Mathematical Perspectives on Insurance for Low-Income Populations

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Abstract

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In this thesis, insurance solutions for low-income populations and their capacity for poverty reduction are considered. Classical risk theory techniques are adopted to the study of the trapping probability, where "trapping" refers to the event at which an economic entity falls below the poverty line and into an area of poverty, from which it is difficult to escape without external help. In the poverty setting, the trapping probability mimics an insurer's probability of ruin. Studying two household-level capital processes that align with risk processes with deterministic investment and (i) random-valued and (ii) multiplicative claims, explicit trapping probabilities are derived. The ability of low-income insurance strategies to reduce trapping probabilities is assessed, with a particular focus on government subsidy schemes. For those closest to the poverty line, insurance without subsidies increases their probability of trapping in both the random-valued and multiplicative cases, in line with the existing literature. The governmental cost of social protection is reduced under subsidisation schemes, with a premium payment barrier strategy additionally ensuring the increased risk associated with insurance purchase is mitigated. Purchase of insurance for multiplicative losses is found to be more affordable than for random-valued losses. A stochastic dissemination model is proposed for the extension of the problem to the group setting, in line with the prevalence of risk sharing and group-based insurance schemes across low-income communities. Consideration of risk sharing suggests that the impact of loss and premium payment is shared throughout a homogeneous group, mitigating the severity of negative wealth transaction events. Subsidisation is also found to support both the insured and the uninsured, further highlighting the benefit of governmentally supported schemes. In the second part of the thesis, the existence of lifetime dependence and the influence of socioeconomic features on its structure are considered. Analysis is undertaken on data sets from Ghana and Egypt, with dependence induced through joint stochastic mortality and copula models, respectively. The impact of dependence on the pricing of a reversionary annuity is derived through implementation of the indifference pricing principle. In general, pricing under the dependence assumption decreases the indifference price of the annuity. In visualising both data sets, dependence is observed to be lower in this alternative socioeconomic environment than previously observed in the existing empirical literature, supporting suggestion of socioeconomic influences on bereavement processes. Studying the existence of pairwise dependence within relationships beyond the classical husband-wife case, dependence within child-parent relationships is also found to be significant. Accounting for this existence, even where reduced, is critical to improving the accuracy of insurance product pricing and to mitigate the mortality risks faced by insurers, particularly given the uncertain nature of the low-income financial environment.

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Contents

Chapter 1

Introduction

Social risk management is defined by the Jorgensen and Siegel (2019) as the way in which society:

(i) manages income variability,

(ii) manages the risks of poverty and vulnerability to poverty,

(iii) builds resilience to poverty over time.

This thesis assesses insurance strategies which span the three components of social risk management as a means for reducing the financial vulnerability of low-income populations living close to the poverty line.

1.1 Microinsurance

Inclusive insurance, commonly referred to as microinsurance, relates to the provision of insurance services to low-income populations with limited, or no access to mainstream insurance or alternative effective risk management strategies. First formally introduced by Dror and Jacquier (1999), the term "microinsurance" has been defined in varying ways. Regulators, academics, donors and governing institutions differ in their definition, with microinsurance interpreted both qualitatively and quantitatively. "Micro" typically refers to (i) the low and lower-middle income target population, (ii) the affordability of the low cost and low coverage products or (iii) the level of society associated with the organisation providing the insurance [Dror, 2019; Ingram and McCord, 2011]. Credit life and life insurance, funeral insurance, health insurance and crop and livestock insurance are the most common types of microinsurance product in the market.

In 2018, 656 million of the worlds population were estimated to live below the international extreme poverty line of $1.90 USD per day (World Bank, 2022) and in 2017, 1.7 billion adults remained unbanked (Demirguc-Kunt et al., 2018). Targeting low-income individuals living close to or below the poverty line, microinsurance products aim to close the protection gap that exists between uninsured and insured losses to life, property and health by providing protection to the poor. However, barriers to microinsurance penetration exist due to constraints on product affordability resulting from fundamental features of the microinsurance environment. These distinct features include the nature of low-income risks, limited consumer financial literacy and experience, product accessibility and data availability. While novel solutions for the supply and distribution of products in this environment exist (see, for example, Microinsurance Network (2020)), it is important to consider the viability of microinsurance uptake for all sectors of the target population, particularly for the most vulnerable.

In this thesis, insurance solutions for low-income couples, households and communities are considered. Divided into two core topics, the first focuses on the impact of microinsurance on poverty reduction and the importance of government partnerships and subsidisation. The second considers life-based public and private products with a specific application to the low-income setting.

1.1.1 Government subsidisation

Premium payments can in fact heighten the risk of falling into poverty for the proportion of the population living just above the poverty line, inducing a balance between protection and loss as a result of insurance coverage which is dependent on the entity's level of capital. This insufficiency of microinsurance alone as a means for poverty reduction for the most exposed necessitates an alternative solution. For this purpose, Chapter 2 considers microinsurance schemes that are supported by social protection strategies, and more specifically, their potential in minimising both the probability of a household falling below the poverty line and the governmental cost of social protection. Microinsurance products typically suffer from ad-hoc design and under-regulation, leaving those with little capital or access to infrastructure open to exploitation. Public-private collaborations are therefore more effective than standalone products. For thorough discussions of microinsurance, the challenges associated with adapting commercial insurance to serve the poor and the insurability of risks in the market, the interested reader may refer to Dror (2019), Churchill (2007) and Biener and Eling (2012), respectively.

Governments in low-income and emerging countries have been increasingly involved in the provision of insurance programmes in recent years. Premium subsidies are the most common form of government support for insurance. In the particular case of the Ugandan Agricultural Insurance Scheme, the Government of Uganda provides subsidies of 30%, 50% or 80% to participating farmers, dependent on whether they are large-scale farmers, small-scale farmers, or farmers in disaster-prone areas, respectively (Erena et al., 2019). While doubts about the role of insurers in alleviating poverty exist among practitioners, adequate coordination between governments and private insurance companies has been shown to enhance the development of sustainable, affordable and cost-effective insurance products (see, for example, Solana (2015)). Insurance premium subsidies should be designed with a clearly stated purpose, target those in need and address market deficiencies or consumer equity concerns (Hill et al., 2014). When designed properly, subsidised insurance schemes represent a powerful and cost-effective way to achieve public policy objectives, while poorly designed premium subsidies are inefficient and can lead to significant economic costs (Hazell et al., 2017).

Previous studies have approached the subsidisation problem from a dynamic stochastic programming perspective. Ikegami et al. (2016), Carter and Janzen (2018) and Janzen et al. (2021) propose dynamic models of household consumption, investment and risk management, considering a social insurance-type mechanism which first prioritises lending aid to the vulnerable non-poor, contingent on their experience of negative shocks, then to those already below the poverty line. Introduction of an index-based insurance market is found to outperform the asset-based vulnerability-targeted protection in poverty reduction, economic growth and the cost of social protection. Although implementation of a vulnerability-targeted strategy induces a short-term increase in poverty, rates are lower than those associated with both in-kind and cash transfers in the medium- and long-term.

Carter and Janzen (2018) and Janzen et al. (2021) compare the impact of insurance when all costs are paid by the policyholder and when targeted-subsidies are provided to the vulnerable and already poor. In the latter study, those in the neighbourhood of the poverty line do not optimally purchase insurance (without subsidies), instead suppressing their consumption and mitigating the probability of falling into poverty. Observing a greater reduction in poverty in comparison to pure cash transfers, Jensen et al. (2017) provide empirical evidence for the benefits of insurance-based social protection through analysis of safety net and drought-based livestock insurance programmes in northern Kenya. Chantarat et al. (2017) consider the welfare impacts of the same index-based insurance programme, using herd size dynamics to address the vulnerability to poverty associated with covariate livestock mortality such that critical herd size mimics the poverty line. Targeted premium subsides are optimised across various herd size groups such that given measures of poverty reduction are maximised. Increases and decreases in household wealth and poverty, respectively, were greater under the optimal strategy than under alternative needs-based subsidisation mechanisms and with no insurance. In the presence of needs-based subsidisation which provides free protection to the most poor, the number of poor continued to increase, thus highlighting the importance of social protection strategies that target those still above but close to the poverty line, in addition to the already poor.

Kovacevic and Pflug (2011) alternatively propose a risk-theoretic model for calculating the probability of falling below the poverty line, aligning the problem with that of an insurer's probability of ruin. Negative consequences of purchasing insurance are observed for members of low-income populations closest to the poverty line, with the most vulnerable experiencing an increase in their probability of falling when insured. Implementing a multiple-equilibrium framework in the dynamic stochastic programming setting, Liao et al. (2020) echo this finding in their analysis of the impact of subsidised and unsubsidised agricultural insurance on poverty rates in rural China. Subsidisation of insurance is not considered in Kovacevic and Pflug (2011). The proportion of coverage and the choice to insure is also fixed across the population, as in Chantarat et al. (2017). Kovacevic and Semmler (2021) instead optimise the level of coverage by permitting a change in the retention rate of policyholders after any catastrophic event, defining a multi-equilibria model for their analysis. Flexibility in the insurance purchase decision is also considered in Janzen et al. (2021). The application of classical risk theory concepts to the context of poverty is the approach adopted in Chapters 2 and 3, for two types of capital growth model.

Besides reducing the impact on household capital growth, the use of subsidies to lower consumer premium payments has the potential to increase microinsurance uptake, with wealth and product price positively and negatively influencing microinsurance demand, respectively, see Eling et al. (2014) and Platteau et al. (2017). Focusing specifically on agricultural insurance, Hazell et al. (2017) present government and donor incentives for subsidisation. As an example, temporary subsidies can enable low-income farmers to bear the risk of adopting innovative technologies which may bring them out of poverty. However, in addition to improving the economic circumstances of the insured, through the provision of insurance experience this strategy mitigates the uncertainties surrounding insurance common among consumers in the microinsurance environment, while improving the quality of consumer data.

Households that live or fall below the poverty line are said to be in a poverty trap, where a poverty trap is a self-perpetuating state of poverty from which it is difficult to escape without external help. Due to the prevalence of the informal sector and, in rural areas, the reliance upon agricultural yields, income processes in low-income economies are typically inconsistent and liquidity constraints are widespread. This limits the ability to save and as a result, the availability of self-sustained financial protection. Exposure to the risks associated with extreme loss events and their potential to induce poverty trapping are therefore heightened among low-income populations, while insurance is often unaffordable and demand is low. An additional strategy for easing liquidity constraints and increasing consumer trust is to allow policyholders to delay premium payments until the end of the insured period, at which time any indemnities are paid. For discussion of this approach, see, for example, Liu et al. (2013) and Liu and Myers (2016), where positive consumer investment effects are also observed.

Poverty trapping is a well-studied topic in development economics, with a large literature focus on why economic stagnation below the poverty line occurs in certain communities. In theory, the poor could readily grow their way out of poverty by adopting profitable strategies including productive asset accumulation, opportunistic exchange and implementation of cost-effective production technologies. However, poverty traps are underlined by reinforcing behaviours induced by the state of being poor (Barrett et al., 2016). Multiple dynamic equilibria models dominate existing poverty trap literature, capturing the existence of both poor and non-poor equilibria. Single equilibrium models are also studied, with the economic entity converging to a unique dynamic equilibrium point which lies below the poverty line.

Although important for poverty alleviation, the behaviour of a household below the poverty line is not considered in this thesis. The analysis in Chapters 2-4 focuses only on low-income behaviours above this critical line. For further discussion, the interested reader may refer to Azariadis and Stachurski (2005), Bowles et al. (2006), Carter and Barrett (2006), Kraay and McKenzie (2014), Barrett et al. (2016) and references therein; see Matsuyama (2008) for a detailed description of the mechanics of poverty traps.

1.1.2 Risk sharing

Given the widespread exclusion of the low-income sector from traditional financial and insurance services, low-income populations often rely on alternative risk mitigation strategies. Risk-sharing is one such strategy for smoothing consumption fluctuations and mitigating the high-risk nature of the low-income environment. Gift exchange, informal credit and rotating savings associations, mutual aid and burial societies are examples of informal risk-sharing mechanisms (see, for example, Bardhan and Udry (1999)) used to lessen exposure to the idiosyncratic risks faced by low-income households. These risks include death, accident, illness, crop damage due to fire, insect and disease, with support provided in the form of monetary payments, labour services and food provisions. The absence of a formal contract, or rules and regulations, characterises the informal nature of these agreements.

A large development economics literature exists on the study of risk-sharing mechanisms, with many studies highlighting the prevalence of risk-sharing among small social groups rather than entire communities. Udry (1990), Townsend (1994) and de Weerdt and Dercon (2006), among others, discuss the presence of subgroups within villages or communities, including burial societies, women's organisations, families, religions and ethnic groups, and their influence on the structure of risk-sharing in a heterogeneous society. Dependent on the risk, groups with similar features (or dissimilar features when diversifying the risk) will be sought as participants in the risk-sharing agreement, where the socioeconomic characteristics determining partnership preferences are shaped by membership of these often overlapping subgroups.

As such, consideration of financial structures within small societal groups is important when studying the wealth dynamics of a system. Grimard (1997), Ligon (2002), Murgai et al. (2002), Fafchamps and Lund (2003), Dercon et al. (2006), Fafchamps and La Ferrara (2012), Attanasio et al. (2012), Mobarak and Rosenzweig (2013) and Dercon et al. (2014) are further examples of empirical work considering risk-sharing at this more granular level, in varying low-income economies.

It is often the case that agents within informal risk-sharing agreements have interchangeable roles, with recipients of capital also acting as capital providers at varying points in time (Coate and Ravallion, 1993). However, some indigenous associations also provide insurance in a more formalised manner. Dercon et al. (2006) study group-based insurance using evidence from Ethiopia and Tanzania. Focusing specifically on funeral associations, explicit rules on membership, insurance schedules and payouts were observed. Additional coverage for risks other than those relating to funeral costs were offered by a typically low number of schemes and as such, consumers were found to increase their coverage by participating in more than one scheme. Applications for short-term credit payments were also permitted in the event of a shock.

Due to their limited financial experience and the prevalence of basis risk, low-income consumers often have little trust in the insurer. In addition, lack of understanding of the workings of insurance causes incorrect use of products and thus induces moral hazard. By increasing communication among consumers, targeting insurance at the group-level contributes towards mitigating these fundamental risks. Chemin (2018) promotes the role of social networks in the uptake of health insurance. To overcome the lack of trust prevalent in the microinsurance environment, the study compares health insurance uptake among members of pre-existing informal groups with uptake among randomly selected sub-groups, where interventions including subsidies, registration assistance and information about the scheme are provided. While full subsidisation increases initial uptake to 45%, almost no retention is observed within the randomly selected group. Promotion of insurance within informal groups on the other hand, results in a 7% one-year retention rate with 12% initial uptake. Risk sharing is also found to be complementary to index-based insurance in overcoming basis risk (Dercon et al., 2014).

Note that, while risk sharing mechanisms mitigate the idiosyncratic risks experienced independently by group members over time, the occurrence of covariate risks, such as natural disasters and epidemics, requires more formal coverage. In Chapter 4, the dissemination model of Chan and Mandjes (2022) is adopted for analysis of the impact of insurance on wealth behaviours within a low-income group of the nature discussed here. In a simulation-based study of a similar nature, Will et al. (2021) assess the impact of the availability of microinsurance on participation in risk-sharing agreements. In this study, an agent-based simulation model is combined the Watts-Strogatz small-world networks model (Watts and Strogatz, 1998), which induces the risk sharing connections between households.

1.2 Lifetime dependence modelling

Two lives involved in the pricing of an insurance contract are traditionally assumed to be mutually independent, inferring there exists no relationship between their remaining lifetimes. This assumption induces greater simplicity in pricing calculations through reduction of the joint life estimation problem to the estimation problem of a single life, however it does not reflect reality. The existence of dependence between individual lifetimes presents the need to refine the independence assumption to improve the accuracy of the pricing and reserving of life insurance products that involve multiple lives and mortality assumptions. Survival probabilities of two individuals whose remaining lifetimes are dependent at the initiation of a policy will vary in line with the life status of the pair. Mortality laws determined at the time of the policy's valuation, which dictate the calculation of prospective premium and benefit payments, must therefore account for the likelihood of future shifts in mortality.

Existing joint lifetime research largely considers dependence between husband and wife. Denuit and Cornet (1999) and Denuit et al. (2001) use data from the Belgian National Institute of Statistics for estimation of the marginal force of mortality in a Markovian model. Here, an increase in mortality among widowed individuals is observed, with bereaved males experiencing a more significant deviation from the non-widowed mortality. Bivariate data is however more difficult to obtain, data for copula estimation in these studies was sampled from the gravestones of couples in Belgian cemeteries. The impact of marriage status on mortality is also presented in Maeder (1995). Many studies including those by Frees et al. (1996), Carriere (2000), Youn and Shemyakin (1999, 2001), Shemyakin and Youn (2001, 2006), Luciano et al. (2008), Spreeuw and Wang (2008); Spreeuw and Owadally (2013), Ji et al. (2011), Luciano et al. (2016), Dufresne et al. (2018), Zhang and Brockett (2020) and Arias and Cirillo (2021) consider a generation-based joint life data set from a large Canadian insurer in their analysis of joint life dependence. Joint annuity data from a French insurer is analysed in Lu (2017), French geneaology data in Cabrignac et al. (2020) and Dutch census data on married couples in Sanders and Melenberg (2016). Henshaw et al. (2020) (Chapter 5) consider Ghanaian survey data and Walter et al. (2021) joint life and last survivor annuity data from a Kenyan insurer, where the latter two studies are the only studies assessing dependence in an alternative socioeconomic context.

Various methods for dependence modelling appear in the literature. Copula-based approaches are widely used to express the joint survival functions of interest. Studies include those by Frees et al. (1996), Youn and Shemyakin (1999), Carriere (2000), Denuit et al. (2001) Spreeuw (2006), Shemyakin and Youn (2006), Luciano et al. (2016) and Dufresne et al. (2018). Copula selection is often found to influence the strength of the dependence observed. To mitigate the influence of copula choice, Sanders and Melenberg (2016) extend the parametric Weibull copula to a more flexible semi-nonparametric model through multiplication of the probability density by squared polynomials. Considering age-related influences on dependence structures, Youn and Shemyakin (1999) and Dufresne et al. (2018) find dependence to be a decreasing function of age difference, while Luciano et al. (2016) observe reduced dependence in younger joint lives through application of a generation-based model. Changes in the value of joint life and last survivor annuities with varying benefits are consistently observed under the dependence assumption. Joint life annuities making payments until the first loss of life are underpriced, while last survivor annuities providing benefits until the final death are overpriced. The observed impact of age difference on product pricing for an individual liability is however mitigated in Dufresne et al. (2018) when considering the total liability of an insurer given a portfolio of policyholders. Kaluszka and Okolewski (2014) alternatively study multiple life products where the joint distribution of future lifetimes is unknown, with the dependence structure assumed to belong to a nonparametric neighbourhood of independence.

Frailty models are an extension of the Cox (1972) proportional hazards model, and an alternative to the copula approach, which account for unobserved heterogeneities between individuals in a population. Hetergeneities may be positively or negatively correlated, dependent on the specification of their distribution. First introduced by Vaupel et al. (1979) and developed by Hougaard (1984), Marshall and Olkin (1988) and Oakes (1989) among others, in the lifetime dependence context, frailty-type models have been used in studies including those by Clayton (1978), Hougaard et al. (1992), Klein (1992), Nielsen et al. (1992) and Gourieroux and Lu (2015), where dependence is captured by the sharing of an unobserved common risk factor (or frailty) which negatively impacts mortality. Walter et al. (2021) apply a frailty model for construction of dependence life tables. Bivariate distributions generated by frailty models are a subclass of the Archimedean copula family popular in joint lifetime modelling (Marshall and Olkin, 1988; Oakes, 1989). Lu (2017) implement a mixed proportional hazards model to account for observed and unobserved frailties, with a treatment effect capturing the mortality jump characteristic of spousal dependence (see Section 1.2.1). The impact of losing a spouse is found to be asymmetric between males and females, a finding also observed in Dufresne et al. (2018). See Frees and Valdez (1998) for a thorough overview of copulas and their link with frailty models in the actuarial setting.

Dependence induced by the occurrence of an event experienced simultaneously by two lifetimes, due to, for example, a car accident or natural disaster can be modelled using the common shock model of Marshall and Olkin (1967). Gobbi et al. (2019) consider the extended Marshall-Olkin model of Pinto and Kolev (2015) which combines the copula and common shock approaches. Common shock models have also been used in the risk theory context to model dependence in the frequency and severity of claims across insurance business classes, see, for example, Ambagaspitiya (2003), Wang and Yuen (2005), Dang et al. (2009) and Wang et al. (2016). Multivariate risk models with common shock and thinning dependence are adopted in Wang and Yuen (2005) and Wang et al. (2016) to additionally capture the possibility of a given claim inducing claims in other insurance classes. Lee and Cha (2018) consider a more flexible common shock model in the joint survival analysis context, accounting for shock events that affect the remaining lifetimes of two individuals but do not necessarily induce the death of both.

Markov chain methods are implemented in a number of works as a basis for joint mortality models. Martikainen and Valkonen (1996) propose the significance of adaptations in the living conditions of bereaved individuals, due to factors including grief and stress, on the interdependence of the lifetimes of paired lives. The state-based approach facilitated in the Markovian setting enables the capturing of such lifestyle changes. Norberg (1989) proposes a four-state Markov mortality model dependent on marital status. Denuit and Cornet (1999) use Fréchet-Hoeffding bounds to estimate the maximum impact of dependence under assumption of the Norberg (1989) model, again observing a notable reduction in premium. This study is further developed in Denuit et al. (2001). Spreeuw and Wang (2008) and Spreeuw and Owadally (2013) extend Norberg's model to account for the typically short-term nature of spousal dependence. Through the inclusion of an additional state, the mortality of the survivor is assumed to be dependent on the time elapsed since the first death.

A limitation of the discrete model of Spreeuw and Wang (2008) is that, in moving between states, the bereaved spouse experiences sudden jumps in their mortality intensity. Semi-Markov chain models have superior flexibility in comparison to the standard Markov chain model. In calculation of transition probabilities, time since previous transition is considered in addition to the current time and state occupied. Ji et al. (2011) define the impact of spousal loss as a smooth, parametric, decreasing function of time since bereavement through implementation of a semi-Markov model, enabling greater information gain in regard to how the effect changes with time. Clearly, model selection influences the pricing and valuation of insurance products. The Markov chain model of Norberg (1989) suggests bereaved mortality increases permanently; however, the semi-Markov model of Ji et al. (2011) allows for the recovery of bereaved mortality after the initial death, thus facilitating duration dependence.

Deterministic mortality intensities, defined as functions of the age of the policyholder, have traditionally been used by actuaries in the pricing and valuation of insurance products. Use of stochastic processes in modelling mortality intensity allows for incorporation of the uncertainty of future mortality development and time dependence in mortality models. This facilitates improvements in the accuracy of pricing calculations, in addition to allowing for quantification of the mortality risk faced by insurance companies. Financial risk, systematic and unsystematic mortality risk, are the fundamental risk factors insurers are exposed to (Dahl, 2004).

Paralleling mathematical approaches for modelling time to default in the credit risk literature, Dahl (2004), Biffis (2005), Luciano and Vigna (2005), Schrager (2006) and Luciano and Vigna (2008) among others, model the remaining lifetime of an individual as a doubly stochastic stopping time, with stochastic intensity given by the force of mortality. The link between credit-sensitive securities and insurance contracts was first proposed by Artzner and Delbaen (1995). This connection enables exploitation of the similarities between time to default and remaining lifetime, and short-term interest rate and force of mortality.

Credit risk models can, in general, be classified into two distinct categories (see Jarrow and Protter (2004) for a detailed comparison of the two). Originating from the approaches of Black and Scholes (1973) and Merton (1974), structural models focus on the structural characteristics of an institution, comparing the market value of a company's assets to their liabilities, with complete knowledge of a comprehensive information set. The approach of the aforementioned mortality literature falls into the class of reduced form models, implemented without the need to account for company specific factors underlying the occurrence of a default, due to assumption of an exogenous cause (Saunders and Allen, 2002).

Although well-established for single cohort studies, use of stochastic mortality models for joint life dependence is limited. In the case of the joint mortality experience of coupled lives, Luciano et al. (2008) again adopt the reduced form credit risk methodology, implementing a continuous time cohort model of affine type with dependence induced through an Archidmedean copula. This study creates the first link between stochastic and copula based approaches. Luciano et al. (2016) find two parameter extensions of Archimedean copulas to be more suitable for representing coupled dependence when investigating the dependence of spouses across generations.

Symmetric bereavement reactions and the staticity of dependence over time are two drawbacks of the use of copulas. Allowing for asymmetric mortality reactions to the occurrence of a death, Gourieroux and Lu (2015) introduce mortality jumps through combination of a Freund model with an unobservable, common, static frailty representing the socioeconomic conditions shared by coupled lives. Dependence between lifetimes further to the contagion effects resulting directly from a loss is also therefore accounted for. In addition, copula density functions are largely continuous, implying significant changes (or jumps) in mortality do not occur.

Jevtić and Hurd (2017) introduce an alternative to copula dependence in the credit risk environment through definition of a probabilistic mechanism which describes the influence of a loss on the dependent lifetime. A stochastic mortality model of affine type is implemented for mortality experience, assuming correlated non-mean-reverting Ornstein–Uhlenbeck (OU) diffusions for the mortality intensities of coupled lives. Inclusion of mean-reversion in mortality modelling requires a diminishing likelihood of future mortality improvements in the event that recent mortality developments occur at a faster rate than anticipated. Uncertainties surrounding medical advances and improvements in the healthcare and pharmaceutical industries highlight the unsuitability of a mortality model with such constraints. In line with this, existing research suggests time-homogeneous mean-reverting affine processes do not fit observed mortality tables, while forces of mortality appear to behave exponentially rather than in a mean-reverting fashion (Luciano and Vigna, 2005). The requirements for a good stochastic mortality model are presented in Cairns et al. (2006).

In relation to credit risk, however, default intensities are generally modelled as mean-reverting processes. As such, implementation of the classical Cox–Ingersoll–Ross (CIR) model is popular within credit risk literature. For examples of applications of CIR processes in the financial and insurance mathematics literature, the interested reader may refer to Liang et al. (2011), Nowak and Romaniuk (2018) and Dassios et al. (2019). Milevsky and Promislow (2001) implement a CIR interest rate process with stochastic force of mortality, incorporating both interest rate risk and systematic mortality risk. The Feller process is a non-mean-reverting adaptation of the classical CIR model. In contrast to OU-type processes, the non-negativity constraint of mortality intensity is not violated through implementation of this model, conditional on the non-negativity of the initial starting point. Calibrating the Feller process to three generations in the UK population, Luciano and Vigna (2008) find the associated survival probability to decrease at every age. Inclusion of a rooted mortality intensity also tempers the volatility of the process. Non-mean reverting CIR and OU stochastic processes are combined in Chapter 5 to construct a joint mortality model applicable to the low-income environment.

An alternative approach to correlating stochastic processes in the credit risk setting is proposed by Zhang and Brockett (2020). In this study, individual mortalities are modelled as Brownian motions with drift and have time indices that move according to correlated subordinators. Dependence is induced through this correlation, where the subordinators are structured to capture both shared frailties and idiosyncratic risks in a similar manner to Jevtić and Hurd (2017) and Henshaw et al. (2020) (Chapter 5). Instead adopting a machine learning perspective, Arias and Cirillo (2021) propose the use of the non-parametric bivariate reinforced urn process which learns from the lifetime experience of individuals and uses the information obtained to make inference about the lifetimes of others. In line with the Bayesian approach, prior knowledge can be incorporated in the model and updated at the end of each lifetime, thus facilitating improvements over time.

1.2.1 Broken-heart syndrome and shared frailty dependence

Establishing the duration of dependence is important in determining the full extent of pricing implications. Hougaard (2000) discusses the classification of dependence across three time frames, with time elapsed since the death of the deceased allowing for differentiation between the three classes.

Dependence between the lifetimes of coupled lives is commonly referred to as broken-heart syndrome. Characterised by an elevated force of mortality which is a decreasing function of time since death, broken-heart syndrome is the most recognised form of short-term dependence. Long-term and instantaneous dependence, also referred to as the common shock effect, constitute the remaining classifications which should be considered by insurers in the pricing of products involving mortality assumptions. For further details on the definition of each dependence structure, see Hougaard (2000).

Spreeuw (2006) investigates the nature of dependence and time-dependent association between lifetimes through implementation of a number of single parameter Archimedean copula models. Almost all of the nine copulas studied exhibit long-term dependence. One copula family presents short-term dependence, however only for young ages or short durations, with pure short-term dependence not recognised in any of the copulas studied. Although these results question the relevance of copulas in the context of broken-heart syndrome, the limitation of single parameter models in capturing all dependence classes is highlighted in line with Luciano et al. (2016).

Historical research into the prevelance of broken-heart syndrome suggests the elevated mortality of the survivor diminishes significantly following an approximate period of between six and twelve months (Rees and Lutkins, 1967; Parkes et al., 1969; Ward, 1976), falling to the commonly lower mortality of the comparative non-widowed population in some cases. Factors influencing the impact of broken-heart syndrome include the cause of death of the deceased spouse (Elwert and Christakis, 2008), the age of the bereaved spouse and the location of the first death (Rees and Lutkins, 1967), with widowers experiencing a greater change in mortality compared to that of widows. Spreeuw and Wang (2008), Ji et al. (2011), Spreeuw and Owadally (2013), Gourieroux and Lu (2015), Jevtić and Hurd (2017) and Henshaw et al. (2020) introduce jumps in the mortality intensity of the survivor after the death of their spouse in line with empirical findings of dependence studies on coupled lives.

The nature of joint life dependence is important to insurers both for pricing and to ensure diversification opportunities are exploited. Dependence between the remaining lifetimes of paired lives exists before the occurrence of a death, through unobserved couple-level heterogeneities (Klein, 1992). This prior association is referred to as spurious risk dependence. In contrast to broken-heart syndrome, which is causal by definition, spurious risk dependence is purely attributable to the sharing of correlated risk factors or frailties. Living conditions, healthcare access, diet habits and mutual emotional stresses are among some of the lifestyle features that are determinants of both health and mortality. Selectivity in the formation of couples also implies the pairing of individuals with equivalent levels of risk, heightening the prevalence of any unobserved correlation and the associated risks. Improvements in the underwriting processes of an insurance company and diversification of insurance portfolios would mitigate dependence risk if correlated unobserved heterogeneities were the main component of spousal mortality dependence. However, identification of a marked causal effect of spousal bereavement would need to be addressed, with assumption of coupled mortality dependence required across all policyholders regardless of their characteristics and socioeconomic status.

Lu (2017) separates the impact of spurious and causal dependence, observing 92.4% and 81% of the mortality jump to be accounted for by broken-heart syndrome among bereaved males and females, respectively. The remaining proportion reflects unobserved heterogeneities. However, despite the dominance of broken-heart syndrome over spurious correlation, disregarding either effect was found to produce significant pricing errors. Both spurious and causal dependence are also captured by van den Berg et al. (2011), where the mortality increase is of greatest significance during the first two and a half years of bereavement, with no effect observed more than five years post loss. This study provides evidence for the causal effect of conjugal bereavement on mortality and health, reporting a reduction in residual life expectancy of 12% on average following the loss of a spouse. In line with Lu (2017), the error in assuming the life status of an individual's spouse to be an exogenous determinant of mortality is here acknowledged. Evaluating the impact of bereavement on premium pricing over time, in Spreeuw and Wang (2008), the mortality drop characteristic of broken-heart syndrome was found to outweigh the initial mortality elevation, with little impact on future pricing.

Although the negative implications of the loss of a spouse on the remaining lifetime of the bereaved partner are widely accepted, differences in cultural patterns of bereavement reactions are observed across socioeconomic environments (Osterwers and Solomon, 1984; Laungani, 1996; Parkes et al., 2015). Cultural differences in outlook on life, behaviours and religious beliefs, the environmental characteristics of a country or region, disease incidence, healthcare access and levels of economic, social or political change each contribute to the mortality and bereavement reactions of individuals.

Knowledge of the importance of insurance in sustaining economic growth has a history of significant length (United Nations, 1964); however, penetration in low and lower-middle income countries remains reduced in comparison to higher-middle and high income countries (Outreville, 2013), with determinants of consumption ranging from income per capita, inflation and banking sector development to religious inclination (Beck and Webb, 2003). As a long-term savings instrument and measure of risk mitigation providing protection against the financial consequences of death, the need for increased life insurance penetration rates in such economies is highlighted by its connection with financial development levels (Outreville, 1996). The importance of acknowledging dependence between coupled lives particularly in low and lower-middle income countries and consequently alleviating the insurance risk associated with inaccurate pricing mechanisms is therefore heightened by this connection, due to the instability of the associated economies.

1.3 Contributions and outline

Although well-studied in the economics and development economics literature, actuarial methods for microinsurance and for traditional insurance with application to low-income settings are lacking. Existing actuarial research largely focuses on empirical analysis of the pricing, demand, renewal and sustainability of products. However, to facilitate movement away from direct replication of traditional insurance products, rigorous mathematical modelling is required. The main objective of this thesis is therefore to contribute with mathematical perspectives on insurance solutions in the low-income environment.

Chapter 2 adopts the insurance risk process with deterministic investment and exponentially distributed claims to the poverty reduction context. Classical risk theory techniques are used to analyse the behaviour of low-income capital processes. Comparing the impact of three microinsurance strategies on the probability of falling below the poverty line, specifically, microinsurance schemes with (i) unsubsidised premiums, (ii) subsidised constant premiums and (iii) subsidised flexible premiums, explicit solutions for the desired probability and for the governmental cost of social protection are obtained. Cost of social protection is defined as the present value of government subsidies plus a supplemental fixed cost that ensures, with a certain level of confidence, that households will not return to poverty, should they fall below the threshold, where previous studies consider only the cost of lifting households to the poverty line. Mimicking the dividend barrier strategy well-studied in risk theory, strategy (iii) is defined such that premium payments are made by consumers only when their capital is above some predefined barrier.

Capital losses in Chapter 2 are captured by negative jumps of random size. Chapter 3 studies an adjustment of the process with households susceptible to random shocks proportional to their level of capital. Kovacevic and Pflug (2011) consider a version of this process discretised at loss event times, performing numerical analysis on the associated probability of falling below the poverty line. In Chapter 3, closed-form expressions of the probability are obtained for special cases of beta distributed remaining proportions of capital using Laplace transform methods. The impact of insurance on the probability of falling below the poverty line is again considered through assumption of proportional insurance coverage. Asymptotic analysis of a similar light to that applied in Constantinescu (2006) is considered. Analysis is undertaken on the infinitesimal generator of the proportional capital process and comparisons made with the uninsured case through simulation.

Chapter 4 adjusts the stochastic dissemination model of Chan and Mandjes (2022) for application to the wealth behaviours of a group. The impact of group membership and wealth interactions on the probability of falling below the poverty line are considered for the first time in the mathematical context. Deriving a system of coupled differential equations for the joint transient distribution of agent wealth and an exogenously evolving Markov background process, which represents the economic state of the system, numerical analysis is facilitated through consideration of the first, reduced and mixed second moments of the wealth process. Group wealth behaviours are investigated through sensitivity analysis and the probability of falling below the poverty line with normal approximation techniques.

Chapters 5 and 6 address the existence of socioeconomic influences on dependence between paired lifetimes and the bereavement processes of survivors. Focusing on dependence between coupled lives, in Chapter 5, an adjustment of the joint mortality model of Jevtić and Hurd (2017) is adopted. Motivated by analysis of a Ghanaian data set collected for the purpose of this study, correlated non-mean-reverting CIR diffusions are proposed to represent the paired mortalities of coupled lives. Unoberserved heterogeneities are accounted for by correlating the Brownian motions present in the CIR processes. A mean-reverting OU process is then selected to represent the broken-heart syndrome effect. In moving from the deterministic bereavement of Jevtić and Hurd (2017) to a stochastic bereavement process, should it exist, any non-diversifiable risk associated with a loss can also be accounted for. Deterministic bereavement processes only capture diversifiable risks. Applying classical methods from stochastic optimal control, the indifference price of a reversionnary annuity is derived and used to determine the impact of the dependence assumption on the pricing of an insurance product requiring mortality assumptions.

Chapter 6 adopts the copula-based approach for lifetime dependence modelling well-used in the literature, fitting four Archimedean copulas to a large sample of Egyptian social pension data. Covering, by law, a policyholder's spouse, children, parents and siblings, this data set enables analysis of pairwise dependence between multiple familial relationships beyond the well-known husband and wife case. Bayesian Markov Chain Monte Carlo (MCMC) techniques are used for parameter estimation with likelihood derived by the two-step inference functions for margins (IFM) method. Given the traditional use of UK mortality tables in the modelling of mortality in countries such as Ghana and Egypt, Chapters 5 and 6 are useful in informing the definition of appropriate joint mortality assumptions that correctly fit the target population.

Concluding remarks on the five studies and notes on future work are presented at the end of the thesis.

Chapter 2

Subsidising inclusive insurance to reduce poverty

In this chapter, the benefits of coordination and partnerships between governments and private insurers are assessed, and further evidence provided for microinsurance products as powerful and cost-effective tools for achieving poverty reduction. Household capital is modelled from a ruin-theoretic perspective to measure the impact of microinsurance on poverty dynamics and the governmental cost of social protection. The model is analysed under four frameworks: uninsured, insured (without subsidies), insured with subsidised constant premiums and insured with subsidised flexible premiums. Although insurance alone (without subsidies) may not be sufficient to reduce the likelihood of falling into the area of poverty for specific groups of households, since premium payments constrain their capital growth, the analysis suggests that subsidised schemes can provide maximum social benefits while reducing governmental costs. This chapter is based on work submitted to a peer-reviewed academic journal, currently under review (Flores-Contró et al., 2022).

2.1 Introduction

Adopting the novel ruin-theoretic approach presented by Kovacevic and Pflug (2011), this chapter studies the impact of insurance on poverty dynamics and the governmental cost of social protection. The aim of this anlaysis is to determine the benefits derived from coordination and partnerships between governments and private insurers, and to highlight the cost-effectiveness of government support for insurance. As discussed in Chapter 1, premium payments may increase the risk of falling into poverty for the proportion of the population living just above the poverty line. Microinsurance schemes which are supported by social protection strategies are therefore considered, and more specifically, their potential in minimising both the probability of a household falling below the poverty line and the governmental cost of social protection.

The piecewise-deterministic Markov process proposed by Kovacevic and Pflug (2011) is adapted such that households are subject to shocks of random size. Where Kovacevic and Pflug (2011) consider a version of the process discretised at the jump times, in this chapter, the full, non-discretised capital process is considered. In line with the poverty trap ideology, the area of poverty is assumed to be an absorbing state and only the state of events above the poverty threshold is considered. Obtaining explicit solutions for trapping probabilities and the governmental cost of social protection using classical risk theory techniques, the influence of three structures of microinsurance is compared. Specifically, microinsurance schemes with (i) unsubsidised premiums, (ii) subsidised constant premiums and (iii) subsidised flexible premiums are considered. Unlike in previous studies, where the cost of social protection is defined as the present value of government subsidies plus the transfers needed to close the poverty gap for all poor households (see, for example, Ikegami et al. (2016) and Janzen et al. (2021)), the ruin-theoretic perspective adopted in the proposed model enables inclusion of a supplemental fixed cost that ensures, with a certain level of confidence, that households will not return to poverty, should they fall below the threshold. In this way, the likelihood that the government will re-incur these costs for the same household is reduced.

Under the first premium framework, premiums are paid by households in full. These payments constrain household capital growth and increase their trapping probability compared to that of uninsured households, as in the existing literature. The cost of social protection remains lower than the corresponding uninsured cost. With the need for an alternative solution to address the observed negative impact on poverty dynamics, under the second premium framework it is assumed that governments provide insurance premium subsidies to all households. Reducing premium payments by means of subsidies has a positive impact on household capital growth and on the associated trapping probabilities. The analytic results enable optimisation of the subsidy level for households with varying degrees of capital, such that a trapping probability equal to that of when uninsured is preserved. The proposed subsidy optimisation aligns with the idea of "smart" subsidies, which are defined as those that provide maximum social benefits while minimising distortions in the insurance market and the mis-targeting of clients (Hill et al., 2014). Under the optimal subsidised microinsurance scheme, comparing with the uninsured case, while non-essential for more privileged households, vulnerable households with capital close to the poverty line are in need of government support. Moreover, the cost of social protection for the most vulnerable is lower than the corresponding uninsured and insured (without subsidies) costs, but is higher for the most privileged.

Mimicking the dividend barrier strategy well-known in risk theory, the third framework considers a novel scheme under which households pay premiums only when their capital is above some pre-defined capital barrier, with the premium otherwise paid by the government. Granting flexibility on premium payments allows households to attain lower trapping probabilities, since they are assisted by government when their capital lies close to the poverty line. In this analysis, the capital barrier level at which governments should begin providing support is optimised. Intuitively, those closest to the poverty line require immediate aid, with optimal barriers lying above their initial capital. On the other hand, those further away from the poverty line have sufficient capital to pay premiums themselves on enrolment to the scheme, yielding optimal barriers that lie below initial capital levels. Under this framework, the cost of social protection remains lower than the corresponding uninsured cost.

The remainder of the chapter is structured as follows. In Section 2.2, the household capital model and its associated infinitesimal generator are introduced. The (trapping) time at which a household falls into the area of poverty is defined in Section 2.3, and subsequently the explicit trapping probability and the expected trapping time are derived for the basic uninsured model. Links between classical ruin models and the trapping model of this chapter are stated in Sections 2.2 and 2.3. Microinsurance is introduced in Section 2.4, where a proportion of household losses are covered by a microinsurance policy. The capital model is redefined and the trapping probability derived. Sections 2.5 and 2.6 consider the case where households are proportionally insured through a government subsidised microinsurance scheme, with the impact of subsidised flexible premiums discussed in Section 2.6. Optimisation of the subsidy and capital barrier levels are presented in Sections 2.5 and 2.6, alongside the associated governmental cost of social protection. Concluding remarks are provided in Section 2.7.

2.2 The capital model

The fundamental dynamics of the model follow those of Kovacevic and Pflug (2011), where the growth in accumulated capital $(X_t)_{t\geq 0}$ of an individual household is given by

$$\frac{dX_t}{dt} = r \cdot [X_t - x^*]^+ , \tag{2.2.1}$$

where $[x]^+ = \max(x, 0)$. These dynamics are constructed under the assumption that a household's income (I_t) is split into consumption (C_t) and savings or investments (S_t), such that at time t,

$$I_t = C_t + S_t, \tag{2.2.2}$$

where consumption is an increasing function of income:

$$C_t = \begin{cases} I_t, & \textit{if } I_t \leq x^* \quad (2.2.3a) \\ I^* + a(I_t - I^*), & \textit{if } I_t > x^* \quad (2.2.3b) \end{cases}$$

for $0 < a < 1$. The critical point below which a household consumes all of their income with no facility for savings or investment, is denoted I^*. Accumulated capital is assumed to grow proportionally to the level of savings, such that

$$\frac{dX_t}{dt} = cS_t, \tag{2.2.4}$$

for $0 < c < 1$, and income is generated through the accumulated capital, such that

$$I_t = bX_t,$$

for $b > 0$. Combining (2.2.2), (2.2.3a), (2.2.3b) and (2.2.4), gives (2.2.1), where the capital growth rate $r = (1 - a) \cdot b \cdot c > 0$ incorporates household rates of consumption (a), income generation (b) and investment or savings (c), while $x^* = \frac{I^*}{b} > 0$ denotes the threshold below which a household lives in poverty.

Reflecting the ability of a household to produce, accumulated capital (X_t) is composed of land, property, physical and human capital, with health a form of capital in extreme cases where sufficient health services and food accessibility are not guaranteed (Dasgupta, 1997). The notion of a household in this model setting may be extended for consideration of poverty trapping within economic units such as community groups, villages and tribes, in addition to the traditional household structure.

The poverty threshold x^* represents the amount of capital required to forever attain a critical level of income, below which a household would not be able to sustain their basic needs, facing elementary problems relating to health and food security. Throughout the chapter, this threshold will be referred to as the critical capital or the poverty line. Since (2.2.1) is positive for all levels of capital greater than the critical capital, all points less than or equal to x^* are stationary (capital remains constant if the critical level is not met). In this basic model, stationary points below the critical capital are not attractors of the system if the initial capital exceeds x^*, in which case the capital process X_t grows exponentially with rate r.

Using capital as an indicator of financial stability over other commonly used measures such as income enables a more effective analysis of a household's wealth and well-being. As, for example, households with relatively high income, considerable debt and few assets would still be vulnerable if any loss of income was to occur, while low-income households may live comfortably on assets previously acquired and saved (see, for example, Gartner et al. (2004)).

In line with Kovacevic and Pflug (2011), the dynamics of (2.2.1) are expanded under the assumption that households are susceptible to the occurrence of capital losses, including severe illness, the death of a household member or breadwinner and catastrophic events such as floods and earthquakes. The occurrence of loss events is assumed to follow a Poisson process with intensity λ, where the capital process follows the dynamics of (2.2.1) in between events. On the occurrence of a loss, the household's capital reduces by a random amount Z_i. The sequence (Z_i) is independent of the Poisson process and independent and identically distributed (i.i.d.) with common distribution function G_Z. In contrast to Kovacevic and Pflug (2011), claims are assumed to be random-valued, rather than a random proportion of the capital itself. This adaptation facilitates analysis of a tractable mathematical model that enables derivation of an analytic solution for the infinite-time trapping probability (see Section 2.3). The analysis of this chapter therefore differs from previous work on the topic, in which numerical methods are employed to estimate the quantities of interest (see, for example, Kovacevic and Pflug (2011) and Azaïs and Genadot (2015)). The core objective of studying the probability of a household falling into the area of poverty remains.

A household reaches the area of poverty if it suffers a loss large enough that the remaining capital is attracted into the poverty trap. Since a household's capital does not grow below the critical capital x^*, households that fall into the area of poverty will never escape without external help. Once below the critical capital, households are exposed to the risk of falling deeper into poverty, with the dynamics of the model allowing for the possibility of negative capital. A reduction in a household's capital below zero could represent a scenario where total debt exceeds total assets, resulting in negative capital net worth. The experience of a household below the critical capital is, however, out of the scope of this chapter.

The stochastic capital process is now formally defined, where the structure of the process between loss events is derived through solution of the first order Ordinary Differential Equation (ODE) (2.2.1). This model is an adaptation of the model proposed by Kovacevic and Pflug (2011).

Definition 2.2.1. Let T_i be the i^{th} event time of a Poisson process $(N_t)_{t\geq 0}$ with parameter λ, where $T_0 = 0$. Let $Z_i \geq 0$ be a sequence of i.i.d. random variables with distribution function G_Z, independent of the process N_t. For $T_{i-1} \leq t < T_i$, the stochastic growth process of the accumulated capital X_t is defined by

$$X_t = \begin{cases} \left(X_{T_{i-1}} - x^*\right) e^{r(t-T_{i-1})} + x^*, & \textit{if } X_{T_{i-1}} > x^* \quad (2.2.5a) \\ X_{T_{i-1}}, & \textit{otherwise.} \quad (2.2.5b) \end{cases}$$

At the jump times $t = T_i$, the process is given by

$$X_{T_i} = \begin{cases} \left(X_{T_{i-1}} - x^*\right) e^{r(T_i-T_{i-1})} + x^* - Z_i, & \textit{if } X_{T_{i-1}} > x^* \\ X_{T_{i-1}} - Z_i, & \textit{otherwise.} \end{cases}$$

The stochastic process $(X_t)_{t\geq 0}$ is a piecewise-determinsitic Markov process (Davis, 1984) with infinitesimal generator:

$$(\mathcal{A}f)(x) = r(x - x^*)f'(x) + \lambda \int_0^\infty [f(x - z) - f(x)]\, \mathrm{d}G_Z(z), \qquad x \geq x^*. \tag{2.2.7}$$

The capital model as presented in Definition 2.2.1 is in fact a model well-studied in ruin theory since the 1940s. As such, well-established techniques can be easily applied to the poverty trapping context of this chapter. In ruin theory, modelling is undertaken from the point of view of an insurance company. Consider an insurer's surplus process $(U_t)_{t\geq 0}$ given by

$$U_t = u + pt + \nu \int_0^t U_s\, ds - \sum_{i=1}^{N_t} Z_i, \tag{2.2.8}$$

where u is the insurer's initial capital, p is the constant premium rate, ν is the risk-free interest rate, N_t is a Poisson process with parameter λ which counts the number of claims in the time interval $[0, t]$ and $(Z_i)_{i=1}^\infty$ is a sequence of i.i.d. claim sizes with distribution function G_Z. This model is also called the insurance risk model with deterministic investment, first proposed by Segerdahl (1942) and subsequently studied by Harrison (1977) and Sundt and Teugels (1995). For a detailed literature review on the model prior to the turn of the century, readers may consult Paulsen (1998).

Observe that when $p = 0$, the insurance model (2.2.8) for positive surplus is equivalent to the capital model of Definition 2.2.1 above the poverty line $x^* = 0$. Subsequently, the capital growth rate r in the model presented here corresponds to the risk-free investment rate ν of the insurer's surplus model. More connections between the two models will be made in the next section, following introduction of the trapping time.

2.3 The trapping time

Let

$$\tau_x := \inf\{t \geq 0 : X_t < x^* \mid X_0 = x\}$$

denote the time at which a household with initial capital $x \geq x^*$ falls into the area of poverty (the trapping time) and let $\psi(x) = \mathbb{P}(\tau_x < \infty)$ be the infinite-time trapping probability. To study the distribution of the trapping time in this chapter, the expected discounted penalty function at ruin, a concept commonly used in actuarial science (Gerber and Shiu, 1998), is adopted. For a force of interest $\delta \geq 0$ and initial capital $x \geq x^*$, consider

$$m_\delta(x) = \mathbb{E}\left[w(X_{\tau_x^-} - x^*, |X_{\tau_x} - x^*|)e^{-\delta\tau_x}\mathbb{1}_{\{\tau_x<\infty\}}\right], \tag{2.3.1}$$

where $\mathbb{1}_{\{A\}}$ is the indicator function of a set A and $w(x_1, x_2)$ for $0 \leq x_1, x_2 < \infty$, is a non-negative penalty function of x_1, the capital surplus prior to the trapping time, and x_2, the capital deficit at the trapping time. For more details on the so-called Gerber-Shiu risk theory, the interested reader may wish to consult Kyprianou (2013).

The probabilistic properties of the trapping time are contained in its distribution function. In studying the distribution of the trapping time in this chapter, the Laplace transform, as defined in Appendix A, is considered, where the Laplace transform characterises the probability distribution uniquely. Note that, specifying the penalty function such that $w(x_1, x_2) = 1$ in (2.3.1), $m_\delta(x)$ is exactly the Laplace transform of the trapping time, or intuitively, the expected present value of a unit payment due at the trapping time.

Throughout the remainder of the chapter capital losses are assumed to be exponentially distributed, i.e. $Z_i \sim Exp(\alpha)$. The following theorem is required for derivation of the uninsured trapping probability, where the trapping time mimics the time of ruin.

Theorem 2.3.1 (Paulsen and Gjessing (1997)). Let $\tau_x = \inf\{t \geq 0 : X_t < 0 \,|\, X_0 = x\}$ denote the time of ruin given initial surplus x, where τ_x is fixed at infinity if $X_t \geq 0 \; \forall t$. Assume $m_\delta(x)$ is a bounded and twice continuously differentiable function on $x \geq 0$, with a bounded first derivative. If $m_\delta(x)$ solves $\mathcal{A}m_\delta(x) = \delta m_\delta(x)$ on $x \geq 0$, together with boundary conditions

$$m_\delta(x) = 1 \quad \text{for} \quad x < 0$$

and

$$\lim_{x\to\infty} m_\delta(x) = 0,$$

then

$$m_\delta(x) = \mathbb{E}[e^{-\delta\tau_x}].$$

Proof. See Paulsen and Gjessing (1997) for proof. □

Proposition 2.3.1. Consider a household capital process (as proposed in Definition 2.2.1) with initial capital $x \geq x^*$, capital growth rate r, loss intensity $\lambda > 0$ and exponentially distributed capital losses with parameter $\alpha > 0$. The Laplace transform of the trapping time is given by

$$m_\delta(x) = \frac{\lambda}{(\lambda+\delta)U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda+\delta}{r}; 0\right)} e^{y(x)} U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda+\delta}{r}; -y(x)\right), \tag{2.3.2}$$

where $\delta \geq 0$ is the force of interest, $y(x) = -\alpha(x - x^*)$ and $U(\cdot)$ is Tricomi's Confluent Hypergeometric Function as defined in Appendix B.

Proof. Using the standard arguments based on the infinitesimal generator presented in Theorem 2.3.1, the expected discounted penalty function at the trapping time $m_\delta(x)$ in (2.3.1), can be characterised as the solution of the IDE

$$r(x-x^*)m_\delta'(x) - (\lambda+\delta)m_\delta(x) + \lambda\int_0^{x-x^*} m_\delta(x-z)dG_Z(z) = -\lambda A(x), \qquad x \geq x^*, \tag{2.3.3}$$

where

$$A(x) := \int_{x-x^*}^{\infty} w(x-x^*, z-(x-x^*))dG_Z(z).$$

When $Z_i \sim Exp(\alpha)$ and $w(x_1, x_2) = 1$, (2.3.3) can be written such that

$$r(x-x^*)m_\delta'(x) - (\lambda+\delta)m_\delta(x) + \lambda\int_0^{x-x^*} m_\delta(x-z)\alpha e^{-\alpha z}dz = -\lambda e^{-\alpha(x-x^*)}, \quad x \geq x^*. \tag{2.3.4}$$

Applying the operator $\left(\frac{d}{dx} + \alpha\right)$ on both sides of (2.3.4), together with a number of algebraic manipulations, yields the second order homogeneous differential equation

$$-\frac{(x-x^*)}{\alpha}m_\delta''(x) + \left[\frac{(\lambda+\delta-r)}{\alpha r} - (x-x^*)\right]m_\delta'(x) + \frac{\delta}{r}m_\delta(x) = 0, \qquad x \geq x^*. \tag{2.3.5}$$

Letting $f(y) := m_\delta(x)$, such that y is associated with the change of variable $y := y(x) = -\alpha(x - x^*)$, (2.3.5) reduces to Kummer's confluent hypergeometric equation (Slater, 1960)

$$y \cdot f''(y) + (b - y)f'(y) - af(y) = 0, \qquad y < 0, \tag{2.3.6}$$

for $a = -\frac{\delta}{r}$ and $b = 1 - \frac{\lambda+\delta}{r}$, with regular singular point at $y = 0$ and irregular singular point at $y = -\infty$ (corresponding to $x = x^*$ and $x = \infty$, respectively). A general solution of (2.3.6) and thus $m_\delta(x)$ is given by

$$f(y) = A_1 M\left(-\frac{\delta}{r}, 1 - \frac{\lambda+\delta}{r}; y(x)\right) + A_2 e^{y(x)} U\left(1 - \frac{\lambda}{r}, 1 - \frac{\lambda+\delta}{r}; -y(x)\right) \tag{2.3.7}$$

for $x \geq x^*$, where $A_1, A_2 \in \mathbb{R}$ are arbitrary constants, $M(a, b; z)$ and $U(a, b; z)$ are the Kummer and Tricomi confluent hypergeometric functions, respectively, as defined in Appendix B, and $m_\delta(x) = f(y) = 1$ for $x < x^*$. Tricomi's function is typically complex-valued when its argument z is negative, i.e. when $x \geq x^*$ in the case of interest. Seeking a real-valued solution of $m_\delta(x)$ over the entire domain, an alternative independent pair of solutions to (2.3.6) is therefore selected for $x \geq x^*$, specifically $M(a, b; z)$ and $e^z U(b - a, b; -z)$, (13.1.12) and (13.1.18) of Abramowitz and Stegun (1972), respectively.

In order to determine the constants A_1 and A_2, consider the boundary conditions for $m_\delta(x)$ at x^* and infinity. Applying Kummer's Transformation: $M(a, b; z) = e^z M(b-a, b; -z)$, (2.3.7) can be written such that

$$m_\delta(x) = e^{y(x)} \left[A_1 M\left(1 - \frac{\lambda}{r}, 1 - \frac{\lambda+\delta}{r}; -y(x)\right) + A_2 U\left(1 - \frac{\lambda}{r}, 1 - \frac{\lambda+\delta}{r}; -y(x)\right)\right], \tag{2.3.8}$$

for $x \geq x^*$. For $z \to \infty$, it is well-known that

$$M(a, b; z) = \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \left[1 + O\left(|z|^{-1}\right)\right] \tag{2.3.9}$$

and

$$U(a, b; z) = z^{-a} \left[1 + O\left(|z|^{-1}\right)\right].$$

Asymptotic behaviours of the first and second terms of (2.3.8) as $y(x) \to -\infty$ are therefore given by

$$\frac{\Gamma\left(1 - \frac{\lambda+\delta}{r}\right)}{\Gamma\left(1 - \frac{\lambda}{r}\right)} (-y(x))^{\frac{\delta}{r}} \left(1 + O\left(|-y(x)|^{-1}\right)\right) \tag{2.3.10}$$

and

$$e^{y(x)} (-y(x))^{\frac{\lambda}{r}-1} \left(1 + O\left(|-y(x)|^{-1}\right)\right), \tag{2.3.11}$$

respectively. See, for example, (13.1.27), (13.1.4) and (13.1.8) of Abramowitz and Stegun (1972) for Kummer's transformation and the asymptotic behaviours of the Kummer and Tricomi functions, respectively. For $x \to \infty$, (2.3.10) is unbounded, while (2.3.11) tends to zero. The boundary condition $\lim_{x\to\infty} m_\delta(x) = 0$, by definition of $m_\delta(x)$ in (2.3.1), thus implies that $A_1 = 0$.

Letting $x = x^*$ in (2.3.4) and (2.3.7) yields

$$\frac{\lambda}{(\lambda+\delta)} = A_2 U\left(1 - \frac{\lambda}{r}, 1 - \frac{\lambda+\delta}{r}; 0\right).$$

Hence, $A_2 = \frac{\lambda}{(\lambda+\delta)U\left(1-\frac{\lambda}{r},1-\frac{\lambda+\delta}{r};0\right)}$ and the Laplace transform of the trapping time for $x \geq x^*$ is given by (2.3.2), as required. □

Remark 2.3.1. Figure 2.1a shows that the Laplace transform of the trapping time (2.3.2) approaches the trapping probability as δ tends to zero. This is clear, since by definition of $m_\delta(x)$,

$$\lim_{\delta \downarrow 0} m_\delta(x) = \mathbb{P}(\tau_x < \infty) \equiv \psi(x).$$

As $\delta \to 0$, (2.3.2) yields

$$\psi(x) = \frac{1}{U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda}{r}; 0\right)} e^{y(x)} U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda}{r}; -y(x)\right). \tag{2.3.12}$$

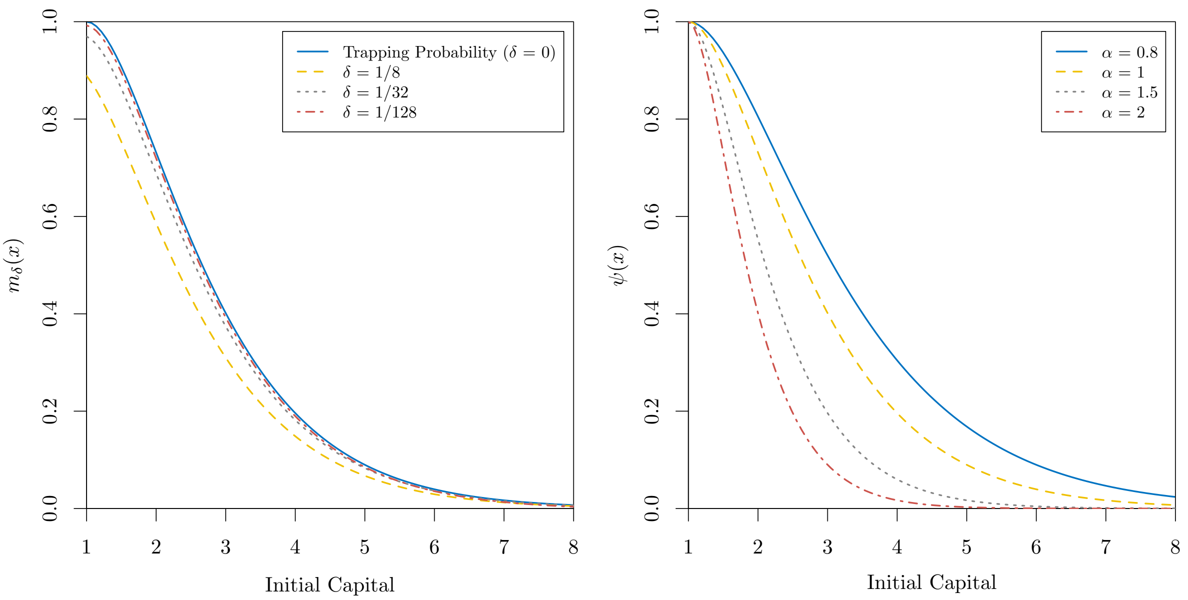
The trapping probability in (2.3.12) can be further simplified through application of the upper incomplete gamma function $\Gamma(a;z) = \int_z^\infty e^{-t}t^{a-1}dt$. Applying the relation

$$\Gamma(a;z) = e^{-z}U(1-a, 1-a; z),$$

(see, for example, (6.5.3) of Abramowitz and Stegun (1972)) and since $\Gamma(a;0) = \Gamma(a)$ for $\text{Re}(a) > 0$, it holds that

$$\psi(x) = \frac{\Gamma\left(\frac{\lambda}{r}; -y(x)\right)}{\Gamma\left(\frac{\lambda}{r}\right)}. \tag{2.3.13}$$

Figure 2.1b presents the trapping probability $\psi(x)$ for the stochastic capital process X_t. Increasing the value of the exponential parameter α, which describes the size of capital losses, reduces the trapping probability for all households, since losses are more likely to take values close to zero and so will have a lesser impact on household capital.



(a) **(b)**

Figure 2.1: (a) Laplace transform $m_\delta(x)$ of the trapping time when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^* = 1$ for $\delta = 0, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}$ (b) Trapping probability $\psi(x)$ when $Z_i \sim Exp(\alpha)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^* = 1$ for $\alpha = 0.8, 1, 1.5, 2$.

Remark 2.3.2. As an application of the Laplace transform of the trapping time, one quantity of interest is the expected trapping time, i.e. the expected time at which a household will fall into the area of poverty. Reducing a household's trapping probability is central to poverty alleviation. However, knowledge of the time at which a household is expected to fall below the poverty line would allow insurers and governments to better prepare for the potential need to lift households out of poverty. It also provides an alternative comparative measure for the performance analysis of different schemes, helping to inform insurance product design and financial education for consumers. For example, a household with a low expected trapping time may be encouraged to adopt certain risk mitigating behaviours to reduce the impact of shock events and hence the likelihood of falling below the poverty line.

The expected trapping time can be obtained through the derivative of $m_\delta(x)$, such that

$$\mathbb{E}\left[\tau_x; \tau_x < \infty\right] = -\left.\frac{d}{d\delta} m_\delta(x)\right|_{\delta=0},$$

where $\mathbb{E}\left[\tau_x; \tau_x < \infty\right]$ is analogous to $\mathbb{E}\left[\tau_x \mathbb{1}_{\{\tau_x<\infty\}}\right]$.

Corollary 2.3.1. The expected trapping time under the household capital model proposed in Definition 2.2.1 with initial capital $x \geq x^*$, capital growth rate r, loss intensity $\lambda > 0$ and exponentially distributed capital losses with parameter $\alpha > 0$ is given by

$$\begin{aligned}\mathbb{E}\left[\tau_x; \tau_x < \infty\right] =& \frac{\Gamma\left(\frac{\lambda}{r}; -y(x)\right)}{\lambda U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda}{r}; 0\right)} - \frac{\Gamma\left(\frac{\lambda}{r}; -y(x)\right) U^{(b)}\left(1-\frac{\lambda}{r}, 1-\frac{\lambda}{r}; 0\right)}{r\left[U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda}{r}; 0\right)\right]^2} \\ &+ e^{y(x)} \frac{U^{(b)}\left(1-\frac{\lambda}{r}, 1-\frac{\lambda}{r}; -y(x)\right)}{rU\left(1-\frac{\lambda}{r}, 1-\frac{\lambda}{r}; 0\right)}, \end{aligned} \tag{2.3.14}$$

where $y(x) = -\alpha(x - x^*)$, $U(\cdot)$ is Tricomi's Confluent Hypergeometric Function, as defined in Appendix B, and $U^{(b)}(\cdot)$ its derivative with respect to the second parameter as in (2.3.16).

Proof. Denote

$$U^{(b)}(a, b; z) \equiv \frac{d}{db} U(a, b; z). \tag{2.3.15}$$

A closed form expression of (2.3.15) can be given in terms of series expansions, such that

$$\begin{aligned}U^{(b)}(a, b; z) =& (\eta(a-b+1) - \pi\cot(b\pi))U(a, b; z) \\ &- \frac{\Gamma(b-1)z^{1-b}\log(z)}{\Gamma(a)} M(a-b+1, 2-b; z) \\ &- \frac{\Gamma(b-1)z^{1-b}}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(a-b+1)_k(\eta(a-b+k+1) - \eta(2-b+k))z^k}{(2-b)_k k!} \\ &- \frac{\Gamma(1-b)}{\Gamma(a-b+1)} \sum_{k=0}^{\infty} \frac{\eta(b+k)(a)_k z^k}{(b)_k k!}, \qquad b \notin \mathbb{Z}, \end{aligned} \tag{2.3.16}$$

where $\eta(z) = \frac{d\ln[\Gamma(z)]}{dz} = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function, see, for example, (6.3.1) of Abramowitz and Stegun (1972). Taking the derivative of (2.3.12) by the standard chain rule and applying (2.3.16) then gives (2.3.14), as required. □

Note that, the conditional expected trapping time given that trapping occurs is given by

$$\mathbb{E}\left[\tau_x | \tau_x < \infty\right] = \frac{\mathbb{E}\left[\tau_x ; \tau_x < \infty\right]}{\psi(x)},$$

see, for example, (4.37) of Gerber and Shiu (1998). In line with intuition, the expected trapping time is an increasing function of both the capital growth rate r and initial capital x. A number of examples of expected trapping times for varying r are displayed in Figure 2.2.

Remark 2.3.3. The ruin probability for the insurance model (2.2.8) given by

$$\xi(u) = \mathbb{P}(U_t < 0 \text{ for some } t > 0 \mid U_0 = u),$$

is found by Sundt and Teugels (1995) to satisfy the Integro-Differential Equation (IDE)

$$(\nu u + p)\xi'(u) - \lambda\xi(u) + \lambda \int_0^u \xi(u - z)\, dG_Z(z) + \lambda(1 - G_Z(u)) = 0, \qquad u \geq 0. \qquad (2.3.17)$$

When $p = 0$, (2.3.17) coincides with the special case of (2.3.3) when $x^* = 0$, $w(x_1, x_2) = 1$ and $\delta = 0$. Thus, the household trapping time can be thought of as the insurer's ruin time. Indeed, the ruin probability in the case of exponential claims when $p = 0$, as shown in Section 6 of Sundt and Teugels (1995), is exactly the trapping probability (2.3.13) when $x^* = 0$.

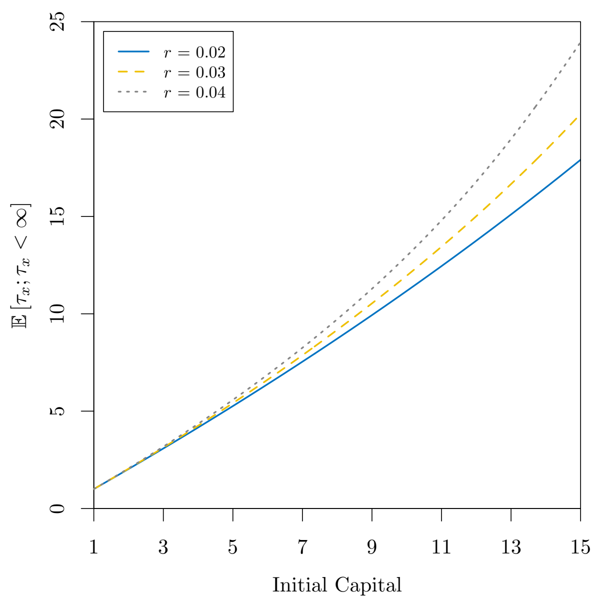


Figure 2.2: Expected trapping time when $Z_i \sim Exp(1)$, $\lambda = 1$ and $x^* = 1$ for $r = 0.02, 0.03, 0.04$.

2.4 Introducing microinsurance

As in Kovacevic and Pflug (2011), in this section, households are assumed to have the option of enrolling in a microinsurance scheme that covers a proportion of the capital losses they encounter. The scheme has proportionality factor $1-\kappa$, where $\kappa \in (0,1]$, such that $100 \cdot (1-\kappa)$ percent of any loss is covered by the microinsurance provider. The premium rate paid by households, calculated according to the expected value principle, is given by

$$\pi(\kappa, \theta) = (1+\theta) \cdot (1-\kappa) \cdot \lambda \cdot \mathbb{E}\left[Z_i\right], \tag{2.4.1}$$

where θ is the loading factor specified by the insurer. The expected value principle is popular due to its simplicity and transparency. When $\theta = 0$, $\pi(\kappa, \theta)$ can be considered as the pure risk premium (see, for example, Albrecher et al. (2017)).

The stochastic capital process of a household covered by a microinsurance policy is denoted by $X_t^{(\kappa)}$. All variables and parameters relating to the original uninsured (Section 2.3) and the insured processes are distinguished through use of the superscript (κ) in the latter case. The basic model parameters are assumed to be unchanged by the introduction of microinsurance coverage (parameters a, b and c introduced in Section 2.2). As mentioned in Chapter 1, Kovacevic and Semmler (2021) derive the retention rate process that maximises the expected discounted capital, by allowing adjustments in the retention rate of the policyholder after each capital loss throughout the lifetime of the insurance contract. Variable coverage is also considered by Janzen et al. (2021). In the study presented in this chapter, households are assumed to select a fixed retention rate.

Since premiums are paid out of household income, the capital growth rate r is adjusted such that it reflects the lower rate of income generation resulting from the need for premium payment. The premium rate is restricted to prevent certain poverty, which would occur should it exceed the rate of income generation. As such, $\pi < b$. The capital growth rate of the insured household $r^{(\kappa)} = (1-a) \cdot (b-\pi) \cdot c > 0$ is lower than that of the uninsured household, while the critical capital is increased. Previous work, such as that of Janzen et al. (2021), allow households to select optimal levels of consumption and insurance coverage over time based on asset holdings and the probability distribution of future assets. Here, all households for which microinsurance is affordable enrol in a scheme; that is, households whose rate of income generation is greater than the insurance premium, thus admitting both optimal and suboptimal insurance decisions with respect to the trapping probability. Although this feature aligns with the low levels of financial literacy that characterise the microinsurance environment (Churchill and Matul, 2006), it could be considered as a limitation of the proposed model. However, one of the core objectives of the subsidised schemes introduced in Sections 2.5 and 2.6 is to diminish the adverse effects that arise with suboptimal insurance decisions and as such any limitation is accounted for.

In between jumps, the insured stochastic growth process $X_t^{(\kappa)}$ behaves in the same manner as (2.2.5a) and (2.2.5b), with parameters corresponding to the proportional insurance case of this section. By enrolling in a microinsurance scheme, a household's capital losses are reduced to $Y_i := \kappa \cdot Z_i$. Considering the case in which losses follow an exponential distribution with parameter $\alpha > 0$, the structure of the IDE in (2.3.3) remains the same. However, acquisition of a proportional microinsurance policy changes the parameter of the random loss distribution G_Y. Namely, $Y_i \sim Exp\left(\alpha^{(\kappa)}\right)$ for $\kappa \in (0,1]$, where $\alpha^{(\kappa)} := \frac{\alpha}{\kappa}$.

Following a similar procedure to that in Proposition 2.3.1, the Laplace transform of the trapping time and thus the trapping probability for the insured process is obtained.

Proposition 2.4.1. Consider the capital process of a household enrolled in a microinsurance scheme with proportionality factor $1-\kappa \in (0,1]$. Assume the household has initial capital

$x \geq x^{(\kappa)*}$, capital growth rate $r^{(\kappa)}$, loss intensity $\lambda > 0$ and exponentially distributed capital losses with parameter $\alpha^{(\kappa)} > 0$. The Laplace transform of the trapping time is given by

$$m_\delta^{(\kappa)}(x) = \frac{\lambda}{(\lambda+\delta) U\left(1 - \frac{\lambda}{r^{(\kappa)}}, 1 - \frac{\lambda+\delta}{r^{(\kappa)}}; 0\right)} e^{y^{(\kappa)}(x)} U\left(1 - \frac{\lambda}{r^{(\kappa)}}, 1 - \frac{\lambda+\delta}{r^{(\kappa)}}; -y^{(\kappa)}(x)\right),$$

where $\delta \geq 0$ is the force of interest and $y^{(\kappa)}(x) = -\alpha^{(\kappa)}\left(x - x^{(\kappa)*}\right)$.

Proof. Proof follows that of Proposition 2.3.1. □

Figure 2.1a presents the Laplace transform $m_\delta^{(\kappa)}(x)$ for varying values of δ. As mentioned in Section 4.4, as $\delta \to 0$, the Laplace transform $m_\delta^{(\kappa)}(x)$ converges to the trapping probability $\psi^{(\kappa)}(x)$.

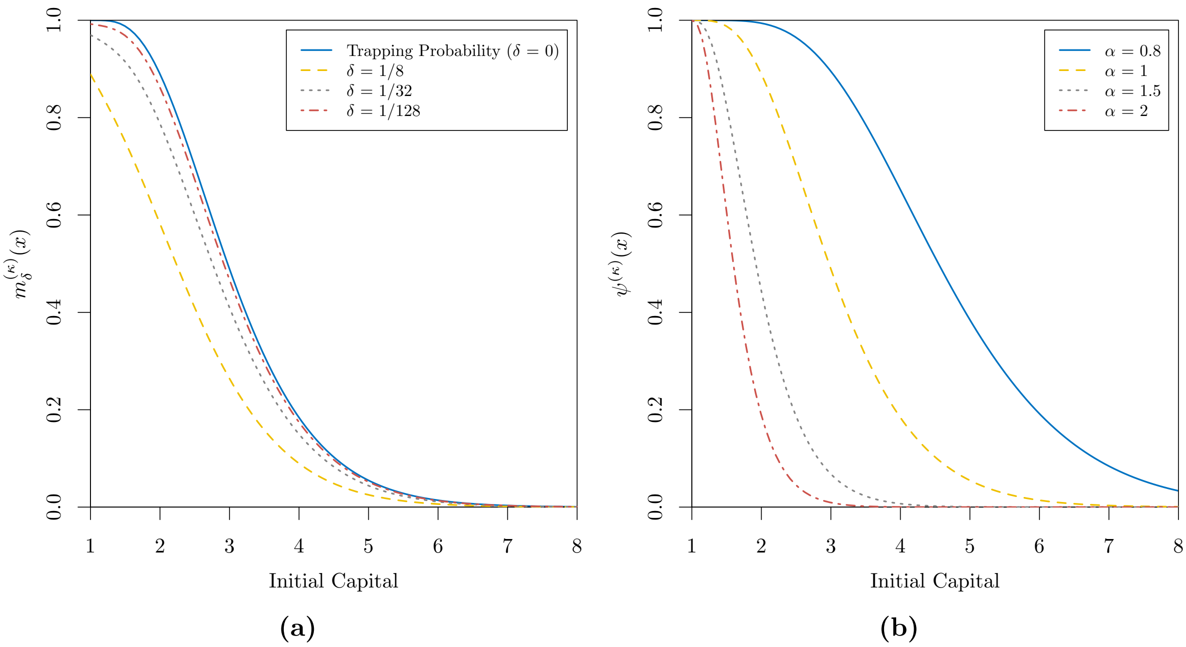


Figure 2.3: (a) Laplace transform $m_\delta^{(\kappa)}(x)$ of the trapping time when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^{(\kappa)*} = 1$, $\kappa = 0.5$ and $\theta = 0.5$ for $\delta = 0, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}$ (b) Trapping probability $\psi^{(\kappa)}(x)$ when $Z_i \sim Exp(\alpha)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^{(\kappa)*} = 1$, $\kappa = 0.5$ and $\theta = 0.5$ for $\alpha = 0.8, 1, 1.5, 2$.

Remark 2.4.1. The trapping probability of the insured process $\psi^{(\kappa)}(x)$, displayed in Figure 2.3b, is given by

$$\psi^{(\kappa)}(x) = \frac{\Gamma\left(\frac{\lambda}{r^{(\kappa)}}; -y^{(\kappa)}(x)\right)}{\Gamma\left(\frac{\lambda}{r^{(\kappa)}}\right)}. \tag{2.4.2}$$

As previously, increasing the value of the exponential parameter α reduces the trapping probability. Equivalently, note that as the proportionality factor $\kappa \to 0$, the parameter $\alpha^{(\kappa)} := \frac{\alpha}{\kappa}$ of the distribution of capital losses Y_i of the insured capital process increases, leading households to experience low impact losses with a higher probability. However, by (2.4.1), this level of coverage induces higher premiums.

Remark 2.4.2. When $\kappa = 0$ the household has full microinsurance coverage, as the microinsurance provider covers the total capital loss experienced by the household. On the other hand, when $\kappa = 1$, no coverage is provided by the insurer, i.e. $X_t = X_t^{(\kappa)}$.

Figure 2.4 presents a comparison between the trapping probabilities of the insured and uninsured processes. As in Kovacevic and Pflug (2011), households with initial capital close to the critical capital (here, the critical capital x^* is fixed at 1), i.e. the most vulnerable households, do not receive real benefits from enrolling in a microinsurance scheme. Although subscribing to a proportional scheme reduces capital losses, premium payments appear to make households more prone to falling into the area of poverty. The intersection point of the two probabilities in Figure 2.4 corresponds to the boundary between households that benefit from the uptake of microinsurance and those who are adversely affected.

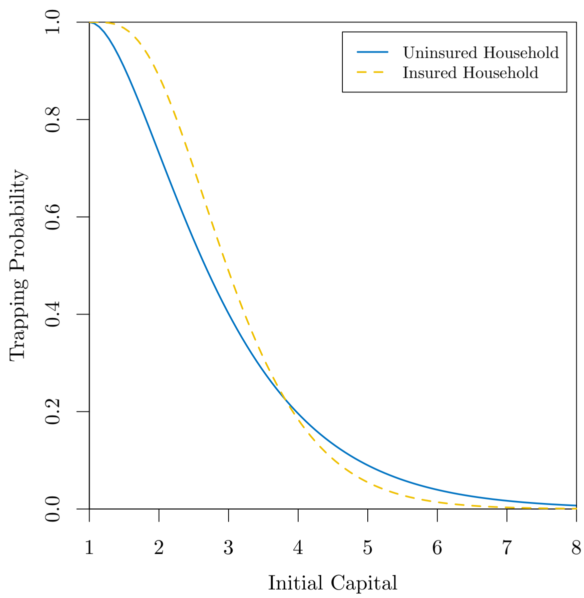


Figure 2.4: Trapping probabilities for the uninsured and insured capital processes for $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $\kappa = 0.5$, $\theta = 0.5$ and $x^* = x^{(\kappa)*} = 1$.

2.5 Microinsurance with subsidised constant premiums

The preliminary results suggest that microinsurance alone is not enough to reduce the likelihood of impoverishment for those closest to the poverty line, and so additional aid is required. In this section, the cost-effectiveness of government subsidised premiums is studied for governments subsidising an amount $\beta = \pi - \pi^*$, such that $\pi^* \geq 0$ is the premium after subsidisation. Lower values of π^* therefore correspond to higher levels of government support. When $\pi^* = 0$ the premium is fully subsidised, when $\pi = \pi^*$ households pay premiums in full with no subsidisation. Previous work largely considers fixed subsidies with limited flexibility. Kovacevic and Pflug (2011) restrict subsidisation to the loading factor, while Janzen et al. (2021) adopt a self-targeted subsidy strategy with subsidies provided uniformly to poor households with demand for unsubsidised insurance. The schemes proposed in Sections 2.5 and 2.6 allow vulnerable non-poor households to benefit from greater subsidisation in addition to minimising the associated governmental costs by optimising subsidy levels.

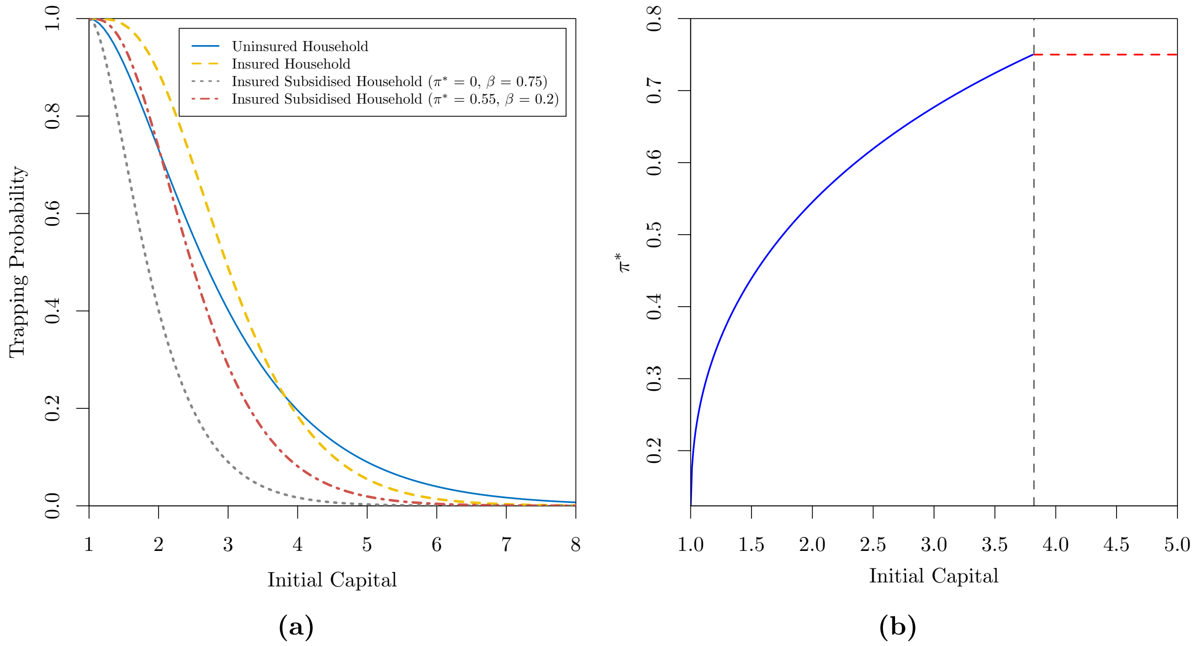


Figure 2.5: (a) Trapping probabilities for the uninsured, insured and insured subsidised capital processes when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^* = x^{(\kappa)*} = x^{\pi(\kappa,\theta)*} = 1$, $\kappa = 0.5$, $\theta = 0.5$ and $\pi = 0.75$ for $\pi^* = 0, 0.55$ (b) Optimal π^* for varying initial capital when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^{\pi(\kappa,\theta)*} = 1$, $\kappa = 0.5$, $\theta = 0.5$ and $\pi = 0.75$.

Government subsidisation of microinsurance should enhance household benefits of enrolling in microinsurance schemes. However, it is also important to gauge the cost-effectiveness of subsidy provision. Although households with capital close to the critical capital will require additional aid, government support is not necessarily essential for more privileged households. In determining the level of subsidy it is therefore intuitively assumed that governments would like to optimise subsidy provision. Since all non-zero values of π^* below the optimal value induce a trapping probability lower than that of the uninsured process through a reduction in premium, one approach to determining the optimal subsidy for households that require government aid is to solve

$$\psi^{\pi^*(\kappa,\theta)}(x) = \psi(x),$$

where $\psi^{\pi^*(\kappa,\theta)}(x)$ and $\psi(x)$ denote the trapping probabilities of the insured subsidised and uninsured capital processes, respectively. The behaviours of these trapping probabilities can be seen in Figure 2.5a. While government support is not needed by the most privileged households, since insurance without subsidies lowers their trapping probability below the uninsured, the most poor require further support. Figure 2.5b illustrates the optimal value of π^* for varying initial capital, verifying that, from the point at which the yellow dashed line (insured household) intersects the blue solid line (uninsured household) in Figure 2.5a, payment of the entire premium is affordable for the most privileged households, with the optimal premium remaining constant at $\pi^* = \pi = 0.75$ after this point (red dashed line in Figure 2.5b).

2.5.1 Cost of social protection

The governmental cost-effectiveness of the provision of microinsurance premium subsidies is now assessed. Let $\tau_x^{\pi^*(\kappa,\theta)}$ denote the trapping time of a household covered by a subsidised microinsurance policy. Let $\delta \geq 0$ be the force of interest and let S denote the present value of all subsidies provided by government until the trapping time, such that

$$S = \beta \int_0^{\tau_x^{\pi^*(\kappa,\theta)}} e^{-\delta t} dt.$$

Further assume that governments provide subsidies according to the strategy introduced in this section, i.e. the government subsidises $\beta = \pi - \pi^*$.

For $x \geq x^{\pi^*(\kappa,\theta)*}$, where $x^{\pi^*(\kappa,\theta)*}$ denotes the critical capital of the insured subsidised process, let $V^{\pi^*(\kappa,\theta)}(x)$ be the expected discounted premium subsidies provided by the government to a household with initial capital x until the trapping time, that is,

$$V^{\pi^*(\kappa,\theta)}(x) = \mathbb{E}\left[S \mid X_0^{\pi^*(\kappa,\theta)} = x\right].$$

Then, the following proposition holds:

Proposition 2.5.1. Consider a household enrolled in a microinsurance scheme with subsidised constant premiums in which the government subsidises an amount $\beta = \pi - \pi^*$, where $\pi \geq \pi^* \geq 0$, with proportionality factor $1 - \kappa \in (0, 1]$. Assume an initial capital $x \geq x^{\pi^*(\kappa,\theta)*}$, capital growth rate $r^{\pi^*(\kappa,\theta)}$, loss intensity $\lambda > 0$ and exponentially distributed capital losses with parameter $\alpha^{(\kappa)} > 0$. The expected discounted premium subsidies provided by the government to the household until the trapping time is given by

$$V^{\pi^*(\kappa,\theta)}(x) = \frac{\beta}{\delta}\left[1 - m_\delta^{\pi^*(\kappa,\theta)}(x)\right], \tag{2.5.1}$$

where $m_\delta^{\pi^*(\kappa,\theta)}(x)$ is the Laplace transform of the trapping time with rate $r^{\pi^*(\kappa,\theta)}$ and critical capital $x^{\pi^*(\kappa,\theta)*}$.

Proof. Since

$$S = \frac{\beta}{\delta}\left[1 - e^{-\delta \tau_x^{\pi^*(\kappa,\theta)}}\right]$$

and $\mathbb{E}[e^{-\delta \tau_x^{\pi^*(\kappa,\theta)}}]$ is exactly the Laplace transform of the trapping time of the insured subsidised process with capital growth $r^{\pi^*(\kappa,\theta)}$ and critical capital $x^{\pi^*(\kappa,\theta)*}$, (2.5.1) holds. □

The governmental cost of social protection is now formally defined.

Definition 2.5.1. Consider the expected discounted penalty function at trapping (2.3.1) for a household enrolled in a subsidised microinsurance scheme with initial capital x. Let $w(x_1, x_2) = x_2 + M^{\pi^*(\kappa,\theta)} - x^{\pi^*(\kappa,\theta)*}$ be the penalty function, where $M^{\pi^*(\kappa,\theta)} \geq x^{\pi^*(\kappa,\theta)*}$. Here, x_2 accounts for the cost of lifting households up to the critical level (or poverty line) $x^{\pi^*(\kappa,\theta)*}$, while $M^{\pi^*(\kappa,\theta)} - x^{\pi^*(\kappa,\theta)*}$ is a constant representing the cost of lifting households away from the area of poverty. The expected discounted penalty function at trapping $m_{\delta,w}^{\pi^*(\kappa,\theta)}(x)$ is therefore the expected present value of the capital deficit at trapping plus the cost $M^{\pi^*(\kappa,\theta)} - x^{\pi^*(\kappa,\theta)*}$ due at the trapping time. The governmental cost of social protection is thus defined as the expected discounted premium subsidies (2.5.1), plus the expected present value of the capital deficit and $M^{\pi^*(\kappa,\theta)} - x^{\pi^*(\kappa,\theta)*}$.

Remark 2.5.1. The government does not provide subsidies for uninsured households. The uninsured expected discounted penalty function at trapping is therefore given by $m_{\delta,w}(x)$, where $w(x_1, x_2) = x_2 + M - x^*$. The choice of this penalty function is based on the idea that in order to lift a household out of poverty the government incurs a cost equivalent to the household's capital deficit at the moment they fall into poverty, plus a fixed cost $M - x^*$ that lifts the household away from the area of poverty, reducing their probability of falling again. This approach differs to those presented in previous research, where the cost of social protection considers only the present value of the transfers required to close the poverty gap (Ikegami et al., 2016; Janzen et al., 2021). Through this alternative specification, the likelihood of re-incurring social protection costs for the same household is reduced. The constant M could be precisely defined in such a way that the government ensures with some probability that households will not fall again into the area of poverty. For instance, consider ϵ to be the most admissible trapping probability for an uninsured household. Then,

$$M := \inf \{x \geq x^* : \psi(x) < \epsilon\} \tag{2.5.2}$$

denotes the minimum initial capital (MIC) required to ensure a trapping probability of less than ϵ. This measure has also been studied from the point of view of an insurance company, where ϵ represents the most admissible probability that the company becomes insolvent (Sattayatham et al., 2013; Constantinescu et al., 2019). In this way, the government is able to define M such that a household's probability of re-entering the area of poverty is less than $\epsilon \in (0, 1)$. Clearly, higher values of M will increase the certainty that households will not return to poverty. Further note that M will differ between uninsured, insured and insured subsidised households due to their distinct trapping probabilities. Figure 2.6 displays the cost incurred by the government at the trapping time when employing the penalty function $w(x_1, x_2) = x_2 + M - x^*$ for an uninsured household.

Remark 2.5.2. Selection of an appropriate force of interest $\delta \geq 0$ is managed by the government and determined by interest rates in the market. For lower force of interest the government discounts future subsidies more heavily, while for higher interest future subsidies tend to zero.

When losses are exponentially distributed with parameter $\alpha^{(\kappa)} > 0$ it is possible to obtain a closed form expression for the cost of social protection. Given the derivation of $V^{\pi^*(\kappa,\theta)}(x)$ in (2.5.1), Proposition 2.5.2 gives an expression for $m_{\delta,w}^{\pi^*(\kappa,\theta)}(x)$.

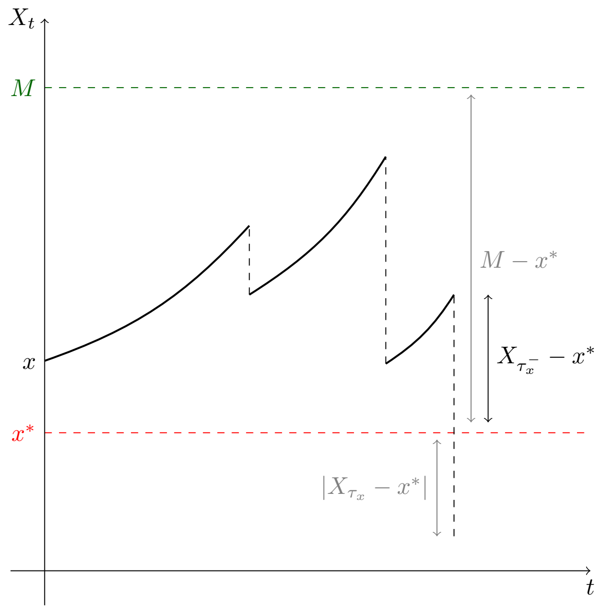


Figure 2.6: The cost incurred by the government at the trapping time is given by $| X_{\tau_x} - x^* |$, the capital deficit at the trapping time, plus $M - x^*$, the cost to lift households away from the area of poverty.

Proposition 2.5.2. Consider a household enrolled in a microinsurance scheme with proportionality factor $1 - \kappa \in (0, 1]$ and subsidised constant premiums, in which the government subsidises an amount $\beta = \pi - \pi^*$, where $\pi \geq \pi^* \geq 0$. Assume an initial capital $x \geq x^{\pi^*(\kappa,\theta)*}$, capital growth rate $r^{\pi^*(\kappa,\theta)}$, loss intensity $\lambda > 0$ and exponentially distributed capital losses with parameter $\alpha^{(\kappa)} > 0$. Let $M^{\pi^*(\kappa,\theta)} - x^{\pi^*(\kappa,\theta)*}$, where $M^{\pi^*(\kappa,\theta)} \geq x^{\pi^*(\kappa,\theta)*}$, be the cost to lift such a household away from the area of poverty. Then, the expected discounted cost incurred by the government at the trapping time is given by

$$m_{\delta,w}^{\pi^*(\kappa,\theta)}(x) = \left[\frac{1}{\alpha^{(\kappa)}} + M^{\pi^*(\kappa,\theta)} - x^{\pi^*(\kappa,\theta)*}\right] m_{\delta}^{\pi^*(\kappa,\theta)}(x), \tag{2.5.3}$$

where $m_{\delta}^{\pi^*(\kappa,\theta)}(x)$ is the Laplace transform of the trapping time with rate $r^{\pi^*(\kappa,\theta)}$ and critical capital $x^{\pi^*(\kappa,\theta)*}$, and $\delta \geq 0$ is the force of interest.

Proof. Following a similar approach to that in the proof of Proposition 2.3.1, consider the integral

$$\begin{aligned} A(x) &:= \int_{x-x^{\pi^*(\kappa,\theta)*}}^{\infty} w(x - x^{\pi^*(\kappa,\theta)*}, z - (x - x^{\pi^*(\kappa,\theta)*}))dG_Z(z) \\ &= \int_{x-x^{\pi^*(\kappa,\theta)*}}^{\infty} \left[z - (x - x^{\pi^*(\kappa,\theta)*}) + M^{\pi^*(\kappa,\theta)} - x^{\pi^*(\kappa,\theta)*}\right] \alpha^{(\kappa)} e^{-\alpha^{(\kappa)} z} dz \\ &= \left(\frac{1}{\alpha^{(\kappa)}} + M^{\pi^*(\kappa,\theta)} - x^{\pi^*(\kappa,\theta)*}\right) e^{-\alpha^{(\kappa)}(x - x^{\pi^*(\kappa,\theta)*})}, \end{aligned}$$

which under the assumption $w(x_1, x_2) = x_2 + M^{\pi^*(\kappa,\theta)} - x^{\pi^*(\kappa,\theta)*}$ yields a modified version of the in IDE (2.3.3) given by

$$\begin{aligned}&r^{\pi^*(\kappa,\theta)}(x - x^{\pi^*(\kappa,\theta)*})m_{\delta,w}^{\pi^*(\kappa,\theta)\prime}(x)\\&-(\lambda+\delta)m_{\delta,w}^{\pi^*(\kappa,\theta)}(x) + \lambda\int_0^{x-x^{\pi^*(\kappa,\theta)*}} m_{\delta,w}^{\pi^*(\kappa,\theta)}(x-z)\alpha^{(\kappa)}e^{-\alpha^{(\kappa)}z}dz\\=&-\lambda\left(\frac{1}{\alpha^{(\kappa)}} + M^{\pi^*(\kappa,\theta)} - x^{\pi^*(\kappa,\theta)*}\right)e^{-\alpha^{(\kappa)}(x-x^{\pi^*(\kappa,\theta)*})}\end{aligned} \tag{2.5.4}$$

for $x \geq x^{\pi^*(\kappa,\theta)*}$.

Solving (2.5.4) in the same manner as (2.3.4) gives (2.5.3), as required. □

Remark 2.5.3. Due to the lack-of-memory property of the exponential distribution, the deficit at trapping, given that trapping occurs, is also exponentially distributed. One can easily verify this by specifying the penalty function such that for any fixed u, $w(x_1, x_2) = \mathbb{1}_{\{x_2 \leq u\}}$ and $\delta = 0$. Similar results to that of Proposition 2.5.2 have been obtained for other risk processes (see, for instance, Example 3.2 of Albrecher et al. (2005)).

Figure 2.7 presents the governmental cost of social protection for varying initial capital. In this example, a high force of interest is considered. Moreover, values of M are given by the MIC (2.5.2) corresponding to each capital process. As supposed, high values of δ hand a lower weight to future government subsidies, and high values of M grant greater certainty that a household will not return to the area of poverty once lifted above the critical capital.

Governments do not benefit from subsidising insurance for the most privileged households since they will subsidise premiums indefinitely, almost surely. As such, as also illustrated in Figure 2.5b, it is favourable for governments to remove subsidies for this particular household group, since their cost of social protection, as presented in Figure 2.7 (red dashed-dotted and gray dotted lines for highest values of initial capital), is higher than when uninsured (blue solid line for highest values of initial capital). This is largely due to the fact that governments are still obliged to subsidise a given amount of the premium despite greater initial capital leading to lower trapping probabilities and therefore a reduction in the likelihood of the government need to lift households away from the area of poverty. Governments perceive a lower cost of social protection when subsidising premiums for households with initial capital lying closer to the critical capital x^*. The cost of social protection for households that do not pay the premium in full is lower than for those paying the premium entirely.

Figure 2.7 further shows that under the optimal subsidy scheme, the cost of social protection incurred for the most vulnerable is reduced, such that only the fully subsidised scheme ($\pi^* = 0$) is of lower cost, with a marginal difference observed between the two (blue circular-marked line just above gray dotted line for the most vulnerable). This behaviour aligns with the high probability of the most vulnerable falling into the poverty trap, and the associated need for governments to lift them out of and away from poverty. The cost of social protection for this particular group is therefore mainly constituted by the capital injection $M^{\pi^*(\kappa,\theta)} - x^{\pi^*(\kappa,\theta)*}$, which under the subsidised scheme reaches its minimum when the scheme is fully subsidised. The cost incurred when providing optimal subsidies to more privileged households lies slightly above that of insured households without subsidies (blue circular-marked line above yellow dashed line for more privileged households). Higher initial capitals reduce the weight of the capital injection in the cost of social protection since households are less likely to fall into the area of poverty. Where the injection weight is small, the capital injection needed for an insured household without subsidies is almost equivalent to that of a household receiving optimal support (both frameworks require the same capital injection for $x > 3.821$), however, under the latter scheme, governments are additionally required to subsidise a given amount of the premium. Note that, the cost of social protection for the insured (yellow dashed line), non-optimal insured subsidised (gray dotted and red dashed-dotted lines) for the most vulnerable (initial capital less than $x = 3.037$ when $\pi^* = 0$ and $x = 4.661$ when $\pi^* = 0.55$) and optimally insured subsidised households (blue circular-marked line) is below that of the uninsured, thus highlighting the significance of insurance as a tool for reducing the governmental cost of social protection.

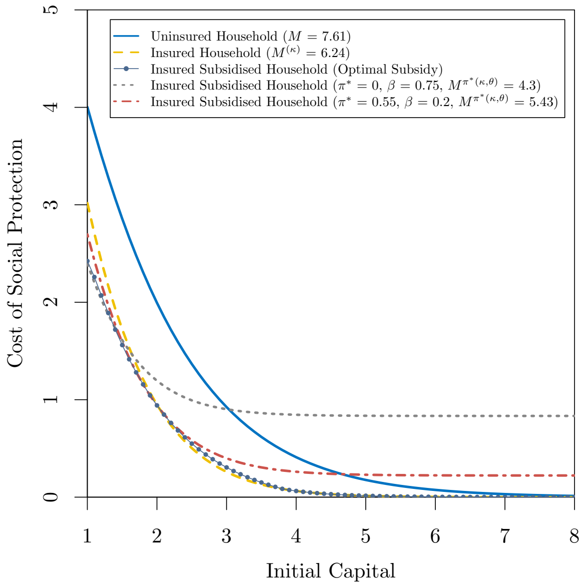


Figure 2.7: Cost of social protection for the uninsured, insured and insured subsidised with $\pi^* = \pi^*_{\texttt{Optimal}}, 0, 0.55$ capital processes when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^* = x^{(\kappa)*} = x^{\pi(\kappa,\theta)*} = 1$, $\kappa = 0.5$, $\theta = 0.5$, $\delta = 0.9$, $\epsilon = 0.01$ and $\pi = 0.75$.

2.6 Microinsurance with subsidised flexible premiums

Microinsurance premiums are typically paid as soon as coverage is purchased. The capital growth of a household could therefore be constrained after joining a scheme, as observed in the results of Sections 2.4 and 2.5. As such, it is important to consider alternative premium payment mechanisms. From the point of view of microinsurance providers, advance premium payments are preferred so that additional income can be generated through investment, naturally leading to lower premium rates. Consumers on the other hand may find it difficult to pay premiums up front. A common problem in low-income populations, research suggests that consumer preference is for payment of smaller installments that are spread over time (Churchill and Matul, 2006). Collecting premiums at a time that is inconvenient for households can be futile, particularly given the liquidity constraints associated with informal and agricultural work. Flexible premium payment mechanisms have been widely adopted by informal funeral insurers in South Africa for instance, with policyholders paying premiums whenever financially feasible, rather than at fixed times (Roth, 2000). As mentioned in Chapter 1, similar alternative insurance designs in which premium payments are delayed until the insured's income is realised and any indemnities are paid have been studied. Under such strategies, insurance take-up increases, due to the relaxing of liquidity constraints and concerns regarding insurer default (Liu and Myers, 2016).

In this section, an alternative microinsurance subsidy scheme with flexible premium payments is introduced. The capital process of a household enrolled in the alternative scheme is denoted by $X_t^{(\mathcal{A})}$. As in Section 2.4, variables and parameters relating to the uninsured, insured and alternative insured processes are distinguished by their respective superscripts, with the superscript $(\mathcal{A})$ applied in the latter case. Under this alternative microinsurance subsidy scheme, households pay premiums only when their capital is above some capital barrier $B \geq x^{(\mathcal{A})*}$, with the premium otherwise paid entirely by the government, through full subsidisation. This form of premium collection could motivate households to maintain a level of capital below the barrier B in order to avoid premium payments. For the purpose of this study, it is therefore assumed that households always pursue capital growth. The aim is to assess how this alternative subsidy scheme could help households to reduce their probability of falling into the area of poverty, while also measuring the cost-effectiveness of the strategy from the governmental perspective.

The intangibility of microinsurance makes it difficult to attract potential consumers. Most policyholders will never experience a claim and so cannot perceive the real value of purchasing coverage, paying more into the scheme through premium payments than the monetary benefit they receive from being covered. It is only when claims are settled that microinsurance becomes tangible. The alternative microinsurance subsidy scheme described here could increase client value, since, for example, households with capital below the barrier B could submit a claim, receive a payout and thus perceive the value of microinsurance when a loss is suffered, regardless of whether they have ever paid a single premium. Further strategies for increasing microinsurance client value include bundling microinsurance with other products and introducing Value Added Services (VAS), which for health schemes, for example, include services such as telephone hotlines for consultation with doctors or remote diagnosis services, offered to clients outside of the microinsurance contract (Madhur and Saha, 2019).

Proposition 2.6.1. Consider a household enrolled in an alternative microinsurance scheme with subsidised flexible premiums, capital barrier $B \geq x^{(\mathcal{A})*}$ and proportionality factor $1-\kappa \in (0,1]$. Assume an initial capital $x \geq x^{(\mathcal{A})*}$, capital growth rates $r^{(\kappa)}$ and r above and below the barrier, respectively, loss intensity $\lambda > 0$ and exponentially distributed capital losses with parameter $\alpha^{(\kappa)} > 0$. The Laplace transform of the trapping time is given by

$$m_\delta^{(\mathcal{A})}(x) = \begin{cases} C_1 M\left(-\frac{\delta}{r}, 1-\frac{\lambda+\delta}{r}; y^{(\mathcal{A})}(x)\right) \\ \quad + C_2 e^{y^{(\mathcal{A})}(x)} U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda+\delta}{r}; -y^{(\mathcal{A})}(x)\right), & \text{if } x^{(\mathcal{A})*} \leq x \leq B \quad (2.6.1a) \\ C_3 M\left(-\frac{\delta}{r^{(\kappa)}}, 1-\frac{\lambda+\delta}{r^{(\kappa)}}; y^{(\mathcal{A})}(x)\right) \\ \quad + C_4 e^{y^{(\mathcal{A})}(x)} U\left(1-\frac{\lambda}{r^{(\kappa)}}, 1-\frac{\lambda+\delta}{r^{(\kappa)}}; -y^{(\mathcal{A})}(x)\right), & \text{if } x \geq B, \quad (2.6.1b) \end{cases}$$

where $y^{(\mathcal{A})}(x) = -\alpha^{(\kappa)}(x - x^{(\mathcal{A})*})$ and constants C_i for $i = 1, 2, 3, 4$ are given by (2.6.4), (2.6.6), (2.6.3) and (2.6.5), respectively.

Proof. Under the alternative microinsurance subsidy scheme, the Laplace transform of the trapping time satisfies the following differential equations:

$$\begin{cases} -\dfrac{(x - x^{(\mathcal{A})*})}{\alpha^{(\kappa)}} m_\delta^{(\mathcal{A})''}(x) \\ \quad + \Big[\dfrac{(\lambda + \delta - r)}{\alpha^{(\kappa)} r} - (x - x^{(\mathcal{A})*})\Big] m_\delta^{(\mathcal{A})'}(x) + \dfrac{\delta}{r} m_\delta^{(\mathcal{A})}(x), & \text{if } x^{(\mathcal{A})*} \leq x \leq B \quad (2.6.2\text{a}) \\ -\dfrac{(x - x^{(\mathcal{A})*})}{\alpha^{(\kappa)}} m_\delta^{(\mathcal{A})''}(x) \\ \quad + \Big[\dfrac{(\lambda + \delta - r^{(\kappa)})}{\alpha^{(\kappa)} r^{(\kappa)}} - (x - x^{(\mathcal{A})*})\Big] m_\delta^{(\mathcal{A})'}(x) + \dfrac{\delta}{r^{(\kappa)}} m_\delta^{(\mathcal{A})}(x), & \text{if } x \geq B \quad (2.6.2\text{b}) \end{cases}$$

As in Proposition 2.3.1, use of the change of variable $y^{(\mathcal{A})} := y^{(\mathcal{A})}(x) = -\alpha^{(\kappa)}(x - x^{(\mathcal{A})*})$ leads to Kummer's confluent hypergeometric equation, and so (2.6.1a) and (2.6.1b) are obtained for arbitrary constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$. Under the boundary condition $\lim_{x\to\infty} m_\delta^{(\mathcal{A})}(x) = 0$ and given the asymptotic behaviour of the Kummer function $M(a, c; z)$ in (2.3.9),

$$C_3 = 0. \tag{2.6.3}$$

Then, since $m_\delta^{(\mathcal{A})}(x^{(\mathcal{A})*}) = \frac{\lambda}{\lambda+\delta}$,

$$C_1 = \frac{\lambda}{\lambda + \delta} - C_2 U\left(1 - \frac{\lambda}{r}, 1 - \frac{\lambda + \delta}{r}; 0\right). \tag{2.6.4}$$

Due to the continuity of the functions $m_\delta^{(\mathcal{A})}(x)$ and $m_\delta^{(\mathcal{A})'}(x)$ at $x = B$ and the differential properties of the confluent hypergeometric functions:

$$\frac{d}{dz} M(a, b; z) = \frac{a}{b} M(a + 1, b + 1; z),$$

$$\frac{d}{dz} U(a, b; z) = -aU(a + 1, b + 1; z),$$

upon simplification,

$$\begin{aligned} C_4 =& \Bigg(\left[\frac{\lambda}{\lambda + \delta} - C_2 U\left(1 - \frac{\lambda}{r}, 1 - \frac{\lambda + \delta}{r}; 0\right)\right] e^{-y^{(\mathcal{A})}(B)} M\left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}; y^{(\mathcal{A})}(B)\right) \\ &\times U\left(1 - \frac{\lambda}{r^{(\kappa)}}, 1 - \frac{\lambda + \delta}{r^{(\kappa)}}; -y^{(\mathcal{A})}(B)\right)^{-1} \Bigg) \\ &+ C_2 U\left(1 - \frac{\lambda}{r}, 1 - \frac{\lambda + \delta}{r}; -y^{(\mathcal{A})}(B)\right) U\left(1 - \frac{\lambda}{r^{(\kappa)}}, 1 - \frac{\lambda + \delta}{r^{(\kappa)}}; -y^{(\mathcal{A})}(B)\right)^{-1} \end{aligned} \tag{2.6.5}$$

and

$$\begin{aligned} C_2 =& \frac{\lambda K^{-1}}{\lambda + \delta} \Big[\frac{\delta \alpha^{(\kappa)}}{(r - \lambda - \delta)} M\left(1 - \frac{\delta}{r}, 2 - \frac{\lambda + \delta}{r}; y^{(\mathcal{A})}(B)\right) \\ &+ M\left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}; y^{(\mathcal{A})}(B)\right) \left(\alpha^{(\kappa)} - D\right) \Big], \end{aligned} \tag{2.6.6}$$

where

$$D := \frac{\alpha^{(\kappa)}\left(\frac{\lambda}{r^{(\kappa)}}-1\right)U\left(2-\frac{\lambda}{r^{(\kappa)}},2-\frac{\lambda+\delta}{r^{(\kappa)}};-y^{(\mathcal{A})}(B)\right)}{U\left(1-\frac{\lambda}{r^{(\kappa)}},1-\frac{\lambda+\delta}{r^{(\kappa)}};-y^{(\mathcal{A})}(B)\right)} \tag{2.6.7}$$

and

$$\begin{aligned} K :&= M\left(-\frac{\delta}{r},1-\frac{\lambda+\delta}{r};y^{(\mathcal{A})}(B)\right)U\left(1-\frac{\lambda}{r},1-\frac{\lambda+\delta}{r};0\right)\left(\alpha^{(\kappa)}-D\right)\\ &+ De^{y^{(\mathcal{A})}(B)}U\left(1-\frac{\lambda}{r},1-\frac{\lambda+\delta}{r};-y^{(\mathcal{A})}(B)\right)\\ &+\frac{\delta\alpha^{(\kappa)}}{(r-\lambda-\delta)}M\left(1-\frac{\delta}{r},2-\frac{\lambda+\delta}{r};y^{(\mathcal{A})}(B)\right)U\left(1-\frac{\lambda}{r},1-\frac{\lambda+\delta}{r};0\right)\\ &-\alpha^{(\kappa)}e^{y^{(\mathcal{A})}(B)}\left(\frac{\lambda}{r}-1\right)U\left(2-\frac{\lambda}{r},2-\frac{\lambda+\delta}{r};-y^{(\mathcal{A})}(B)\right). \end{aligned} \tag{2.6.8}$$

□

Remark 2.6.1. The trapping probability $\psi^{(\mathcal{A})}(x)$ for the alternative microinsurance subsidy scheme is given by

$$\begin{cases} 1-\dfrac{\Gamma\left(\frac{\lambda}{r}\right)-\Gamma\left(\frac{\lambda}{r};-y^{(\mathcal{A})}(x)\right)}{\left(-y^{(\mathcal{A})}(B)\right)^{\lambda\left(\frac{1}{r}-\frac{1}{r^{(\kappa)}}\right)}\Gamma\left(\frac{\lambda}{r^{(\kappa)}};-y^{(\mathcal{A})}(B)\right)+\Gamma\left(\frac{\lambda}{r}\right)-\Gamma\left(\frac{\lambda}{r};-y^{(\mathcal{A})}(B)\right)}, & \textit{if } x^{(\mathcal{A})*}\le x\le B\\ \dfrac{\left(-y^{(\mathcal{A})}(B)\right)^{\lambda\left(\frac{1}{r}-\frac{1}{r^{(\kappa)}}\right)}\Gamma\left(\frac{\lambda}{r^{(\kappa)}};-y^{(\mathcal{A})}(x)\right)}{\left(-y^{(\mathcal{A})}(B)\right)^{\lambda\left(\frac{1}{r}-\frac{1}{r^{(\kappa)}}\right)}\Gamma\left(\frac{\lambda}{r^{(\kappa)}};-y^{(\mathcal{A})}(B)\right)+\Gamma\left(\frac{\lambda}{r}\right)-\Gamma\left(\frac{\lambda}{r};-y^{(\mathcal{A})}(B)\right)}, & \textit{if } x\ge B. \end{cases}$$

In a similar manner to that of the subsidised case, the optimal barrier B can be found by determining the solution of the equation

$$\psi^{(\mathcal{A})}(x)=\psi(x),$$

where $\psi^{(\mathcal{A})}(x)$ and $\psi(x)$ denote the trapping probabilities of the capital process under the alternative microinsurance subsidy scheme and in the uninsured case, respectively. A number of examples for varying initial capital are presented in Figure 2.9a.

Remark 2.6.2. When $B\to x^{(\mathcal{A})*}$, the trapping probability for the alternative microinsurance subsidy scheme is equivalent to the trapping probability obtained in the insured case without subsidies $\psi^{(\kappa)}(x)$:

$$\lim_{B\to x^{(\mathcal{A})*}}\psi^{(\mathcal{A})}(x)=\frac{\Gamma\left(\frac{\lambda}{r^{(\kappa)}};-y^{(\kappa)}(x)\right)}{\Gamma\left(\frac{\lambda}{r^{(\kappa)}}\right)}.$$

Moreover, when $B\to\infty$, the trapping probability is given by

$$\lim_{B\to\infty}\psi^{(\mathcal{A})}(x)=\frac{\Gamma\left(\frac{\lambda}{r};-y^{(\kappa)}(x)\right)}{\Gamma\left(\frac{\lambda}{r}\right)},$$

which is exactly the trapping probability of the insured subsidised process $\psi^{\pi^*(\kappa,\theta)}(x)$ with $\pi^*=0$.

Remark 2.6.3. Figure 2.8 presents the expected trapping time under the alternative microinsurance subsidy scheme for varying initial capital. Again, in line with intuition, the expected trapping time is an increasing function of both the capital level B and initial capital x. Steps for obtaining the expected trapping time under the alternative scheme are analogous to those used to derive (2.3.14) and are thus not presented here.

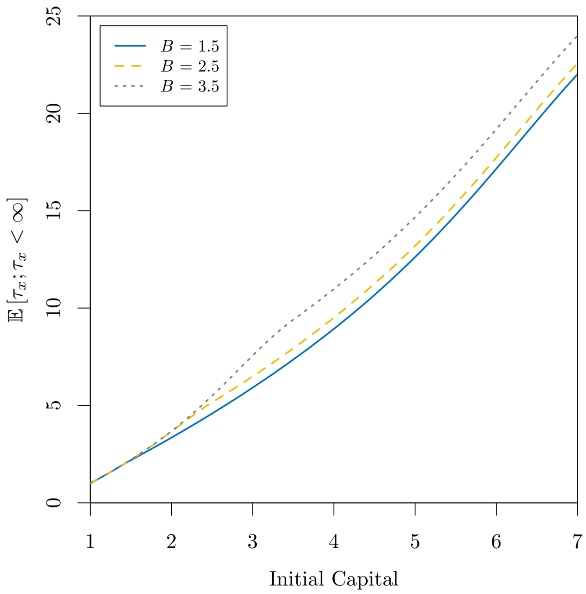


Figure 2.8: Expected trapping time when $Z_i \sim Exp(1)$, $a = 0.8$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^{(\mathcal{A})*} = 1$, $\kappa = 0.5$, and $\theta = 0.5$ for $B = 1.5, 2.5, 3.5$.

Figure 2.9a presents trapping probabilities for varying initial capital under the uninsured, insured, insured subsidised and insured alternatively subsidised schemes. As expected, increasing the capital barrier B helps almost all households reduce their probability of falling into the area of poverty, since support from the government is received when their capital resides in the region between the critical capital $x^{(\mathcal{A})*}$ and the capital level B. Furthermore, as in Section 2.5, since insurance without subsidies decreases their trapping probability to a level below the uninsured, government support is not required for households with higher levels of initial capital (capital greater than or equal to the point at which the yellow short-dashed line intersects the blue solid line). The optimal barrier for such households is in fact the critical capital, i.e. $B = x^{(\mathcal{A})*}$. This household group can afford to cover the costs of microinsurance coverage themselves.

Figure 2.9b shows that in order to remove the capital growth constraints associated with premium payments experienced by the most vulnerable, governments should fix the barrier level above their initial capital. This level should be selected until the household reaches a capital level that is adequate in ensuring their trapping probability is equal to that of an uninsured household. On the other hand, for more privileged households (central area of Figure 2.9b), governments should establish barriers below their initial capital, with households paying premiums as soon as they enrol in the microinsurance scheme. This observation is largely due to the distance of the associated capital levels from the critical capital $x^{(\mathcal{A})*}$. These households are not likely to fall into the area of poverty after suffering one (non-catastrophe) capital loss, and are instead likely to fall into the region between the critical capital $x^{(\mathcal{A})*}$ and the barrier level B, where full government subsidisation occurs, before entering the area of poverty. A such, this region acts as a "buffer", where households benefit from coverage without the need for premium payments. Increasing initial capital leads to a decrease in the size of the "buffer" region until its disappearance when the optimal barrier $B = x^{(\mathcal{A})*}$, as shown by the red dashed line in the right area of Figure 2.9b.

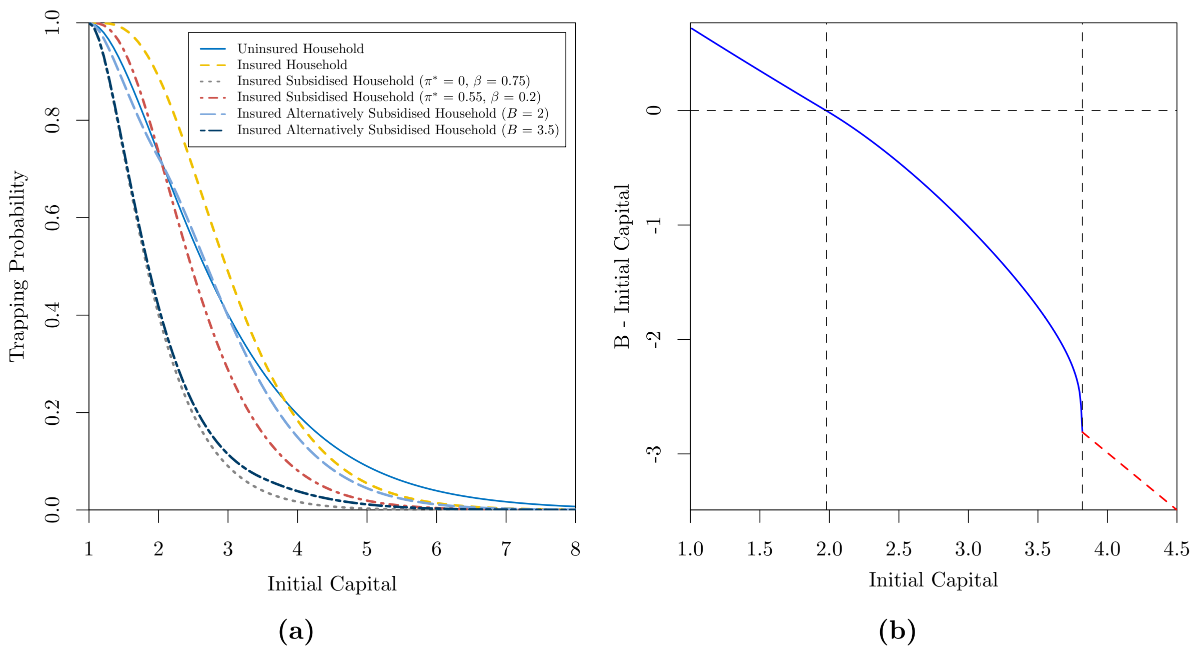


Figure 2.9: (a) Trapping probabilities for the uninsured, insured, insured subsidised with $\pi^* = 0, 0.55$ and insured alternatively subsidised with $B = 2, 3.5$ capital processes when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^* = x^{(\kappa)*} = x^{\pi(\kappa,\theta)*} = x^{(\mathcal{A})*} = 1$, $\kappa = 0.5$, $\theta = 0.5$ and $\pi = 0.75$ (b) Difference between the optimal barrier and the initial capital, i.e. $B - x$, for varying initial capital, when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^{(\mathcal{A})*} = 1$, $\kappa = 0.5$, $\theta = 0.5$ and $\pi = 0.75$.

2.6.1 Cost of social protection

To ensure sustainability, it is again important to measure the governmental cost-effectiveness of providing microinsurance premium subsidies to households under the alternative subsidy scheme. For this purpose, let $\tau_x^{(\mathcal{A})}$ be the trapping time of a household covered by the alternative scheme and $V^{(\mathcal{A})}(x)$ the expectation of the present value of all subsidies provided by the government to the household until the trapping time, that is

$$V^{(\mathcal{A})}(x) := \mathbb{E}\left[\int_0^{\tau_x^{(\mathcal{A})}} \pi e^{-\delta t} \mathbb{1}_{\left\{X_t^{(\mathcal{A})} < B\right\}} dt \,\middle|\, X_0^{(\mathcal{A})} = x\right].$$

The following proposition then holds:

Proposition 2.6.2. Consider a household enrolled in an alternative microinsurance scheme with subsidised flexible premiums, capital barrier $B \geq x^{(\mathcal{A})*}$ and proportionality factor $1 - \kappa \in (0, 1]$. Assume an initial capital $x \geq x^{(\mathcal{A})*}$, capital growth rates $r^{(\kappa)}$ and r above and below the barrier, respectively, loss intensity $\lambda > 0$ and exponentially distributed capital losses with parameter $\alpha^{(\kappa)} > 0$. Then, the expected discounted premium subsidies provided by the government to the household until the trapping time is given by

$$V^{(\mathcal{A})}(x) = \begin{cases} R_1 M\left(-\frac{\delta}{r}, 1-\frac{\lambda+\delta}{r}; y^{(\mathcal{A})}(x)\right) \\ \quad + R_2 e^{y^{(\mathcal{A})}(x)} U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda+\delta}{r}; -y^{(\mathcal{A})}(x)\right) + \frac{\pi}{\delta}, & \textit{if } x^{(\mathcal{A})*} \leq x \leq B \quad (2.6.10a) \\ R_3 M\left(-\frac{\delta}{r^{(\kappa)}}, 1-\frac{\lambda+\delta}{r^{(\kappa)}}; y^{(\mathcal{A})}(x)\right) \\ \quad + R_4 e^{y^{(\mathcal{A})}(x)} U\left(1-\frac{\lambda}{r^{(\kappa)}}, 1-\frac{\lambda+\delta}{r^{(\kappa)}}; -y^{(\mathcal{A})}(x)\right), & \textit{if } x \geq B \quad (2.6.10b) \end{cases}$$

where $y^{(\mathcal{A})}(x) = -\alpha^{(\kappa)}(x - x^{(\mathcal{A})*})$ and the constants R_i for $i = 1, 2, 3, 4$ are given by (2.6.13), (2.6.16), (2.6.14) and (2.6.15), respectively.

Proof. If the derivative of $V^{(\mathcal{A})}(x)$ exists, then using the standard arguments based on the infinitesimal generator presented in Theorem 2.3.1 for $X_t^{(\mathcal{A})}$, under the barrier B, the following IDE for $V^{(\mathcal{A})}(x)$ is obtained:

$$r(x-x^{(\mathcal{A})*})V^{(\mathcal{A})\prime}(x)-(\lambda+\delta)V^{(\mathcal{A})}(x)+\lambda\int_0^{x-x^{(\mathcal{A})*}} V^{(\mathcal{A})}(x-z)dG_Z(z)+\pi = 0, \quad x^{(\mathcal{A})*} \leq x \leq B. \tag{2.6.11}$$

Assuming $Z_i \sim Exp(\alpha^{(\kappa)})$, $V^{(\mathcal{A})}(x)$ therefore satisfies the nonhomogeneous differential equation

$$-\frac{\left(x-x^{(\mathcal{A})*}\right)}{\alpha^{(\kappa)}}V^{(\mathcal{A})\prime\prime}(x) + \left[\frac{(\lambda+\delta-r)}{\alpha^{(\kappa)}r} - \left(x - x^{(\mathcal{A})*}\right)\right]V^{(\mathcal{A})\prime}(x) + \frac{\delta}{r}V^{(\mathcal{A})}(x) - \frac{\pi}{r} = 0 \tag{2.6.12}$$

under application of the operator $(\frac{d}{dx} + \alpha^{(\kappa)})$, where $x^{(\mathcal{A})*} \leq x \leq B$.

Letting $V_h^{(\mathcal{A})}(x)$ be the homogeneous solution of (2.6.12), which is an ODE of the same form as (2.6.2a), it holds that

$$V_h^{(\mathcal{A})}(x) = R_1 M\left(-\frac{\delta}{r}, 1-\frac{\lambda+\delta}{r}; y^{(\mathcal{A})}(x)\right) + R_2 e^{y^{(\mathcal{A})}(x)} U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda+\delta}{r}; -y^{(\mathcal{A})}(x)\right)$$

for $x^{(\mathcal{A})*} \leq x \leq B$ and arbitrary constants $R_1, R_2 \in \mathbb{R}$, where $y^{(\mathcal{A})}(x) = -\alpha^{(\kappa)}(x - x^{(\mathcal{A})*})$ as previously.

The general solution of (2.6.12) can be written as

$$V^{(\mathcal{A})}(x) = V_h^{(\mathcal{A})}(x) + V_p^{(\mathcal{A})}(x),$$

where $V_p^{(\mathcal{A})}(x)$ is a particular solution. It can therefore be verified that $V_p^{(\mathcal{A})}(x) = \frac{\pi}{\delta}$ for all $x^{(\mathcal{A})*} \leq x \leq B$. Letting $x = x^{(\mathcal{A})*}$ in (2.6.11) then yields $V^{(\mathcal{A})}(x^{(\mathcal{A})*}) = \frac{\pi}{\lambda+\delta}$ and subsequently

$$R_1 = -\left[\frac{\lambda\pi}{(\lambda+\delta)\delta} + R_2 U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda+\delta}{r}; 0\right)\right]. \tag{2.6.13}$$

Above the barrier B, $V^{(\mathcal{A})}(x)$ satisfies (2.6.2b) for $x \geq B$ and so

$$V^{(\mathcal{A})}(x) = R_3 M\left(-\frac{\delta}{r^{(\kappa)}}, 1-\frac{\lambda+\delta}{r^{(\kappa)}}; y^{(\mathcal{A})}(x)\right) + R_4 e^{y^{(\mathcal{A})}(x)} U\left(1-\frac{\lambda}{r^{(\kappa)}}, 1-\frac{\lambda+\delta}{r^{(\kappa)}}; -y^{(\mathcal{A})}(x)\right)$$

for arbitrary constants $R_3, R_4 \in \mathbb{R}$. Since $\lim_{x\to\infty} V^{(\mathcal{A})}(x) = 0$ by definition, it holds that

$$R_3 = 0. \tag{2.6.14}$$

Using the continuity of the functions $V^{(\mathcal{A})}(x)$ and $V^{(\mathcal{A})\prime}(x)$ at $x = B$ and the differential properties of the confluent hypergeometric functions,

$$\begin{aligned} R_4 =& \frac{-\left[\frac{\lambda\pi}{(\lambda+\delta)\delta} + R_2 U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda+\delta}{r}; 0\right)\right] M\left(-\frac{\delta}{r}, 1-\frac{\lambda+\delta}{r}; y^{(\mathcal{A})}(B)\right)}{e^{y^{(\mathcal{A})}(B)} U\left(1-\frac{\lambda}{r^{(\kappa)}}, 1-\frac{\lambda+\delta}{r^{(\kappa)}}; -y^{(\mathcal{A})}(B)\right)} \\ &+ \frac{R_2 e^{y^{(\mathcal{A})}(B)} U\left(1-\frac{\lambda}{r}, 1-\frac{\lambda+\delta}{r}; -y^{(\mathcal{A})}(B)\right) + \frac{\pi}{\delta}}{e^{y^{(\mathcal{A})}(B)} U\left(1-\frac{\lambda}{r^{(\kappa)}}, 1-\frac{\lambda+\delta}{r^{(\kappa)}}; -y^{(\mathcal{A})}(B)\right)} \end{aligned} \tag{2.6.15}$$

and

$$\begin{aligned} R_2 =& -\frac{\lambda\pi K^{-1}}{\lambda+\delta}\Big[\frac{\alpha^{(\kappa)}}{(r-\lambda-\delta)} M\left(1-\frac{\delta}{r}, 2-\frac{\lambda+\delta}{r}; y^{(\mathcal{A})}(B)\right) \\ &+ \frac{1}{\delta} M\left(-\frac{\delta}{r}, 1-\frac{\lambda+\delta}{r}; y^{(\mathcal{A})}(B)\right)\left(\alpha^{(\kappa)} - D\right)\Big] - \frac{\pi K^{-1}}{\delta}\left(D - \alpha^{(\kappa)}\right), \end{aligned} \tag{2.6.16}$$

where D and K are (2.6.7) and (2.6.8), respectively. □

As in Section 2.5.1, it is possible to obtain an explicit expression for $m_{\delta,w}^{(\mathcal{A})}(x)$, the expected discounted cost incurred by the government at the trapping time under the alternative microinsurance scheme:

Proposition 2.6.3. Consider a household enrolled in an alternative microinsurance scheme with subsidised flexible premiums, capital barrier $B \geq x^{(\mathcal{A})*}$ and proportionality factor $1-\kappa \in (0,1]$. Assume an initial capital $x \geq x^{(\mathcal{A})*}$, capital growth rates $r^{(\kappa)}$ and r above and below the barrier, respectively, loss intensity $\lambda > 0$, exponentially distributed capital losses with parameter $\alpha^{(\kappa)} > 0$ and a cost to lift households further away from the area of poverty $M^{(\mathcal{A})} - x^{(\mathcal{A})*}$, with $M^{(\mathcal{A})} \geq x^{(\mathcal{A})*}$. Then, the expected discounted cost incurred by the government at the trapping time is

$$m_{\delta,w}^{(\mathcal{A})}(x) = \left[\frac{1}{\alpha^{(\kappa)}} + M^{(\mathcal{A})} - x^{(\mathcal{A})*}\right] m_{\delta}^{(\mathcal{A})}(x), \tag{2.6.17}$$

where $m_{\delta}^{(\mathcal{A})}(x)$ is given by (2.6.1a) and (2.6.1b).

Proof. Proof follows that of Proposition 2.5.2. □

As for the subsidised scheme, under the alternative scheme, the cost of social protection incurred by the government is defined as the expected discounted subsidies provided until trapping plus the expected discounted cost incurred at trapping, here given by (2.6.10a), (2.6.10b) and (2.6.17), respectively.

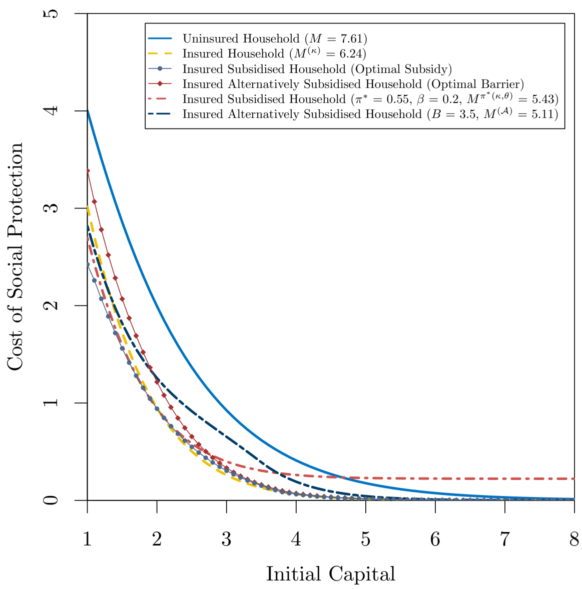


Figure 2.10: Cost of social protection for the uninsured, insured, insured subsidised with $\pi^* = \pi^*_{\texttt{Optimal}}, 0.55$ and insured alternatively subsidised capital processes with $B = B_{\texttt{Optimal}}, 3.5$ when $Z_i \sim Exp(1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $x^* = x^{(\kappa)*} = x^{\pi(\kappa,\theta)*} = x^{(\mathcal{A})*} = 1$, $\kappa = 0.5$, $\theta = 0.5$, $\delta = 0.9$, $\epsilon = 0.01$ and $\pi = 0.75$.

Figure 2.10 compares the cost of social protection for the uninsured, insured, insured subsidised and insured alternatively subsidised households. Cost of social protection for the most vulnerable is reduced with all forms of microinsurance coverage (yellow dashed, red dashed-dotted, blue dashed-dashed, blue circular-marked and red diamond-marked lines below the blue solid line for initial capitals close to the critical capital x^*). As in relation to Figure 2.7, this aligns with the high trapping probability associated with this portion of the population when uninsured, with governments almost surely needing to lift households out of the area of poverty. Although already eliminated when providing optimal subsidies under the insured subsidised scheme (blue circular-marked line below the red dashed-dotted line for the most privileged), the aforementioned drawback of governments subsidising premiums indefinitely, almost surely under suboptimal subsidised schemes is also eliminated under both optimal and suboptimal alternative subsidy schemes due to the ceasing of subsidies on households reaching sufficient capital (red diamond-marked and blue dashed-dashed lines below red dashed-dotted line for households with higher levels of capital). Furthermore, as observed in Figure 2.9a, when the barrier level is sufficiently high, households of all capital levels experience a decrease in their trapping probability, almost reaching that of a household enrolled in a fully subsidised insurance scheme. Governments are not required to subsidise premiums indefinitely even for schemes with high barrier levels, since households will pay the entire premium as soon as their capital reaches a sufficient level. The scheme therefore facilitates a reduction in the trapping probability of all households, while reducing the cost of social protection incurred by the government, highlighting the cost-efficiency of this alternative strategy.

2.7 Concluding remarks

Comparing the impact of three microinsurance frameworks on the trapping probabilities of low-income households, this chapter provides evidence for the importance of governmentally supported inclusive insurance in the strive towards poverty alleviation. The results of Sections 2.4 and 2.5 support those of Kovacevic and Pflug (2011), highlighting a threshold below which insurance could increase the probability of trapping. Motivated by the recent increased involvement of governments in the support of insurance programmes and maintaining the idea of "smart" subsidies, a transparent method with a mathematical foundation for calculating optimal subsidies that can strengthen government social protection programmes while lowering the associated costs, is introduced.

Numerical examples indicate that while many of the proposed insurance mechanisms, both with and without subsidies, reduce the cost of social protection for the most vulnerable, they do not reduce their probability of trapping. This undermines the faculty of inclusive insurance as a cost-effective social protection strategy for poverty alleviation and brings to light questions as to its capability in reducing both the probability of households falling below the poverty line and the associated social protection costs. However, analysis of a subsidised microinsurance scheme with a premium payment barrier suggests that in general, the trapping probability of a household covered by such a scheme is reduced in comparison to when covered by unsubsidised and (for the most vulnerable) partially subsidised microinsurance schemes, in addition to when uninsured, alleviating this limitation.

More significant influence is observed in relation to the governmental cost of social protection, with the cease of subsidy payments when household capital is sufficient facilitating government savings and therefore increasing social protection efficiency. This provides evidence for the relevance of the alternative scheme as a cost-effective social protection strategy for poverty reduction. The cost of social protection for those closest to the area of poverty remains lower than the corresponding uninsured cost in both subsidised frameworks considered, achieving similar results to those obtained with the targeted-subsidisy scheme proposed by Janzen et al. (2021). In this analysis, governments must account for their support of premium payments, the likely need for household removal from poverty and an extra capital injection to ensure that they will not return to poverty with some level of confidence. Total subsidies paid by the government for the most poor have a small weight within the cost of social protection due to the fact that those closest to the poverty line will fall into the area of poverty almost surely. The capital injection on trapping is also much lower in comparison to that of uninsured households. Each of these factors enhances the reduction in the cost of social protection, particularly for the most vulnerable.

Chapter 3

On a low-income capital process with deterministic growth and multiplicative jumps

Risk theory techniques are adopted in this chapter for the study of a capital growth process with multiplicative losses proportional to the level of accumulated capital, where proportional jumps reflect the high-risk nature of the low-income environment. Explicit expressions for the trapping probability are obtained via analysis of the Laplace transform of the infinitesimal generator of the process, where trapping occurs when a household's capital falls into the area of poverty, from which it is difficult to escape without external help. Introduction of an insurance product offering proportional coverage is presented and the Frobenius method considered for asymptotic analysis of the associated Laplace and trapping side ODEs. Classical results from risk theory are used to derive constraints on the rate parameters of the process. Comparisons are made between the trapping probabilities of uninsured and insured households, and with those under random-valued losses.

3.1 Introduction

As discussed in Chapter 2, capital levels in low-income economies include all forms of capital that enable production, whether for trade or self-sustaining purposes. The prevalence of agricultural work in such environments means that the threat of catastrophic events, including floods, droughts, earthquakes and disease, is of great concern due to the typically large-scale nature of their impact. In contrast to losses relating to health, life or death, agricultural losses could immediately eliminate a high proportion of a household's ability to produce through loss of land and livestock.

In this chapter, the capital growth model of Chapter 2 is adjusted to account for such heavy capital losses. At loss events, accumulated capital is reduced by a random proportion of itself, rather than by an amount of random value. In addition to capturing the threat of catastrophic events, small-scale proportional losses can be used to represent losses of lesser severity also faced by households. A process of this structure is referred to as a growth-fragmentation or growth-collapse process, characterised by their growth in between the random collapse times at which downwards jumps occur, where jumps are of random size dependent on the state of the process immediately before the jump.

The growth-collapse process of this chapter is also studied in the poverty trap setting by Kovacevic and Pflug (2011), where estimates of the infinite-time trapping probability of a discretised capital process are obtained through numerical simulation. Azaïs and Genadot (2015) perform further numerical analysis on the same model, with mention of applications to the capital setting of Kovacevic and Pflug (2011) and to population dynamics, where the critical level denotes extinction. In both cases, derivation of analytical solutions of the infinitesimal generator is not attempted. Research on growth-collapse processes with applications outside the field of actuarial mathematics includes studies by Altman et al. (2002) and Löpker and Van Leeuwaarden (2008) for congestion control in data networks, Eliazar and Klafter (2004) and Eliazar and Klafter (2006) for phenomena in physical systems and Derfel et al. (2009) for cell growth and division. Löpker and Van Leeuwaarden (2008) obtain the Laplace transform of the transient moments of a growth-collapse process, while Eliazar and Klafter (2004) consider the state of a growth-collapse process at equilibrium, computing Laplace transforms of the system and of the high- and low-levels of the growth-collapse cycle. In this chapter, Laplace transform methods are applied to derive the trapping probabilities of the capital process with proportional losses.

The capital growth process is again considered both with and without proportional insurance coverage, and derivation of the associated explicit trapping probabilities undertaken. However, due to the proportionality of losses, generators of the capital process no longer directly align with those of classical models used to describe the surplus process of an insurer. Obtaining the solution of the infinitesimal generator equation in this case is therefore nontrivial. Traditionally a sum of independent random variables, random absolute losses are now correlated with one another, and with the inter-arrival times of the loss event. In addition, only the surplus of a household's capital above the critical capital grows exponentially, where in the classical context, it is the surplus above zero that grows.

An outline of the remainder of the chapter is as follows. Section 3.2 introduces the capital growth model and its alignment with the classical Crámer-Lundberg model. This connection enables derivation of parameter constraints and an upper bound on the trapping probability. Derivation of the infinitesimal generator is also presented in this section. Derivation of trapping probabilities for uninsured losses with $Beta(1,1)$ (Section 3.3.1) and $Beta(\alpha,1)$ (Section 3.3.2) distributed remaining proportions of capital is presented in Section 3.3. Section 3.4 discusses the introduction of microinsurance and presents analysis on the associated infinitesimal generator. Concluding remarks are presented in Section 3.5.

3.2 The capital model

As in Chapter 2, a piecewise deterministic Markov process of compound Poisson-type represents household capital, where accumulated capital is deterministic in-between the randomly occurring jump times at which large capital losses occur. The growth dynamics of the model are analogous to those in Section 2.2. Accumulated capital $(X_t)_{t\geq 0}$ grows exponentially with rate $r = (1-a)\cdot b\cdot c > 0$, where $0 < a < 1$, $b > 0$ and $0 < c < 1$ are household rates of consumption, income generation and investment or savings, respectively. Capital growth occurs only when the process is above the critical level of capital $x^* > 0$, below which a household struggles to meet their basic needs.

It is again assumed that households are susceptible to large capital losses, which follow a Poisson process with intensity λ. At the i-th loss event, the capital process experiences a downwards jump to $X_{T_i} \cdot Z_i$, where $Z_i \in (0,1]$ is the random proportion determining the remaining capital after loss i and X_{T_i} the level of capital accumulated up to the loss time. The sequence $\{Z_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables with common distribution function G_Z, independent of the Poisson process. In this chapter, the proportion of capital remaining after each loss event Z_i is assumed to follow a Beta distribution with parameters $\alpha > 0$ and $\beta > 0$. The stochastic capital process with deterministic exponential growth and multiplicative losses is formally defined as follows:

Definition 3.2.1 (Kovacevic and Pflug (2011)). Let T_i be the i^{th} event time of a Poisson process $(N_t)_{t\geq 0}$ with parameter λ, where $T_0 = 0$. Let $Z_i \geq 0$ be a sequence of i.i.d. random variables with distribution function G_Z, independent of the process N_t. For $T_{i-1} \leq t < T_i$, the stochastic growth process of the accumulated capital X_t is defined as

$$X_t = \begin{cases} \left(X_{T_{i-1}} - x^*\right) e^{r(t-T_{i-1})} + x^*, & \textit{if } X_{T_{i-1}} > x^* \quad (3.2.1a) \\ X_{T_{i-1}}, & \textit{otherwise.} \quad (3.2.1b) \end{cases}$$

At the jump times $t = T_i$, the process is given by

$$X_{T_i} = \begin{cases} \left[\left(X_{T_{i-1}} - x^*\right) e^{r(T_i-T_{i-1})} + x^*\right] \cdot Z_i, & \textit{if } X_{T_{i-1}} > x^* \quad (3.2.2a) \\ X_{T_{i-1}} \cdot Z_i, & \textit{otherwise.} \quad (3.2.2b) \end{cases}$$

As in Chapter 2, the aim of this chapter is to study the probability of a household falling below the poverty line, i.e. the trapping probability. As defined previously, the infinite-time trapping probability describes the distribution of the trapping time $\tau_x := \inf\{t \geq 0 : X_t < x^* | X_0 = x\}$ and is given by

$$\psi(x) = \mathbb{P}(\tau_x < \infty),$$

where x is the initial capital of a household. When deriving the trapping probabilities of the capital processes of this chapter, following the definition of the infinitesimal generator at the end of this section, ψ will be denoted by f.

Consider an adjustment of the capital process in Definition 3.2.1 discretised at loss event times such that $\tilde{X}_i = X_{T_i}$, i.e. the capital process studied in Kovacevic and Pflug (2011). Taking the logarithm of the adjusted process with critical capital x^* fixed at 0 yields

$$L_i = L_{i-1} + r(T_i - T_{i-1}) + \log(Z_i),$$

such that

$$L_i = \log x + rT_i + \sum_{i=1}^{N_t} \log(Z_i), \tag{3.2.3}$$

where L_i is the logarithm of the i-th step in the discretised process $\tilde{X}_i$ and $\log(Z_i) < 0$. The model in (3.2.3) is a version of the classical Crámer-Lundberg model introduced by Lundberg (1903) and Cramér (1930), which assumes an insurance company collects premiums continuously and pays claims of random size at random times. The corresponding surplus process is given by

$$U_t = u + ct - \sum_{k=1}^{N_t} X_k, \tag{3.2.4}$$

where u is the initial capital, c the constant premium rate, $X_1, X_2, ..., X_{N_t}$ the random claim sizes and N_t the number of claims in the interval $[0, t]$. Claim sizes are assumed to be independent and identically distributed, N_t a homogeneous Poisson process and the sequence of claim sizes $\{X_k\}_{k\geq 1}$ and N_t independent.

The relationship between the capital model of this chapter and the Crámer-Lundberg model (3.2.4) enables specification of an upper bound for the trapping probability of the logarithmic process, and hence the capital growth process X_t, through Lundberg's inequality, derived in Lundberg (1926).

Theorem 3.2.1 (Lundberg's inequality)**.** Assume that there exists a constant $s > 0$ such that the process $\{e^{sU(t)}\}_{t\in\geq 0}$ is a martingale. Then, for all $u \geq 0$,

$$\psi(u) \leq e^{-Ru},$$

where R is the unique positive root (in s) of Lundberg's equation, which is given by

$$M_X(s)M_T(-cs) = 1,$$

where $M_X(s)$ and $M_T(s)$ are the moment generating functions of the claim size and waiting time distributions, respectively. If R exists, it is referred to as the adjustment coefficient.

Theorem 3.2.2 (The Crámer-Lundberg approximation)**.** Assume that the adjustment coefficient R exists and that $M'_X(R) < \infty$. Then,

$$\lim_{u\to\infty} \psi(u)e^{Ru} = \frac{c - \lambda\mu}{\lambda M'_X(R) - c},$$

where $\mu = \int_0^\infty (1 - G(x))dx$ and $G(x)$ is the claim size distribution.

For proof of Theorems 3.2.1 and 3.2.2, see, for example, Schmidli (2017).

Proposition 3.2.1. Consider a household capital process as proposed in Definition 3.2.1. For initial capital $x \geq x^*$, capital growth rate r, loss intensity $\lambda > 0$ and remaining proportions of capital with distribution $Beta(\alpha, 1)$, the adjustment coefficient of the corresponding Lundberg equation exists if

$$\frac{\lambda}{r} < \alpha.$$

Proof. By Theorem 3.2.1, the Lundberg equation corresponding to the logarithmised process in (3.2.3) is

$$\mathbb{E}[e^{-s\log(Z_i)}]\mathbb{E}[e^{-sr\tilde{T}_i}] = \mathbb{E}[e^{-s(\log(Z_i)+r\tilde{T}_i)}] = 1, \tag{3.2.5}$$

for $s > 0$, where $\tilde{T}_i = T_i - T_{i-1}$. In order for R to exist, the following condition must therefore hold:

$$\mathbb{E}[\log(Z_i) + r\tilde{T}_i] > 0 \iff \mathbb{E}[\log(Z_i)] + r\mathbb{E}[\tilde{T}_i] > 0,$$

such that for remaining proportions of capital with distribution $Beta(\alpha, 1)$,

$$\frac{\lambda}{r} < \alpha \tag{3.2.6}$$

must hold, as required, where $\mathbb{E}[\log(Z_i)] = \alpha \int_0^1 \log(z)z^{\alpha-1}dz$. □

This constraint ensures that the adjustment coefficient and thus an upper bound on the trapping probability of the logarithmic process L_i exists. This adjustment coefficient is in fact also the adjustment coefficient for the capital growth process defined in Definition 3.2.1.

Proposition 3.2.2. If $\mathbb{E}[\log(Z_i) + r\tilde{T}_i] > 0$, then there exists an adjustment coefficient R for the capital growth process X_t in Definition 3.2.1 which is identical to that of the logarithmised process L_i, where R is the unique positive solution of (3.2.5).

Proof. The proof of Proposition 3.2.2 follows that of Proposition 2 of Kovacevic and Pflug (2011).

Consider the process L_i in (3.2.3), where $i \geq 0$ denotes the i-th loss event. Aligning with the classical Crámer-Lundberg model, by Theorem 3.2.1, the trapping probability of L_i satisfies

$$\mathbb{P}(L_i \leq 0 | L_0 = \log x) \leq e^{-Ru}, \tag{3.2.7}$$

where $\mathbb{E}[\log(Z_i) + r\tilde{T}_i] > 0$ such that R is the adjustment coefficient. For $x^* = 0$, the trapping probability in (3.2.7) is equivalent to $\mathbb{P}(\tilde{X}_i \leq 1 | \tilde{X}_0 = x)$. Therefore,

$$\mathbb{P}(\tilde{X}_i \leq 0 | \tilde{X}_0 = x) \leq \mathbb{P}(\tilde{X}_i \leq 1 | \tilde{X}_0 = x) \leq e^{-Ru}. \tag{3.2.8}$$

In order to prove that the adjustment coefficient for $\tilde{X}_i$ at $x^* = 0$ is equivalent to that of the process for which x^* is an unknown constant, consider the fact that for $r > 0$ and $t \geq 0$, $(x - x^*)e^{rt} + x^* \leq xe^{rt}$, where the left-hand side of the inequality corresponds to the process $\tilde{X}_i$ for x^* constant, and the right-hand side for $x^* = 0$, which has upper bound as in (3.2.8). By the Crámer-Lundberg approximation in Theorem 3.2.2,

$$R = -\lim_{u \to \infty} \frac{1}{u} \log \psi(u), \tag{3.2.9}$$

where for constant $C > 0$, $\lim_{u \to \infty} \frac{\log C}{u} = 0$. Comparing the two processes in line with (3.2.9), it therefore holds that

$$-\limsup_{x \to \infty} \frac{1}{x} \log \mathbb{P}(\tilde{X}_i \leq x^* | \tilde{X}_0 = x) \leq -\lim_{x \to \infty} \frac{1}{x} \log \mathbb{P}(\tilde{X}_i \leq 0 | \tilde{X}_0 = x) = R(r), \tag{3.2.10}$$

where $R(r)$ is the adjustment coefficient of the process $\tilde{X}_i$ with $x^* = 0$.

Similarly consider the case $x \leq \frac{r}{r - r_1}$, where r_1 is close to but less than r. For sufficiently large x, i.e. for

$$x \geq \frac{x^*(1 - e^{-rt})}{1 - e^{-(r - r_1)t}},$$

the inequality $(x - x^*)e^{rt} + x^* \geq xe^{r_1 t}$ holds. Then,

$$-\liminf_{x \to \infty} \frac{1}{x} \log \mathbb{P}(\tilde{X}_i \leq x^* | \tilde{X}_0 = x) \geq -\lim_{x \to \infty} \frac{1}{x} \log \mathbb{P}(\tilde{X}_i \leq 0 | \tilde{X}_0 = x) = R(r_1).$$

Since r_1 approaches r from below, it also holds that $R(r_1)$ approaches $R(r)$ from below. As such,

$$-\liminf_{x \to \infty} \frac{1}{x} \log \mathbb{P}(\tilde{X}_i \leq x^* | \tilde{X}_0 = x) \geq \lim_{r_1 \to r} R(r_1) = R(r). \tag{3.2.11}$$

Combining (3.2.11) with (3.2.10) gives that the adjustment coefficient for the discretised process $\tilde{X}_i$ with critical capital x^* is the same as that of the same process with $x^* = 0$. In addition, since $\tilde{X}_i$ is simply a version of the original capital process X_t discretised at loss events, this adjustment coefficient also holds for X_t. □

Note that, the Crámer-Lundberg approximation is applicable in the proof of Proposition 3.2.2 since losses of size $-\log Z_i$, as in (3.2.3), have exponential tails.

The infinitesimal generator $\mathcal{A}$ of the original stochastic process $(X_t)_{t\geq 0}$ as in Definition 3.2.1 is derived by Definition 3.2.2:

Definition 3.2.2. The infinitesimal generator $\mathcal{A}$ of a Markov process $(X_t)_{t\geq 0}$ is defined by

$$\mathcal{A}f(x) = \lim_{h\to 0} \frac{\mathbb{E}[f(X_h)|X_0 = x] - f(x)}{h},$$

for $x \in \mathbb{R}$. The domain of the generator $\mathcal{A}$ is given by the class of all real-valued, bounded, Borel-measurable functions f defined on S such that the generator exists, i.e.

$$\mathcal{D}_{\mathcal{A}} = \{f \in \mathcal{B}(S) \mid \mathcal{A}f(x) \text{ exists for all } x\}.$$

Consider the expected value of a function f of the capital process, conditional on the initial starting point:

$$\begin{aligned}\mathbb{E}[f(X_h)|X_0 = x] =& \sum_{k=0}^{\infty} \frac{(\lambda h)^k}{k!} e^{-\lambda h}\mathbb{E}[f(X_h)|N(h) = k] \\ =& e^{-\lambda h} f((x - x^*)e^{rh} + x^*) \\ &+ \lambda h e^{-\lambda h}\mathbb{E}[f((((x - x^*)e^{rT} + x^*) \cdot z) - x^*)e^{r(h-T)} + x^*)|N(h) = 1] \\ &+ \mathcal{O}(h), \end{aligned} \tag{3.2.12}$$

where $T \leq h$ is the jump time. Given $\lim_{h\to\infty} \frac{\mathcal{O}(h)}{h} = 0$, expressing the exponential components of the first term of (3.2.12) as their Power series representation and expanding the resulting product, it holds that

$$\begin{aligned}\lim_{h\to 0} \frac{\mathbb{E}[f(X_h)|X_0 = x] - f(x)}{h} &= \lim_{h\to 0} \frac{e^{-\lambda h} f((x - x^*)e^{rh} + x^*) - f(x)}{h} + \lambda\mathbb{E}[f(x \cdot z)] \\ &= r(x - x^*)f'(x) - \lambda f(x) + \lambda \int_0^1 f(x \cdot z)dG(z).\end{aligned}$$

By Definition 3.2.2, the infinitesimal generator of X_t is therefore given by

$$(\mathcal{A}f)(x) = r(x - x^*)f'(x) + \lambda \int_0^1 [f(x \cdot z) - f(x)]dG(z), \qquad x \geq x^*. \tag{3.2.13}$$

Multiplication of the capital process by the random proportion in the integral function makes the Laplace transform methods typically used in risk theory no longer straightforward.

3.3 Derivation of trapping probability for uninsured losses

In this section, the analytic trapping probabilities associated with (3.2.13) are derived for $Beta(1, 1)$, i.e. uniform (Section 3.3.1) and $Beta(\alpha, 1)$ (Section 3.3.2) distributed remaining proportions of capital.

3.3.1 Uniformly distributed Z_i

Assume Z_i is uniformly distributed, such that $\alpha = \beta = 1$, and let $u = x \cdot z$. Then, (3.2.13) reduces to

$$(\mathcal{A}f)(x) = r(x - x^*)f'(x) - \lambda f(x) + \frac{\lambda}{x}\int_0^x f(u)d(u)$$

for $x \geq x^*$. The behaviour of the capital process above the critical capital x^* determines a household's trapping probability, with only surplus capital above the critical capital growing exponentially. Thus, consider the change of variable $h(x) = f(x + x^*)$ such that for $\tilde{x} = x - x^* > 0$, the infinitesimal generator is instead given by

$$(\mathcal{A}h)(\tilde{x}) = r\tilde{x}h'(\tilde{x}) - \lambda h(\tilde{x}) + \frac{\lambda}{\tilde{x} + x^*}\int_0^{\tilde{x}+x^*} h(u - x^*)du. \tag{3.3.1}$$

Analysis of the trapping probability can be undertaken through study of the infinitesimal generator. Theorem 3.3.1 provides a link between the generator and the probability of interest in the classical ruin theoretic context. This theorem is discussed in detail in Constantinescu (2006). The change of variable adopted here aligns the domain of the generator in (3.3.1) with the domain of the generator in Theorem 3.3.1.

Theorem 3.3.1 (Paulsen and Gjessing (1997))**.** Let $\tau_x = \inf\{t \geq 0 : X_t < 0 \,|\, X_0 = x\}$ denote the time of ruin given initial surplus x, where τ_x is fixed at infinity if $X_t \geq 0 \ \forall t$. Assume $f(x)$ is a bounded and twice continuously differentiable function on $x \geq 0$, with a bounded first derivative. If $f(x)$ solves $\mathcal{A}f = 0$ on $x \geq 0$, together with boundary conditions

$$f(x) = 1 \quad \text{for} \quad x < 0$$

and

$$\lim_{x\to\infty} f(x) = 0,$$

then

$$f(x) = \mathbb{P}(\tau_x < \infty)$$

such that $f(x)$ is the ruin probability.

Proof. See Paulsen and Gjessing (1997) for proof. □

Theorem 3.3.1 holds analogously in the trapping probability context. The remainder of this section therefore works towards solving $\mathcal{A}h = 0$.

Closed-form expressions for Laplace transforms of ruin (trapping) probabilities are often more easily obtained than for the probability itself. In this section, the trapping probability of an uninsured household is obtained by inverting the solution of the Laplace side ODE.

Solution of the integro-differential equation in (3.2.13) has so far only been undertaken numerically, see, for example, Kovacevic and Pflug (2011). In the proof of Proposition 3.3.1, the explicit solution is derived.

Proposition 3.3.1. Consider a household capital process (as proposed in Definition 3.2.1) with initial capital $x \geq x^*$, capital growth rate r, loss intensity $\lambda > 0$ and remaining proportions of capital with distribution $Beta(1, 1)$. The trapping probability is given by

$$f(x) = 1 - \frac{1}{\Gamma(1 + \frac{\lambda}{r})\Gamma(1 - \frac{\lambda}{r})}\Big(\frac{x - x^*}{x^*}\Big)^{\frac{\lambda}{r}} {}_2F_1\Big(1, \frac{\lambda}{r}; 1 + \frac{\lambda}{r}; -\frac{(x - x^*)}{x^*}\Big), \tag{3.3.2}$$

for $\frac{\lambda}{r} < 1$, where ${}_2F_1(\cdot)$ is the Gauss hypergeometric function as defined in Appendix B.

Proof. Fix $(\mathcal{A}h)(\tilde{x}) = 0$ in line with Theorem 3.3.1 such that

$$r\tilde{x}h'(\tilde{x}) - \lambda h(\tilde{x}) + \frac{\lambda}{\tilde{x}+x^*}\int_0^{\tilde{x}+x^*} h(u-x^*)du = 0$$
$$\iff r\tilde{x}(\tilde{x}+x^*)h'(\tilde{x}) - \lambda(\tilde{x}+x^*)h(\tilde{x}) + \lambda\int_{-x^*}^{\tilde{x}} h(v)dv = 0$$

where $v = u - x^*$,

$$\iff r\tilde{x}(\tilde{x}+x^*)h'(\tilde{x}) - \lambda(\tilde{x}+x^*)h(\tilde{x}) + \lambda\int_{-x^*}^{0} h(v)dv + \lambda\int_0^{\tilde{x}} h(v)dv = 0$$
$$\iff r\tilde{x}(\tilde{x}+x^*)h'(\tilde{x}) - \lambda(\tilde{x}+x^*)h(\tilde{x}) + \lambda x^* + \lambda\int_0^{\tilde{x}} h(v)dv = 0. \tag{3.3.3}$$

The Laplace transform of (3.3.3) is

$$s^2F''(s) + s\Big(2 + \frac{\lambda}{r} - x^*s\Big)F'(s) + \Big(\frac{\lambda}{r} - x^*s\Big(1+\frac{\lambda}{r}\Big)\Big)F(s) = -\frac{\lambda x^*}{r}. \tag{3.3.4}$$

Let $F(s) = s^{-1}w(s)$ such that (3.3.4) reduces to

$$sw''(s) + \Big(\frac{\lambda}{r} - x^*s\Big)w'(s) - \frac{\lambda x^*}{r}w(s) = -\frac{\lambda x^*}{r}, \tag{3.3.5}$$

which by (108) of Zaitsev and Polyanin (2002) has homogeneous solution

$$w_h(s) = e^{x^*s}\mathcal{J}\left(0, \frac{\lambda}{r}; -x^*s\right),$$

where $\mathcal{J}(a,b;z)$ is an arbitrary solution of the degenerate hypergeometric equation. Selecting two of the eight solutions of the degenerate hypergeometric equation, the following general solution of (3.3.5) is constructed:

$$w_h(s) = C_1e^{x^*s} + C_2(-x^*s)^{1-\frac{\lambda}{r}}U\left(1, 2-\frac{\lambda}{r}, x^*s\right),$$

where $y_1(s)$ and $y_2(s)$ are (13.1.12) and (13.1.19) of Abramowitz and Stegun (1972), respectively. Proposition of a constant Ansatz $w_p(s) = A$ then gives the particular solution $w_p(s) = 1$, such that

$$w(s) = C_1e^{x^*s} + C_2(-x^*s)^{1-\frac{\lambda}{r}}U\left(1, 2-\frac{\lambda}{r}, x^*s\right) + 1.$$

By the initial value theorem,

$$\lim_{\tilde{x}\to 0} h(\tilde{x}) = \lim_{s\to\infty} sF(s) = 1. \tag{3.3.6}$$

Take the limit of $w(s) = sF(s)$ and apply (3.3.6). The asymptotic behaviour of the Tricomi function $U(a,c,z)$ as z tends to infinity is given by $z^{-a}[1+\mathcal{O}(|z|^{-1})]$. As such,

$$\lim_{s\to\infty} s^{1-\frac{\lambda}{r}}U\left(1, 2-\frac{\lambda}{r}, x^*s\right) = s^{1-\frac{\lambda}{r}}(x^*s)^{-1}(1+\mathcal{O}(|x^*s|^{-1})) = 0$$

and so $C_1 = 0$. The Tricomi function $U(a, b; z)$ (see Appendix B) can be expressed in terms of the Kummer function $M(a, b; z)$ in the following way:

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} M(a,b;z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(1+a-b, 2-b; z).$$

As such, since

$$(-x^* s)^{1-\frac{\lambda}{r}} U\left(1, 2-\frac{\lambda}{r}, x^* s\right) = (-1)^{1-\frac{\lambda}{r}} U\left(\frac{\lambda}{r}, \frac{\lambda}{r}, x^* s\right)$$

by the Kummer transformations (see, for example, Abramowitz and Stegun (1972)), it holds that $\lim_{s\to 0} U\left(\frac{\lambda}{r}, \frac{\lambda}{r}, x^* s\right) = \Gamma\left(1-\frac{\lambda}{r}\right)$. Applying the final value theorem:

$$\lim_{\tilde{x}\to\infty} h(\tilde{x}) = \lim_{s\to 0} sF(s) = 0$$

therefore yields $C_2 = \frac{(-1)^{\frac{\lambda}{r}}}{\Gamma\left(1-\frac{\lambda}{r}\right)}$. Alternatively, C_2 could be determined by inverting $F(s)$ as follows:

By Section (3.34.1) of Prudnikov et al. (1992), the inverse Laplace transform of $s^{-v}U(a, b, ws)$, for $\mathrm{Re}(a + v), \mathrm{Re}\, s > 0$; $|\arg(w)| < \pi$, is

$$\frac{w^{-a} x^{a+v-1}}{\Gamma(a+v)} {}_2F_1\left(a, a-b+1; a+v; -\frac{x}{w}\right).$$

Thus, inverting

$$F(s) = C_2(-x^*)^{1-\frac{\lambda}{r}} s^{-\frac{\lambda}{r}} U\left(1, 2-\frac{\lambda}{r}, x^* s\right) + s^{-1}$$

gives

$$h(\tilde{x}) = \frac{C_2(-1)^{1-\frac{\lambda}{r}}}{\Gamma\left(1+\frac{\lambda}{r}\right)} \left(\frac{\tilde{x}}{x^*}\right)^{\frac{\lambda}{r}} {}_2F_1\left(1, \frac{\lambda}{r}; 1+\frac{\lambda}{r}; -\frac{\tilde{x}}{x^*}\right) + 1.$$

Applying the second boundary condition on $h(\tilde{x})$: $\lim_{\tilde{x}\to\infty} h(\tilde{x}) = 0$, consider the asymptotic behaviour of ${}_2F_1(a, b; c; z)$. By (15.3.7) of Abramowitz and Stegun (1972),

$$\begin{aligned} {}_2F_1(a,b;c;z) =& \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} {}_2F_1\left(a, 1-c+a; 1-b+a; \frac{1}{z}\right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b} {}_2F_1\left(b, 1-c+b; 1-a+b; \frac{1}{z}\right) \end{aligned} \tag{3.3.7}$$

for $|\arg(-z)| < \pi$. Therefore,

$$\begin{aligned} \lim_{\tilde{x}\to\infty} h(\tilde{x}) =& C_2(-1)^{1-\frac{\lambda}{r}} \lim_{\tilde{x}\to\infty} \left(\frac{\Gamma\left(\frac{\lambda}{r}-1\right)}{\Gamma\left(\frac{\lambda}{r}\right)^2} \left(\frac{\tilde{x}}{x^*}\right)^{\frac{\lambda}{r}-1} {}_2F_1\left(1, 1-\frac{\lambda}{r}; 2-\frac{\lambda}{r}; -\frac{x^*}{\tilde{x}}\right) \right. \\ &\left. + \Gamma\left(1-\frac{\lambda}{r}\right) {}_2F_1\left(\frac{\lambda}{r}, 0; \frac{\lambda}{r}; -\frac{x^*}{\tilde{x}}\right) \right) + 1 \\ =& C_2(-1)^{1-\frac{\lambda}{r}} \Gamma\left(1-\frac{\lambda}{r}\right) + 1 \end{aligned}$$

since $\lim_{z\to 0} {}_2F_1(a,b;c,z) = 1$ and $\frac{\lambda}{r} < 1$ by (3.2.6). Then,

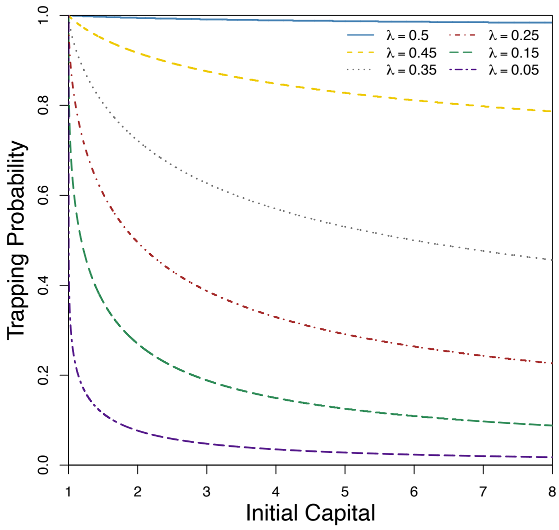
$$\lim_{\tilde{x}\to\infty} h(\tilde{x}) = 0 \iff C_2 = \frac{(-1)^{\frac{\lambda}{r}}}{\Gamma\left(1-\frac{\lambda}{r}\right)}.$$

The explicit trapping probability under the change of variable $h(x)$ is therefore given by

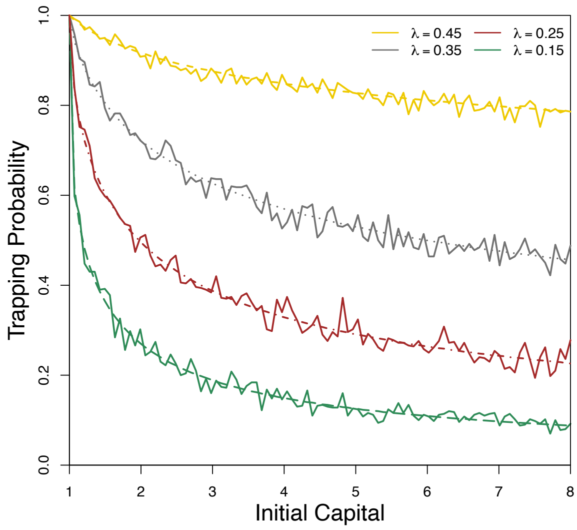
$$h(\tilde{x}) = 1 - \frac{1}{\Gamma\left(1+\frac{\lambda}{r}\right)\Gamma\left(1-\frac{\lambda}{r}\right)}\left(\frac{\tilde{x}}{x^*}\right)^{\frac{\lambda}{r}} {}_2F_1\left(1, \frac{\lambda}{r}; 1+\frac{\lambda}{r}; -\frac{\tilde{x}}{x^*}\right)$$

for $\tilde{x} > 0$. Then, since $h(\tilde{x}) = f(x)$, it holds that $f(x)$ is given by (3.3.2) for $x > x^*$, as required. □

Figure 3.1a presents the trapping probability (3.3.2) for varying initial capital x and loss rate parameter λ. Note that the trapping probability tends to 1 as $\frac{\lambda}{r}$ tends to 1. Parameters a, b and c are selected to correspond with those in Chapter 2. The low value of the rate parameter λ reflects the vulnerability of low-income households to both high and low frequency loss events, while aligning with the constraint in Proposition 3.3.1. Figure 3.1b presents a comparison of the trapping probabilities derived explicitly in Proposition 3.3.1 with trapping probabilities obtained from simulations of the capital growth process.



(a)



(b)

Figure 3.1: Trapping probability when $Z_i \sim Beta(1,1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$ ($r = 0.504$) and $x^* = 1$ for (a) $\lambda = 0.5, 0.45, 0.35, 0.25, 0.15, 0.05$, (b) $\lambda = 0.45, 0.35, 0.25, 0.15$ with comparison to the corresponding simulated trapping probability of the capital growth process with $N = 500$ households, terminal time $T = 1000$ and time-step $dt = 0.1$, evaluated at 100 initial capital levels.

Particularly high levels of accumulated capital are not relevant in the microinsurance and poverty trapping context. However, the asymptotic behaviour of the trapping probability in (3.3.2) at infinity is interesting for understanding the behaviour of the function. Applying the transform in (3.3.7), (3.3.2) is equivalent to

$$-\frac{\Gamma\left(\frac{\lambda}{r}-1\right)}{\Gamma\left(\frac{\lambda}{r}\right)^2\Gamma\left(1-\frac{\lambda}{r}\right)}\left(\frac{x-x^*}{x^*}\right)^{\frac{\lambda}{r}-1}{}_2F_1\left(1,1-\frac{\lambda}{r};2-\frac{\lambda}{r};-\frac{x^*}{x-x^*}\right),$$

which behaves asymptotically like the power function

$$-\frac{\Gamma\left(\frac{\lambda}{r}-1\right)}{\Gamma\left(\frac{\lambda}{r}\right)^2\Gamma\left(1-\frac{\lambda}{r}\right)}\left(\frac{x-x^*}{x^*}\right)^{\frac{\lambda}{r}-1}. \tag{3.3.8}$$

As such, the uninsured trapping probability (3.3.2) has power-law asymptotic decay as $x \to \infty$.

Comparing the decay of the household-level trapping probability under proportional losses with that of the random-valued losses in Chapter 2, note that the equivalent uninsured trapping probability in (2.3.12) decays at a faster rate, following

$$\alpha^{\frac{\lambda}{r}-1}(x-x^*)^{\frac{\lambda}{r}-1}e^{-\alpha(x-x^*)}(1+\mathcal{O}(|\alpha(x-x^*)|^{-1})) \tag{3.3.9}$$

asymptotically, where α is the exponential loss parameter. The ratio of (3.3.9) to (3.3.8) is

$$Ae^{-\alpha(x-x^*)}(1+\mathcal{O}(|\alpha(x-x^*)|^{-1})),$$

for constant $A = -(\alpha x^*)^{\frac{\lambda}{r}-1}\Gamma\left(\frac{\lambda}{r}-1\right)\Gamma\left(\frac{\lambda}{r}\right)^{-2}\Gamma\left(1-\frac{\lambda}{r}\right)^{-1}$, such that the trapping probability in the random-valued case decays exponentially faster than when a household experiences proportional losses. This result is intuitive, since proportional losses are more risky than random-valued losses at high capital levels, due to the non-zero probability of a household losing all (or a high proportion) of its wealth. This is particularly severe in the uniform case, where high and low levels of proportional losses are equally likely.

Remark 3.3.1. The result presented in Remark 2.3.1 of Chapter 2, for the uninsured trapping probability under random-valued losses, can also be derived in the Laplace transform manner of this chapter. This alternative approach is presented in Proposition 3.3.2.

Proposition 3.3.2. Consider a household capital process (as proposed in Definition 2.2.1) with initial capital $x \geq x^*$, capital growth rate r, loss intensity $\lambda > 0$ and exponentially distributed capital losses with parameter $\alpha > 0$. The trapping probability is given by

$$f(x) = \frac{\Gamma\left(\frac{\lambda}{r}; -y(x)\right)}{\Gamma\left(\frac{\lambda}{r}\right)}, \tag{3.3.10}$$

where $y(x) = -\alpha(x-x^*)$.

The trapping probability in (3.3.10) is plotted in comparison to the proportional case of this chapter in Figure 3.3 of Section 3.3.2.

Proof. The infinitesimal generator of the capital growth process under random-valued losses is given by

$$(\mathcal{A}f)(x) = r(x-x^*)f'(x) + \lambda\int_0^\infty [f(x-z)-f(x)]\,dG(z), \qquad x \geq x^* \tag{3.3.11}$$

as in (2.2.7). Applying the change of variable $h(x) = f(x + x^*)$ under exponential loss events, (3.3.11) becomes

$$(\mathcal{A}h)(\tilde{x}) = r\tilde{x}h'(\tilde{x}) - \lambda h(\tilde{x}) + \alpha\lambda \int_0^\infty h(\tilde{x} - z)e^{-\alpha z}dz \tag{3.3.12}$$

$$= r\tilde{x}h'(\tilde{x}) - \lambda h(\tilde{x}) + \alpha\lambda \int_0^{\tilde{x}} h(\tilde{x} - z)e^{-\alpha z}dz + \lambda e^{-\alpha\tilde{x}} \tag{3.3.13}$$

for $\tilde{x} \geq 0$. For $H_1(\tilde{x}) = h(\tilde{x})$ and $H_2(\tilde{x}) = e^{-\alpha\tilde{x}}$, the integral term in (3.3.13) is a convolution, where the convolution of two functions f and g is denoted $f * g$ and given by

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau$$

for $f, g : [0, \infty) \to \mathbb{R}$. As such,

$$(\mathcal{A}h)(\tilde{x}) = r\tilde{x}h'(\tilde{x}) - \lambda h(\tilde{x}) + \alpha\lambda(H_1 * H_2)(\tilde{x}) + \lambda e^{-\alpha\tilde{x}}. \tag{3.3.14}$$

It now remains to solve $(\mathcal{A}h)(\tilde{x}) = 0$ in line with Theorem 3.3.1. Taking the Laplace transform of $(\mathcal{A}h)(\tilde{x}) = 0$ for $(\mathcal{A}h)(\tilde{x})$ in (3.3.14) gives

$$sF'(s) + \left(1 + \frac{\lambda}{r} - \frac{\alpha\lambda}{r(s+\alpha)}\right)F(s) = \frac{\lambda}{r(s+\alpha)}, \tag{3.3.15}$$

where $\mathcal{L}((H_1 * H_2)(\tilde{x})) = \mathcal{L}(H_1(\tilde{x})) \times \mathcal{L}(H_2(\tilde{x}))$. Considering the homogeneous part of (3.3.15):

$$\begin{aligned}
\frac{d}{ds} \ln F_h(s) &= \left(\frac{\alpha\lambda}{r(s+\alpha)} - \frac{\lambda}{r} - 1\right) \cdot \frac{1}{s} \\
\iff \ln F_h(s) &= \frac{\alpha\lambda}{r} \int \frac{1}{\alpha s} - \frac{1}{\alpha(s+\alpha)} ds - \left(\frac{\lambda}{r} + 1\right) \ln s + C \\
&= \ln\left(s^{-1}(s+\alpha)^{-\frac{\lambda}{r}}\right) + C \\
\iff F_h(s) &= As^{-1}(s+\alpha)^{-\frac{\lambda}{r}}.
\end{aligned}$$

To obtain a particular solution, select an Ansatz of $F_p(s) = \frac{A}{s}$. Substitution into (3.3.15) yields $A = 1$, such that the Laplace side solution of the IDE (3.3.12) is

$$F(s) = As^{-1}(s+\alpha)^{-\frac{\lambda}{r}} + \frac{1}{s}.$$

By Section (2.1.2) of Prudnikov et al. (1992), the inverse Laplace transform of the function $s^\mu(s+a)^v$ is given by

$$\frac{x^{-\mu-v-1}}{\Gamma(-\mu-v)} M(-v, -\mu-v; -ax),$$

for $\mathrm{Re}(\mu + v) < 0; \mathrm{Re}\, s > 0, -\mathrm{Re}\, a$, where $\mu, v \notin \mathbb{Z}_0^+$. As such,

$$h(\tilde{x}) = A \cdot \frac{\tilde{x}^{\frac{\lambda}{r}}}{\Gamma\left(1 + \frac{\lambda}{r}\right)} M\left(\frac{\lambda}{r}, 1 + \frac{\lambda}{r}; -\alpha\tilde{x}\right) + 1.$$

The constant A can then be determined through the boundary condition of the trapping probability at infinity: $\lim_{\tilde{x}\to\infty} h(\tilde{x}) = 0$. Applying the identity $M(a,b;z) = e^z M(b-a, b, -z)$, see, for example, (13.1.27) of Abramowitz and Stegun (1972), and by the asymptotic behaviour of $M(a,b;z)$ at infinity in (2.3.9), it holds that

$$A = -\alpha^{\frac{\lambda}{r}}.$$

Then, by the integral representation of $M(a,b;z)$ and by substitution of $v = \alpha\tilde{x}u$,

$$\begin{aligned} M\left(\frac{\lambda}{r}, 1+\frac{\lambda}{r}; -\alpha\tilde{x}\right) &= \frac{\Gamma\left(1+\frac{\lambda}{r}\right)}{\Gamma\left(\frac{\lambda}{r}\right)} \int_0^1 e^{-\alpha\tilde{x}u} u^{\frac{\lambda}{r}-1} du \\ &= \frac{\lambda}{r}(\alpha\tilde{x})^{-\frac{\lambda}{r}} \int_0^{\alpha\tilde{x}} e^{-v} v^{\frac{\lambda}{r}-1} dv \\ &= \frac{\lambda}{r}(\alpha\tilde{x})^{-\frac{\lambda}{r}} \gamma\left(\frac{\lambda}{r}, \alpha\tilde{x}\right), \end{aligned}$$

where $\gamma(a;z) = \int_0^z e^{-t} t^{a-1} dt$ is the lower incomplete gamma function. Therefore,

$$f(x) = h(\tilde{x}) = 1 - \frac{\gamma\left(\frac{\lambda}{r}, \alpha\tilde{x}\right)}{\Gamma\left(\frac{\lambda}{r}\right)},$$

which by the identity $\Gamma(a) = \Gamma(a;x) + \gamma(a;x)$ gives (2.3.13), as required. □

Remark 3.3.2. Due to the specification of exponential losses in Chapter 2, adjusting to the proportional microinsurance case requires only a change of parameters. Derivation of the trapping probability in (2.4.2) using Laplace transform methods is therefore analogous to the proof of Proposition 3.3.2.

3.3.2 Power distributed Z_i

Under assumption of remaining proportions of capital with distribution $Z_i \sim Beta(\alpha, 1)$, the infinitesimal generator of the capital growth process in (3.2.13) is given by

$$(\mathcal{A}f)(x) = r(x - x^*)f'(x) - \lambda f(x) + \frac{\lambda\alpha}{x^\alpha} \int_0^x f(u) u^{\alpha-1} du, \tag{3.3.16}$$

for $x > x^*$ and $u = x \cdot z$. Aligning with the structure of the density function in this special case, throughout the remainder of the chapter, remaining proportions $Z_i \sim Beta(\alpha, 1)$ will be referred to as power distributed proportions. Application of the change of variable $h(x) = f(x + x^*)$ in (3.3.16) induces a $(x + x^*)^\alpha$ term, for which obtaining the Laplace transform is nontrivial. As such, in the power distributed case, the Laplace transform of the generator itself is considered.

Proposition 3.3.3. Consider a household capital process (as proposed in Definition 3.2.1) with initial capital $x \geq x^*$, capital growth rate r, loss intensity $\lambda > 0$ and remaining proportions of capital with distribution $Beta(\alpha, 1)$. The trapping probability is given by

$$f(x) = \frac{(\alpha - \frac{\lambda}{r})^{-1}\Gamma(\alpha)}{\Gamma\left(\frac{\lambda}{r}\right)\Gamma\left(\alpha - \frac{\lambda}{r}\right)} \left(\frac{x}{x^*}\right)^{\frac{\lambda}{r}-\alpha} {}_2F_1\left(\alpha - \frac{\lambda}{r}, 1 - \frac{\lambda}{r}; \alpha - \frac{\lambda}{r} + 1; \frac{x^*}{x}\right), \tag{3.3.17}$$

for $\frac{\lambda}{r} < \alpha$, where ${}_2F_1(\cdot)$ is the Gauss hypergeometric function as in Appendix B.

Proof. Fix $(\mathcal{A}f)(x) = 0$. Taking the Laplace transform of (3.3.16), where the infinitesimal generator of the process for $x \leq x^*$ is zero, gives

$$s^2 F^{(\alpha+1)}(s) + s\Big(\Big(\alpha + 1 + \frac{\lambda}{r}\Big) + x^* s\Big) F^{(\alpha)}(s) + \alpha\Big(x^* s + \frac{\lambda}{r}\Big) F^{(\alpha-1)}(s) = 0. \tag{3.3.18}$$

Let $y(s) = F^{(\alpha-1)}(s)$, such that $y'(s) = F^{(\alpha)}(s)$ and $y''(s) = F^{(\alpha+1)}(s)$. Then, (3.3.18) is equivalent to

$$s^2 y''(s) + s\Big(\Big(\alpha + 1 + \frac{\lambda}{r}\Big) + x^* s\Big) y'(s) + \alpha\Big(x^* s + \frac{\lambda}{r}\Big) y(s) = 0. \tag{3.3.19}$$

For $y(s) = s^{-\alpha} w(s)$, (3.3.19) reduces to give

$$s w''(s) + \Big(\Big(1 + \frac{\lambda}{r} - \alpha\Big) + x^* s\Big) w'(s) = 0,$$

which has solution

$$\begin{aligned} w(s) &= C_1 \int_0^s e^{-x^* t} t^{-(1+\frac{\lambda}{r}-\alpha)} dt + C_2 \\ \iff y(s) &= C_1 s^{-\alpha} \int_0^s e^{-x^* t} t^{-(1+\frac{\lambda}{r}-\alpha)} dt + C_2 s^{-\alpha}. \end{aligned}$$

Under the substitution $u = x^* t$,

$$y(s) = F^{(\alpha-1)}(s) = C_1 x^{*(\frac{\lambda}{r}-\alpha)} s^{-\alpha} \gamma\left(\alpha - \frac{\lambda}{r}, x^* s\right) + C_2 s^{-\alpha}. \tag{3.3.20}$$

for $\frac{\lambda}{r} < \alpha$.

Since $\frac{d}{ds} F(s) = -\mathcal{L}(x f(x))$, it can be proven by induction that

$$\frac{d^n}{ds^n} F(s) = (-1)^n \mathcal{L}(x^n f(x)). \tag{3.3.21}$$

As such, applying the inverse Laplace transform to (3.3.20) and by (3.3.21), it holds that the general solution of (3.3.16) is

$$f(x) = \begin{cases} C_2 \dfrac{(-1)^{1-\alpha}}{\Gamma(\alpha)} + C_1 x^{*(\frac{\lambda}{r}-\alpha)} \dfrac{\Gamma\left(\alpha - \frac{\lambda}{r}\right)}{\Gamma(\alpha)} (-1)^{1-\alpha}, & 0 < x < x^* \\ C_2 \dfrac{(-1)^{1-\alpha}}{\Gamma(\alpha)} & \\ \quad + C_1 \dfrac{\left(\alpha - \frac{\lambda}{r}\right)^{-1}}{\Gamma\left(\frac{\lambda}{r}\right)} (-1)^{1-\alpha} x^{\frac{\lambda}{r}-\alpha} {}_2F_1\Big(\alpha - \dfrac{\lambda}{r}, 1 - \dfrac{\lambda}{r}; \alpha - \dfrac{\lambda}{r} + 1; \dfrac{x^*}{x}\Big), & x^* < x, \end{cases}$$

where the Laplace transform of the piecewise function

$$f(x) = \begin{cases} \dfrac{\Gamma(v)}{\Gamma(\mu)} x^{\mu-1}, & 0 < x < a \\ \dfrac{a^v x^{\mu-v-1}}{v\Gamma(\mu - v)} {}_2F_1\left(v, v - \mu + 1; v + 1; \dfrac{a}{x}\right), & a < x \end{cases}$$

is $s^{-\mu}\gamma(v, as)$ for $\text{Re}(v-\mu) < 1$ and $\text{Re}(\mu, a), \text{Re}(s) > 0$ (see, for example, Section (3.10) of Prudnikov et al. (1992)).

The boundary conditions on $f(x)$:

$$\lim_{x\to\infty} f(x) = 0 \quad \text{and} \quad \lim_{x\to 1} f(x) = 1,$$

yield

$$C_2 = 0 \quad \text{and} \quad C_1 = \frac{\Gamma(\alpha)}{\Gamma\left(\alpha - \frac{\lambda}{r}\right)}(-1)^{\alpha-1}x^{*(\alpha-\frac{\lambda}{r})},$$

such that the analytic trapping probability is given by (3.3.17), as required. □

Note that, substitution of $\alpha = 1$ into (3.3.17) and application of the hypergeometric transform:

$$\begin{aligned} {}_2F_1(a,b;c;z) =& \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} z^{-a} {}_2F_1\left(a, a-c+1; a+b-c+1; 1-\frac{1}{z}\right) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} z^{a-c} {}_2F_1\left(c-a, 1-a; c-a-b+1; 1-\frac{1}{z}\right) \end{aligned}$$

which holds for $|\arg z| < \pi$ and $|\arg(1-z)| < \pi$ (see, for example, (15.3.9) of Abramowitz and Stegun (1972)), yields exactly (3.3.2). Here, the gamma function is extended to negative non-integer values by definition of

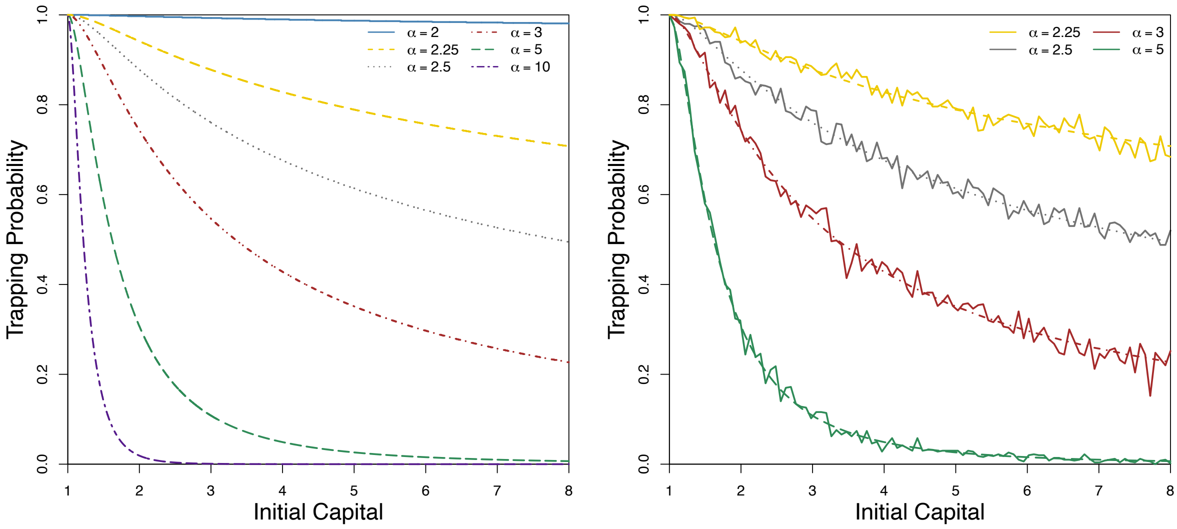
$$\Gamma(x) := \frac{1}{x}\Gamma(x+1),$$

for $x < 0, x \notin \mathbb{Z}$.

The analytic trapping probability for households susceptible to proportional losses with power distributed remaining proportions of capital, as derived in Proposition 3.3.3, is presented in Figure 3.2a for varying shape parameter α. Note that in the power distributed case, the trapping probability tends to 1 as $\frac{\lambda}{r}$ tends to α. Figure 3.2b compares realisations of the explicit trapping probability of Proposition 3.3.3 with trapping probabilities obtained via simulation.

Figure 3.3 compares the trapping probabilities in (3.3.17) and (2.4.2) for proportional and random-valued losses, respectively, for a given set of parameters. Note that the parameter of the exponential distribution in Chapter 2 and Proposition 3.3.2, has been changed to μ for clarity of presentation. Trapping probabilities for a number of exponential claim size distributions are compared to that of proportional losses with an expected value of approximately 16.7%. For random-valued claim sizes with an expected value of 0.5 ($\mu = 2$), the trapping probability is greater in comparison to that of proportional losses for the most vulnerable, however as capital increases, the trapping probability under proportional losses exceeds the random-valued case. If the expected claim size increases to 1 ($\mu = 1$), the trapping probability for proportional losses is significantly lower than in the random-valued case, exceeding the random-valued trapping probability only at the highest levels of capital. Compared to the mean loss associated with beta distributed remaining proportions with $\alpha = 5$, an expected claim size of 1 is low with respect to the highest levels of initial capital considered. For $x = 6$, the two loss rates coincide. This therefore suggests that for equivalent loss size, the trapping probability for proportional losses is reduced in comparison to random-valued losses. However, for capital levels below this point, random-valued losses account for a greater proportion of capital than the proportional loss case selected for comparison, and thus the increased trapping probability is intuitive.

Further analysis is needed to validate the consistency in the reduction of the probability for equivalent losses.



(a) **(b)**

Figure 3.2: Trapping probabilities for parameters $a = 0.1$, $b = 1.4$, $c = 0.4$ ($r = 0.504$), $\lambda = 1$ and $x^* = 1$ with $Z_i \sim Beta(\alpha, 1)$ for (a) $\alpha = 2, 2.25, 2.5, 3, 5, 10$, (b) $\alpha = 2.25, 2.5, 3, 5$ with comparison to the corresponding simulated trapping probability of the capital growth process with $N = 500$ households, terminal time $T = 1000$ and time-step $dt = 0.1$, evaluated at 100 initial capital levels.

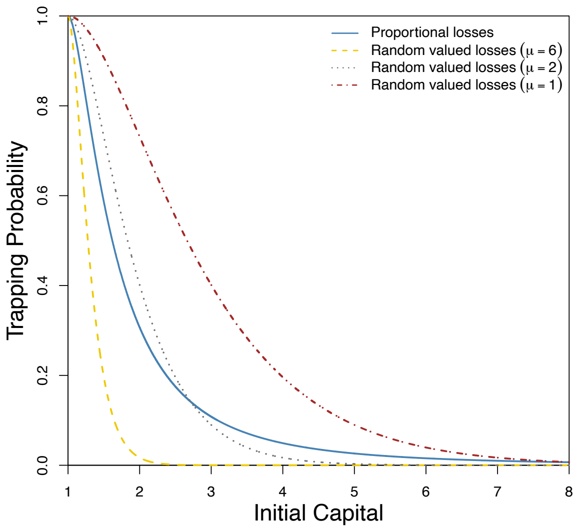


Figure 3.3: (a) Trapping probabilities for parameters $a = 0.1$, $b = 1.4$, $c = 0.4$ ($r = 0.504$), $\lambda = 1$ and $x^* = 1$ for proportional losses with $Z_i \sim Beta(5, 1)$ and random-valued losses with $Z_i \sim Exp(\mu)$ for $\mu = 6, 2, 1$.

3.4 Introducing microinsurance

In line with Section 2.4 of Chapter 2, the presence of a fixed premium insurance product that covers $100 \cdot (1-\kappa)$ percent of all household losses is now considered, where $1-\kappa$ for $\kappa \in (0,1]$ is the proportionality factor. Assume that coverage is purchased by all households. The capital growth process has an analogous structure to that of Definition 3.2.1, with the remaining proportion of capital after each loss event instead denoted Y_i, where $Y_i = 1-\kappa(1-Z_i) \in (1-\kappa, 1)$. As such, in between loss events, where $T_{i-1} \leq t < T_i$, the capital growth process again follows (3.2.1a) and (3.2.1b), while at event times $t = T_i$ the process is given by

$$X_{T_i} = \begin{cases} [\left(X_{T_{i-1}} - x^{*(\kappa)}\right) e^{r^{(\kappa)}(T_i - T_{i-1})} + x^{*(\kappa)}] \cdot Y_i, & \text{if } X_{T_{i-1}} > x^{*(\kappa)} \quad (3.4.1a) \\ X_{T_{i-1}} \cdot Y_i, & \text{otherwise.} \quad (3.4.1b) \end{cases}$$

The critical capital (or poverty line) and capital growth rate associated with an insured household are denoted $x^{*(\kappa)}$ and $r^{(\kappa)} = (1-a) \cdot (b-\pi) \cdot c > 0$, respectively, where π denotes the premium rate. The premium is calculated according to the expected value principle as in (2.4.1), such that

$$\pi = \pi(\kappa, \theta) = (1+\theta) \cdot (1-\kappa) \cdot \lambda \cdot \mathbb{E}\left[1 - Z_i\right]. \tag{3.4.2}$$

As discussed in Chapter 2, due to the need for premium payments, the critical capital in the insured case is greater than that of an uninsured household, while the growth rate is reduced.

Note that for $\kappa = 1$, the capital model in (3.4.1a) and (3.4.1b) exactly corresponds to that of an uninsured household, as discussed in Section 3.3.

Proposition 3.4.1. Consider a household capital process that follows (3.2.2a) and (3.2.2b) in between loss events and (3.4.1a) and (3.4.1b) at loss event times. For initial capital $x \geq x^{*(\kappa)}$, capital growth rate $r^{(\kappa)}$, loss intensity $\lambda > 0$ and remaining proportions of capital Z_i with distribution $Beta(1,1)$, the adjustment coefficient of the corresponding Lundberg equation exists if

$$\frac{\lambda}{r^{(\kappa)}} < \frac{2}{{}_2F_1(1,2;3;\kappa) \cdot \kappa}, \tag{3.4.3}$$

where ${}_2F_1(\cdot)$ is the Gauss hypergeometric function as defined in Appendix B.

Proof. The Lundberg equation corresponding to the logarithmised process of the capital process of Proposition 3.4.1 at $x^{*(\kappa)} = 0$ is

$$\mathbb{E}[e^{-s\log(Z_i)}]\mathbb{E}[e^{-sr^{(\kappa)}\tilde{T}_i}] = \mathbb{E}[e^{-s(\log(Z_i)+r^{(\kappa)}\tilde{T}_i)}] = 1.$$

The condition that must hold in order for the adjustment coefficient R in Theorem 3.2.1 to exist is therefore

$$\mathbb{E}[r^{(\kappa)}\tilde{T}_i + \log(1-\kappa(1-Z_i))] > 0 \iff r^{(\kappa)}\mathbb{E}[\tilde{T}_i] + \mathbb{E}[\log(1-\kappa(1-Z_i))] > 0. \tag{3.4.4}$$

For power distributed remaining proportions, integrating by parts:

$$\mathbb{E}[\log(1-\kappa(1-Z_i))] = -\frac{\kappa}{\alpha}\int_0^1 (1-\kappa+\kappa z)^{-1} z^{\alpha} dz.$$

Since the Mellin transform of the piecewise function

$$\begin{cases}(1+ax)^{-1}, & 0<x<b\\ 0, & x>b\end{cases}$$

for $|\arg(1-ab)|<\pi$ is given by

$$b^s s^{-1}{}_2F_1(1,s;1+s;-ab)$$

for $\mathrm{Re}(s)>0$ by definition of the Mellin transform in Appendix A (see, for example, Chapter 6.2 of Erdelyi et al. (1954)), it holds that

$$\mathbb{E}[\log(1-\kappa(1-Z_i))] = -\frac{\kappa}{\alpha(\alpha+1)(1-\kappa)}{}_2F_1\left(1,\alpha+1;\alpha+2;-\frac{\kappa}{1-\kappa}\right).$$

The constraint in (3.4.4) is therefore equivalent to

$$\frac{r^{(\kappa)}}{\lambda} > \frac{\kappa}{\alpha(\alpha+1)(1-\kappa)}{}_2F_1\left(1,\alpha+1;\alpha+2;-\frac{\kappa}{1-\kappa}\right),$$

which for uniformly distributed remaining capital $\alpha=1$, reduces to

$$\frac{\lambda}{r^{(\kappa)}} < \frac{2(1-\kappa)}{{}_2F_1\left(1,2;3;\frac{-\kappa}{1-\kappa}\right)\cdot\kappa} = \frac{2}{{}_2F_1\left(1,2;3;\kappa\right)\cdot\kappa},$$

by the identity

$${}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)$$

(see, for example, (15.3.4) of Abramowitz and Stegun (1972)), as required. □

Note that for $\kappa=1$, since ${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, (3.4.3) reduces to the uninsured constraint in (3.2.6).

For ease of presentation, in the remainder of this section, the critical capital will be denoted by x^* and the capital growth rate by r in all trapping probability computations. Superscript (κ) will be included if comparisons with the uninsured case of Section 3.3 are made.

Derivation of the infinitesimal generator of the insured process is analogous to that of Section 3.2. The only adjustment appears in the domain of the random variable capturing the remaining proportion of capital. The generator $(\mathcal{A}f)(x)$ is therefore given by

$$r(x-x^*)f'(x)-\lambda f(x)+\lambda\int_{1-\kappa}^{1} f(x\cdot y)d\tilde{G}(y)=0,$$

where $\tilde{G}(y)$ is the distribution function of Y_i, such that $\tilde{G}(y)=G\left(1-\frac{1}{\kappa}(1-y)\right)$ and $\tilde{g}(y)=\frac{1}{\kappa}g\left(1-\frac{1}{\kappa}(1-y)\right)$.

Note that if $y\le\frac{x^*}{x}$, trapping has occurred with the first loss. In order to account for this boundary on x, redefine the infinitesimal generator as a piecewise function with boundary at $x(1-\kappa)=x^*$, where $1-\kappa$ is the lower bound of Y_i. Then, $(\mathcal{A}f)(x)$ is given by

$$\begin{cases} r(x-x^*)f'(x)-\lambda f(x)+\dfrac{\lambda}{\kappa x}\displaystyle\int_{x(1-\kappa)}^{x} f(u)du, & x>\dfrac{x^*}{1-\kappa} \quad (3.4.5a)\\ r(x-x^*)f'(x)-\lambda f(x)+\dfrac{\lambda}{\kappa x}\displaystyle\int_{x^*}^{x} f(u)du+\dfrac{\lambda x^*}{\kappa x}+\dfrac{\lambda(\kappa-1)}{\kappa}, & x^*<x<\dfrac{x^*}{1-\kappa}. \quad (3.4.5b)\end{cases}$$

As in Section 3.2, only the behaviour of the process above the critical capital x^* is of interest. Therefore, making the same change of variable $h(x) = f(x + x^*)$, the infinitesimal generator in (3.4.5a) and (3.4.5b) is reformulated such that $(\mathcal{A}h)(\tilde{x})$ is

$$\begin{cases} (\tilde{x}+x^*)(r\tilde{x}h'(\tilde{x})-\lambda h(\tilde{x}))+\dfrac{\lambda}{\kappa}\displaystyle\int_{(\tilde{x}+x^*)(1-\kappa)}^{\tilde{x}+x^*} h(u-x^*)du, & \tilde{x}>\dfrac{x^*\kappa}{1-\kappa} \quad (3.4.6a)\\ (\tilde{x}+x^*)(r\tilde{x}h'(\tilde{x})-\lambda h(\tilde{x}))+\dfrac{\lambda}{\kappa}\displaystyle\int_{x^*}^{\tilde{x}+x^*} h(u-x^*)du+\lambda x^* & \\ -\dfrac{\lambda\tilde{x}(1-\kappa)}{\kappa}, & \tilde{x}<\dfrac{x^*\kappa}{1-\kappa}, \quad (3.4.6b) \end{cases}$$

where $\tilde{x} = x - x^*$.

Solution of $(\mathcal{A}h)(\tilde{x}) = 0$ is again sought to obtain the trapping probability of the insured process. For this purpose, two approaches are considered. In line with the computations presented so far in this chapter, in the first approach, Laplace transform methods are implemented. The second approach alternatively considers the derivative of the piecewise IDE in (3.4.6a) and (3.4.6b).

Approach 1: The Laplace transform of the piecewise IDE in (3.4.6a) and (3.4.6b) is given by

$$\begin{aligned}\mathcal{L}(r\tilde{x}(\tilde{x}+x^*)h'(\tilde{x})-\lambda(\tilde{x}+x^*)h(\tilde{x}))+\frac{\lambda}{\kappa}\int_{\frac{x^*\kappa}{1-\kappa}}^{\infty} e^{-s\tilde{x}}\int_{(\tilde{x}+x^*)(1-\kappa)}^{\tilde{x}+x^*} h(u-x^*)dud\tilde{x}\\ +\frac{\lambda}{\kappa}\int_0^{\frac{x^*\kappa}{1-\kappa}} e^{-s\tilde{x}}\int_{x^*}^{\tilde{x}+x^*} h(u-x^*)dud\tilde{x}+\int_0^{\frac{x^*\kappa}{1-\kappa}} e^{-s\tilde{x}}\Big(\lambda x^*-\frac{\lambda\tilde{x}(1-\kappa)}{\kappa}\Big)d\tilde{x}. \qquad (3.4.7)\end{aligned}$$

Applying integration by parts on all double integral terms:

$$\begin{aligned}\int_{\frac{x^*\kappa}{1-\kappa}}^{\infty} e^{-s\tilde{x}}\int_{(\tilde{x}+x^*)(1-\kappa)}^{\tilde{x}+x^*} h(u-x^*)dud\tilde{x} =&\frac{1}{s}e^{-\frac{sx^*\kappa}{1-\kappa}}\int_{x^*}^{\frac{x^*\kappa}{1-\kappa}+x^*} h(u-x^*)du+\frac{1}{s}\int_{\frac{x^*\kappa}{1-\kappa}}^{\infty} e^{-s\tilde{x}}h(\tilde{x})d\tilde{x}\\ &-\frac{1-\kappa}{s}\mathcal{L}(h(\tilde{x}(1-\kappa)-x^*\kappa))-\frac{1-\kappa}{s^2}\Big(e^{-\frac{sx^*\kappa}{1-\kappa}}-1\Big),\end{aligned}$$

$$\int_0^{\frac{x^*\kappa}{1-\kappa}} e^{-s\tilde{x}}\int_{x^*}^{\tilde{x}+x^*} h(u-x^*)dud\tilde{x} = -\frac{1}{s}e^{-\frac{sx^*\kappa}{1-\kappa}}\int_{x^*}^{\frac{x^*\kappa}{1-\kappa}+x^*} h(u-x^*)du+\frac{1}{s}\int_0^{\frac{x^*\kappa}{1-\kappa}} e^{-s\tilde{x}}h(\tilde{x})d\tilde{x}$$

and

$$\int_0^{\frac{x^*\kappa}{1-\kappa}} \lambda x^*e^{-s\tilde{x}}-\frac{\lambda(1-\kappa)}{\kappa}\tilde{x}e^{-s\tilde{x}}d\tilde{x}=\frac{\lambda x^*}{s}+\frac{\lambda(1-\kappa)}{\kappa s^2}\Big(e^{-\frac{sx^*\kappa}{1-\kappa}}-1\Big),$$

such that (3.4.7) reduces to

$$\mathcal{L}(r\tilde{x}(\tilde{x}+x^*)h'(\tilde{x})-\lambda(\tilde{x}+x^*)h(\tilde{x}))+\frac{\lambda}{\kappa s}\mathcal{L}(h(\tilde{x}))-\frac{(1-\kappa)\lambda}{\kappa s}\mathcal{L}(h(\tilde{x}(1-\kappa)-x^*\kappa))+\frac{\lambda x^*}{s}.$$

Evaluating Laplace transforms and solving for $(\mathcal{A}h)(\tilde{x}) = 0$ then gives the following second order ODE:

$$\begin{aligned}s^2F''(s)+\Big(\Big(2+\frac{\lambda}{r}\Big)s-x^*s^2\Big)F'(s)+\Big(-x^*\Big(1+\frac{\lambda}{r}\Big)s+\frac{\lambda}{r\kappa}\Big)F(s)-\frac{\lambda e^{\frac{-sx^*\kappa}{1-\kappa}}}{r\kappa}F\Big(\frac{s}{1-\kappa}\Big)\\ =-\frac{(1-\kappa)\lambda}{r\kappa s}\Big(e^{\frac{-sx^*\kappa}{1-\kappa}}-1\Big)-\frac{\lambda x^*}{r}. \qquad (3.4.8)\end{aligned}$$

Note that, the presence of $1-\kappa$ in the lower limit of the IDE (3.4.6a) induces a scaled function in the ODE obtained in the evaluation of Laplace transforms. Since $s=0$ is a regular singular point of the homogeneous part of (3.4.8), consider that the homogeneous solution has the form

$$F_h(s)=\sum_{n=0}^{\infty}A_n s^{n+m}. \tag{3.4.9}$$

The homogeneous part of (3.4.8) is therefore equivalent to

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+m)(n+m-1)A_n s^{n+m-2}+\Big(2+\frac{\lambda}{r}\Big)\sum_{n=0}^{\infty}(n+m)A_n s^{n+m-2}\\
&-x^*\sum_{n=0}^{\infty}(n+m)A_n s^{n+m-1}-x^*\Big(1+\frac{\lambda}{r}\Big)\sum_{n=0}^{\infty}A_n s^{n+m-1}+\frac{\lambda}{r\kappa}\sum_{n=0}^{\infty}A_n s^{n+m-2}\\
&-\frac{\lambda e^{\frac{-sx^*\kappa}{1-\kappa}}}{r\kappa}\sum_{n=0}^{\infty}A_n\frac{s^{n+m-2}}{(1-\kappa)^{n+m}}=0\\
\iff&\sum_{n=0}^{\infty}\Big((n+m)(n+m-1)+\Big(2+\frac{\lambda}{r}\Big)(n+m)+\frac{\lambda}{r\kappa}\Big(1-e^{\frac{-sx^*\kappa}{1-\kappa}}(1-\kappa)^{-(n+m)}\Big)\Big)A_n s^{n+m-2}\\
&-x^*\sum_{n=1}^{\infty}\Big((n+m-1)+\Big(1+\frac{\lambda}{r}\Big)\Big)A_{n-1}s^{n+m-2}=0\\
\iff&\Big(m(m-1)+\Big(2+\frac{\lambda}{r}\Big)m+\frac{\lambda}{r\kappa}\Big(1-e^{\frac{-sx^*\kappa}{1-\kappa}}(1-\kappa)^{-m}\Big)\Big)A_0 s^{m-2}\\
&+\sum_{n=1}^{\infty}\Big(\Big((n+m)\Big(n+m+1+\frac{\lambda}{r}\Big)+\frac{\lambda}{r\kappa}\Big(1-e^{\frac{-sx^*\kappa}{1-\kappa}}(1-\kappa)^{-(n+m)}\Big)\Big)A_n\\
&-x^*\Big(n+m+\frac{\lambda}{r}\Big)A_{n-1}\Big)s^{n+m-2}=0.
\end{aligned} \tag{3.4.10}$$

Since the exponential terms in (3.4.10) are also functions of s, consider their power series representation:

$$e^{\frac{-sx^*\kappa}{1-\kappa}}=\sum_{k=0}^{\infty}\frac{(-\frac{x^*\kappa}{1-\kappa}\cdot s)^k}{k!}.$$

Then, $e^{\frac{-sx^*\kappa}{1-\kappa}} = 1 - \frac{x^*\kappa}{1-\kappa}\cdot s + \mathcal{O}(s^2)$ such that (3.4.10) can be expressed as

$$\begin{aligned}
&\Big(m\Big(m+1+\frac{\lambda}{r}\Big) + \frac{\lambda}{r\kappa}\Big(1-\Big(1-\frac{x^*\kappa}{1-\kappa}\cdot s+\mathcal{O}(s^2)\Big)(1-\kappa)^{-m}\Big)\Big)A_0 s^{m-2}\\
&+\sum_{n=1}^{\infty}\Big(\Big((n+m)\Big(n+m+1+\frac{\lambda}{r}\Big)+\frac{\lambda}{r\kappa}\Big(1-\Big(1-\frac{x^*\kappa}{1-\kappa}\cdot s+\mathcal{O}(s^2)\Big)(1-\kappa)^{-(n+m)}\Big)\Big)A_n\\
&-x^*\Big(n+m+\frac{\lambda}{r}\Big)A_{n-1}\Big)s^{n+m-2}=0\\
\iff &\Big(m\Big(m+1+\frac{\lambda}{r}\Big)+\frac{\lambda}{r\kappa}\Big(1-(1-\kappa)^{-m}\Big)\Big)A_0 s^{m-2}+\sum_{n=1}^{\infty}\Big(\Big((n+m)\Big(n+m+1+\frac{\lambda}{r}\Big)\\
&+\frac{\lambda}{r\kappa}\Big(1-(1-\kappa)^{-(n+m)}\Big)\Big)A_n - x^*\Big(n+m+\frac{\lambda}{r}\Big)A_{n-1}-\frac{\lambda}{r\kappa}\frac{(\frac{-x^*\kappa}{1-\kappa})^n}{n!}(1-\kappa)^{-m}A_0\Big)s^{n+m-2}\\
&-\frac{\lambda}{r\kappa}(1-\kappa)^{-(n+m)}\sum_{k=1}^{\infty}\frac{(\frac{-x^*\kappa}{1-\kappa})^k}{k!}A_n s^{k+n+m-2}.
\end{aligned}\tag{3.4.11}$$

The coefficients of s^n in (3.4.11) must vanish for all n. Considering the coefficient of the smallest power of s, the indicial equation is given by

$$m\Big(m+1+\frac{\lambda}{r}\Big)+\frac{\lambda}{r\kappa}\Big(1-(1-\kappa)^{-m}\Big)=0, \tag{3.4.12}$$

since $A_0 \neq 0$. Solution of (3.4.12) gives the value of the power m in (3.4.9).

If the ratio of coefficients A_n/A_{n-1} were a rational function, the series in (3.4.9) could be written as the product of a power function and a generalised hypergeometric series, aligning with the structure of the uninsured trapping probability in (3.3.2). Obtaining the coefficients A_n from (3.4.11) and (3.4.12) analytically, is however intractable.

The limit of the series as x^* goes to 0 provides an upper bound on the trapping probability of the process with critical capital x^*. However, reverting back to (3.4.10), this assumption gives

$$\begin{aligned}
&\Big(m(m-1)+\Big(2+\frac{\lambda}{r}\Big)m+\frac{\lambda}{r\kappa}\Big(1-(1-\kappa)^{-m}\Big)\Big)A_0 s^{m-2}\\
&\quad+\sum_{n=1}^{\infty}\Big((n+m)\Big(n+m+1+\frac{\lambda}{r}\Big)+\frac{\lambda}{r\kappa}\Big(1-(1-\kappa)^{-(n+m)}\Big)\Big)A_n s^{n+m-2}=0,
\end{aligned}$$

which again has indicial equation (3.4.12).

The approach considered here is that of the Frobenius method. Discussed in detail in Fedoryuk (1993), the algebraic properties of a homogeneous solution of the form (3.4.9) with regular singular point at $s=0$ are known, enabling asymptotic analysis of the desired solution. As attempted in the approach presented here, Albrecher et al. (2012) consider the asymptotic behaviour of a renewal risk process with stochastic investment through application of the Frobenius method. The asymptotic decay of the ruin probability is then obtained through application of the Karamata-Tauberian theorems.

Approach 2: In this alternative approach, each component of the piecewise function in (3.4.6a) and (3.4.6b) is considered separately.

Proposition 3.4.2. Consider a household capital process defined by (3.2.1a) and (3.2.1b) in between losses and by (3.4.1a) and (3.4.1b) at loss event times, with coverage proportionality factor $1-\kappa \in (0,1]$. Assume capital growth rate r, loss intensity $\lambda > 0$ and remaining proportions of capital with distribution $Beta(1,1)$. Then, for initial capital $x^* < x < \frac{x^*}{1-\kappa}$, the general form of the trapping probability is given by

$$f(x) = C \cdot \left(\frac{x-x^*}{x^*}\right)^{\frac{\lambda}{r}} {}_2F_1\left(a_1 + \frac{\lambda}{r}, b_1 + \frac{\lambda}{r}, 1 + \frac{\lambda}{r}; -\frac{x-x^*}{x^*}\right) + 1, \tag{3.4.13}$$

for constant C and $\frac{\lambda}{r} < 1$, where $a_1 + b_1 + 1 = 2 - \frac{\lambda}{r}$, $a_1 \cdot b_1 = \lambda(1-\kappa)/r\kappa$ and ${}_2F_1(\cdot)$ is the Gauss hypergeometric function as in Appendix B.

Proof. Consider (3.4.6b). By Theorem 3.3.1, the trapping probability for $\tilde{x} < \frac{x^*\kappa}{1-\kappa}$ again satisfies $(\mathcal{A}h)(\tilde{x}) = 0$. As such, the IDE that must be solved is

$$r\tilde{x}(\tilde{x}+x^*)h'(\tilde{x}) - \lambda(\tilde{x}+x^*)h(\tilde{x}) + \frac{\lambda}{\kappa}\int_{x^*}^{\tilde{x}+x^*} h(u-x^*)du + \lambda x^* - \frac{\lambda\tilde{x}(1-\kappa)}{\kappa} = 0. \tag{3.4.14}$$

Taking the derivative of (3.4.14) with respect to x gives

$$\tilde{x}(\tilde{x}+x^*)h''(\tilde{x}) + \left(\left(2-\frac{\lambda}{r}\right)\tilde{x} + x^*\left(1-\frac{\lambda}{r}\right)\right)h'(\tilde{x}) + \frac{\lambda(1-\kappa)}{r\kappa}h(\tilde{x}) = \frac{\lambda(1-\kappa)}{r\kappa}, \tag{3.4.15}$$

the homogeneous part of which is exactly Gauss' hypergeometric equation, as defined in Appendix B. Under the change of variable $g(z) := h(\tilde{x})$, where $z = -\frac{\tilde{x}}{x^*}$, (3.4.15) yields

$$z(z-1)g''(z) + [(a_1+b_1+1)z - \gamma]g'(z) + a_1b_1g(z) = 0,$$

for $a_1 + b_1 + 1 = 2 - \frac{\lambda}{r}$, $a_1 \cdot b_1 = \frac{\lambda(1-\kappa)}{r\kappa}$ and $\gamma = 1 - \frac{\lambda}{r}$, with general solution

$$g(z) = C_1 \cdot {}_2F_1\left(a_1, b_1, 1-\frac{\lambda}{r}; z\right) + C_2 \cdot z^{\frac{\lambda}{r}} {}_2F_1\left(a_1+\frac{\lambda}{r}, b_1+\frac{\lambda}{r}, 1+\frac{\lambda}{r}; z\right).$$

Proposing an Ansatz of $g_p(\tilde{x}) = A$ for the particular solution of (3.4.15) yields $A = 1$, such that the general solution of $h(\tilde{x}) = g\left(-\frac{\tilde{x}}{x^*}\right)$ for $\tilde{x} < \frac{x^*\kappa}{1-\kappa}$ is

$$h(\tilde{x}) = C_1 \cdot {}_2F_1\left(a_1, b_1, 1-\frac{\lambda}{r}; -\frac{\tilde{x}}{x^*}\right) + C_2 \cdot \left(-\frac{\tilde{x}}{x^*}\right)^{\frac{\lambda}{r}} {}_2F_1\left(a_1+\frac{\lambda}{r}, b_1+\frac{\lambda}{r}, 1+\frac{\lambda}{r}; -\frac{\tilde{x}}{x^*}\right) + 1.$$

The lower boundary condition for $h(\tilde{x})$ in this interval:

$$\lim_{\tilde{x}\to 0} h(\tilde{x}) = 1,$$

then holds if and only if $C_1 = 0$. Then, since $h(\tilde{x}) = f(x)$ and letting $C = C_2 \cdot (-1)^{\frac{\lambda}{r}}$, (3.4.13) holds, as required. ☐

In order to analyse the upper interval of the infinitesimal generator IDE, consider (3.4.6a). Solving $(\mathcal{A}h)(\tilde{x}) = 0$, it holds that

$$r\tilde{x}(\tilde{x}+x^*)h'(\tilde{x}) - \lambda(\tilde{x}+x^*)h(\tilde{x}) + \frac{\lambda}{\kappa}\int_{(\tilde{x}+x^*)(1-\kappa)}^{\tilde{x}+x^*} h(u-x^*)du = 0 \tag{3.4.16}$$

for $\tilde{x} > \frac{x^*\kappa}{1-\kappa}$, which has derivative

$$\tilde{x}(\tilde{x}+x^*)h''(\tilde{x}) + \Big(\Big(2-\frac{\lambda}{r}\Big)\tilde{x} + x^*\Big(1-\frac{\lambda}{r}\Big)\Big)h'(\tilde{x}) + \frac{\lambda(1-\kappa)}{r\kappa}h(\tilde{x})$$
$$-\frac{\lambda(1-\kappa)}{r\kappa}h(\tilde{x}(1-\kappa)-x^*\kappa) = 0. \tag{3.4.17}$$

Inclusion of $(1-\kappa)$ in the lower interval limit in (3.4.16) again induces a non-trivial function in the resulting second order ODE.

Since $\tilde{x} = 0$ is a regular singular point of (3.4.17), consider that $h(\tilde{x})$ has the following form:

$$h(\tilde{x}) = \sum_{n=0}^{\infty} a_n \tilde{x}^{n+m}.$$

Substitution into (3.4.17) gives

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+m)(n+n-1)a_n\tilde{x}^{n+m} + x^*\sum_{n=0}^{\infty}(n+m)(n+m-1)a_n\tilde{x}^{n+m-1}\\
&+\Big(2-\frac{\lambda}{r}\Big)\sum_{n=0}^{\infty}(n+m)a_n\tilde{x}^{n+m} + x^*\Big(1-\frac{\lambda}{r}\Big)\sum_{n=0}^{\infty}(n+m)a_n\tilde{x}^{n+m-1}\\
&+\frac{\lambda(1-\kappa)}{r\kappa}\sum_{n=0}^{\infty}a_n\tilde{x}^{n+m} - \frac{\lambda(1-\kappa)}{r\kappa}\sum_{n=0}^{\infty}a_n(\tilde{x}(1-\kappa)-x^*\kappa)^{n+m} = 0\\
\iff &\sum_{n=0}^{\infty}\Big((n+m)\Big(n+m+1-\frac{\lambda}{r}\Big)+\frac{\lambda(1-\kappa)}{r\kappa}\Big)a_n\tilde{x}^{n+m}\\
&+x^*\sum_{n=0}^{\infty}(n+m)\Big(n+m-\frac{\lambda}{r}\Big)a_n\tilde{x}^{n+m-1} - \frac{\lambda(1-\kappa)}{r\kappa}\sum_{n=0}^{\infty}a_n(1-\kappa)^{n+m}\Big(\tilde{x}-\frac{x^*\kappa}{1-\kappa}\Big)^{n+m}\\
&=0\\
\iff &\sum_{n=0}^{\infty}\Big(\Big((n+m)\Big(n+m+1-\frac{\lambda}{r}\Big)+\frac{\lambda(1-\kappa)}{r\kappa}\Big)a_n\\
&+x^*(n+m+1)\Big(n+m+1-\frac{\lambda}{r}\Big)a_{n+1}\Big)\tilde{x}^{n+m} + x^*m\Big(m-\frac{\lambda}{r}\Big)a_0\tilde{x}^{m-1}\\
&-\frac{\lambda(1-\kappa)}{r\kappa}(\tilde{x}(1-\kappa)-x^*\kappa)^m\sum_{n=0}^{\infty}a_n\sum_{k=0}^{n}\binom{n}{k}(-x^*\kappa)^{n-k}(1-\kappa)^k\tilde{x}^k = 0,
\end{aligned} \tag{3.4.18}$$

by the binomial expansion of $\Big(\tilde{x}-\frac{x^*\kappa}{1-\kappa}\Big)^n$. The final term in (3.4.18) contains all powers of $\tilde{x}$ from 0 to nm. Since the value of m is unknown, it cannot be said that there are no coefficients of $\tilde{x}^{m-1}$ in this latter term. Equating coefficients to 0 and obtaining the indicial equation is therefore not possible in the absence of knowledge on the value of m.

Taking the limit of (3.4.18) as x^* goes to 0 results in an indicial equation of a similar structure to that in (3.4.11). However, it is important to note, that the parameters m in (3.4.11) and (3.4.19) do not necessarily coincide. In Approach 2, m is the power multiplying the series in the trapping probability defined only for $x > \frac{x^*}{1-\kappa}$. Whereas in Approach 1, m is the power term that appears in the Laplace transform of this probability over the whole interval $x > x^*$. The series in (3.4.18) reduces to

$$\sum_{n=0}^{\infty}\Big((n+m)\Big(n+m+1-\frac{\lambda}{r}\Big)+\frac{\lambda(1-\kappa)}{r\kappa}\Big(1-(1-\kappa)^{n+m}\Big)\Big)a_n\tilde{x}^{n+m}=0,$$

which holds if and only if

$$m\Big(m+1-\frac{\lambda}{r}\Big)+\frac{\lambda(1-\kappa)}{r\kappa}\Big(1-(1-\kappa)^{m}\Big)=0, \qquad (3.4.19)$$

for $a_0 \neq 0$.

Given the analytically intractable structure of the indicial equations obtained in Approaches 1 and 2, future research will involve determining the constant m numerically.

Figure 3.4 presents a comparison of trapping probabilities for the uninsured and insured capital processes, where the insured trapping probability is obtained through simulation of $N = 1000$ realisations of the capital process at 100 initial capital levels, with terminal time $T = 2000$ and observation intervals $dt = 0.1$. The increase in the trapping probability associated with the most vulnerable when covered by insurance is again observed. However, this increase occurs for a much smaller proportion of the low-income sample in comparison to when covered for random-valued losses as in Chapter 2. In simulating the insured trapping probability, the increase in the critical capital associated with the need for premium payment is accounted for through specification of $x^{*(\kappa)}$, such that an insured household is deemed to be trapped when their capital falls below $I^*/(b-\pi)$, where the critical income I^* (as defined in Chapter 2) is equal to b when $x^* = 1$. As such, in the insured case, households with capital just above $x^* = 1$ have already become trapped. However, when comparing the uninsured trapping probability with the insured trapping probability for $x^{*(\kappa)} = 1$, as in Chapter 2, the trapping probability is lower for insured households for all levels of capital considered. These observations suggest that for the most poor, purchase of microinsurance for coverage of proportional losses is more affordable than classical coverage for random-valued losses.

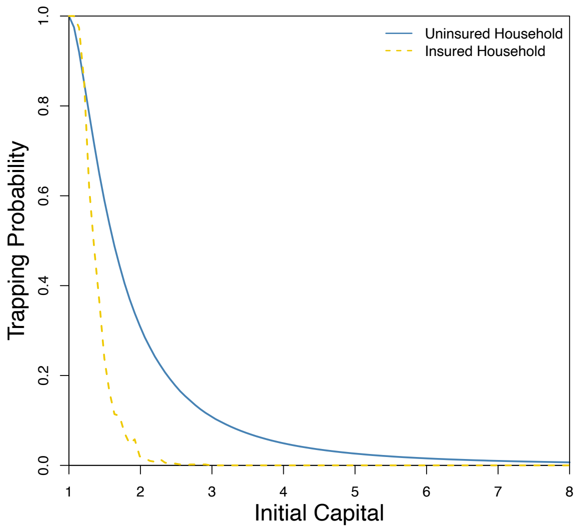


Figure 3.4: Trapping probabilities for the uninsured and insured capital processes when $Z_i \sim Beta(5, 1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$, $\kappa = 0.5$, $\theta = 0.5$, $x^* = 1$ and $x^{(\kappa)*} = b/(b - \pi)$, with simulation parameters for the insured process: $N = 1000$, $T = 2000$, $dt = 0.1$.

3.5 Concluding remarks

This chapter considers an adjustment of the capital process of Chapter 2 in which low-income households are susceptible to losses proportional to their capital level, as in Kovacevic and Pflug (2011). Using Laplace transform methods, explicit trapping probabilities for $Beta(1, 1)$ (uniform) and $Beta(\alpha, 1)$ distributed remaining proportions of capital were obtained. In comparison to the corresponding trapping probability for random-valued losses, in the uniform case, the proportional trapping probability exhibits a slower rate of decay, in line with the non-zero probability of high income households losing a large proportion of their wealth.

Consideration of proportional insurance coverage requires redefinition of the infinitesimal generator of the process. For $x < x^*/(1 - \kappa)$ and under assumption of uniformly distributed remaining proportions of capital, use of Laplace transform methods again yields the general form of the explicit trapping probability. This trapping probability tends to the uninsured case as κ tends to 1. However, for $x \geq x^*/(1 - \kappa)$, the structure of the proportional insurance product considered in this analysis induces an unknown constant term in the integral limits of the associated IDE, which further induces both scaled and shifted functional terms in the Laplace and trapping side ODEs, making them intractable to solve analytically. Future work will involve solving the insured IDE numerically.

Simulation analysis of the insured trapping probability suggests that the increase in trapping probability observed for random-valued losses is less severe in this proportional case. Given the results of Kovacevic and Pflug (2011), where an increase in trapping probability similar to that of Chapter 2 is observed, this is likely to be highly dependent on the specification of parameters. It should however be noted that, the distribution of the remaining proportion of capital considered in the numerical example of Kovacevic and Pflug (2011) is such that losses have an expected value of 88%, an extremely high proportion given the loss rate parameter of 1. In turn, the associated premium rates are high and will constrain capital growth more significantly. The lower rate associated with the distribution selected for presentation in the analysis of this chapter captures losses of varying severity, as is the experience of a low-income population, and will necessitate reduced premiums.

The findings of this chapter therefore suggest that insurance for proportional losses is more affordable than coverage for losses of random value. This aligns with the idea that premiums are normalised to wealth under the proportional loss structure, thus improving the variability in the affordability of the associated products. As such, if the assumption of proportionality is correct, in the context of subsidisation, the proportion of the low-income population requiring full government support may be narrower than anticipated. However, for those closest to the poverty line, as in Chapter 2 and in the findings of existing studies, insurance and the associated need for premium payments again increases the probability of becoming trapped.

Chapter 4

A group-based approach to inclusive insurance

Risk sharing mechanisms are widespread in low-income communities, mitigating the impact of financial losses that are otherwise uninsured. In this chapter, the group-based nature of financial vulnerability is addressed. Adopting a highly flexible stochastic dissemination model to the context of poverty reduction, the wealth of a group containing both uninsured and insured agents is analysed. The model captures four types of transaction events: external arrivals, internal redistribution events, wealth losses and premium payments, where the modelling of premium payments is analogous to loss events, with increased frequency and reduced severity. The model is underlined by an exogenously evolving Markov background process that represents the economic state of the system. A system of coupled differential equations for the joint transient distribution of agent wealth is derived and is reduced to a linear system of differential equations through consideration of the moments of agent wealth. Sensitivity analysis is performed to establish the impact of each component of the system's construction on the wealth of the group. The probability of falling below the poverty line is then determined through application of a normal approximation and the impact of insurance in reducing this probability considered under varying levels of subsidisation.

4.1 Introduction

Addressing the prevalence of risk sharing among low-income communities (see Section 1.1.2), this chapter focuses on the wealth dynamics of a group. Considering a system of wealth in which both transactions and losses occur, an adjustment of the stochastic dissemination model proposed by Chan and Mandjes (2022) is adopted to analyse the behaviour of group wealth over time. This model is a highly general model describing the spread of wealth over a population of agents and accounts for the occurrence of two types of wealth transactions. Specifically, external arrivals of wealth, reflecting, for example, the payment of salaries from agents outside the population or government cash-transfers, and internal wealth redistributions, reflecting transactions such as the purchase of commodities or services from other agents within the population, gift exchange or the informal provision of credit. Transaction rates are affected by an exogenously evolving Markov background process, which represents the state of the economy. In the numerical analysis, a two-state background process is chosen to mimic the fluctuation of the system between good and bad states, where a good state may represent a period of economic growth and a bad state the occurrence of a pandemic or economic recession. The background process is of particular significance in the low-income setting due to the instability of the associated markets.

The goal in adopting this model is to study the financial vulnerability of low-income groups. For this purpose, the probability of an agent falling below the poverty line is again considered, given they are a member of such a group. All existing theoretical studies consider this probability on an individual agent basis. This chapter therefore considers, for the first time mathematically, the impact of group membership and the associated group interactions on the probability of falling below the poverty line. Risk-sharing arrangements are useful in mitigating idiosyncratic risks, however correlated risks, such as those resulting from natural disasters, often affect agents simultaneously. Formal insurance provides more robust protection against such correlated risks. In addition to informal risk-sharing arrangements represented by internal transactions within the group, the impact of insurance on the probability of falling below this critical line is therefore assessed. To do this, an adjustment of the model of Chan and Mandjes (2022) is proposed which further encompasses the susceptibility of agents to wealth losses, due to, for example, catastrophic natural disasters, severe illness and the loss of a household member or breadwinner. On the occurrence of a loss, agent wealth is binomially thinned, such that losses are proportional to wealth as in Chapter 3. Insurance coverage is captured by analogously defining a frequently occurring loss event that represents premium payments.

As discussed in Chapter 1, research on the impact of microinsurance mechanisms on the probability of falling below the poverty line has largely been undertaken through application of multi-equilibrium models and dynamic stochastic programming, or from a ruin-theoretic perspective, where the probability mimics an insurer's ruin probability. Notably, these studies suggest that purchase of insurance and the associated need for premium payment increases the risk of falling below the poverty line for the most vulnerable. As such, public private partnerships between government and private microinsurers are proposed to minimise the severity of premium payments for the most vulnerable. Government support may be provided in the form of cash transfers, or more cost-effectively, through premium subsidies, where subsidies can be targeted such that only the most vulnerable receive support.

In the numerical study of Will et al. (2021), the influence of the availability of insurance on the survival of households, particularly those who cannot afford coverage, above a critical budget is observed. For households with budget below the critical level, transfers are received from one randomly selected household with which they have an established connection. Due to premium payments, insured households have a lower budget than uninsured households and thus have less capacity to provide risk-sharing support, reducing the resilience of the uninsured to income shocks as a consequence. Formal insurance is however found to be complementary to informal insurance in the event of covariate shocks.

To analyse the wealth dynamics of a group in this chapter, as in Chan and Mandjes (2022), a system of coupled differential equations for the joint transient distribution of agent wealth that incorporates the state of the Markov background process is derived. As in Chapters 2 and 3, the concept of wealth reflects the ability of an agent to produce and so encompasses land, property, livestock, physical, health and human capital. Derivation of the moments of the wealth process enables numerical analysis through reduction of the system to a system of linear differential equations. Sensitivity analysis is undertaken to provide insight into the impact of the structure of the system on the wealth of the group. Mean and variance wealth processes are presented to compare the vulnerability of those with and without insurance coverage and in line with Chapter 2, the impact of both fixed and flexible (targeted) studies are considered.

By the central limit theorem, a normal approximation is used to determine the probability of falling below the poverty line and the distribution of the number of agents that fall, where the parameters of the distribution are combinations of the stationary means and reduced second moments of the wealth process.

The fundamental concepts forming this stochastic dissemination model follow from the large literature on queuing network and population models. Fiems et al. (2018) implement a similar approach in the queuing context, with randomly occurring linear transformations of the population vector of the network aligning with the internal redistribution events of this model. Transformations are multiplicative as in the case presented here. For thorough discussion of stochastic networks with application to queuing and population models, see Kelly (2011) and Serfozo (2012).

The model studied in this chapter can easily be adjusted to account for large groups or economic entities, if, for example, risk-sharing among cities or countries is of interest. Risk-sharing studies at the international level include those by Devereux and Smith (1994), Sørensen and Yosha (1998) and Gardberg (2019) and largely appear in the economics literature. Considering risk sharing in the form of risk pooling, Ni et al. (2020) discuss the pooling of flood risk exposure at the continental and global levels, diversifying flood exposure through consideration of a large geographical area. Although interesting to assess, in this analysis, the focus remains on the subgroups prevalent in low-income communities.

The remainder of the chapter is structured as follows. Section 4.2 presents the stochastic dissemination model and derivation of the system of coupled differential equations for the joint distribution of agent wealth. Time-dependent first and reduced second moments are derived in Sections 4.4 and 4.5, respectively. Sensitivity analysis is presented in Section 4.6 and the normal approximation of the probability of falling below the poverty line in Section 4.7. Concluding remarks are provided in Section 4.8.

4.2 Wealth dissemination model

In this chapter, the stochastic behaviour of $\boldsymbol{M}(t) \equiv (M_1(t), ..., M_I(t))$ is studied under varying levels of insurance coverage, where $M_i(t)$ is the wealth of an agent i at time t for $i = 1, ..., I \in \mathbb{N}$. Agent wealth takes only discrete values and is regarded as being composed of $M_i(t)$ wealth units. As proposed in Chan and Mandjes (2022), the dynamics of the wealth model are affected by an autonomously evolving continuous-time Markov background process $(X(t))_{t\geq 0}$, which is irreducible and has state space $\{1, ..., d\}$ for $d \in \mathbb{N}$.

Over time, the background process moves between states according to the transition rate matrix $Q = \{q_{ij}\}_{i,j=1}^{d}$, where

$$\mathbb{P}(X(t) = l | X(0) = k) = (e^{Qt})_{k,l}.$$

See, for example, Norris (1997) for a thorough presentation of continuous-time Markov chains. Concepts relevant to this chapter are defined in Appendix C.

Changes in an agent's wealth are triggered by the occurrence of four types of transaction events. Namely, external arrivals, internal redistribution events, catastrophic losses and premium payments. Given the background process is in state $k \in \{1, ..., d\}$, the joint probability generating function must therefore capture the following components, where events (i) and (ii) are defined as in Chan and Mandjes (2022):

(i) All agents in receipt of an external arrival of wealth of type j, i.e. all agents $i \in S_j \subseteq \{1, ..., I\}$, experience an increase of one wealth unit, where $j \in \{1, ..., J\}$ for $J \in \mathbb{N}$. External arrivals occur at exponentially distributed times with rate $\lambda_{jk} > 0$.

(ii) At internal shock times, transfers of wealth occur such that every wealth unit of agent i transfers $W_{ijk} \in \mathbb{N}_0$ wealth units to agent j, where W_{ijk} is randomly distributed. As such, the number of wealth units at agent j after an internal shock at time $t > 0$ is

$$\sum_{i=1}^{I} \sum_{n=1}^{m_i} W_{ijkn},$$

where $\boldsymbol{M}(t-) = \boldsymbol{m}$ is the wealth vector immediately before the shock and $(W_{ijkn})_{n \in \mathbb{N}}$ a sequence of independent and identically distributed random variables. In the case considered here, it is assumed that the number of wealth units transferred by each wealth unit is Bernoulli distributed. Under this specification, each wealth unit can transfer at most one wealth unit to any other agent. However, if transfers were instead assumed to be binomially distributed with parameter $n > 1$, the possibility of wealth creation would be captured, where wealth may be created through, for example, investment or the adoption of productive technologies.

Each W_{ijkn} shares the same distribution as W_{ijk}. The random variables W_{ijk} for $i = 1, ..., I$ and $k = 1, ...d$ are independent, however dependence in j is permitted. Agents therefore redistribute their wealth independently of one another but with dependence between the transactions made by a single agent.

Wealth is redistributed within the population at exponentially distributed times with rate $\gamma_k > 0$. As such, transaction events affect the entire system simultaneously. Definition of transaction events that affect only a subset of agents is also possible through specification of multiple transaction rate parameters.

The associated probability generating functions for $\boldsymbol{z} = (z_1, ..., z_I)$ with $\max\{|z_1|, ..., |z_I|\} \leq 1$ are defined by

$$g_{ik}(\boldsymbol{z}) = \mathbb{E}\Big[\prod_{j=1}^{I} z_j^{W_{ijk}}\Big],$$

where the probability generating function of a discrete random variable gives a power series representation of its probability mass function.

(iii) An uninsured loss forces a proportion of an affected agent's wealth to leave the system. For a loss of type l_n, all affected agents, i.e. all agents $i \in S^l_{nk} \subseteq \{1, ..., I\}$, experience a reduction in wealth of the corresponding claim size, where $n \in \{1, ..., N_l\}$ for a given number of loss types $N_l \in \mathbb{N}$. In the event such a shock occurs at time $t > 0$, the number of wealth units after the shock at agent $i \in S^l_{nk}$ is

$$m_i - \sum_{m=1}^{m_i} L_{inkm},$$

where $\boldsymbol{M}(t-) = \boldsymbol{m}$ is again the wealth vector immediately before the shock and $(L_{inkm})_{m \in \mathbb{N}}$ is a sequence of independent and identically distributed random variables representing the number of wealth units contributed to payment of the loss by each of the $M_i(t)$ wealth units of the agent. Random contributions per wealth unit follow a binomial distribution with parameters 1 and p^u_{nk}, such that the wealth of the affected agent is binomially thinned with thinning operator p^u_{nk}. Losses occur at exponentially distributed times with rate $\mu^u_{nk} > 0$.

By construction, the model permits the occurrence of losses with varying frequency and severity through respecification of the binomial probability and exponential rate parameters. This is particularly useful in the microinsurance setting, where coverage for commonly occurring events, including hospital admissions and household deaths, is needed in addition to low frequency, high severity events such as natural disasters.

For $|z| \leq 1$, the associated probability generating function is given by

$$l_{ink}(z) = \mathbb{E}[z^{L_{ink}}] = \sum_{n=0}^{\infty} p(n)z^n = 1 - (1-z)p^u_{nk},$$

where L_{ink} denotes the random contribution of each wealth unit by agent i to cover an n-type loss when the background process is in state k.

(iv) Premium payments force a proportion of an agent's wealth to leave the system. In an analogous manner to uninsured losses, for a premium payment of type p_n, all affected agents, i.e. all agents $i \in S^p_{nk} \subseteq \{1, ..., I\}$, experience a reduction in wealth of the corresponding premium, where $n \in \{1, ..., N_p\}$ for $N_p \in \mathbb{N}$. Then, on the premium payment date ($t > 0$), given the wealth vector just before time t is $\boldsymbol{M}(t-) = \boldsymbol{m}$, the number of wealth units after the shock at agent i is

$$m_i - \sum_{m=1}^{m_i} P_{inkm},$$

where $(P_{inkm})_{m\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables representing the number of wealth units contributed to premium payment by each of the $M_i(t)$ wealth units of the agent. Random wealth unit contributions again follow a binomial distribution, with parameters 1 and p^p_{nk}. Premium payments occur at exponentially distributed times with rate $\mu^p_{nk} > 0$, such that $\mu^p_{nk} \cdot p^p_{nk}$ constitutes the premium rate. Given the randomness of premium payment events, policyholders are assumed to pay non-constant premiums. This specification aligns with the prevalence of the informal sector in low-income economies and the associated pay-as-you-go nature of insurance, with high levels of income uncertainty determining when an agent is able to purchase coverage. Flexibility in the model allows for inclusion of varying premium rates as for uninsured losses.

Note that, μ^p_{nk} should be low relative to μ^l_{nk} to reflect the increased frequency of premium payments in comparison to loss events.

For $|z| \leq 1$, the associated probability generating function is given by

$$p_{ink}(z) = \mathbb{E}[z^{P_{ink}}] = 1 - (1-z)p^p_{nk},$$

where the contribution of each wealth unit by agent i to cover an n-type premium in state k is denoted P_{ink}.

4.3 Derivation of joint probability generating function

To analyse the transient wealth of the system $\boldsymbol{M}(t)$ and its behaviour as the Markov background process $X(t)$ evolves, the multivariate time-dependent joint probability generating function (PGF) of $(\boldsymbol{M}(t), X(t))_{t\geq 0} \in \mathbb{N}^I \times \{1, ..., d\}$ is used. This PGF defines the joint distribution of $(\boldsymbol{M}(t), X(t))_{t\geq 0}$ and is given by

$$f_k(\boldsymbol{z}, t) := \mathbb{E}\Big[\prod_{j=1}^{I} z_j^{M_j(t)} \mathbb{1}_{\{X(t)=k\}}\Big], \tag{4.3.1}$$

for $\boldsymbol{z} = (z_1, ..., z_I)$ with $\max\{|z_1|, ..., |z_I|\} \leq 1$.

In the remainder of this section, a system of coupled differential equations for the time-dependent joint PGF in (4.3.1) is established. Considering an interval of length Δt, the system is defined through observation of the change in the wealth vector $\boldsymbol{M}(t)$ or the background state, and the associated change in $f_k(\boldsymbol{z}, t)$ induced by each event that could occur between t and $t + \Delta t$. The contribution of each event to the system is as follows, where the background process, external arrival and internal redistribution components are as in Chan and Mandjes (2022):

- Background process change. The background process transitions from state l to state $k \neq l$ between times t and Δt, where the process is in state l at time t:

$$\sum_{l\neq k}^{d} q_{lk} \Delta t f_l(\boldsymbol{z}, t).$$

- External arrival. An external arrival of type j occurs, such that the wealth of all agents $i \in S_j$ increases by one wealth unit:

$$\sum_{j=1}^{J} \lambda_{jk} \Delta t \Big(\prod_{i\in S_j} z_i\Big) f_k(\boldsymbol{z}, t).$$

- Internal redistribution. An internal shock triggers the redistribution of the wealth of all agents within the system. Let $R_k(t)$ denote the occurrence of an internal redistribution shock between times t and $t + \Delta t$ when the background process is in state k. Then, by the law of total expectation:

$$\begin{aligned}
&\mathbb{E}\Big[\prod_{j=1}^{I} z_j^{M_j(t+\Delta t)} \mathbb{1}_{\{X(t+\Delta t=k)\}} \,\Big|\, \mathcal{R}_k(t)\Big] \\
&= \sum_{\boldsymbol{m}\in\mathbb{N}^I} \mathbb{E}\Big[\prod_{j=1}^{I} z_j^{M_j(t+\Delta t)} \mathbb{1}_{\{X(t+\Delta t)=k\}} \,\Big|\, \boldsymbol{M}(t) = \boldsymbol{m}, \mathcal{R}_k(t)\Big] \mathbb{P}(\boldsymbol{M}(t) = \boldsymbol{m}) \\
&= \sum_{\boldsymbol{m}\in\mathbb{N}^I} \prod_{j=1}^{I} (g_{jk}(\boldsymbol{z}))^{m_j}\, \mathbb{P}(\boldsymbol{M}(t) = \boldsymbol{m}, X(t) = k) \\
&= \mathbb{E}\Big[\prod_{j=1}^{I} (g_{jk}(\boldsymbol{z}))^{M_j(t)}\, \mathbb{1}_{\{X(t)=k\}}\Big] \\
&= f_k(\boldsymbol{h}_k(\boldsymbol{z}), t),
\end{aligned} \tag{4.3.2}$$

where $\boldsymbol{h}_k(\boldsymbol{z}) := (g_{1k}(\boldsymbol{z}), ..., g_{Ik}(\boldsymbol{z}))$.

- Uninsured loss. An uninsured loss causes each wealth unit of all agents $i \in S^l_{nk}$ to decrease by L_{ink} (0 or 1), independently of one another. In a similar manner to that of the internal shock case, by the law of total expectation:

$$\begin{aligned}
&\mathbb{E}\Big[\prod_{j=1}^{I} z_j^{M_j(t+\Delta t)} \mathbb{1}_{\{X(t+\Delta t=k)\}} \,\Big|\, \mathcal{L}_k(t)\Big] \\
&= \sum_{\boldsymbol{m}\in\mathbb{N}^I} \mathbb{E}\Big[\prod_{j=1}^{I} z_j^{M_j(t+\Delta t)} \mathbb{1}_{\{X(t+\Delta t)=k\}} \,\Big|\, \boldsymbol{M}(t)=\boldsymbol{m}, \mathcal{L}_k(t)\Big]\mathbb{P}(\boldsymbol{M}(t)=\boldsymbol{m}) \\
&= \sum_{n=1}^{N^l} \mu^l_{nk}\Delta t \sum_{\boldsymbol{m}\in\mathbb{N}^I} \prod_{i\in S^l_{nk}} (l_{ink}(z_i))^{-m_i} \prod_{j=1}^{I} z_j^{m_j} \mathbb{P}(\boldsymbol{M}(t)=\boldsymbol{m}, X(t)=k) \\
&= \sum_{n=1}^{N_l} \mu^l_{nk}\Delta t \mathbb{E}\Big[\prod_{i\in S^l_{nk}} (l_{ink}(z_i))^{-M_i(t)} \prod_{j=1}^{I} z_j^{M_j(t)} \mathbb{1}_{\{X(t)=k\}}\Big],
\end{aligned}$$

where $\mathcal{L}_k(t)$ denotes the occurrence of an uninsured loss event in the interval $(t, t+\Delta t)$.

- Premium payment. A premium payment event causes each wealth unit of all agents $i \in S^p_{nk}$ to decrease by P_{ink} (0 or 1), independently of one another. Since premium payments are modelled analogously to uninsured losses, the associated term is derived in the same way:

$$\sum_{n=1}^{N_p} \mu^p_{nk}\Delta t \mathbb{E}\Big[\prod_{i\in S^p_{nk}} (p_{ink}(z_i))^{-M_i(t)} \prod_{j=1}^{I} z_j^{M_j(t)} \mathbb{1}_{\{X(t)=k\}}\Big].$$

- No event. There are no transitions of the background process, external arrivals, internal shocks, losses (for the uninsured) or premium payments (for the insured) between the times t and $t+\Delta t$.

As such,

$$\begin{aligned}
f_k(\boldsymbol{z}, t+\Delta t) =& \sum_{l\neq k}^{d} q_{lk}\Delta t f_l(\boldsymbol{z}, t) + \sum_{j=1}^{J} \lambda_{jk}\Delta t\Big(\prod_{i\in S_j} z_i\Big) f_k(\boldsymbol{z}, t) + \gamma_k \Delta t f_k(\boldsymbol{h}_k(\boldsymbol{z}), t) \\
&+ \sum_{n=1}^{N_l} \mu^l_{nk}\Delta t \mathbb{E}\Big[\prod_{i\in S^l_{nk}} (l_{ink}(z_i))^{-M_i(t)} \prod_{j=1}^{I} z_j^{M_j(t)} \mathbb{1}_{\{X(t)=k\}}\Big] \\
&+ \sum_{n=1}^{N_p} \mu^p_{nk}\Delta t \mathbb{E}\Big[\prod_{i\in S^p_{nk}} (p_{ink}(z_i))^{-M_i(t)} \prod_{j=1}^{I} z_j^{M_j(t)} \mathbb{1}_{\{X(t)=k\}}\Big] \\
&+ \Big(1 - \sum_{l\neq k}^{d} q_{lk}\Delta t - \sum_{j=1}^{J} \lambda_{jk}\Delta t - \gamma_k\Delta t - \sum_{n=1}^{N_l} \mu^l_{nk}\Delta t - \sum_{n=1}^{N_p} \mu^p_{nk}\Delta t\Big) f_k(\boldsymbol{z}, t) \\
&+ o(\Delta t). \qquad (4.3.3)
\end{aligned}$$

The system of differential equations for the joint PGF is then obtained by subtracting $f_k(\boldsymbol{z}, t)$ from both sides of (4.3.3), dividing by Δt and taking the limit as $\Delta t \to 0$, such that

$$\begin{aligned}\frac{\partial}{\partial t} f_k(\boldsymbol{z}, t) =& \sum_{l=1}^{d} q_{lk} f_l(\boldsymbol{z}, t) + \sum_{j=1}^{J} \lambda_{jk} \Big(\prod_{i \in S_j} z_i - 1 \Big) f_k(\boldsymbol{z}, t) + \gamma_k (f_k(\boldsymbol{h}_k(\boldsymbol{z}), t) - f_k(\boldsymbol{z}, t)) \\ &+ \sum_{n=1}^{N_l} \mu_{nk}^l \Big(\mathbb{E}\Big[\prod_{i \in S_{nk}^l} (l_{ink}(z_i))^{-M_i(t)} \prod_{j=1}^{I} z_j^{M_j(t)} \mathbb{1}_{\{X(t)=k\}} \Big] - f_k(\boldsymbol{z}, t) \Big) \\ &+ \sum_{n=1}^{N_p} \mu_{nk}^p \Big(\mathbb{E}\Big[\prod_{i \in S_{nk}^p} (p_{ink}(z_i))^{-M_i(t)} \prod_{j=1}^{I} z_j^{M_j(t)} \mathbb{1}_{\{X(t)=k\}} \Big] - f_k(\boldsymbol{z}, t) \Big), \end{aligned} \tag{4.3.4}$$

using the fact that the row sums of Q are 0. The following proposition therefore holds:

Proposition 4.3.1. For $t \geq 0$, $f_k(\boldsymbol{z}, t)$ satisfies the system of differential equations in (4.3.4), with initial condition

$$f_k(\boldsymbol{z}, 0) = \prod_{i=1}^{I} z_i^{m_{0,i}} \mathbb{1}_{\{X_0=k\}},$$

where $\boldsymbol{M}(0) = \boldsymbol{m}_0$ and $X(0) = X_0$.

The internal redistribution component (4.3.2) causes the system to be non-linear due to the $\boldsymbol{h}_k(\boldsymbol{z})$ argument. While it is possible to solve (4.3.4) numerically, for the remainder of the chapter, the time-dependent first, reduced and mixed second moments are considered. By reducing the system to a system of linear differential equations, this enables tractable analysis of the evolution of wealth over time and for analytical expressions for the moments of transient wealth to be obtained.

4.4 Derivation of first moments of transient wealth

Proposition 4.4.1. Let $m_{ik}(t) := \mathbb{E}[M_i(t)\mathbb{1}_{\{X(t)=k\}}]$ and $w_{ijk} := \mathbb{E}[W_{ijk}]$. For $t \geq 0$, the transient mean wealth of the system $\boldsymbol{m}(t)$ satisfies a system of dI coupled, non-homogeneous linear differential equations of the following form:

$$\boldsymbol{m}'(t) = A\boldsymbol{m}(t) + \Lambda \boldsymbol{\pi}(t), \tag{4.4.1}$$

with initial condition

$$m_{ik}(0) = m_{0,i} \mathbb{1}_{\{X_0=k\}}, \tag{4.4.2}$$

where $\boldsymbol{M}(0) = \boldsymbol{m}_0$ and $X(0) = X_0$.

Proof. Taking the derivative of (4.3.4) with respect to z_i and evaluating at $\boldsymbol{z} = 1$ yields

$$\begin{aligned} m'_{ik}(t) =& \sum_{l=1}^{d} q_{lk} m_{il}(t) + \sum_{j:i \in S_j} \lambda_{jk} \pi_k(t) + \gamma_k \Big(\sum_{j=1}^{I} w_{jik} m_{jk}(t) - m_{ik}(t) \Big) \\ &- \Big(\sum_{n:i \in S_{nk}^l} \mu_{nk}^l p_{nk}^l + \sum_{n:i \in S_{nk}^p} \mu_{nk}^p p_{nk}^p \Big) m_{ik}(t), \end{aligned}$$

where $\pi_k(t) = \mathbb{P}(X(t) = k)$ and use of the chain rule permits computation of the derivative of $f_k(\boldsymbol{h}_k(\boldsymbol{z}), t)$:

$$\frac{\partial f_k(\boldsymbol{h}_k(\boldsymbol{z}), t)}{\partial z_i} = \sum_{j=1}^{I} \frac{\partial f_k(\boldsymbol{x}, t)}{\partial x_j}\Big|_{\boldsymbol{x}=\boldsymbol{h}_k(\boldsymbol{z})} \frac{\partial (h_k(\boldsymbol{z}))_j}{\partial z_i}$$

for $i = 1, ..., I$, where

$$\frac{\partial (h_k(\boldsymbol{z}))_j}{\partial z_i}\Big|_{\boldsymbol{z}=\boldsymbol{1}} = w_{jik}.$$

Define vectors $\boldsymbol{m}(t) \in \mathbb{R}^{dI}$ and $\boldsymbol{\pi}(t) \in \mathbb{R}^{dI}$ such that $\boldsymbol{m}_i(t) \equiv (m_{i1}(t), ..., m_{id}(t))^T$ and $\boldsymbol{\pi}_i(t) \equiv (\pi_1(t), ..., \pi_d(t))^T$ for each $i \in I$. Let

$$G_{ji} := \mathrm{diag}\{\gamma_1 w_{ji1}, ..., \gamma_d w_{jid}\} - \mathrm{diag}\{\gamma_1, ..., \gamma_d\}\mathbb{1}_{\{i=j\}}$$

and $M_i := \mathrm{diag}\{\bar{\mu}^l_{i1} + \bar{\mu}^p_{i1}, ..., \bar{\mu}^l_{id} + \bar{\mu}^p_{id}\}$, where $\bar{\mu}^l_{ik} = \sum_{n:i\in S^l_{nk}} \mu^l_{nk} p^l_{nk}$ and $\bar{\mu}^p_{ik}$ is analogously defined. Let

$$A := \begin{pmatrix} Q^T + G_{11} - M_1 & G_{21} & G_{31} & \cdots & G_{I1} \\ G_{12} & Q^T + G_{22} - M_2 & G_{32} & \cdots & G_{I2} \\ G_{13} & G_{23} & Q^T + G_{33} - M_3 & \cdots & G_{I3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{1I} & G_{2I} & G_{3I} & \cdots & Q^T + G_{II} - M_I \end{pmatrix}$$

and

$$\Lambda := \begin{pmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda_I \end{pmatrix},$$

where $\Lambda_i := \mathrm{diag}\{\bar{\lambda}_{i1}, ..., \bar{\lambda}_{id}\}$ and $\bar{\lambda}_{ik} := \sum_{j:i\in S_j} \lambda_{jk}$. Then, in matrix-vector form, $\boldsymbol{m}'(t)$ satisfies (4.4.1) as required. The initial condition in (4.4.2) follows trivially by definition of $m_{ik}(t)$. □

The vector of transient state probabilities of the background process $X(t)$ satisfies

$$\boldsymbol{\pi}'(t) = (\mathbb{I} \otimes Q^T)\boldsymbol{\pi}(t),$$

where $\otimes$ is the Kronecker product and $\mathbb{I}$ the identity matrix of dimension I, such that for $\bar{Q} = \mathbb{I} \otimes Q^T$, it holds that

$$\boldsymbol{\pi}(t) = e^{\bar{Q}t}\boldsymbol{\pi}(0).$$

Solving for $\boldsymbol{m}(t)$ in (4.4.1) for $t \geq 0$ then gives

$$\boldsymbol{m}(t) = e^{At}\boldsymbol{m}(0) + \int_0^t e^{A(t-s)}\Lambda e^{\bar{Q}s}\boldsymbol{\pi}(0)ds,$$

as in Proposition 4.2 of Chan and Mandjes (2022).

The probability of falling below the poverty line in Section 4.7 is considered in infinite time. As such, the normal approximation relies upon the steady-state mean and variance of agent wealth. Let $\boldsymbol{\pi} := \boldsymbol{\pi}(\infty)$ be the unique solution of $\boldsymbol{\pi Q} = \mathbf{0}$, such that it is the stationary distribution of $X(t)$. Then, if the Markov chain $(\boldsymbol{M}(t), X(t))_{t\geq 0}$ is stable, $\boldsymbol{m}' = 0$ and so the steady-state mean vector $\boldsymbol{m}$ satisfies

$$\boldsymbol{m} = -A^{-1}\Lambda\boldsymbol{\pi}. \tag{4.4.3}$$

Proposition 4.4.2. Let ω be the eigenvalue of A in (4.4.3) with the largest real part, defining the spectral abscissa of A. The Markov chain $(\boldsymbol{M}(t), X(t))_{t\geq 0}$ is ergodic if $\omega < 0$.

Proof. See Fiems et al. (2018) and Chan and Mandjes (2022) for proof. □

The dissemination model can be adjusted to fit many constructions of a population simply through varying the selection of parameters. In the following subsection, one special case reflecting the question of this chapter is discussed.

4.4.1 Special case

Consider a population that consists of a single distinct agent ('leader') and two internally homogeneous interacting subpopulations ('followers'), one insured and one uninsured. Focusing on the impact of insurance on the financial vulnerability of the group, with the exception of the disparities in insurance coverage, the two follower groups are assumed to be homogeneous. The insured subpopulation consists of I_p agents and the uninsured subpopulation $I_u := I - I_p - 1$ agents. The indices u and p are used throughout the remainder of the chapter to reflect the uninsured group and those making premium payments, respectively.

Let $m_{L,k}(t)$, $m_{u,k}(t)$ and $m_{p,k}(t)$ denote the mean wealth of the leader, an arbitrary uninsured agent and an arbitrary insured agent, respectively, at time t, with background process in state k. Due to the homogeneity within the follower groups, mean wealth is equal across all agents in each group. Let $J = I$, $N_l = I_u$ and $N_p = I_p$ such that there are I, I_u and I_p different types of external arrivals, losses and premium payments, respectively, and let $S_j = \{j\}$, $S_n^l = \{n\}$ and $S_n^p = \{n\}$ such that a single agent is affected by each event type. For the purpose of this example, agents affected by each type of loss are assumed to be independent of the background state. Since only the insurance coverage status differs between the two groups of followers, let $\lambda_{F,k}$ denote the rate of external arrival for the uninsured and insured subpopulations with background process in state k. Equivalently, let the leader's external arrival rate be denoted by $\lambda_{L,k}$.

Within subpopulations, the random variable W_{ijk} describing the internal redistribution of wealth now only depends on the state of the background process k. Therefore, let $w_{LL,k}$, $w_{Lu,k}$, $w_{Lp,k}$, $w_{uL,k}$, $w_{pL,k}$, $w_{up,k}$, $w_{pu,k}$, $w_{uu,k}$ and $w_{pp,k}$ denote the expected values of all possible internal transactions (per wealth unit) within the population, when the background process is in state k.

Since the two follower groups are internally homogeneous, all agents within each group must experience loss and premium payment events in the same way. As such, the rate and probability of loss parameters corresponding to the event experience of each agent are fixed within the two subpopulations. Fixing the set of affected agents S such that $S = \{1, ..., I\}$, where all agents are affected by each event, would also facilitate the homogeneous assumption. Let μ_k^u and p_k^u denote the loss rate and probability of loss (per wealth unit) for uninsured followers with background process in state k. Similarly, let μ_k^p and p_k^p denote the premium payment rate and probability of payment for insured followers with background process in state k.

The system in Proposition 4.4.1 therefore reduces to the following:

$$
\begin{aligned}
m'_{L,k}(t) =& \sum_{l=1}^{d} q_{lk} m_{L,l}(t) + \lambda_{L,k}\pi_k(t) + \gamma_k((w_{LL,k}-1)m_{L,k}(t) + I_u w_{uL,k} m_{u,k}(t) \\
&+ I_p w_{pL,k} m_{p,k}(t)), \\
m'_{u,k}(t) =& \sum_{l=1}^{d} q_{lk} m_{u,l}(t) + \lambda_{F,k}\pi_k(t) + \gamma_k(w_{Lu,k} m_{L,k}(t) + (I_u w_{uu,k}-1)m_{u,k}(t) \\
&+ I_p w_{pu,k} m_{p,k}(t)) - \mu_k^u p_k^u m_{u,k}(t), \\
m'_{p,k}(t) =& \sum_{l=1}^{d} q_{lk} m_{p,l}(t) + \lambda_{F,k}\pi_k(t) + \gamma_k(w_{Lp,k} m_{L,k}(t) + I_u w_{up,k} m_{u,k}(t) \\
&+ (I_p w_{pp,k}-1)m_{p,k}(t)) - \mu_k^p p_k^p m_{p,k}(t),
\end{aligned}
$$

such that $\boldsymbol{m}'(t) = A\boldsymbol{m}(t) + \Lambda\boldsymbol{\pi}(t)$ for

$$
\boldsymbol{m}(t) = \begin{pmatrix} \boldsymbol{m}_L(t) \\ \boldsymbol{m}_u(t) \\ \boldsymbol{m}_p(t) \end{pmatrix} \quad \text{and} \quad \boldsymbol{\pi}(t) = \begin{pmatrix} \boldsymbol{\pi}_L(t) \\ \boldsymbol{\pi}_u(t) \\ \boldsymbol{\pi}_p(t) \end{pmatrix},
$$

where each $\boldsymbol{m}_i(t) = (m_{i,1}(t), ..., m_{i,d}(t))^T$ and $\boldsymbol{\pi}_i(t) = (\pi_1(t), ..., \pi_d(t))^T$ for $i \in \{L, u, p\}$ is a d-dimensional vector,

$$
A := \begin{pmatrix} Q^T + G_{LL} & G_{uL} & G_{pL} \\ G_{Lu} & Q^T + G_{uu} - M_u & G_{pu} \\ G_{Lp} & G_{up} & Q^T + G_{pp} - M_p \end{pmatrix}
$$

where $G_{ji} := I_j \text{diag}\{\gamma_1 w_{ji,1}, ..., \gamma_d w_{ji,d}\} - \text{diag}\{\gamma_1, ..., \gamma_d\}\mathbb{1}_{\{i=j\}}$ and $M_i := \text{diag}\{\mu_1^i p_1^i, ..., \mu_d^i p_d^i\}$ for $I_L = 1$, and

$$
\Lambda := \begin{pmatrix} \Lambda_L & 0 & 0 \\ 0 & \Lambda_u & 0 \\ 0 & 0 & \Lambda_p \end{pmatrix}.
$$

Remark 4.4.1. Expressions required to analyse an agent's mean wealth can also be obtained when increasing the heterogeneity in the population in an analogous manner. The resulting differential equations simply become more complex due to an increased number of terms.

For heterogeneous event experience, multiple agents could still be affected by the same loss (or premium payment) through the specification of S assumed in the special case considered. This would encompass the potential for covariate losses due to, for example, natural disasters, in addition to individually experienced losses. However, μ_{nk}^u and p_{nk}^u (respectively μ_{nk}^p and p_{nk}^p) would need to coincide for all affected.

Now, consider the system presented in this special case with background process reflecting the state of the economy. As such, let $k = 1, 2$, where the process fluctuates been good ($k = 1$) and bad ($k = 2$) economic states, representing growth and recession, respectively. In this setting, the leader could, for example, be the facilitator of the insurance scheme, an informal employer or the leader of a societal group. The leader's wealth is obtained from outside of the system, while followers obtain their wealth from the leader. External arrival parameters are therefore defined as follows:

$$\lambda_{L,1} = \lambda_1, \quad \lambda_{L,2} = \lambda_2, \quad \text{and} \quad \lambda_{F,k} = 0 \text{ for } \quad k = 1, 2.$$

Here, $0 < \lambda_2 < \lambda_1$, such that the leader receives additional income at a higher rate during periods of economic growth.

Due to the binomial specification of wealth unit transfer at redistribution events, for each agent, transfer of wealth to the system is multinomially distributed with parameters corresponding to the vectors $(W_{j1k}, ..., W_{jIk})$, for $j = 1, ..., I$ and $k = 1, 2$. Let the redistribution of leader wealth be distributed with parameters 1 and $(p_k, r_k, ..., r_k)$, and redistribution of follower wealth be distributed with parameters 1 and $(0, s_k, ..., s_k)$, where $r_k = (1 - p_k)/(I - 1)$ and $s_k = 1/(I - 1)$. Here, follower wealth is distributed evenly throughout the follower group. The multinomial distribution could also be specified such that it captures variation in the proportion of wealth saved by a follower and the proportion consumed. This is considered in the sensitivity analysis of Section 4.6.

Remark 4.4.2. Specifying $r_k \leq (1-p_k)/(I-1)$ or $s_k \leq 1/(I-1)$ would enable the possibility that wealth leaves the system at transaction events, with multinomial probabilities $1 - p_k - (I - 1)r_k \in [0, 1]$ and $1 - (I - 1)s_k \in [0, 1]$, respectively.

Although, in reality, it is likely that insurance covers only a proportion of consumer wealth, in this example, the uninsured and insured groups are assumed to be mutually exclusive. As such, the uninsured group experiences losses in full with no premium payments and the insured group makes premium payments with no loss experience. The associated net profit condition is therefore

$$\mu^p_{nk} \cdot p^p_{nk} > \mu^l_{nk} \cdot p^l_{nk}.$$

One application of a wealth system of this structure would be to a pension or funeral insurance scheme, where each agent represents a household. In such a case, the insured group could be considered to be workers, or those with a funeral insurance contract, who make contributions to the scheme but do not experience losses. Retired or non-workers would then constitute the uninsured group, where the agent (household) experiences a loss on the death of a household member. Although in this chapter, external transactions other than those to the leader are not considered, pension payments or insurance payouts could be captured by the receipt of external arrivals of wealth by uninsured agents.

The case in which agents are proportionally insured, experiencing both losses and premium payments, will be mentioned in the discussion of future work.

The system of differential equations corresponding to the special case of this section is given by:

$$\begin{aligned} m'_{L,k}(t) =& \sum_{l=1}^{2} q_{lk} m_{L,l}(t) + \lambda_{L,k}\pi_k(t) + \gamma_k((p_k-1)m_{L,k}(t), \\ m'_{u,k}(t) =& \sum_{l=1}^{2} q_{lk} m_{u,l}(t) + \gamma_k(r_k m_{L,k}(t) + (I_u s_k - 1)m_{u,k}(t) + I_p s_k m_{p,k}(t)) \\ & - \mu_k^u p_k^u m_{u,k}(t), \\ m'_{p,k}(t) =& \sum_{l=1}^{2} q_{lk} m_{p,l}(t) + \gamma_k(r_k m_{L,k}(t) + I_u s_k m_{u,k}(t) + (I_p s_k - 1)m_{p,k}(t) \\ & - \mu_k^p p_k^p m_{p,k}(t). \end{aligned}$$

Then, where $(\boldsymbol{M}(t), X(t))_{t\geq 0}$ is stable and the stationary distribution of $X(t)$ exists, the steady-state mean wealth of the leader, uninsured follower and insured follower in background states 1 and 2 are as follows:

$$\begin{pmatrix} m_{L,1} \\ m_{L,2} \end{pmatrix} = -\begin{pmatrix} -q_1 + \gamma_1(p_1-1) & q_2 \\ q_1 & -q_2 + \gamma_2(p_2-1) \end{pmatrix}^{-1} \cdot \begin{pmatrix} \lambda_1\pi_1 \\ \lambda_2\pi_2 \end{pmatrix}$$

and

$$\begin{pmatrix} m_{u,1} \\ m_{u,2} \\ m_{p,1} \\ m_{p,2} \end{pmatrix} = -\begin{pmatrix} F_u & \bar{F}_p \\ \bar{F}_u & F_p \end{pmatrix}^{-1} \cdot \begin{pmatrix} \gamma_1 r_1 & 0 & 0 & 0 \\ 0 & \gamma_2 r_2 & 0 & 0 \\ 0 & 0 & \gamma_1 r_1 & 0 \\ 0 & 0 & 0 & \gamma_2 r_2 \end{pmatrix} \begin{pmatrix} m_{L,1} \\ m_{L,2} \\ m_{L,1} \\ m_{L,2} \end{pmatrix},$$

where

$$F_i := \begin{pmatrix} -q_1 + \gamma_1(I_i s_1 - 1) - \mu_1^i p_1^i & q_2 \\ q_1 & -q_2 + \gamma_2(I_i s_2 - 1) - \mu_2^i p_2^i \end{pmatrix}$$

and $\bar{F}_i = I_i \mathrm{diag}\{\gamma_1 s_1, \gamma_2 s_2\}$.

4.5 Derivation of reduced and mixed second moments of transient wealth

Denote the reduced second moments of $M_i(t)$ and the mixed second moments of $M_i(t)$ and $M_{i'}(t)$ by

$$v_{iik}(t) := \mathbb{E}[M_i(t)(M_i(t)-1)\mathbb{1}_{\{X(t)=k\}}] = \frac{\partial^2 f_k(\boldsymbol{z},t)}{\partial z_i^2}$$

and

$$v_{ii'k}(t) := \mathbb{E}[M_i(t)M_{i'}(t)\mathbb{1}_{\{X(t)=k\}}] = \frac{\partial^2 f_k(\boldsymbol{z},t)}{\partial z_i \partial z_{i'}},$$

respectively. Then, Proposition 4.5.1 holds:

Proposition 4.5.1. Let $m_{ik}(t) := \mathbb{E}[M_i(t)\mathbb{1}_{\{X(t)=k\}}]$ and $w_{ijk} := \mathbb{E}[W_{ijk}]$. For $t \geq 0$, the transient second moment of the wealth of the system $\boldsymbol{v}(t)$ satisfies a system of dI^2 coupled, non-homogeneous linear differential equations of the following form:

$$\boldsymbol{v}'(t) = (\gamma(\bar{A} - \mathbb{I}) + I)\boldsymbol{v}(t) + A_m \boldsymbol{m}(t), \tag{4.5.1}$$

with initial conditions

$$v_{ii}(0) = m_{0,i}(m_{0,i} - 1)\mathbb{1}_{\{X_0=k\}} \tag{4.5.2}$$

and

$$v_{ii'}(0) = m_{0,i}m_{0,i'}\mathbb{1}_{\{X_0=k\}}, \tag{4.5.3}$$

where $\boldsymbol{M}(0) = \boldsymbol{m}_0$ and $X(0) = X_0$.

Proof. Taking the derivative of (4.3.4) with respect to z_i and $z_{i'}$ and evaluating at $\boldsymbol{z} = 1$ yields

$$\begin{aligned}
v'_{ii'k}(t) =& \sum_{l=1}^{d} q_{lk} v_{ii'l}(t) + \mathbb{1}_{\{i\neq i'\}} \sum_{j:i,i'\in S_j} \lambda_{jk}\pi_k(t) + \sum_{j:i\in S_j} \lambda_{jk} m_{i'k}(t) + \sum_{j:i'\in S_j} \lambda_{jk} m_{ik}(t) \\
&+ \gamma_k \Big(\sum_{j=1}^{I}\sum_{j'=1}^{I} v_{jj'k}(t) w_{jik} w_{j'i'k} + \sum_{j=1}^{I} m_{jk}(t) w^{(2)}_{jii'k} - v_{ii'k}(t) \Big) \\
&+ \Big(\mathbb{1}_{\{i\neq i'\}} \Big(\sum_{n^c:i,i'\in S^c_n} \mu^c_{nk}(p^c_{nk})^2 + \sum_{n^p:i,i'\in S^p_n} \mu^p_{nk}(p^p_{nk})^2 \Big) \\
&- \Big(\sum_{n^c:i\in S^c_n} \mu^c_{nk}p^c_{nk} + \sum_{n^p:i\in S^p_n} \mu^p_{nk}p^p_{nk} \Big) - \Big(\sum_{n^c:i'\in S^c_n} \mu^c_{nk}p^c_{nk} + \sum_{n^p:i'\in S^p_n} \mu^p_{nk}p^p_{nk} \Big) \Big) v_{ii'k}(t) \\
&+ \mathbb{1}_{\{i'=i\}} \Big(\sum_{n^c:i\in S^c_n} \mu^c_{nk}(p^c_{nk})^2 + \sum_{n^c:i\in S^c_n} \mu^p_{nk}(p^p_{nk})^2 \Big) (v_{iik}(t) + 2m_{ik}(t)),
\end{aligned}$$

where $v_{iik}(t)+2m_{ik}(t) = \mathbb{E}[M_i(t)(M_i(t)+1)\mathbb{1}_{\{X(t)=k\}}]$ and the second derivative of $f_k(\boldsymbol{h}_k(\boldsymbol{z}), t)$ is obtained by:

$$\begin{aligned}
\frac{\partial^2 f_k(\boldsymbol{h}_k(\boldsymbol{z}), t)}{\partial z_i \partial z_{i'}} =& \frac{\partial}{\partial z_i} \Big(\sum_{j'=1}^{I} \frac{\partial f_k(\boldsymbol{x}, t)}{\partial x_{j'}}\Big|_{\boldsymbol{x}=\boldsymbol{h}_k(\boldsymbol{z})} \frac{\partial (h_k(\boldsymbol{z}))_{j'}}{\partial z_i} \Big) \\
=& \sum_{j=1}^{I}\sum_{j'=1}^{I} \frac{\partial^2 f_k(\boldsymbol{x}, t)}{\partial x_j \partial x_{j'}}\Big|_{\boldsymbol{x}=\boldsymbol{h}_k(\boldsymbol{z})} \frac{\partial (h_k(\boldsymbol{z}))_j}{\partial z_i} \frac{\partial (h_k(\boldsymbol{z}))_{j'}}{\partial z_{i'}} \\
&+ \sum_{j'=1}^{I} \frac{\partial f_k(\boldsymbol{x}, t)}{\partial x_{j'}}\Big|_{\boldsymbol{x}=\boldsymbol{h}_k(\boldsymbol{z})} \frac{\partial^2 (h_k(\boldsymbol{z}))_{j'}}{\partial z_i \partial z_{i'}}
\end{aligned}$$

for $i, i' = 1, ..., I$. For $i \neq j$,

$$w^{(2)}_{jii'k} = \frac{\partial^2 (h_k(\boldsymbol{z}))_j}{\partial z_i \partial z_{i'}}\Big|_{\boldsymbol{z}=\boldsymbol{1}} = \mathbb{E}[W_{jik}W_{ji'k}] \tag{4.5.4}$$

and for $i = j$

$$w^{(2)}_{jii'k} = \frac{\partial^2 (h_k(\boldsymbol{z}))_j}{\partial z_i \partial z_{i'}}\Big|_{\boldsymbol{z}=\boldsymbol{1}} = \mathbb{E}[W_{jik}(W_{jik} - 1)].$$

Note that the random variables W_{ijk} are dependent in j, therefore for all $i \neq j$, the expectation in (4.5.4) is equivalent to

$$\begin{aligned}\mathbb{E}[W_{jik}W_{ji'k}] &= Cov(W_{jik}, W_{ji'k}) + \mathbb{E}[W_{jik}]\mathbb{E}[W_{ji'k}] \\ &= n(n-1)w_{jik}w_{ji'k},\end{aligned}$$

where n is the number of independent trials and $Cov(X,Y) = -np_xp_y$ for two binomially distributed random variables X and Y. In the case considered here, since $n = 1$, $\mathbb{E}[W_{jik}W_{ji'k}] = 0$. Similarly, using the fact that the second moment of a binomially distributed random variable is $np(1-p) + n^2p^2$, it holds that

$$\begin{aligned}\mathbb{E}[W_{jik}(W_{jik}-1)] &= \mathbb{E}[W_{jik}^2] - \mathbb{E}[W_{jik}] \\ &= np^2(n-1),\end{aligned}$$

such that for $n = 1$, $\mathbb{E}[W_{jik}(W_{jik}-1)] = 0$.

For ease of presentation, consider a population consisting of I_L leaders, I_u uninsured followers and I_p insured followers as in Section 4.4.1. Define the matrix

$$\bar{A} := \begin{pmatrix}\bar{A}^1_{LL} & \bar{A}^1_{uu} & \bar{A}^1_{pp} & \bar{A}^2_{uu'} & \bar{A}^2_{pp'} & \bar{A}^3_{Lu} & \bar{A}^3_{Lp} & \bar{A}^3_{up}\end{pmatrix},$$

where

$$\bar{A}^n_{jj'} := (\bar{A}^n_{LLjj'}, \bar{A}^n_{uujj'}, \bar{A}^n_{ppjj'}, \bar{A}^n_{uu'jj'}, \bar{A}^n_{pp'jj'}, \bar{A}^n_{Lujj'}, \bar{A}^n_{Lpjj'}, \bar{A}^n_{upjj'})^T$$

for $n = 1, 2, 3$, and

$$\begin{aligned}\bar{A}^1_{ii'jj'} &:= I_j\mathrm{diag}\{w_{ji1}w_{ji'1}, ..., w_{jid}w_{ji'd}\}, \\ \bar{A}^2_{ii'jj'} &:= I_j(I_j-1)\mathrm{diag}\{w_{ji1}w_{ji'1}, ..., w_{jid}w_{ji'd}\}, \\ \bar{A}^3_{ii'jj'} &:= I_jI_{j'}\mathrm{diag}\{w_{ji1}w_{j'i'1} + w_{j'i1}w_{ji'1}, ..., w_{jid}w_{j'i'd} + w_{j'id}w_{ji'd}\}.\end{aligned}$$

Let

$$\begin{aligned}M^1_{ii'} &:= -2\mathrm{diag}\{\mu^i_{n1}p^i_{n1}, ..., \mu^i_{nd}p^i_{nd}\} + \mathrm{diag}\{\mu^i_{n1}(p^i_{n1})^2, ..., \mu^i_{nd}(p^i_{nd})^2\}\mathbb{1}_{\{i'=i\}}, \\ M^2_i &:= -\mathrm{diag}\{\mu^i_{n1}p^i_{n1}, ..., \mu^i_{nd}p^i_{nd}\}\end{aligned}$$

and

$$\bar{I} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M^1_{uu} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M^1_{pp} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M^1_{uu'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M^1_{pp'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M^2_u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M^2_p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^2_u + M^2_p \end{pmatrix},$$

such that

$$I = \bar{I} + \mathbb{I} \otimes Q^T,$$

where $\mathbb{I}$ denotes the 8-dimensional identity matrix. Finally, for

$$W_{ijj'} := I_i\mathrm{diag}\{\gamma_1 w^{(2)}_{ijj',1}, ..., \gamma_d w^{(2)}_{ijj',d}\}$$

and

$$M_i^3 := \text{diag}\{\mu_{n1}^i (p_{n1}^i)^2, ..., \mu_{nd}^i (p_{nd}^i)^2\},$$

define

$$A_m := \begin{pmatrix} 2\Lambda_L + W_{LLL} & W_{uLL} & W_{pLL} \\ W_{Luu} & 2\Lambda_c + W_{uuu} + 2M_u^3 & W_{puu} \\ W_{Lpp} & W_{upp} & 2\Lambda_p + W_{ppp} + 2M_p^3 \\ W_{Luu'} & 2\Lambda_u + W_{uuu'} & W_{puu'} \\ W_{Lpp'} & W_{upp'} & 2\Lambda_p + W_{ppp'} \\ \Lambda_u + W_{LLu} & \Lambda_L + W_{uLu} & W_{pLu} \\ \Lambda_p + W_{LLp} & W_{uLp} & \Lambda_L + W_{pLp} \\ W_{Lup} & \Lambda_p + W_{uup} & \Lambda_u + W_{pup} \end{pmatrix}.$$

Then, in matrix-vector form, $\boldsymbol{v}'(t)$ satisfies (4.5.1) as required, where

$$\boldsymbol{v}(t) := \big(\boldsymbol{v}_{LLk}(t), \boldsymbol{v}_{uuk}(t), \boldsymbol{v}_{ppk}(t), \boldsymbol{v}_{uu'k}(t), \boldsymbol{v}_{pp'k}(t), \boldsymbol{v}_{Luk}(t), \boldsymbol{v}_{Lpk}(t), \boldsymbol{v}_{upk}(t)\big)^T$$

and each $\boldsymbol{v}_{ijk}(t) = (v_{ij1}(t), ..., v_{ijd}(t))^T$ is a d-dimensional vector. The initial conditions (4.5.2) and (4.5.3) follow by definition of $v_{iik}(t)$ and $v_{ii'k}(t)$.

An equivalent system for a population of ungrouped agents can also be obtained. In this case, matrices $\bar{A}, I$ and A_m are of much higher dimension and so are not presented here. □

In the following section, sensitivity analysis of the mean and variance of agent wealth in a system aligning with the special case of Section 4.4.1 is presented.

4.6 Sensitivity analysis

Sensitivity analysis performed in this section helps to ascertain the impact of changes in the structure of the population and of wealth dissemination events on the state of a system. Parameter selection is aligned with that of Chapter 3 due to the proportional nature of losses. The loss proportion parameter p_k^u is fixed at $1 - \frac{\alpha}{\alpha+\beta}$, the expected value of a beta distribution with parameters $\alpha > 0$ and $\beta > 0$. The premium proportion p_k^u required to cover losses of this size in full, such that the insured group does not experience any loss of wealth at loss events, is specified as in (3.4.3), for $\kappa = 0$, such that

$$p_k^u = (1+\theta) \cdot \mu_k^u \cdot \left(1 - \frac{\alpha}{\alpha+\beta}\right),$$

where θ is the loading factor set by the insurer.

In Figures 4.1-4.11, initial state ($X_0 = 1$), transition rate ($q_{11} = 0.3, q_{22} = 0.7$), redistribution event rate ($\gamma_k = 52$), loss frequency ($\mu_k^u = 1$), loss proportion ($\alpha = 10, \beta = 1$), premium frequency ($\mu_k^p = 12$), loading factor ($\theta = 0.5$), insurance coverage ($\kappa = 0$) and leader retention rate ($p_k = 0.4$) parameters remain fixed, unless otherwise specified. Loss, premium and redistribution rate parameters are also fixed across good (state 1) and bad (state 2) economic states.

Figures 4.1 and 4.2 consider homogeneous preference for wealth retention at redistribution events across uninsured and insured agents. In each of these cases, differences in the mean and variance of wealth between the two groups are negligible. Rates of loss and premium payment are approximately 0.091 and 0.14, respectively. Coinciding mean wealth and variance processes therefore suggests that for equal wealth retention levels, the excess cost of premium payment is distributed throughout the group at transaction events, such that its impact on the wealth of insured followers is unobservable.

Interestingly, a reduction in the number of agents in the system in Figure 4.1 increases the mean wealth level, whereas for larger groups, mean wealth decreases over time. This finding aligns with the high rate of wealth redistribution per agent, with each agent distributing approximately 90% of their wealth, evenly across all other agents, at transaction events. Thus, the greater the number of agents, the lower the contribution from each agent to every other agent. Given the low-income level of agents in the system, such a high level of dissemination reflects the limited facility for savings. Variability of wealth in the system is however much greater when the sample size is small.

As expected, increasing the initial wealth of agents increases the mean wealth in the system, with the impact more significant at the beginning of the observation (see Figures 4.2a and 4.2b). In Figure 4.2c, the wealth of the leader is increased to ten times the number of follower agents. Although mean follower wealth experiences an initial jump as a result of this parameter change, the overall pattern in both mean and variance is the same. Increased variation appears among the wealth of agents with greater initial wealth. However, this variation again decreases over time. The decreasing variance observed in Figures 4.1 and 4.2 infers that the even redistribution of wealth by each follower agent to all other followers prevents the long-term presence of any agents of particularly high or low wealth in the system.

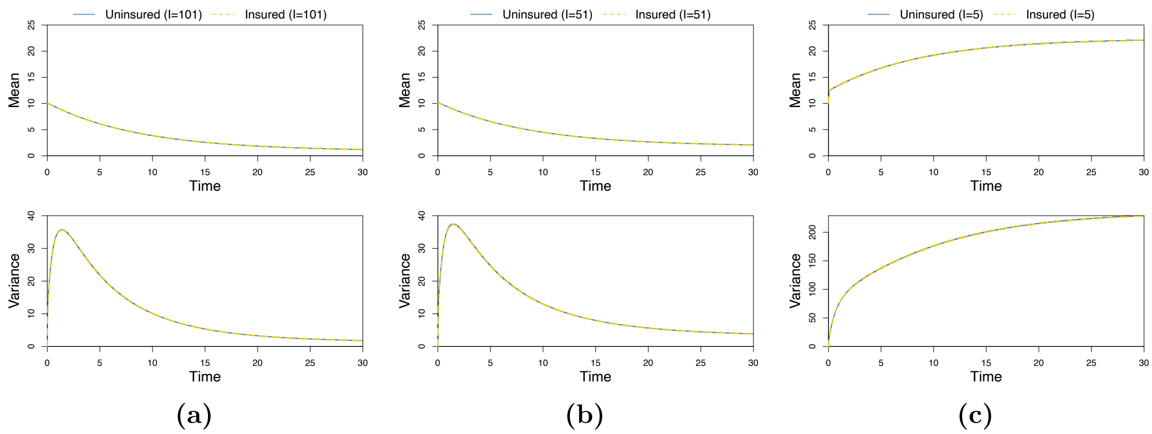
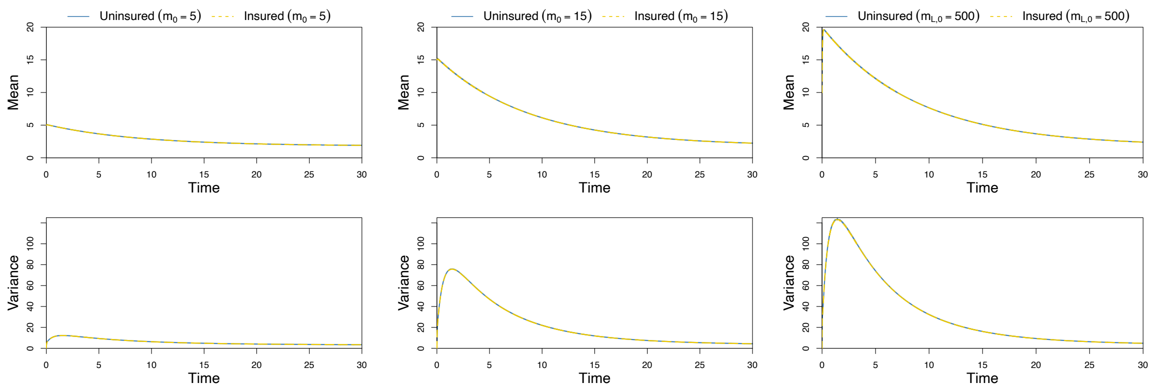


Figure 4.1: Mean and variance of uninsured and insured agent wealth for $\boldsymbol{m}_0 = 10, \boldsymbol{\lambda} = (12, 6), \boldsymbol{s_{uu}} = \boldsymbol{s_{pp}} = 0.1$ with (a) 50, (b) 25 and (c) 2 agents in each follower group.



(a) (b) (c)

Figure 4.2: Mean and variance of uninsured and insured agent wealth for $I = 51, \boldsymbol{\lambda} = (12, 6), \boldsymbol{s_{uu}} = \boldsymbol{s_{pp}} = 0.1$ with (a) $\boldsymbol{m}_0 = 5$, (b) $\boldsymbol{m}_0 = 15$, (c) $m_{u,0} = m_{p,0} = 10$ and $m_{L,0} = m_{F,0} \cdot (I - 1)$.

To gain insight into the impact of premium subsidisation on the wealth of a group-based system, Figures 4.3-4.9 present the mean and variance of uninsured and insured agent wealth with (a) unsubsidised insurance, (b) insurance with a 50% subsidy and (c) insurance with a 100% subsidy, where subsidies are provided homogeneously across all insured agents. Computations for derivation of the first, reduced and mixed second moments of transient wealth in Sections 4.4 and 4.5 are analogous when accounting for premium subsidies provided uniformly to all agents in this manner, with only a change in parameter required. Insurance premiums are specified as for Figures 4.1 and 4.2, with subsidies applied through a reduction in the premium payment probability p_k^p by either 50% (case (b)) or 100% (case (c)). In this way, subsidies may be considered to be random and are proportional to agent wealth. Structuring subsidy provision in this way captures the uncertainty typically associated with provision of financial support and social security in the low-income economies relevant to the microinsurance and risk sharing context. Random subsidies could also reflect the prevalence of informal, pay-as-you-go coverage in the low-income setting, due to variability in income levels aligning with high rates of informal sector work.

Figures 4.3-4.5 assess the impact of the proportion of wealth retained by agents at transaction events, with all other parameters fixed. A wealth retainment parameter of $1/(I-1)$ causes an agent to distribute their wealth evenly across all other agents, without retaining a more significant portion of wealth for themselves. Note that, altering the probability of an agent retaining wealth at transaction events requires an adjustment of the transient mean and variance differential equations corresponding to the special case of Section 4.4.1. Mean and variance are again observed to be equivalent for uninsured and insured agents with the same retainment preference (Figure 4.3). If financially feasible, uninsured agents may be likely to save a proportion of their wealth to protect against potential future losses. As such, Figures 4.4 and 4.5 consider an increased uninsured retainment parameter. In both cases, mean uninsured wealth lies above that of the insured, with variance increasing with increasing wealth in all plots.

Since uninsured and insured agent wealth coincide under homogeneous retainment parameters, the increased wealth observed in Figures 4.4a and 4.5a among the uninsured, where the premium is paid in full, is accountable to the increased level of savings. The reduction in the distribution of uninsured wealth lessens the tempering of excess premium payment costs at redistribution events. Insured wealth therefore decreases with increasing uninsured retention. Intuitively, premium subsidies increase agent wealth in all cases, while the greater the disparity in the proportion of retained wealth, the greater the difference in the mean wealth of the two follower groups. The sharing nature of group-based wealth systems is further exemplified by the increase in mean wealth observed among uninsured agents when subsidies are provided to the insured.

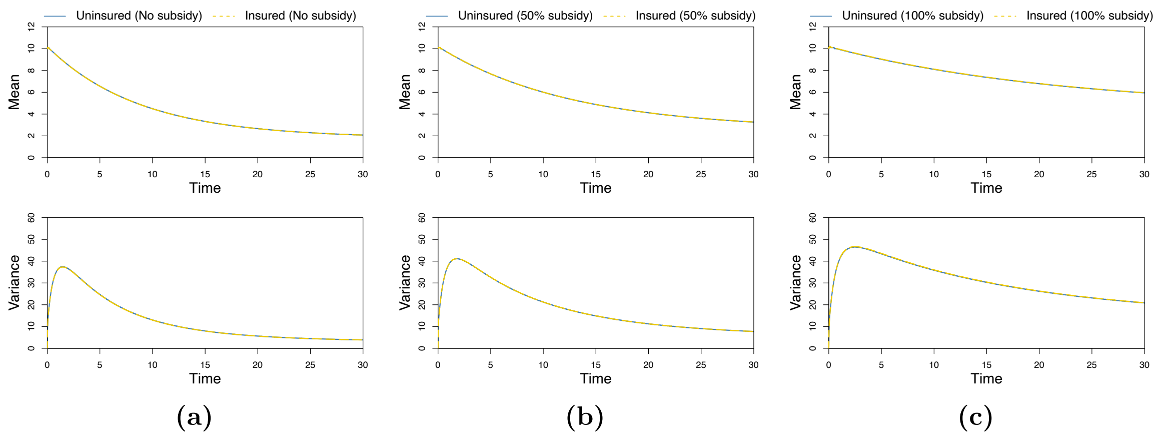


Figure 4.3: Mean and variance of uninsured and insured agent wealth for $I = 51, \boldsymbol{m}_0 = 10, \boldsymbol{\lambda} = (12, 6), \boldsymbol{s_{uu}} = \boldsymbol{s_{pp}} = 1/(I-1)$ with subsidisation of (a) 0%, (b) 50% and (c) 100% of the premium for all insured agents.

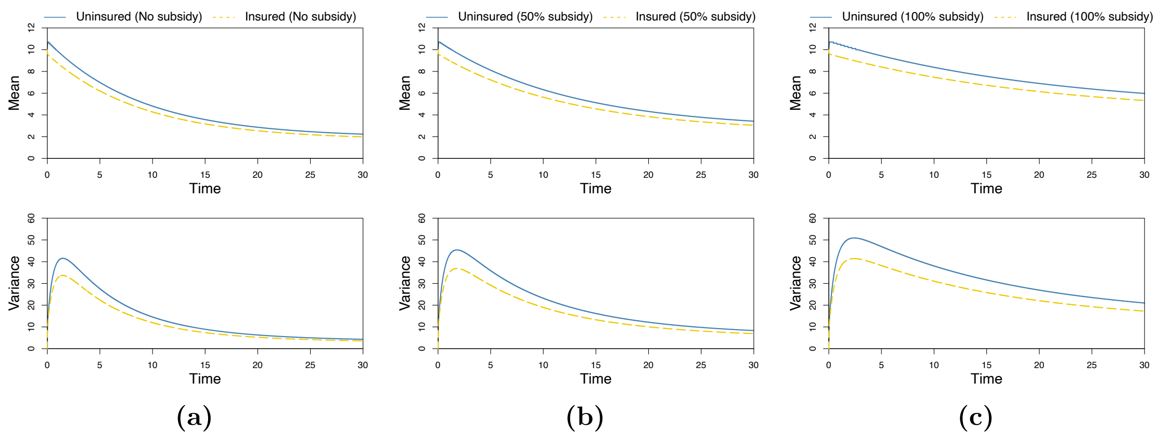
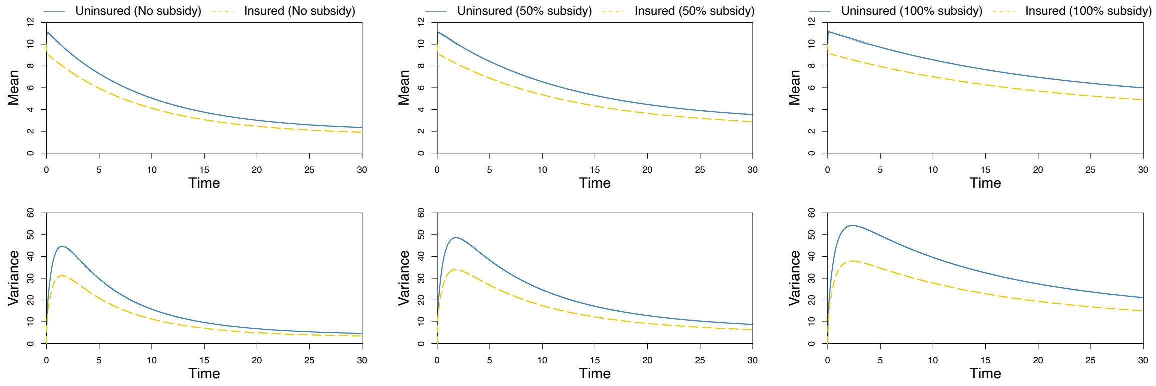


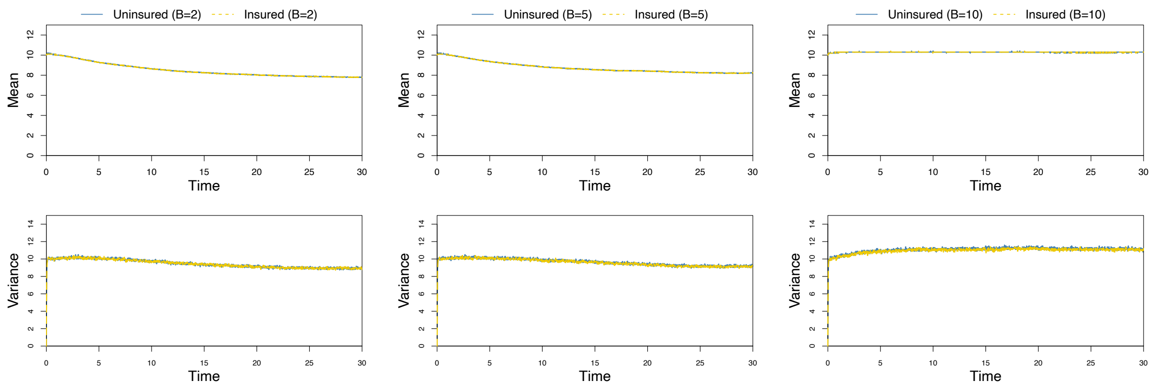
Figure 4.4: Mean and variance of uninsured and insured agent wealth for $I = 51, \boldsymbol{m}_0 = 10, \boldsymbol{\lambda} = (12, 6), \boldsymbol{s_{uu}} = 0.2, \boldsymbol{s_{pp}} = 0.1$ with subsidisation of (a) 0%, (b) 50% and (c) 100% of the premium for all insured agents.



(a) **(b)** **(c)**

Figure 4.5: Mean and variance of uninsured and insured agent wealth for $I = 51, \boldsymbol{m}_0 = 10, \boldsymbol{\lambda} = (12, 6), \boldsymbol{s_{uu}} = 0.2, \boldsymbol{s_{pp}} = 1/(I-1)$ with subsidisation of (a) 0%, (b) 50% and (c) 100% subsidy of the premium for all insured agents.

As discussed in Chapter 2, consideration of the cost of social protection is important when designing government subsidisation schemes. Figures 4.6-4.8 present the mean and variance of wealth within the two follower groups under a subsidisation scheme aligning with the barrier strategy of Chapter 2, Section 2.6, simulated for $N = 500$ realisations of the wealth system with time-step $dt = 0.01$ over an observation period of $T = 30$. Premium subsidies are provided only to those with wealth below a critical level B. Case (a), for barrier $B = 2$, coincides with the optimal barrier derived in Chapter 2. Note that the decrease in wealth observed in Figures 4.1-4.5 appears with much less severity under the barrier scheme, while wealth variation within the system remains almost constant throughout the observation period. This suggests that the severity of catastrophic loss and premium payment events is mitigated by the subsidisation barrier. The high probability of falling below the poverty line observed in Chapter 2 is eliminated in this case since agents experience neither losses or premium payments when below the barrier under full insurance coverage. Adjusting the retainment level as in Figures 4.3-4.5 yields analogous results in regard to the disparities between uninsured and insured wealth, with uninsured wealth increasing with increasing retention.



(a) **(b)** **(c)**

Figure 4.6: Mean and variance of uninsured and insured agent wealth for $I = 51, \boldsymbol{m}_0 = 10, \boldsymbol{\lambda} = (12, 6), \boldsymbol{s_{uu}} = \boldsymbol{s_{pp}} = 1/(I-1)$ and subsidisation barrier (a) $B = 2$, (b) $B = 5$ and (c) $B = 10$.

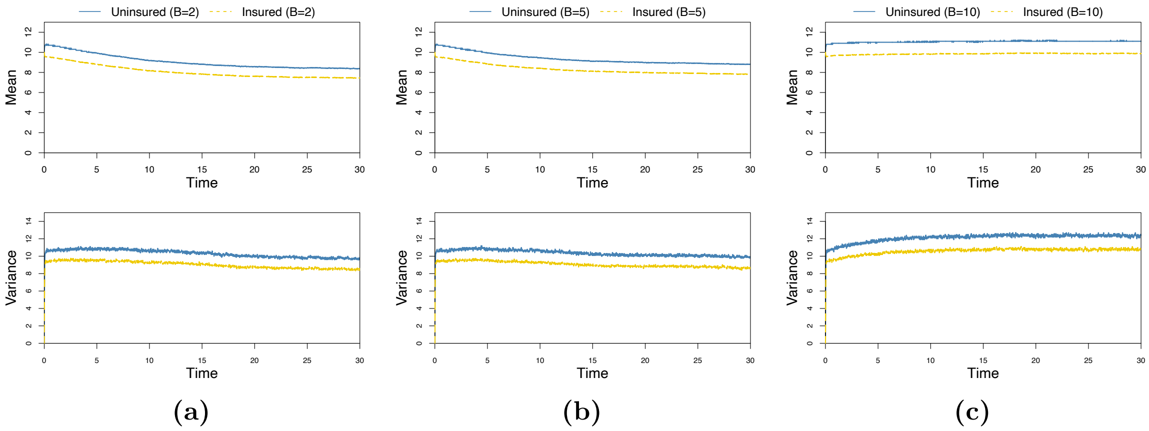


Figure 4.7: Mean and variance of uninsured and insured agent wealth for $I = 51, \boldsymbol{m}_0 = 10, \boldsymbol{\lambda} = (12, 6), \boldsymbol{s_{uu}} = 0.2, \boldsymbol{s_{pp}} = 0.1$ and subsidisation barrier (a) $B = 2$, (b) $B = 5$ and (c) $B = 10$.

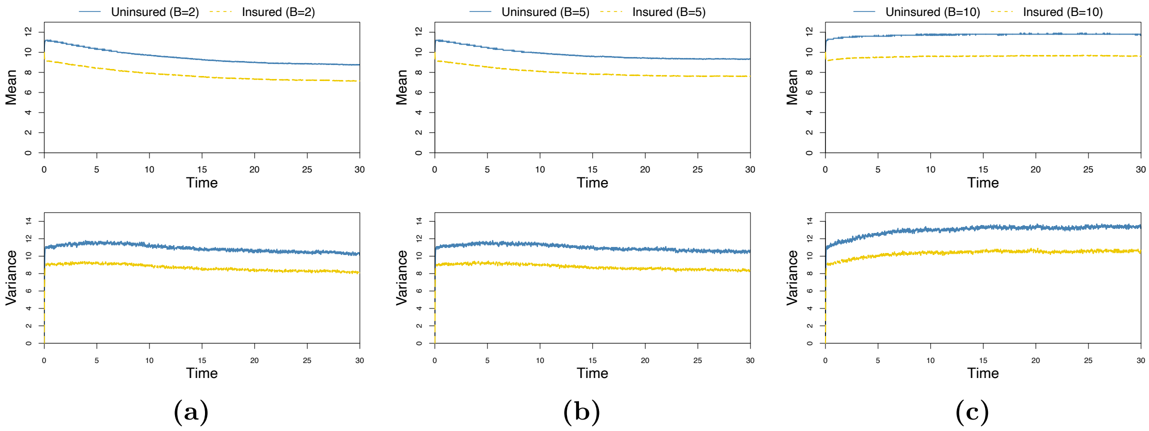
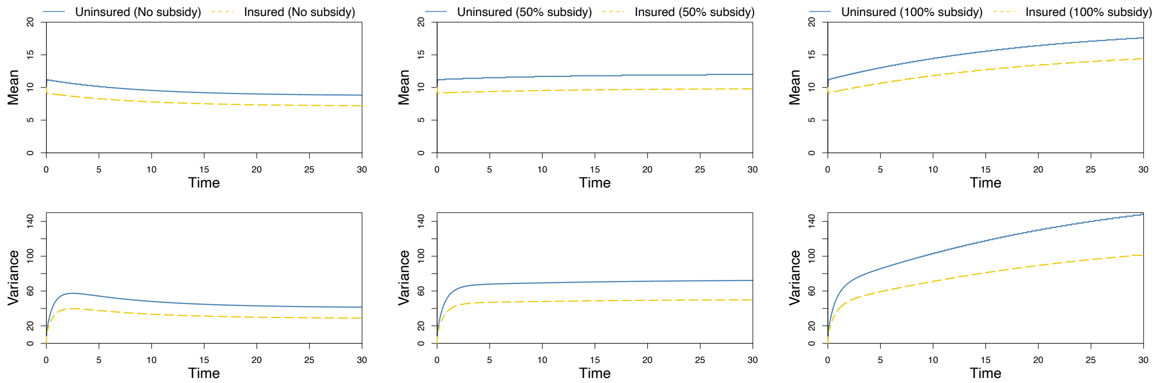


Figure 4.8: Mean and variance of uninsured and insured agent wealth for $I = 51, \boldsymbol{m}_0 = 10, \boldsymbol{\lambda} = (12, 6), \boldsymbol{s_{uu}} = 0.2, \boldsymbol{s_{pp}} = 1/(I-1)$ and subsidisation barrier (a) $B = 2$, (b) $B = 5$ and (c) $B = 10$.

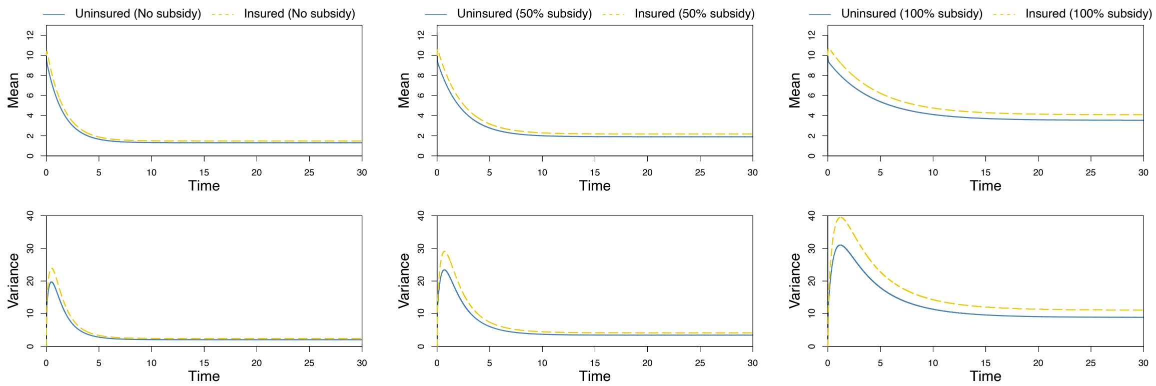
The decreasing mean wealth behaviour observed in many of the figures presented aligns with the low level of external arrivals of wealth to the system. Wealth is added to the system only through external transactions experienced by the leader, which induce an increase in leader wealth of just one wealth unit, negligible when spread evenly across the follower groups. As such, mean wealth decreases in line with greater rates of loss and premium payment. Figure 4.9 considers the impact of increasing leader wealth on the overall state of the system under the fixed subsidy scheme. Comparing with Figure 4.5, increasing the rate of external arrivals decreases the rate of wealth decay in the case of no subsidies. For fully subsidised insurance, despite the continued experience of uninsured losses and the largely even distribution of wealth, the system's wealth increases.



(a) (b) (c)

Figure 4.9: Mean and variance of uninsured and insured agent wealth for $I = 51, \boldsymbol{m_0} = 10, \boldsymbol{\lambda} = (52, 26), \boldsymbol{s_{uu}} = 0.2, \boldsymbol{s_{pp}} = 1/(I-1)$ with subsidisation of (a) 0%, (b) 50% and (c) 100% of the premium for all insured agents.

The behaviour of the system is however highly dependent on the rate of loss. For $\alpha = \beta = 1$, an uninsured agent loses, on average, 50% of their wealth at each loss event. At such a high rate, the increase in wealth received by the leader is ineffective in enabling growth in the wealth of follower groups (see Figures 4.10 and 4.11). Even with such high rates of loss, insurance premiums decrease the wealth of the insured below that of the uninsured group when retainment parameters are inhomogeneous. Figures 4.10 and 4.11 present the homogeneous retainment case where it can be seen that the mean and variance of uninsured wealth falls marginally below that of the insured, with the difference increasing with increasing subsidy.



(a) (b) (c)

Figure 4.10: Mean and variance of uninsured and insured agent wealth for $I = 51, \boldsymbol{m_0} = 10, \boldsymbol{\lambda} = (52, 26), \boldsymbol{s_{uu}} = 0.2, \boldsymbol{s_{pp}} = 1/(I-1), \alpha = 1$ with subsidisation of (a) 0%, (b) 50% and (c) 100% of the premium for all insured agents.

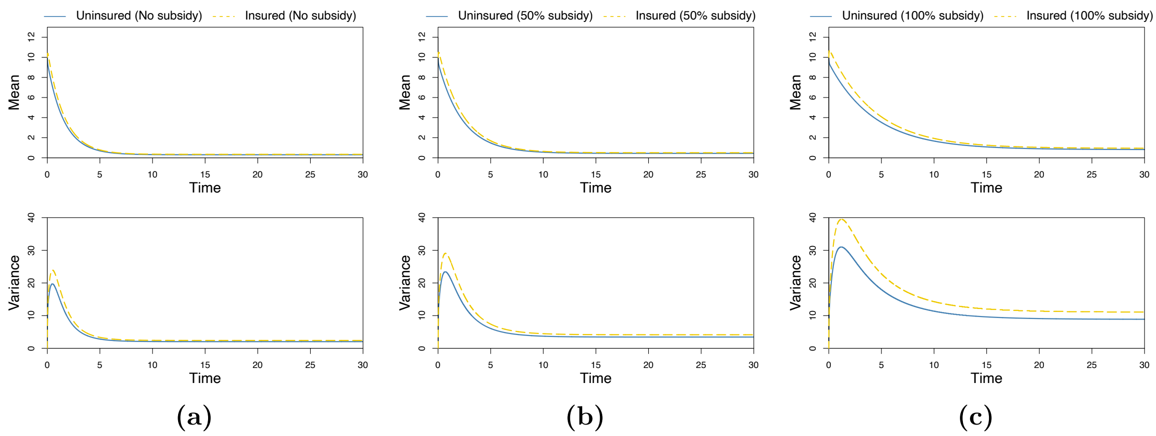


Figure 4.11: Mean and variance of uninsured and insured agent wealth for $I = 51, \boldsymbol{m}_0 = 10, \boldsymbol{\lambda} = (12, 6), \boldsymbol{s_{uu}} = 0.2, \boldsymbol{s_{pp}} = 1/(I-1), \alpha = 1$ with subsidisation of (a) 0%, (b) 50% and (c) 100% of the premium for all insured agents.

The implication of this exploratory analysis is that the even redistribution of wealth at internal transaction events and the limited facility for wealth retention (or savings), mitigates the impact of loss and premium payments. The more even the redistribution of wealth, the more negative financial events are felt by the group as a whole, rather than at the individual level. Although in reality, wealth redistribution may not be completely uniform across all agents, when considering a societal group of agents with similar socioeconomic backgrounds, such an assumption aligns with the prevalence of risk sharing and is therefore not so out of place.

4.7 The trapping probability

As in Chapters 2 and 3, the aim of this chapter is to determine the impact of insurance on the financial vulnerability of low-income individuals through analysis of the trapping probability, where the term “trapping” refers to the event at which an agent falls into an area of poverty from which it is difficult to escape without external help. In order to assess this measure with the stochastic dissemination model presented here, as in Chan and Mandjes (2022), a normal approximation is used to estimate the probability of agent wealth falling below a given critical level. Aligning with the poverty trapping context, this critical level represents the critical level of wealth below which an agent would struggle to meet their basic needs, i.e. the poverty line. The critical level of wealth assessed here is equivalent to the critical capital discussed in Chapters 2 and 3.

Let ψ denote the probability that the stationary wealth of an arbitrary follower lies below the critical capital x^*. As in Chapters 2 and 3, this probability represents the trapping probability of an individual agent. Given the group-based nature of the dissemination model, under consideration of a sufficiently large number of agents, a standard normal approximation may be used to estimate this probability. As such, let the trapping probability of an arbitrary uninsured follower be given by

$$\psi_u = \mathbb{P}(M_j \leq x^*) \approx \psi_{N,u} := \Phi\Big(\frac{x^* - m_u}{\sqrt{v_{uu}^o}}\Big), \tag{4.7.1}$$

where M_j denotes the stationary wealth of follower j for $j = 2, ..., I_u + 1$. The mean and reduced second moments of stationary wealth are given by

$$m_u := m_{u,1} + m_{u,2}, \quad v_{uu} := v_{uu,1} + v_{uu,2},$$

where, by definition of the variance of the random variable M_j,

$$\begin{aligned} v^o_{uu} :&= \mathbb{E}[(M_j - \mathbb{E}[M_j])^2] \\ &= \mathbb{E}[M_j(M_j - 1)] + \mathbb{E}[M_j] - \mathbb{E}^2[M_j] \\ &= v_{uu} + m_u - m_u^2, \end{aligned}$$

which can be obtained by (4.4.3). The trapping probability for an insured follower is defined analogously for $j = I_u + 2, ..., I$, with subscripts p replacing u.

Although the trapping probabilities of all agents are identically distributed, due to the reliance of agent wealth on the shared background process, they are not independent. As such, in order to analyse trapping within the full uninsured follower group, define the following random variable:

$$B_u := \sum_{j=2}^{I_u+1} B_j,$$

where $B_j = \mathbb{1}_{\{M_j \le x^*\}}$. Then, for $j, j' = 2, ..., I_u + 1$ and $j \neq j'$, let

$$\psi'_u := \mathbb{P}(M_j \le x^*, M_{j'} \le x^*),$$

which has normal approximation

$$\psi'_{N,u} := \mathbb{P}(M^o_j \le x^*, M^o_{j'} \le x^*), \tag{4.7.2}$$

for bivariate normal $(M^o_j, M^o_{j'})$ with mean (m_u, m_u) and covariance matrix

$$\Sigma = \begin{pmatrix} v^o_{uu} & v^o_{uu'} \\ v^o_{uu'} & v^o_{uu} \end{pmatrix}.$$

The covariance of the wealth of two distinct uninsured follower agents is derived in the standard manner:

$$\begin{aligned} v^o_{uu'} :&= \mathbb{E}[(M_j - \mathbb{E}[M_j])(M_{j'} - \mathbb{E}[M_{j'}])] \\ &= \mathbb{E}[M_j M_{j'}] - \mathbb{E}^2[M_j] \\ &= v_{uu'} - m_u^2. \end{aligned}$$

The corresponding probability for insured followers is again defined analogously.

In this analysis, the approximation presented by Cox and Wermuth (1991) for the distribution of bivariate normal random variables (X, Y), with zero means, unit variances and correlation coefficient ρ, is adopted for estimation of the joint trapping probability in (4.7.2). The approximation is given as follows:

$$\mathbb{P}(X > a, Y > b) \simeq \Phi(-a)[\Phi\Big(\frac{\rho\mu(a) - b}{\sqrt{1-\rho^2}}\Big) - \frac{1}{2}\frac{\rho^2(\rho\mu(a) - b)}{(1-\rho^2)^{3/2}}\phi\Big(\frac{\rho\mu(a) - b}{\sqrt{1-\rho^2}}\Big)\sigma^2(a)],$$

where $\phi(x)$ is the standardised normal density,

$$\mu(a) = \mathbb{E}[X|X > a] = \frac{\phi(a)}{\Phi(-a)}$$

is used to approximate the random variable X and

$$\sigma^2(a) = Var[X|X > a] = 1 + \mu(a) + \mu^2(a).$$

This approximation relies on the near linearity of $(\rho X - b)/\sqrt{1-\rho^2}$ over the range of appreciable probability, which is satisfied in the case of interest here, where $a = b = (x^* - m_u)/\sqrt{v^o_{uu}} > 0$ in all examples considered.

The mean and variance of the number of trapped agents are derived by definition of B:

$$\mathbb{E}[B] = \mathbb{E}\left[\sum_{j=2}^{I_u+1} B_j\right] = \sum_{j=2}^{I_u+1} \mathbb{E}[B_j] = I_u \psi_{N,u}.$$

Similarly,

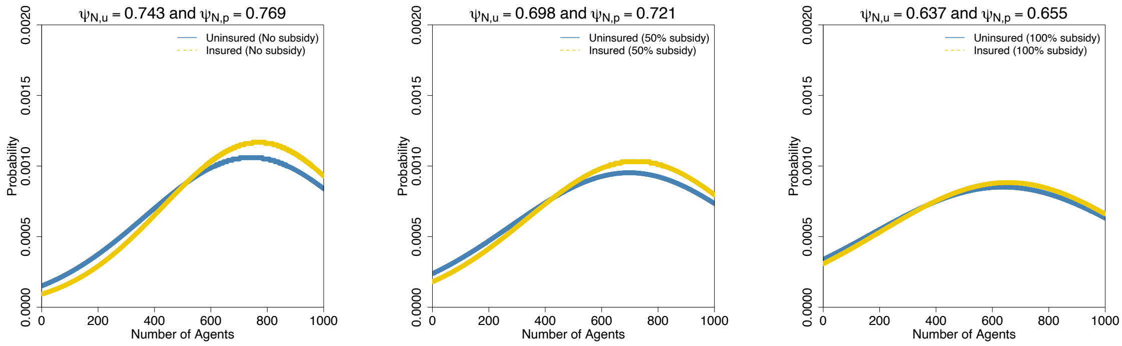
$$\begin{aligned}\mathbb{E}\left[\left(\sum_{j=2}^{I_u+1} B_j\right)^2\right] &= \sum_{j=2}^{I_u+1} \mathbb{E}\left[B_i^2\right] + \sum_{j=2}^{I_u+1} \sum_{i=2, i\neq j}^{I_u+1} \mathbb{E}[B_i B_j] \\ &= I_u \psi_{N,u}(1 - \psi_{N,u}) + I_u \psi^2_{N,u} + I_u(I_u - 1)\psi'_{N,u},\end{aligned}$$

such that $Var[B] = I_u \psi_{N,u}(1 - \psi_{N,u}) + I_u(I_u - 1)(\psi'_{N,u} - \psi^2_{N,u})$. Then, applying a normal approximation with $\mu_B \approx \mathbb{E}[B], \sigma^2_B \approx Var[B]$ and accounting for discontinuity, it holds that for $n = 1, ..., I_u$,

$$\mathbb{P}(B = n) \approx \Phi\left(\frac{n + \frac{1}{2} - \mu_B}{\sigma_B}\right) - \Phi\left(\frac{n - \frac{1}{2} - \mu_B}{\sigma_B}\right).$$

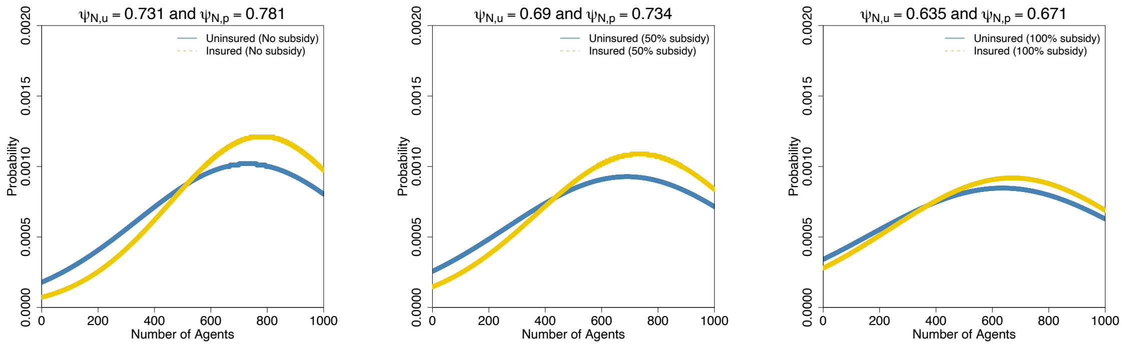
Figures 4.12-4.14 present density plots for the number of trapped uninsured and insured agents for three levels of subsidisation with wealth systems as in Figures 4.3-4.5. A large sample of size 2001 is selected to increase the accuracy of the normal approximation. The sample constitutes one leader, and 1000 followers in each of the two groups. Initial wealth ($\boldsymbol{m}_0 = 10$) and the rate of external arrivals ($\boldsymbol{\lambda} = (12, 6)$) remain fixed, with all other parameters as stated at the beginning of Section 4.6. Trapping probabilities calculated by (4.7.1) are presented at the top of each plot.

Note that, the shape of the count distribution is similar across all plots. Trapping probabilities consistently decrease with increasing subsidy, along with the spread of the number of trapped agents. In addition, disparities in the densities corresponding to the uninsured and insured groups decrease with increasing subsidy. This again suggests that subsidisation positively impacts the wealth of the uninsured. Trapping probabilities and corresponding density plots are identical when retention rates are equal (Figure 4.14). When the uninsured agent retains a greater proportion of wealth, the trapping probability of the insured group is consistently higher.



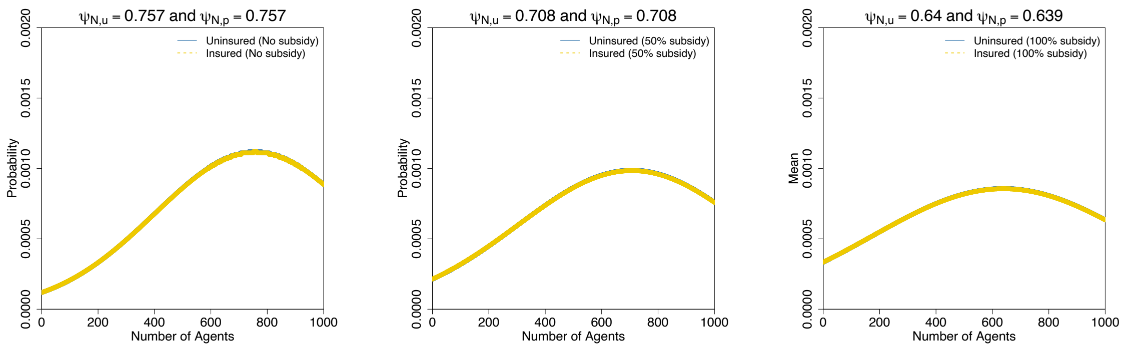
(a) **(b)** **(c)**

Figure 4.12: Distribution of number of trapped for $\boldsymbol{s_{uu}} = 0.2, \boldsymbol{s_{pp}} = 0.1$, with subsidisation of (a) 0%, (b) 50% and (c) 100% of the premium for all insured agents.



(a) **(b)** **(c)**

Figure 4.13: Distribution of number of trapped for $\boldsymbol{s_{uu}} = 0.2, \boldsymbol{s_{pp}} = 1/(I-1)$, with subsidisation of (a) 0%, (b) 50% and (c) 100% of the premium for all insured agents.



(a) **(b)** **(c)**

Figure 4.14: Distribution of number of trapped for $\boldsymbol{s_{uu}} = \boldsymbol{s_{pp}} = 1/(I-1)$, with subsidisation of (a) 0%, (b) 50% and (c) 100% of the premium for all insured agents.

Based on the steady-state mean and variance of agent wealth, trapping probabilities are high in comparison to those observed in Chapters 2 and 3 for equivalent initial wealth. However, this could be accountable to the consideration of full, or no coverage, rather than due to the impact of the group structure. Chapters 2 and 3 consider a single agent (household) with proportional insurance coverage, thus reducing the impact of wealth losses in comparison to those experienced by the uninsured in this group setting.

4.8 Concluding remarks

This chapter addresses a fundamental feature of the low-income environment for the first time in a rigorous mathematical context. Participation in risk sharing mechanisms and their prevalence across the societal groups of low-income communities influences many components of the agent decision to insure. These components, which include insurer trust and financial education levels, must be understood in order to increase microinsurance penetration and thus financial inclusion in the low-income market.

Observations in the sensitivity analysis presented in Section 4.6 bring to light the possibility that within the group structure, the financial impact of both losses and premium payments are in fact largely shared among participating agents. This aligns with the findings of Will et al. (2021). Although this observation is dependent on the level of sharing within the group, when considering a group of agents of the same socioeconomic background, the assumption of even (or close to even) redistribution of wealth is not such a strict assumption. The limited capacity for savings also increases the likelihood of agents retaining only a small proportion of their wealth, if any wealth is retained.

In addition, subsidisation of insurance premiums benefits both the insured and the uninsured, providing further evidence for the benefit of government-insurer partnerships. This finding could be due to the increased liquidity of wealth associated with the spread of premium payments over time. Insured agents are likely to have greater access to wealth at redistribution event times, thus increasing the wealth of all agents. Trapping probabilities decrease with increasing subsidisation as expected, however, in comparison to the results of Chapters 2 and 3, they remain high. The assumption of full or no coverage considered in this study likely contributes to this observation. Consideration of proportional insurance in the group setting is discussed in the presentation of future work.

While the selected model yields intuitive results, it is important to highlight its limitations. Premium payments are required to be proportional to agent wealth. In reality, this is an unlikely feature of insurance, requiring premiums to be consistently reviewed over the course of the insured period. However, premium payments that increase with increasing wealth align with the proportional structure of losses, with those of greater wealth susceptible to higher losses and thus encountering higher premium payments. On the other hand, adjusting classical risk theory models, such as those considered in Chapters 2 and 3, to capture randomly occurring interactions between multiple capital processes is difficult. The flexibility of the stochastic dissemination model of this chapter facilitates re-specification of the population structure and wealth transaction events through simple parameter adjustments, enabling analysis of an unlimited number of scenarios.

Chapter 5

Stochastic mortality modelling for dependent coupled lives

Broken-heart syndrome is the most common form of short-term dependence, inducing a temporary increase in an individual's force of mortality upon the occurrence of extreme events, such as the loss of a spouse. Socioeconomic influences on bereavement processes allow for suggestion of variability in the significance of short-term dependence between couples in countries of differing levels of economic development. Motivated by analysis of a Ghanaian data set, a stochastic mortality model of the joint mortality of paired lives and the causal relation between their death times is proposed, in the context of a less economically developed country than those considered in existing studies. The paired mortality intensities are assumed to be non-mean-reverting Cox–Ingersoll–Ross processes, reflecting the reduced concentration of the initial loss impact apparent in the data set. The effect of the death on the mortality intensity of the surviving spouse is given by a mean-reverting Ornstein–Uhlenbeck process which captures the subsiding nature of the mortality increase characteristic of broken-heart syndrome. Inclusion of a population wide volatility parameter in the Ornstein–Uhlenbeck bereavement process gives rise to a significant non-diversifiable risk, heightening the importance of the dependence assumption in this case. Applying the proposed model to an insurance pricing problem, the appropriate premium under consideration of dependence between coupled lives is obtained through application of the indifference pricing principle. This chapter is based on Henshaw et al. (2020).

5.1 Introduction

In this chapter, as addressed for the first time in Henshaw et al. (2020), the existence of socioeconomic influences on dependence between coupled lives and the bereavement processes of surviving spouses in less economically developed populations is considered. Analysis of a Ghanaian dataset within which a lesser initial concentration of broken-heart syndome is observed is utilised to inform the proposal of a joint mortality model. In line with observations in the data which fit the nature of a reduced volatility, and following the results of Luciano and Vigna (2008), correlated non-mean-reverting Cox–Ingersoll–Ross (CIR) diffusions are introduced as paired mortality intensities. A model of the joint mortality of a couple assumed to share the same socioeconomic environment is defined based on the stochastic mortality model proposed by Jevtić and Hurd (2017). A mean-reverting Ornstein–Uhlenbeck process is selected to represent the influence of the loss of a spouse on the remaining lifetime of the surviving partner. In moving from deterministic (as presented in Jevtić and Hurd (2017)) to stochastic bereavement, the proposed model has the ability to capture the existence of a non-diversifiable risk, requiring a premium that accounts for the change in mortality observed in the data. Risks associated with a deterministic bereavement process are, on the other hand, diversifiable in nature.

Observation of an increased mortality during the first period of bereavement paired with the findings of Lu (2017) suggests that dependence between lifetimes within the sample is mostly causal in nature. Existence of a cause-and-effect relationship between the remaining lifetimes of paired individuals is therefore captured in the proposed model through definition of the bereavement effect, while couple specific unobserved heterogeneities are accounted for through inclusion of correlated Brownian motions in the paired mortality processes.

An outline of the chapter is as follows, in Section 5.2, data supporting the modification of mortality processes is discussed, before introduction of the proposed mortality model in Section 5.3. One example of the pricing of a life insurance product incorporating the dependence model is provided in Section 5.4, where the indifference pricing principle is adopted. A numerical pricing example is presented in Section 5.5 and concluding remarks in Section 5.6.

5.2 Data set

Evaluation of the existence of the broken-heart syndrome effect requires data which details the time of death of both members of a couple. Since relevant data in this socioeconomic setting is lacking, a questionnaire was designed and distributed to students at a university in Ghana. The questions obtained information including the subject's current age, or age at death and the number of years since their death, where applicable. Details of potential determinants of dependence, including numbers of children, living circumstances and circumstances of the death, were also obtained. Each complete questionnaire contributed two data points to the sample, with questions relating to both maternal and paternal grandparents. A sample of the questionnaire is provided in Appendix E.

All couples for which life status (alive or dead) or number of years since death, if dead, were left blank or answered "I don't know" were removed from the sample, leaving 1117 out of 1652 initial data points. Couples in which both spouses were alive do not provide any survival information, and so 188 couples with no death experience were removed. The maximum number of years since death included in the questionnaire was "60+". Since survival data cannot be obtained if both members of a couple died "60+" years ago, 8 couples were removed. Finally, for numerical analysis purposes, if either member of the couple died "60+" years ago and their spouse was alive, "60+" was fixed at 60 (6 couples). If the spouse died more than 40 years ago, "60+" was fixed at 60 (14 couples), since a survival time of 20 years would not be indicative of broken-heart syndrome. If the spouse died less than 40 years ago but the participant selected the grandparent that was first to die inconsistently with their time of death response, the couple was removed (3 couples). The remaining 3 couples with a "60+" response were removed since the corresponding survival time was too short to make any assumption that would not affect the results in regard to the severity of the dependence. Data cleaning therefore left 915 couples, consisting of 459 double deaths and 456 single deaths, with 386 female survivors.

The terminal time of the observation is determined by the survey completion date. The data is therefore right-censored, with a number of bereaved spouses surviving the observation period. Survival curves for the full sample and for the samples of female and male first deaths, comprising 189 and 656 couples, respectively, are displayed in Figure 5.1, with censored data points indicated. In the first year of bereavement, the survival probability is greater in the sample of bereaved males than in that of bereaved females. At all subsequent time points, an increased survival probability is observed among bereaved females. Disparities in the sample sizes of male and female first deaths should however be noted.

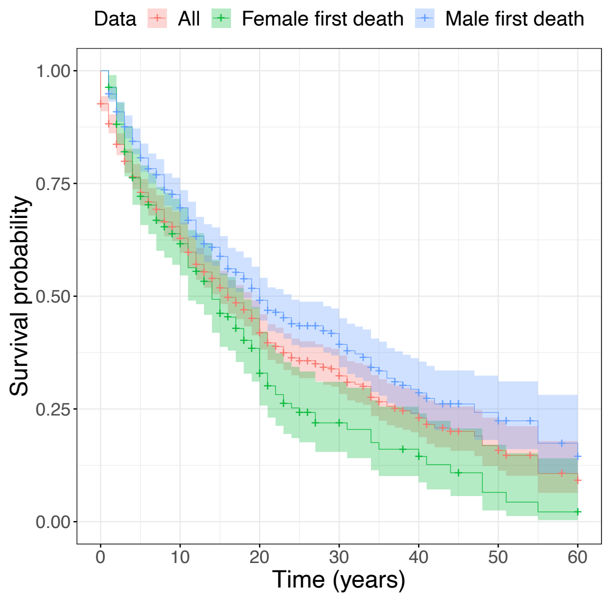
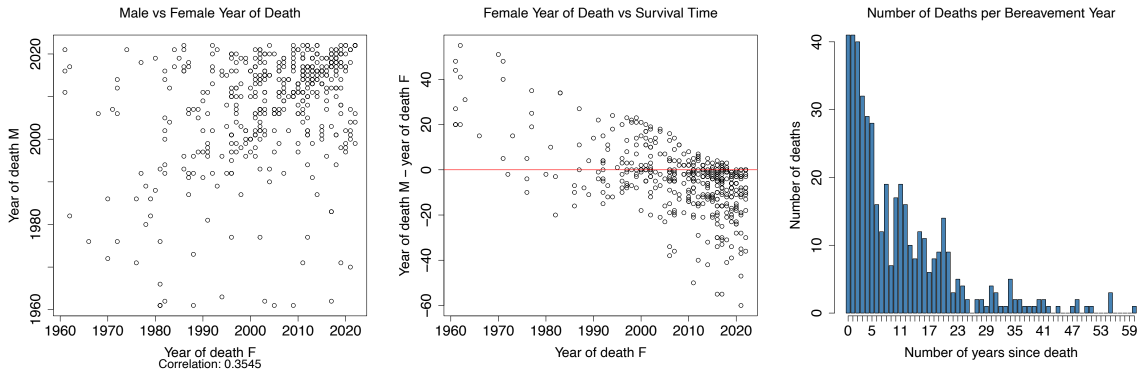


Figure 5.1: Kaplan–Meier curves for the full, male and female widowed samples.

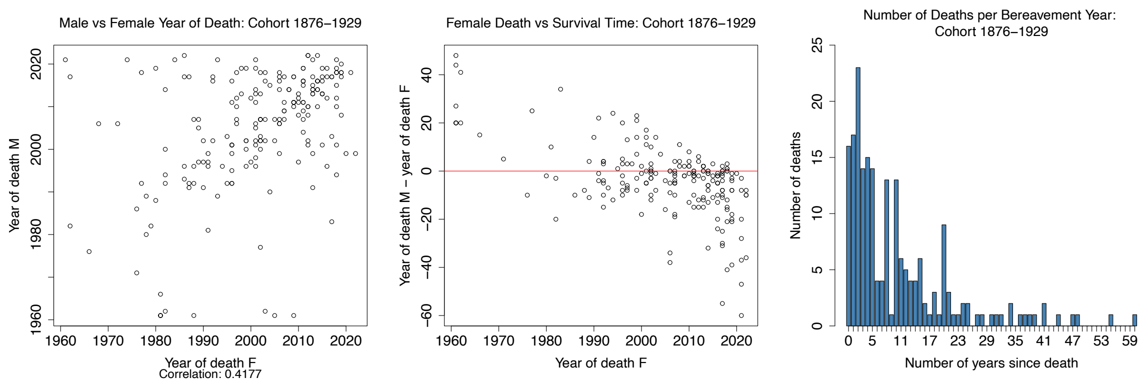
Figures 5.2a and 5.2b present the distribution of male and female death times within a couple and Figure 5.2c the number of deaths per year of bereavement. Here, data for the 459 samples in which both spouses have died is presented, focusing on the waiting time between deaths as the variable of interest. A total of 41 bereaved spouses died in both the first and second years of bereavement, corresponding to 17.86% of bereaved individuals in the sample. The death rate steadily decreases after this time. Couples along the red line in Figure 5.2b are those exhibiting the classical features of broken-heart syndrome, with survival time less than one year.



(a) (b) (c)

Figure 5.2: Male versus female year of death plot, (b) female year of death versus death interrarival time, and (c) number of deaths per year of bereavement.

The Pearson correlation between male and female deaths is 0.3545 when considering the total sample of couples that experience two deaths; however, since the range of birth years for the male spouse subset spans approximately 95 years, the data was split into two cohorts to test for existence of a generational effect. Data splitting is undertaken by male year of birth, however female splitting could analogously be considered. In the questionnaire, age at death was recorded categorically through selection of an age range. The central age of each range was therefore selected to determine an individual's age at birth. Figures 5.3 and 5.4 correspond to the cohorts of paired lives with male year of birth in the intervals 1876–1929 and 1930–1971, respectively. After removal of data points with incomplete male date of birth and splitting the data at the median birth year, 202 couples remained in cohort one and 192 couples in cohort two.



(a) (b) (c)

Figure 5.3: (a) Male versus female year of death plot, (b) female year of death versus death interrarival time, and (c) number of deaths per year of bereavement, for cohort of couples with male spouse born between 1876 and 1929.

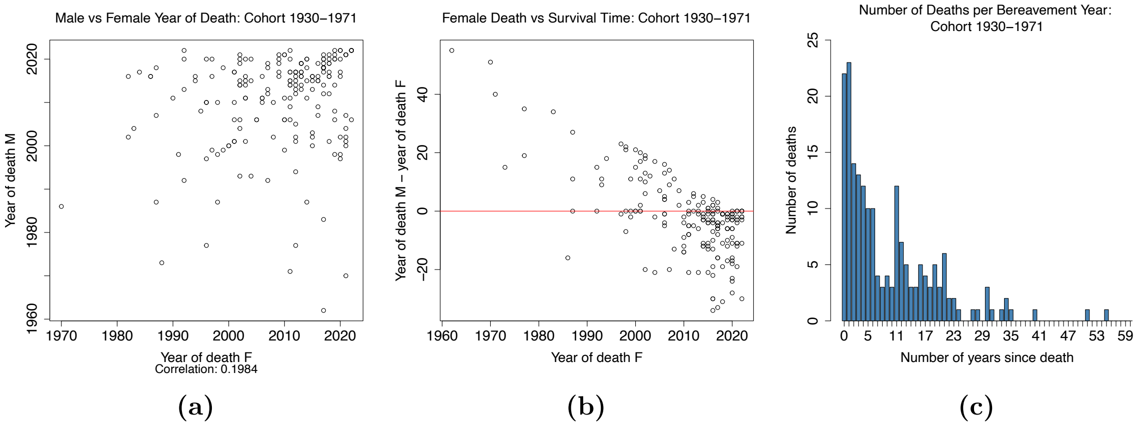


Figure 5.4: Male versus female year of death plot, (b) female year of death versus death interrarival time, and (c) number of deaths per year of bereavement, for cohort of couples with male spouse born between 1930 and 1971.

During the first year of bereavement, 16 surviving spouses, representing 7.921% of the bereaved sample, died in cohort one. However, the peak in the number of bereaved deaths appeared in the third year of bereavement, with 23 (11.39%) deaths observed. The proportion of bereaved deaths experienced in the second cohort was 11.46% (22 deaths) during the first year of bereavement, with a similar rate of death in year two (11.98% with 23 deaths). In addition to the differing patterns of bereaved deaths observed in Figures 5.3 and 5.4, with the older cohort achieving its peak frequency later in the bereavement process, the Pearson correlation between male and female deaths within the sub-samples varies. Correlation coefficients are 0.4177 and 0.1984 in the first and second cohorts respectively, suggesting the existence of a potential change in reactions to bereavement across generations.

Previous investigations into the impacts of the effect have largely considered a high-income group as their sample population, with the recent paper by Walter et al. (2021) the only known exception. Rees and Lutkins (1967) present analysis on the mortality of bereaved close relatives. The number of observed deaths vary significantly from the control group in the first year of bereavement, with 11.6% of deaths followed by the death of a close relative in comparison to just 1.6% in the control. Subsequently, total deaths in the bereaved group falls to a rate of 1.99%, differing negligibly from the comparative non-bereaved rate.

Focusing specifically on the bereavement effect in a married couple, Rees and Lutkins (1967) find the severity of increases in mortality to be greatest during the first year of bereavement, after which the magnitude of rate elevation diminishes. The mortality rate of (male) widowers within the first year after the loss is observed to be 19.6%, sizeably greater than that of (female) widows (8.5%). For widowers, the pattern of changing mortality differs slightly from the general findings, with 13.7% dying within the first six months of bereavement and just 5.9% in the second, a difference in mortality between widows and widowers found to be significant at the 1% level. Observations in the well-studied Canadian data mentioned in Chapter 1 further highlight the increased mortality of both widows and widowers, particularly in the first year of bereavement.

Although the Ghanaian survey data supports suggestion that broken-heart syndrome exists in countries of differing levels of development, comparison with existing literature prompts suggestion that behaviours under broken-heart syndrome differ. The significant decrease in mortality following survival of the first year of bereavement is a characteristic prevalent in much of the research in this area; however, the decreasing trend of mortality with increasing year since bereavement, although apparent, cannot be identified with such high initial concentration and decay in the Ghanaian data. Such dissimilarities lead to the definition of a mortality model representing the impact of a dependence with less immediate significance. The proposed model is introduced in the following section.

5.3 Model description

Inclusion of the probabilistic framework in the stochastic mortality model proposed by Jevtić and Hurd (2017) prompted the decision to implement a similar model within the investigation of this chapter. While multi-state models such as the semi-Markov chain model presented in Ji et al. (2011) offer transparency and the ability to observe whether the level of risk changes following a death event, the probabilistic mechanism further enables incorporation of the health of both members of a couple prior to the primary death. This feature increases the accuracy of the dependence model by permitting varied responses to the initial death, where dependence may be irregular across the population, determined by the health circumstances of each couple under consideration. Although only the short-term dependence of coupled lives is considered in this chapter, the model proposed is capable of addressing both short- and long-term structures, with the ability to encompass the existence of dependence between two lifetimes before the death of one spouse.

Fundamental in modelling mortality risk, the survival function of an individual aged x, denoted (x), specifies the probability that the individual survives for at least t years, and is defined by

$$S_x(t) = \mathbb{P}(\tau_x > t), \tag{5.3.1}$$

where τ_x is the remaining lifetime of (x). Manipulation of this function allows for calculation of the force of mortality, such that

$$\lambda_{x+t} = -\frac{d}{dt} \log S_x(t),$$

where λ_{x+t} is the force of mortality of $(x+t)$ for $t > 0$, describing the instantaneous rate at which the individual experiences death. The force of mortality of an individual (x) at time 0 is given by

$$\lambda_x = -\frac{d}{dx} \log S_0(x).$$

Analogous to the pricing at time t of a default-free zero-coupon bond with maturity $s > t$, under assumption of a reduced-form credit risk setting, the conditional probability of a stopping time τ exceeding some arbitrary time $s \geq t$, where τ is doubly stochastic with intensity $\lambda(t)$ (or equivalently λ_t), can be shown to satisfy

$$\mathbb{P}(\tau > s \mid \mathcal{G}_t) = \mathbb{E}[e^{-\int_t^s \lambda(u)du} \mid \mathcal{G}_t],$$

where $\mathcal{G}_t$ represents the information at time t. In the mortality modelling context, the stopping time τ represents the remaining lifetime of an individual.

5.3.1 Probabilistic mechanism

Fundamental concepts required to set up the probabilistic mechanism are defined in Appendix D.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration satisfying the usual conditions of right continuity and completeness, large enough to carry a d-dimensional Brownian motion W, two exponentially distributed random variables E_1 and E_2, and a single uniformly distributed random variable U. Within this space, the set $\{W, E_1, E_2, U\}$ is fully independent, with a realisation of the time of death of each partner following from every realisation of the randomly generated elements. Allowing T to represent a finite time horizon of suitable length, the Brownian filtration is defined over the interval $t \in [0, T]$ and is given by $\mathcal{G}_t = \sigma(W_s)$ for $s \leq t$, where $\mathcal{G}_t$ is a sub-filtration of $\mathcal{F}_t$. When applying the model to a sizeable population, an index $n \in \{1, ..., N\}$, where N is the number of couples in the sample, should be added to every element whose properties are specific to a given pair.

Consider two coupled lives aged x and y at time 0 with future lifetimes τ_x and τ_y, respectively. The instantaneous forces of mortality at time t given by $\lambda_i(t)$ for $i \in \{x, y\}$, are predictable $\mathcal{G}_t$-adapted processes driven by the Brownian motion W. The spouse whose death occurs first is identified as the deceased partner and denoted $p \in \{x, y\}$. Equivalently, the spouse who survives the first death is denoted q and labelled the bereaved partner. The remaining lifetime of spouse p, conditional on the information set $\mathcal{F}_0 \cap \mathcal{G}_t$, is the first jump-time of a nonexplosive inhomogeneous Poisson counting process N with parameter $\int_0^t \lambda_x(u) + \lambda_y(u)du$, where N_t counts the number of deaths at time t, for $t \geq 0$. The remaining lifetime of spouse q is defined in an analogous manner, with both doubly stochastic stopping times representing τ_p and τ_q driven by the sub-filtration $\mathcal{G}_t$.

The first time of death τ_p is given by

$$\tau_p := \inf\left\{t \geq 0 \,\middle|\, \int_0^t \lambda_x(u) + \lambda_y(u)du \geq E_1\right\},$$

while the uniform random variable U allows for identification of the deceased spouse through comparison with a function of the forces of mortality at the instant of the primary death. Recalling that p is the label given to the partner who dies first, it holds that

$$\begin{aligned} \{x = p\} = \{\tau_x = \tau_p\} &= \left\{U \leq \frac{\lambda_x(\tau_p)}{\lambda_x(\tau_p) + \lambda_y(\tau_p)}\right\}, \\ \{y = p\} = \{\tau_y = \tau_p\} &= \left\{U > \frac{\lambda_x(\tau_p)}{\lambda_x(\tau_p) + \lambda_y(\tau_p)}\right\}. \end{aligned} \tag{5.3.2}$$

In line with the belief that the loss of a spouse has an impact on the mortality of the surviving spouse, $\tilde{\lambda}_q(t)$ is defined for $t \geq \tau_p$ as the mortality intensity of the bereaved partner following the initial death. This force of mortality is an adjustment of the original process $\lambda_q(t)$ and the association between the two rates reflects the influence that losing a partner has on the bereaved spouse's health and hence their remaining lifetime. The bereavement effect is therefore defined by

$$r_q(t) := \tilde{\lambda}_q(t) - \lambda_q(t),$$

the change in mortality process, where the modified process $\tilde{\lambda}_q(t)$ is inclusive of a structural break at τ_p representing the instant effect on the bereaved spouse's mortality. The instantaneous rise at the first death time is given by a linear combination of the mortality of each spouse at time τ_p^-, directly before the death, such that

$$r_q(\tau_p) := \tilde{\lambda}_q(\tau_p) - \lambda_q(\tau_p^-) = \delta^q + \epsilon^q \lambda_q(\tau_p^-) + \zeta^q \lambda_p(\tau_p^-), \tag{5.3.3}$$

where coefficients δ^q, ϵ^q and ζ^q are assumed to be non-negative. Intuitively, the mortality jump reflects the short-term dependence structure of broken-heart syndrome and the modification of λ_q the adaptations in the mortality intensity of the surviving spouse due to adjustments in the life circumstances of the bereaved. Inclusion of the mortality intensity of both spouses in the estimation of the bereavement jump in (5.3.3) allows for incorporation of unobserved shared frailties.

Determined using a similar approach to the first time of death τ_p, the second time of death τ_q is given by

$$\tau_q := \inf \Big\{ t > \tau_p \, \Big| \int_{\tau_p}^{t} \tilde{\lambda}_q(u) du \geq E_2 \Big\}.$$

The model proposed is an adjustment of the reduced form modelling approach frequently used in the study of credit risk to model default as a stopping time whose occurrence is unexpected. Implementation of this method in regard to lifetime dependencies in coupled lives suggests that a change in the remaining lifetime of the bereaved spouse does not occur following the primary death, since random variables used in the determination of the time of death of each spouse $\{E_1, E_2, U\}$, are required to be independent across the index. Inclusion of the modified intensity $\tilde{\lambda}_q(t)$ resolves this limitation.

Determination of the structure of dependence across a population may also be of interest, in addition to dependence within a couple. To model the dependence relationship among a population, risk factors experienced commonly by all individuals and risks specific to each member of the population should be considered. These factors are labelled systematic and idiosyncratic risks, respectively, and are an independent collection of factors by construction, with correlation between individuals induced by the risks shared among those under consideration.

The objective of the probabilistic mechanism is to determine the joint probability density function of the remaining lifetimes of two coupled lives (τ_x, τ_y). Theorem 5.3.1 provides an expression for the joint density proposed with proof in Jevtić and Hurd (2017), where expectations are taken under the probability measure $\mathbb{P}$ and it is assumed that the death events do not occur simultaneously.

Theorem 5.3.1 (Jevtić and Hurd (2017))**.**

1. The joint probability density function $\rho(t_x, t_y)$ of the remaining lifetimes of two coupled lives (τ_x, τ_y) is given by the reduced form expression

$$\rho(t_x, t_y) = \begin{cases} \mathbb{E}\Big[\lambda_p(t_x) e^{-\int_0^{t_x} \lambda_x(u) + \lambda_y(u) du} \mathbb{E}\Big[\tilde{\lambda}_q(t_y) e^{-\int_{t_x}^{t_y} \tilde{\lambda}_q(u) du} \mid \mathcal{G}_T\Big]\Big], & t_x < t_y \\ \mathbb{E}\Big[\lambda_p(t_y) e^{-\int_0^{t_y} \lambda_x(u) + \lambda_y(u) du} \mathbb{E}\Big[\tilde{\lambda}_q(t_x) e^{-\int_{t_y}^{t_x} \tilde{\lambda}_q(u) du} \mid \mathcal{G}_T\Big]\Big], & t_x > t_y. \end{cases}$$

2. The marginal probability density function $\rho_p(t)$ for the time of the first occurring death τ_p is

$$\rho_p(t) = \mathbb{E}\Big[\lambda_p(t) e^{-\int_0^t \lambda_x(u) + \lambda_y(u) du}\Big],$$

for $p \in \{x, y\}$.

Proof. For proof, see Jevtić and Hurd (2017). □

5.3.2 Stochastic mortality model with non-mean-reverting Cox–Ingersoll–Ross mortality processes

It is common practice in financial modelling to assume the stochastic mortality intensity $\lambda(t)$ to be an affine process, due to their analytical tractability. Under sufficient technical conditions, the affine assumption gives rise to the expression

$$\mathbb{E}[e^{-\int_t^T \lambda(u)du} \mid \mathcal{G}_t] = e^{A(T-t)+B(T-t)\lambda(t)}, \tag{5.3.5}$$

where $A(t)$ and $B(t)$ are unique functions satisfying generalised Riccati ordinary differential equations. Owing to the convenience of affine jump-diffusions, the stochastic mortality model presented here is proposed under the assumption of affine mortality intensities, assuming a cohort of single-life mortality models in continuous time with correlated, non-mean-reverting Cox–Ingersoll–Ross (CIR) processes representing the paired mortality intensities $\lambda_x(t)$ and $\lambda_y(t)$. For further discussion and treatment of affine processes, see Duffie et al. (2003).

The adapted CIR processes are defined by

$$d\lambda_p(t) = \mu^p \lambda_p(t)dt + \sigma^p \sqrt{\lambda_p(t)}dW_p(t) \quad \text{for} \quad p \in \{x, y\}, \tag{5.3.6}$$

where the parameters $\lambda_p(0)$, μ^p, and σ^p are positive. Let $B = (B_x, B_y, B_z)$ be a three-dimensional Brownian motion; each Brownian motion $W_x(t)$ and $W_y(t)$ is then considered as a linear combination of two independent Brownian motions, such that

$$W_x(t) = \gamma^x B_x(t) + \bar{\gamma}^x B_z(t),$$
$$W_y(t) = \gamma^y B_y(t) + \bar{\gamma}^y B_z(t),$$

which gives

$$d\lambda_p(t) = \mu^p \lambda_p(t)dt + \sigma^p \sqrt{\lambda_p(t)}d(\gamma^p B_p(t) + \bar{\gamma}^p B_z(t)), \quad \text{for} \quad p \in \{x, y\}.$$

Here, $B_x(t)$ and $B_y(t)$ represent the random idiosyncratic risk factors specific to each member of the couple and $B_z(t)$ reflects the random couple specific risk factors commonly experienced by both members of the pair, an example of which is the mutual living environment often shared by coupled lives. Weights $\gamma^p \in [-1, 1]$ and $\bar{\gamma}^p$ are selected in order to satisfy $\rho = \bar{\gamma}^x \bar{\gamma}^y$, where $\bar{\gamma}^p := \sqrt{1 - (\gamma^p)^2}$ and $\rho \in [-1, 1]$ is the Pearson correlation between $W_x(t)$ and $W_y(t)$. Introducing correlation between the two Brownian motions in this way allows for dependence prior to the initial death and enables the capturing of unobserved heterogeneities assumed to be shared between coupled lives.

Remark 5.3.1. Selection of population specific risk factors for representation in the $B_z(t)$ component of the Brownian motions $W_x(t)$ and $W_y(t)$, rather than the couple specific risks assumed in this chapter, initiates a non-diversifiable risk to insurers, creating with certainty, a long-term effect for insurance companies that should be considered in the pricing and valuation of insurance products.

The final step in establishing the model is to define the bereavement effect explicitly, since determination of the second death time requires the modified process $\tilde{\lambda}_q(t)$, for $t \geq \tau_p$. With correlation between coupled lives reflected in the paired Brownian motions, the bereavement model explains the causal relation between remaining lifetimes and the true contagion effect of the loss of a spouse. Specification of the bereavement process determines the dependence structure assumed to exist between the lives of interest. Such flexibility in the model allows for consideration of all dependence classifications, where required. Jevtić and Hurd (2017) define $r_q(t)$ as a deterministic function with dynamics given by

$$dr_q(t) = -\kappa^q r_q(t)dt \quad \text{with} \quad r_q(\tau_p) = \epsilon^q \lambda_q(\tau_p^-),$$

for values of t greater than the initial death time τ_p. In the deterministic case, the law of large numbers implies diversification of the risk associated with the loss of a spouse. In this chapter, an alternative approach to modelling the bereavement jump is proposed. The approach facilitates the existence of a non-diversifiable bereavement risk, such that the associated premiums must account for the change in mortality experienced after the first death. Coefficients δ^q and ζ^q of (5.3.3) are fixed at zero for computational simplicity; however, selection of positive values for δ^q and ζ^q would allow for the incorporation of the mortality intensities of both lives, prior to the first death, in the initial value of the bereavement effect at time τ_p.

The proposed bereavement effect is presented in Definition 5.3.1.

Definition 5.3.1. The change in mortality process $r_q(t)$ has dynamics given by an Ornstein–Uhlenbeck process, such that

$$dr_q(t) = -\kappa^q r_q(t)dt + \sigma_r^q dW(t) \quad \text{with} \quad r_q(\tau_p) = \epsilon^q \lambda_q(\tau_p^-), \tag{5.3.8}$$

where $W(t)$ is an independent d-dimensional Brownian motion, $\epsilon^q, \kappa^q, \sigma_r^q \geq 0$ and $q \in \{x, y\}$.

The explicit solution of the bereavement process for $t \geq \tau_p$ is given by

$$r_q(t) = \epsilon^q \lambda_q(\tau_p^-) e^{-\kappa^q (t-\tau_p)} + \sigma_r^q e^{-\kappa^q t} \int_{\tau_p}^{t} e^{\kappa^q s} dW(s). \tag{5.3.9}$$

The bereavement process in Definition 5.3.1 is again assumed to be of affine type to allow for computation of the joint probability density function.

Remark 5.3.2. The volatility of the bereavement process driven by the Brownian motion $dW(t)$ in (5.3.9) is a determining feature of the nature of dependence. Fixing σ_r^q across the population infers the occurrence of an event experienced individually by all bereaved spouses at some future point in time. Such an event induces a non-diversifiable risk that poses a significant threat to the insurance industry in practice, creating the need for premiums that account for the observed change in mortality. On the other hand, assumption of a couple specific σ_r^q means that the future event risk is diversifiable, through the insuring of a large and varied sample of couples.

Selection of σ_r^q as either fixed or varying should be determined through observation of data. Establishing a more detailed underwriting process would help in the identification of unobserved heterogeneities, reducing the dependence risk associated with this component of the bereavement process. Estimation of the volatilities in (5.3.6) and (5.3.8) could also be facilitated through increased data collection; however, since the occurrence of a future event common to all lives is not apparent in the data set analysed, σ_r^q is assumed to be couple specific, acknowledging the possibility that a more populated data set may support the need for a change in this assumption.

When pricing a reversionary annuity in Section 5.4, the volatility coefficient σ_r^q does not appear in the indifference price. Since the risks associated with the correlated Brownian motions are diversifiable through inclusion of only couple specific risk factors, this independence of σ_r^q is of no concern in the case considered here. In the non-diversifiable case discussed in Remark 5.3.1, however, the initial value of the bereavement effect in (5.3.8) should be redefined to incorporate both σ_r^q and the causal dependence between the members of each couple, to ensure the price of the insurance product covers the full risk associated with a spousal loss.

Remark 5.3.3. The Brownian motion of the bereavement process is assumed to be independent of the paired Brownian motions associated with mortality intensity. If the Brownian motion in (5.3.8) is instead given by $W_q(t)$, the change in mortality of the surviving spouse upon the death of their partner is assumed to be correlated with their mortality before the primary death. The independence assumption adopted in this chapter increases the significance of random risks, such as environmental factors, in determining the impact of the loss, rather than historical mortality. The existence of dependence between the bereavement process and the mortality of the surviving spouse before the death at time τ_p, although an interesting concept, is not considered in this chapter.

In the proposed Ornstein-Uhleneck model of bereavement, the mean reversion parameter is fixed at zero. This aligns with the decreasing significance of the mortality gap over time, due to the subsiding nature of mortality elevation characteristic of broken-heart syndrome. In contrast to the exponential model of bereavement assumed in Jevtić and Hurd (2017), mean reversion allows for the process to take both positive and negative values, accounting for instances when the mortality of the bereaved improves in comparison to a non-widowed mortality.

Having established the structure of the bereavement effect, expectations required for the joint probability density and survival function calculations can be computed through application of the affine framework with term structure equation determined by the Feynman–Kac formula. In the affine setting, the conditional formula whose aforementioned specific form (5.3.5) is used in the calculation of bond prices (see, for example, Grasselli and Hurd (2015)), is given by

$$\mathbb{E}[e^{-c_1 \int_t^T \lambda(u)du - c_2\lambda(T)} \mid \mathcal{G}_t] = e^{A(T-t;\theta,c_1,c_2)+B(T-t;\theta,c_1,c_2)\lambda(t)}, \tag{5.3.10}$$

where c_1 and c_2 are constant and $\theta_p = (\lambda_p(0), \mu^p, \sigma^p)$ for $p \in \{x, y\}$.

In order to derive the explicit forms of $A(T-t;\theta,c_1,c_2)$ and $B(T-t;\theta,c_1,c_2)$, the Feynman-Kac formula is first defined:

Definition 5.3.2. The existence of a function $f(t, x)$ for $t \in [0, T]$, where $T > 0$, $x \in \mathbb{R}$ and

$$f(t, X_t) = \mathbb{E}[F(X_T)e^{\int_t^T \Phi(s,X_s)ds} + \int_t^T e^{\int_t^u \Phi(s,X_s)ds} G(u, X_u)du \mid \mathcal{F}_t]$$

for any functions $F(x), G(t, x)$ and $\Phi(t, x)$ that are sufficiently integrable is implied by the Markov property.

The Feynman–Kac formula states that the solution of the non-homogeneous parabolic partial differential equation

$$\begin{cases} \partial_t f(t,x) + \mathcal{L}[f](t,x) + \Phi(t,x)f(t,x) + G(t,x) = 0, & t < T \\ f(T,x) = F(x) \end{cases}$$

is given by the function f.

Proposition 5.3.1. Application of the Feynman–Kac formula to (5.3.10) for a non-mean-reverting Cox–Ingersoll–Ross (CIR) process gives

$$A(T-t;\theta,c_1,c_2) = 0$$

and

$$B(T-t;\theta,c_1,c_2) = \frac{-4c_1^2(1-e^{\gamma(t-T)}) - 2c_1c_2(\gamma-\mu)(1+e^{\gamma(t-T)}) - 4c_1c_2\mu}{(\gamma-\mu)(2c_1+c_2(\gamma+\mu)) + (\gamma+\mu)(2c_1-c_2(\gamma-\mu))e^{\gamma(t-T)}}, \tag{5.3.12}$$

where $\gamma = \sqrt{\mu^2 + 2c_1\sigma^2}$.

The derivative of (5.3.10) with respect to c_2 is then

$$\begin{aligned}\mathbb{E}[\lambda(T)e^{-c_1\int_t^T \lambda(u)du - c_2\lambda(T)} \mid \mathcal{G}_t] = &-[\tilde{A}(T-t;\theta,c_1,c_2) + \tilde{B}(T-t;\theta,c_1,c_2)\lambda(t)] \\ &\times e^{A(T-t;\theta,c_1,c_2)+B(T-t;\theta,c_1,c_2)\lambda(t)},\end{aligned}$$

where $\tilde{A}(t)$ and $\tilde{B}(t)$ are given by $\frac{\partial A(t)}{\partial c_2}$ and $\frac{\partial B(t)}{\partial c_2}$, respectively, such that

$$\tilde{A}(T-t;\theta,c_1,c_2) = 0$$

and

$$\tilde{B}(T-t;\theta,c_1,c_2) = -\frac{16c_1^2\gamma^2 e^{\gamma(t-T)}}{(\gamma-\mu)(2c_1+c_2(\gamma+\mu)) + (\gamma+\mu)(2c_1-c_2(\gamma-\mu))e^{\gamma(t-T)}}. \tag{5.3.13}$$

Proof. By (5.3.10) and in the CIR case of interest, the general form of the function $f(t,x)$ in Definition 5.3.2 is

$$f(t,\lambda(t)) = \mathbb{E}[e^{-c_1\int_t^T \lambda(u)du - c_2\lambda(T)} \mid \mathcal{G}_t],$$

where c_1 and c_2 are real-valued constants. It therefore holds that $F(\lambda(T)) = \exp[-c_2\lambda(T)]$, $\Phi(t,\lambda(t)) = -c_1\lambda(t)$ and $G(t,\lambda(t)) = 0$, such that

$$\partial_t f(t,\lambda(t)) + \mu\lambda(t)\partial_{\lambda(t)} f(t,\lambda(t)) + \frac{1}{2}\sigma^2\lambda(t)\partial_{\lambda(t)\lambda(t)} f(t,\lambda(t)) - c_1\lambda(t)f(t,\lambda(t)) = 0, \tag{5.3.14}$$

where $f(t,\lambda(t)) = \exp[A(T-t;\theta,c_1,c_2) + B(T-t;\theta,c_1,c_2)\lambda(t)]$. Substitution of $f(t,\lambda(t))$ into the partial differential equation (5.3.14) gives

$$A'(T-t;\theta,c_1,c_2) = 0,$$

and

$$B'(T-t;\theta,c_1,c_2) = -\mu B(T-t;\theta,c_1,c_2) - \frac{1}{2}\sigma^2 B^2(T-t;\theta,c_1,c_2) + c_1,$$

where $A(T-t;\theta,c_1,c_2)$ and $B(T-t;\theta,c_1,c_2)$ have boundary condition $A(T-t;\theta,c_1,c_2) = B(T-t;\theta,c_1,c_2) = 0$ at $t = T$.

Solution of the first order ODE for $B(T-t;\theta,c_1,c_2)$ involves a calculation of significant length; however, following a number of algebraic steps,

$$B(T-t;\theta,c_1,c_2) = \frac{-4c_1^2(1-e^{\gamma(t-T)}) - 2c_1c_2(\gamma-\mu)(1+e^{\gamma(t-T)}) - 4c_1c_2\mu}{(\gamma-\mu)(2c_1+c_2(\gamma+\mu)) + (\gamma+\mu)e^{\gamma(t-T)}(2c_1-c_2(\gamma-\mu))},$$

is derived, under application of the initial conditions of the Feynman–Kac formula: $f(T,\lambda(t)) = F(\lambda(t))$. Then,

$$\tilde{B}(T-t;\theta,c_1,c_2) = \frac{\partial B(T-t;\theta,c_1,c_2)}{\partial c_2}$$

simplifies to

$$-\frac{16c_1^2\gamma^2 e^{\gamma(t-T)}}{(\gamma-\mu)(2c_1+c_2(\gamma+\mu)) + (\gamma+\mu)e^{\gamma(t-T)}(2c_1-c_2(\gamma-\mu))}.$$

Since $A(t) = 0$, it also holds that $\tilde{A}(t) = 0$. □

Three corollaries follow Proposition 5.3.1. The explicit form of the expectations of interest under conditions appropriate in the mortality context are given in Corollary 5.3.1, while Corollaries 5.3.2 and 5.3.3 provide expressions for the joint probability density and survival functions, respectively.

Corollary 5.3.1. The specific form of the conditional formula required for calculation of both the probability density and survival functions occurs when constants c_1 and c_2 are fixed at 1 and 0, respectively. This gives

$$\mathbb{E}[e^{-\int_t^T \lambda(u)du} \mid \mathcal{G}_t] = e^{B(T-t;\theta,1,0)\lambda(t)}$$

and

$$\mathbb{E}[\lambda(T)e^{-\int_t^T \lambda(u)du} \mid \mathcal{G}_t] = -\tilde{B}(T-t;\theta,1,0)\lambda(t) \times e^{B(T-t;\theta,1,0)\lambda(t)},$$

where

$$B(T-t;\theta,1,0) = -\frac{2(1-e^{\gamma(t-T)})}{(\gamma-\mu)+(\gamma+\mu)e^{\gamma(t-T)}} \tag{5.3.15}$$

and

$$\tilde{B}(T-t;\theta,1,0) = -\frac{8\gamma^2 e^{\gamma(t-T)}}{(\gamma-\mu)+(\gamma+\mu)e^{\gamma(t-T)}}.$$

Throughout the remainder of the chapter, for ease of presentation, functions $A(T-t;\theta,1,0)$ and $B(T-t;\theta,1,0)$ will be denoted $A(T-t;\theta)$ and $B(T-t;\theta)$, respectively.

Corollary 5.3.2. The joint probability density function of the remaining lifetimes τ_x and τ_y of individuals x and y with bereavement process of Ornstein–Uhlenbeck type is given by the expression

$$\rho(t_x,t_y) =[\tilde{A}^r(t_y-t_x;\theta_q^r)+\tilde{B}^r(t_y-t_x;\theta_q^r)r_q(t_x)+\tilde{B}(t_y-t_x;\theta_q)\lambda_q(t_x)]\tilde{B}(t_x;\theta_x)\lambda_x(0)$$
$$\times e^{A^r(t_y-t_x;\theta_q^r)+B^r(t_y-t_x;\theta_q^r)r_q(t_x)+B(t_y-t_x;\theta_q)\lambda_q(t_x)+B(t_x;\theta_y)\lambda_y(0)+B(t_x;\theta_x)\lambda_x(0)},$$

for $t_x < t_y$, where $\theta_q^r = (r_q(0), \kappa^q, \sigma_r^q)$, $B(t)$ and $\tilde{B}(t)$ are as defined in Corollary 5.3.1, and $A^r(t)$ and $B^r(t)$ are unique functions satisfying the generalised Riccati ordinary differential equations for the Ornstein–Uhlenbeck bereavement process $r_q(t)$, such that

$$A^r(T-t;\theta) = \frac{\sigma^2}{2\kappa^2}((T-t)+\frac{2}{\kappa}(e^{-\kappa(T-t)}-1)-\frac{1}{2\kappa}(e^{-2\kappa(T-t)}-1)) \tag{5.3.16}$$

and

$$B^r(T-t;\theta) = \frac{1}{\kappa}(e^{-\kappa(T-t)}-1).$$

Evaluating the derivatives of $A^r(T-t;\theta,c_1,c_2)$ and $B^r(T-t;\theta,c_1,c_2)$ with respect to c_2 in line with Proposition 5.3.1 and Corollary 5.3.2, gives

$$\tilde{A}^r(T-t;\theta) = -\frac{\sigma^2}{2\kappa^2}(2(e^{-\kappa(T-t)}-1)-(e^{-2\kappa(T-t)}-1))$$

and

$$\tilde{B}^r(T-t;\theta) = -e^{-\kappa(T-t)}.$$

Proof. Consider the case $t_x < t_y$, then by Theorem 5.3.1,

$$\begin{aligned}\rho(t_x,t_y) &= \mathbb{E}[\lambda_p(t_x)e^{-\int_0^{t_x}\lambda_x(u)+\lambda_y(u)du}\mathbb{E}[\tilde{\lambda}_q(t_y)e^{-\int_{t_x}^{t_y}\tilde{\lambda}_q(u)du} \mid \mathcal{G}_T]] \\ &= \mathbb{E}[\lambda_p(t_x)e^{-\int_0^{t_x}\lambda_x(u)du}]\times\mathbb{E}[e^{-\int_0^{t_x}\lambda_y(u)du}\mathbb{E}[\tilde{\lambda}_q(t_y)e^{-\int_{t_x}^{t_y}\tilde{\lambda}_q(u)du} \mid \mathcal{G}_T]]\end{aligned}$$

holds since the mortality intensities λ_x and λ_y are independent under $t_x < t_y$ when conditioning on the information set $\mathcal{G}_T$. Proposition 5.3.1 implies that the first component of the joint probability density function is given by

$$\mathbb{E}[\lambda_p(t_x)e^{-\int_0^{t_x}\lambda_x(u)du}] = -\tilde{B}(t_x;\theta_x,1,0)\lambda_x(0)\times e^{B(t_x;\theta_x,1,0)\lambda_x(0)},$$

where $B(t)$ and $\tilde{B}(t)$ are as defined in (5.3.12) and (5.3.13), respectively. For the second component of the joint probability density function, consider

$$\begin{aligned}\tilde{\lambda}_q(u) &= \lambda_q(u)+\epsilon^q\lambda_q(t_x)e^{-\kappa^q(u-t_x)}+\sigma_r^q e^{-\kappa^q u}\int_{t_x}^{u}e^{\kappa^q s}dW(s) \\ &= \lambda_q(u)+r_q(u),\end{aligned}$$

where $u \geq t_x$, then

$$\begin{aligned}\mathbb{E}[\tilde{\lambda}_q(t_y)e^{-\int_{t_x}^{t_y}\tilde{\lambda}_q(u)du} \mid \mathcal{G}_T] &= \mathbb{E}[(\lambda_q(t_y)+r_q(t_y))e^{-\int_{t_x}^{t_y}\lambda_q(u)du}e^{-\int_{t_x}^{t_y}r_q(u)} \mid \mathcal{G}_T] \\ &= \mathbb{E}[\lambda_q(t_y)e^{-\int_{t_x}^{t_y}\lambda_q(u)du} \mid \mathcal{G}_T]\mathbb{E}[e^{-\int_{t_x}^{t_y}r_q(u)du} \mid \mathcal{G}_T] \\ &\quad +\mathbb{E}[r_q(t_y)e^{-\int_{t_x}^{t_y}r_q(u)du} \mid \mathcal{G}_T]\mathbb{E}[e^{-\int_{t_x}^{t_y}\lambda_q(u)du} \mid \mathcal{G}_T]\end{aligned} \tag{5.3.17}$$

by the independence of λ_q and r_q, given their independent Brownian motions. Since the bereavement process is assumed to be of affine type, functions $A^r(t)$ and $B^r(t)$ satisfying generalised Riccati ordinary differential equations can be obtained, such that a closed form solution of the required conditional expectation and its derivative with respect to c_2 is given as follows for $\theta = (r(0), \kappa, \sigma)$:

$$\mathbb{E}[e^{-c_1 \int_t^T r(u)du - c_2 r(T)} \mid \mathcal{G}_t] = e^{A^r(T-t;\theta,c_1,c_2)+B^r(T-t;\theta,c_1,c_2)r(t)},$$

where

$$B^r(T-t;\theta,c_1,c_2) = \frac{1}{\kappa}((c_1-\kappa c_2)e^{-\kappa(T-t)} - c_1)$$

and

$$A^r(T-t;\theta,c_1,c_2) = \frac{\sigma^2}{2\kappa^2}[c_1^2(T-t) + \frac{2c_1}{\kappa}(c_1-\kappa c_2)(e^{-\kappa(T-t)}-1) - \frac{1}{2\kappa}(c_1-\kappa c_2)^2(e^{-2\kappa(T-t)}-1)]$$

(see Jevtić and Hurd (2017)). Application of the affine framework therefore allows for (5.3.17) to be expressed as

$$-[\tilde{A}^r(t_y-t_x;\theta_q^r,1,0) + \tilde{B}^r(t_y-t_x;\theta_q^r,1,0)r_q(t_x) + \tilde{B}(t_y-t_x;\theta_q,1,0)\lambda_q(t_x)] \\ \times e^{A^r(t_y-t_x;\theta_q^r,1,0)+B^r(t_y-t_x;\theta_q^r,1,0)r_q(t_x)+B(t_y-t_x;\theta_q,1,0)\lambda_q(t_x)},$$

where $B(t)$ and $\tilde{B}(t)$ are defined in Corollary 5.3.1. As in the CIR case of Proposition 5.3.1, functions $\tilde{A}^r(t)$ and $\tilde{B}^r(t)$ are given by $\frac{\partial A^r(t)}{\partial c_2}$ and $\frac{\partial B^r(t)}{\partial c_2}$, respectively. The second component of the joint probability density function is then

$$\begin{aligned}
&\mathbb{E}[e^{-\int_0^{t_x}\lambda_y(u)du} \times -[\tilde{A}^r(t_y-t_x;\theta_q^r,1,0) + \tilde{B}^r(t_y-t_x;\theta_q^r,1,0)r_q(t_x) + \tilde{B}(t_y-t_x;\theta_q,1,0)\lambda_q(t_x)] \\
&\times e^{A^r(t_y-t_x;\theta_q^r,1,0)+B^r(t_y-t_x;\theta_q^r,1,0)r_q(t_x)+B(t_y-t_x;\theta_q,1,0)\lambda_q(t_x)}] \\
=& -[\tilde{A}^r(t_y-t_x;\theta_q^r,1,0) + \tilde{B}^r(t_y-t_x;\theta_q^r,1,0)r_q(t_x) + \tilde{B}(t_y-t_x;\theta_q,1,0)\lambda_q(t_x)] \\
&\times e^{A^r(t_y-t_x;\theta_q^r,1,0)+B^r(t_y-t_x;\theta_q^r,1,0)r_q(t_x)+B(t_y-t_x;\theta_q,1,0)\lambda_q(t_x)+B(t_x;\theta_y,1,0)\lambda_y(0)},
\end{aligned}$$

where $\theta_q^r = (r_q(0), \kappa^q, \sigma_r^q)$ and thus, it holds that the joint probability density function of remaining lifetimes t_x and t_y is given by (5.3.16), as required. □

Remark 5.3.4. The joint probability density function for $t_x > t_y$ is analogous to Corollary 5.3.2, with indices $i \in \{x, y\}$ interchanged.

Under the assumption of independent coupled lives, the convenience of the affine environment allows for the survival probability of an individual aged x to be given by

$$S_x(t) = \mathbb{P}(\tau_x > t \mid \mathcal{G}_0) = \mathbb{E}[e^{-\int_0^t \lambda_x(u)du} \mid \mathcal{G}_0] = e^{A(t)+B(t)\lambda_x(0)}. \tag{5.3.18}$$

Due to the changing mortality of the bereaved spouse upon the initial death at time τ_p, consideration of dependent lives requires the redefinition of the survival function in (5.3.18). For $t \geq \tau_p$, the survival function is instead regarded as the product of two survival functions, split at the first jump time τ_p such that

$$S_x(t) = S_x(\tau_p)S_{x+\tau_p}(t-\tau_p).$$

In accordance with the affine process selection for mortality intensity, the expression

$$S_x(t) = S_x(\tau_p) \times \mathbb{E}[e^{-\int_{\tau_p}^{t} \tilde{\lambda}_{x+\tau_p}(u)du} \mid \mathcal{G}_{\tau_p}]$$

therefore holds for $t \geq \tau_p$, leading to Corollary 5.3.3.

Corollary 5.3.3. The survival probability of an individual (x) assuming a mortality intensity of non-mean reverting Cox–Ingersoll–Ross type is given by

$$S_x(t) = \begin{cases} e^{B(t)\lambda_x(0)}, & t < \tau_p \\ e^{B(\tau_p)\lambda_x(0)} \times e^{B(t-\tau_p)\tilde{\lambda}_{x+\tau_p}(\tau_p)}, & t \geq \tau_p, \end{cases}$$

where $B(t)$ is defined as in (5.3.15).

Note that τ_p is unknown, the survival probability is therefore presented as in Corollary 5.3.3 purely to highlight the change in the survival probability of the surviving spouse at the first death time. In reality, it is not possible to know, prior to the first death, whether an arbitrary survival time $t > 0$ lies above or below τ_p. In addition, mortality information $\mathcal{G}_{\tau_p}$ can only be observed after the first death.

If the first death has already occurred and the identity of the survivor is known, the survival probability of the surviving spouse, here $(x + \tau_p)$, could instead be redefined as

$$S_{x+\tau_p}(t) = e^{B(t)\tilde{\lambda}_{x+\tau_p}(\tau_p)},$$

for $t > 0$.

Remark 5.3.5. The survival probability of spouse (y) is analogous to the survival probability of spouse (x), with indices $i \in \{x, y\}$ interchanged.

One example of incorporating the proposed model in the pricing of a joint-life insurance contract using the indifference pricing principle is presented in the following section.

5.4 Indifference price calculation for a joint-life insurance product

When pricing in the incomplete financial market setting, the utility indifference principle is an approach introduced by Hodges and Neuberger (1989) which compares the maximal expected utilities of an investor with and without taking a given risk. Initially implemented in the pricing of European options and motivated by the unrealistic assumption of no transaction costs in the pure Black–Scholes model, extensions of the method have since been developed by Davis et al. (1993) and Ludkovski and Young (2008) among others, with the latter considering mortality contingent claims in a fully stochastic setting. In relation to life insurance, such an approach involves equating the expected utility of an insurer when a certain number of insurance policies are written to the expected utility when the same policies are not written. The indifference premium of interest to this research is therefore the change in premium that should be charged when dependence of coupled lives is assumed.

Various articles detail application of the indifference principle to pricing in the life insurance sector. Young and Zariphopoulou (2002) implement the approach in the valuation of insurance risks in the dynamic financial market setting. Extensions of their results are presented by Delong (2009) and Liang and Lu (2017) who follow similar procedures in order to determine indifference premiums. Liang and Lu (2017) apply a jump-diffusion model of Black–Scholes type to model the stochastic price of a risky asset with jumps given by a shot-noise process, while Delong (2009) makes use of a Lévy process in order to drive price dynamics. In both cases, the mortality intensity is assumed to be a stochastic process of diffusion type. Further distinction between the two papers appears in the definition of the policyholder benefits, with Delong (2009) defining benefits as fixed rates and Liang and Lu (2017) proposing the indifference premium for an equity-linked life insurance contract with benefits dependent on the value of the underlying asset.

Choi (2016) adapts further work by Young (2003) in line with Liang and Lu (2017) through implementation of the equivalent utility principle for valuation of equity-linked life insurance to obtain the indifference price of an insurance contract in both the deterministic and stochastic mortality cases. Solution of a stochastic optimisation problem determined by solving the associated Hamilton–Jacobi–Bellman equation enables calculation of the indifference premium in each of Delong (2009), Choi (2016) and Liang and Lu (2017). Explicit optimal solutions are then obtained under the assumption of an exponential utility function.

Blanchet-Scalliet et al. (2019) consider the indifference principle in pricing life insurance portfolios under the assumption of contingent lives, with dependence introduced through correlation of policyholders' lifetimes with a Farlie–Gumbel–Morgenstern copula. Medical breakthroughs and environmental features are suggested factors associated with dependence structures between the lifetimes of individuals within a population. When restricting the model by Blanchet-Scalliet et al. (2019) to consider just two policyholders, the surviving policyholder is said to experience a jump in mortality intensity when the other dies, in line with the assumption of the model proposed in Section 5.3.

An example of the pricing of a life insurance product under the assumption of dependence between the two coupled lives involved in the contract is now provided, implementing the indifference principle to obtain the result. To illustrate how dependence between coupled lives influences the pricing and valuation of insurance products involving mortality assumptions, consider a reversionary annuity which insures the life of an individual (x). The annuity pays a value of 1 to individual (y) at the end of each year with the initial payment due at the end of the year of (x)'s death, where the beneficiary (y) is the surviving spouse of (x). The contract terminates on the final payment at the end of the year preceding (y)'s death. If the individual (y) dies before (x), the contract terminates before any payment is made.

In order to compute the price of such an annuity, the classical model by Merton (1969) is first introduced. This model optimises the investment strategies of an individual seeking to maximise their expected utility of terminal wealth given some value of initial wealth. The insurer may trade between a risky asset and a risk-free asset. A geometric Brownian motion is used to model the price of the risky asset, such that

$$\begin{cases} dS_s = \mu S_s ds + \sigma\ S_s dB_s, \\ S_t = S > 0, \end{cases}$$

for some $s > t$, where t is fixed and S_s gives the price of the risky asset at time s. The mean rate of return μ and volatility σ are positive constants and the process B_s is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with probability measure $\mathbb{P}$ and filtration $\mathcal{F}$ containing information about the financial market. The price of the risk-free asset R_s with rate of return r at time $s > t$ is modelled such that

$$dR_s = rR_s ds,$$

where it is assumed that $\mu > r > 0$.

Suppose the insurer trades dynamically between the risky asset and the risk-free asset given initial wealth $w \geq 0$ at time $t > 0$. Defining W_s as the wealth of the insurer for $s \in [t, T]$, where $T > 0$ is the terminal time, the insurer invests π_s^{rf} in the risk-free asset and π_s in the risky asset at time s, such that $W_s = \pi_s^{rf} + \pi_s$. The wealth process then satisfies the dynamics

$$\begin{cases} dW_s = (rW_s + (\mu - r)\pi_s)ds + \sigma\pi_s dB_s, & t \leq s \leq T \\ W_t = w. \end{cases}$$

Under the assumption of an absence of any additional insurance risk, the investor wishes to maximise their expected utility of terminal wealth such that the value function $V(w, t)$ satisfies

$$V(w,t) = \sup_{\pi_t \in \mathcal{A}} \mathbb{E}[u(W_T) \mid W_t = w],$$

where $\mathcal{A}$ is the set of admissible policies and $u : \mathbb{R} \to \mathbb{R}$ is the utility function assumed to be increasing, concave and smooth. The value function without insurance risk has been shown by Björk (2009) to satisfy the Hamilton–Jacobi–Bellman (HJB) equation

$$\begin{cases} V_t(w,t) + \max_{\pi_t}[(\mu - r)\pi_t V_w(w,t) + \frac{1}{2}\sigma^2\pi_t^2 V_{ww}(w,t)] + rwV_w(w,t) = 0, \\ V(w,T) = u(w), \end{cases} \tag{5.4.3a}$$

which has optimal investment process given by

$$\pi^*(w,t) = -\frac{(\mu - r)}{\sigma^2}\frac{V_w(w,t)}{V_{ww}(w,t)}.$$

The maximum of the HJB equation exists due to the linearity of the wealth process dynamics with respect to the wealth and portfolio process and the concavity of the utility function u, which is inherited by the value function.

Assumption of an exponential utility function reduces technical difficulties associated with general utility functions and so enables determination of the indifference price. Considering an exponential utility function of the form $u(w) = -\frac{1}{a}\exp[-aw]$, where $w \in \mathbb{R}$ and $a > 0$ is the coefficient of risk aversion, substitution into the HJB (5.4.3a) gives the following closed form solution:

$$V(w,t) = -\frac{1}{a}\exp\left[-awe^{r(T-t)} - \frac{(\mu - r)^2}{2\sigma^2}(T - t)\right].$$

Now, suppose the insurer has the opportunity to insure an individual aged x. If the insured individual $(x+t)$ dies in the interval $[t, t+h]$, the insurer pays the expected present value (EPV) of the reversionary annuity to the surviving spouse $(y+t)$ at the end of the year of the primary death at time $T = \tau_p$, where τ_p is the first death time within the couple as defined in the model proposed in Section 5.3. The expected present value of the annuity is given by

$$\text{EPV} = \sum_{s=0}^{\tau_q - \tau_p - 1}\left({}_s p_{y+\tau_p} q_{y+\tau_p+s}\sum_{i=1}^{s+1} e^{-r((s+1)-i)}\right),$$

where τ_q is the death time of the surviving spouse under the assumption of dependent coupled lives, ${}_tp_y$ denotes the survival function $S_y(t)$ and $1 - {}_tp_y = {}_tq_y$ the distribution function $F_y(t)$ of τ_y (or equivalently in this case, τ_q). The expectation of the expected present value given τ_x represents the remaining lifetime of (x) is then

$$\mathbb{E}[\text{EPV}] = \Big(\sum_{s=0}^{\tau_q-\tau_p-1} {}_sp_{y+\tau_p} q_{y+\tau_p+s} \sum_{i=1}^{s+1} e^{-r((s+1)-i)} \Big) \times \mathbb{P}(\tau_x = \tau_p),$$

since the insurance contract terminates if the beneficiary dies before the insured. Note that, in the following calculations, τ_x and τ_y could be used interchangeably with τ_p and τ_q, respectively. However, for clarity, use of τ_p and τ_q remains. Using the survival functions derived in Section 5.3, the expectation of the expected present value of the reversionary annuity can be expressed such that

$$\mathbb{E}[\text{EPV}] = \sum_{s=0}^{\tau_q-\tau_p-1} \Big(\Big(e^{B(s;\theta_y)\tilde{\lambda}_{y+\tau_p}(\tau_p)} - e^{B(s+1;\theta_y)\tilde{\lambda}_{y+\tau_p}(\tau_p)} \Big) \sum_{i=1}^{s+1} e^{-r((s+1)-i)} \Big) \times \mathbb{P}(\tau_x = \tau_p), \tag{5.4.4}$$

where by (5.3.2)

$$\mathbb{P}(\tau_x = \tau_p) = \mathbb{P}\Big(U \leq \frac{\lambda_x(\tau_p)}{\lambda_x(\tau_p) + \lambda_y(\tau_p)} \Big) = \frac{\lambda_x(\tau_p)}{\lambda_x(\tau_p) + \lambda_y(\tau_p)}.$$

The value charged at time t to cover this payout is

$$\mathbb{E}[\text{EPV}] \times e^{-r(\tau_p - t)}.$$

Since the insurance contract remains standing if the individual $(x+t)$ survives until time $t+h$ and continues under the value function without the claim if $(x+t)$ dies between time t and $t+h$, the insurer's optimisation problem is defined by

$$\begin{aligned} U(w,t) \geq & \mathbb{E}[V(W^*_{t+h} - e^{-r(\tau_p-(t+h))}\mathbb{E}[\text{EPV}], t+h) \mid W_t = w]_h q_{x+t} \\ & + \mathbb{E}[U(W^*_{t+h}, t+h) \mid W_t = w]_h p_{x+t}, \end{aligned} \tag{5.4.5}$$

where W^*_s is the wealth of the insurer under the optimal strategy π^*_s for $t \leq s \leq t+h$.

Proposition 5.4.1. Under assumption of the appropriate conditions of regularity and integrability on the value functions discussed by Björk (2009), for an insurer following optimal investment strategy π^*_s for $t \leq s \leq t+h$, the HJB equation corresponding to the optimisation in (5.4.5) is given

$$\begin{cases} U_t(w,t) + rwU_w(w,t) + [V(w - e^{-r(\tau_p-(t+h))}\mathbb{E}[\text{EPV}], t) - U(w,t)]\lambda_x(t) \\ \quad + \max_{\pi_t}\{(\mu - r)\pi_t + \frac{1}{2}\sigma^2\pi_t^2 U_{ww}(w,t)\} = 0, \\ U(w,T) = u(w). \end{cases} \tag{5.4.6a}$$

Proof. Under the assumption that functions $U(w,t)$ and $V(w,t)$ are sufficiently smooth

$$U(W_{t+h}, t+h) = U(W_t, t) + \int_t^{t+h} dU,$$

where by Itô's formula,

$$dU = (U_s + U_w(rW_s + (\mu - r)\pi_s) + \frac{1}{2}\sigma^2\pi_s^2 U_{ww})ds + \sigma\pi_s U_w dB,$$

Then,

$$\begin{aligned} U(W^*_{t+h}, t+h) =& U(W_t, t) + \int_t^{t+h} U_t(W^*_s, s) + U_w(W^*_s, s)(rW^*_s + (\mu - r)\pi^*_s) \\ &+ \frac{1}{2}\sigma^2 \pi_s^{*^2} U_{ww}(W^*_s, s)ds + \int_t^{t+h} \sigma\pi_s U_w(W^*_s, s)dB \end{aligned}$$

and so

$$\begin{aligned} \mathbb{E}[U(W^*_{t+h}, t+h) \mid W_t = w] =& \mathbb{E}[U(w, t) + \int_t^{t+h} U_t(W^*_s, s) + U_w(W^*_s, s)(rW^*_s + (\mu - r)\pi^*_s) \\ &+ \frac{1}{2}\sigma^2 \pi_s^{*^2} U_{ww}(W^*_s, s)ds \mid W_t = w]. \end{aligned}$$

Carrying out similar computations for $\mathbb{E}[V(W^*_{t+h} - e^{-r(\tau_p - (t+h))}\mathbb{E}[\text{EPV}], t+h) \mid W_t = w]$ gives the following expression for the insurer's optimisation problem:

$$\begin{aligned} U(w, t) \geq& \mathbb{E}[U(w, t) + \int_t^{t+h} U_t(W^*_s, s) + U_w(W^*_s, s)(rW^*_s + (\mu - r)\pi^*_s) \\ &+ \frac{1}{2}\sigma^2 \pi_s^{*^2} U_{ww}(W^*_s, s)ds \mid W_t = w] \;{}_h p_{x+t} \\ &+ \mathbb{E}[V(w - e^{-r(\tau_p - (t+h))}\mathbb{E}[\text{EPV}], t) + \int_t^{t+h} V_t(W^*_s - e^{-r(\tau_p - (s+h))}\mathbb{E}[\text{EPV}], s) \\ &+ V_w(W^*_s - e^{-r(\tau_p - (s+h))}\mathbb{E}[\text{EPV}], s)(rW^*_s + (\mu - r)\pi^*_s) \\ &+ \frac{1}{2}\sigma^2 \pi_s^{*^2} V_{ww}(W^*_s - e^{-r(\tau_p - (s+h))}\mathbb{E}[\text{EPV}], s)ds \mid W_t = w] \;{}_h q_{x+t}, \end{aligned}$$

such that

$$\begin{aligned} U(w, t)\frac{{}_h q_{x+t}}{h} \geq& V(w - e^{-r(\tau_p - (t+h))}\mathbb{E}[\text{EPV}], t)\frac{{}_h q_{x+t}}{h} \\ &+ \frac{1}{h}\mathbb{E}[\int_t^{t+h} U_t(W^*_s, s) + U_w(W^*_s, s)(rW^*_s + (\mu - r)\pi^*_s) \\ &+ \frac{1}{2}\sigma^2 \pi_s^2 U_{ww}(W^*_s, s)ds \mid W_t = w] \;{}_h p_{x+t} \\ &+ \frac{1}{h}\mathbb{E}[\int_t^{t+h} V_t(W^*_s - e^{-r(\tau_p - (s+h))}\mathbb{E}[\text{EPV}], s) \\ &+ V_w(W^*_s - e^{-r(\tau_p - (s+h))}\mathbb{E}[\text{EPV}], s)(rW^*_s + (\mu - r)\pi^*_s) \\ &+ \frac{1}{2}\sigma^2 \pi_s^{*^2} V_{ww}(W^*_s - e^{-r(\tau_p - (s+h))}\mathbb{E}[\text{EPV}], s)ds \mid W_t = w] \;{}_h q_{x+t}, \end{aligned}$$

when subtracting $U(w, t)_h p_{x+t}$ and dividing by h. Taking the limit as $h \to 0$ then yields

$$\begin{aligned} U(w, t)\lambda_x(t) \geq& V(w - e^{-r(\tau_p - t)}\mathbb{E}[\text{EPV}], t)\lambda_x(t) \\ &+ U_t(w, t) + U_w(w, t)(rw + (\mu - r)\pi_t) + \frac{1}{2}\sigma^2\pi_t^2 U_{ww}(w, t), \end{aligned}$$

since $\lim_{h\to 0} \frac{{}_h q_{x+t}}{h} = \lambda_x(t)$, where $\lambda_x(t)$ is the force of mortality of an individual aged x at time t, as previously defined, and $\lim_{h\to 0} {}_h q_{x+t} = 0$. Then, since the investment is optimal only if there exists an equality, (5.4.6a) is obtained as required. □

Proposition 5.4.2. The indifference price $P(w,t)$ that should be charged by an insurer with coefficient of risk aversion $a > 0$ and rate of return on investment $r > 0$, to insure an individual $(x+t)$ at time t under a reversionary annuity scheme that pays out until the death of the dependent spouse $(y+t)$, is given by

$$P(w,t) = \frac{1}{a} e^{-r(\tau_p - t)} \ln\Big(e^{a\mathbb{E}[\mathrm{EPV}]}(1 - e^{-\int_t^{\tau_p} \lambda_x(s)ds}) + e^{-\int_t^{\tau_p} \lambda_x(s)ds} \Big), \tag{5.4.7}$$

where τ_p is the remaining lifetime of the insured and $\mathbb{E}[\mathrm{EPV}]$ is defined as in (5.4.4).

Proof. Assume exponential utility of the form $u(w) = -\frac{1}{a}\exp[-aw]$ for $a > 0$ and consider the solution of the HJB equation to be of the form $U(w,t) = V(w,t)\phi(t)$, as in Young and Zariphopoulou (2002), where $\phi(\tau_p) = 1$. Then, by substitution into (5.4.6a),

$$\begin{aligned} V_t(w,t)\phi(t) + V(w,t)\phi_t(t) + rwV_w(w,t)\phi(t) - \frac{(\mu - r)^2}{2\sigma^2} \frac{V_w^2(w,t)}{V_{ww}(w,t)} \\ + [V(w - e^{-r(\tau_p - t)}\mathbb{E}[\mathrm{EPV}], t) - V(w,t)\phi(t)]\lambda_x(t) = 0, \end{aligned}$$

which reduces to

$$V(w,t)\phi_t(t) + [V(w - e^{-r(\tau_p - t)}\mathbb{E}[\mathrm{EPV}], t) - V(w,t)\phi(t)]\lambda_x(t) = 0,$$

as $V(w,t)$ satisfies the HJB equation for the value function under no additional insurance risk given by (5.4.3a). Since it is possible to show that

$$V(w - e^{-r(\tau_p - t)}\mathbb{E}[\mathrm{EPV}], t) = V(w,t) \times e^{a\mathbb{E}[\mathrm{EPV}]},$$

further simplification yields a first order ODE with respect to $\phi(t)$, which can be solved explicitly under application of the boundary condition $\phi(\tau_p) = 1$ such that

$$\phi(t) = e^{a\mathbb{E}[\mathrm{EPV}]}(1 - e^{-\int_t^{\tau_p} \lambda_x(s)ds}) + e^{-\int_t^{\tau_p} \lambda_x(s)ds},$$

where

$$e^{-\int_t^{\tau_p} \lambda_x(s)ds} = S_{x+t}(\tau_p - t) = \frac{S_x(\tau_p)}{S_x(t)}$$

and $S_{x+t}(\tau_p - t)$ denotes the probability of the insured individual $(x+t)$ surviving $\tau_p - t$ more years.

By the indifference principle, the minimum premium the insurer should charge in order to insure the individual $(x+t)$ at time t, for a reversionary annuity which pays in arrears from the moment of death of $(x+t)$ until the death of $(y+t)$, is the indifference price $P(w,t)$ which satisfies

$$V(w,t) = U(w + P(w,t), t),$$

where $U(w + P(w,t), t) = V(w + P(w,t), t)\phi(t)$ by substitution. Then,

$$P(w,t) = \frac{1}{a} e^{-r(\tau_p - t)} \cdot \ln \phi(t),$$

as required. □

Observe that the indifference price is independent of the wealth of the insurer. Specification of an exponential utility enables this desirable property due to the constant absolute risk aversion incorporated in the optimal investment process. Dependence on risk aversion, however, cannot be eliminated when applying the indifference principle approach, unlike with Black–Scholes pricing. Intuitively, this makes sense as it is impossible to completely hedge the risks priced due to the inexistence of a relationship between tradable assets and the associated uncertainties in relation to mortality.

Remark 5.4.1. Note that, in the indifference pricing setting, the maximum premium that the buyer of insurance $(x+t)$ should be willing to pay is given by solution of the expression

$$V(w - P^B(w,t), t) = U(w,t)$$

with respect to $P^B(w,t)$. This is the indifference price of the insurance buyer. Although unrealistic, in the event of equivalent risk aversion of insurer and buyer, the indifference prices $P(w,t)$ and $P^B(w,t)$ will be the same in this case, with price increasing with increasing risk aversion.

Under the assumption of independent lifetimes, the minimum premium to be charged by an insurer is again in the form of (5.4.7); however, the mortality process incorporated in the expected present value is unadapted and independent of the mortality intensity of the deceased spouse, such that

$$\mathrm{EPV}^I = \sum_{s=0}^{\tau_q^I - \tau_p^I - 1} \left(\left(e^{B(s;\theta_y)\lambda^I_{y+\tau_p^I}(\tau_p^I)} - e^{B(s+1;\theta_y)\lambda^I_{y+\tau_p^I}(\tau_p^I)} \right) \sum_{i=1}^{s+1} e^{-r((s+1)-i)} \right),$$

where τ_p^I and τ_q^I are the remaining lifetimes of individuals (x) and (y), respectively, and λ_y^I is the mortality intensity of (y), given the independence of coupled lives.

Corollary 5.4.1. The difference in premium that the insurer should charge when incorporating dependence between coupled lives, and thus covering the risk of unexpected claim rates during the first period of bereavement, is given by

$$P^*(w,t) = \frac{1}{a} e^{-r(\tau_p - t)} (\ln \phi(t) - e^{-r(\tau_p^I - \tau_p)} \ln \phi^I(t)),$$

where

$$\phi^I(t) = e^{a\mathbb{E}[\mathrm{EPV}^I]} (1 - e^{-\int_t^{\tau_p^I} \lambda_x^I(s)ds}) + e^{-\int_t^{\tau_p^I} \lambda_x^I(s)ds}$$

and the premium $P^*(w,t)$ is the difference between the indifference price for dependent and independent coupled lives assuming constant risk aversion.

5.5 Numerical example

Having obtained an expression for the indifference price of a reversionnary annuity, numerical results are presented to illustrate the significance of the dependence assumption. Luciano and Vigna (2008) calibrate a non-mean reverting CIR (or Feller) process to a number of generations in the UK population. Comparison of historical Ghanaian life expectancies with those of the

UK populations considered in Luciano and Vigna (2008) alongside the parameter choices of Jevtić and Hurd (2017), allows for selection of the parameters of the paired mortality processes considered in this example. Parameters κ and ϵ determining the nature of the bereavement effect were chosen through sensitivity analysis, utilising observations in the Ghanaian dataset of Section 5.2 to inform the selection. Table 5.1 presents numerical results for the indifference price of the insurance product for three levels of risk aversion.

	Dependent Price	**Independent Price**
$a = 2.0$	0.8199	1.2005
$a = 1.0$	0.6953	0.9291
$a = 0.1$	0.5376	0.5764

Table 5.1: Comparison of indifference price with and without dependence assumption.

For each level of risk aversion in Table 5.1, a reduced indifference price is observed under the assumption of dependent coupled lives. Increasing the risk aversion coefficient in order to compare the risk neutral and risk averse insurance standpoints reveals increasing variation in the two prices. This should be expected since a risk averse insurer would consider the impact of dependence on mortality more significantly than a risk neutral insurer, hence charging at a more extreme rate.

During the simulation process, cases in which insurance priced higher under the dependence assumption were also observed. Due to the large sample size considered, it is also possible that the difference between death interarrival times of a number of couples is greater in the dependent case, with not all bereaved spouses experiencing such a significant mortality jump in relation to the causal nature of broken-heart syndrome. Consideration of potential improvements in mortality following the loss of a spouse, due to factors such as the stress associated with caring for an ill partner, supports the occurrence of such findings in reality. Although limitations on the accurate estimation of parameters due to the size of the data set may have an influence on the results obtained, observation of the need for a change in price under the assumption of dependent lives is consistent throughout all simulations.

5.6 Concluding remarks

In this chapter, the existence of short-term dependence between coupled lives in a lower-middle income country is considered. A stochastic mortality model with non-mean-reverting Cox–Ingersoll–Ross (CIR) mortality processes of affine type is proposed to represent mortality experience. Observation of a different pattern of deaths in comparison to the findings of existing literature prompts suggestion of the existence of socioeconomic influences on the structure of dependence. Proposal of a CIR type model, defined by a rooted process, fits the nature of the Ghanaian data set analysed during the investigation. The tempered volatility induced by the process appears to be more relevant to such a sample than the non-mean-reverting Ornstein Uhlenbeck mortality processes implemented in previous research.

Reflecting the influence of the loss of a spouse on the remaining lifetime of the surviving partner, the bereavement effect is defined as an Ornstein–Uhlenbeck process with zero mean-reversion parameter. The mean-reverting nature of the process captures the classical features of short-term dependence and allows for improvements in the mortality intensity of the surviving spouse to rates above a non-widowed population. This aligns with empirical research and is therefore more realistic than the assumption of a positive bereavement effect throughout the remaining bereaved lifetime. Although a couple specific volatility is assumed within the bereavement process, it is important to note, that when the volatility is common across a population, the assumption of dependence carries a non-diversifiable risk that should be considered by insurers.

Through application of the indifference pricing principle, the price at which an insurer is indifferent between taking on the risk of insuring an individual and not taking the risk was obtained for a reversionary annuity. An expression for the appropriate price change under the assumption of dependent coupled lives, in comparison to the traditional assumption of independence was presented. Although an Ornstein–Uhlenbeck bereavement process appears to fit the pattern of observed data well, when pricing in the indifference principle setting, an equivalent result is obtained under assumption of a simpler exponential bereavement, with only the adjusted mortality intensity at the moment of the initial death incorporated in the survival function. Increasing the sophistication of the bereavement process model is therefore not essential when applying this pricing method, under assumption of couple specific volatility parameters. If volatility parameters are population specific, however, redefinition of the initial adjusted mortality intensity is required in order to obtain a price which accounts for the non-diversifiable risk.

Chapter 6

Dependence modelling of paired lifetimes in Egyptian families

In this chapter, analysis of a large sample of Egyptian social pension data that covers, by law, the policyholder's spouse, children, parents and siblings is undertaken. This data set uniquely enables study and comparison of pairwise dependence between multiple familial relationships beyond the well-known husband and wife case. Applying Bayesian Markov Chain Monte Carlo (MCMC) estimation techniques with the two-step inference functions for margins (IFM) method, dependence between lifetimes in spousal, parent-child and child-parent relationships, is modelled, using copulas to capture the strength of association. Dependence is observed to be strongest in child-parent relationships and, in comparison to the high-income countries of data sets previously studied, of lesser significance in the husband and wife case, commonly referred to as broken-heart syndrome. Given the traditional use of UK mortality tables in the modelling of mortality in Egypt, the findings of this chapter will help to inform appropriate mortality assumptions specific to the unique structure of the Egyptian scheme. This chapter is based on work submitted to a peer-reviewed academic journal, currently under review, with data provided, through Dalia Khalil, by the National Authority for Social Insurance of Egypt.

6.1 Introduction

As discussed in Chapters 1 and 5, the existence of dependence between joint lifetimes presents the need to refine the independence assumption traditionally used in the pricing and reserving of life insurance products that involve multiple lives and mortality assumptions. Joint lifetime research in the existing literature largely considers dependence between husband and wife. Previous research beyond the context of lifetimes paired through marriage includes the study of dependence in disease incidence among fathers and sons (Clayton, 1978), lifetime dependence and disease heritability in adult twins (Hougaard et al. 1992; Wienke et al. 2002; van den Berg and Drepper 2022, (Denmark); Iachine et al. 1998; Lichtenstein et al. 2000, (Denmark, Sweden, Finland)), where there is a large genetics literature, and familial dependence and its impact on child mortality among siblings (Zenger 1993, (Bangladesh); Guo 1993, (Guatemala); Sastry 1997, (Brazil)). However, the implications for insurance are not specifically considered in these works.

In this chapter, dependence between the lifetimes of multiple family members is therefore assessed for the first time on this scale and in this socioeconomic context. Pairwise dependence between the lifetimes of husband, wife, son, daughter, father and mother is considered through analysis of Egyptian pension data. Dependence within these relationships spans each of the three classifications of dependence presented in Chapter 1.

Socioeconomic influences on determinants of an individual's lifetime, including living circumstances, health, education, religious beliefs and the associated approaches to bereavement and loss are widely accepted. Yet the study of dependence and its impact on insurance is limited to high-income countries. Given that many low and lower-middle income countries rely on mortality tables from high-income countries for the pricing of their mortality-based products, it is critical to know whether patterns observed in samples from countries such as the UK and Canada can also be seen in different socioeconomic environments. In Chapter 5, a coupled stochastic mortality model with a tempered volatility is proposed to reflect the impact of close familial and community structures in low-income countries on the severity of broken-heart syndrome. Findings in this chapter provide evidence in support of the propositions of Chapter 5, where a Ghanaian data set is considered.

Differences in familial structures across socioeconomic environments are highlighted by the structure of the Egyptian social pension scheme. In the event of a policyholder's death, social pension schemes typically pay out to a spouse or child, however, in Egypt, siblings and parents are also listed as beneficiaries (see Section 6.2 for details). This policy aligns with the fact that children often live with their parents until marriage, and for male children, even after marriage in some cases. In addition, many families remain financially dependent on the main income provider or breadwinner, typically the father or eldest son. Emotional ties and living circumstances that influence dependence between family members are strongly reinforced by such traditions and norms. Wide age differences in marital relationships, polygamous partnerships and large families are further features of the environment that may change the strength of dependence in comparison to the samples considered in previous studies.

This chapter contributes to the dependence literature by expanding the study of dependence within marital relationships through analysis of a large data set in a previously unstudied socioeconomic context. The study focuses on male policyholders and their beneficiaries, assessing the impact of the death of a father or son, i.e. the main income provider, on the lifetimes of their relatives. Five samples are included in the analysis, where each sample considers a different relationship. Samples are collected from the Egyptian social pension scheme for pensioners covered under the General Social Insurance System (Law 79, 1975), a compulsory scheme with two funds, covering the government sector and the public and private sectors, respectively. Throughout the chapter, those covered by the pension scheme will be referred to as the policyholder or pensioner and the beneficiaries.

Copula-based analysis is used to capture the dependence between the lifetimes in each relationship. Comparisons are made between four Archimedean copulas, in line with the widespread use of the Archimedean family in the modelling of bivariate lifetimes. As in Dufresne et al. (2018), the Clayton, Frank, Gumbel and Joe copulas are assumed. Copula dependence parameters determining the level of association between two lifetimes are estimated using the two-step inference functions for margins (IFM) method. In each of the five samples, the marginal distribution parameters are first estimated independently, before estimation of the copula dependence parameter. All marginal distributions are fitted with the informative reparametrised Gompertz law (Carriere, 1992, 1994).

For parameter estimation, the Bayesian Markov Chain Monte Carlo (MCMC) Metropolis-Hastings (MH) algorithm is implemented. Classical estimation techniques such as maximum likelihood estimation (MLE) provide point estimates of unknown parameters. However, Bayesian MCMC algorithms treat the unknown parameters as random variables and derive estimates of their distribution using random sampling techniques, thus capturing parameter uncertainty. MCMC methods enable inclusion of prior parameter information and reduce the risk of obtaining local rather than global maxima or minima when random walk sampling is not used, a benefit particularly useful for high-dimensional problems. Computation time can however be high for problems with many parameters and complex likelihood functions in comparison to MLE. For a more detailed discussion of the workings of MCMC, the interested reader may refer to Robert and Casella (1999), Roberts and Rosenthal (2004) and van Ravenzwaaij et al. (2018). In the analysis of this chapter, all MCMC results are compared with MLE point parameter estimates.

Bayesian MCMC techniques are well-used in the literature on copula-based dependence analysis. Huard et al. (2006) and Silva and Lopes (2008) adopt Bayesian analysis to study copula selection criteria, reparameterising the problem such that the prior distribution is on the Kendall's tau correlation coefficient rather than the copula parameter. Comparison of one and two-step Bayesian estimation techniques is made in Silva and Lopes (2008) and Ausin and Lopes (2010). Almeida and Czado (2012) use MCMC sampling to estimate parameters of a stochastic copula autoregressive model with time-varying dependence, again reparametrising to estimate Kendall's tau. A common approach in the copula literature, this reparametrisation enables clear comparison of dependence across copula families by unifying the domains of the estimated dependence parameters. Following the IFM two-step method with MH in the second step, Thongkairat et al. (2019) observe a more accurate estimation of mixed copula models when using Bayesian rather than ML estimation. MCMC methods have also been applied in problems including claim reserve and loss prediction (de Alba, 2002; Ntzoufras and Dellaportas, 2002; da Rocha Neves and Migon, 2007; Hong and Martin, 2017); survival analysis (Arjas and Gasbarra, 1996, use a coupled MH algorithm with joint prior distribution to account for stochastic ordering with known differences in the lifetimes of samples) and mortality modelling (Czado et al., 2005; Cairns et al., 2011; Antonio et al., 2015; Li and Lu, 2018; Fung et al., 2019). For a non-exhaustive list of early use of MCMC techniques in actuarial modelling, see Scollnik (2001).

The remainder of the chapter is organised as follows. In Section 6.2, the data set is introduced and empirical correlation measures for the five samples presented. Section 6.3 describes the Gompertz survival model and the copula models used for dependence estimation. The MCMC algorithm is introduced in Section 6.4 and the IFM method in Section 6.5. Results are presented in Section 6.6 and concluding remarks in Section 6.7.

6.2 Data set

Between 1975 and 1980, a number of fundamental laws were issued to ensure coverage of all working Egyptian citizens, both inside and outside of Egypt. These laws provide compulsory coverage for employees in government, public and private sectors (Law 79, General Social Insurance System, Egyptian Social Insurance and Pension (1975)), pay-as-you-go (PAYG) coverage for employers and the self-employed (Law 108, Egyptian Social Insurance and Pension (1976)), regulation of the voluntary social insurance system for Egyptians working abroad (Law 50, Egyptian Social Insurance and Pension (1978)) and PAYG coverage for all working individuals excluded under the three aforementioned laws (Law 112, Comprehensive Social Insurance System, Egyptian Social Insurance and Pension (1980)). Each law covers beneficiaries against old age, disability and death. The data analysed in this chapter consists of lifetime data for individuals covered by Law 79, i.e. those working in the government, public and private sectors. Law 79 is a defined benefit system that provides additional benefits including injury at work, health, unemployment and social patronage insurance, which offers benefits such as provision of housing and monetary discounts. Law 79 was restructured under Law 148 in 2019 to cover all four social security laws. Although no significant changes were observed, in this study, only Law 79 is relevant.

The laws determining the structure of the Egyptian social security system reflect the nature of living circumstances within families in Egypt. The Egyptian social pension scheme is designed to provide benefits to participating workers when they become of pension age, where contributions are made by the worker throughout their employment. A worker exits the scheme through death, partial permanent disability, total disability, or reaching retirement age, where retirement age is to be increased from 60 (Egyptian Social Insurance and Pension, 1975, Section 18(3)) to 65 in 2040, in line with Section 41 of Egyptian Social Insurance and Pension (2019).

Following the death of a pensioner, benefits are distributed among their beneficiaries. By law, beneficiaries are defined as the widow or widower, sons, daughters, parents, brothers and sisters (Egyptian Social Insurance and Pension, 2019, Section 98). Payments cease and beneficiaries exit the scheme through, for example, death, marriage for a widow, daughter or sister, and reaching age 21 for a son or brother, except for those incapable of earning, students not yet aged 26 and unemployed, university degree holders not yet aged 26 and unemployed and those with lower-level qualifications not yet aged 24 (Egyptian Social Insurance and Pension, 2019, Section 105). In the event an individual is listed as a beneficiary of multiple policyholders, they receive only one benefit. The order in which the selected benefit is received is: personal pension, spouse's pension, parents' pension, son's pension, brother or sister's pension (Egyptian Social Insurance and Pension, 2019, Section 102).

Data for this analysis was collected from the Social Egyptian Pension scheme, with an observation period of 10 years, from 2010 to 2019. A pair is included in the data set only if the policyholder dies within the observation period. The observed distribution of the survival time of the policyholder is therefore conditional on their death within this period. The sample consists of 20,863 male pensioners (the policyholders) and their dependents, where dependents are either a spouse, parents, sons or daughters. On average, the male policyholder dies at age 62.9, with 80% dying between the ages of 53 and 74. Further descriptive statistics for the full sample are given in Table 6.1.

Count	10th Quantile	25th Quantile	50th Quantile	75th Quantile	90th Quantile	Mean	SD
20683	53	57	62	68	74	62.9	8.6

Table 6.1: Descriptive statistics of the male pensioners.

Classifying the data according to the pensioner-beneficiary relationship, five samples are observed. Specifically, Husband & Wife (H,S), Father & Son (F,S), Father & Daughter (F,D), Son & Father (S,F) and Son & Mother (S,M). The most commonly studied relationship in the existing literature, the husband and wife sample contains 19,475 males and 19,937 females. The discrepancy in size indicates instances of polygamy, where the law in Egypt permits one man to have up to four wives. Participation in such a relationship could be a determinant of the strength of dependence between husband and wife. However, although an interesting feature in the Egyptian social context and one that does not appear among the subjects of previous research in this area, due to the small sample size (462 duplicated husbands) polygamous relationships are removed, such that only one spouse beneficiary is considered. Average entry ages are approximately 58 and 50.5 for husbands and wives, respectively, with corresponding deaths at ages 62.6 and 65. Of the 19,937 wives in the sample, just 955 (4.79%) died within the 10-year observation period.

76 sons (0.56%) and 57 daughters (0.40%) exited the observation due to death, where 13,655 father-son and 14,274 father-daughter relationships were included in the sample. The average ages at death of these beneficiaries were just 28 and 36, respectively. Son-father and son-mother constitute the smallest observed samples, with 218 son-father relationships and 1067 son-mother relationships. Since the data is from a pension scheme, age at entry and age at death of the child in child-parent relationships are relatively high in comparison to parent-child relationships. However, within the child-parent samples, in comparison to the average age at death of the parent the average age at death of a son is low. This is likely due to the fact that only policyholders that die within the observation period with a parent that is still alive are included in these samples. As such, pensioner sons dying at older ages outside of this period and those who have already lost the relevant parent are not accounted for. Full summary statistics for all five samples are provided in Table 6.2.

		Sample	Count	10th Quantile	25th Quantile	50th Quantile	75th Quantile	90th Quantile	Mean	SD
(H,W)	Husband	Entry	19475	49	52	57	63	68	58.02	8.03
		Death	19475	53	57	62	68	73	62.57	8.27
		Death*	955	56	61	66	73	79	67.02	8.92
	Wife	Entry	19937	39	44	50	57	62	50.52	9.13
		Death	955	53	59	65	71	77	64.92	9.46
(F,S)	Father	Entry	13655	47	50	53	58	63	54.07	7.01
		Death	13655	50	53	57	62	67	57.99	7.19
		Death*	76	51.5	54	58	65.25	78.5	61.22	11.16
	Son	Entry	13655	6	10	15	18	22	14.47	7.16
		Death	76	16.5	19.75	23.5	35.25	51	28.34	16.16
(F,D)	Father	Entry	14274	47	50	55	61	67	56.17	8.55
		Death	14274	51	54	59	65	71	60.17	8.7
		Death*	57	52.6	60	64	73	88	67.3	12.88
	Daughter	Entry	14274	6	11	16	23	31	17.52	10.39
		Death	57	19	25	33	41	58.4	36.12	14.66
(S,F)	Son	Entry	218	43	48	51	55	58	49.84	8.06
		Death	218	45.7	50	54	58	61.3	53.14	8.23
		Death*	119	49	51	55	58	61	54.67	5.13
	Father	Entry	218	68	74	78	83	86	77.35	8.23
		Death	119	78	82	86	89	92	85.71	5.71
(S,M)	Son	Entry	1067	44	48	52	56	60	51.58	7.43
		Death	1067	47	51	56	60	64	55.16	7.56
		Death*	429	49	53	57	60	65	56.61	6.39
	Mother	Entry	1076	66	71	76	81	85	75.67	8.33
		Death	429	76	80	85	89	93	84.63	7.16

Table 6.2: Descriptive statistics for age at entry (Entry) and age at death (Death) for each of the five samples. "Death*" gives the descriptive statistics for policyholders whose beneficiaries have also died.

	Sample	Count	Pearson	Spearman	Kendall
(H,W)	$X_h > X_w$	807	0.819	0.817	0.659
	$X_h \leq X_w$	148	0.905	0.911	0.767
	Total	955	0.769	0.771	0.604

Table 6.3: Empirical dependence measures for the husband and wife sample split by the sex of the elder spouse, where X_h represents the lifetime of the husband and X_w the lifetime of the wife.

Empirical dependence measures for the relationships in each sample are provided in Table 6.4. Here, the Pearson, Spearman and Kendall's tau correlation coefficients for the lifetimes of each pair are calculated. Note that in obtaining these measures only pairs in which both members die within the observation period are considered.

Age differences among married couples vary significantly in the data. The sample is therefore split by age difference (d) to test whether there is an observed impact in the respective correlations, where a positive age difference indicates that the policyholder is the elder member of the pair. In the husband and wife data set, correlation measures are also provided for samples split by the sex of the elder spouse (Table 6.3). Although the minimum age difference between husband and wife is 0 years and the maximum 59 years, 80% of the sample differ in age by between 1 and 15 years, where the husband is the elder spouse. While the impact of age difference may be less significant in parent-child relationships, comparison is also made in these cases.

High positive correlation is observed between the lifetimes of husband and wife, father and son, father and daughter, and son and father. Correlation decreases with increasing age difference in each of the four samples in line with the results for spousal dependence in the literature (Youn and Shemyakin, 1999; Dufresne et al., 2018). In contrast, correlation between the lifetimes of son and mother increases with increasing age difference, indicating a greater reliance of mother upon son with age. Although the two sample sizes are not comparable, couples in which the wife is the elder spouse exhibit an increased correlation, aligning with the findings of asymmetric mortality experience in Lu (2017) and Dufresne et al. (2018).

The distribution of age difference and survival time within each of the samples is presented in Figure 6.1. Increased correlation between lifetimes in child-parent relationships can also be observed in the survival time distribution plots, with a greater proportion of bereaved deaths occurring in the early years of bereavement, specifically years 2 and 3. This trend also appears with less significance in the husband and wife data set, however in both parent-child relationships, the association between survival probability and years since bereavement is less clear. This perhaps aligns with their fairly small sample sizes. The father-son sample experiences a gradual increase in mortality, which falls after the fifth year of bereavement, while the father-daughter sample experiences the same year three peak as observed in the three other samples, with an additional peak much later in the bereavement. In comparison to the data set of Chapter 5, lifetime dependence is of much greater initial significance in the full Ghanaian data set presented in Figure 5.2, with 17.86% of the sample dying across the first and second years of bereavement. In the Egyptian data set presented here, the impact of losing a spouse appears to be delayed.

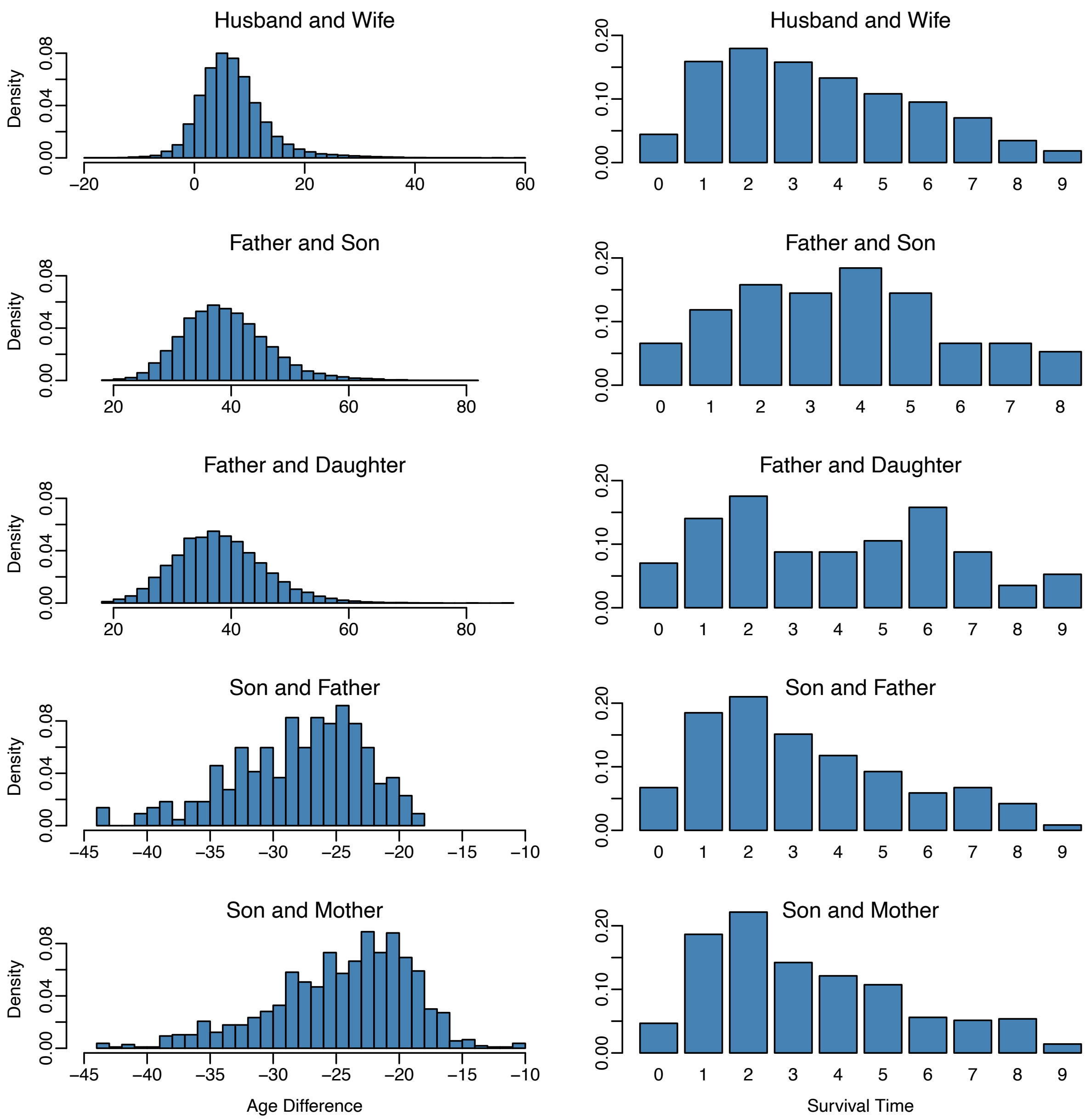


Figure 6.1: Age difference and survival time distributions for all samples.

	Sample	Count	Pearson	Spearman	Kendall
(H,W)	$0 \leq d < 4$	288	0.946	0.942	0.819
	$4 \leq d < 8$	343	0.899	0.881	0.742
	$d \geq 8$	324	0.776	0.803	0.655
(F,S)	$18 \leq d < 35$	28	0.962	0.923	0.771
	$d \geq 35$	48	0.892	0.779	0.647
	Total	76	0.881	0.781	0.610
(F,D)	$18 \leq d < 35$	31	0.971	0.924	0.834
	$d \geq 35$	26	0.916	0.825	0.688
	Total	57	0.871	0.771	0.621
(S,F)	$18 \leq d < 25$	34	0.891	0.871	0.743
	$25 \leq d < 35$	74	0.876	0.742	0.597
	$d \geq 35$	11	0.758	0.704	0.594
	Total	119	0.544	0.513	0.385
(S,M)	$18 \leq d < 25$	222	0.820	0.775	0.618
	$25 \leq d < 35$	181	0.842	0.822	0.654
	$d \geq 35$	26	0.932	0.943	0.832
	Total	429	0.612	0.575	0.425

Table 6.4: Empirical dependence measures for each of the five samples, split by age difference d.

6.3 Model description

In this section, the survival and copula models for dependence are presented. As in Chapter 5, let (x) denote an individual aged x. Then, the survival function of (x) is again defined by (5.3.1), such that

$$S_x(t) = \mathbb{P}(\tau_x > t),$$

where τ_x is the remaining lifetime of (x) given their survival to age x and X denotes the remaining lifetime of an individual at birth.

Many mortality models exist and are implemented in the literature. For the purpose of this study, Gompertz's law of mortality is adopted. Gompertz's law is a classical model of mortality experience first proposed in Gompertz (1825) which states that after a given age, the logarithm of mortality intensity is a linear function of age. The law is specified to reflect mortality behaviours above a sufficiently high level (observed to be approximately 30 years of age), the suitability of Gompertz's law for old-age mortality is however also widely debated. Many studies argue the existence of a deceleration in the increase in mortality at the highest ages (above approximately 80-90 years), with mortality observed to curve away from the Gompertzian trend and to plateau at very high ages, see, for example, Thatcher et al. (1998) and Thatcher (1999). However, more recently, developments in the reporting of age and mortality data have been proposed as improvements that could contest the non-Gompertzian nature of old-age mortality, see, for example Gavrilov and Gavrilova (2019).

In line with this ongoing debate, extensions of the classical Gompertz model and alternative mortality models have been developed to allow for flexibility in the modelling of mortality behaviours. The simplest extension of the Gompertz model is the Gompertz-Makeham model (Makeham, 1860, 1867), where the addition of a constant term is introduced to capture age-independent mortality. Willemse and Koppelaar (2000) and Willemse and Kaas (2007) propose generalisations of the Gompertz distribution in the context of frailty-based mortality models, extending the model beyond classical age dependent considerations. In more recent work, El-Gohary et al. (2013) propose an alternative generalised Gompertz distribution that allows for flexibility in the specification of the hazard rate to overcome the monotonic requirement of the Gompertz hazard function. Li et al. (2021) alternatively capture the old-age mortality curvature and plateau through proposition of a multi-factor exponential model based on the approximation of mortality measures with Laguerre functions. For a thorough overview of mortality models and their suitability for capturing the mortality experience of different age ranges, see, for example, Booth and Tickle (2008) and the references therein.

With the exception of survival data in the son and daughter samples, Table 6.2 shows that age at entry largely lies within the Gompertz range. As such, the marginal distributions of the individuals studied in this analysis are assumed to follow the Gompertz law. MCMC (see Section 6.4) relies on the idea that the Markov chain describing the transient behaviour of the parameters accepted by the algorithm converges to its stationary distribution after a sufficient number of iterations, where the stationary distribution resembles the desired probability distribution of the estimated parameters. For the small samples of sons and daughters with ages largely outside of the Gompertz range, inaccurate estimates and thus greater parameter uncertainty are more likely to appear. This is exemplified in the results of Table 6.7 and is further noted in the discussion of Figure 6.3. However, the parameter distribution obtained in the first IFM step captures this uncertainty. Sampling from this distribution to estimate the marginal parameters for the second IFM step paired with the assumed convergence of the Markov chain to its stationary distribution therefore mitigates the significance of errors in the marginal estimation. Future work could involve fitting a more appropriate model for the age range of child beneficiaries. In addition, while the simple construction of the Gompertz model provides a good starting point for the exploration of mortality experience, it would be interesting to consider the impact on the dependence results of fitting a more comprehensive model of mortality.

The force of mortality λ_x and survival function $S(x)$ associated with X are given by

$$\lambda_x = Bc^x \quad \text{and} \quad S(x) = \exp\left(-\frac{B}{\ln c}(c^x - 1)\right),$$

respectively, for all samples, where $B > 0, c > 1$ and $x \geq 0$. Reparametrising the Gompertz law such that the estimated parameters are informative (Carriere, 1992, 1994), let

$$e^{\frac{-m}{\sigma}} = \frac{B}{\ln c} \quad \text{and} \quad e^{\frac{1}{\sigma}} = c,$$

where $m > 0$ is the modal density and $\sigma > 0$ the dispersion of the density about the mode. Then,

$$\lambda_{x+t} = \frac{1}{\sigma}\exp\left(\frac{x+t-m}{\sigma}\right)$$

and

$$_tp_x = \exp\left(e^{\frac{x-m}{\sigma}}(1 - e^{\frac{t}{\sigma}})\right), \tag{6.3.1}$$

where $_tp_x = \mathbb{P}(X > x + t | X > x) = \frac{S(x+t)}{S(x)}$.

The probability that (x) dies at a given time t, i.e. the probability density function of the remaining lifetime of (x), is then derived by

$$f_x(t) = {}_tp_x\lambda_{x+t}.$$

Copulas are widely used across a broad set of disciplines for the study of dependence between random variables. Since this chapter focuses on the estimation of dependence between two lifetimes, bivariate copula functions will be used throughout the analysis. (Bivariate) copula functions are defined as in Definition 6.3.1:

Definition 6.3.1. A copula function is a bivariate function $C : [0,1]^2 \to [0,1]$ that satisfies the following three properties:

1. $C(u,0) = C(0,v) = 0$ for every $u, v \in [0,1]$.
2. $C(u,1) = u$ and $C(1,v) = v$ for every $u, v \in [0,1]$.
3. For every $u_1, u_2, v_1, v_2 \in [0,1]$ with $u_1 \le u_2$ and $v_1 \le v_2$,

$$C([u_1,v_1] \times [u_2,v_2]) = C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \ge 0.$$

First introduced by Sklar (1959), copula functions provide a link between the marginal and bivariate distributions of two random variables, thus facilitating tractable analysis of the associated dependence structures. The structure of this link (or coupling), is illustrated in Theorem 6.3.1.

Theorem 6.3.1 (Sklar's theorem (Sklar, 1959))**.** Let H be a joint distribution function with univariate marginal distribution functions F and G. Then there exists a copula C such that for all $x, y \in R$,

$$H(x,y) = C(F(x), G(y)). \tag{6.3.2}$$

On the other hand, for any univariate marginal distribution functions F and G and any copula C, the function H in (6.3.2) is a joint distribution function with marginals F and G. In addition, if F and G are continuous, then C is unique. Otherwise, C is uniquely determined on RanF×RanG, where Ran denotes the range of the distribution.

Definition 6.3.1 and Theorem 6.3.1 can be extended for definition of multivariate copulas of dimension greater than two. Details of these extensions and a thorough discussion of copula families, their definitions and properties are provided in Nelsen (2006).

The Archimedean copula family is a class of copulas well-used in the modelling of bivariate survival functions due to their analytical tractability and their relation with informative measures of association, such as Kendall's tau. Copulas in the Archimedean family are particularly useful in high-dimensional studies as they facilitate the modelling of dependence with a single parameter. In addition, in their theoretical study, Genest and Kolev (2021) introduce an extension of the law of uniform seniority to two dependent lives, proving that for a bilinear averaging function, paired lifetimes exhibit Archimedean dependence and have marginal distributions from the same scale family. In two dimensions, Definition 6.3.2 holds.

Definition 6.3.2. Let $\phi : [0,1] \to [0,\infty)$ be a continuous, strictly decreasing and convex function such that $\phi(1) = 0$ and let

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t), & \text{if } 0 \leq t \leq \phi(0) \\ 0, & \text{if } t \geq \phi(0) \end{cases}$$

be the pseudo-inverse of the copula generator function ϕ. Then,

$$C(u,v) = \phi^{[-1]}(t)(\phi(u) + \phi(v)),$$

for $u, v \in [0,1]$, is an Archimedean copula.

The analysis in the remainder of this chapter focuses on the Clayton, Frank, Gumbel and Joe copulas. Details of the structure of each copula are given in Table 6.5, where all copulas are single parameter models and α is the dependence parameter to be estimated.

	Copula	Generator	Domain
Clayton	$(u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}$	$t^{-\alpha} - 1$	$\alpha > 0$
Frank	$-\frac{1}{\alpha} \ln\left(1 + \frac{(e^{-\alpha u}-1)(e^{-\alpha v}-1)}{(e^{-\alpha}-1)}\right)$	$-\ln\left(\frac{e^{-\alpha t}-1}{e^{-\alpha}-1}\right)$	$\alpha \neq 0$
Gumbel	$\exp\left\{-[(-\ln(u))^{\alpha} + (-\ln(v))^{\alpha}]^{1/\alpha}\right\}$	$(-\ln(t))^{\alpha}$	$\alpha > 1$
Joe	$1 - [(1-u)^{\alpha} + (1-v)^{\alpha} - (1-u)^{\alpha}(1-v)^{\alpha}]^{1/\alpha}$	$-\ln(1-(1-t)^{\alpha})$	$\alpha > 1$

Table 6.5: Copula function, generator and domain for the Clayton, Frank, Gumbel and Joe Archimedean copulas.

To improve the interpretability of the results, estimates of the Kendall's tau correlation coefficient given the dependence parameter estimate for each copula are also provided in Section 6.6. Table 6.6 presents the corresponding relationship for each copula.

	Kendall's tau
Clayton	$\frac{\alpha}{\alpha+2}$
Frank	$1 + \frac{4}{\alpha}(D_1(\alpha) - 1)$
Gumbel	$\frac{\alpha-1}{\alpha}$
Joe	$1 + \frac{4}{\alpha^2} \int_0^1 t \log(t)(1-t)^{2(1-\alpha)/\alpha} dt$

Table 6.6: Kendall's tau correlation coefficient as a function of the copula dependence parameter α, where $D_1(x) = x^{-1} \int_0^x t(e^t - 1)^{-1} dt$ is the Debeye function of order 1.

To fit the copulas in Table 6.5 to the Egyptian pension data set of Section 6.2, let τ_{x_1} and τ_{x_2} denote the remaining lifetimes of the first (pensioner) and second (beneficiary) member of each pair, respectively, given their current ages x_1 and x_2. Then, by Sklar's theorem Sklar (1959), if τ_{x_1} and τ_{x_2} are positive and continuous, there exists a unique copula $C : [0,1]^2 \to [0,1]$ that describes the joint distribution function of the bivariate pair of random variables (τ_{x_1}, τ_{x_2}), such that

$$\mathbb{P}(\tau_{x_1} \leq t_1, \tau_{x_2} \leq t_2) = C(F_{\tau_{x_1}}(t_1), F_{\tau_{x_2}}(t_2)),$$

where $F_{\tau_{x_1}}(t_1)$ and $F_{\tau_{x_2}}(t_2)$ are the marginal distribution functions of τ_{x_1} and τ_{x_2}, respectively. The joint survival function of (τ_{x_1}, τ_{x_2}) is similarly given by

$$\begin{aligned}\mathbb{P}(\tau_{x_1} > t_1, \tau_{x_2} > t_2) &= \tilde{C}(S_{\tau_{x_1}}(t_1), S_{\tau_{x_2}}(t_2)) \\ &= S_{\tau_{x_1}}(t_1) + S_{\tau_{x_2}}(t_2) - 1 + C(F_{\tau_{x_1}}(t_1), F_{\tau_{x_2}}(t_2)).\end{aligned}$$

Considering marginal distributions conditional on survival to observation means that lifetimes are coupled at the beginning of the observation period. Coupling lifetimes at an earlier date would infer the existence of dependence prior to the observation. In the husband and wife case, date of marriage would therefore be an appropriate starting point, however this data is not readily available. Similarly, coupling from the date of birth of the child would be relevant in the parent-child and child-parent relationships, however for consistency, in this chapter, coupling is assumed from the outset of the observation for all samples.

6.4 Metropolis-Hastings MCMC

Model parameters are estimated using Bayesian Markov Chain Monte Carlo (MCMC) techniques. Through this approach, Bayes' theorem is used to update the conditional probability of an event given some known information, as more information is obtained. Given a sample of observed data $\boldsymbol{y} \in \mathbb{R}^n$, with distribution $p(\boldsymbol{y}, \boldsymbol{\theta})$, Bayes' theorem states that

$$p(\boldsymbol{\theta}|\boldsymbol{y}) = \frac{p(\boldsymbol{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\boldsymbol{y})} = \frac{p(\boldsymbol{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{\int p(\boldsymbol{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}},$$

where $\boldsymbol{\theta}$ is the vector of parameters to be estimated. Since $p(\boldsymbol{y})$ does not change with $\boldsymbol{\theta}$, it holds that

$$p(\boldsymbol{\theta}|\boldsymbol{y}) \propto p(\boldsymbol{y}|\boldsymbol{\theta})p(\boldsymbol{\theta}). \tag{6.4.1}$$

The posterior distribution $p(\boldsymbol{\theta}|\boldsymbol{y})$ which describes the distribution of the parameters given the observed data is therefore proportional to the product of the likelihood $p(\boldsymbol{y}|\boldsymbol{\theta})$ and the prior distribution of parameters $p(\boldsymbol{\theta})$. Analytical and numerical analysis of the normalising constant $p(\boldsymbol{y})$ is however largely intractable in higher dimensions, enforcing restrictions on full estimation of the posterior.

MCMC methods provide algorithms for constructing Markov chains with stationary distributions replicating that of the posterior. Ensuring convergence to the target distribution, these Markov chains are ergodic and stationary with respect to the posterior distribution (see Appendix C). As such, the state of the chain after a sufficient number of steps can be used to approximate the target distribution, the quality of which increases with the number of iterations. Early chain values are highly dependent on the initial value of the chain due to the Markovian nature of the algorithm and are thus typically discarded. Through estimation of (6.4.1), MCMC algorithms enable random sampling from any probability distribution defined up to a normalisation factor, thus eliminating limitations associated with integral evaluation.

The Metropolis-Hastings (MH) MCMC algorithm (Robert and Casella, 1999) proposes a simple method for constructing such a Markov chain $(X_t)_{t\geq 0}$ on the state space of the posterior distribution, where $X_t \in \mathbb{R}^d$ for a parameter vector of dimension d. The algorithm explores the state space of the posterior, progressively constructing an approximation of the target distribution. To implement the algorithm, a proposal kernel $q(\boldsymbol{\theta'}|\boldsymbol{\theta})$ must first be selected as the distribution from which potential parameters are sampled. This kernel describes the probability of transitioning to a new point in space $\boldsymbol{\theta'}$ given the chain is currently in state $\boldsymbol{\theta}$ and thus describes the movement of the Markov chain. Once specified, the algorithm proceeds as follows:

- Initalise, i.e. $X_0 = \boldsymbol{\theta}_0$.
- For $t = 1, 2, ..., N$
  - Sample the proposal $\boldsymbol{\theta'} \in \mathbb{R}^d$ for $\boldsymbol{\theta}_t$ from $q(\boldsymbol{\theta'}|\boldsymbol{\theta}_{t-1})$.
  - Compute
  $$A = \min\left(1, \frac{p(\boldsymbol{\theta'}|\boldsymbol{y})q(\boldsymbol{\theta}_{t-1}|\boldsymbol{\theta'})}{p(\boldsymbol{\theta_{t-1}}|\boldsymbol{y})q(\boldsymbol{\theta'}|\boldsymbol{\theta}_{t-1})}\right),$$
  where A defines the acceptance probability.
  - Draw $u \sim U(0, 1)$. If $U < A$, accept the proposal, fixing $\boldsymbol{\theta}_t = \boldsymbol{\theta'}$. Else, fix $\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1}$.

Note that if the proposal kernel is specified such that it is symmetric in distribution, the acceptance probability simplifies to

$$A = \min\left(1, \frac{p(\boldsymbol{\theta'}|\boldsymbol{y})}{p(\boldsymbol{\theta_{t-1}}|\boldsymbol{y})}\right),$$

since $q(\boldsymbol{\theta}|\boldsymbol{\theta'}) = q(\boldsymbol{\theta'}|\boldsymbol{\theta})$ for all $\boldsymbol{\theta}, \boldsymbol{\theta'}$.

In the analysis of this chapter, a normal proposal distribution is selected such that $\boldsymbol{\theta'} = X_t + N(0, \sigma)$, where σ is the standard deviation (step-wise) parameter selected by the user to ensure sufficient exploration of the parameter space. Due to the inclusion of the noise term, such a proposal is referred to as a random walk proposal. Prior distributions in all simulations undertaken in the estimations of this study are assumed to be non-informative Uniform priors.

The proportion of parameters sampled from the proposal distribution that are accepted by the MH algorithm is the acceptance rate. This measure is used to assess the efficiency of the algorithm, with an acceptance rate of 0.234 considered optimal (Gelman et al., 1997). The integrated autocorrelation (IAT) score is a further indicator of the robustness of an MCMC simulation. The IAT estimates the number of iterations, on average, needed for an independent sample to be drawn. When running the analysis, an estimate was therefore selected if the acceptance rate of the chain was sufficiently close to the optimal level and if the associated IAT score was low. For the purpose of this study, parameter estimates are given by the mean of the estimated probability distributions.

The standard error of the MCMC sampler is given by

$$\sigma = \sqrt{\frac{\text{IAT}}{N}}\hat{\sigma},$$

where N is the number of iterations of the MCMC algorithm and $\frac{N}{\text{IAT}}$ the effective sample size, which provides an estimate of the sample size required to achieve the same level of precision as if the sample was a random sample.

6.5 Inference functions for margins

The inference functions for margins (IFM) approach of Joe and Xu (1996) is adopted to specify the likelihood function for maximisation. Use of IFM for dependence estimation in copula-based models in the actuarial literature has been observed in studies including those by da Silva Filho et al. (2012) and Brechmann et al. (2013) for dependence between international financial markets, Krämer et al. (2013) and Lee and Shi (2019) for dependence between the number and size of insurance claims and Wang et al. (2015), Lin et al. (2015) and Dufresne et al. (2018) for dependence in mortality models. IFM for estimation of dependence between mortalities modelled with affine processes, as in Chapter 5, is also implemented by Xu et al. (2020). For a d-dimensional multivariate distribution, IFM involves first estimating vectors of marginal distribution parameters $\boldsymbol{\theta}_1, ..., \boldsymbol{\theta}_d$, then substituting the marginal estimates to maximise the associated likelihood function for the parameters of the joint distribution, which is given by

$$L(\boldsymbol{\alpha}, \boldsymbol{\theta}_1, ..., \boldsymbol{\theta}_d) = \prod_{i=1}^{N} f(\boldsymbol{x}_i; \boldsymbol{\theta}_1, ..., \boldsymbol{\theta}_d, \boldsymbol{\alpha}),$$

where $\boldsymbol{x}_i$ is the observed data, $\boldsymbol{\alpha}$ the vector of parameters of the joint distribution and, for joint distributions captured with copula-based models,

$$f(\boldsymbol{x}_i; \boldsymbol{\theta}_1, ..., \boldsymbol{\theta}_d, \boldsymbol{\alpha}) = c(F_1(x_1; \boldsymbol{\theta}_1), ..., F_d(x_d; \boldsymbol{\theta}_d); \boldsymbol{\alpha}) \prod_{j=1}^{d} f_j(x_j; \boldsymbol{\theta}_j),$$

where $c(F_1(x_1; \boldsymbol{\theta}_1), ..., F_d(x_d; \boldsymbol{\theta}_d); \boldsymbol{\alpha})$ is the copula density and $f_j(x_j; \boldsymbol{\theta}_j)$ the marginal density for variate j. Splitting parameter estimation in this way is particularly useful for reducing computation time for multivariate problems in which large numbers of parameters are to be estimated.

Following this two-step approach, two sets of parameter pairs, $\boldsymbol{\theta}_1 = (m_1, \sigma_1)$ and $\boldsymbol{\theta}_2 = (m_2, \sigma_2)$, are estimated for the marginal distributions in each of the five samples, where subscripts differentiate between the first and second members of each pair. The univariate likelihood for estimation of $\boldsymbol{\theta}_k$, where $k = 1, 2$ is given by

$$L(\boldsymbol{\theta}_k) = \prod_{i=1}^{N} [{}_{c_k^i} p_{x_k^i}(\boldsymbol{\theta}_k)]^{1-\delta_k^i} [f_{x_k^i}(t_k^i, \boldsymbol{\theta}_k)]^{\delta_k^i}, \tag{6.5.1}$$

where N is the number of pairs in the sample and x_k^i, t_k^i and c_k^i the age at entry, remaining lifetime and censoring point of member k of pair i, respectively, where the censoring point marks the time between entry into the sample and the terminal time of the observation, such that $\delta_k^i = \mathbb{1}_{\{t_k^i \leq c_k^i\}}$. The remaining lifetime of an individual (x_k^i) in the observed period is then $\min(t_k^i, c_k^i)$. Inclusion of the censoring point and conditioning on an individual's survival to their age at entry ensures left truncation and right censoring in the data are accounted for.

In estimating the dependence between lifetimes, the survival time is the variable of interest. As such, the likelihood function for estimation of the copula dependence parameters is constructed in relation to the joint survival function $\tilde{C}(u_i, v_i)$, where

$$u_i = S_{\tau_{x_1^i}}(t_1^i)|_{\hat{\boldsymbol{\theta}}_1} \quad \text{and} \quad v_i = S_{\tau_{x_2^i}}(t_2^i)|_{\hat{\boldsymbol{\theta}}_2},$$

for member 1 and member 2 marginal estimates $\hat{\boldsymbol{\theta}}_1$ and $\hat{\boldsymbol{\theta}}_2$, respectively. Having obtained the parameter estimates for the marginal distributions of each pair, copula dependence parameters are estimated through maximisation of the following likelihood function:

$$L(\alpha) = \prod_{i=1}^{N} \left[\frac{\partial^2 \tilde{C}_\alpha(u_i, v_i)}{\partial u_i \partial v_i}\right]^{\delta_1^i \delta_2^i} \left[\frac{\partial \tilde{C}_\alpha(u_i, v_i)}{\partial u_i}\right]^{\delta_1^i(1-\delta_2^i)} \left[\frac{\partial \tilde{C}_\alpha(u_i, v_i)}{\partial v_i}\right]^{(1-\delta_1^i)\delta_2^i} [\tilde{C}_\alpha(u_i, v_i)]^{(1-\delta_1^i)(1-\delta_2^i)}. \tag{6.5.2}$$

The four terms in (6.5.2) correspond to the likelihood of the death of both (x_1^i) and (x_2^i), the death of (x_1^i) and survival of (x_2^i), the survival of (x_1^i) and death of (x_2^i), and the survival of both (x_1^i) and (x_2^i), respectively, where

$$\frac{\partial \tilde{C}_\alpha(u_i, v_i)}{\partial u_i} = \mathbb{P}(\tau_{x_2^i} > t_2^i | \tau_{x_1^i} = t_1^i) f_{\tau_{x_1^i}}(t_1^i) \tag{6.5.3}$$

and

$$\frac{\partial^2 \tilde{C}_\alpha(u_i, v_i)}{\partial u_i \partial v_i} = f_{\tau_{x_1^i}, \tau_{x_2^i}}(t_1^i, t_2^i).$$

The partial derivative of $\tilde{C}(u_i, v_i)$ with respect to v_i is analogous to (6.5.3). Given that all pensioners die within the observation period, for the data considered in this study, (6.5.2) reduces to the product of only the first and second terms.

When presenting the results in Section 6.6, labels $k = 1, 2$ will be replaced by labels corresponding to the identity of each family member. The likelihood functions (6.5.1) and (6.5.2) are also used in the comparison of the MCMC estimation with MLE. In this case, the standard error of the parameter estimates is calculated via the inverse of the Information matrix $I(\theta)$, where $I(\theta) = -\mathbb{E}[H(\theta)]$, the negative of the expected value of the Hessian matrix.

6.6 Results

Table 6.7 displays the MCMC and MLE marginal parameter estimation results, with acceptance rate, IAT score and standard error (SE) as defined in Section 6.4. Note that in all cases, MCMC and MLE produce almost the same results. Standard errors are generally low for both estimation techniques but are lower when MCMC is used. In all samples, the modal age at death of the beneficiary is greater than that of the pensioner. In the husband and wife case, this reflects the higher life expectancy of females. Modal age at death is particularly high among beneficiaries in the son and father, and son and mother samples. This observation could be due to the fact that here, the parent is alive at the time of the child's (pensioner's) death and so may already be of high age. Each of the marginal estimates may also be influenced by the level of censoring, with many survivors observed relative to the respective sample sizes (see Table 6.2). The effect of censoring and the associated small sample sizes can also be seen in the IAT score, with Markov chains corresponding to samples with fewer data points exhibiting higher scores.

Comparison between the non-parametric Kaplan-Meier distribution and the Gompertz distribution obtained from the survival function in (6.3.1) with MCMC parameters as in Table 6.7 is made in Figure 6.2. Note that in all cases, the Gompertz and Kaplan-Meier distributions fit more closely for the marginals of the beneficiaries. Bias in the data induced by the fact that all pensioners die within the observation period (otherwise neither pensioner or beneficiary is observed) could be a determinant of this observation. Confidence intervals for the son and daughter samples are also much larger at higher ages, aligning with their small sample sizes and thus increased uncertainty. The limited number of observation points resulting from the data's annual reporting of deaths could also be associated with inaccuracies in the fitting of the continuous marginals.

		MCMC					MLE	
		Estimate	Acceptance	SD	IAT	SE	Estimate	SE
(H,W)	m_h	66.79	0.2573	0.06765	9.303	0.002918	66.80	0.06923
	σ_h	9.076	0.2573	0.04583	6.609	0.001666	9.078	0.04436
	m_w	86.65	0.2486	0.3638	25.77	0.02612	86.66	0.3671
	σ_w	6.958	0.2486	0.1342	18.42	0.008143	6.955	0.1342
(F,S)	m_f	62.61	0.2769	0.1016	9.590	0.004448	62.61	0.1004
	σ_f	8.973	0.2769	0.06157	7.239	0.002342	8.973	0.05980
	m_s	75.38	0.2438	2.942	93.84	0.4030	74.48	2.583
	σ_s	9.244	0.2438	0.6400	78.79	0.08033	9.053	0.5774
(F,D)	m_f	65.73	0.2529	0.1091	10.36	0.004967	65.73	0.1158
	σ_f	10.65	0.2529	0.07023	7.125	0.002651	10.64	0.07252
	m_d	89.91	0.2348	3.649	118.9	0.5628	89.29	3.598
	σ_d	10.15	0.2348	0.7810	99.46	0.1101	10.03	0.7776
(S,F)	m_s	59.69	0.2478	0.4648	11.26	0.02206	56.70	0.4417
	σ_s	6.263	0.2478	0.3441	9.471	0.01498	6.199	0.3225
	m_f	91.73	0.2360	0.5434	8.767	0.02275	91.70	0.5100
	σ_f	5.636	0.2360	0.3838	7.649	0.01501	5.554	0.3708
(S,M)	m_s	58.67	0.2601	0.2184	10.94	0.01022	58.67	0.2144
	σ_s	6.640	0.2601	0.1482	8.864	0.006238	6.621	0.1488
	m_m	94.23	0.2488	0.3679	9.668	0.01617	94.20	0.3578
	σ_m	7.289	0.2488	0.2385	7.572	0.009281	7.248	0.2323

Table 6.7: Marginal distribution parameter estimation results for all five data sets. MCMC: estimate, acceptance rate, standard deviation (SD), integrated autocorrelation score (IAT) and standard error (SE); MLE: estimate, SE.

Figure 6.3 displays the marginal posterior density and accepted parameter traceplots for two of the ten individual samples. Large differences in the marginal sample sizes occur due to the nature of the observed relationships. In the 10-year observation period, 19,475 husbands exited the scheme through death, in comparison to only 57 daughters. As such, the traceplot displaying parameters accepted in estimation of the modal age at death of a daughter does not exhibit the same stationary behaviour as the equivalent plot in the sample of husbands. In line with this, as mentioned previously, an increased IAT score is observed in Table 6.7 for the marginal daughter simulation. The plots in Figure 6.3 were selected to exemplify this result, however the same is observed in the estimation of all parameters for which only small samples are available. The impact of any inaccuracies in parameter estimation associated with a limited sample size is overcome in the second MCMC step due to the stationarity of the chain.

Empirical dependence measures presented in Table 6.4 suggest that the lifetimes of family members in all relationships considered exhibit strong dependence, aligning with the findings in the literature. However, a large number of censored data points appear in all samples, particularly husband and wife, father and son, and father and daughter. In contrast to the

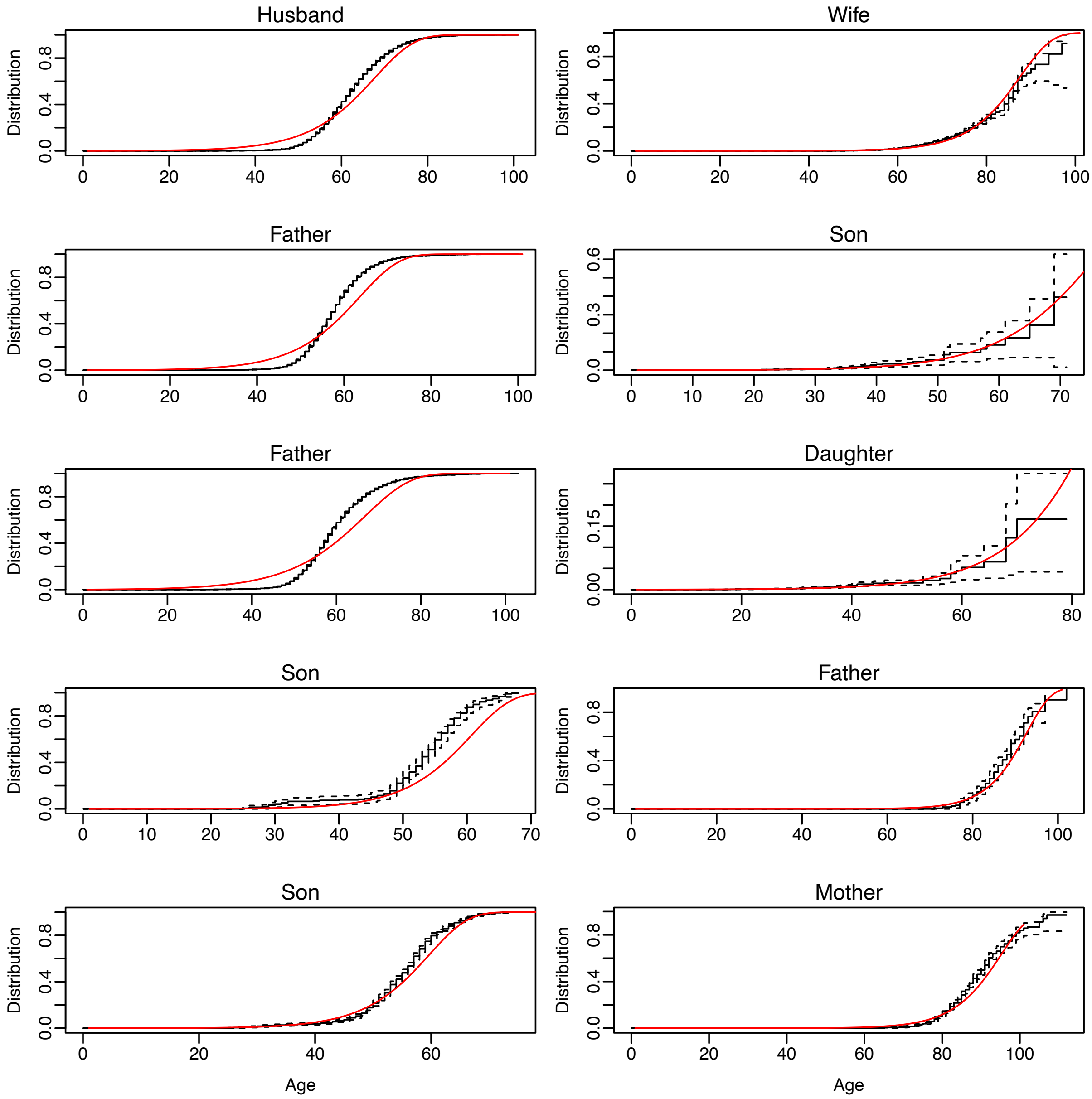


Figure 6.2: Comparison of Kaplan-Meier (black) and Gompertz (red) distribution functions for MCMC marginal parameter estimates.

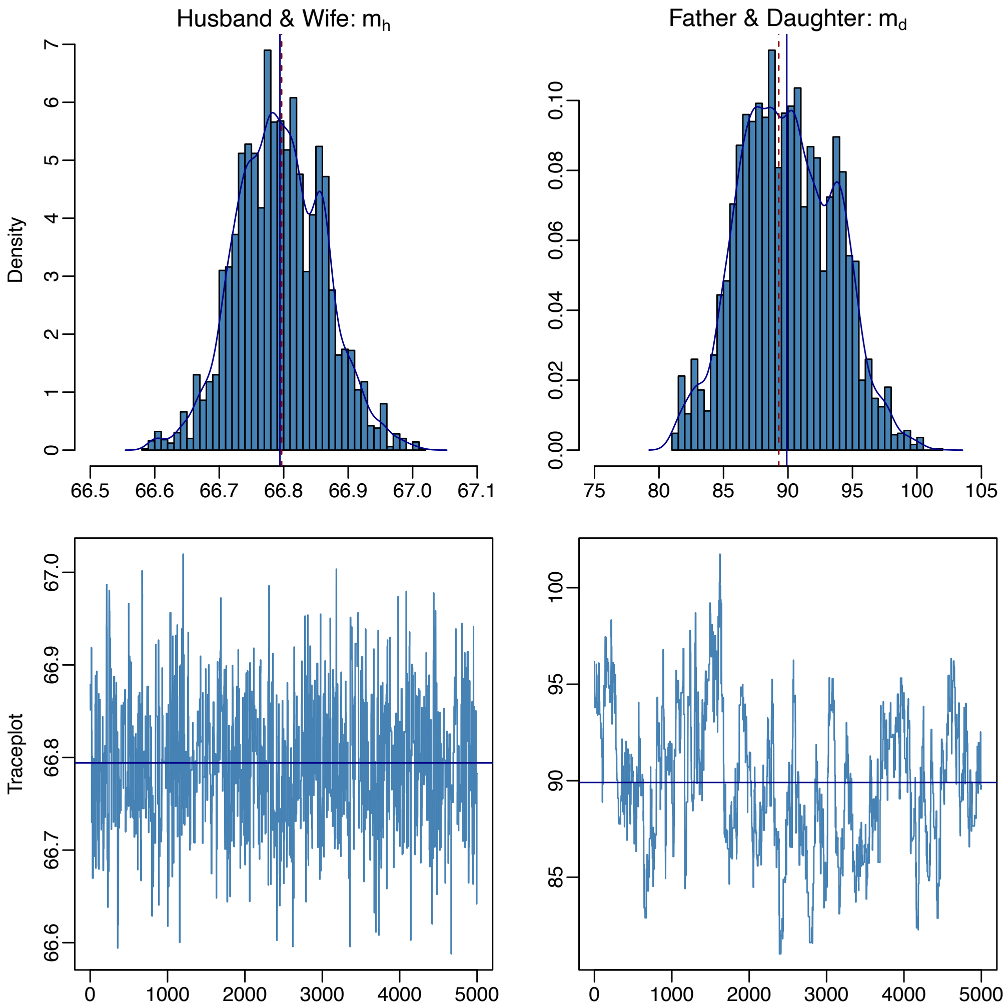


Figure 6.3: MCMC posterior density accepted m traceplots for husband (H,W) and daughter (F,D) marginal distributions.

empirical correlation estimates that consider only those who have died, and so a biased sample of the data, assumption of copula models for dependence enables censoring in the data to be captured.

Estimation results for the dependence parameters of the Clayton, Frank, Gumbel and Joe copulas defined in Section 6.3 are presented in Table 6.8. MCMC and MLE results are compared, with estimates aligning consistently as in the marginal case. The IAT and SE are low for all MCMC estimates, with the increased errors in the marginal distribution estimates in Table 6.7 unobservable in the copula parameter estimation as expected.

In comparison to the findings of Dufresne et al. (2018), dependence parameters are relatively low across all samples. Dependence is greatest between the lifetimes of son and mother or father, which may be expected due to the typically unnatural ordering of the deaths. In addition, with increasing age, elder members of Egyptian families are traditionally taken care of by their children. As such, the loss of a son could impact the living circumstances of the bereaved parent, particularly in cases where the son is the breadwinner. Dependence between the lifetimes of husband and wife is stronger than parent-child and weaker than child-parent relationships. Focusing on age at death dependence in historical French genealogy data, Cabrignac et al. (2020) consider parent-child and grandparent-child dependencies in addition to the classical marital case, noting a very weak but significant association between lifetimes in the alternative relationships, in line with the parent-child findings of this study.

Kendall's tau correlation coefficient estimates obtained from the MCMC dependence parameter estimates in Table 6.8 are given in Table 6.9. Here, when comparing between relationships, the same trends in dependence strength as those discussed for the copula estimation are observed. Correlation between lifetimes modelled with a Clayton copula is much lower than for the Frank, Gumbel and Joe copulas. This may suggest that the Clayton copula is not the most appropriate copula for estimation of dependence within the data set of this chapter. This finding is also observed in Dufresne et al. (2018) through comparison of IFM with the omnibus semi-parametric procedure (or pseudo-maximum likelihood) approach.

		MCMC					MLE	
		Estimate	Acceptance	SD	IAT	SE	Estimate	SE
(H,W)	Clayton	0.1557	0.2523	0.01013	6.106	0.0003539	0.1553	0.01056
	Frank	2.474	0.2603	0.1390	6.445	0.004991	2.470	0.1392
	Gumbel	1.322	0.2777	0.02173	6.472	0.0007817	1.321	0.02053
	Joe	1.677	0.2448	0.06180	5.745	0.002095	1.676	0.05869
(F,S)	Clayton	0.06314	0.2703	0.01551	5.111	0.0004960	0.06225	0.01523
	Frank	1.799	0.2474	0.3997	5.848	0.01367	1.759	0.3983
	Gumbel	1.179	0.2484	0.04224	5.819	0.001441	1.175	0.04116
	Joe	1.412	0.2719	0.1536	5.052	0.004882	1.386	0.1456
(F,D)	Clayton	0.05751	0.2757	0.01816	5.360	0.0005944	0.05773	0.01859
	Frank	1.738	0.2410	0.4699	6.600	0.01707	1.738	0.4613
	Gumbel	1.172	0.2348	0.05206	5.314	0.001697	1.174	0.05060
	Joe	1.435	0.2947	0.1720	5.101	0.005493	1.427	0.1763
(S,F)	Clayton	0.2863	0.2541	0.06058	5.790	0.002061	0.2774	0.06242
	Frank	3.498	0.2721	0.5054	5.680	0.01703	3.457	0.4852
	Gumbel	1.534	0.2444	0.09218	6.370	0.003290	1.512	0.09009
	Joe	2.243	0.2835	0.2214	5.101	0.007071	2.177	0.2167
(S,M)	Clayton	0.3205	0.2480	0.03032	5.476	0.001003	0.3179	0.03019
	Frank	3.040	0.2817	0.2325	5.213	0.007506	3.029	0.2360
	Gumbel	1.459	0.2611	0.04415	5.875	0.001513	1.454	0.04267
	Joe	1.832	0.2555	0.09921	6.277	0.003515	1.814	0.09694

Table 6.8: Copula dependence parameter estimation results for all five data sets. MCMC: estimate, acceptance rate, standard deviation (SD), integrated autocorrelation score (IAT) and standard error (SE). MLE: estimate, SE.

	Clayton	Frank	Gumbel	Joe
Husband & Wife	0.07225	0.2593	0.2435	0.2738
Father & Son	0.03061	0.1937	0.1518	0.1881
Father & Mother	0.02795	0.1873	0.1469	0.1970
Son & Father	0.1252	0.3484	0.3480	0.4046
Son & Mother	0.1381	0.3100	0.3145	0.3154

Table 6.9: Kendall's tau from MCMC α dependence parameter estimates.

Figure 6.4 presents a selection of MCMC simulation results for the copula dependence parameter estimation. The density of the estimated parameter distribution, the traceplot of accepted parameters and the copula likelihood (6.5.2) with estimated parameter indicated are presented, where the traceplot depicts the behaviour of the Markov chain and is thus a plot of the parameters accepted by the MH algorithm. Aligning with Figure 6.3, results for the husband and wife and father and daughter samples are selected for all four copulas. In contrast to the husband, wife and father marginal samples, the increased IAT score of the daughter marginal parameters (Table 6.7) induces non-stationary behaviour in the chain. However, despite the risk of inaccurate marginal estimation resulting from this non-convergent behaviour, stationarity in the Markov traceplots presented in Figure 6.4 is observed in both data sets for all copulas. In addition, plotting the likelihood function (6.5.2) for varying α shows that the algorithm maximises the likelihood well in all cases. The MCMC estimate consistently lies close to the ML estimate, with the ML estimate always within its distribution.

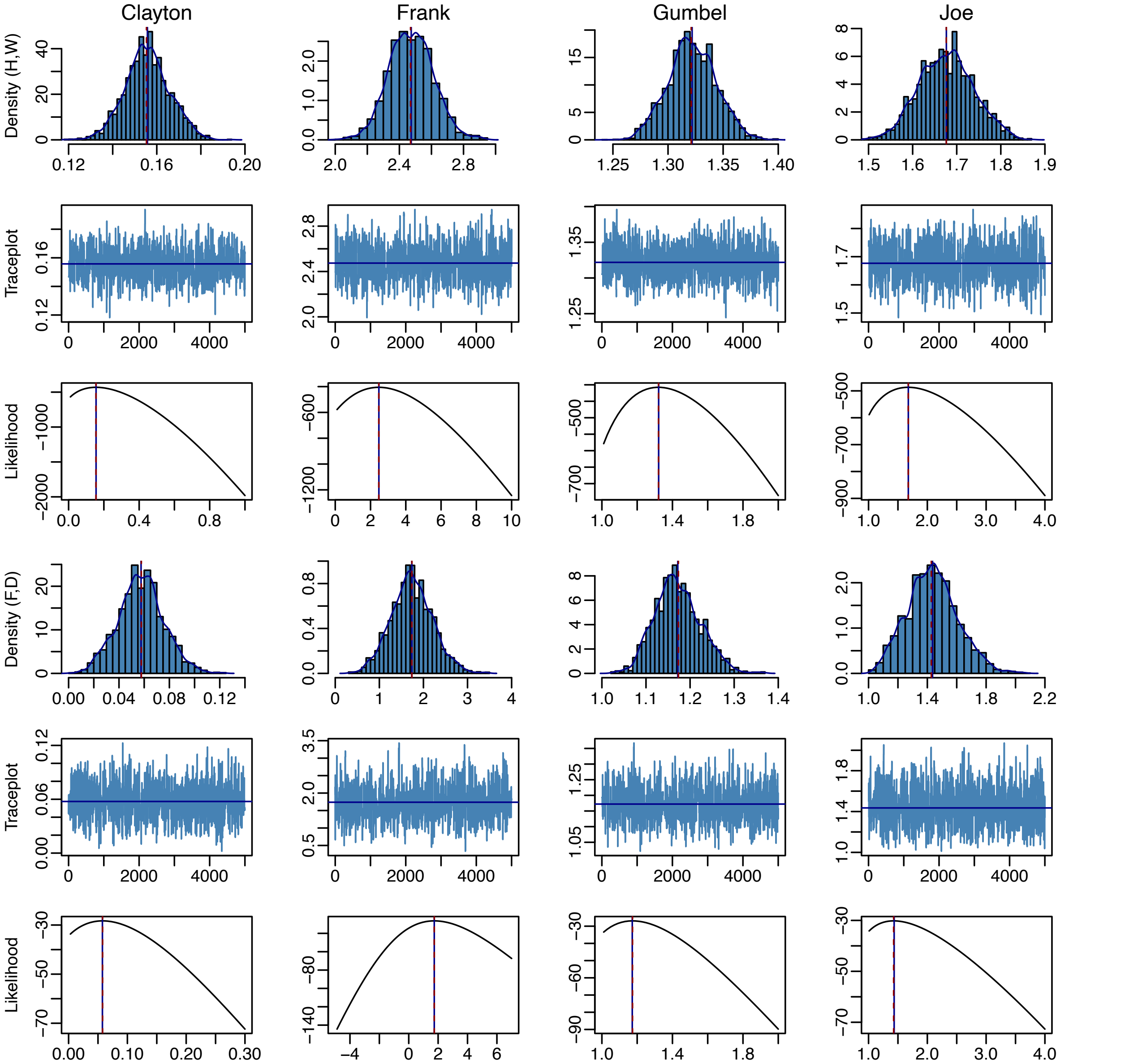


Figure 6.4: MCMC posterior density, accepted parameter (α) traceplots and likelihood function for estimation of the Clayton, Frank, Gumbel and Joe dependence parameters. Results for (H,W) and (F,D) given in rows 1-3 and 4-6, respectively. MCMC estimates given by blue solid line, MLE estimates by red dashed line.

6.7 Concluding remarks

In this chapter, copula dependence parameters were estimated for five different relationships within Egyptian families, using data from the Egyptian social pension scheme. MCMC techniques with likelihood specified using IFM were implemented and compared with classical MLE. Copula dependence parameters were found to be low in comparison to those in the literature for all five relationships considered. However, the corresponding Kendall's tau correlation estimates imply that dependence in this data set and in this socioeconomic context should not be ignored when pricing the associated pension products. Dependence is greatest among child-parent relationships, with non-negligible correlation estimates of between 0.3 and 0.4. Dependence between husband and wife is lower than that of child-parent, with parent-child relationships exhibiting the lowest levels of dependence.

Results presented in this chapter cannot be compared with those of previous studies for all samples, due to the absence of research into dependence between lifetimes of varying family members. However, in the husband and wife case, the Canadian insurance data largely considered in previous studies exhibits higher levels of dependence than the Egyptian sample. Dependence is also of less significance here than in the Ghanaian data set of Chapter 5. The socioeconomic influences on dependence discussed in Chapter 1 alongside the characteristics specific to Egypt introduced in Sections 6.1 and 6.2 of this chapter likely contribute to this observed difference.

Furthermore, joint life data, such as the joint and last-survivor annuity data of the Canadian insurer, consists of the lifetime data of individuals who specifically sought a joint life policy. In contrast to the compulsory nature of the Egyptian pension scheme, this optional participation in such a policy, over a single life policy, implies the existence of a relationship (and hence dependence) between the policyholders, which may align with the increased dependence observed in the data set. This supports the findings of Sanders and Melenberg (2016), where a reduced significance of dependence and the associated pricing impacts is observed among married couples under analysis of census data. Since the Egyptian pension scheme is compulsory for all working individuals, data also spans all social classes. Although this cannot be considered in detail here given the accessible data it may further impact the strength of lifetime dependence and is an interesting area for further study.

Future research

- Chapter 3: Given the intractable nature of the ODEs (3.4.8) and (3.4.17) associated with the IDE of the infinitesimal generator in the insured case, future work will involve solving the IDE for $\mathcal{A}f = 0$ numerically, in order to determine the true behaviour of the underlying trapping probability. In addition, insurance coverage structured differently to the proportional case considered here may allow for the analytic solution to be obtained. As such, alternative insurance mechanisms relevant to the microinsurance environment will be explored.

- Chapter 4: Insurance mechanisms are typically designed such that only a proportion of losses experienced by policyholders are covered by the insurer. Following the sensitivity analysis performed in this chapter, future work will involve adjusting the stochastic dissemination model such that it contains a follower group that both pays premiums and experiences losses. Subsidisation mechanisms will again be considered.

  In addition, the impact of the background process will be explored further. Altering the transaction rate parameters associated with each state, in line with the likely reality when moving between periods of growth and recession, will provide insight into the impact on the wealth of each type of agent and their elasticity to the change in state. A further measure of interest is the speed of default, i.e. the time to trapping, and the way in which this quantity changes after a change in the background process. Knowledge of the reactions of wealth to economic change in group-based systems will help to inform practitioners on the long-term impact and resilience of microinsurance.

  Further examples of the construction of the wealth system will also be tested alongside the impact of a changing role of the leader. In moving between economic states, it could be the case that the leadership role also switches between agents. Finally, in line with the barrier strategy of Chapter 2, the system of differential equations corresponding to targeted subsidy provision will be derived.

- Chapter 5: In order to improve the estimation of the impact of the dependence assumption on insurance products involving mortality assumptions which are targeted towards Ghanaian consumers, it would be beneficial to calibrate the proposed mortality model with the data set collected.

  Furthermore, it would be interesting to consider an extension of the stochastic mortality model that captures the existence of short-term dependence beyond the coupled setting. Such a model could be applied to the study of multivariate dependence due to the increased risk of death following the spread of a communicable disease or for dependence modelling across multiple family relationships, in line with Chapter 6.

- Chapter 6: Performing goodness-of-fit tests on the copula estimates would enable selection of the most appropriate dependence model. Although MCMC estimation mitigates errors associated with the miss-fitting of marginal distributions, selection of a mortality model that fits the age range of child beneficiaries well would help to improve the accuracy of the complete estimation procedure. The impact of age difference on the strength of the dependence is also a quantity of interest, with wide age ranges observed in all samples.

  In order to determine the impact of the varying levels of dependence observed, it would be useful to adjust the pricing of the insurance and pension products of the Egyptian social security scheme to account for the increased risk where present, and to compare with existing prices and premium rates under the assumption of independence.

Final remarks

Penetration of insurance and microinsurance in low-income economies is typically low, leaving the most financially vulnerable without sufficient protection. Addressing the affordability constraints associated with microinsurance and the accuracy of insurance pricing is therefore crucial to improving financial inclusion and in turn, the rate of poverty reduction. Motivated by the absence of adequate insurance coverage, this thesis adopts classical techniques from risk theory, queuing and population modelling, credit risk and dependence modelling to explore fundamental questions in the low-income insurance environment, with the overarching goal of improving protection for the poor.

Chapters 2-4 explicitly focus on the impact of insurance for poverty reduction. Chapters 2-3 consider the probability of falling below the poverty line on an individual agent basis at the household level. Classical risk theory techniques are adopted to study this critical probability, which mimics an insurer's probability of ruin. Chapter 4 extends consideration to the group-based setting prevalent in the low-income environment through implementation of a highly flexible stochastic wealth dissemination model. Addressing a feature of the mortality environment traditionally disregarded by practitioners, Chapters 5-6 consider the existence of lifetime dependence and its implications for insurance pricing. Largely unstudied in this socioeconomic context. Chapter 5 considers dependence between coupled lives with application to private mortality-based insurance products. Chapter 6 considers pairwise dependence between policyholder and beneficiary in a public insurance scheme, where beneficiaries are uniquely defined, by Egyptian law, as the spouse, parents, children and siblings of policyholders. A summary of the findings of each chapter is as follows:

In Chapter 2, the impact of microinsurance frameworks with (i) unsubsidised premiums, (ii) subsidised constant premiums and (iii) subsidised flexible premiums, on the probability of low-income households falling below the poverty line (the trapping probability) is assessed through introduction of a capital model with determinstic growth and random-valued claims. This chapter highlights the importance of governmentally supported inclusive insurance. In line with the existing literature, a level of capital below which unsubsidised insurance increases the probability of falling below the poverty line is observed. Under fixed premium subsidisation, this finding is again observed with reduced severity. The subsidised microinsurance scheme with premium payment barrier reduces a household's trapping probability in comparison to when covered by unsubsidised and (for the most vulnerable) partially subsidised microinsurance, in addition to when uninsured. Social protection costs associated with the provision of government subsidies in the strategies considered in this chapter are accounted for by optimising the subsidy level with respect to the uninsured trapping probability. Costs are reduced under the constant subsidy and barrier strategies.

Chapter 3 adjusts the capital process of Chapter 2 such that households are subject to losses proportional to their level of capital, with accumulated capital given by a risk process with deterministic investment and multiplicative jumps. Laplace transform analysis of the infinitesimal generator of the capital process enables derivation of the explicit trapping probability in the uninsured case. Intuitively, this probability decays more slowly than that of Chapter 2. Proportional insurance is again introduced and its impact on the probability of trapping considered. Simulation analysis suggests that the increase in trapping probability observed for random-valued losses is less severe in this proportional case. Thus inferring that insurance for proportional losses is more affordable than coverage for losses of random value. Establishing the true loss experience of low-income consumers would therefore be beneficial to understanding the need for government subsidisation and to improve the efficiency of social protection schemes. Government support for microinsurance in the proportional loss environment remains of importance. Although less significant, the increase in the trapping probability associated with the most financially vulnerable is again observed when proportional insurance coverage is purchased.

Chapter 4 considers wealth behaviours in a low-income group and addresses, for the first time mathematically, the prevalence of risk sharing and group-based insurance in the low-income setting. Subsets of agents are assumed to be susceptible to wealth transaction events, including wealth losses and premium payments, while an exogeneously evolving Markov background process reflects the state of the economy. As in Chapter 3, losses are proportional to agent wealth. In this group setting, the implications of loss events and the increased costs associated with premium payment are mitigated by the sharing of wealth among all agents. Increasing homogeneity within a group increases the level of sharing. This implies that for risk-sharing groups of agents with the same socioeconomic background, the severity of negative wealth transaction events is lessened. In addition, premium subsidisation supports both the insured and the uninsured, providing further evidence for the benefit of government-insurer partnerships, while trapping probabilities decrease with increasing subsidisation, as expected.

In Chapter 5, credit risk methods are applied to the study of joint-life dependence. Although well-studied in the high-income setting, dependence analysis in alternative socioeconomic environments is lacking. In this chapter, data collected from a Ghanaian sample motivates selection of the joint mortality model. Paired mortality processes are assumed to be correlated non-mean-reverting Cox-Ingersoll-Ross processes, with bereavement effect, reflecting the influence of the loss of a spouse on the remaining lifetime of the surviving partner, given by an Ornstein–Uhlenbeck process with zero mean-reversion parameter. Comparing the Ghanaian data set with observations of empirical research in the literature, a different mortality pattern appears. The initial increase in mortality is of lesser significance than previously observed, supporting suggestion of socioeconomic influences on the structure of dependence. The impact of the dependence assumption on the pricing of a reversionary annuity is determined through derivation of the indifference price, which is equivalent under Ornstein-Uhlenbeck and deterministic bereavement processes for equivalent jump parameters. As in the literature, insurance products capturing the dependence assumption are, in general, priced lower than those for independent lives.

Chapter 6 considers the existence of lifetime dependence beyond the classical husband and wife case. Using a large data set from the Egyptian social pension scheme, pairwise dependence within multiple familial relationships is analysed through estimation of copula dependence parameters. Adopting the two-step inference functions for margins method for specification of the likelihood function, parameter estimation is undertaken through implementation of the Metropolis-Hastings Markov Chain Monte Carlo algorithm. Dependence is greatest among child-parent relationships, with lifetimes in parent-child relationships exhibiting the lowest levels of association. Although reduced in comparison to previous studies, the findings of this chapter confirm that dependence in this data set and in this socioeconomic context should not be ignored when pricing pension products involving beneficiaries and mortality assumptions.

In this thesis, analysis of public and private, life and non-life insurance products for couples, families and communities in low-income populations is presented, with two key takeaways.

The first surrounds public-private-partnerships for subsidisation of insurance. Insurance without subsidies increases the risk of falling below the poverty line for the most poor. However, well-designed, targeted subsidies have the potential to mitigate this additional risk, while decreasing the associated governmental costs. Thus, increasing the efficiency of social support and lessening the financial vulnerability of the low-income target population. In addition, it is important to consider the impact of risk sharing on the benefits of insurance coverage and subsidisation, in order to support design of appropriate subsidy schemes.

Secondly, there is evidence that lifetime dependence exists across socioeconomic environments and within relationships other than those typically studied. The results of this thesis highlight the importance of understanding the lifetime behaviours of all of those protected by a given insurance product. In order to improve insurance product pricing, particularly in the understudied mortality environments such as those considered in this thesis, the existence of dependence cannot be ignored.

Through the chapters of this thesis, we advocate both for the provision of affordable, accurately priced insurance to those who need it most, and for the place for rigorous mathematical analysis and classical theoretical concepts in the strive to reduce the financial vulnerability of the poor.

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Appendix A: Integral transforms

Definition A.1 *(Laplace transform)* Let $f(t)$ be a function of t for $t \in \mathbb{R}_+$. Then, the Laplace transform of $f(t)$, denoted $\mathcal{L}\{f(t)\}$, is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$$

for $s \in \mathbb{C}$. The Laplace transform of $f(t)$ exists if the integral converges for some s.

Definition A.2 *(Mellin transform)* Let $f(t)$ be a function of t for $t \in \mathbb{R}_+$. Then, the Mellin transform of $f(t)$ denoted $\mathcal{M}\{f(t)\}$, is defined by

$$\mathcal{M}\{f(t)\} = F(s) = \int_0^\infty t^{s-1} f(t) dt,$$

for $s \in \mathbb{C}$, with $a_1 < \mathrm{Re}(s) < a_2$. The Mellin transform of $f(t)$ exists if $f(t)$ is piecewise continuous in every closed interval $[a, b] \subset (0, \infty)$ and if

$$\int_0^1 t^{a_1 - 1} |f(t)| dt < \infty \quad \text{and} \quad \int_1^\infty t^{a_2 - 1} |f(t)| dt < \infty.$$

Appendix B: Hypergeometric functions

Definition B.1 *(Confluent hypergeometric functions)* Let a and b be real numbers, the ODE

$$x\frac{d^2y}{dx^2} + (b - x)\frac{dy}{dx} - ay = 0$$

is the confluent hypergeometric equation or Kummer's equation. The simplest solution to this ODE is Kummer's function, given by

$$M(a, c; z) = {}_1F_1(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!},$$

where $(\cdot)_n$ is the Pochhammer symbol defined by $(a)_n \equiv a(a + 1)(a + n - 1)$ for $n \geq 1$ and $(a)_0 = 1$. The series is convergent for all $a, b, x \in \mathbb{R}$ excluding $b = 0, -1, -2, ...$ Kummer's function has integral representation

$$M(a, c; z) = \frac{\Gamma(b)}{\Gamma(b - a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1 - t)^{b-a-1} dt.$$

A second solution of Kummer's equation is Kummer's function of the second kind, or Tricomi's function, given by

$$U(a, b, z) = \frac{\pi}{\sin \pi b} \left[\frac{M(a, b; z)}{\Gamma(1 + a - b)\Gamma(b)} - z^{1-b} \frac{M(1 + a - b, 2 - b; z)}{\Gamma(a)\Gamma(2 - b)} \right],$$

and defined even as $b \to \pm n, 0$. Tricomi's function has integral representation

$$U(a, c; z) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-zt} t^{a-1} (1 + t)^{b-a-1} dt.$$

For full details of the properties of the confluent hypergeometric equations, see Abramowitz and Stegun (1972).

Definition B.2 *(Gauss hypergeometric function)* Let $a, b, c \in \mathbb{R}$ and $c \notin \mathbb{Z}_{\leq 0}$. The Gauss hypergeometric function is defined by the following hypergeometric series:

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

which converges for all $|z| < 1$ (given $c \notin \mathbb{Z}_{\leq 0}$, as specified) and on the unit circle $|z| = 1$ if $\mathrm{Re}(c - a - b) > 0$. This function is a solution of the Gauss hypergeometric equation, given by

$$x(1-x)\frac{d^2y}{dx^2} + [c - (a+b+a)x]\frac{dy}{dx} - abx = 0$$

and has integral representation

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt,$$

for $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$. For full details of the properties of the hypergeometric function, see Abramowitz and Stegun (1972).

Appendix C: Markov chains

Definition C.1 Suppose $(X(t))_{t\geq 0}$ is a Markov chain with state space S, such that all states $i, j \in S$. Then,

(i) The chain is *irreducible* if all states belong to a single closed communicating class, such that each state can be reached from every other state with a non-zero probability, i.e. for all pairs of states i, j, if the chain is in state i at time 0, the probability of being in state j at time $t > 0$ is non-zero.

(ii) The chain is *stationary* with stationary distribution $\boldsymbol{\pi}$ if:

a) (continuous-time Markov chain) $\boldsymbol{\pi Q} = \mathbf{0}$, where $\boldsymbol{Q} = \{q_{ij}\}_{i,j=1}^{d}$ is the transition rate matrix and

$$\mathbb{P}(X(t) = l | X(0) = k) = (e^{\boldsymbol{Q}t})_{k,l}.$$

b) (discrete-time Markov chain) $\boldsymbol{\pi P} = \boldsymbol{\pi}$, where $\boldsymbol{P} = \{p_{ij}\}_{i,j=1}^{d}$ is the transition probability matrix and

$$\mathbb{P}(X(t) = l | X(0) = k) = (\boldsymbol{P}^t)_{k,l}.$$

(iii) The chain is *ergodic* if it is irreducible and aperiodic, where a chain is *aperiodic* if for all states i, the set $\{n \geq 0 : p_{ii}^{(n)} > 0\}$ has no common divisor other than 1, where $p_{ij}^{(n)}$ is the the n-step transition probability from i to j.

Appendix D: Stochastic processes

Definition D.1 *(Martingale)* The process $\{X_n : n \in \mathbb{N}\}$ is a martingale with respect to the filtration $\mathcal{F}_n$ if the following three properties hold:

(i) $\mathbb{E}(|X_n|) < \infty$, $n \in \mathbb{N}$,

(ii) X_n is adapted to $\mathcal{F}_n$,

(iii) $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$.

Definition D.2 *(Sigma-algebra)* A sigma-algebra (σ-algebra or σ-field) $\mathcal{F}$ is a set of subsets $\omega \in \Omega$ that satisfies the following three conditions:

(i) $\emptyset \in \mathcal{F}$,

(ii) if $B \in \mathcal{F}$ then its complement $B^c \in \mathcal{F}$,

(iii) if $B_1, B_2, ...$ is a countable collection of sets in $\mathcal{F}$, then their union $\bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$.

Definition D.3 *(Sub-sigma-algebra)* Let $\mathcal{A}$ and $\mathcal{B}$ be σ-algebras. Then, $\mathcal{A}$ is said to be a sub-sigma-algebra (or sub-σ-algebra) of $\mathcal{B}$ if and only if $\mathcal{A} \subseteq \mathcal{B}$.

Definition D.4 *(Borel sigma-algebra)* The Borel σ-algebra of $\mathbb{R}$, denoted $\mathcal{B}(\mathbb{R})$, is the smallest σ-algebra containing the open subsets of $\mathbb{R}$.

Definition D.5 *(Measurable)* A random variable X is said to be measurable with respect to the σ-algebra $\mathcal{F}$ if for every Borel set $B \in \mathcal{B}(\mathbb{R})$

$$X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$

Definition D.6 *(Filtration)* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(X_t)_{t \geq 0}$ be a stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then, the following definitions hold:

(i) A set $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is called a filtration if

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n : \mathcal{F}_t = \sigma\{X(s); s \leq t\},$$

i.e. a filtration is a non-decreasing family of sub-σ-algebras of $\mathcal{F}$.

(ii) A set of random variables $\{X_n : n \in \mathbb{N}\}$ is said to be adapted to the filtration $\{\mathcal{F}_n : n \in \mathbb{N}\}$ if X_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$.

Appendix E: Questionnaire

Title: An Investigation Into the Existence of Broken-Heart Syndrome in Ghana

Participants were presented with questions 1-3, where appropriate, for each of their four grandparents. Questions relating to the paternal grandmother are presented below. If the participant answered that both maternal or paternal grandparents had died, they were additionally presented with question 4. The questionnaire was completed through an online platform.

1. Is your **paternal grandmother** alive? (Select the appropriate answer).

    - Yes
    - No

    *(**If yes**, the online platform presents the participant with question 2, **if no**, the online platform presents question 3).*

2. a) How old is your **paternal grandmother**? *(Response options presented in drop-down menu).*

    b) How many children does your **paternal grandmother** have today? *(Response options presented in drop-down menu).*

    c) What are your **paternal grandmother's** living circumstances? (Select all appropriate responses)

    - Lives alone
    - Lives with paternal grandfather
    - Lives with partner (other than paternal grandfather)
    - Lives with children
    - Lives with other family (someone who is not your grandmother's partner or children)
    - Lives in care home
    - Hospital patient
    - I don't know
    - Other (please specify)

3. a) How many years ago did your **paternal grandmother** die? Note: Select 0 if your grandmother died less than 1 year ago. *(Response options presented in drop-down menu).*

b) How old was your **paternal grandmother** when she died? *(Response options presented in drop-down menu).*

c) How would you describe the circumstances of your **paternal grandmother's** death? (Circumstances of the death should be classified as accidental if the death was due to the occurrence of an unexpected shock event such as a car accident).

- Following a long period of illness (3 months or more)
- Following a short period of illness (less than 3 months)
- Old age
- Accidental
- I don't know
- Other (please specify)

d) How many children did your **paternal grandmother** have at the time of her death? *(Response options presented in drop-down menu).*

e) What were your **paternal grandmother's** living circumstances at the time of her death? (Select all appropriate responses).

- Lived alone
- Lived with paternal grandfather
- Lived with partner (someone who is not your paternal grandfather)
- Lived with children
- Lived with other family (someone who is not your grandmother's partner or children)
- Lived in residential care
- Hospital patient
- I don't know
- Other (please specify)

4. If there was less than one year between the deaths of your paternal (or maternal) grandmother and your paternal (or maternal) grandfather, which grandparent was the first to die?

- Grandmother
- Grandfather
- I don't know
- Other, i.e. if your grandparents died in a car accident, they may have died at the same time (please specify)