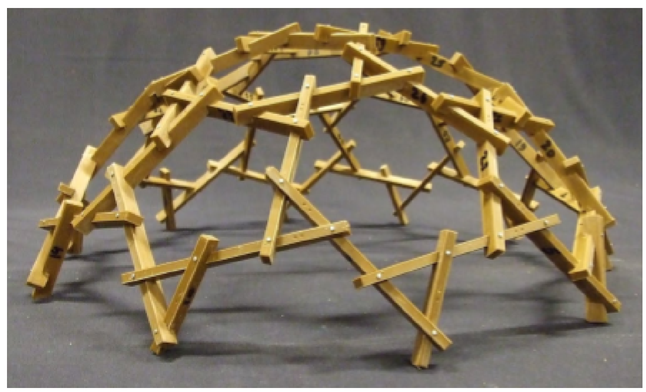

Figure 2. Three-Dimensional Reciprocal Frame (Godthelp 2019)

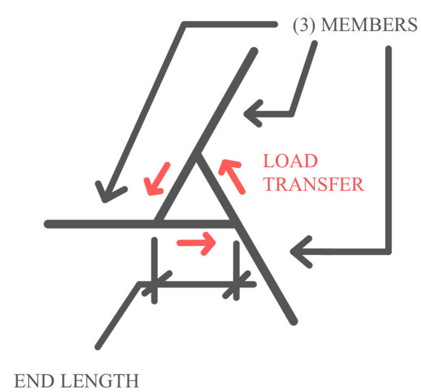

Figure 3. Triangular Connection Load Transfer

II. Historical Background

Although not many reciprocal frame structures were capable of standing the test of time, for reasons such as decay and fire, the idea in itself is scattered throughout history in various parts of the world. It seems clear that many great minds from across all civilizations came upon the possibility of creating a structure, solely through the use of limited resources at the time, that could stand as a mutually supported system. Several examples that hint at the fundamental concept of reciprocal frames are listed below.

i. Cases in Native America

Case #1: Native American Teepee

The Native American Teepee, shown in Figure 4 (Hautman), is a primitive example of how the concept of reciprocal frames was implemented in the history of the Americas. It consists of a simple circular plan where all members are staked to the ground, and then slope upwards where they meet at the center of the teepee as shown on Figure 5. At the tip of the teepee all members support each other to keep the structure upright without having a central support. This is what is means for a structure to be considered self-supported as the members behave as a reciprocal frame.

Figure 4. Native American Teepee (Hautman 2020)

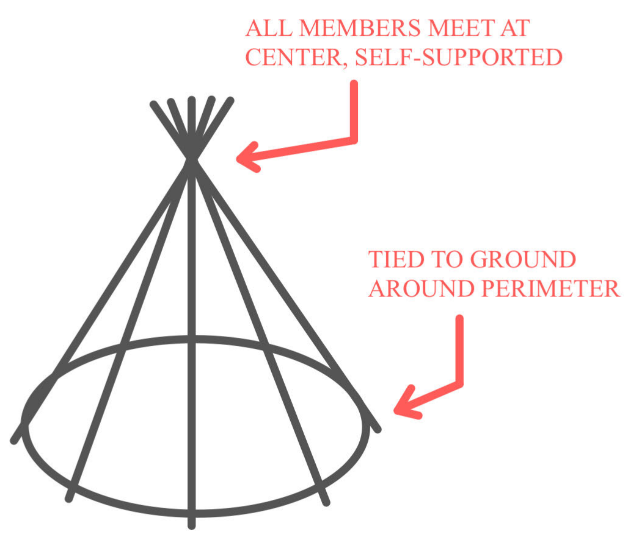

Figure 5. Native American Teepee Structure

Case #2: Navajo Hogan Dwelling

The Navajo Hogan Dwelling is far more complex in comparison to the Native American Teepee. It carries many similarities with the reciprocal framing scheme prevalent in East Asian culture; members are arranged along an octagonal plan around its center. Shown in Figure 7 (Null 2021), the framing scheme of the reciprocal frame structure within a Hogan Dwelling can be seen as if a person where standing right below its center. Several members are aligned side by side where they span parallel to each other; the ends are supported by grouped adjacent members. Load flow staggers radially from the center to the perimeter of the plan. It can be seen that the structure is mutually supported as it closes into the center where there is no central support. Instead of using members that span across the octagon, smaller members that were more available and were more easily transportable were used to form the intricate pattern shown.

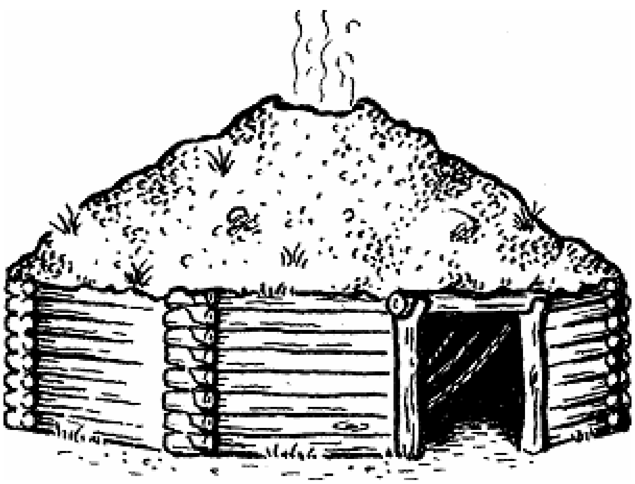

Figure 6. Hogan Dwelling (Webster 2022)

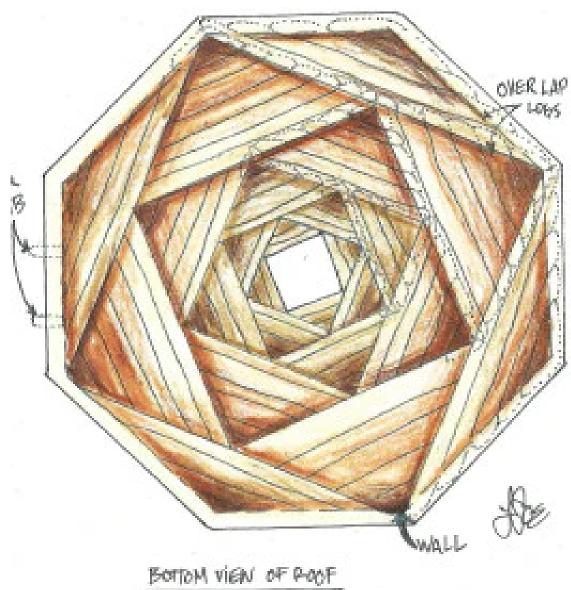

Figure 7. Hogan Dwelling Reciprocal Framing Scheme (Null 2021)

ii. Cases in East Asia

Case #1: Buddhist Monk Chogen Temples

"There is evidence (Ishii 1992/3) that in the twelfth century, the Buddhist Monk Chogen established a technique of spiral layering of wood beams which was used in construction of temples and shrines' '(Larsen 2008). As mentioned previously, due to a lack of documentation and external factors that influenced the structural life of these temples and shrines, there, unfortunately, are no known structures that were capable of overcoming the test of time. There are, however, several structural examples in Japan at present time that follow the premise of Chogen's teachings. Demonstrated in Figure 8, shows an East

Asian reciprocal framing pattern inspired by such teachings.This framing scheme displayed has been implemented in structural form in present day Japan, and will be mentioned in the following case. All members are supported at the perimeter of the structure by walls or columns; similarly to the Native American Teepee, the members end up forming a reciprocal connection at the center of the plan without a central support. This can be more easily seen on Figure 10 (Ishii 2004).

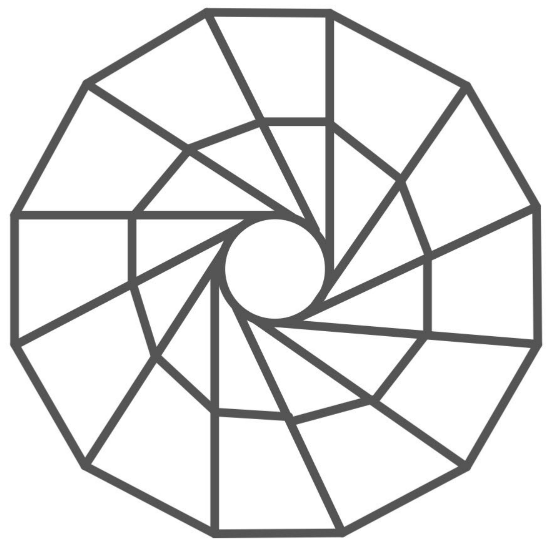

Figure 8. East Asian Reciprocal Framing Scheme

Case #2: Seiwa Bunraku Puppet Theater

The Seiwa Bunraku Puppet Theater complex, located in Seiwa Kumamoto Prefecture, Japan, is a modern day complex inspired by the teachings of the Buddhist Monk Chogen. Kazuhiro Ishii studied Chogens teachings and used his findings to design several structures that embody the reciprocal frame concept throughout the complex. These structures represent the legacy of reciprocal frame structures that were relevant in East Asian culture throughout history. Although there are many examples of reciprocal frames throughout the compound, there is one in particular that correlates with the method that Chogen promoted throughout his life. This method is the spiral layering of timbers which can be seen on the roof of the Seiwa Exhibition Hall. In Figures 9 and 10 (Ishii 2004), two different perspectives are shown of the construction of the circular reciprocal frame that encases the roof of the exhibition hall. What is shown is a present day example of a circular reciprocal frame, inspired by the teachings of past Buddhist monk, that is assumed to form the basis of all East Asian temples and shrines throughout the 12th century.

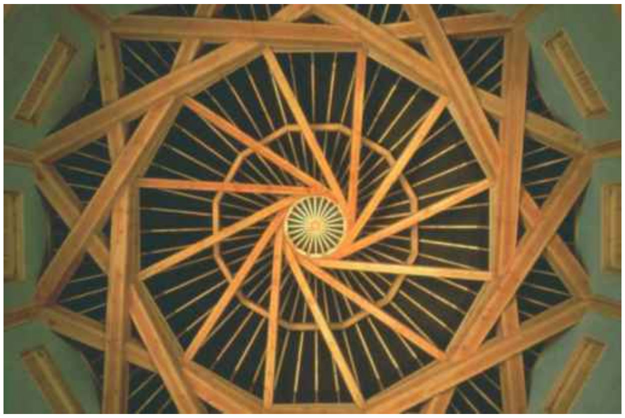

Figure 9. Seiwa Exhibition Hall Roof (Ishii 2004)

Figure 10. Seiwa Exhibition Hall Roof Under Construction (Ishii 2004)

iii. Cases in Medieval Europe

Case #1: Leonardo da Vinci's Planar Grillage Sketches

During the Renaissance, the acclaimed thinker, Leonardo da Vinci, had several accounts of attempting to create a planar grillage during the medieval times. It is recorded in both the Codex Madrid and the Codex Atlantico, where Leonardo da Vinci explores several flat beam grillage assemblies. Figure 9 (Larsen 2007) consists of a flat beam grillage consisting of four members meeting the midpoint of the adjacent supporting member; this is the most simple flat beam grillage as it consists of only a few members that act orthogonal to each other on a square plan. Figure 12 (Larsen 2007) expands the initial four member pattern and repeats across a plane to create a far more complex reciprocal frame system. Figure 13 (Larsen 2007) is the most complicated pattern out of the three consisting members that are angled in plan and meet in both triangular and hexagonal shapes.

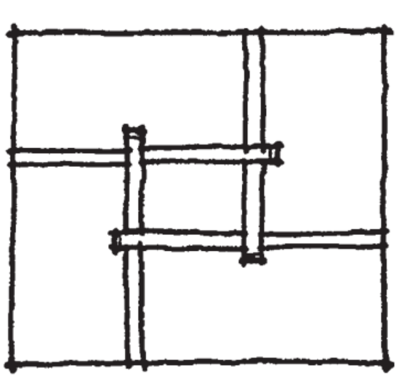

Figure 11. Leonardo da Vinci's Sketch I (Larsen 2007)

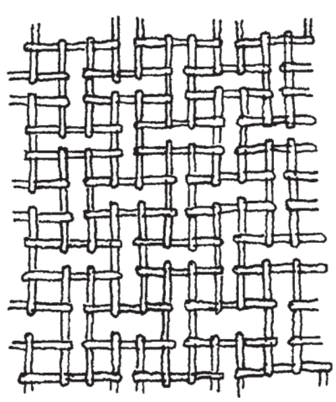

Figure 12. Leonardo da Vinci's Sketch II (Larsen 2007)

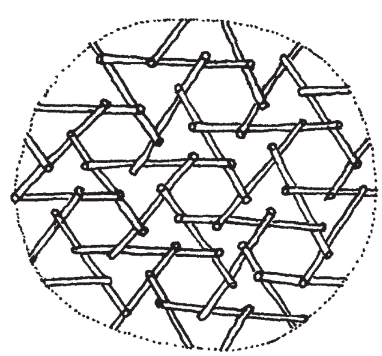

Figure 13. Leonardo da Vinci's Sketch III (Larsen 2007)

Case #2: John Wallis Flat Beam Grillage

John Wallis created his own take of a flat beam grillage within the Opera Matematica. Wallis states that his creations was his own and did not take any form of inspiration from previous thinkers. The assembly shown in Figure 14 (Beguin 2018) consists of a square plan with orthogonal members following a similar pattern shown in Leonardo da Vinci's sketch in Figure 16 (Larsen 2007). This pattern should be noted as it is the pattern used to create Model II in this research report

Case #3: Sebastiano Serlio Flat Beam Grillage

Sebastiano Serlio, similarly to both John Wallis and Leonardo da Vinci, created his own flat beam grillage assembly. It consists of a total of 16 members forming a squared shaped reciprocal connection. The pattern used is the same as some of Leonardo da Vinci's sketches. In plan it is difficult to visualize whether this floor system would function . It is also important to understand the importance of the connections where members meet. To promote reciprocity the connection must consist of a spiral where each member is supported by a subsequent member below it. This will be more clear in the next few pages were models of this flat beam grillages have been created.

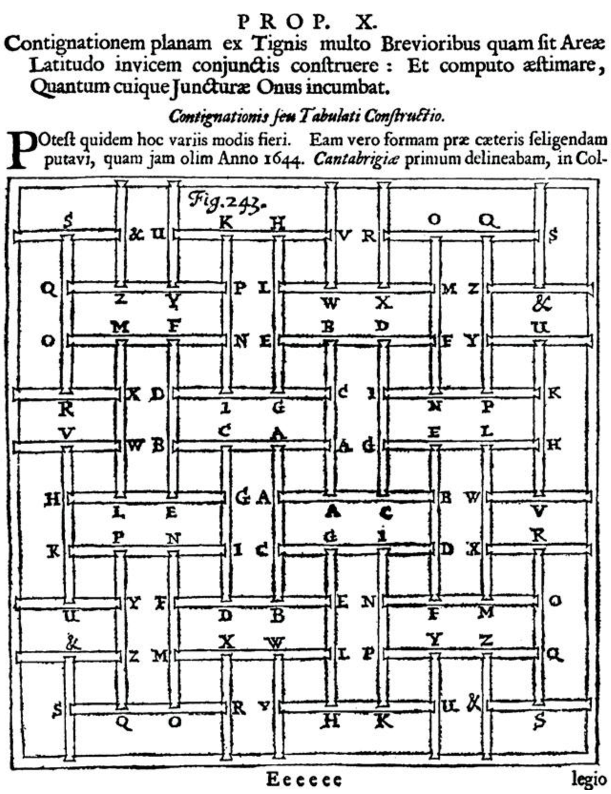
PROP. X.
Contignationem planam ex Tignis multo Brevioribus quam sit Areæ Latitudo invicem conjunctis construere: Et computo æstimare, Quantum cuique Juncturæ Onus incumbat.
Contignationis seu Tabulati Constructio.
Potest quidem hoc variis modis fieri. Eam vero formam præ cæteris seligendam putavi, quam jam olim Anno 1644. *Cantabrigiæ* primum delineabam, in Col-

Figure 14. John Wallis Flat Beam Grillage (Beguin 2018)

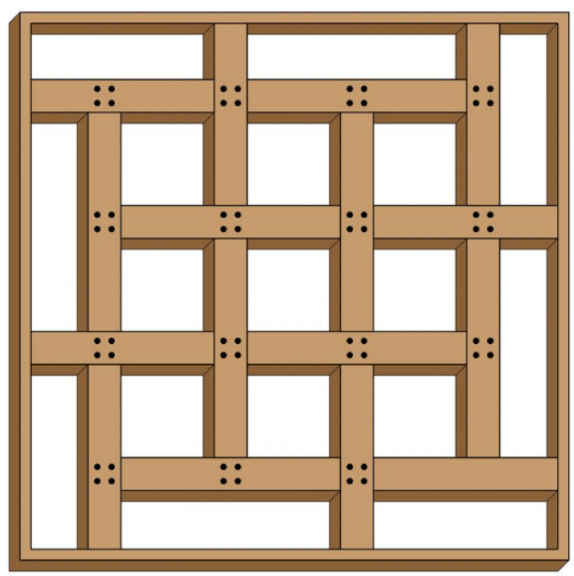

Figure 15. Sebastiano Serlio Flat Beam Grillage (Larsen 2022)

III. Scope of Work

The tasks that will proceed delve deeper within one specific type of reciprocal frame structure that occurred throughout history. This structure will consist of the flat beam grillages that were discussed in the historical background and prevalent through Medieval Europe; as mentioned previously, most of the information available regarding these structures consist only of sketches and diagrams without any real built examples that could potentially provide insight in their design. Is it possible to construct a flat beam grillage with relatively short members that could span large distances as a cohesive system? Were Leonardo da Vinci's and Sebastiano Serlio's sketches and plans possible to construct during the medieval time period? Many questions arise regarding this topic and this report aspires to answer these questions.

IV. Flat Beam Grillage Models

Inspired by the medieval sketches created by Leonardo da Vinci, John Wallis, and Sebastiano Serlio, the models displayed within this report will consist of several flat beam grillage assemblies that differ in layout, shape, form, and size. Reciprocal frame structures are mutually supported, meaning that if any member fails the entire system fails. During construction of these structures, shoring will be required to temporarily support the ends of members that are supported by adjacent members until the configuration is complete. The assembly is complete once all members throughout the span are self supporting each other and members at the perimeter are supported. To create the models the method used was a false floor system. Members were properly aligned into their configuration on the false floor which was temporarily being supported by wooden dowels underneath. Once all members are self supported and supported at the perimeter, dowels are removed dropping the false floor below and leaving behind a stable flat beam grillage assembly.

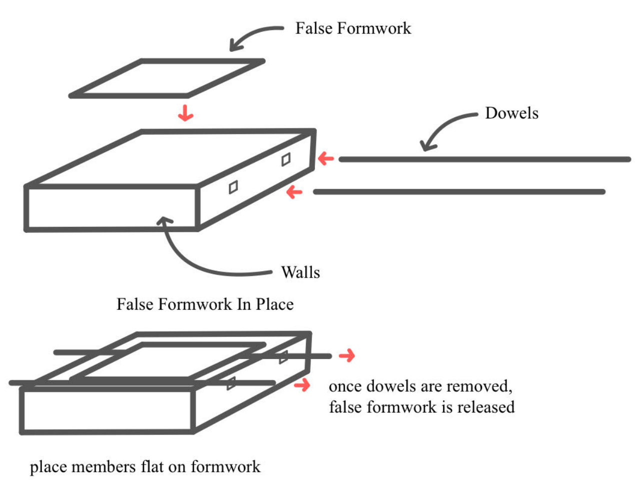

Figure 16. False Formwork Method

Before displaying the flat beam grillage models, it is crucial to explain the members and the connections that will be formed between them. Each model has slight differences in the members from other models to accommodate changes to the flat beam assembly; however, they all follow a similar design. Members consist of pinned-roller supports at the ends, shown in both Figures 17 and 18, to promote a determinate system. Furthermore, to maintain reciprocity at connections, where load transfer occurs in a spiral, the end connections are offset higher than that of the length of the beam. This allows a supported member to lay on top of the supporting member and so on. If it is difficult to visualize Figure 19 shows how the connections will function.

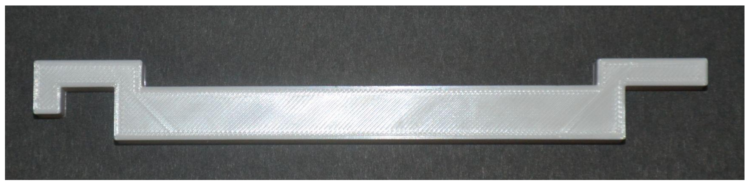

Figure 17. Member Plan View

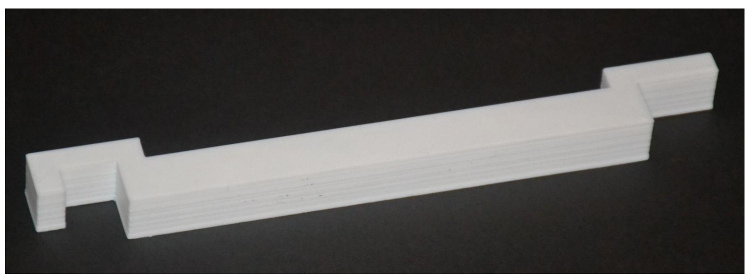

Figure 18. Member Isometric View

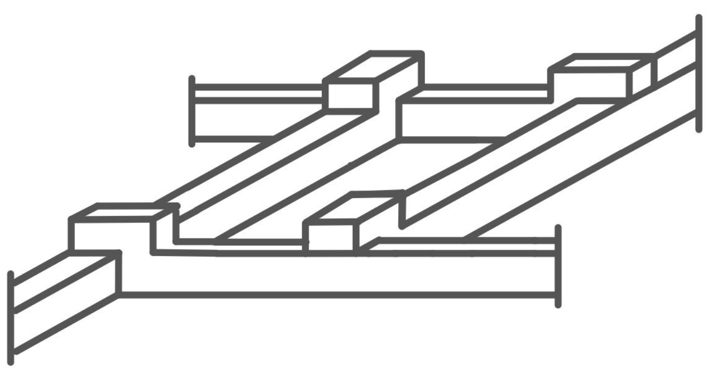

Figure 19. Four Member Connection Oblique View

i. Model I

Model I consists of a flat beam grillage composed of only four members on a square plan. The members align orthogonal to each other and transfer the load along a square path; the pattern created is thus a square at the center of the floor system. Model I was the only model out of the four that did not use the false floor system and instead was simply constructed by hand with columns along the perimeter. Model I is not only the most simple model out of the four, but is one, if the not the most, simple flat beam grillage that could be constructed. Inspired by Leonardo da Vinci's sketch Figure 11 (Larsen 2007)

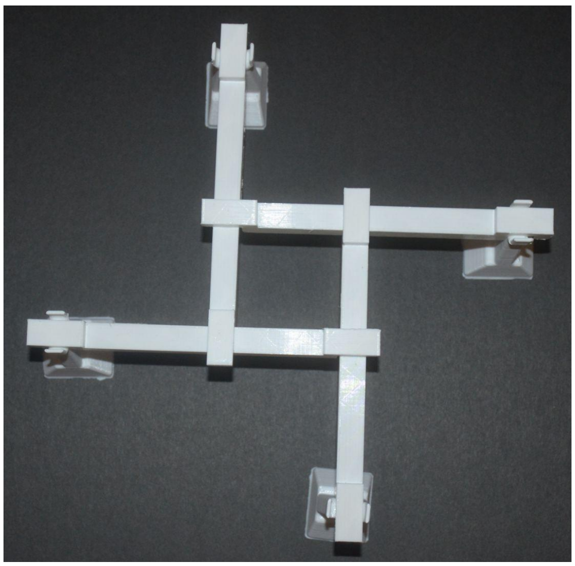

Figure 20. Model I Plan View

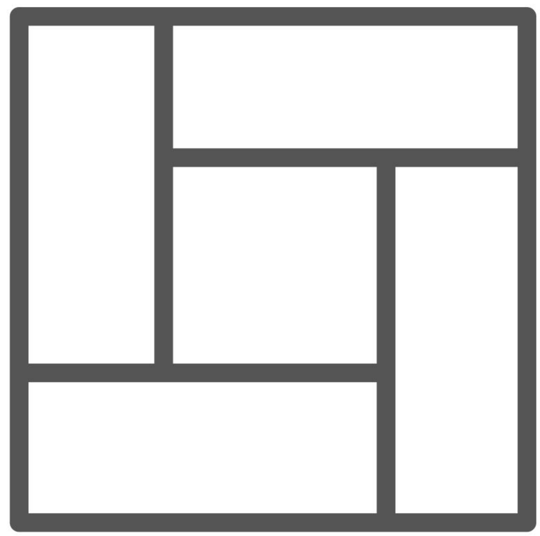

Figure 21. Model I Sketch

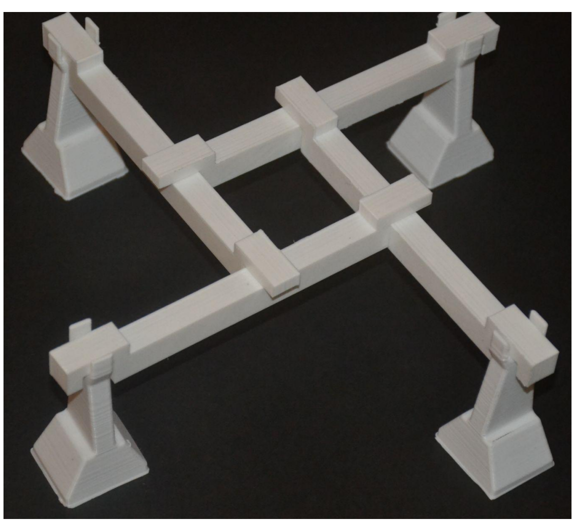

Figure 22. Model I Additional Image I

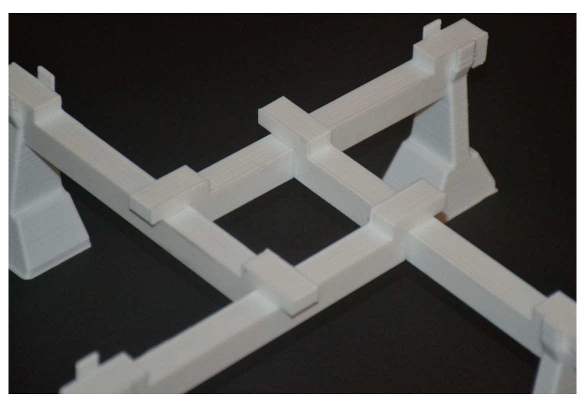

Figure 23. Model I Additional Image II

ii. Model II

Model II consists of a total twenty-two members that act on a rectangular plane. Similarly to Model I, Model II uses the simply four beam reciprocal connection where four members mutually support each other at a connection. Unlike Model I, altering where members end along an adjacent member can influence the pattern created by the assembly. In Model I, all members ended approximately around 40% of the total length of the supporting member from its end. However, in Model II multiple members where being supported by one member, in this case members not being supported by the perimeter walls were supporting two different members along their length. This in effect alters the geometry of the pattern displayed on the floor system. Model II shows several rectangles and squares in its assembly. To make modeling easier with multiple members, Model II reduced the overall size of the members that were used in Model I. Inspired by John Wallis flat beam grillage Figure 14 (Beguin 2018)

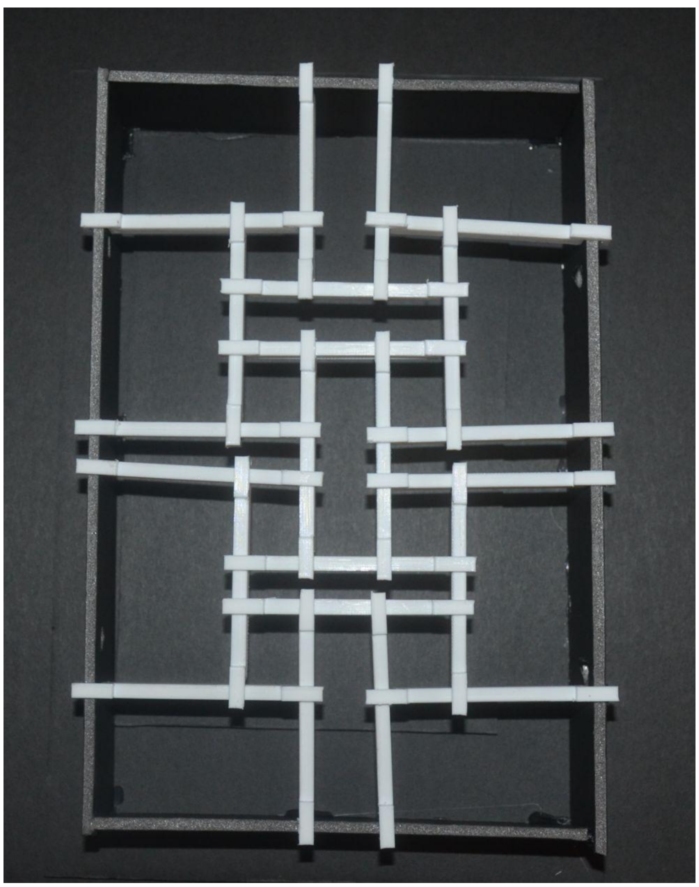

Figure 24. Model II Plan View

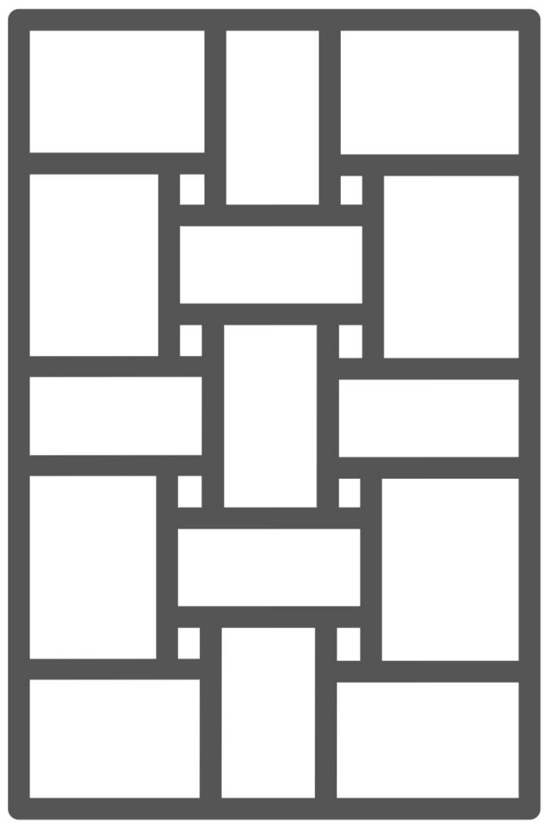

Figure 25. Model II Sketch

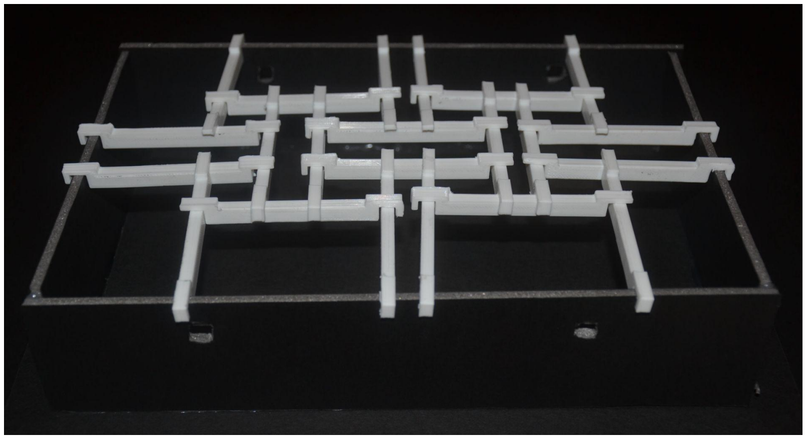

Figure 26. Model II Additional Image I

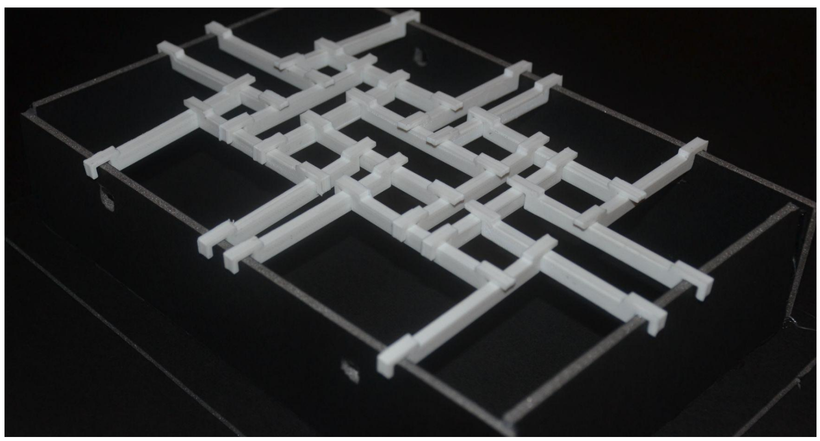

Figure 27. Model II Additional Image II

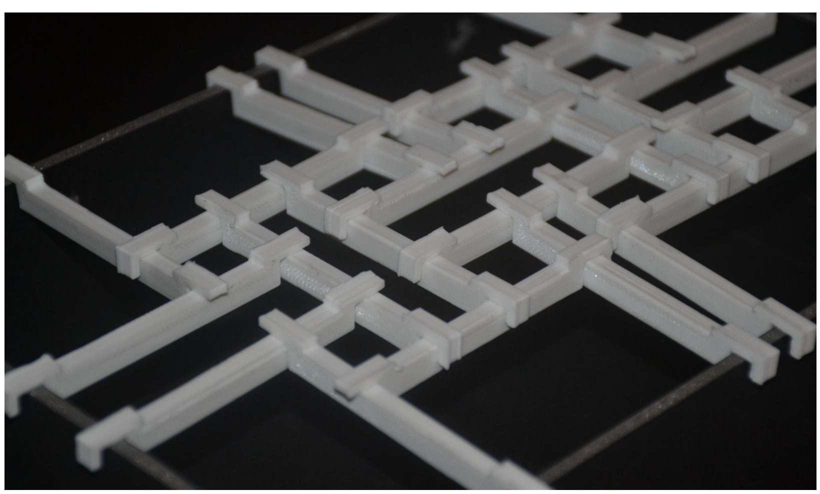

Figure 28. Model II Additional Image III

iii. Model III

Model III consists of a total of thirty-six members acting on a square plan. The members' form pattern consists of squares that alter in size. The members are also not orthogonal to the perimeter walls; the members are angled at a sixty degree angle allowing for a square grid to be formed. Members at the perimeter required slight modification which would allow the member to be pinned to the perimeter wall, but with the sixty degree rotational allowance. Model III out of the four models is considered the longest spanning model which provided thorough insight on the possibility for a flat beam grillage to span large distances.

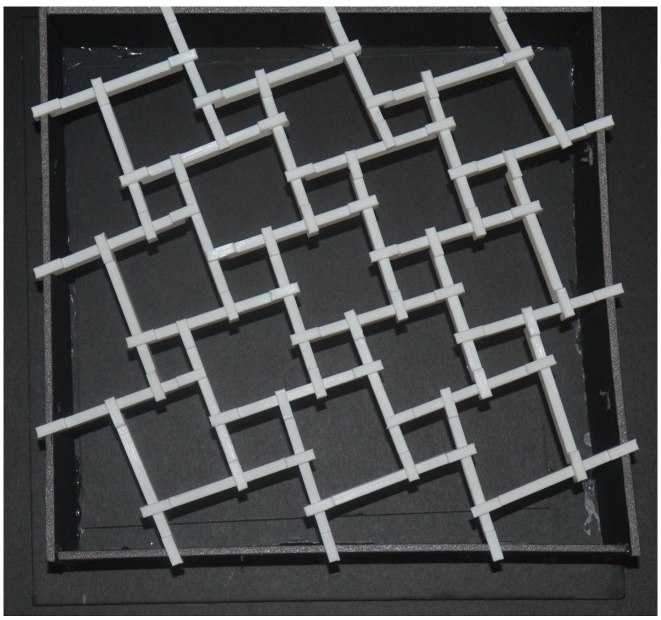

Figure 29. Model III Plan View

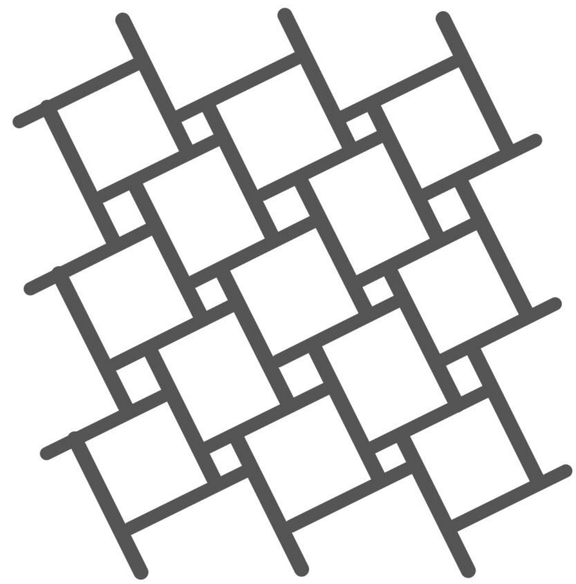

Figure 30. Model III Sketch

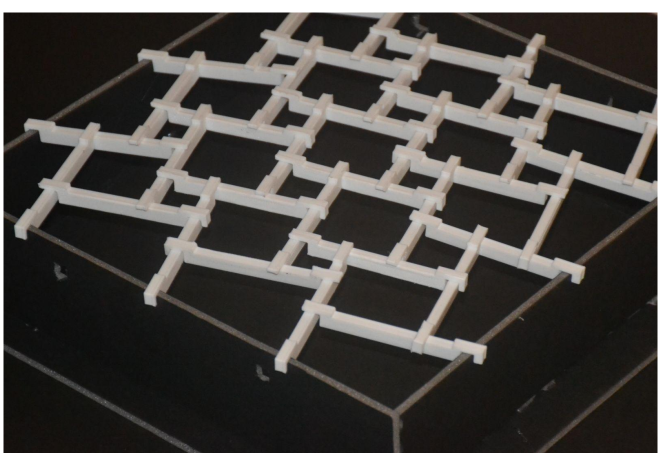

Figure 31. Model III Additional Image I

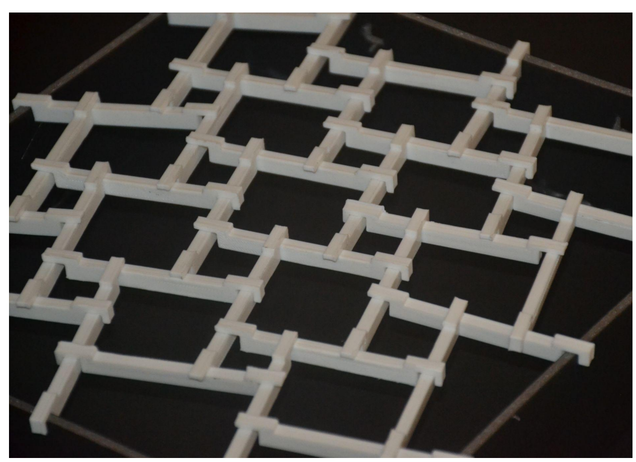

Figure 32. Model III Additional Image II

iv. Model IV

Model IV consists of a total of twelve members acting on a hexagonal plan. The members form triangular connections which overall forms a hexagonal shape at the center of the flat beam grillage. The members used in Model III, which had been adjusted to allow for members to be attached to other members or supports at an non-orthogonal angle, were used in this layout. This planar assembly was not common in Medieval Europe, but more closely resembles the reciprocal frames within the Hogan Dwelling, Native American Teepee, and East Asian Spiraling Timbers.

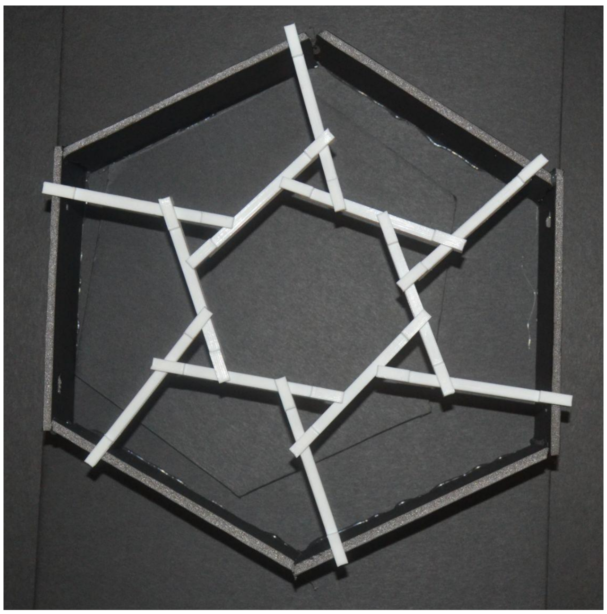

Figure 33. Model IV Plan View

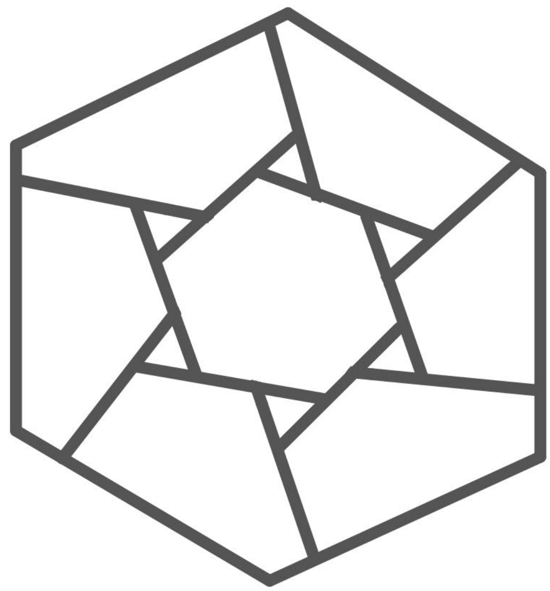

Figure 34. Model IV Sketch

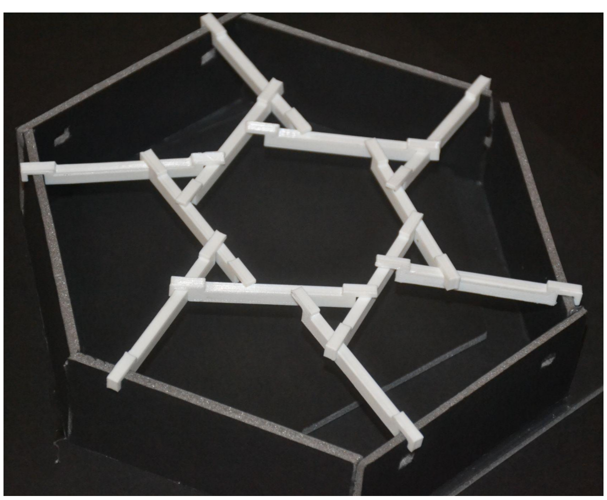

Figure 35. Model III Additional Image I

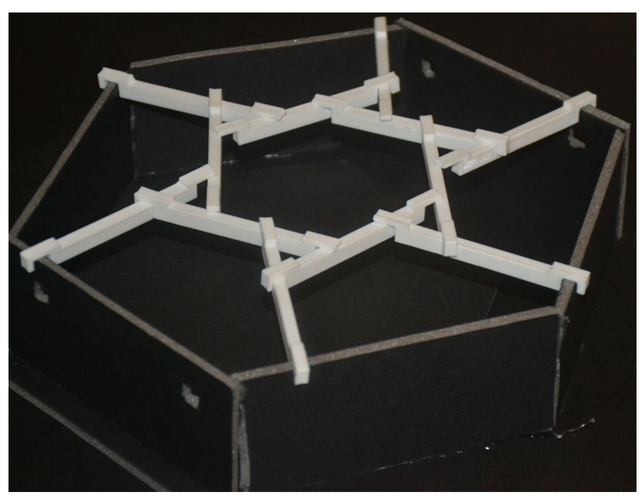

Figure 36. Model IV Additional Image II

V. Computational Analysis

i. Analysis Goals

In attempting to further understand the structural behavior of reciprocal frames, a computational analysis was undertaken. The goal of this analysis was to determine the system feasibility and efficiency in order to provide general design guidelines. As mentioned previously, these systems can be arranged in a variety of framing geometry and forms, and, to retain scope, the analysis of these systems is limited to square planar grillages.

ii. Assumptions/Criteria

In analyzing planar reciprocal framing systems, it was essential to understand the basic model hypotheses and analyze system behavior through a variety of methods. To begin, a basic model consisting of four equal members was analyzed by hand, using python, and through both RISA and SAP2000 software. This framing system is shown in the figure below.

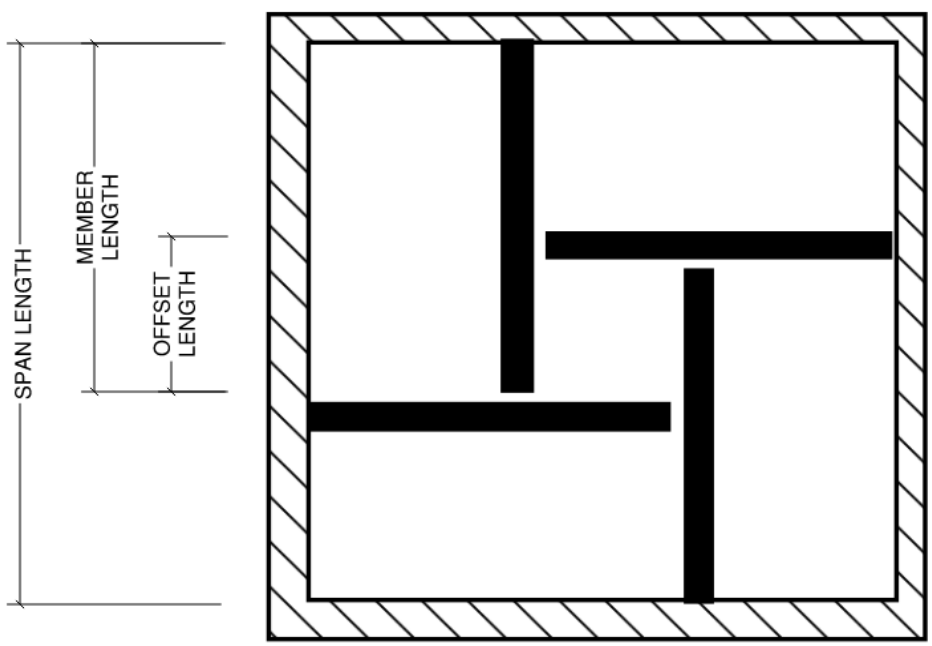

Figure 37. Initial Framing Pattern

As seen in this figure, the concept of being able to span long distances with short members is illustrated. The system members are all analyzed members with one pinned and one bearing connection (pin/roller). This modeling technique causes the system to gain independence from axial forces. Upon closer inspection it can be seen that the system is loosely symmetric with a given span length and repeating members lengths and offsets. As hypothesized, the adjustments of these dimensions will change the force distribution within the system. Extrapolating from this concept, a larger planar grillage model can be created. Examples of these larger systems can be seen in the figures below.

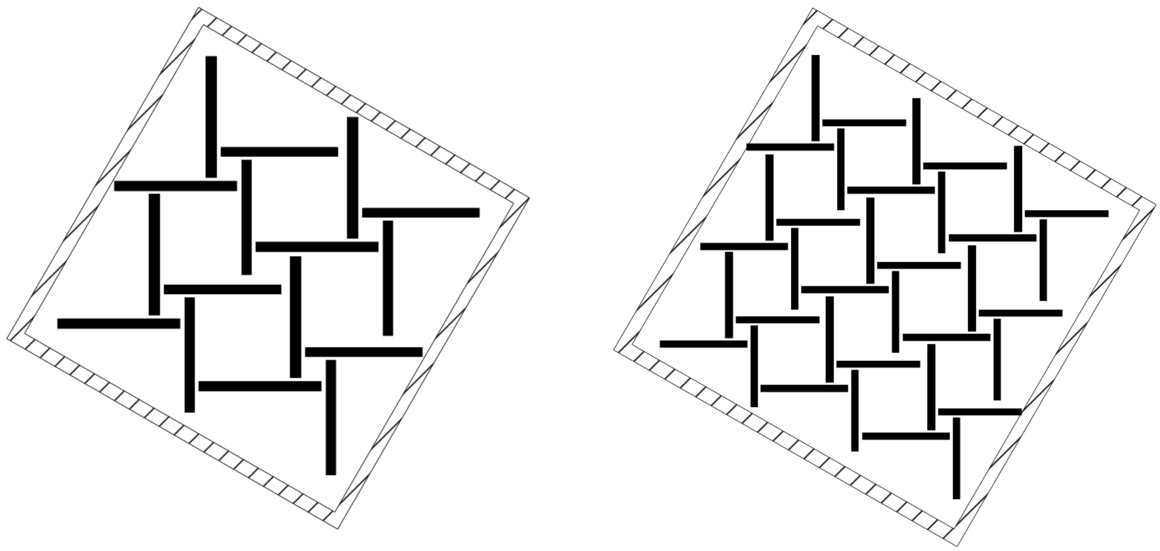

Figure 38. Repetitive Patterns of Initial Framing

As can be seen from the above figures, the grillage framing is a patterned repetition of the original simplified frame. When performing analysis the span length, member length, and offset length in each respective system similarly affect the overall load flow. In the following analysis, these values are isolated to determine efficiency/feasibility generalizations.

iii. Load Flow

Although the majority of the analysis was completed using structural modeling software (RISA and SAP2000), the results were first vetted through a hand analysis of flexure, shear, and deflection. Performing this hand analysis allowed for a greater understanding of load flow in complicated systems. The hand analysis of this load flow assumed a repeating system with members of identical lengths receiving equal amounts of load. The initial hand analysis was based on a simple four-member reciprocal frame as shown in Figure 37. From this assumption, the model shown in the figure below was analyzed as follows.

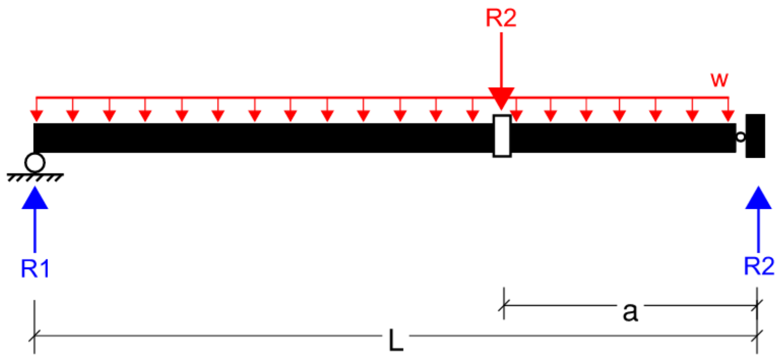

Figure 39. Hand Analysis Model

In the above model, the load was assumed to be uniformly distributed between four members of a reciprocal system. The offset distance (shown in Figure 37) is represented by the variable 'a', the uniform load by the variable 'w', and the member length by the variable 'L'. Because the member support at R2 is identical to the point load along the member, a set of equations was able to be solved to provide determinacy to the system. Through this analysis, the following equations were developed for the exterior support reaction, R1, and the internal reaction, R2.

$$R1 = wL \text{ (System Support Reaction)} \qquad \text{Equation 1}$$

$$R2 = \frac{w*L^2}{2(L-a)} \text{ (Internal Reaction)} \qquad \text{Equation 2}$$

With these end reactions the following equations for shear and moment were developed using beam tables and superposition. Please note that the variable 'x' is the position along the member where data is taken.

$$V1 = \frac{3wL}{2} - wx \text{ (System Edge Shear Equation)} \qquad \text{Equation 3}$$

$$V2 = \frac{3wL}{2} - wx - \frac{w*L^2}{2(L-a)} \text{ (Internal Shear Equation)} \qquad \text{Equation 4}$$

$$M = \frac{wx*(L-x)}{2} + \frac{R2*(L-a)*x}{L} \text{ (Member Moment Equation)} \qquad \text{Equation 5}$$

Although these equations were valuable in solving for the reactions and member forces in the indeterminate system, they require further expansion for larger frames. An interactive deflection process then used to determine governing deflection of the framing system. These values had a deviation of around up to 3% from the values acquired using digital model software.

After successfully validating the use of digital software to analyze the systems, several digital models were created to understand load flow through large planar systems as shown below. The values for the flexural and shear system had a parabolic distribution as shown below.

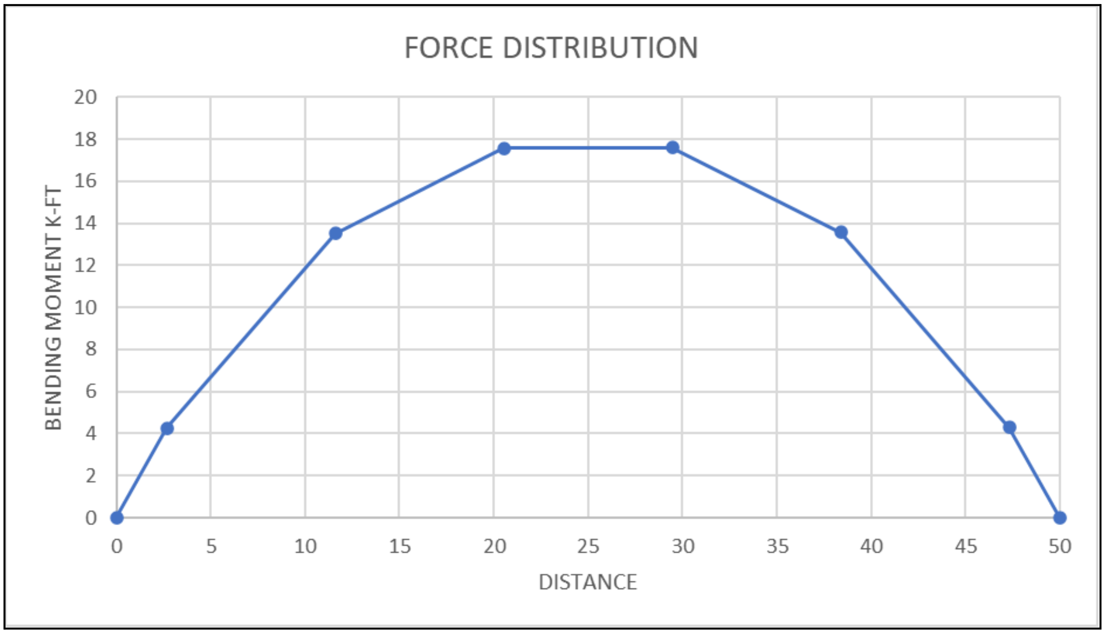

Figure 40. Bending Moment Force Distribution

The bending moments data shown in the figure above was collected along a line across the planar system. The corresponding values for flexure collected along this line progress parabolically for the system. This system is similar to that of a two-way slab system where bending moment similarly distributes parabolically. Although, internally, reciprocal framing load redistributes through a circular load flow, the force distribution can be idealized as a two-way slab. Additionally, all of the load in these framing systems flow through the parabolic shear and moment distribution. As a result, there is no axial force in the system. The digital models demonstrated this through analysis completed with axial releases. Similarly, as noted above, the physical models verified this assumption having "pin-roller" end conditions.

Because reciprocal framing systems can take an infinite number of shapes, it was important to understand how changing parameters affected the load in the system and ultimately how this impacts the overall efficiency. The first analysis attempted to understand how changing the number of members affected the system. In performing this analysis a total of 5 frames were modeled and all had different iterations of the initial framing pattern. In order to keep the analysis constant, arbitrary values were assigned to the framing systems. All of the frames were modeled to span 50 feet in both directions and had a uniform load applied of 100 psf. Additionally, the frames were modeled to emulate a timber-like material with a base dimension of 10 inches. The corresponding material was assigned a bending stress of 1,000 psi and a modulus of elasticity of 1,700,000 psi. The analyzed frames can be seen below.

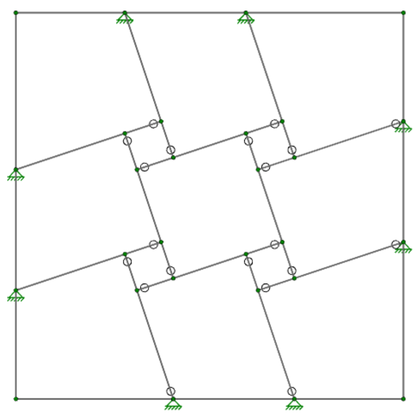

Figure 41. Analysis A Model I

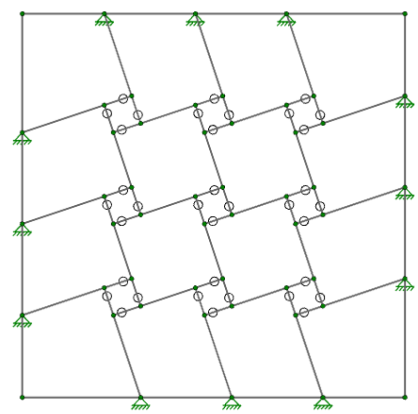

Figure 42. Analysis A Model II

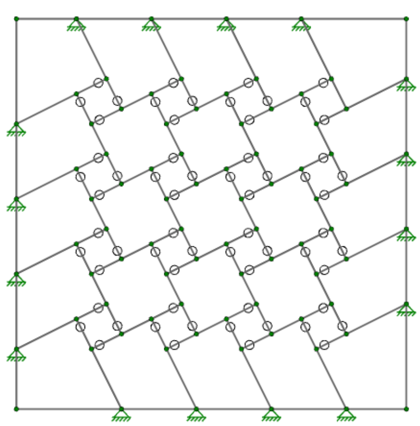

Figure 43. Analysis A Model III

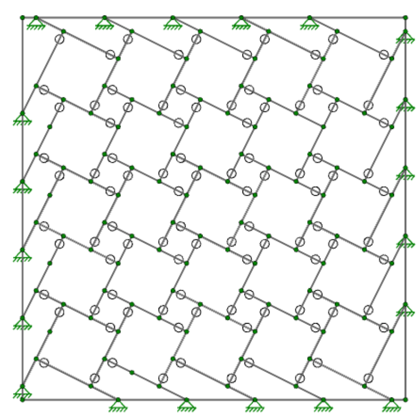

Figure 44. Analysis A Model IV

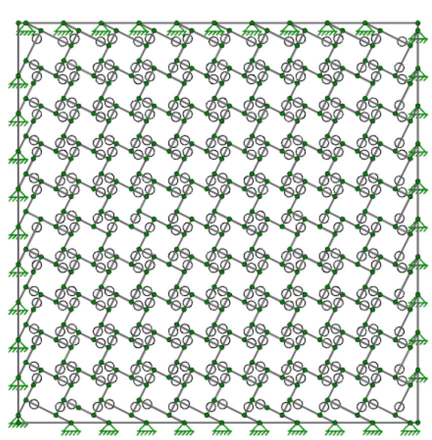

Figure 45. Analysis A Model V

The framing systems above contain 12, 24, 40, 60, and 220 members respectively. The first and the last frames, containing 12 and 220 members, were used as outliers to better understand how the framing behaved at the extents of this analysis. As noted previously, these framing systems all were iterations of the initial framing scheme shown in Figure 37. All members were modeled to have pinned end conditions with an axial release at one side of the member. These releases, as shown in the model, are representative of the ideal system where load does not transfer through axial load.

iv. System Efficiency

As a preface to analysis on these frames, it should be noted that a portion of load on each of the above systems was distributed to the perimeter members. This is due to the two-way load distribution assignments used during RISA analysis. In order to account for variance in load distribution, an efficiency modifier was used to equate systems. This system modifier is dependent on the number of members in a system and the offset length of members. The figure below shows the results of this efficiency modifier.

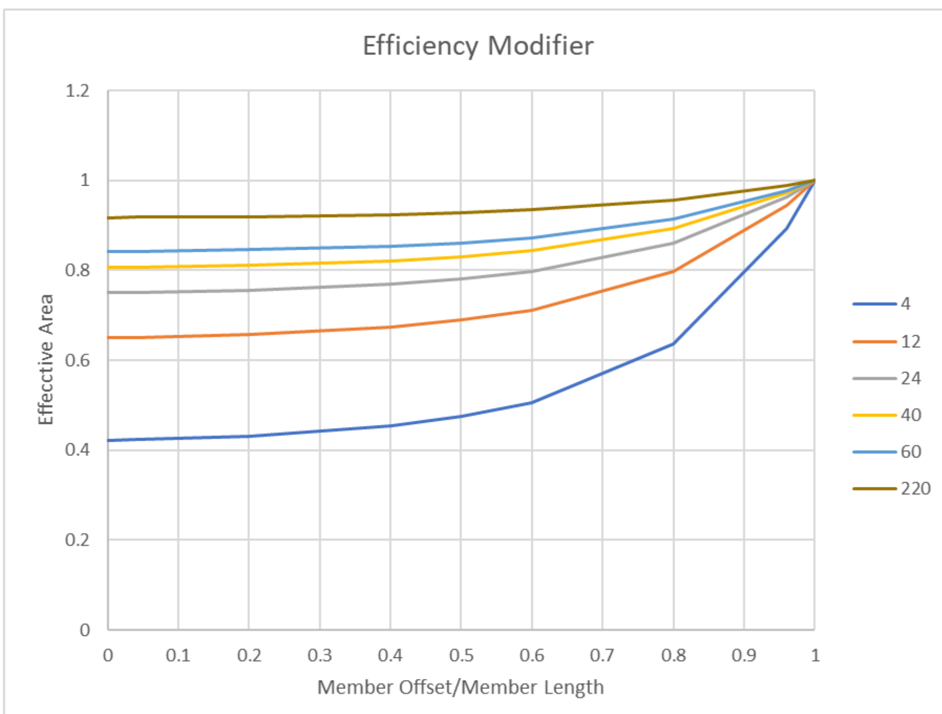

Figure 46. Efficiency Modifier by Number of Members

The effective area was calculated as a function of the ratio of the member offset to the member length and the total system length. This data was collected and extrapolated for use in systems with a different number of members. The number of members, shown to the right of the figure can be used in conjunction with the offset-length ratio to find the effective area value. This value is used to modify efficiency in the analysis to follow. As expected, planar grillages with a finer mesh pattern (more members) sustained more of the overall system load because they contained less spaces around the edges. At a limit state, an infinitely fine mesh (infinite number of members) would have an effective area/efficiency modifier of 1. Additionally, as the ratio of member offset to member length increased all systems approach an effective area of 1. For these analyses, the efficiency modifiers were used to normalize the data to compare systems that sustain relatively identical amounts of load.

In order to complete the analysis, the system was modeled twice, once to obtain maximum efficiency for flexure and again for deflection. Although shear impacts the design, the placement of members in these systems eliminate the governance of shear and prioritized flexural demand. To design a system with maximum capacity, elastic design was used to determine the member depth required to equate the flexural demand to the arbitrary capacity. Similarly, the system deflection limit was measured in terms of the total system length over 360. The member depth was in-turn determined to satisfy the total system deflection limits.

Table 1: Flexure and Deflection Design for Systems with Varying Members

System	Members	Flexure Depth (in)	Deflection Depth (in)
1	12	51.495	44.25
2	24	41.09	37.25
3	40	36.209	33.5
4	60	34.65	32.75
5	220	26.46	27.5

From observation, as the number of members in each of these systems increased the maximum demand in each member decreased. As shown above, the ultimate demand for both flexure and deflection was shown in terms of member depth required. The calculated values are far from realistic, but they serve as a relative comparison between the given systems. From the basis of historical precedent and construction feasibility, the systems were idealized as timber (noted previously) and are assumed to have members of uniform depth. With this in mind, the above required depths show the system feasibility for the given span and loading constraints. To understand a general design guideline for developing this system, the total volume was used to determine the system efficiency. The calculated relationship for efficiency is the inverse of total material volume (as the amount of material increases, the system efficiency decreases). The relationship between this efficiency and the number of members can be seen below for these systems.

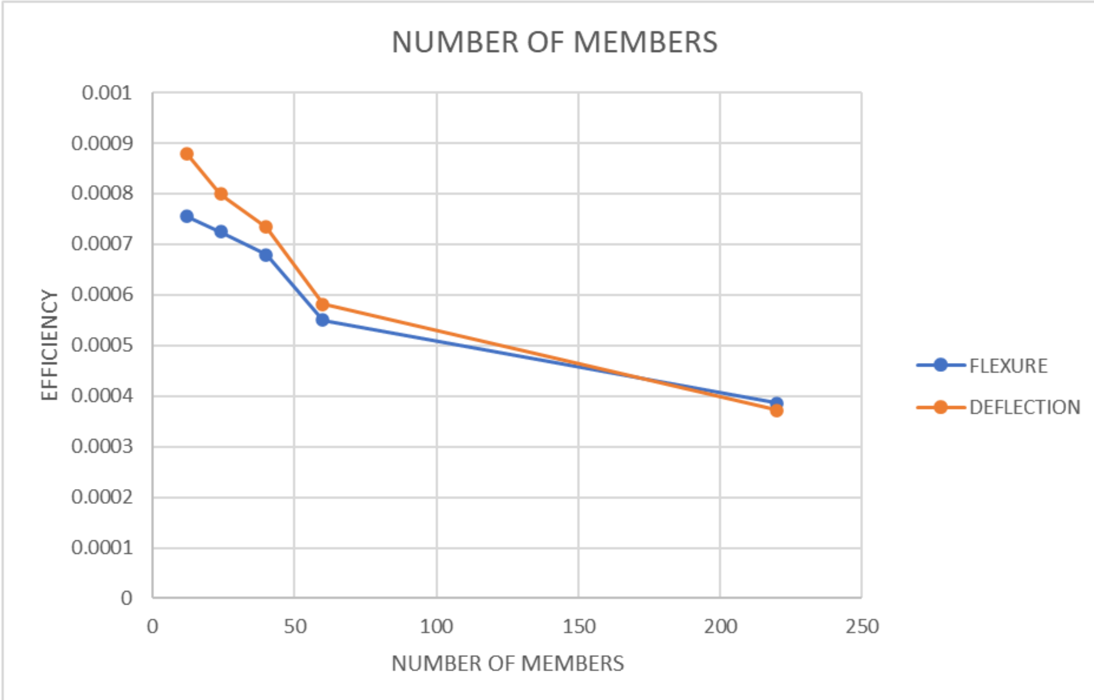

Figure 47. Efficiency to Number of Members

As noted, the above system shows the total efficiency against the number of members for the demand limits of flexure and deflection. In analyzing each system independently, it is apparent that as the number of members increases, the total system efficiency decreases. Both flexure and deflection follow this trend. System 1, containing only 12 members had the highest relative efficiency, and of the frames, System 5 had the lowest relative efficiency. In further understanding the relationship between the two demand limits, it can be seen that flexural efficiency governs in a system. This means that a system requires more total material to meet the demands of flexure than that of deflection. This relationship holds true until the system has a number of members closer to the assigned limit condition. This efficiency was modified for each system using the modifiers as calculated and shown in Figure 47 above.

The relative system efficiency used for comparison between the modeled systems is dependent on the given system size and loading parameters. While system loading has a scaled efficiency relationship (identical relative efficiency), as system size and member configuration change, the individual member lengths can change based on independent parameters such as offset distance and angle of rotation. To encompass the difference in member placement, the data for a system with an arbitrary number of members over an arbitrary span was normalized. The resulting variable used was the length of a member over the overall system length. This relationship can be further shown in the figure below.

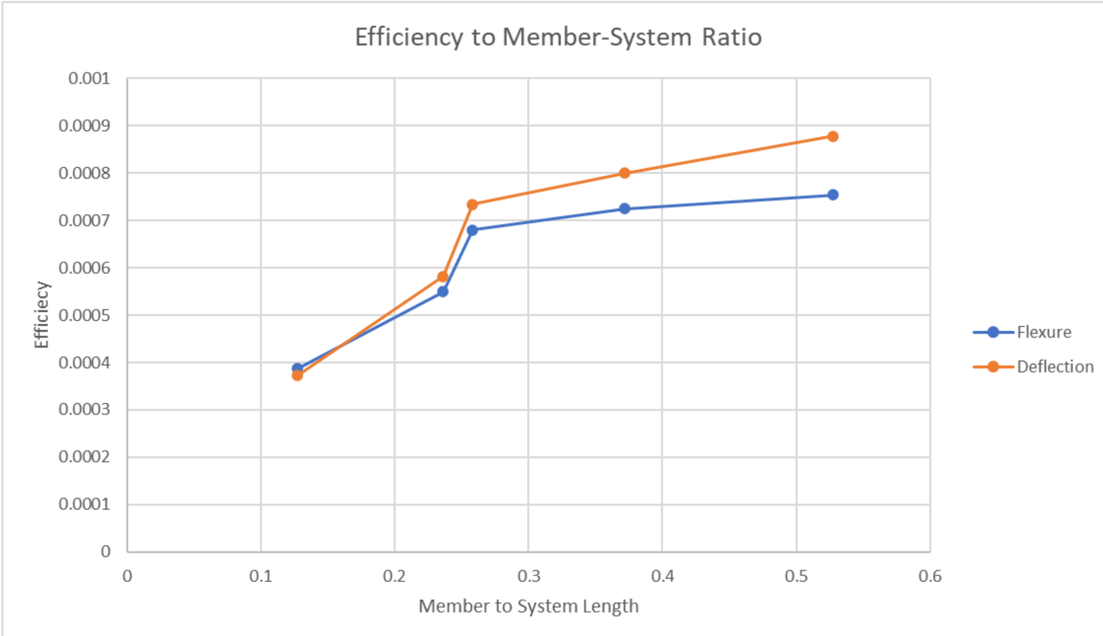

Figure 48. Efficiency to Member Length to Span Ratio

The figure shown above indirectly relates the number of members to the system efficiency. However, instead of a direct correlation between the number of members and the efficiency, a member-to-system length was used. This value relates the overall system length to the length of the member used. As a result, as the member-to-system length increases, the number of members in a system decreases. This shows an identical relationship to the previous figure, but can be applied to any system. As in the previous analysis, these values for efficiency were adjusted to normalize the amount of load sustained by each of the five systems. Similar to before, as a given member spans more of a system (fewer members) the system efficiency increases, and as a member spans less of a system (more members) the total efficiency decreases. Additionally, flexural design governs the system unless the member-to-system ratio is less than 0.16. The precise results from this data are dependent upon the system configurations but generally represent the planar grillages with a change in member configuration. The adjustment of this configuration shifts the relative system efficiency in accordance with the data and figures in the analysis to follow.

Similarly to the fact that the number of members in a system is arbitrary, the position, and angle of a set of members is also arbitrary. As can be seen in figures below, the offset length and member length can be adjusted while keeping the number of members in a system constant. In altering these values, the angle of rotation of the members changes as well. To understand the consequences of adjusting this variable, the following 7 systems were modeled and analyzed.

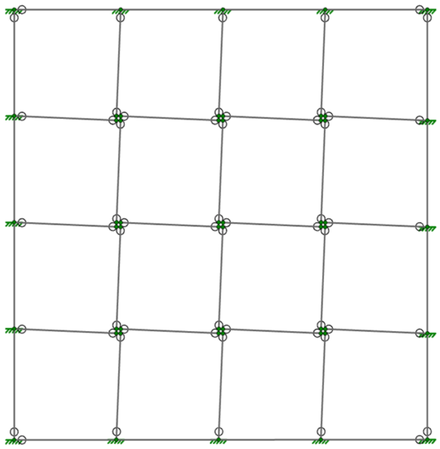

Figure 49. Analysis B Model I

Number of Members = 24
Member Length = 12.88 ft
Offset Length = 0.52 ft
Relative Offset = 0.04
Angle of Members = 2.39°
Efficiency Modifier = 0.751

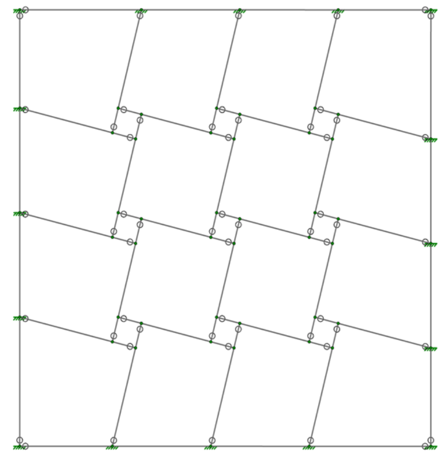

Figure 50. Analysis B Model II

Number of Members = 24
Member Length = 14.52 ft
Offset Length = 2.90 ft
Relative Offset = 0.20
Angle of Members = 14.05°
Efficiency Modifier = 0.756

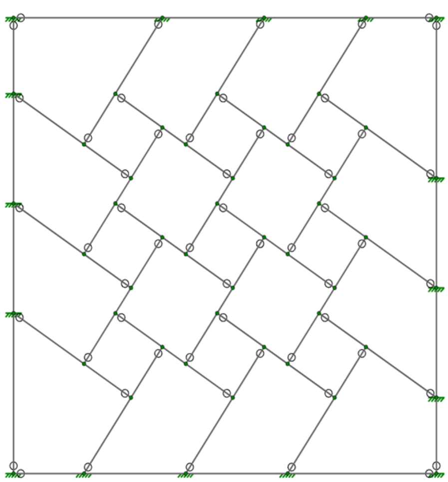

Figure 51. Analysis B Model III

Number of Members = 24
Member Length = 16.69 ft
Offset Length = 6.68 ft
Relative Offset = 0.40
Angle of Members = 33.69°
Efficiency Modifier = 0.767

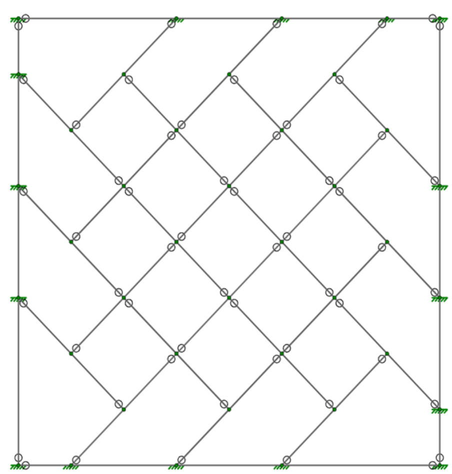

Figure 52. Analysis B Model IV

Number of Members = 24
Member Length = 17.75 ft
Offset Length = 8.87 ft
Relative Offset = 0.50
Angle of Members = 45.00°
Efficiency Modifier = 0.780

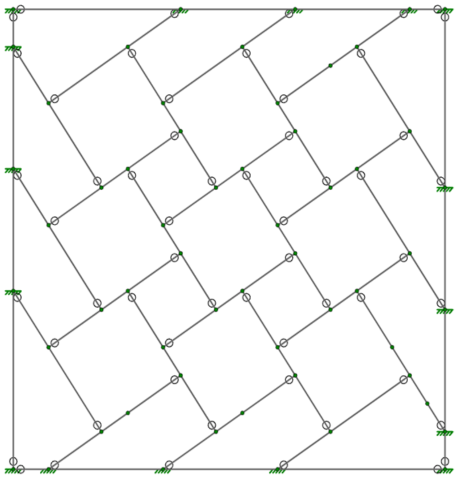

Figure 53. Analysis B Model V

Number of Members = 24
Member Length = 18.40 ft
Offset Length = 11.04 ft
Relative Offset = 0.60
Angle of Members = 56.31°
Efficiency Modifier = 0.797

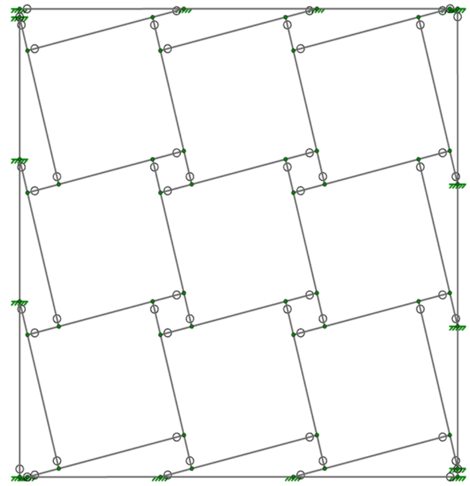

Figure 54. Analysis B Model VI

Number of Members = 24
Member Length = 18.41 ft
Offset Length = 14.73 ft
Relative Offset = 0.80
Angle of Members = 75.95°
Efficiency Modifier = 0.860

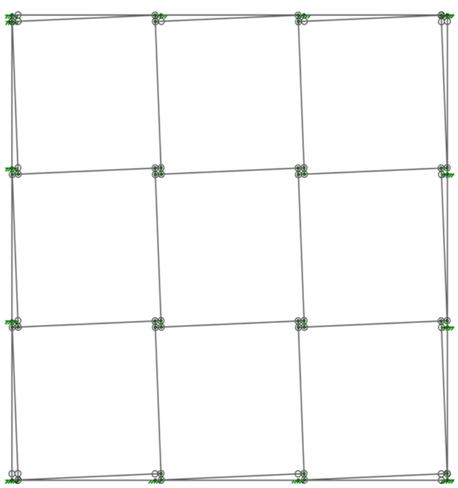

Figure 55. Analysis B Model VII

Number of Members = 24
Member Length = 17.10 ft
Offset Length = 16.42 ft
Relative Offset = 0.96
Angle of Members = 87.61°
Efficiency Modifier = 0.963

As can be seen from the above systems, the total number of members remains constant at 24. The first and last systems, Figure 48 and Figure 54, are modeled as the extents of rotation in either direction. As an initial analysis of the modeled frames, a relationship between the member length, offset distance, and angle of rotation can be made. Starting at frame 1, as a system is rotated, the length of the member increases and the corresponding offset distance increases as well. The relationship between these values can be expressed by the following equation.

$$\theta = \tan^{-1}(a/(L-a)) \quad \text{Equation 6}$$

This relationship allows for the understanding of the rotational impact of adjusting the offset length (variable 'a') on a planar system. Evidence of this can be seen in Figures 48-54 and can serve as a design guideline for system aesthetics and ultimately, system modeling.

Using these modeled framing systems and their corresponding efficiency modifiers, an analysis of varying efficiency was conducted. Similar to the previous analysis, the frames each span 50 feet and support a uniformly distributed load of 100 psf. The members were modeled with a timber-like material with a base dimension of 10 inches. The arbitrary material was assigned an allowable bending stress of 1,000 psi, an allowable shear stress of 180 psi, and a modulus of elasticity of 1,700,000 psi. Using the given capacities, the section was modeled using elastic design to determine system efficiency and the governing demand mechanism. The inverse of the volume needed to support the system demand was used as a measure of efficiency. These values were then adjusted by the system's given efficiency factor.

Table 2: Relative Offset Design for Shear Flexure and Deflection

System	Relative Offset	Shear Depth (in)	Flexure Depth (in)	Deflection Depth (in)
1	0.04	316.75	38.46	48.00
2	0.20	58.42	38.03	37.75
3	0.40	30.50	40.88	36.50
4	0.50	27.08	43.75	36.75
5	0.60	27.50	40.91	37.00
6	0.80	46.50	38.57	37.50
7	0.96	245.08	38.51	42.00

From observation and verified by system results, the adjustment of the load on a member results in the adjustment of required member depth. For a system with members framing at an offset distance near supports (offsets close to 0 or 1), shear governed the system design. As the additional point loads moved toward the center of the members, the flexural demand governed because of the increasing moment arm. Finally, deflection limits did not govern the design with the given system parameters. Because the section width was assumed, the governing member heights created smaller overall deflections. However, a change in height will have exponentially greater adverse impact from shear to deflection. In this analysis, the base dimension was constrained and the given results provided. Because deflection of reciprocal frames contains multi-order effects, the demands of deflection are not intuitive. As a member offset is placed closer to an end condition, there are greater multi-order effects. Inversely, as a member is placed closer to the center there are greater initial deflection demands. The efficiency results based on the data in Table 2 are shown below.

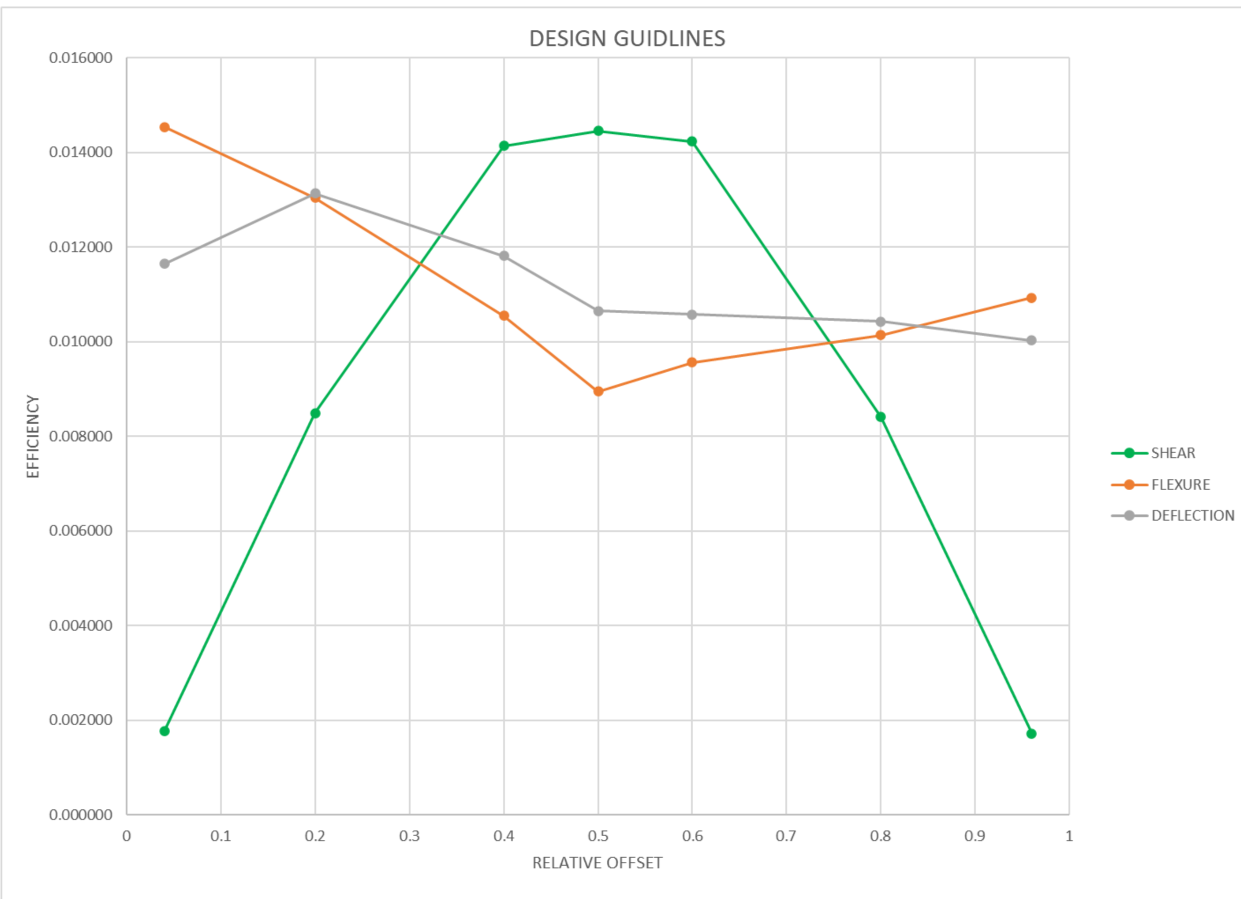

Figure 56. Efficiency to Relative Offset

The figure above compares the efficiencies of the modeled systems based on the demands of shear, flexure, and deflection. For shear analysis, the design becomes more efficient as the secondary members frame into the main member closer to midspan (offset = 0.5). On the other hand, at this relative offset distance, the flexural efficiency is the least and governs the system design. Inversely, with secondary members framing into the main member at supports (offsets = 0 or 1), there is high concentrations of shear demand and negligible flexural demand. When the offsets reach the end conditions the system fails to transfer force through shear and instead acts as an unstable system. This instability is caused by the alignment of 3 or more “pinned” end conditions. At this state, the system does not transfer forces through bending or shear and instead would act similar to a net. In this case, the forces would be entirely axial and deflection demand would solely control design. Regardless, the effects of axial are released in the modeling environment, and the system results in instability at these end conditions.

As a final analysis of the above figure, the envelope diagram consists of the area under the above curves. For the given section, the shear design governs for relative offsets less than 0.32 and greater than 0.74. Conversely, flexural design governs between 0.32 and 0.74. As noted previously, a change in the assigned section properties will shift the demand curves based on the exponential degree of impact. This adjustment of section properties was not analyzed, and, as a result, the demands of shear and flexure govern at the above framing limits.

VI. Discussion & Conclusions

i. Design Guidelines

This analysis provides a general framework for reciprocal framing usage and design. Based on the above data and modeling assumptions, several valuable conclusions can be made about both load flow and system efficiency. Although these conclusions require more verification, they are helpful in creating a system hypothesis and a set of design guidelines.

As a preface, the analysis performed did not cover all systems of reciprocal framing and focused on planar grillages, similar to that of Leonardo Da Vinci. Additionally, these planar grillage systems were modeled to be a timber like material. This material choice was modeled based on traditional and contemporary material usage in reciprocal frames. This assumption affects governing system demands and may adjust system efficiencies. Model members were only considered as having "pin-roller" end conditions. The use of materials with fixity, such as concrete would create torsion in system members that was not considered in the above analysis. Similarly, through elastic design, a base dimension was chosen for members that adjusts member design properties. This, in turn, affects the impact of shear, flexure and deflection. Additional analysis should be done to determine the extent of how this affects system efficiency and performance.

Of these framing analysis, load flow analysis was performed to understand overall system behavior. This system behavior focused primarily on the forces of vertical shear and bending moments. These forces behaved differently depending on the framing configuration and framing offset. Although planar grillage frames of other configurations (triangles, rectangles, hexagons, etc.) may behave similarly to the above constraints, further analysis should be done to confirm system behavior. Additionally, the forces varied based on the number of members included in a system. The analysis performed analyzed all systems from an elastic design perspective for shear, flexure, and deflection to determine the optimal system. From these results, the following conclusions can be made based on both load flow and system efficiency.

1. Reciprocal frames contain circular load flow that redistributes load.
2. Simple systems can be analyzed by hand for shear, flexure, and deflection.
3. Complicated systems should be analyzed using either coding, or structural analysis software (RISA, SAP2000, etc.)
4. Vertical shear and bending moment flow parabolically through a system.
5. Frames with more members have smaller maximum shear and moment values, but greater deflections.
6. Planar grillages are more efficient with fewer members.

7. Members with small reciprocal offsets have the highest shear values while frames with central reciprocal offsets have the highest moment values.
8. Material consideration is important in governing system demands.

These conclusions were based on system analysis and the noted assumptions. From these conclusions, a strong case can be made for the use of reciprocal frames. Although engineering judgment and modeling analysis should be used in reciprocal framing design, the design parameters below can be used for reciprocal framing design. First, frames should be designed with the fewest members possible for material efficiency. Second, framing offsets should be at approximately ⅓ of the member length to maximize shear and flexure efficiency. These design guidelines are based on the analysis and conclusions as noted above.

The design aids used in this analysis were not available to early inventors of the reciprocal frame, but can be used to aid in the evolution or use of planar framing. Historical context has provided depth to the usage and meaning of these systems in the past, and the analysis has given direction to the design of reciprocal systems. With past and present knowledge, the reciprocal framing system has a potential to influence and impact global engineering standards and efforts.

VII. Influence & Impact

i. Global Implications

Reciprocal framing has played an important role in historical structural systems due to the accessible and transportable use of short timber members. This method of framing has the potential to be incorporated in the architectural, engineering, and construction industries in current times. Although few present standards exist, the usage of reciprocal framing has the potential to add to the aesthetics and environment of a structure, the design of circular load flow systems, and the improvement of temporary shoring systems. The further advance in the use and understanding of reciprocal framing systems will add to the knowledge and art of structures for both engineers and designers, but also for the people living and/or working within them.

ii. Cultural Implications

History has a profound impact on culture through the forms of traditions and heritage. Our past influences and distinguishes our culture. Structures can alter our environment which can influence our livelihoods. The reciprocal frame structures that were created in our history formed a space that impacted the lives of our ancestors which would affect their decisions and change our culture today. It is important to study our history because, whether we like it or not, it is a part of who we are. The hogan dwelling in Native America, the temples and shrines in East Asia, and the flat beam grillages throughout Medieval Europe were all instances of reciprocal frames impacting the lives of our predecessors. The design and analysis of these structures will determine their construction in our present; these structures will create new spaces that we can experience. For some of us these experiences will have long lasting impacts on our own beliefs which could ultimately create cultural changes.

iii. Social Implications

Reciprocal frame structures consist of short timber members that are easily accessible and transportable. This makes these structures extremely viable for temporary forms of shelter. Natural disasters can occur throughout the world when least expected; whether it it is an earthquake, hurricane, or a flood, it is uncertain how many people will need some form of structural refuge. The study of reciprocal framing shows that, in its most simplistic form, some form of lodging can be constructed and dismantled with relative ease via the use of an abundant resource. Other social implications, include the potential to shape the environment and space in which an inhabitant resides in. Due to the elegance reciprocal frame structures provide, social experiences can benefit from occurring alongside these structures.

iv. Environmental Implications

The use of reciprocal framing can have several potential environmental benefits. Although reciprocal framing uses more material than traditional framing, it makes use of smaller members. This framing can have the potential to minimize material waste by using even the smallest of members to span floors. Although this may not apply to the usage of steel and concrete, it is beneficial in the timber industry where field cuts are more routine. Based on the material usage, reciprocal framing systems are primarily timber; these systems have the potential to impact the mass timber industry which has the greatest environmental impact with buildings using renewable resources and having a net-zero impact. Reciprocal framing can add to the growing methods of mass-timber floor plates. As mentioned previously, this system would utilize material in small sections to minimize construction waste. These combined benefits of reciprocal framing can have a positive environmental effect.

v. Economic Implications

As interest in reciprocal frame structures rises, the economic impact that reciprocal frames create will increase. Structures undergo a construction process which influences economic transaction. A client who wishes to pursue a reciprocal frame structure will seek to use their funds to provide for material acquisition and construction costs. These expenses will not only promote jobs, but more importantly, will support work that is also benefiting environmental and social implications.

VIII. References

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VIV. Appendix A. (List of Figures)

VV. Appendix B (Analysis Data)

Table 3: Bending Moment Distribution from Linear Data Collection

Linear Distance	Bending Moment		Linear Distance	Bending Moment
0	0		0	0
2.679	4.266		5.893	2.001
11.608	13.517		14.311	6.906
20.537	17.557		18.52	8.232
29.466	17.589		26.938	12.486
38.395	13.574		31.147	12.982
47.324	4.289		39.565	12.982
50	0		43.774	12.486
			52.192	8.232
			56.401	6.906
			64.819	2.001
			70.712	0

Table 4: Efficiency Modifier by Number of Members

Relative Offset	**Number of Members**					
	4	**12**	**24**	**40**	**60**	**220**
0	0.421875	0.649519	0.75	0.8059274	0.8414664	0.917315
0.04	0.423226	0.650559	0.7508	0.8065721	0.8420048	0.917608
0.2	0.431738	0.657068	0.7558	0.8105973	0.8453647	0.919437
0.4	0.454047	0.67383	0.7686	0.8208717	0.853926	0.924081
0.5	0.474552	0.688877	0.7800	0.8299864	0.8615029	0.928172
0.6	0.506262	0.711521	0.7970	0.8435168	0.8727201	0.934195
0.8	0.636278	0.79767	0.8601	0.8931239	0.9135438	0.955795
0.96	0.891944	0.944428	0.9626	0.9718168	0.9773892	0.98863
1	1	1	1.0000	1	1	1

Table 5: Analysis A: Efficiency to Number of Members

Bending Moment Analysis				
Number of Members	**Member Length**	**Height (in)**	**Volume (ft^3)**	**Efficiency (1/ft^3)**
12	26.390	51.495	1132.460875	0.000754463
24.000	18.585	41.09	1272.76275	0.000725037
40.000	12.900	36.209	1297.489167	0.000679466
60.000	11.797	34.65	1703.191875	0.000549556
220.000	6.349	26.46	2566.58325	0.000386506
Deflection Analysis				
Number of Members	**Member Length**	**Height (in)**	**Volume (ft^3)**	**Efficiency (1/ft^3)**
12	26.390	44.25	973.13125	0.000877991
24.000	18.585	37.25	1153.81875	0.000799779
40.000	12.900	33.5	1200.416667	0.000734412
60.000	11.797	32.75	1609.798958	0.000581439
220.000	6.349	27.5	2667.461806	0.000371889

Table 4: Analysis B, Offset Result Demands

System	Length	Offset	Nominal	Angle	Maximum V	MaximumM
1	12.88	0.515	0.04	2.39	0.001765	68.79
2	14.518	2.903	0.2	14.04	0.008490	76.69
3	16.692	6.676	0.4	33.69	0.014142	94.77
4	18.395	9.197	0.5	45	0.014452	111.77
5	18.395	11.03	0.6	56.31	0.014233	104.53
6	18.407	14.72	0.8	75.95	0.008412	98.60
7	17.1	16.41	0.96	87.61	0.001718	91.47

Table 4: Analysis B, Offset Result Efficiencies

System	Length	Offset	Nominal	Angle	Vol V	Eff V	Vol M	Eff M	Vol Δ	Eff Δ	Eff Total
1	12.88	0.515	0.04	2.39	566.63	0.001765	68.79	0.01454	85.87	0.0116	0.001765
2	14.518	2.903	0.2	14.04	117.79	0.008490	76.69	0.01304	76.12	0.0131	0.008490
3	16.692	6.676	0.4	33.69	70.71	0.014142	94.77	0.01055	84.62	0.0118	0.010552
4	18.395	9.197	0.5	45	69.19	0.014452	111.77	0.00895	93.89	0.0106	0.008947
5	18.395	11.03	0.6	56.31	70.26	0.014233	104.53	0.00957	94.53	0.0105	0.009567
6	18.407	14.72	0.8	75.95	118.88	0.008412	98.60	0.01014	95.87	0.0104	0.008412
7	17.1	16.41	0.96	87.61	582.07	0.001718	91.47	0.01093	99.75	0.0100	0.001718