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Homogeneous Finite-gain $\mathcal{L}_p-$stability Analysis on Homogeneous Systems

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Abstract

In this thesis, it is shown that the classical $\mathcal{L}_p-$stability and $\mathcal{L}_p-$gain is not well-defined for arbitrary continuous weighted homogeneous systems. By modifying the classical $\mathcal{L}_p-$norm of signals to be homogeneous w.r.t. some weight vectors, which is called homogeneous $\mathcal{L}_p-$norm, it is possible to show that every internally stable homogeneous system has a globally defined finite homogeneous $\mathcal{L}_p-$gain, for p sufficiently large.

With the help of a homogeneous Lyapunov function, homogeneous $\mathcal{L}_p-$stability can be characterized by a homogeneous partial differential inequality, which in the input affine case can be transformed to a homogeneous Hamilton-Jacobi inequality. Furthermore, in this thesis some detailed methods to calculate upper estimates for the homogeneous $\mathcal{L}_p-$gain are provided from theses inequalities. This also includes the homogeneous $\mathcal{L}_\infty-$gain and homogeneous Input-to-State gain.

For feedback interconnected systems, if the weight vectors between plants are matched, the additive inequality for homogeneous $\mathcal{L}_p-$norm allows the introduction of the homogeneous small gain theorem for each p, enabling stability analysis on the closed loop system.

Finally, some homogeneous $\mathcal{H}_\infty-$controllers can be designed, if the system is affine in the control input. Without the convenient tools for the linear systems, such homogeneous $\mathcal{H}_\infty-$controllers can only guarantee that the closed loop system has homogeneous $\mathcal{L}_p-$gain less than some derivable numbers, its optimality can not be guaranteed.

Several short examples are presented within each chapter to illustrate how such homogeneous $\mathcal{L}_p-$gain can be calculated. In particular a detailed analysis on the Continuous Super-Twisting Like Algorithm is included with deeper insight for interested readers.

Kurzfassung

In dieser Arbeit wird gezeigt, dass die klassische $\mathcal{L}_p-$Stabilität und $\mathcal{L}_p-$Verstärkung für beliebige stetige, gewichtete homogene Systeme nicht wohldefiniert ist. Indem die klassische $\mathcal{L}_p-$Norm von Signalen zu einer homogenen $\mathcal{L}_p-$Norm so angepasst wird, dass diese bezüglich der Gewichtsvektoren homogen ist, ist es möglich zu zeigen, dass jedes intern stabile homogene System für hinreichend große p eine global definierte endliche homogene $\mathcal{L}_p-$Verstärkung besitzt.

Mit Hilfe einer homogenen Lyapunov-Funktion kann die homogene $\mathcal{L}_p-$Stabilität durch eine homogene partielle Differentialungleichung charakterisiert werden, die sich im eingangsaffinen Fall in eine homogene Hamilton-Jacobi-Ungleichung transformieren lässt. Des Weiteren werden in dieser Arbeit detaillierte Methoden zur Abschätzung von oberen Schranken für homogene $\mathcal{L}_p-$Verstärkungen aus diesen Ungleichungen abgeleitet. Dies schließt die homogene $\mathcal{L}_\infty-$Verstärkung und die homogene Eingangs-Zustands-Verstärkung ebenfalls ein.

Bei rückgekoppelten homogenen Systemen, bei denen die Gewichtsvektoren zwischen den Systemen zueinander passend sind, erlaubt die additive Ungleichung für die homogene $\mathcal{L}_p-$Norm die Einführung des homogenen Small-Gain Theorems für beliebige p, wodurch eine Stabilitätsanalyse des geschlossenen Regelkreises ermöglicht wird.

Weiterhin können homogene $\mathcal{H}_\infty-$Regler entworfen werden, wenn das System eingangsaffin ist. Da die konventionellen Werkzeuge der linearen Systemtheorie nicht zur Verfügung stehen, können solche homogenen $\mathcal{H}_\infty-$Regler nur garantieren, dass der geschlossene Regelkreis eine homogene $\mathcal{L}_p-$Verstärkung hat, die kleiner als ein bestimmbarer Wert ist. Ihre Optimalität kann hingegen nicht garantiert werden.

In jedem Kapitel werden mehrere kurze Beispiele vorgestellt, um zu veranschaulichen, wie eine solche homogene $\mathcal{L}_p-$Verstärkung berechnet werden kann. Insbesondere ist eine detaillierte Analyse des “Continuous Super-Twisting Like Algorithm” mit tieferen Einblicken für interessierte Leser enthalten.

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Contents

List of Symbols

The following lists provide an overview of the used variables, functions and abbreviations. Sporadically and specially used symbols are omitted.

Notation

$\mathbb{R}$ and $\mathbb{C}$	Sets of real and complex numbers
$\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$	Sets of non-negative and positive real numbers
$A \triangleq B$	A is defined as B
$A \in B$	Set A belongs to set B
$\forall A$	For all A
$\exists A$	There exists A
$A \Leftrightarrow B$	A is equivalent to B, A if and only if B, or A is a sufficient and necessary condition of B
$A \Rightarrow B$	A is a sufficient condition of B, or A implies B
$A \Leftarrow B$	A is a necessary condition of B, or B implies A

$A^\top$	Transpose of A.
A^H	Hermitian transpose, element-wise transpose and complex conjugate of A.
A^{-1}	inverse such that AA^{-1} or $A^{-1}A$ equals identity or identity matrix.
$\det(A)$	determinant of a matrix, which is the multiplication of all the eigenvalues of A.
$\mathrm{trace}(A)$	trace of a matrix, which is the summation of all the eigenvalues of A.
$\rho\{A\}$	spectral radius of a matrix, which is the maximal magnitude of eigenvalue of A.
$\sigma\{A\}$	singular value of a transfer function, which is the square root of eigenvalue of A^HA or AA^H. $\overline{\sigma}$ means the biggest such singular value, $\underline{\sigma}$ the smallest.
$\mu\{A\}$	structured singular value of A, equivalently conditioned $\mathcal{H}_\infty$-norm of a plant, detailed in the thesis.
$\mathrm{diag}(E_1, \ldots, E_n)$	an $n \times n$ diagonal matrix with E_i as its ith diagonal element
$\lceil x \rfloor^p$	Sign preserving power of a scalar variable x to the power of p, i.e. $\lceil x \rfloor^p = \lvert x \rvert^p \,\mathrm{sign}(x)$.
$\nu_\kappa^\tau(x)$	homogeneous dilation of a vector x with weight vector τ and variable κ.
$\lvert x \rvert$	absolute value of a scalar x.
$\lVert x \rVert$	any norm of vector x, that satisfies the property of the norm.

$\|x\|_p$	$p-$norm of vector x, with $p > 0$. In particular, $p = \infty$ is also allowed.
$\langle x, y \rangle$	inner product of two vectors, which equals to $x^\top y$.
$\|x\|_{\tau,p}$	τ-homogeneous $p-$norm of vector x, with weight vector τ.
$\|x\|_{\mathcal{L}_p}$	$\mathcal{L}_p$ norm of a signal x
$\langle x, y \rangle_s$	inner product of two signals, which equals to $\int_0^\infty x^\top(t)y(t)\mathrm{d}t$.
$\|x\|_{\tau,\mathcal{L}_p}$	homogeneous $\mathcal{L}_p$ norm of a signal x, with weight vector τ
$\mathcal{L}_{\tau,p}$	homogeneous $\mathcal{L}_p$ space, with weight vector τ
$\mathcal{L}_{\tau,pe}$	extended homogeneous $\mathcal{L}_p$ space, with weight vector τ
$\|G\|_\infty$	$\mathcal{H}_\infty$ norm of a plant.
$\|G\|_{\tau,\infty}$	homogeneous $\mathcal{H}_\infty$ norm of a homogeneous plant, with weight vector τ.

Abbreviations

ARE Algebraic Riccati Equation

ARI Algebraic Riccati Inequality

CLF Control Lyapunov Function

CSTLA Continuous super-twisting like algorithm

FTC Finite time convergence

HJI Hamilton-Jacobi inequality

IOM Input-Output Map

LMI Linear matrix inequality

LTI Linear time-invariant

MIMO Multiple-Input Multiple-Output system

RS Robust stability

RP Robust performance

SISO Single-Input Single-Output system

SMA Sliding mode algorithm

SOS Sum of square

SSM State space model

STA Super-twisting algorithm

SMA Sliding mode algorithm

PDE Partial differential Equation

PDI Partial differential Inequality

1 Introduction

In this thesis, the traditional $\mathcal{L}_p-$norm as well as $\mathcal{H}_\infty-$norm are modified to be suitable for continuous homogeneous systems. They are called homogeneous $\mathcal{L}_p-$norm and homogeneous $\mathcal{H}_\infty-$norm. Such tools allow the tuning of continuous homogeneous systems to achieve a better performance in the sense of the homogeneous $\mathcal{H}_\infty-$norm or the homogeneous $\mathcal{L}_p-$gain. Also, the homogeneous small gain theorem with the homogeneous $\mathcal{L}_p-$gain allows closed-loop stability results for feedback interconnected system.

1.1 History of the $\mathcal{H}_\infty-$norm and Homogeneous Systems

The $\mathcal{H}_\infty-$norm of a linear time-invariant (LTI) system can be defined by the maximal singular value of its transfer function along the imaginary axis [19]. When the state space representation (state space model) is available, it can be calculated by the smallest real number such that the associated Hamiltonian matrix has no eigenvalues on the imaginary axis [65].

Unlike the frequency domain analysis from the first method, the latter method comes from a more fundamental definition, i.e. the $\mathcal{H}_\infty-$norm (or equivalently $\mathcal{L}_2-$gain) of a system is the supremal ratio of the $\mathcal{L}_2-$norm of the output over the $\mathcal{L}_2-$norm of the input in time domain [4]. Whereas the frequency domain analysis only allows for the calculation of the $\mathcal{H}_\infty-$norm, the time domain analysis allows the design of $\mathcal{H}_\infty-$optimal controllers, $\mathcal{H}_\infty-$optimal observers or a combination of both. Detailed methods of such time-domain game theorem (mini-max design problem) can be found in [4].

Whereas the frequency domain analysis applies only for LTI systems, the time domain analysis allows the application of the $\mathcal{H}_\infty-$norm (or $\mathcal{L}_p-$gain, similarly defined with $\mathcal{L}_p-$norm) to nonlinear systems [29, 50, 42]. On the one hand, with the help of the $\mathcal{H}_\infty-$norm ($\mathcal{L}_2-$gain) small gain theorem one can guarantee the closed loop stability for an interconnected system [29]. On the other hand, reducing the respective $\mathcal{H}_\infty-$norm when designing a controller or observer, the worst case response of the system subject to worst disturbance can be mitigated, thus its performance can be improved [65].

Furthermore, if an LTI system shows a structured uncertainty (i.e. the uncertainty in vector field, e.g. the uniformly bounded uncertainty in input, output or state matrix), by taking such uncertainty out of the nominal vector field (through defining new input and output for such uncertainty), the small gain theorem with $\mathcal{H}_\infty-$norm can guarantee robust stability (RS), which naturally indicate nominal stability when the uncertainty is zero, for the closed loop system [54, 62]. When the performance requirement is set, timing the inverse of such requirement prior input or after output, the small gain theorem with $\mathcal{H}_\infty-$norm can guarantee further robust performance (RP), i.e. the performance requirement is met under worst uncertainty [54, 62].

The small gain theorem with $\mathcal{H}_\infty-$norm alone introduces some conservativeness in RS or RP, unless the uncertainty is of full rank and complex. The $\mathcal{H}_\infty-$norm of the plant pre-multiplied by a D-scaling matrix, which should commute with the structured uncertainty, and post-multiplied by its inverse allows a less conservative estimate, which is called structured singular value of the plant. The structured singular value allows a bigger structured uncertainty and achieve closed loop stability. Thereafter, the so called $\mu-$synthesis (or D-K iteration) of controller design or observer design provides a possibility of further optimizing the closed loop system, guaranteeing RS and RP [62].

Homogeneous systems, to which linear systems also belong, are becoming more popular. Homogeneous systems are widely used in nonlinear controller and observer design, e.g. in [46, 1, 40, 6]. Unforced asymptotically stable linear time invariant (LTI) systems, whose dynamics have homogeneous degree zero, provide exponential convergence globally. When the homogeneous degree of the dynamics is negative, then such unforced stable homogeneous system guarantees global finite time convergence (FTC) [3, 42]. Such homogeneous system, when equipped with a discontinuous vector field (through discontinuous controller or observer), might further provide FTC in the presence of non-vanishing input. This property is widely used in sliding mode algorithms (SMA) [32, 33]. Moreover, some nonlinear systems can be approximated more accurately by homogeneous systems than by local linearization, thus homogeneous systems are studied more intensively [21, 27].

In most recent studies of homogeneous systems, e.g.[18, 53, 40, 6, 32, 33, 35, 39, 56, 13, 57, 38, 37, 47], besides the structure of the closed loop system, where some gains are tunable, a storage function is also provided to prove asymptotic stability with the Lyapunov method (for a discontinuous homogeneous system, the non-vanishing input rejection might also be proven). There the range of gains such that the storage functions serve as Lyapunov functions are studied, i.e. the range of gains such that the closed loop system is guaranteed to be stable (or reject some non-vanishing input) is studied. Within this “stable” range of gains, there exists little preference for the choice of gain, except some experience preference, e.g. the gains often used in [32] for the Super Twisting Algorithm (STA). In the recent work of [45], the authors derive some gain preferences based on the describing function approach in order to achieve chattering reduction for the STA. However, such approach, when applied to a homogeneous system, is restricted to discontinuous design such as sliding mode algorithms.

To this end, the $\mathcal{H}_\infty-$norm analysis for possible preferred choice of gain is of interest to provide some preferred choice of gains within the “stable” range.

1.2 State of the Art: $\mathcal{L}_p-$stability on Homogeneous Systems

In previous work of [48], the author shows that for smooth and classical homogeneous systems (i.e. with weight vectors of the states as well as input equal to one and with non-negative homogeneity degree, in this case the weight vectors for input and output are equal), having the state variable as output, asymptotic stability of the origin of the unforced system implies $\mathcal{L}_p-$stability (for p sufficiently large) and Input-to-State Stability (ISS), both with linear gain. However, [48] does not discuss how to characterize and estimate the $\mathcal{L}_p-$gain and the ISS-gain.

The authors of [25] considers also continuous classical homogeneous systems (i.e. with weight vectors of the states equal to one and non-negative homogeneous degree), which are affine in the input with constant input matrix and whose weight vectors of the input and the output being equal. It is shown that internal Lyapunov stability of the unforced system implies $\mathcal{L}_2-$stability with finite $\mathcal{L}_2-$gain. The novelty consists in its characterization by means of a homogeneous Hamilton-Jacobi Inequality (HJI). Meanwhile, a continuous homogeneous $\mathcal{H}_\infty-$controller is developed by using the HJI, similar to the derivation for LTI systems, whose $\mathcal{L}_2-$gain from input to extended output of the closed loop system can be mitigated under a calculable constant.

In [24], the author extends the material in [25] by covering a class of continuous weighted homogeneous systems of arbitrary homogeneity degree, which are still affine in the input and whose weight vectors of the input and the output being equal. It is shown that internal stability implies again $\mathcal{L}_2-$stability with finite $\mathcal{L}_2-$gain, and this can be characterized by using a homogeneous HJI. Moreover, internal Lyapunov stability is also shown to imply $\mathcal{L}_p-$stability (for sufficiently large p) with finite $\mathcal{L}_p-$gain (including $p = \infty$). However, these results are obtained by imposing severe restrictions on the weight vectors of the inputs, outputs and states. Again, estimation of the $\mathcal{L}_p-$gains is not discussed. The continuous homogeneous $\mathcal{H}_\infty-$controller, similar to that in [25] is extended for the weighted homogeneous system.

The authors in [1] and [7] discuss ISS and other related properties for general weighted homogeneous systems, generalizing the results of [48] relating the internal stability of the unforced system and ISS. However, [1, 7] do not consider $\mathcal{L}_p-$stability for any value of p, and the linear ISS-gain is not clearly stated. This linear ISS gain issue is clarified in [5] by using solely homogeneous norms instead of mixture of classical norms and homogeneous norms. Furthermore, the homogeneous small gain theorem is considered in both [1, 5]. In [1], by using the classical norm the nonlinear ISS-gain is a $\mathcal{K}$ function, and in [5] external inputs are not introduced.

Authors of [64, 63], with a state transformation, derive a local $\mathcal{H}_\infty-$norm on the homogeneous and discontinuous Super-Twisting Algorithm (STA). There the $\mathcal{H}_\infty-$norm is noted as a local norm, thus, for the norm to be valid the input must be restricted within some sets. The reason for such $\mathcal{H}_\infty-$norm being local is that the weight vector of the input and the output are different, therefore the $\mathcal{H}_\infty-$norm is of non-zero homogeneous degree. The authors of [61] notice this deficiency. Therefore, they introduce for the first time a homogeneous $\mathcal{H}_\infty-$norm of degree zero for the Continuous Super-Twisting Like Algorithm (CSTLA).

In [28] it is shown that if a system is finite-gain $\mathcal{L}_2-$stable (or $\mathcal{L}_2-$stable), then it is ISS (or integral ISS). And if it is ISS (or integral ISS), then it is finite-gain $\mathcal{L}_2-$stable (or $\mathcal{L}_2-$stable) under some homeomorphic change of coordinate on input and output. That is, the study in [28] assumes one stability and derives the other stability.

1.3 Contribution of this thesis

In this thesis, extending the results from [25, 24, 61], a new homogeneous $\mathcal{L}_p-$stability concept (including the case of $p = \infty$) as well as Input-to-State Stability (ISS) is introduced for an arbitrary continuous weighted homogeneous system, based on homogeneous $\mathcal{L}_p-$norms for input and output signals. Every stable homogeneous system has finite homogeneous $\mathcal{L}_p-$gains with p sufficiently large, that relates linearly and globally the homogeneous norms of the input and the output variables. This extends the well-known situation for linear systems. The so defined homogeneous $\mathcal{L}_p-$norms can then be used in the traditional manner for e.g. controller design to minimize the effect of perturbations, as it happens in the $\mathcal{H}_\infty-$control problem, or for parameter optimization, as illustrated in our previous work [61].

Then contrary to [25, 24], for all homogeneous $\mathcal{L}_2-$stable systems, a systematic method to calculate an upper estimate for such homogeneous $\mathcal{L}_p-$gain (including homogeneous ISS gain and homogeneous $\mathcal{L}_\infty-$gain) is proposed. A simplified method, which corresponds to solving the Hamilton-Jacobi inequality, is also provided, when the system is input affine. In LTI systems, where the method for homogeneous systems also applies, it enables the calculation of the classical $\mathcal{L}_p-$gain for $p > 1$.

Furthermore, similar to the traditional $\mathcal{H}_\infty-$controller design, a homogeneous $\mathcal{H}_\infty-$controller design based on a control Lyapunov function (CLF) is proposed. We propose a static state-feedback homogeneous $\mathcal{H}_\infty-$controller, which is highly dependent on the choice of the CLF, when the system is affine in control input.

Within each Chapter, some examples are included to show how such homogeneous $\mathcal{L}_p-$gain can be calculated for some simple homogeneous systems. Also a detailed analysis on the Continuous Super-Twisting Like Algorithm (CSTLA) is extended from [61].

The final aspects of this work are a discussion of the inverse of homogeneous systems, the homogeneous $\mathcal{L}_p-$gain for cascaded homogeneous system as well as the closed-loop stability for feedback homogeneous systems (homogeneous small gain theorem). Whereas both [1, 5] use the (linear or nonlinear) ISS-gain to check the closed loop stability, the homogeneous small gain theorem in this thesis adopts any homogeneous $\mathcal{L}_p-$gain (when it exists, including $\mathcal{L}_{\infty h}-$gain and also homogeneous ISS-gain) to verify closed loop stability.

In summary, in this thesis

1. Modifications of the traditional $\mathcal{L}_p-$norm and the finite-gain $\mathcal{L}_p-$stability to be suitable for homogeneous systems are proposed, defined as the homogeneous $\mathcal{L}_p-$norm and the finite-gain homogeneous $\mathcal{L}_p-$stability.

2. All continuous memoryless homogeneous Input-Output Maps are shown to have a finite and global homogeneous $\mathcal{L}_p-$gain for $p \geq 1$.

3. With the help of the state space model, all continuous asymptotically stable homogeneous systems are shown to have a finite and global homogeneous $\mathcal{L}_p-$gain for some p sufficiently large, including homogeneous $\mathcal{L}_\infty-$gain and homogeneous ISS gain.

4. When a control input is present, all continuous homogeneously stabilizable systems are shown to have finite and global homogeneous $\mathcal{L}_p-$gain with any homogeneous stabilizing controller. If the vector field is affine in the control input, a

continuous stabilizing homogeneous $\mathcal{H}_\infty-$controller based on some choice of CLF is proposed.

5. Systematic methods to calculate an upper estimate for the above mentioned homogeneous $\mathcal{L}_p-$gain are proposed under various assumptions with the help of a homogeneous partial differential inequality (PDI) (or so called dissipation inequality in [50]) and a storage function, which serves as Lyapunov function or control Lyapunov function without disturbance input. A method to calculate the homogeneous $\mathcal{L}_\infty-$gain and ISS-gain based on some homogeneous $\mathcal{L}_p-$gain is also included.

6. A homogeneous small gain theorem for feedback interconnected system is proposed together with a homogeneous $\mathcal{L}_p-$gain for cascaded homogeneous systems.

1.4 Outline

The structure of this thesis is arranged in the following:

- In Chapter 2, the traditional $\mathcal{H}_\infty-$norm of an LTI system as well as its application on robustness (i.e. RS, RP, μ synthesis) are revisited. Especially, the role of storage functions (serving as Lyapunov function when undisturbed) on the prediction of worst disturbance (as well as the construction of an optimal $\mathcal{H}_\infty-$controller) is emphasized. Meanwhile, a state-feedback method based on the HJI and storage function is proposed, to achieve a preset actual ratio of $\mathcal{L}_2-$norm of output over input less than the $\mathcal{H}_\infty-$norm of the closed loop system without monitoring the ratio.

- In Chapter 3, the definitions of the traditional $\mathcal{L}_p-$norm and $\mathcal{L}_p-$gain, which are revisited and proven to be unsuitable in the general for homogeneous systems, are modified to be suitable for arbitrary continuous homogeneous systems. Properties of this homogeneous $\mathcal{L}_p-$norm are discussed, e.g. the additive inequality. All continuous homogeneous memoryless Input-Output Maps are shown to have finite and global homogeneous $\mathcal{L}_2-$gain (homogeneous $\mathcal{H}_\infty-$norm). Furthermore, the inverse of the homogeneous Input-Output Map and the homogeneous small gain theorem (using homogeneous $\mathcal{L}_p-$gain) for interconnected systems are also included.

- In Chapter 4, after the introduction of intermediate states and a homogeneous storage function (which serves as Lyapunov function without input), all asymptotically stable continuous homogeneous systems are shown to have a finite and global homogeneous $\mathcal{L}_2-$gain for some $L-$scaled weight vector and degree (or have finite and global homogeneous $\mathcal{L}_p-$gain for p sufficiently large). Further, the methods of calculating an upper estimate of the homogeneous $\mathcal{H}_\infty-$norm by solving a PDI (or so called dissipation inequality in [50]) are presented. For the input-affine case the method consists of solving the HJI.

- In Chapter 5, it is shown that any continuous stabilizing homogeneous controller for a continuous homogeneous system leads to a finite homogeneous $\mathcal{L}_2-$gain in the closed loop system for some $L-$scaled weight vector and degree (a natural extension from Chapter 4). Furthermore, when the system is affine in control input, then similar to the $\mathcal{H}_\infty-$controller design for LTI systems, an optimal continuous homogeneous $\mathcal{H}_\infty-$controller for continuous stabilizable homogeneous systems w.r.t. a particular CLF is proposed. That is, for each CLF that satisfies the PDI (or so called dissipation inequality) for some $L-$scaled weight vector and degree, the proposed stabilizing homogeneous controller achieves the minimum of such PDI w.r.t. each state. Thus it achieves the smallest upper estimate for the homogeneous $\mathcal{L}_2-$gain for the closed loop system from input to the extended output w.r.t. the chosen CLF.

- In Chapter 6.1, as a counter example, a local $\mathcal{H}_\infty-$norm, unpublished derivation similar to that in [64, 63], on the continuous Super-Twisting like algorithm (CSTLA) is included. The CSTLA is a continuous extension of the Super-Twisting algorithm (STA) studied in [64, 63], whose model, with homogeneous degree set free to a range, includes both the linear case and the STA case. Then in Chapter 6.2, the homogeneous $\mathcal{L}_p-$gain for the CSTLA, material in [60], is included. Here the homogeneous $\mathcal{L}_p-$gain is well defined for all cases except for STA. Therefore it provides a good comparison with the results from Chapter 6.1 in the linear case. Some interesting intuitions and insights from figures are also included for the interested reader.

- In Appendix A, the method for reconstructing the storage function out of a Hamiltonian matrix for LTI systems, solution of discontinuous vector field in the sense of Filippov, some inequalities used throughout this thesis, detailed derivation of the example in Chapter 6.1, and some interesting figures for example in Chapter 6.2 are included.

2 A Review of the $\mathcal{L}_p-$stability, $\mathcal{H}_\infty-$norm and Robustness for LTI Systems

The idea to apply the $\mathcal{H}_\infty-$norm to optimize the feedback controller for uncertain plants has been first brought up in [59]. Since then, frequency domain analysis, operator and approximation theory, spectral factorization, and Youla–Kučera parametrization are developed for the $\mathcal{H}_\infty-$norm analysis [4, 41].

The $\mathcal{H}_\infty-$norm is associated with robustness. For an LTI system, where uncertainty of the state model exists, the uncertainty can be dispatched into additive or multiplicative parts (e.g. higher order dynamics or parameter variations) around a nominal model. Such uncertainty, represented as Δ, can be taken out from the nominal system by introducing extra in- and outputs. Thereafter with the small gain theorem, one can decide which degree of the uncertainty is tolerated without breaking the stability of the system, equivalently ensuring Robust Stability [54]. Furthermore, after scaling the inverse of the desired performance requirement before input or after output, Robust Performance can also be guaranteed for the LTI system under the worst case uncertainty [62]. Moreover, when the uncertainty is structured, then using the structured singular value, which involves $D-$scaling around the nominal plant, can improve the accuracy of the uncertainty allowance [65].

With the state space model, the mini-max design problem can be solved for the optimal linear feedback controller ($\mathcal{H}_\infty-$controller) in the sense that it minimizes the worst $\mathcal{L}_2-$gain from input to output with the help of a partial differential inequality (PDI, or so called dissipation inequality in [50]). Detailed analysis can be found in [4]. Similarly, a $\mathcal{H}_\infty-$observer as well as a combination of both $\mathcal{H}_\infty-$controller and $\mathcal{H}_\infty-$observer can be derived from the corresponding HJI [4]. Combing such $\mathcal{H}_\infty-$controller or $\mathcal{H}_\infty-$observer design with the structured uncertainty, the $\mu-$synthesis is developed, which can further increase the uncertainty allowance and optimize the robust performance of the $\mathcal{H}_\infty-$controller and $\mathcal{H}_\infty-$observer design.

In this Chapter, we shall revisit some basic concepts on the traditional $\mathcal{H}_\infty-$norm (also $\mathcal{L}_p-$gain analysis and synthesis methods) and the robustness concept derived from it. The traditional $\mathcal{H}_\infty-$norm when applied on a nonlinear system usually gives local results for small signal, therefore most analysis of robustness of nonlinear systems guarantee local robustness around an equilibrium [29]. Yet with the introduction of the global homogeneous $\mathcal{H}_\infty-$norm, such concept or approaches could be directly migrated to continuous homogeneous system, where some results similar to the LTI system can be derived.

2.1 The $q-$norm, $\mathcal{L}_p-$stability and $\mathcal{H}_\infty-$norm

First of all, a norm for a linear space S_L should satisfy the following properties

Definition 2.1 (Norms on linear spaces [15])**.** *A function* $\|\cdot\| : S_L \to \mathbb{R}_{\geq 0}$ *is a norm on* S_L *if it satisfies the following properties:*

positive definiteness: $u \in S_L$ *and* $u \neq 0 \Rightarrow \|u\| > 0$.

homogeneity: $\|\kappa u\| = |\kappa| \, \|u\|\,, \quad \forall\, \kappa \in \mathbb{R}, \forall\, u \in S_L$.

triangle inequality: $\|u_1 + u_2\| \leq \|u_1\| + \|u_2\|\,, \quad \forall\, u_1, u_2 \in S_L$.

2.1.1 Vector Norm: The $q-$norm

Among all norms for vector spaces, the vector $q-$norm is adopted in this thesis, namely

Definition 2.2 (Vector norm: $q-$norm [29, 50])**.** *For $q \geq 1$, the $q-$norm of a vector $u \in \mathbb{R}^m$ is defined as*

$$\|u\|_q \triangleq \left(\sum_{i=1}^{m} |u_i|^q \right)^{\frac{1}{q}} . \tag{2.1}$$

For the case of $q = 2$, it can also be written as

$$\|u\|_2 = \sqrt{u^\top u} \triangleq \sqrt{\langle u, u \rangle} ,$$

which is associated with the concept of inner product of two vectors. When $q = \infty$, the $q-$norm is further defined as

$$\|u\|_\infty \triangleq \max_i |u_i| \ , i = 1, \cdots, m .$$

All vector norms give a measure of the size of the vector in some sense. In particular, the $2-$norm of a vector $\|u\|_2$, which is also called Euclidean norm, gives the Euclidean distance between the point u and the origin.

Remark 2.1 (Equivalence between all vector norms [15])**.** *All norms on a linear space are equivalent [15]. For the $q-$norm, this means that for any $q_1, q_2 \geq 1$, there exist two positive numbers c_l and c_u, s.t.*

$$c_l \|u\|_{q_1} \leq \|u\|_{q_2} \leq c_u \|u\|_{q_1} , \quad \forall\, u \in \mathbb{R}^m .$$

Remark 2.2 (Inequality between $q-$norm)**.** *From* (A.5)*, we have for $q \geq 1$, $\|u\|_q \leq \|u\|_1$. When $0 < q < 1$, the operator $\|\cdot\|_q$ can be defined similar to* (2.1)*, yet it is no longer a norm, since it violates the triangle inequality in Definition 2.1. For such operator $\|\cdot\|_q$ defined for all $q > 0$, we have $\|u\|_q \geq \|u\|_1$ when $0 < q < 1$. In general, we have $\|u\|_{q_2} \geq \|u\|_{q_1}$ for $q_1 > q_2 > 0$.*

2.1.2 Signal Norm: The $\mathcal{L}_p-$norm

Before defining the $\mathcal{L}_p-$norm for a signal u, in order for the norm to exist, we first introduce the $\mathcal{L}_p-$space as

Definition 2.3 (The $\mathcal{L}_p-$space [50]). *For $p \geq 1$ the set $\mathcal{L}_p$ consists of all signals $u : [0, \infty) \to \mathbb{R}$, which are measurable and satisfy $\int_0^\infty |u(t)|^p \, \mathrm{d}t < \infty$.*
For multivariable signals $u : [0, \infty) \to \mathbb{R}^n$ the signal space $\mathcal{L}_p^n$ consists of all measurable signals such that

$$\int_0^\infty \|u(t)\|^p \, \mathrm{d}t < \infty \,,$$

where $\|\cdot\|$ is any norm in $\mathbb{R}^n$. The $\mathcal{L}_p-$spaces are Banach spaces (i.e. complete normed linear spaces)[50].

In the rest of the thesis, we omit the superscript of n in $\mathcal{L}_p^n$ when there is no ambiguity. Then a norm for signal $u \in \mathcal{L}_p$ can be similarly defined as Definition 2.1 by

Definition 2.4 (Signal norm: $\mathcal{L}_p-$norm [29, 50]). *For $p \geq 1$, the $\mathcal{L}_p-$norm of a signal $u \in \mathcal{L}_p$ is defined as*

$$\|u\|_{\mathcal{L}_p} \triangleq \left(\int_0^\infty \|u(t)\|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \,,$$

where $\|\cdot\|$ can be any vector norm. When $p = \infty$, the $\mathcal{L}_\infty-$norm is defined as

$$\|u\|_{\mathcal{L}_\infty} \triangleq \operatorname*{ess\,sup}_{t \geq 0} \|u(t)\| \,,$$

that is the supremal value of some norm of the vector $u(t)$ along all time. In this thesis, the vector $q-$norm in (2.1) is adopted inside the integral, i.e.

$$\|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \|u(t)\|_q^p \, \mathrm{d}t \right)^{\frac{1}{p}} = \left[\int_0^\infty \left(\sum_i^m |u_i(t)|^q \right)^{\frac{p}{q}} \mathrm{d}t \right]^{\frac{1}{p}} . \tag{2.2}$$

Since all vector norms are equivalent from Remark 2.1, the $\mathcal{L}_p-$norm in (2.2) exists and is finite. When $p = q$ is chosen, this becomes

$$\|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \|u(t)\|_p^p \, \mathrm{d}t \right)^{\frac{1}{p}} = \left(\int_0^\infty \sum_i^m |u_i(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} ,$$

here the summation operator and integral operator can interchange. For the case of $p = q = 2$, it can also be written as

$$\|u\|_{\mathcal{L}_2} = \left(\int_0^\infty u^\top(t)u(t)\,\mathrm{d}t\right)^{\frac{1}{2}} \triangleq \langle u, u\rangle_s^{\frac{1}{2}} \,.$$

i.e. the $\mathcal{L}_p-$norm of a measurable signal u equals the square root of the inner product of the same signal. When $p = q = \infty$, the $\mathcal{L}_\infty-$norm is defined as

$$\|u\|_{\mathcal{L}_\infty} = \operatorname*{ess\,sup}_{t\geq 0} \|u(t)\|_\infty = \max_i \operatorname*{ess\,sup}_{t\geq 0} |u_i(t)| \;, \quad i = 1, \cdots, m\,.$$

In particular, the $\mathcal{L}_2-$norm of a measurable signal $\|u\|_{\mathcal{L}_2}$ when $q = 2$ gives the measure of the energy of such signal in the engineering sense. Note that when a different q is chosen in (2.2), all the conclusions around the $\mathcal{L}_p-$norm remain valid, yet their value for the same signal u can vary. This is shown in the following example.

Example 2.1. *For the signal*

$$u(t) = \frac{1}{t + t_0}\,, \; t_0 > 0\,,$$

its $\mathcal{L}_p-$norm is

$$\|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \frac{1}{|t + t_0|^p}\mathrm{d}t\right)^{\frac{1}{p}} = \left(\int_{t_0}^\infty \frac{1}{s^p}\mathrm{d}s\right)^{\frac{1}{p}} \,,$$

with $s = t + t_0$. Since the signal is a scalar signal, all $q-$norms equal the absolute value. Thus we have

$$\|u\|_{\mathcal{L}_p} = \begin{cases} \left(\frac{s^{1-p}}{1-p}\Big|_{t_0}^\infty\right)^{\frac{1}{p}} & \text{when } p > 1 \\ \left(\ln(s)\big|_{t_0}^\infty\right)^{\frac{1}{p}} & \text{when } p = 1 \end{cases} = \begin{cases} \frac{t_0^{\frac{1-p}{p}}}{(p-1)^{\frac{1}{p}}} & \text{when } p > 1 \\ \infty & \text{when } p = 1 \end{cases}$$

For $p = \infty$, $\|u\|_{\mathcal{L}_p} = \left|\frac{1}{t_0}\right|$ by definition.

It is apparent that the $\mathcal{L}_p-$norm for the same signal u is different under different p. Further, for the same p, using different $q-$norm inside the $\mathcal{L}_p-$norm as in (2.2) gives also a different value. Yet, they are equivalent, this is shown in the following remark.

Remark 2.3 (Equivalency of $\mathcal{L}_p-$norm with different q)**.** *The values of the $\mathcal{L}_p-$norm defined by* (2.2) *with different choice of q are equivalent, since for any $q_1, q_2 \geq 1$*

$$\left(\int_0^\infty \|u(t)\|_{q_1}^p \,\mathrm{d}t\right)^{\frac{1}{p}} \leq \left(\int_0^\infty c_2^p \|u(t)\|_{q_2}^p \,\mathrm{d}t\right)^{\frac{1}{p}} = c_2 \left(\int_0^\infty \|u(t)\|_{q_2}^p \,\mathrm{d}t\right)^{\frac{1}{p}}, \quad \forall\, u \in \mathcal{L}_p\,.$$

This is clear from Remark 2.1. Similarly, we have the other side of the inequality as

$$\beta \left(\int_0^\infty \|u(t)\|_{q_2}^p \,\mathrm{d}t\right)^{\frac{1}{p}} \leq \left(\int_0^\infty \|u(t)\|_{q_1}^p \,\mathrm{d}t\right)^{\frac{1}{p}}, \quad \forall\, u \in \mathcal{L}_p\,.$$

Thus the $\mathcal{L}_p-$norm adopting different q are equivalent.

Definition 2.5 (Extended $\mathcal{L}_p-$space [50])**.** *Define truncated signal u_T of the signal u to the interval $[0, T]$ as*

$$u_T(t) = \begin{cases} u(t) & 0 \leq t \leq T \\ 0 & t > T \end{cases}$$

for each $T \in \mathbb{R}_{\geq 0}$. Then the extended $\mathcal{L}_p-$space ($\mathcal{L}_{pe}-$space in short) consists of all measurable signals u, s.t. $u_T \in \mathcal{L}_p$ for all $0 \leq T < \infty$. Naturally $\mathcal{L}_p \subset \mathcal{L}_{pe}$, since the $\mathcal{L}_p-$space is restricted to the case of $T = \infty$ and the latter is not [50]. Note that $\mathcal{L}_{pe}$ is a linear space, but not a Banach space.

Remark 2.4 (Relationship between $\mathcal{L}_p-$space [15])**.** *If a signal u belongs to the $\mathcal{L}_1-$space and the $\mathcal{L}_\infty-$space, then it also belongs to the $\mathcal{L}_p-$space for all $p \in [1, \infty]$ [15, Fact in Page 17]. Figure 2.1 from [15, Fig. II.1 in Page 17] shows the Venn diagram of relationship of $\mathcal{L}_1-$space, $\mathcal{L}_2-$space and $\mathcal{L}_\infty-$space.*

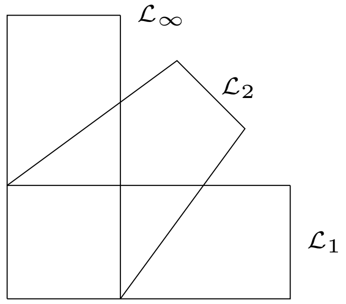

Figure 2.1: Venn diagram from [15].

Further in $\mathcal{L}_{pe}-$space, we have $\mathcal{L}_{\infty e} \subset \mathcal{L}_{pe} \subset \mathcal{L}_{1e}$ [15, Exercise 4 in Page 17]. In other

words, if a signal u belongs to the extended $\mathcal{L}_{p_1}-$space, then it also belongs to the extended $\mathcal{L}_p-$space for any $1 \leq p \leq p_1$.
An example is the step function, whose $\mathcal{L}_\infty-$norm exists, yet any $\mathcal{L}_p-$norm is unbounded. For extended space, since the $\mathcal{L}_\infty-$norm for its truncated signal exists, the step function belongs to all extended $\mathcal{L}_p-$space for any $p \geq 1$.

2.2 The Input-Output Map

A system can be represented as an Input-Output Map, or when necessary can even be generalized to Input-Output Relation [50]. In this thesis, the Input-Output Maps are adopted when the intermediate state for the system is not available for some reason.

Definition 2.6 (Input-Output Map [50])**.** *An Input-Output Map $G : y = G(u)$ is defined as a mapping from input signal $u \in \mathcal{L}_{pe}$ to an output signal $y \in \mathcal{L}_{pe}$.*

Naturally, all Input-Output Maps in this thesis are assumed to be causal and time-invariant, i.e.

Definition 2.7 (Causal Input-Output Map [50])**.** *An Input-Output Map $G : y = G(u)$ is called causal if and only if*

$$\forall\, u, v \in \mathcal{L}_{pe} \ s.t.\ u_T = v_T \Rightarrow (G(u))_T = (G(v))_T \ , \quad \forall\, T \in \mathbb{R}_{\geq 0} \, .$$

That is the output of the Input-Output Map G in any interval $[0, T]$ depends only on the input over the same interval $[0, T]$.

Definition 2.8 (Time-invariant Input-Output Map [50])**.** *An Input-Output Map G : $y = G(u)$ is called time-invariant if*

$$y = G(u) \Rightarrow \check{y} = G(\check{u}),\ where\ \check{u}(t) = u\,(t + T_d)\ , \check{y}(t) = y\,(t + T_d)\ ,$$

for all $u \in \mathcal{L}_{pe}$ and $T_d \in \mathbb{R}_{\geq 0}$. That is, for all inputs and its outputs of the Input-Output Map G, the delayed inputs produce the same outputs delayed by the same constant.

2.3 The $\mathcal{L}_p-$stability

After defining the $\mathcal{L}_p-$norm for signals as well as Input-Output Map, the ratio of the $\mathcal{L}_p-$norm of the output over the input is of interest.

2.3.1 The $\mathcal{L}_p-$stability for Input-Output Maps

With the tool of the $\mathcal{L}_p-$norm, we can introduce the concept of $\mathcal{L}_p-$stability for an Input-Output Map G, namely

Definition 2.9 ($\mathcal{L}_p-$stability [50])**.** *An Input-Output Map $G : y = G(u)$ is called $\mathcal{L}_p-$stable if*

$$u \in \mathcal{L}_p \Rightarrow y = G(u) \in \mathcal{L}_p \,.$$

It is called finite-gain $\mathcal{L}_p-$stable if there exists a finite constant $\gamma > 0$ and $\beta \geq 0$, s.t.

$$\|(G(u))_T\|_{\mathcal{L}_p} \leq \gamma \, \|u_T\|_{\mathcal{L}_p} + \beta \,, \quad \forall \, T \in \mathbb{R}_{\geq 0} \,, \forall \, u \in \mathcal{L}_{pe} \,.$$

It is clear that if G has a finite $\mathcal{L}_p-$gain then it is $\mathcal{L}_p-$stable, i.e. by limiting $u \in \mathcal{L}_p$ and letting $T \to \infty$, we have

$$\|G(u)\|_{\mathcal{L}_p} \leq \gamma \, \|u\|_{\mathcal{L}_p} + \beta \,, \quad \forall \, u \in \mathcal{L}_p \,. \tag{2.3}$$

The smallest value of $\gamma = \gamma'$ s.t. (2.3) *is satisfied is called the $\mathcal{L}_p-$gain of the Input-Output Map G.*

The $\mathcal{L}_p-$stability for general nonlinear systems might have a nonlinear gain, i.e. the linear gain of $\gamma \, \|u_T\|_{\mathcal{L}_p}$ in (2.3) is replaced by a $\mathcal{K}_\infty$ function (radially unbounded strictly increasing function) $\gamma \left(\|u_T\|_{\mathcal{L}_p} \right)$ [28].

Definition 2.10 (Linear Input-Output Map)**.** *An Input-Output Map G is linear if*

$$y = G(u) \Rightarrow G(\kappa u) = \kappa y \,, \quad \forall \, u \in \mathcal{L}_{pe} \,, \forall \, \kappa \in \mathbb{R} \,.$$

This definition is solely from the input-output perspective, i.e. the process inside the Input-Output Map G is omitted. Then we have

Proposition 2.1 (Finite-gain $\mathcal{L}_p$ with zero bias [50])**.** *If a linear Input-Output Map G is $\mathcal{L}_p$−stable, then $\beta = 0$ in* (2.3)*.*

Proof. If an Input-Output Map G is linear, then a linearly scaled input κu $(\kappa \neq 0)$ results in an output that is scaled by the same constant as κy, thus (2.3) becomes

$$\|\kappa y\|_{\mathcal{L}_p} \leq \gamma \|\kappa u\|_{\mathcal{L}_p} + \beta \,, \quad \forall\, u \in \mathcal{L}_p \,,$$

which is

$$\|y\|_{\mathcal{L}_p} \leq \gamma \,\|u\|_{\mathcal{L}_p} + \beta/\kappa, \quad \forall\, u \in \mathcal{L}_p \,. \tag{2.4}$$

From (2.3), the inequality (2.4) is satisfied for any $\kappa \neq 0$, we conclude $\beta = 0$ for a linear Input-Output Map. □

2.3.2 The $\mathcal{L}_p$−stability for State Space Models

If the Input-Output Map G can be described by a state space model (SSM) Σ, e.g.

$$\Sigma : \begin{cases} \dot{x}(t) = f\left(x(t), u(t)\right) \,, \\ y(t) = h\left(x(t), u(t)\right) \,, \end{cases} \tag{2.5}$$

with the intermediate state $x(t) \in \mathbb{R}^n$, then it is necessary to include the initial value of the state $x_0 = x(0)$ in Definition 2.9 for $\mathcal{L}_p$−stability. In the rest of the thesis, the dependency on time in SSM might be dropped without ambiguity.

Definition 2.11 ($\mathcal{L}_p$−stability for the state space model [50])**.** *The system* Σ (2.5) *is finite-gain $\mathcal{L}_p$−stable if there exists a nonnegative constant γ, and for each initial value x_0 there exists a constant $\beta(x_0)$ s.t.*

$$\|y\|_{\mathcal{L}_p} \leq \gamma \,\|u\|_{\mathcal{L}_p} + \beta(x_0) \,, \quad \forall\, u \in \mathcal{L}_p \,. \tag{2.6}$$

The smallest value of $\gamma = \gamma'$ *s.t.* (2.6) *is satisfied is called the* $\mathcal{L}_p-$*gain of the system* Σ *of* (2.5)*.*

Note that from definition of $\mathcal{L}_p-$stability for state space models, γ should be independent of the initial state. Only the term β depends on the initial state.

Remark 2.5 (Properties of $\beta(x_0)$ for linear SSM)**.** *The SSM* Σ (2.5) *is called linear if its vector field and output function are linear functions, i.e.*

$$f(\kappa x, \kappa u) = \kappa f(x,u)\,, \quad h(\kappa x, \kappa u) = \kappa h(x,u)\,, \quad \forall\,\kappa \in \mathbb{R}\,, \forall\,x \in \mathbb{R}^n\,, \forall\,u \in \mathbb{R}^m\,.$$

In the rest of the thesis, $\forall\,x \in \mathbb{R}^n\,, \forall\,u \in \mathbb{R}^m$ *is denoted as* $\forall\,(x,u) \in \mathbb{R}^{n+m}$*. Under such case, the linearly scaled input* κu *together with linearly scaled initial value* κx_0 *result in linearly scaled trajectory* κx *as well as output* κy *for all inputs and initial values. Note that this does not stand if the initial value is not scaled by the same constant* κ*, unless* $x_0 = 0$*. Then with similar reasoning from Proposition 2.1, It is easy to show that, if SSM* Σ (2.5) *is linear and finite-gain* $\mathcal{L}_p-$*stable, then* $\beta\,(\kappa x_0) = \kappa\beta(x_0)$ *for all* κ *(i.e.* β *is a linear function). Especially when* $\kappa = 0$*,* $\beta(0) = 0$*.*

It is also possible to build a linear Input-Output Map with a SSM, whose vector field and output function are not linear, but homogeneous. This shall be brought up in Example 4.1. Under such case, $\beta(x_0)$ is a homogeneous function of degree 1, and we still have $\beta(0) = 0$ for Remark 2.5.
In this Chapter, we revisit the classical concept of finite-gain $\mathcal{L}_p-$stability, thus we assume that the vector field and output function of SSM Σ (2.5) are linear.

Moreover, the Input-Output Map G obtained from the LTI SSM (2.5) is linear only for $x_0 = 0$. The corresponding fact for LTI systems $\dot{x} = Ax + Bu, y = Cx + Du$ is well-known (A, B, C, D being constant matrices of appropriate dimension), since the output of the Input-Output Map G: $y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-s)}Bu(s)\mathrm{d}s + Du(t)$ is linear in input $u(\cdot)$ only when $x_0 = 0$.

2.3.3 The $\mathcal{H}_\infty-$norm

The $\mathcal{H}_\infty-$norm of an LTI system can be interpreted either in frequency domain (maximum amplitude of square root of transfer function times its complex conjugate) or as the maximum $\mathcal{L}_2-$gain from input to output in the time domain [4]. As discussed in Remark 2.5, when the state starts from origin, we have $\beta(0) = 0$, which could spare us the problem of dealing with the initial value related constant term. Therefore the $\mathcal{H}_\infty-$norm can be defined from (2.6) in a fractional form with the state starting from the origin.
As an extension of Definition 2.9 and Definition 2.11, the $\mathcal{H}_\infty-$norm is defined as

Definition 2.12 ($\mathcal{H}_\infty-$norm [29, 50])**.** *The $\mathcal{H}_\infty-$norm for an Input-Output Map (a SSM) $G : y = G(u)$ (when $x_0 = 0$) is defined as*

$$\gamma' = \|G\|_\infty \triangleq \inf \left\{ \gamma \,\middle|\, \sup_{\|u\|_{\mathcal{L}_2} \neq 0, u \in \mathcal{L}_2} \frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}} \leq \gamma \right\}.$$

That is, the $\|G\|_\infty$ is the smallest γ s.t. (2.3) and (2.6) are satisfied. It is also the supremum of all ratios of the $\mathcal{L}_2-$norm of the output over the input for the Input-Output Map (SSM) when $\|u\|_{\mathcal{L}_2} \neq 0$. In nonlinear systems, such norm might be valid for a neighbourhood around the equilibrium, i.e. serves as local norm for small signals [29, 50, 42].
Therefore, minimizing the $\mathcal{H}_\infty-$norm for an LTI system suppresses the biggest ratio of the $\mathcal{L}_2-$norm of the output over the input under a worst input [65].

2.4 The Hamilton-Jacobi Inequality and Algebraic Riccati Equation

In this section, we include a method to calculate the $\mathcal{H}_\infty$ norm for an LTI SSM in the time domain. This method does not involve frequency domain analysis, which is not applicable to nonlinear systems. For an LTI SSM

$$\Sigma_l : \begin{cases} \dot{x} = Ax + Bu\,, \\ y = Cx + Du\,, \end{cases} \tag{2.7}$$

where each constant matrix A, B, C, D meets its appropriate dimensions we have

Lemma 2.1 (The $\mathcal{H}_\infty-$norm for LTI SSM from the algebraic Riccati equation [65])**.** *For the asymptotically stable system Σ_l (2.7) when $u \equiv 0$, if there exists a number $\gamma > \overline{\sigma}(D)$ and a symmetric and positive definite (p.d.) matrix P, s.t. the following algebraic Riccati equation (ARE) is satisfied*

$$P\left(A+BR^{-1}D^\top C\right)+\left(A+BR^{-1}D^\top C\right)^\top P+PBR^{-1}B^\top P+C^\top\left(I+DR^{-1}D^\top\right)C = 0\,, \tag{2.8}$$

where $R \triangleq \gamma^2 I - D^\top D$, then $\|G\|_\infty < \gamma$ from Definition 2.12.

Proof. Since the LTI system Σ_l (2.7) is assumed to be asymptotically stable, therefore there exists a p.d. symmetric and constant matrix $P = P^\top > 0$ from the converse Lyapunov Theorem [3], such that $V(x) \triangleq x^\top P x > 0, \quad \forall x \in \mathbb{R}^n \backslash \{0\}$ and its Lie derivative along the vector field of system Σ_l (2.7), i.e. $\frac{\mathrm{d}V(x)}{\mathrm{d}t} = V_x(Ax + Bu)$ is negative definite when $u \equiv 0$ and for all $x \in \mathbb{R}^n \backslash \{0\}$. Here V_x represents the partial derivative of $V(x)$ w.r.t. x, i.e. $V_x = \partial V(x)/\partial x = 2x^\top P$. We define a value function with such $V(x)$ as

$$J(V_x, x, u, \gamma) \triangleq V_x(Ax + Bu) + y^\top y - \gamma^2 u^\top u\,. \tag{2.9}$$

When there is no ambiguity, we might omit some parameter for readability, e.g. denoting as $J(V_x, x, u)$ when γ is fixed. Expanding $J(V_x, x, u)$, we have

$$\begin{aligned} J(V_x, x, u) &= 2x^\top P(Ax + Bu) + (Cx + Du)^\top (Cx + Du) - \gamma^2 u^\top u \\ &= x^\top\left(PA + A^\top P\right)x + 2x^\top PBu + x^\top C^\top Cx + u^\top D^\top Du + 2x^\top C^\top Du - \gamma^2 u^\top u\,. \end{aligned} \tag{2.10}$$

Here we replace the $D^\top D - \gamma^2 I$ with $-R$, (2.10) becomes

$$J(V_x, x, u) = x^\top\left(PA + A^\top P + C^\top C\right)x + 2x^\top\left(PB + C^\top D\right)u - u^\top Ru\,. \tag{2.11}$$

Note that R is p.d. and invertible since $\overline{\sigma}(D) < \gamma$. Here, the inequality $J(V_x, x, u) \leq 0$

leads to a partial differential inequality (PDI).

If we can maintain $J(V_x, x, u) \leq 0$, $\forall\,(x,u) \in \mathbb{R}^{n+m}$ with some constant γ, then

$$\int_0^\infty J(V_x, x, u)\,\mathrm{d}t = V\left(x(\infty)\right) - V(x_0) + \int_0^\infty \|y(t)\|_2^2\,\mathrm{d}t - \gamma^2 \int_0^\infty \|u(t)\|_2^2\,\mathrm{d}t \leq 0\,. \quad (2.12)$$

Suppose that $u \in \mathcal{L}_2$, then from the above inequality derived from $J(V_x, x, u) \leq 0$, we have

$$\|y\|_{\mathcal{L}_2}^2 - V(x_0) \leq V\left(x(\infty)\right) - V(x_0) + \|y\|_{\mathcal{L}_2}^2 \leq \gamma^2 \|u\|_{\mathcal{L}_2}^2\,,$$

from Jensens' inequality (A.6), this is equivalent to

$$\|y\|_{\mathcal{L}_2} \leq \gamma \|u\|_{\mathcal{L}_2} + \sqrt{V(x_0)}\,, \quad \forall\, u \in \mathcal{L}_2\,,$$

which indicates that if $u \in \mathcal{L}_2$ then $y = G(u) \in \mathcal{L}_2$, combined with Definition 2.9 and Definition 2.11, then the system Σ_l (2.7) is finite-gain $\mathcal{L}_2-$stable with $\beta(x_0) = \sqrt{V(x_0)}$. Thus guaranteeing $J(V_x, x, u) \leq 0$, $\forall\,(x,u) \in \mathbb{R}^{n+m}$ with some finite γ indicates that the $\mathcal{L}_2-$gain of system Σ_l (2.7) is upper bounded by such γ. From now on, the integral problem $\|y\|_{\mathcal{L}_2} \leq \gamma \|u\|_{\mathcal{L}_2} + \beta(x_0)$ of signal space is transferred into a problem $J(V_x, x, u) \leq 0$ in the vector space of $(x,u) \in \mathbb{R}^{n+m}$.

Again inspecting inequality (2.11), since the matrix R is p.d., there exists an input $u^\star(V_x, x, \gamma)$ that maximizes the value of $J(V_x, x, u, \gamma)$ w.r.t. u, i.e.

$$J(V_x, x, u, \gamma) \leq J\left(V_x, x, u^\star(V_x, x, \gamma)\right)\,, \quad \forall\, x \in \mathbb{R}^n$$

Setting the partial derivative of $J(V_x, x, u)$ against u to zero leads to

$$0 = \frac{\partial J(V_x, x, u)}{\partial u} = \frac{V_x\left(Ax + Bu\right) + y^\top y - \gamma^2 u^\top u}{\partial u} = 2\left(PB + C^\top D\right)^\top x - 2Ru\,,$$

which has one unique solution since R is invertible, thus $u^\star(V_x, x, \gamma)$ is

$$u^\star(V_x, x, \gamma) = R^{-1}\left(PB + C^\top D\right)^\top x\,. \quad (2.13)$$

Plugging $u^\star(V_x, x, \gamma)$ back into (2.11) we have the Hamilton-Jacobi inequality (HJI)

$$J\left(V_x, x, u^\star(V_x, x)\right) = x^\top \left(PA + A^\top P + C^\top C\right) x + x^\top \left(PB + C^\top D\right) R^{-1} \left(PB + C^\top D\right)^\top x = x^\top \left(P\left(A + BR^{-1}D^\top C\right) + \left(A + BR^{-1}D^\top E\right)^\top P + PBR^{-1}B^\top P + C^\top \left(I + DR^{-1}D^\top\right) C\right) x\,.$$

In order to maintain $J(V_x, x, u, \gamma) \le J\left(V_x, x, u^\star(V_x, x, \gamma), \gamma\right) \le 0$, $\forall (x, u) \in \mathbb{R}^{n+m}$, we need to satisfy the following algebraic Riccati inequality (ARI)

$$P\left(A + BR^{-1}D^\top C\right) + \left(A + BR^{-1}D^\top E\right)^\top P + PBR^{-1}B^\top P + C^\top \left(I + DR^{-1}D^\top\right) C \le 0\,. \tag{2.14}$$

From (2.8), the inequality (2.14) is satisfied with γ and P. And for any $\check{\gamma} > \gamma$ the inequality (2.14) still stands, since from (2.10) all other terms in $J(V_x, x, u, \gamma)$ are the same except the last term, which satisfies $-\check{\gamma}^2 u^\top u < -\gamma^2 u^\top u$ for $u \neq 0_m$.

Therefore, when ARE (2.8) is satisfied for some γ and P, then for all $\check{\gamma} \ge \gamma$, we have $J(V_x, x, u, \check{\gamma}) < J(V_x, x, u, \gamma) \le 0$, $\forall\,(x, u) \in \mathbb{R}^{n+m}$. The system Σ_l (2.7) is finite-gain $\mathcal{L}_2-$stable from Definition 2.9 and Definition 2.11 with $\mathcal{L}_2-$gain upper bounded by such γ, i.e. $\|G\|_\infty < \gamma$. □

The above proof of Lemma 2.1 only shows half the story, i.e. for γ s.t. the ARE (2.8) is satisfied for some p.d. and symmetric P, then any $\check{\gamma} \ge \gamma$ also satisfies the ARI (2.14), apparently with the same P.
The other half of the story is that when the system is asymptotically stable, then there exists a γ', s.t. the ARE (2.8) is satisfied for some p.d. and symmetric P, and for any $\check{\gamma} < \gamma$, there does not exist any p.d. and symmetric P, s.t. the ARE (2.8) is satisfied [4]. And such value is the $\mathcal{H}_\infty-$norm, i.e. $\gamma' = \|G\|_\infty$.

2.5 Computational Solution of the Algebraic Riccati Equation

There are several methods to calculate the solution γ and P of the ARE (2.8). Here we include two well known methods, which involve only matrix calculation [65]. Thereafter, some frequently used solvers are also presented.

2.5.1 Linear Matrix Inequality for Algebraic Riccati Inequality

We can use the linear matrix inequality (LMI) to solve for γ and matrix P in ARI (2.14) , i.e. $J(V_x, x, u, \gamma) \le 0$ [65] by solving

$$\min \gamma \ge 0 \quad \text{s.t. } \exists P = P^\top > 0 \text{ and } \begin{bmatrix} A^\top P + PA & PB & C^\top \\ B^\top P & -\gamma I & D^\top \\ C & D & -\gamma I \end{bmatrix} \le 0\,. \tag{2.15}$$

All eigenvalues of the above matrix should have negative real part. Note that the matrix can be partitioned into

$$\left[\begin{array}{c|cc} A^\top P + PA & PB & C^\top \\ \hline B^\top P & -\gamma I & D^\top \\ C & D & -\gamma I \end{array}\right] \le 0\,.$$

From Schur complement [65, 54], it is equivalent to the following both matrices being negative semi-definite (n.s.d.)

$$\begin{bmatrix} -\gamma I & D^\top \\ D & -\gamma I \end{bmatrix} \le 0\,,$$

$$A^\top P + PA + \begin{bmatrix} PB & C^\top \end{bmatrix} \begin{bmatrix} -\gamma I & D^\top \\ D & -\gamma I \end{bmatrix}^{-1} \begin{bmatrix} B^\top P \\ C \end{bmatrix} \le 0\,.$$

First by simple row addition of the first matrix, we have

$$\begin{bmatrix} -\gamma I & D^\top \\ 0 & -\gamma I + \gamma^{-1} D^\top D \end{bmatrix} = \begin{bmatrix} -\gamma I & D^\top \\ 0 & -\gamma^{-1} R \end{bmatrix} \le 0\,,$$

which is always true from p.d. of R (i.e. $\overline{\sigma}(D) < \gamma$). Further the inverse of the following matrix can be rewritten as [54]

$$\begin{bmatrix} -\gamma I & D^\top \\ D & -\gamma I \end{bmatrix}^{-1} = \begin{bmatrix} -\gamma R^{-1} & -R^{-1} D^\top \\ -DR^{-1} & -\gamma^{-1} I - \gamma^{-1} D R^{-1} D^\top \end{bmatrix}\,.$$

Then the second matrix can be expanded as

$$A^\top P + PA + \begin{bmatrix} PB & C^\top \end{bmatrix} \begin{bmatrix} -\gamma R^{-1} & -R^{-1}D^\top \\ -DR^{-1} & -\gamma^{-1}I - \gamma^{-1}DR^{-1}D^\top \end{bmatrix} \begin{bmatrix} B^\top P \\ C \end{bmatrix} \leq 0\,,$$

which gives us exactly (2.14) after further expansion.

Another way is to take the LMI directly from (2.10), i.e. $J(V_x, x, u, \gamma) \leq 0$ without taking supremum w.r.t. u, i.e.

$$\min \gamma \geq 0 \quad \text{s.t. } \exists P = P^\top > 0 \text{ and } \begin{bmatrix} A^\top P + PA + C^\top C & PB + C^\top D \\ B^\top P + D^\top C & D^\top D - \gamma^2 I \end{bmatrix} \leq 0\,. \tag{2.16}$$

This can be verified by directly multiplying two vectors before and after, i.e.

$$\begin{bmatrix} x^\top & u^\top \end{bmatrix} \begin{bmatrix} A^\top P + PA + C^\top C & PB + C^\top D \\ B^\top P + D^\top C & D^\top D - \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0\,,$$

which is equivalent to (2.10), the dimension of the LMI in (2.16) is actually smaller than that in (2.15).

2.5.2 Hamiltonian Matrix for Algebraic Riccati Equation

Compared to solving the LMI, a simpler method of solving the ARE is to check simply the eigenvalues of the Hamiltonian matrix, associated with ARE (2.8), given by

$$H \triangleq \begin{bmatrix} A + BR^{-1}D^\top C & BR^{-1}B^\top \\ -C^\top \left(I + DR^{-1}D^\top\right) C & -\left(A + BR^{-1}D^\top C\right)^\top \end{bmatrix}\,. \tag{2.17}$$

When H has no eigenvalue on the imaginary axis, the solution of the symmetric and p.d. P can be derived from the eigenvectors of the Hamiltonian matrix H. This is shown in Appendix A.1 [65].

The advantage of the Hamiltonian matrix compared to the LMI is that the Hamiltonian matrix H is devoid of the storage function P, then only an iteration of γ is sufficient, whereas in the same time the LMI need to solve for both γ and P which satisfies the LMI (2.15) or (2.16).

Not only the value of the $\mathcal{H}_\infty$−norm γ is of interest to us, but also the matrix P (i.e. the storage function $V(x) = x^\top Px$) is of importance, since the input $u^\star(V_x, x, \gamma)$ that maximizes $J(V_x, x, u, \gamma)$ in (2.13) depends also on P. This is more apparent in controller or observer design.
Note that γ' is the infimum of all γ s.t. the Hamiltonian matrix H has no eigenvalue on the imaginary axis, then the γ' when plugged in H might push some of its eigenvalues onto the imaginary axis, under such case the magnitude of elements in P might explode to infinite. Therefore for practical reason, a sub-optimal ARE can be computed instead, i.e.

$$P\left(A+BR^{-1}D^\top C\right)+\left(A+BR^{-1}D^\top C\right)^\top P+PBR^{-1}B^\top P+C^\top\left(I+DR^{-1}D^\top\right)C+\epsilon I=0\,, \tag{2.18}$$

where ϵ is a tiny number, e.g. 10^{-7} is enough for approximate computation of γ', then plugging this γ' back to the ARE (2.8), the matrix P solved from Appendix A.1 is bounded. Now the Hamiltonian matrix is

$$H \triangleq \begin{bmatrix} A + BR^{-1}D^\top C & BR^{-1}B^\top \\ -C^\top\left(I + DR^{-1}D^\top\right)C - \epsilon I & -\left(A + BR^{-1}D^\top C\right)^\top \end{bmatrix}.$$

Or the γ' for ARE (2.8) can be first derived, then for the same ARE (2.8) plug in the $\gamma = \epsilon + \gamma'$ and solve for P as in Appendix A.1.

2.5.3 Matlab© Solver

In MATLAB©, ARE (2.8) can be solved by function `icare`, which is available after 2019a version. Yet during development of [62], its predecessor function `care` fails when the Hamiltonian matrix persistently succeeds to give correct results.

The function `mincx` in MATLAB©'s robust control toolbox can solve the above LMI. Yet, the matrix P need to be solved during LMI verification for each γ. This is computationally inferior than the Hamiltonian method in (2.17), whose matrix P can be recovered as shown in A.1 after finding γ' first.

Therefore, we recommend checking the eigenvalue of Hamiltonian matrix H and build the P as shown in A.1, the Hamiltonian matrix H has no unknown P in it, so the speed of convergence on γ is fast. The only function needed in this approach is `eig` in MATLAB©, which provides higher precision than the `icare`, `care` and `mincx`. The building of P afterwards according to A.1 is also more precise.

A frequency domain solver `norm` can also be used to derive the $\mathcal{H}_\infty-$norm for an LTI SSM.

2.6 Upper estimates for the $\mathcal{L}_p-$gain for Linear Systems

One such upper estimate of the $\mathcal{L}_p-$gain for linear systems is provided in [15, Theorem 22, Page 113], which involves the state-transition matrix of linear systems. For example, if the state-transition matrix for a system $y = G(u)$ is $\Pi(t, s)$, then the output can be derived by the convolution of the state-transition matrix and input as

$$y(t) = \int_{-\infty}^{\infty} \Pi(t, s)u(s)\,\mathrm{d}s\,, \quad \forall\, t \in \mathbb{R}\,.$$

For such system to be causal, we must have $\Pi(t, s) = 0$ when $t < s$. And when $u(s) = \delta\,(s - \lambda)\,, \lambda \geq 0$, it is clear that $y(t) = \Pi(t, \lambda)$ is also the impulse response of output from time λ for such linear system. Then suppose there exist two finite positive numbers as

$$\begin{aligned}
&\int_{-\infty}^{\infty} \|\Pi(t, s)\|\;\mathrm{d}s \leq c_\infty < \infty\,, &&\forall\, t \in \mathbb{R}\,,\\
&\int_{-\infty}^{\infty} \|\Pi(t, s)\|\;\mathrm{d}t \leq \beta < \infty\,, &&\forall\, s \in \mathbb{R}\,,
\end{aligned}$$

where here the $\|\cdot\|$ is any matrix norm [15]. Then for any fixed p, the $\mathcal{L}_p-$gain defined in (2.3) is upper bounded by $\gamma < \beta^{\frac{1}{p}} c_\infty^{\frac{p-1}{p}}$. For LTI systems, such state-transition matrix can be written as $\Pi(t, s) = \Pi\,(t - s)$, thus $\beta = c_\infty$, and [15] predicts that all $\mathcal{L}_p-$gains for such LTI systems are upper bounded by the $\mathcal{L}_1-$norm of the state-transition matrix, i.e.

$$\gamma < \int_{-\infty}^{\infty} \|\Pi(t)\|\;\mathrm{d}t\,. \tag{2.19}$$

Such upper estimate of the $\mathcal{L}_p-$gain is valid for all $p \geq 1$, in Figure 2.2 of the next example, such upper estimate is a pretty good one for a linear system.

Another upper estimate for the $\mathcal{L}_p-$gain is provided in [29, Theorem 5.1] for time-varying **exponentially stable** nonlinear systems, with Lipschitz-continuous vector field and continuous output. For example, if the nonlinear system

$$\dot{x} = f(t, x, u)\,, \quad y = h(t, x, u)$$

is exponentially stable at equilibrium $x = 0$ when $u \equiv 0$, there exists a Lyapunov function $V(t, x)$, s.t. for all $x \in \mathbb{R}^n$

$$\begin{aligned} \beta \left\|x\right\|^2 \le V(t,x) \le c_2 \left\|x\right\|^2 \,, \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) \le -c_3 \left\|x\right\|^2 \,, \\ \left\|\frac{\partial V}{\partial x}\right\| \le c_4 \left\|x\right\| \,. \end{aligned} \tag{2.20}$$

And further for all $(x, u) \in \mathbb{R}^{n+m}$

$$\begin{aligned} \left\|f(t,x,u) - f(t,x,0)\right\| \le L \left\|u\right\| \,, \\ \left\|h(t,x,u)\right\| \le \eta_1 \left\|x\right\| + \eta_2 \left\|u\right\| \,, \end{aligned}$$

then for all $p \ge 1$ the $\mathcal{L}_p-$gain is upper bounded by

$$\gamma \le \eta_2 + \frac{\eta_1 c_2 c_4 L}{\beta c_3}\,. \tag{2.21}$$

Note that, similar to the upper estimate (2.19), the upper estimate (2.21) is valid for all $p \ge 1$. Yet, unlike the optimal quadratic Lyapunov function solvable from a PDE (2.8), the Lyapunov function in (2.20) does not have a clear preference for a smaller upper estimate in (2.21).

Example 2.2. *Take an example of the state space model as follows*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -k & 1 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} u = Ax + Bu\,, \\ y &= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} x = Cx\,, \end{aligned}$$

when $k \in [1.1, 2]$, the state matrix remains Hurwitz.
Figure 2.2 shows the $\mathcal{L}_2-$gain by using `norm` *in* MATLAB© *that provides the $\mathcal{H}_\infty-$norm*

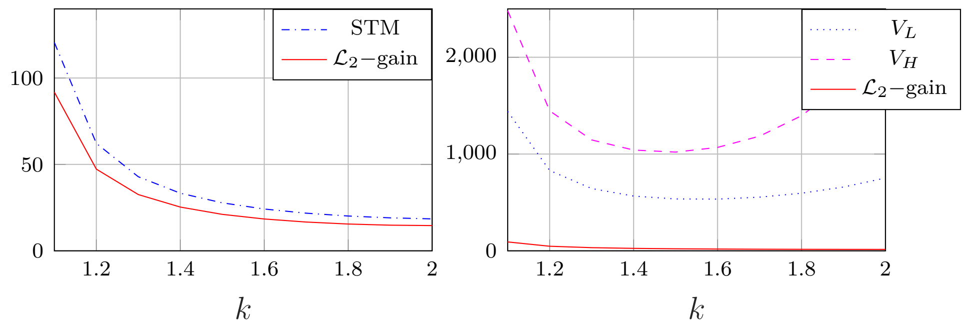

Figure 2.2: Comparison between upper estimates of $\mathcal{L}_p-$gain for Example 2.2.

of the above system. In the left sub-figure, the comparison between $\mathcal{L}_2-$gain and the upper estimate for all $\mathcal{L}_p-$gain from (2.19) *(denoted as STM, from state transition matrix) are plotted, where the state transition matrix for the above system is*

$$\Pi(t) = \begin{cases} Ce^{At}B\,, & t \geq 0 \\ 0\,, & t < 0 \end{cases}.$$

We use the matrix $2-$norm in (2.19)*. On the other hand, the right sub-figure shows two other upper estimates from* (2.21) *with two different choices of Lyapunov function $V(x)$. The one denoted as V_L adopts the Lyapunov function $V(x) = x^\top Px$ from the solution of the following Lyapunov equation*

$$PA + A^\top P = -I\,.$$

In MATLAB© this is solved with **`lyap`** *function. And the other V_H adopts the quadratic Lyapunov function from the solution of the PDE* (2.8) *with γ being the $\mathcal{L}_2-$gain from the system. It is clear that the choice of Lyapunov function affects the value of upper estimate, yet both provides bigger upper estimates than* (2.19)*.*

2.7 Relationship Between the Worst Input and the Storage Function

In the proof of Lemma 2.1, the state-feedback input $u^\star(V_x, x, \gamma)$ in (2.13) is shown to always maximize $J(V_x, x, u, \gamma)$ for each $x \in \mathbb{R}^n$ when the storage function $V(x) = x^\top Px$

is in quadratic form.
Among which, when the $\gamma = \gamma'$, i.e. γ takes the value of the $\mathcal{L}_2-$gain of the system, and the matrix P is the solution to the ARE (2.8) with the same γ, $u^\star(V_x, x, \gamma')$ in (2.13) serves also as the **worst input in time domain** with nonzero initial value, i.e. with this input the ratio of $\mathcal{L}_2-$norm of the output over the input equals to the $\mathcal{L}_2-$gain [65].

Definition 2.13 (Worst input in the sense of $\mathcal{L}_2-$gain)**.** *The set of **worst inputs** for SSM* (2.5) *with initial value x_0 in the sense of $\mathcal{L}_2-$gain is defined as*

$$\mathcal{U}(x_0) = \left\{ u(t, x_0) \in \mathcal{L}_2 \,\middle|\, \frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}} = \gamma' \right\}, \tag{2.22}$$

i.e. the ratio of $\mathcal{L}_2-$norm of output over the worst input for system (2.5) *equals the $\mathcal{L}_2-$gain.*

Suppose that a worst input $u^\star(x_0, t)$ is known, then $\kappa u^\star(\kappa x_0, t)$ is still the worst input for the κ-scaled initial value, since the magnitude of the output is also κ-scaled from linearity and the ratio of (4.13) is unchanged. With the choice of quadratic storage function $V(x)$, the worst input $u^\star(V_x, x, \gamma')$ from (2.13) is linear in the state. On the contrary, if we choose a **non-quadratic** storage function, then the input which maximizes the $J(V_x, x, u)$ in (2.9) is

$$u(V_x, x, \gamma) = R^{-1}\left(B^\top V_x^\top + D^\top Cx\right)$$

stops being the worst input for an LTI system for the same reason. In this case, the PDI $V_x(Ax + Bu) + y^\top y - \gamma^2 u^\top u \leq 0$ cannot be transformed into the ARI (2.14).
In the development of homogeneous $\mathcal{H}_\infty-$norm in later Chapters, the $u(V_x, x, \gamma)$ is linear in state as long as the homogeneous storage function is of degree 2, not necessarily being quadratic.

2.7.1 The $\mathcal{L}_2-$gain and the Optimal Storage Function

In previous sections, some methods to calculate the value of $\mathcal{H}_\infty-$norm of LTI systems (2.7) are shown, e.g. by solving the Hamiltonian matrix associated with the ARE. In the proof of Lemma 2.1, a state dependent worst input $u^\star(V_x, x, \gamma)$ from (2.13) that maximizes the value function $J(V_x, x, u, \gamma)$ is derived, such input using $\gamma = \gamma'$ and P as the solution of ARE (2.8) with the same γ gives the actual worst input from any initial value [65, 4]. This enables us to verify the corresponding γ, since the $\mathcal{H}_\infty-$norm (or $\mathcal{L}_2-$gain) is the supremal value of the ratio of of $\mathcal{L}_2-$norm of the output over the input from Definition 2.9 and Definition 2.11.

From [4] it is clear that any $\gamma > \gamma' = \|G\|_\infty$ serves as an upper bound of the $\mathcal{L}_2-$gain, with which value the inequality in Definition 2.9 and Definition 2.11 is still true. However, the uniqueness of γ' is that such $\mathcal{L}_2-$gain can be actually reached, e.g. through the worst input from (2.13) with the optimal γ and P. An interesting dilemma exists here: If the system is asymptotically stable, when

$\gamma < \gamma'$ There does exist no p.d. P, s.t. the ARE (2.8) is satisfied [4]. Yet some input can incur such ratio of the $\mathcal{L}_2-$norm of the output over the input equal to $\gamma < \gamma'$.

$\gamma > \gamma'$ There exists a p.d. $P(\gamma)$, s.t. the ARE (2.8) is satisfied [4]. However, since γ' is already the upper bound of all ratios of the $\mathcal{L}_2-$norm of the output over the input for all $u \in \mathcal{L}_2$, there exists no input u that can achieve the ratio of the $\mathcal{L}_2-$norm of the output over the input with $\gamma > \gamma'$. So the input $u^\star(V_x, x, \gamma)$ from (2.13) using the $P(\gamma)$ is not actually the worst input in the sense of $\mathcal{L}_2-$gain, i.e. it can not incur the ratio of the $\mathcal{L}_2-$norm of the output over the input equals to γ' let alone $\gamma > \gamma'$.

$\gamma = \gamma'$ There exists a p.d. $P(\gamma')$, s.t. the ARE (2.8) is satisfied. The worst input then is predicted by (2.13) using $P(\gamma')$, which is the worst input in the sense of $\mathcal{L}_2-$gain for any initial value.

Clearly, the optimal $P(\gamma')$ serves a special role in predicting the state-feedback worst input. The problem is: Is it possible to construct an input in state space such that we can maintain its ratio of the $\mathcal{L}_2-$norm of the output over the input equals to some predetermined $\gamma < \gamma'$?

In order to show the idea, we need to introduce two functions. The first function is the ratio of the $\mathcal{L}_2-$norm of output y, incurred by a particular input u, over the $\mathcal{L}_2-$norm of the input u, both truncated to $t \in [0, T]$ as

$$\Gamma(t) = \frac{\|y_T\|_{\mathcal{L}_2}}{\|u_T\|_{\mathcal{L}_2}} = \left| \frac{\int_0^T y^\top(t)y(t)\,\mathrm{d}t}{\int_0^T u^\top(t)u(t)\,\mathrm{d}t} \right|^{\frac{1}{2}} , \quad \text{when } u(0) \neq 0\,, T > 0\,, u \in \mathcal{L}_{pe}\,.$$

Then we can replace the truncation time constant T by t, then $\Gamma(t)$ serves as the running record of ratio of the $\mathcal{L}_2-$norm of output y over input u up to time t. Note that $\Gamma(t)$ is upper bounded by γ', which is the worst $\mathcal{L}_2-$gain achievable, when $x_0 = 0$ [50], since the $\mathcal{L}_2-$gain also applies to truncated input and output from Definition 2.9 and Definition 2.11. $\Gamma(t)$ exists for every uniformly bounded input in finite dimension. The second function is defined as

$$\zeta^2\left(V_x, t\right) = \max\left\{ \frac{V_x(x(t))\left(Ax(t) + Bu(t)\right) + y(t)^\top y(t)}{u^\top(t)u(t)}, 0 \right\} , \quad \text{when } u(t) \neq 0\,. \quad (2.23)$$

Function $\zeta(V_x, t)$ is a memoryless function of both storage function and time (it is also a function of the state, input and output, but we stress the importance of dependency on storage function). It is also the number that ensures $J\left(V_x(x(t)), x(t), u(t)\right) = 0$ when $\gamma = \zeta(V_x, t)$ for any $u(t) \neq 0$. When $\zeta(V_x, t) \neq 0$, $\gamma = \zeta(V_x, t)$ in (2.23) guarantees $J\left(V_x(x(t)), x(t), u(t)\right) = 0$, when $\zeta(V_x, t) = 0$, then $V_x(x(t))\left(Ax(t) + Bu(t)\right) + y(t)^\top y(t) < 0$, thus in both case $J\left(V_x(x(t)), x(t), u(t)\right) \leq 0$ with $\gamma = \zeta(V_x, t)$.

Note that $\zeta(V_x, t)$ does not allow $u(t) = 0$. For $\Gamma(t)$, we need only the measurable input's initial value $u(0) \neq 0$.

Remark 2.6 (Relationship between $\zeta(V_x, t)$ and $\Gamma(t)$)**.** *For LTI systems* (2.7) *whose $\mathcal{L}_2-$gain is γ', let the storage function $V(x) = x^\top P x$ adopt the matrix P as the solution of ARE* (2.8) *with $\gamma = \gamma'$. Then suppose $\zeta(V_x, t)$ takes the value of constant $\gamma < \gamma'$ along trajectory of system for time $t \in [0, T]$ (by adopting some particular $u(t)$), then from definition*

$$V_x(x(t))\left(Ax(t) + Bu(t)\right) + y(t)^\top y(t) - \gamma^2 u^\top(t)u(t) = 0\,.$$

This indicates that $J\left(V_x(x(t)), x(t), u(t)\right) = 0$ *for* $t \in [0, T]$*. From the inequality* (2.12)*, we have*

$$\int_0^T J(V_x, x, u)\,\mathrm{d}t = V(x(t)) - V(x_0) + \int_0^T \|y(t)\|_2^2\,\mathrm{d}t - \gamma^2 \int_0^T \|u(t)\|_2^2\,\mathrm{d}t = 0\,.$$

Under such assumption, the function $\Gamma(t)$ *equals*

$$\Gamma^2(t) = \frac{\|y_T\|_{\mathcal{L}_2}^2}{\|u_T\|_{\mathcal{L}_2}^2} = \frac{\gamma^2\,\|u_T\|_{\mathcal{L}_2}^2 + V(x_0) - V(x(t))}{\|u_T\|_{\mathcal{L}_2}^2} = \gamma^2 + \frac{V(x_0) - V(x(t))}{\|u_T\|_{\mathcal{L}_2}^2}\,.$$

When

$x_0 = 0$ *, it is clear that* $\Gamma(t) \leq \gamma$*, since* $V(x(t)) \geq 0$*. Furthermore, if the state trajectory is uniformly bounded (i.e.* $V(x(t)) < \infty$ *for all* $t \in [0, T]$*) and* $\|u_T\|_{\mathcal{L}_2} \to \infty$ ***or*** *if the state trajectory reaches origin periodically, then* $\Gamma(t)$ *converges to* γ *from below or reaches it from below periodically. This is shown in Example 2.3.*

$x_0 \neq 0$ *, if* $|V(x_0) - V(x(t))| < \infty$ *for* $t \in [0, T]$ *and* $\|u_T\|_{\mathcal{L}_2} \to \infty$ ***or*** $V(x_0) - V(x(t)) = 0$ *periodically, then* $\Gamma(t)$ *converges to the value of* γ *or touches it periodically. This is shown in Example 2.4.*

When $\zeta(V_x, t)$ *is not kept constant, then the relationship between* $\zeta(V_x, t)$ *and* $\Gamma(t)$ *is not clear.*

The introduction of $\zeta(V_x, t)$ provides a possibility to obtain the ratio of the $\mathcal{L}_2-$norm of output y over input u at $\gamma < \gamma'$. This is shown in Example 2.5.

The function $\Gamma(t)$ has the physical meaning of a running record of the ratio of the $\mathcal{L}_2-$norm of output y over input u up to time t when $x_0 = 0$. When $x_0 \neq 0$, the effect of initial value on the output might nullify this physical meaning. In order to retain such physical interpretation for non-zero initial value, it is important to energize the state s.t. either the initial value is periodically reached **or** state is uniformly bounded and $\|u_T\|_{\mathcal{L}_2} \to \infty$.

2.7.2 Single-Input Single-Output LTI System using Sinusoidal Worst Input

For Single-Input Single-Output (SISO) LTI systems, the bode plot may inform us about the biggest gain in amplitude for sinusoidal input, which also equals the $\mathcal{L}_2-$gain, as well as the frequency of such worst sinusoidal input. This is shown in the next example.

Example 2.3 (SISO system with frequency worst input)**.** *Take a SISO second order system as*

$$\dot{x} = \begin{bmatrix} -2 & -7 \\ 1 & -1.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \, .$$

Calculating the $\mathcal{H}_\infty-$norm from u to y using function `norm` *from* MATLAB© *(frequency domain analysis) or by the method of the Hamiltonian matrix leads to* $\gamma' = 0.7593$. *From the latter method, the optimal storage function that $\zeta(V_x, t)$ uses is*

$$V(x) = x^\top P(\gamma')x = x^\top \begin{bmatrix} 0.2847 & 0.0656 \\ 0.0656 & 1.9953 \end{bmatrix} x \, .$$

From the bode plot, the biggest gain happens at $\omega = 1.9895$ *rad/s. All simulations in this chapter adopt sampling period of* $10^{-5}s$ *using forward Euler integration.*

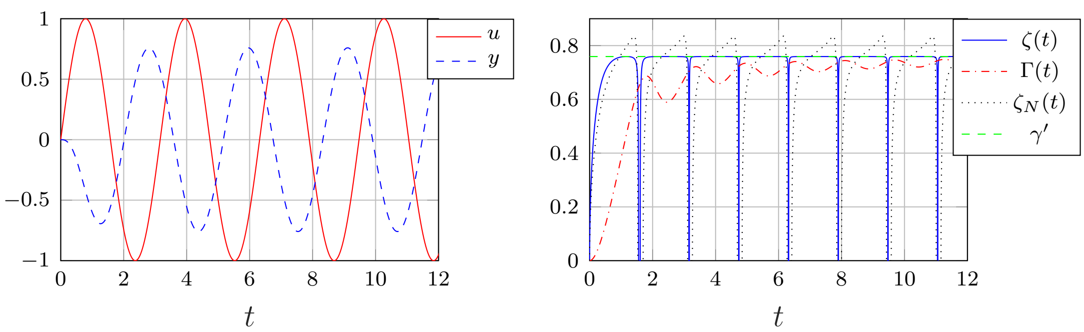

Figure 2.3: The worst sinusoidal input achieving $\gamma' = 0.7593$ for Example 2.3.

The evolution of the trajectory when $u_T = \sin(\omega t), t \in [0, 10s]$ is shown in Figure 2.3, where the newly defined function $\Gamma(t)$ and $\zeta(t) = \zeta\,(V_x, t)$ is also shown on the right sub-figure. In Figure 2.3, it is clear that with the worst input $\zeta\left(2x^\top P(\gamma'), t\right)$ stays

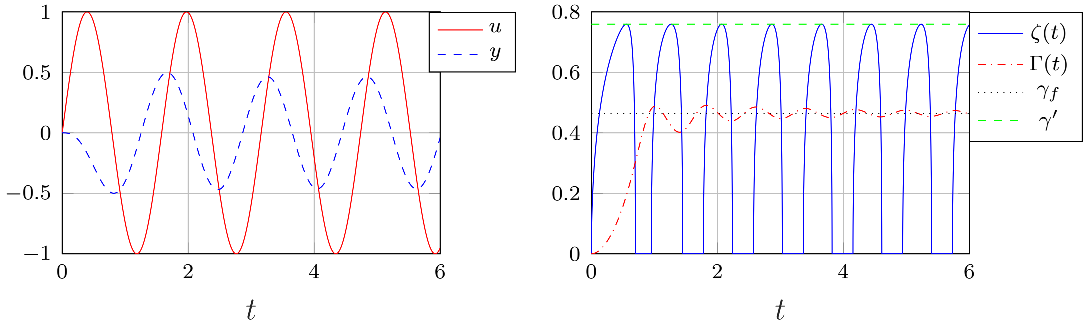

Figure 2.4: The other sinusoidal input achieving $\Gamma = 0.4636$ for Example 2.3.

almost always at γ' except for the first half period and several short time slot where u is close to zero. On the other hand, $\Gamma(t)$ is rising slowly towards γ' and is fluctuating below it. Another $\zeta_N(t)$ is added for comparison reason. It shows $\zeta\left(2x^\top P\left(1.1\gamma'\right), t\right)$, i.e. with the storage function that solves the ARE (2.8) for $\gamma = 1.1\gamma'$, which is

$$V_N(x) = x^\top P(\gamma)x = x^\top \begin{bmatrix} 0.1995 & -0.1097 \\ -0.1097 & 1.2012 \end{bmatrix} x\,, \quad \gamma = 1.1\gamma'\,.$$

Interestingly, ζ_N reaches the value of $\gamma = 1.1\gamma' = 0.8353$, which reflect some similarity to Figure 2.4 introduced later. It is clear that such non-optimal storage function does not help understanding or predicting $\Gamma(t)$ with worst sinusoidal input.

Another simulation with $u_T = \sin(2\omega t), t \in [0, 6\text{s}]$ is shown in Figure 2.4. From the function `bode` from MATLAB©, the gain of amplitude, which is also the ratio of the $\mathcal{L}_2$−norm of output over this sinusoidal input, is $\gamma_f = 0.4636$. Now $\Gamma(t)$ is fluctuating around γ_f (sometimes trespassing it). Here $\zeta(t)$ is behaving quite differently. It still reaches γ', yet more of the time it is staying below that value.

2.7.3 Multi-Input Multi-Output LTI System using State Space Worst Input

For multi-input multi-output (MIMO) systems the worst sinusoidal input is not that simple to find. Although the value of the $\mathcal{H}_\infty$−norm ($\mathcal{L}_2$−gain) is readily derivable from the singular value of the transfer function in frequency domain, the phase shift and amplitude for worst sinusoidal input in each channel is not apparent. Under such case, the worst input $u^\star(V_x, x, \gamma)$ from (2.13) can be a useful tool.

Example 2.4 (MIMO system with initial value and state space worst input)**.** *Here we extend Example 2.3 to MIMO by simply introducing another input and output,*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -2 & -7 \\ 1 & -1.5 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \,. \end{aligned} \tag{2.24}$$

Calculating the $\mathcal{H}_\infty-$norm from u to y using function **norm** *from* MATLAB© *or by the method of Lemma 2.1 leads to $\gamma' = 0.8678$. The optimal storage function is*

$$V(x) = x^\top P(\gamma')x = x^\top \begin{bmatrix} 0.3254 & 0.0766 \\ 0.0766 & 2.2807 \end{bmatrix} x \,.$$

The worst input that can achieve such $\mathcal{L}_2-$gain $\gamma' = 0.8678$ is

$$u^\star(V_x, x, \gamma') = R^{-1}(\gamma') \left(P(\gamma')B + C^\top D\right)^\top x = \begin{bmatrix} 0.4320 & 0.1017 \\ 0.1017 & 3.0282 \end{bmatrix} x \,. \tag{2.25}$$

When we use sinusoidal input for both inputs, we could also achieve similar figure as Figure 2.4. Yet we wish to show some different approaches.

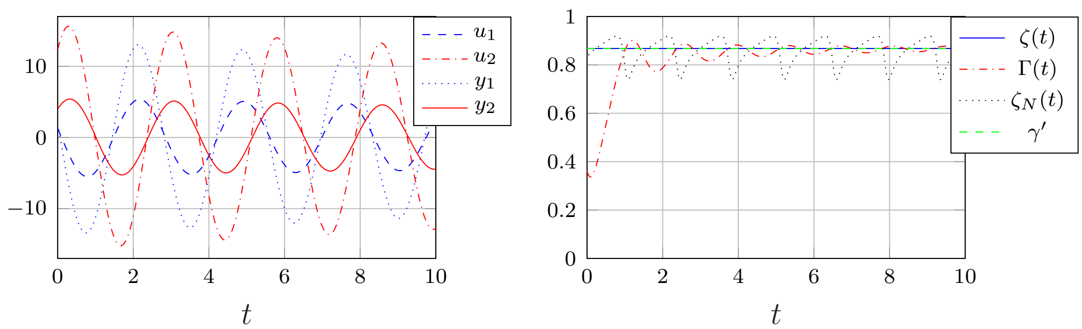

Figure 2.5: Using worst input $u^\star(V_x, x, \gamma')$ with initial value achieving $\gamma' = 0.8678$ for Example 2.4.

The simulation of the system using worst input $u^\star(V_x, x, \gamma')$ (2.25) with initial value at $x_0 = [2, 4]^\top$ is shown in Figure 2.5. It is clear that using the worst input, $\zeta(t) =$

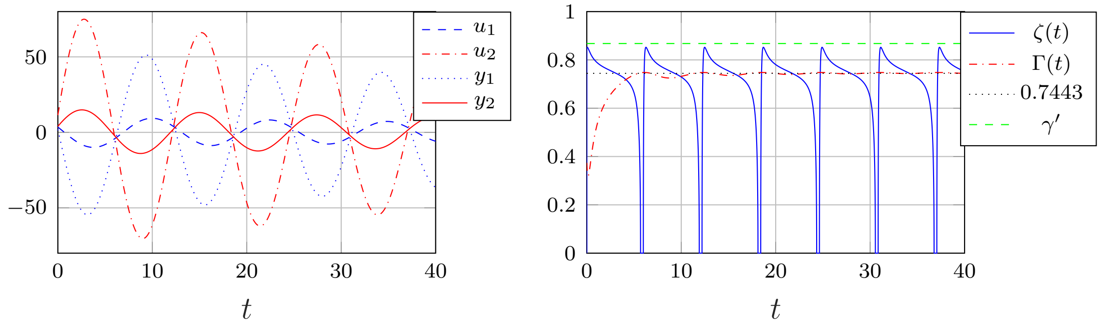

Figure 2.6: Using input $u(x)$ in (2.26) with initial value achieving $\Gamma = 0.7443$ for Example 2.4.

$\zeta\left(2x^\top P(\gamma'), t\right)$ equals to γ' for all time. Again, $\zeta_N(t) = \zeta\left(2x^\top P\left(1.1\gamma'\right), t\right)$ is plotted with storage function

$$V_N(x) = x^\top P\left(\gamma\right) x = x^\top \begin{bmatrix} 0.2291 & -0.0735 \\ -0.0735 & 1.4848 \end{bmatrix} x\,, \quad \gamma = 1.1\gamma'\,.$$

In contrast to $\zeta(t) = \gamma'$ for all time (which stands out against Figure 2.3, whose $\zeta(t)$ does not always stay at γ' with sinusoidal input), $\zeta_N(t)$ is fluctuating around γ'.

In Figure 2.5 we can see that the magnitude of y is shrinking slowly. This is because the closed loop dynamics

$$\dot{x} = Ax + Bu^\star(V_x, x, \gamma) = \left(A + BR^{-1}\left(PB + C^\top D\right)^\top\right) x$$

are stable. The eigenvalues of the closed loop matrix have small negative real part. Another contributor is the simulation discretization of the forward Euler algorithm. The shrinking magnitude of the state contributes to the function $\Gamma(t)$, trespassing the value of γ' from time to time.

In both examples, we wish to stress that with the optimal storage function $V(x)$ for γ' we might be able to better understand the general behaviour of the dynamics.

With this in mind, we develop another input by

$$\begin{aligned} u(x) &= \left(R^{-1}(1.1\gamma') \left(P(\gamma')B + C^\top D \right)^\top + 0.62 B^{-1} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right) x \\ &= \begin{bmatrix} 0.3570 & 0.7040 \\ -0.5360 & 3.1226 \end{bmatrix} x \,. \end{aligned} \tag{2.26}$$

This feedback control keeps the poles of the closed loop dynamics at $-0.0102 \pm 0.5055j$. If we use the $R^{-1}(1.1\gamma') \left(P(\gamma')B + C^\top D \right)^\top$ only, then the state might converge fast. On the other hand if we use $R^{-1}(0.9\gamma') \left(P(\gamma')B + C^\top D \right)^\top$, the closed loop system might diverge if u is not saturated.
The simulation with $u(x)$ from (2.26) is shown in Figure 2.6. Actually, it is more similar to that of Figure 2.3 than 2.4. Based on the figures above, we could say that $\zeta\,(V_x, t)$ might not be helpful in predicting $\Gamma(t)$ (which converges to 0.7443), unless the input achieves the worst gain γ'. Now the problem turns to be whether we can construct an input that achieves a gain $\gamma < \gamma'$ with the help of function $\zeta\,(V_x, t)$.

2.7.4 Achieving Preset Ratio of the $\mathcal{L}_2-$norm of the output over the input

As brought up in Remark 2.6, we would like to try maintain $J(V_x, x, u, \gamma) = 0$ for a certain $\gamma < \gamma'$ by a feedback input $u(x)$, where

$$\begin{aligned} &J(V_x(\gamma'), x, u(x), \gamma) = 2x^\top P(\gamma') \left(Ax + Bu(x) \right) + y^\top y - \gamma^2 u^\top(x) u(x) \\ &= x^\top \left(P(\gamma')A + A^\top P(\gamma') + C^\top C \right) x + 2x^\top \left(P(\gamma')B + C^\top D \right) u(x) - u^\top(x) R(\gamma) u(x) \,. \end{aligned} \tag{2.27}$$

We stress again that the storage function $P(\gamma')$ is from the optimal storage function and $R(\gamma)$ uses $\gamma < \gamma'$. It is clear that, since $J(V_x(\gamma'), x, u(x), \gamma)$ is a scalar value function, when $u \in \mathbb{R}^m$ and $m > 1$ the solution of $u(x)$ that maintains $J(V_x(\gamma'), x, u(x), \gamma) = 0$ from (2.27), when it exists, might not be unique. Under such circumstance, we need to add some restriction on $u(x)$ for easier implementation.

Example 2.5 (MIMO system starting from origin)**.** *For system* (2.24) *in Example 2.4,* *we first include the simulation from the input*

$$u_1(t) = 0.1 \sin(2.2926\, t)\ , u_2(t) = \sin(2.2926\, t + \pi/2)\ , \quad t \in [0,\ 20\,\mathrm{s}]\ .$$

Its simulation is shown in Figure 2.7.
Then after the first two periods, we replace the sinusoidal input with the input $u(x)$ *that maintains* $J(V_x(\gamma'), x, u(x), \gamma) = 0$ *from* (2.27) *for* $t \in [T_c, 20\,\mathrm{s}]\,, T_c = 4\pi/2.2926\,\mathrm{s}$. *To simplify the problem, after setting* $u_1(t) = 0, t \in [T_c\,, 20\,\mathrm{s}]$, $J(V_x(\gamma'), x, u(x), \gamma) = 0$ *could again give us two solutions for* $u_2(x)$*, i.e.*

$$u_{2s} = \frac{-x^\top \left(PB_2 + C^\top D_2\right) + \sqrt{x^\top Q x}}{D_2^\top D_2 - \gamma^2}\ , \quad u_{2b} = \frac{-x^\top \left(PB_2 + C^\top D_2\right) - \sqrt{x^\top Q x}}{D_2^\top D_2 - \gamma^2}\ ,$$
$$Q = \left(PB_2 + C^\top D_2\right)\left(PB_2 + C^\top D_2\right)^\top - \left(D_2^\top D_2 - \gamma^2\right)\left(PA + A^\top P + C^\top C\right)\ ,$$

Note that from Lemma 2.1, $\gamma' > \overline{\sigma}\{D\}$. *Yet with* $\gamma < \gamma'$*, the relationship between* γ *and* $\overline{\sigma}\{D_2\}$ *is undetermined. Since matrix* $PA + A^\top P + C^\top C \leq 0$ *(*$J(V_x(\gamma'), x, 0, \gamma) \leq 0$*), thus when* $\gamma \leq \overline{\sigma}\{D_2\}$*, matrix* $Q > 0$. *On the contrary, when* $\gamma > \overline{\sigma}\{D_2\}$*, matrix* Q *can be indefinite since it is a summation of a p.s.d. matrix and a n.s.d. matrix. When* $x^\top Q x$ *is negative, this part is discarded in the solution* u_{2s}, u_{2b} *to maintain a realizable input, which means* $u_{2s} = u_{2b}$ *under this case. A better yet more complicated method can be designed, e.g. without nullifying* u_1*.*
As mentioned above, the system needs to be energized instead of being allowed to converge for $\Gamma(t)$ *to counteract the effect of initial value. Therefore, we use the Algorithm 1.*

Algorithm 1 Procedure of simulation for Example 2.5

```
Record the y_m = max_{t∈[0,T_c]} ‖y(t)‖_∞.            ▷ Such y_m remains unchanged.
repeat
    repeat
        u_2(t) = u_2s(x)
    until |u_2(t)| < 0.1          ▷ Take the smaller input until the input is too small.
    repeat
        u_2(t) = u_2b(x)
    until |y(t)|_∞ > y_m          ▷ Take the bigger input until the output is too big.
until t > 20 s
```

Two simulations of $\gamma < \gamma'$ are shown in Figure 2.8 and 2.9. The difference is that $x^\top Q x$

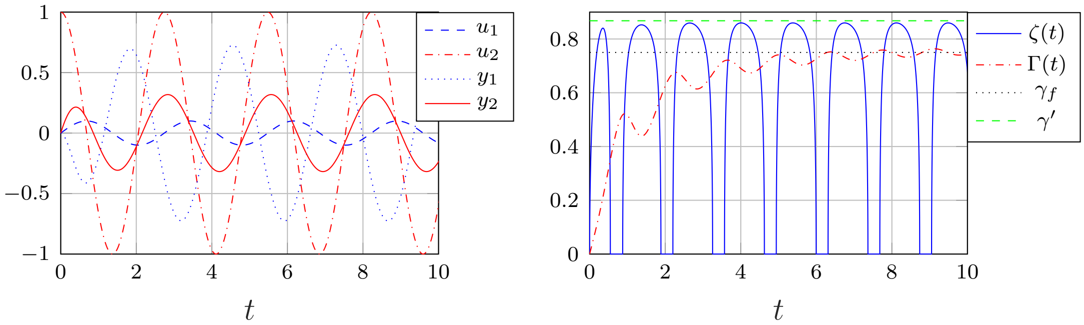

Figure 2.7: Using sinusoidal input u achieving $\Gamma = 0.7499$ for Example 2.5.

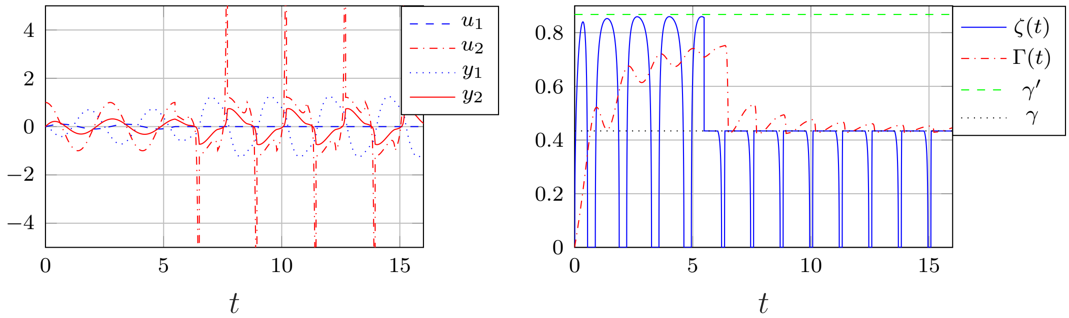

Figure 2.8: Using value function induced input u achieving $\Gamma = 0.4339$ for Example 2.5.

is sometimes negative in the simulation for Figure 2.8, and it is in practice positive (i.e. Q is still indefinite, but the trajectory of x avoids the case of $x^\top Qx < 0$) for Figure 2.9. We can see that $\Gamma(t)$ converges towards γ in both simulations. This is achieved without setting any error feedback of error $\Gamma(t) - \gamma$, but by solving for the input that maintains $J(V_x, x, u, \gamma) = 0$, which is equivalently keeping $\zeta\,(V_x, t) = \gamma$.

Another interesting observation is that this method also works with non-optimal $P(\kappa\gamma')$, $\kappa > 1$, where the cases of $\kappa = 1.1\,, 2\,, 10$ are simulated. That is, the ARE (2.8) is solved with $\gamma = \kappa\gamma'$, $\kappa > 1$. With $u(x)$ ensuring $J(V_x(\kappa\gamma'), x, u(x), \gamma) = 0$ for $\gamma < \gamma'$, $\Gamma(t)$ still converges towards γ with a bigger error than when $\kappa = 1$. The error though it exists, remains insignificant. This method might be quite tolerant with $\kappa > 1$ for LTI SSM.

We introduce this example in order to show that the value function does not only give the upper bound of all running $\mathcal{L}_2-$gains. In fact it is able to guide us through the behaviour of the dynamics. This properties will be brought up again in homogeneous systems.

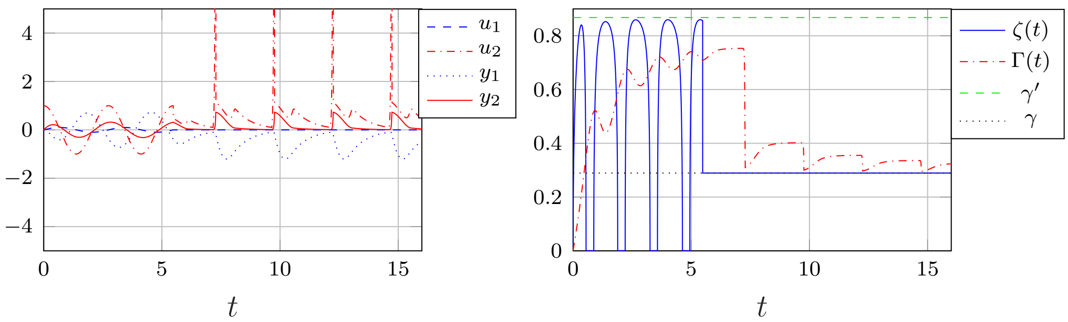

Figure 2.9: Using value function induced input u achieving $\Gamma = 0.2893$ for Example 2.5.

2.8 $\mathcal{H}_\infty$−optimal Controller Design

For a controller design problem, suppose the LTI SSM with control input u as

$$\Sigma_c : \begin{cases} \dot{x} = Ax + B_u u + B_w w \\ y = Cx + Dw\,. \end{cases}$$

Here $x \in \mathbb{R}^n$ is still the state, $u \in \mathbb{R}^m$ now the control input, $w \in \mathbb{R}^o$ the disturbance, and $y \in \mathbb{R}^r$ still the output to be minimized. Let each matrix A, B_u, B_w, C, D satisfy their respective dimension. A stabilizable controller can be designed if the pair (A, B_u) is stabilizable [50]. Further among such stabilizing controllers, an $\mathcal{H}_\infty$−optimal controller can be designed by solving a mini-max optimization problem [4]. The name is derived from the viewpoint that the disturbance input w serves as a maximizer in the sense of maximizing the value function, since the variable w is a downward parabola in the value function $J(V_x, x, u, w)$, similarly built as (2.10) by

$$J(V_x, x, u, w) \triangleq V_x\left(Ax + B_u u + B_w w\right) + y^\top y + \theta^2 u^\top u - \gamma^2 w^\top w\,, \quad V = x^\top P x\,, \tag{2.28}$$

where $\theta > 0$ is some positive constant. The extremum of (2.28) w.r.t. w is now a maximum due to the continuity of $J(V_x, x, u, w)$ as well as for each fixed x, u, $\lim_{w\to\pm\infty} J(V_x, x, u, w) \to -\infty$. And the control input u serves as minimizer, since the variable u forms an upward parabola in J, and from the continuity of $J(V_x, x, u, w)$ as well as for each fixed x, w, $\lim_{u\to\pm\infty} J(V_x, x, u, w) \to \infty$. This allows the existence of saddle point solution for J for the worst disturbance $w^\star$ and the optimal $u^\star$ [4, 41], i.e.

$$J(V_x, x, u^\star, w) \leq J(V_x, x, u, w) \leq J(V_x, x, u, w^\star)\,, \quad \forall\ (x, u, w) \in \mathbb{R}^{n+m+o}\,.$$

Here, the inequality $J\left(V_x, x, u, w^\star\right) \leq 0$ also consists a PDI [4].

Note that the value function (2.28) includes the control input u in the extended output $z^\top = \left[y^\top, \theta u^\top\right], \theta \neq 0$. Hereafter, we denote $[y^\top, \theta u^\top]^\top$ by $(y, \theta u)$ when there is no ambiguity. The controlled output z need to include the control input, since for system with infinite gain margin (i.e. the loop transfer function does not cross the negative real axis in Nyquist plot or the pole lotus does not enter the right-hand plane), for any stabilizing $u(x) = Kx$ multiplied by a large constant $\alpha > 0$, then $A - \alpha BK$ is also asymptotically stable. However, such u is ineffective in an energy sense, and should not be considered as a good design. On the other hand, including control input u in the extended output $z^\top = [y^\top, \theta u^\top], \theta \neq 0$ allows a saddle point solution for u.
Therefore using the $\mathcal{H}_\infty-$optimal controller provides a power-effective control that can suppress the $\mathcal{L}_2-$gain from the worst w to extended output z.

Similar as the analysis from Lemma 2.1, taking the partial derivative from J w.r.t u leads to the optimal feedback controller as

$$u^\star(V_x, x) = \theta^{-2} B_u^\top P(\gamma')x\,, \tag{2.29}$$

which depends on the optimal storage function corresponding to the worst gain γ'. The worst disturbance in the time domain remains

$$w^\star(V_x, x, \gamma') = R^{-1}(\gamma') \left(P(\gamma')B_w + C^\top D\right)^\top x\,.$$

Therefore, reflecting Section 2.7 where the worst state-feedback disturbance is studied, we have a natural incentive to find the smallest γ' and its corresponding optimal storage function. They clearly determine the structure of the $\mathcal{H}_\infty-$optimal controller $u^\star(V_x, x)$ through the solution of V that solves $J(V_x, x, u, w) \leq 0$ for the smallest γ'. More analysis on the topic of the mini-max algorithm based on different knowledge of the state as well as in the discrete-time case can be found in [4].

2.8.1 Solution of Partial Differential Equation for Controller Design

Similar to Section 2.5.2, the method to solve the PDE $J(V_x, x, u, w) = 0$ (2.28) by the Hamiltonian matrix is included here. The value function in (2.28) expands into

$$\begin{aligned} J(V_x, x, u, w) &= 2x^\top P\,(Ax+B_u u+B_w w)+(Cx+Dw)^\top\,(Cx+Dw)+\theta^2 u^\top u-\gamma^2 w^\top w \\ &= x^\top \left(PA + A^\top P\right) x + 2x^\top P B_u u + 2x^\top P B_w w + x^\top C^\top C x + w^\top D^\top D w \\ &+ 2x^\top C^\top D w + \theta^2 u^\top u - \gamma^2 w^\top w\,. \end{aligned} \tag{2.30}$$

It is easy to verify that

$$\begin{aligned} J(V_x, x, u^\star, w^\star) = x^\top \Big(& P\left(A + B_w R^{-1} D^\top C\right) + \left(A + B_w R^{-1} D^\top C\right)^\top P \\ & + P\left(B_w R^{-1} B_w^\top - \theta^{-2} B_u B_u^\top\right) P + C^\top \left(I + D R^{-1} D^\top\right) C\Big)\, x\,, \end{aligned}$$

where we still have $R = \gamma^2 I - D^\top D$, and the corresponding Hamiltonian matrix is

$$H = \begin{bmatrix} A + B_w R^{-1} D^\top C & B_w R^{-1} B_w^\top - \theta^{-2} B_u B_u^\top \\ -C^\top \left(I + D R^{-1} D^\top\right) C & -\left(A + B_w R^{-1} D^\top C\right)^\top \end{bmatrix}.$$

After solving for the smallest γ', s.t. the Hamiltonian matrix has no eigenvalue on the imaginary axis, then matrix P can be recovered as shown in Appendix A.1.

Similar to the LMI in (2.16), $J(V_x, x, u, w) \le 0$ can also be solved by the LMI

$$\min \gamma \ge 0 \quad \text{s.t. } \exists P = P^\top > 0 \text{ and } \begin{bmatrix} A^\top P + PA + C^\top C & PB & PM + C^\top D \\ B^\top P & \theta^2 I & 0 \\ M^\top P + D^\top C & 0 & D^\top D - \gamma^2 I \end{bmatrix} \le 0\,.$$

This can be seen by directly applying two vectors before and after the matrix, i.e.

$$\begin{bmatrix} x^\top & u^\top & w^\top \end{bmatrix} \begin{bmatrix} A^\top P + PA + C^\top C & PB & PM + C^\top D \\ B^\top P & \theta^2 I & 0 \\ M^\top P + D^\top C & 0 & D^\top D - \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ u \\ w \end{bmatrix} \le 0\,,$$

which is equivalent to $J(V_x, x, u, w) \le 0$ from (2.30).

2.9 Robustness Analysis based on the Small Gain Theorem

Aside from the stability analysis of the nominal plant, robust stability studies the stability of the uncertain dynamics. Such uncertainty can be fluctuation of voltage, failure of implementation, or false measurement. Then we are interested in the question, to which degree such uncertainty is tolerated without breaking the stability of the closed loop system, and when such uncertainty's bound is known, how to design the state-feedback controller, s.t. the margin of stability is the largest.

2.9.1 Robust Stability

The concept of robust stability is illustrated by an example.

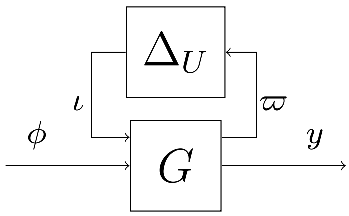

Figure 2.10: Robust stability

Example 2.6 (Robust stability under structured uncertainty)**.** *Extending Example 2.4 with structured uncertain dynamics*

$$\begin{aligned} \dot{x}_1 &= (-2 + \delta_1(t))\, x_1 - 7x_2 + u_1\,, \\ \dot{x}_2 &= (1 + \delta_1(t))\, x_1 + (-1.5 + \delta_2(t))\, x_2 + u_2\,, \\ y &= x\,, \end{aligned} \tag{2.31}$$

where $\delta_1(t), \delta_2(t)$ *represent time-varying, yet uniformly bounded uncertainties in plant with bounds* $|\delta_1(t)| < \overline{\delta}_1, |\delta_2(t)| < \overline{\delta}_2$*.*

In [62] it is shown that such structured uncertainty can always be extracted from the nominal dynamics by introduction of extra inputs and outputs, denoted here as ι and ϖ, as in Figure 2.10. For system (2.31), the plant can be written in the following SSM

$$G : \begin{cases} \dot{x} = \begin{bmatrix} -2 & -7 \\ 1 & -1.5 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \iota + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \,, \\ \varpi = x \,. \end{cases}$$

There the paramterized uncertainty form a diagonal matrix Δ_U, i.e.

$$\Delta_U : \iota = \begin{bmatrix} \delta_1(t) & 0 \\ 0 & \delta_2(t) \end{bmatrix} \varpi = \Delta_U \varpi \quad .$$

As shown in Figure 2.10, ϖ is the output of the plant to Δ_U and ι is the input from Δ_U to the plant. Calculating the $\mathcal{H}_\infty-$norm from ι to ϖ using function `norm` from MATLAB© leads to 1.1109. Therefore from the small gain theorem, if $\|G_{\iota\to\varpi}\Delta_U\|_\infty \leq \|G_{\iota\to\varpi}\|_\infty \|\Delta_U\|_\infty < 1$, then the system is robustly stable under such uncertainty. In this sense, the uncertainty $\|\Delta_U\|_\infty < 1/1.1109 = 0.9002$ is allowed.

Remark 2.7 (Value of $\|\Delta_U\|_\infty$ in Example 2.6)**.** *Since Δ_U is a time-varying Input-Output Map, we can define the $\mathcal{H}_\infty-$norm for the structured uncertainty as*

$$\|\Delta_U\|_\infty \triangleq \sup_{\|\varpi\|_2 \neq 0, \Delta_U} \frac{\|\Delta_U \varpi\|_2}{\|\varpi\|_2} \quad ,$$

where we also take the supremum of Δ_U.

Remark 2.8 (Importance of estimating the nominal plant)**.** *Note that the $\delta_1(t), \delta_2(t)$ represent the uncertain dynamics, which should be centered at the nominal dynamics. That means we should expect $\delta_1(t)$ actually being able to reach its limits $\pm\overline{\delta}_1$ as close as possible, not necessarily zero-centering in statistical sense. Such estimation impacts the performance of the $\mathcal{H}_\infty-$controller or observer design. A huge overestimation of the bound of the uncertainty might force the $\mathcal{H}_\infty-$optimal design to cope with an unrealistic worst case. Then its overall performance might be compromised to some degree.*
Therefore accurate estimations of nominal dynamics as well as the bound of the uncertainty are important for $\mathcal{H}_\infty-$controller or observer design.

Remark 2.9 (For $\mathcal{H}_\infty-$analysis, the controller K should not be taken out)**.** *Some readers might also find an other depiction of block diagram Figure 2.11 in a form that includes the plant P, controller K and uncertainty Δ_U. Yet as discussed in Section 2.8, the roles of controller u and disturbance w or uncertainty Δ_U are different. Therefore for the robust stability in the $\mathcal{H}_\infty-$analysis in Figure 2.11, the controller should be integrated into the plant to form G.*
In an LTI system with unity feedback controller design forming the loop gain PK, the small gain theorem indicates that if $\|PK\|_\infty < 1$, then the closed loop system is stable. This is a sufficient but not necessary condition for closed-loop stability.
First of all, with $\|PK\|_\infty < 1$ the gain margin exists, yet phase margin does not exist. It is more demanding for K to stabilize plant P.
Secondly, when using inequality $\|PK\|_\infty \leq \|P_{u\to y}\|_\infty \|K\|_\infty < 1$, s.t. demanding $\|K\|_\infty < 1/\|P_{u\to y}\|_\infty$ is again more restrictive for controller design from the first inequality.
Therefore, in most $\mathcal{H}_\infty-$analysis, the controller K should be integrated within the plant to form a G, and the robust stability in $\mathcal{H}_\infty-$analysis shall focus mostly on the uncertainty Δ_U and the nominal plant G.

From the small gain theorem, $\|G_{\iota\to\varpi}\Delta_U\|_\infty < 1$ is sufficient for robust stability. Yet the inequality $\|G_{\iota\to\varpi}\Delta_U\|_\infty \leq \|G_{\iota\to\varpi}\|_\infty \|\Delta_U\|_\infty$ might introduce great conservativeness, when Δ is a real diagonal matrix (structured uncertainty) instead of being a full complex matrix. This is where the structured singular value $\mu_\Delta(G)$ and the μ synthesis comes into play [62]. This shall be discussed in the next section.

2.9.2 Robust Performance

Extending the idea of robust stability, we are able to impose a performance requirement into the ΔG diagram, that is, if we wish e.g. the transfer function from input u to output y to be shaped below a transfer function ω_P. When the subscript P stands for performance, then we can put a block ω_P^{-1} after output y or before input u as shown in Figure 2.11. Here we include such additional scaling into the dynamic of G to form G'. Then Δ is

$$\Delta = \begin{bmatrix} \Delta_U & 0 \\ 0 & \Delta_P \end{bmatrix}. \tag{2.32}$$

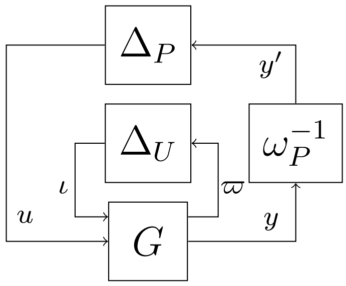

Figure 2.11: Robust performance

By using the small gain theorem to guarantee robust performance, we need to scale Δ_P and Δ_U to the same $\mathcal{H}_\infty-$norm, e.g. to the number 1.

Example 2.7 (Robust stability and performance under structured uncertainty)**.** *Extend Example 2.6. If we adopt ω_P after output y as*

$$\omega_P = \begin{bmatrix} \frac{0.1s+15}{s+6} & 0 \\ 0 & \frac{0.2s+20}{s+10} \end{bmatrix},$$

which means that by robust performance we are asking for

$$\left\|G(s)\omega_P^{-1}\right\|_\infty = \left\|\begin{bmatrix} \frac{s+6}{0.1s+15}G_{u_1\to y_1}(s) & \frac{s+6}{0.1s+15}G_{u_1\to y_2}(s) \\ \frac{s+10}{0.2s+20}G_{u_2\to y_1}(s) & \frac{s+10}{0.2s+20}G_{u_2\to y_2}(s) \end{bmatrix}\right\|_\infty < 1\,,$$

under the condition of $\|\Delta\|_\infty < 1$. Then one necessary condition is that each entry of $G(s)\omega_P^{-1}$ has $\mathcal{H}_\infty-$norm less than 1, so that we must have e.g. $G_{u_1\to y_1}(s) < \frac{0.1s+15}{s+6}$ [65]. In the nominal state, the nominal performance requirement can be fulfilled by applying root locus, pole placement or others method. Yet robust performance is much more demanding with $\|\Delta\|_\infty < 1$. We need to include the intermediate states in the dynamics

ω_P^{-1} *and extend the system into*

$$G' : \begin{cases} \dot{x} = \begin{bmatrix} -2 & -7 & 0 & 0 \\ 1 & -1.5 & 0 & 0 \\ 1 & 0 & -150 & 0 \\ 0 & 1 & 0 & -100 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \iota \\ u \end{bmatrix} \\ \begin{bmatrix} \varpi \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 10 & 0 & -1440 & 0 \\ 0 & 5 & 0 & -450 \end{bmatrix} x \end{cases}$$

Calculating the $\mathcal{H}_\infty$−norm of G' using function `norm` *from* MATLAB© *leads to* 1.5411. *However, this requirement for the $\|\Delta\|_\infty < 1/1.5411$ is not realistic, $\|\Delta_P\|_\infty$ can not be compromised for robust performance. Thus we could extract $\overline{\delta}_1, \overline{\delta}_2$ to the input matrix of ι or output matrix of ϖ, and examine to which value is the uncertainty tolerated for robust performance. The SSM is now*

$$\dot{x} = \begin{bmatrix} -2 & -7 & 0 & 0 \\ 1 & -1.5 & 0 & 0 \\ 1 & 0 & -150 & 0 \\ 0 & 1 & 0 & -100 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \iota \\ u \end{bmatrix},$$

$$\begin{bmatrix} \varpi \\ y' \end{bmatrix} = \begin{bmatrix} \overline{\delta}_1 & 0 & 0 & 0 \\ 0 & \overline{\delta}_2 & 0 & 0 \\ 10 & 0 & -1440 & 0 \\ 0 & 5 & 0 & -450 \end{bmatrix} x \,.$$

After setting $\overline{\delta}_1 = 0.553$, $\overline{\delta}_2 = 0.553$, $\|G'\|_\infty = 1$ by using function `norm` *from* MATLAB©. *Note that the allowance of $\overline{\delta}_1, \overline{\delta}_2$ is smaller than that in Example 2.6, since we are requiring robust performance and robust stability at the same time. Also, allowing $\overline{\delta}_1 \neq \overline{\delta}_2$ might give a better bound, since their impacts are different in the uncertain model.*

Contrary to Δ_U being structured, e.g. being a diagonal real matrix, Δ_P is a full complex matrix. There is no conservativeness in the robust performance requirement.

2.10 Structured Singular Value and μ Synthesis

The concept of structured singular value μ is brought up by Doyle in [16]. At the same issue of the same journal, Safonov [49] introduces a similar idea called "multivariable stability margin" k_m, which is exactly the inverse of $\mu_\Delta(G)$ [54].

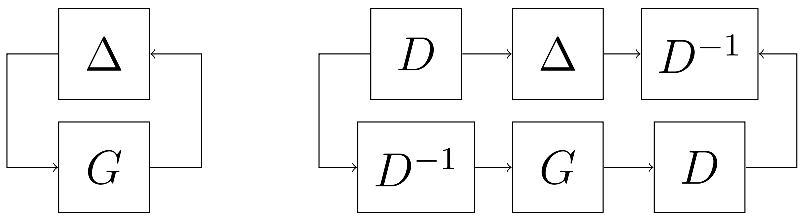

Figure 2.12: $D-$scaling in μ synthesis

2.10.1 Structured Singular Value (Conditioned $\mathcal{H}_\infty-$norm)

From the small gain theorem from left sub-figure in Figure 2.12, for closed loop stability the sufficient condition is $\|G\Delta\|_\infty < 1$. Yet, instead of asking for $\|G\|_\infty < 1/\|\Delta\|_\infty$, the singular structured value is defined as

Definition 2.14 (Structured singular value [54])**.** *Let G be a complex matrix (transfer function in Fourier domain) and let $\Delta = \operatorname{diag}\{\Delta_i\}$ denote a set of complex matrices with $\overline{\sigma}(\Delta) \leq 1$ and with a given block diagonal structure (in which some of the blocks may be repeated and some may be restricted to be real). The real non-negative function $\mu_\Delta(G)$, called the structured singular value, is defined by*

$$\mu_\Delta(G) \triangleq \frac{1}{\max\{k_m : \det(I - k_m G\Delta) = 0\}}, \text{ for structured } \Delta, \overline{\sigma}(\Delta) \leq 1. \quad (2.33)$$

If no such structured Δ exists then $\mu_\Delta(G) = 0$.

It is shown in [65] that in LTI systems $\rho(G) \leq \mu_\Delta(G) \leq \overline{\sigma}(G) = \|G\|_\infty$. When $\Delta = \operatorname{diag}\{\delta I : \delta \in \mathcal{C}\}$, then $\mu_\Delta(G) = \rho(G)$ and when Δ is a full complex matrix, then $\mu_\Delta(G) = \|G\|_\infty$ [65].

If we have $\mu_\Delta(G) < 1/\|\Delta\|_\infty$, then robust performance and robust stability can be guaranteed at the same time. When $\|\Delta\|_\infty = 1$, this is simply $\mu_\Delta(G) < 1$. The value of $\mu_\Delta(G)$ is not simple to find, thus an upper bound for it is proposed by using the so-called $D-$scaling.

2.10.2 The $D-$scaling

When there exist more than one δ in uncertainty, e.g. in Example 2.6 and 2.7, the value of $\mu_\Delta(G)$ is between the two bounds. In order to calculate $\mu_\Delta(G)$ in (2.33), a $D-$scaling tool is developed [54] for matrices D that commute with Δ, namely $D\Delta = \Delta D$. Adding D and D^{-1} between each side of G and Δ, $D^{-1}\Delta D = \Delta$ from the property of commutation, the transformed system is shown in right sub-figure of Fig 2.12. Therefore we have an upper bound of $\mu_\Delta(G)$ as

$$\mu_\Delta(G) \leq \inf_D \overline{\sigma}\left(D^{-1}GD\right) .$$

The function `mussv` from Robust Control toolbox in MATLAB© may help calculate such value. Note that since Δ is in diagonal form as in (2.32), the $D-$scaling is also in diagonal form as

$$D = \begin{bmatrix} D_U & 0 \\ 0 & I \end{bmatrix} .$$

which means the $D-$scaling that commutes with the full complex matrix Δ_P can be picked as identity matrix.

Example 2.8 ($D-$scaling)**.** *Revisiting Example 2.7, by using MATLAB© function* `mussv`*, we see that* $\overline{\delta}_1 \leq 0.8208$*,* $\overline{\delta}_2 \leq 0.8208$ *can guarantee the* $\mu_\Delta(G) < 1$*. The allowance of* $\overline{\delta}_1, \overline{\delta}_2$ *is bigger as shown in Example 2.7.*

As shown in Figure 2.13, collected during development of [62], the $D-$scaling redistributes the transfer function from u to ϖ and that from ι to y to achieve a smaller peak in both transfer functionw along the imaginary axis. This in turn reduces the singular value of $D^{-1}GD$, i.e.

$$\begin{aligned} D^{-1}GD &= \begin{bmatrix} D_U^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} G_{\iota\to\varpi} & G_{u\to\varpi} \\ G_{\iota\to y} & G_{u\to y} \end{bmatrix} \begin{bmatrix} D_U & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} D_U^{-1}G_{\iota\to\varpi}D_U & D_U^{-1}G_{u\to\varpi} \\ G_{\iota\to y}D_U & G_{u\to y} \end{bmatrix}. \end{aligned}$$

The $D-$scaling should satisfy $G_{u\to\varpi}(j\omega) < D_U(j\omega) < 1/G_{\iota\to y}(j\omega)$, under the assumption of $G_{\iota\to y}(j\omega)G_{u\to\varpi}(j\omega) < 1, \quad \forall\omega > 0$ [62]. If there exists some ω, s.t. $G_{\iota\to y}(j\omega)G_{u\to\varpi}(j\omega) \geq 1$, then robust performance can not be met. The $D-$scaling might also adjust $G_{\iota\to\varpi}$ to $D_U^{-1}G_{\iota\to\varpi}D_U$. The effect depends on the choice of $D-$scaling. The $D-$scaling in Figure 2.13 from [62] is fitted by using `fitmagfrd` function with order of 2 in MATLAB© to mimic the transfer function of $D_U(j\omega) = \alpha G_{u\to\varpi}(j\omega) + (1-\alpha)/G_{\iota\to y}(j\omega)$, with $\alpha \in (0,1)$ by a transfer function of order 2. Increasing the intermediate state of $D-$scaling might help in accelerating the D-K iteration, yet the dimension of the combined system also grows bigger, which increases the calculation expense during the ARE calculation in the $\mathcal{H}_\infty-$controller design or observer design remarkably.

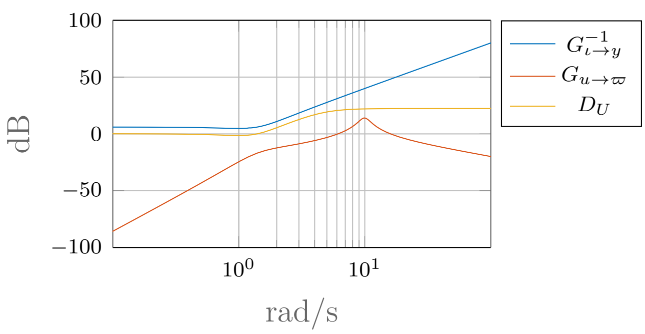

Figure 2.13: $D-$scaling between $G_{\iota\to y}^{-1}$ and $G_{u\to\varpi}$ in [62]

Remark 2.10. *The reason for* $\mu_\Delta(G) \leq \|G\|_\infty$ *is that, when* u *might be freely chosen to be the worst input as shown in proof of Lemma 2.1, the input* ι *is correlated to* ϖ*. Since for* $\iota, u \in \mathcal{L}_2$

$$\|G\|_\infty \triangleq \sup_{\|(\iota,u)\|_{\mathcal{L}_2}\neq 0} \frac{\|(y,\varpi)\|_{\mathcal{L}_2}}{\|(\iota,u)\|_{\mathcal{L}_2}}.$$

On the other hand with a structured uncertainty Δ_U, for $u \in \mathcal{L}_2$

$$\mu_\Delta(G) \leq \inf_{D_U} \sup_{\|u\|_{\mathcal{L}_2} \neq 0, \iota = \Delta_U \varpi} \frac{\left\| \left(y, D_U^{-1} \varpi \right) \right\|_{\mathcal{L}_2}}{\| (D_U \iota, u) \|_{\mathcal{L}_2}}.$$

The extra input ι is depending on the structure of Δ_U, highly correlated with output ϖ. Therefore, under such ΔG diagram from Figure 2.12, the $D-$scaling provides a less conservative estimate of robustness.

2.10.3 The μ Synthesis

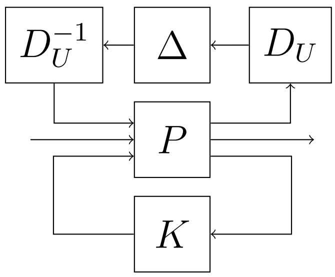

Figure 2.14: $D-$scaling in μ synthesis.

In previous sections and examples, we do not emphasize on controller or observer design with G. Instead we treat the system as if the controller or observer is already included. Here the μ synthesis or so-called D-K iteration involves an iteration for controller (K) or observer (O) design process and $D-$scaling design process. This is shown in Algorithm 2 together with Figure 2.14 [54]. The design of the controller $K = \arg\min_D \|G(P_D, K)\|_\infty$ can be chosen as the $\mathcal{H}_\infty-$optimal controller design in section 2.8. The design of observer alone can be found in [62], and the design of the observer together with the controller can be found in [4], where the solution of two AREs are needed. Note that the introduction of the $D-$scaling inevitably increases the dimension of K or O, since the SSM of the plant is enlarged with extra state from D_U and D_U^{-1} and thus the state-feedback controller in (2.29) has also a bigger dimension [62].

Algorithm 2 D-K iteration for μ synthesis

$D = I$ ▷ Initialize the D matrix
$P_D = D^{-1}PD$
$K = \arg\min_D \|G(P_D, K)\|_\infty$ ▷ Controller or Observer design
$\mu_c = \mu_\Delta\left(G(P, K)\right)$
$D = \arg\min_D \|D^{-1}G(P, K)D\|_\infty$ ▷ Find the new $D-$scaling matrix D for the $G(P, K)$
repeat
  $\mu_p = \mu_c$ ▷ Save the μ_c of last round
  $P_D = D^{-1}PD$
  $K = \arg\min_D \|G(P_D, K)\|_\infty$ ▷ Controller or Observer design
  $\mu_c = \mu_\Delta\left(G(P, K)\right)$
  $D = \arg\min_D \|D^{-1}G(P, K)D\|_\infty$ ▷ Find the new $D-$scaling matrix D for the $G(P, K)$
until $\left|\mu_c - \mu_p\right| / \mu_p < 10^{-7}$

3 Finite-gain Homogeneous $\mathcal{L}_p-$stability for Input-Output Maps

As pointed out in [29, 50], the $\mathcal{H}_\infty-$norm (the linear $\mathcal{L}_2-$gain) can be applied to nonlinear systems. Though with the traditional definition, such $\mathcal{H}_\infty-$norm is usually valid in a neighbourhood of the equilibrium for small signal input, namely being a local norm.

On the other hand, homogeneous dynamics as a special class of nonlinear systems have some interesting properties. In [25, 24], the authors apply a global and constant $\mathcal{H}_\infty-$norm on continuous homogeneous dynamics, whose weight vectors of the input and the output are all equal. Yet this case is a rather special case of all continuous homogeneous systems, therefore few further studies can be found on this aspect.

In [61], the authors noticing the deficiency of [64, 63], build a family of homogeneous $\mathcal{H}_\infty-$norms that provides a measurement of the input-output relationship globally for all continuous homogeneous systems with arbitrary in- and output weights.

In this Chapter, we introduce the concept of the homogeneous $\mathcal{L}_p-$norm for a signal and the homogeneous $\mathcal{L}_p-$gain for homogeneous Input-Output Maps, among which homogeneous $\mathcal{L}_2-$gain (homogeneous $\mathcal{H}_\infty-$norms) is emphasized. Actually the homogeneous $\mathcal{L}_p-$gain can be correlated to the homogeneous $\mathcal{L}_2-$gain by scaling the weight vector. The properties of the homogeneous $\mathcal{L}_p-$norm are studied and stability analysis for feedback interconnected systems is also included in the homogeneous small gain theorem.

The materials in this Chapter are published in [60].

3.1 Weighted Homogeneity for Input-Output Maps

First of all, we recall the definition of the weighted homogeneity for functions.

Definition 3.1 (Weighted homogeneity for functions [3]). *Fix a set of coordinates $u = (u_1, \cdots, u_m)^\top$ in $\mathbb{R}^m$ and $y = (y_1, \cdots, y_r)^\top$ in $\mathbb{R}^r$. Let $\tau_u = (\tau_{u_1}, \cdots, \tau_{u_m})^\top$, $\tau_y = (\tau_{y_1}, \cdots, \tau_{y_r})^\top$ be an m-tuple and r-tuple of positive real vectors.*

- *The one-parameter family of dilations $\nu_\kappa^{\tau_u}, \nu_\kappa^{\tau_y}$ (associated with weight τ_u, τ_y) are defined by*

$$\begin{aligned} \nu_\kappa^{\tau_u}(u) &\triangleq \left(\kappa^{\tau_{u_1}} u_1\,, \cdots, \kappa^{\tau_{u_m}} u_m\right)^\top, \quad \forall\, u \in \mathbb{R}^m\,, \forall\, \kappa > 0\,, \\ \nu_\kappa^{\tau_y}(y) &\triangleq \left(\kappa^{\tau_{y_1}} y_1\,, \cdots, \kappa^{\tau_{y_r}} y_r\right)^\top, \quad \forall\, y \in \mathbb{R}^r\,, \forall\, \kappa > 0\,. \end{aligned}$$

  The positive real numbers τ_{u_i}, τ_{y_j} are called the weight vectors of u_i, y_j, respectively.

- *A function $y_i = h_i(u) : \mathbb{R}^m \to \mathbb{R}$ is said to be τ_u- homogeneous of degree $k_{h_i} = \tau_{y_i}$ if*

$$h_i\left(\nu_\kappa^{\tau_u}(u)\right) = \kappa^{\tau_{y_i}} h_i(u)\,, \quad \forall\, u \in \mathbb{R}^m\,, \forall\, \kappa > 0\,.$$

The functions $h_i(u)$ as in Definition 3.1 remains homogeneous, when the weight vector τ_u is scaled by any positive real constant L, yet with degree Lk_{h_i} scaled by the same constant. This can be verified with simple replacement of $\kappa = \tilde{\kappa}^L$. In another word, the function $y_i = h_i(u) : \mathbb{R}^m \to \mathbb{R}$ is $L\tau_u$- homogeneous of degree Lk_{h_i}. This scaling L of weight vector and degree is frequently used in the rest of this thesis.

Then we shall define a homogeneous Input-Output Map as

Definition 3.2 (Homogeneous Input-Output Map). *An Input-Output Map G is called homogeneous with degree $d = -\tau_t \in \mathbb{R}$, if for each input u and its output y, the dilated input together with scaled time $\tilde{t}$ for all $\kappa > 0$ as*

$$\begin{aligned} u(t) &\quad\to\quad \tilde{u}(\tilde{t}) = \nu_\kappa^{\tau_u}(u(t))\,, \\ t &\quad\to\quad \tilde{t} = \kappa^{\tau_t} t\,, \end{aligned} \tag{3.1}$$

result in the corresponding dilated output with scaled time as

$$\begin{array}{lll} y(t) & \rightarrow & \tilde{y}(\tilde{t}) = \nu_\kappa^{\tau_y}(y(t))\,, \\ t & \rightarrow & \tilde{t} = \kappa^{\tau_t} t\,. \end{array} \tag{3.2}$$

Here the same weights τ_u and τ_y apply for signals u and y.

Similarly, the homogeneous degree d can be scaled by any positive real number L together with the weight vectors τ_u, τ_y, verifiable again with simple replacement of $\kappa = \tilde{\kappa}^L$.

The definition of homogeneous Input-Output Map by (3.1) and (3.2) is unprecedented. When the weighted homogeneous Input-Output Map is dynamical, most previous works define its homogeneous degree from the vector field [3], and those definitions meet Definition 3.2 [35] (to be shown in next Chapter). Therefore, the definition of the homogeneous Input-Output Map is extended as in Definition 3.2. Particularly, when $\tau_t = 0$, $\tau_{u_i} = \ell$ for all $i = 1, \cdots, m$ ($\tau_u = \ell \mathbf{1}_m$ in short) and $\tau_y = \ell \mathbf{1}_r$, then the Input-Output Map G is a linear Input-Output Map.

3.2 Incompatibility of Traditional $\mathcal{H}_\infty$−norm for Homogeneous Input-Output Maps

For motivation purpose, we shall probe how the traditional $\mathcal{H}_\infty$−norm behaves when applied to the homogeneous Input-Output Map by the next example.

Example 3.1 (Traditional $\mathcal{H}_\infty$−norm on scalar homogeneous Input-Output Map)**.** *We take a simple scalar homogeneous Input-Output Map G from input u to output y with weights $L\tau_u, L\tau_y$ and degree of $-L\tau_t$. We apply the traditional $\mathcal{H}_\infty$−norm on this Input-Output Map, i.e.*

$$\gamma^\dagger \triangleq \sup_{\|u\|_{\mathcal{L}_2} \neq 0\,, u \in \mathcal{L}_2} \frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}}\,. \tag{3.3}$$

Hereafter, we shall apply superscript † when the traditional $\mathcal{H}_\infty$−norm is used. Each particular input $\|u_e\|_{\mathcal{L}_2} \neq 0$ results in an output $y_e(\cdot) = G\,(u_e(\cdot))$, the ratio of the $\mathcal{L}_2$

norm of output over the $\mathcal{L}_2$ norm of the input is denoted as $\Gamma_e = \|y_e\|_{\mathcal{L}_2}/\|u_e\|_{\mathcal{L}_2}$. From (3.1) *and* (3.2) *in Definition 3.1, the dilated input $\tilde{u}_e$ with dilated time $\tilde{t}$ produce the output $\tilde{y}_e = G(\tilde{u}_e)$, whose $\mathcal{L}_2-$norms are*

$$\|\tilde{u}_e\|_{\mathcal{L}_p} = \left(\int_0^\infty \left|\tilde{u}_e(\tilde{t})\right|^p \mathrm{d}\tilde{t}\right)^{\frac{1}{p}} = \kappa^{\tau_u + \frac{\tau_t}{p}} \left(\int_0^\infty |u_e(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} = \kappa^{\tau_u + \frac{\tau_t}{p}} \|u_e\|_{\mathcal{L}_p} ,$$
$$\|\tilde{y}_e\|_{\mathcal{L}_p} = \kappa^{\tau_y + \frac{\tau_t}{p}} \|y_e\|_{\mathcal{L}_p} .$$

The ratio of the $\mathcal{L}_2$ norm of output $\tilde{y}_e$ over the $\mathcal{L}_2$ norm of the input $\tilde{u}_e$ is now

$$\Gamma_{\kappa e} = \frac{\|\tilde{y}_e\|_{\mathcal{L}_2}}{\|\tilde{u}_e\|_{\mathcal{L}_2}} = \kappa^{L(\tau_y - \tau_u)}\gamma_e \,. \tag{3.4}$$

When

$\tau_y = \tau_u$ *then $\Gamma_{\kappa e}$ is constant under homogeneous dilation. A continuous homogeneous system in [25, 24] restricts the weight vectors similar as $\tau_y = \ell\mathbf{1}_r, \tau_u = \ell\mathbf{1}_m$.*

$\tau_y \neq \tau_u$ *then $\Gamma_{\kappa e}$ is unbounded. This is apparent when $\tau_y > \tau_u$, $\kappa \to \infty$ or when $\tau_y < \tau_u$, $\kappa \to 0$. Thus the traditional $\mathcal{H}_\infty-$norm* (3.3)*, which is the upper bound for all $\Gamma_{\kappa e}$, is also unbounded. This means that when $\tau_y < \tau_u$, a smaller signal incurs a bigger value of Γ_e, and when $\tau_y > \tau_u$ a bigger signal does the same.*

□

Apparently, the traditional $\mathcal{H}_\infty-$norm does not provide us any beneficial information about this simple scalar homogeneous Input-Output Map G, unless the scalar Input-Output Map G is linear, namely when $\tau_y = \tau_u$.

Further, for a multi-input single-output Input-Output Map, if the weight vectors of the input $\tau_u \neq \ell\mathbf{1}_m$ for any $\ell > 0$, then $\|u\|_{\mathcal{L}_2}$ is no longer homogeneous w.r.t. the weight by choosing one of the input u_i (which should be able to impact the output y, i.e. $\|y\|_{\mathcal{L}_2} \neq 0$ when $\|u_i\|_{\mathcal{L}_2} \neq 0$, or else this input is non-relevant for this Input-Output Map), where $\tau_{u_i} \neq \tau_y$. Similar to (3.4), the gain from such input $\|u_i\|_{\mathcal{L}_2} \neq 0, \|u_j\|_{\mathcal{L}_2} = 0$ for all $j \neq i$ to output y is unbounded. Since the traditional $\mathcal{H}_\infty-$norm is taking the supremum of all inputs, where the above input is only a subset of it, the traditional

$\mathcal{H}_\infty-$norm is also unbounded.
For multi-input multi-output (MIMO) Input-Output Maps, only note that from definition $\|y\|_{\mathcal{L}_2}$ is smaller if some output channels are excluded, i.e. $\|y_i\|_{\mathcal{L}_2} \leq \|y\|_{\mathcal{L}_2}, \forall\, y \in \mathcal{L}_2$ from definition. Then choose one output y_i, (whose weight vector is different from at least one of the input u_j, and such input u_j should be able to impact this output y_i, or else this Input-Output Map can be considered as a combination of linear Input-Output Maps), then the $\mathcal{H}_\infty-$norm of such MIMO system from u to y is greater or equal to Γ_e from such $\|u_j\|_{\mathcal{L}_2} \neq 0$ to its output y_i, which is already unbounded from the analysis above.
Therefore, the traditional $\mathcal{H}_\infty-$norm (3.3) for the homogeneous Input-Output Map can be bounded only if $\tau_u = \ell \mathbf{1}_m, \tau_y = \ell \mathbf{1}_r$ for some $\ell > 0$, in accordance to the studies in [25, 24]. This is also true for the $\mathcal{L}_p-$gain from (2.3).

3.3 Homogeneous $q-$norm

During studies of homogeneous systems, some important results appear more concise under a so-called homogeneous vector norm, which is modified from the traditional norm similar to Definition 2.2 to be suitable for homogeneous systems, namely considering the weight vector [3].

Definition 3.3 (Homogeneous vector norm: $\tau-$homogeneous $q-$norm [3])**.** *A τ_u-homogeneous $q-$norm (qh$-$norm for short) for a vector $u \in \mathbb{R}^m$ is a mapping from a vector space into a non-negative real number $\mathbb{R}^m \mapsto \mathbb{R}_{\geq 0}$, i.e. for any $q \geq 1$*

$$\|u\|_{\tau_u,q} \triangleq \left(\sum_{i=1}^{m} |u_i|^{\frac{q}{\tau_{u_i}}} \right)^{\frac{1}{q}} . \tag{3.5}$$

In particular, when $q = \infty$

$$\|u\|_{\tau_u,\infty} \triangleq \max_i \left\{ |u_i|^{\frac{1}{\tau_{u_i}}} \right\} . \tag{3.6}$$

Remark 3.1. *The τ-homogeneous norm $\|\cdot\|_{\tau,q}$ is τ-homogeneous of degree 1 and it is positive definite. If $q \geq \max \tau_u$, the qh$-$norm $\|u\|_{\tau_u,q}$ is continuously differentiable* *on* $\mathbb{R}^n \setminus \{0\}$*. This is not needed in this thesis. However,* $\|\cdot\|_{\tau,q}$ *is in general not a norm in the usual sense, since it is not* $\mathbf{1}-$*homogeneous in the classical sense, i.e. with* $\tau = \mathbf{1}_n$*. When* $\tau = \mathbf{1}_n$ *the* τ*-homogeneous* $q-$*norm becomes the usual* $q-$*norm in* $\mathbb{R}^n$*. In that case, for* $q \geq \max \tau_x = 1$ *the triangle inequality in Definition 2.1 is valid, i.e.* $\|x+y\|_{\mathbf{1}_n,q} \leq \|x\|_{\mathbf{1}_n,q} + \|y\|_{\mathbf{1}_n,q}$*. In the general case, as it will be shown later in Lemma 3.3, the triangle inequality is replaced by the additive inequality* (3.10).

Remark 3.2 (Effect of weight vector scaling on $\tau-$homogeneous $q-$norm)**.** *For most studies about homogeneous systems, scaling the weight vector and degree by* $L > 0$ *has no impact on their results (e.g. stability results). Yet in this thesis such scaling does matter, e.g. after applying such scaling the* $L\tau$*-homogeneous* $q-$*norm is related to the original* $\tau-$*homogeneous* ${}^{q}/_{L}-$*norm by*

$$\|u\|_{L\tau_u,q} = \left(\sum_{i=1}^{m} |u_i|^{\frac{q}{L\tau_{u_i}}}\right)^{\frac{1}{q}} = \left(\sum_{i=1}^{m} |u_i|^{\frac{q/L}{\tau_{u_i}}}\right)^{\frac{1/L}{q/L}} = \|u\|_{\tau_u,q/L}^{1/L} . \tag{3.7}$$

for $0 \leq L \leq q$*. Usually, there is no need to relate* $L\tau-$*homogeneous* $q-$*norm back to* $\tau-$*homogeneous* $q-$*norm, so any scaling* $L > 0$ *on weight vectors is allowed. Therefore, in this thesis, it is important to fix the weight vector and degree from the beginning, in order for the results to be consistent.*

Remark 3.3 (Companion vector and relationship between $\tau-$homogeneous $q-$norm and traditional $q-$norm)**.** *First, we introduce the sign preserving power of a scalar* u *to the power of* $\vartheta \in \mathbb{R}\setminus\{0\}$ *as*

$$\lceil u \rfloor^{\vartheta} = |u|^{\vartheta} \operatorname{sign}(u) .$$

Note that the partial derivative of the sign preserving powers $\lceil u \rfloor^{\vartheta}$ *and absolute value* $|u|^{\vartheta}$ *with* $\vartheta, u \in \mathbb{R} \setminus \{0\}$ *is*

$$\frac{\partial \lceil u \rfloor^{\vartheta}}{\partial u} = \vartheta\, |u|^{\vartheta-1} , \quad \frac{\partial |u|^{\vartheta}}{\partial u} = \vartheta \lceil u \rfloor^{\vartheta-1} .$$

Apparently $u = 0$ *is a singular point for the partial derivative when* $\vartheta < 1$*, elsewhere the partial derivative is smooth. When* $\vartheta = 0$*, the sign preserving power should be* *understood as a set-valued function [53]*

$$\lceil u \rfloor^0 = \begin{cases} 1 & \textit{when } u > 0 \\ [-1, 1] & \textit{when } u = 0 \\ -1 & \textit{when } u < 0 \end{cases} .$$

Such perception is necessary to derive a solution for discontinuous systems (Solution in Filippov's sense, refer to A.2), yet in this thesis we study in most case the continuous version, i.e. $\vartheta \neq 0$.

For a vector $u \in \mathbb{R}^m$ *and a (weight) vector* $\tau_u > 0$, *define the* companion vector *of* u *as*

$$u^{\frac{1}{\tau_u}} \triangleq \left[\lceil u_1 \rfloor^{\frac{1}{\tau_{u1}}}, \cdots, \lceil u_m \rfloor^{\frac{1}{\tau_{um}}} \right]^\top .$$

Note that the mapping $u \mapsto u^{\frac{1}{\tau_x}}$ *is a homeomorphism, and when* $\min_i \{\tau_{u_i}\} \leq 1$ *it is a diffeomorphism. For homogeneous norms it is possible to associate the* $qh-$*norm* (3.5) *of* u *with the* $q-$*norm of its companion vector by*

$$\|u\|_{\tau_u,q} = \left\| u^{\frac{1}{\tau_u}} \right\|_p . \tag{3.8}$$

Note also the relation of the $qh-$*norm with the inner product of companion vector by*

$$\|u\|_{\tau_u,q}^q = \sum_{i=1}^n |u_i|^{\frac{q}{\tau_{u_i}}} = \left\langle u^{\frac{q}{2\tau_u}}, u^{\frac{q}{2\tau_u}} \right\rangle = \left\| u^{\frac{q}{2\tau_u}} \right\|_2^2 .$$

This bears a relationship with the homeomorphic coordinate change in [28, Definition 7].

Lemma 3.1 (Equivalence between $\tau-$homogeneous $q-$norms)**.** *All* $\tau-$*homogeneous* $q-$*norms, with the same weight vector* τ, *are* equivalent, *in the sense that if* $\|\cdot\|_{\tau_u,\alpha}$ *and* $\|\cdot\|_{\tau_u,\beta}$ *are two different homogeneous* $q-$*norms with* $\alpha \geq 1, \beta \geq 1$, *then there exist positive constants* c_1 *and* c_2 *such that for all* $u \in \mathbb{R}^n$

$$c_1 \|u\|_{\tau_u,\beta} \leq \|u\|_{\tau_u,\alpha} \leq c_2 \|u\|_{\tau_u,\beta} .$$

Proof. This is easily seen using relation (3.8) in Remark 3.3. Since $q-$norms are equivalent [15] as brought up in Remark 2.1, i.e. $c_1\|u\|_\beta \leq \|u\|_\alpha \leq c_2\|u\|_\beta$, then

$$c_1\|u\|_{\tau_u,\beta} = c_1\left\|u^{\frac{1}{\tau_u}}\right\|_\beta \leq \|u\|_{\tau_u,\alpha} = \left\|u^{\frac{1}{\tau_u}}\right\|_\alpha \leq c_2\left\|u^{\frac{1}{\tau_u}}\right\|_\beta = c_2\|u\|_{\tau_u,\beta}\,.$$

Interestingly, the constants c_1 and c_2 relating two homogeneous $q-$norms are the same as the ones for the $q-$norms. □

A natural question arises about the relationship between different homogeneous norms with possibly different weight vectors. The following Lemma clarifies this (and this is a special case of [5, Lemma 9]).

Lemma 3.2 (Relationship between $\tau-$homogeneous $q-$norm with different r)**.** *Consider two homogeneous norms $\|\cdot\|_{\tau_1,\alpha}$ and $\|\cdot\|_{\tau_2,\beta}$, with (possibly) different weight vectors τ_1 and τ_2. Then, there exist two $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that*

$$\alpha_1\left(\|x\|_{\tau_2,\beta}\right) \leq \|x\|_{\tau_1,\alpha} \leq \alpha_2\left(\|x\|_{\tau_2,\beta}\right)\,, \quad \forall\, x \in \mathbb{R}^n\,. \tag{3.9}$$

Proof. Since homogeneous norms are continuous, positive definite and radially unbounded, it follows from a classical result [29, Lemma 4.3] that there exist $\mathcal{K}_\infty$ functions $\mu_{1,\alpha}(\cdot)$, $\mu_{2,\alpha}(\cdot)$ and $\mu_{1,\beta}(\cdot)$, $\mu_{2,\beta}(\cdot)$ such that $\forall\, x \in \mathbb{R}^n$

$$\begin{aligned}\mu_{1,\alpha}\left(\|x\|_\beta\right) \leq \|x\|_{\tau_1,\alpha} \leq \mu_{2,\alpha}\left(\|x\|_\beta\right)\,,\\ \mu_{1,\beta}\left(\|x\|_\beta\right) \leq \|x\|_{\tau_2,\beta} \leq \mu_{2,\beta}\left(\|x\|_\beta\right)\,.\end{aligned}$$

Using the properties of $\mathcal{K}_\infty$ functions it follows that

$$\mu_{1,\alpha}\circ\mu_{2,\beta}^{-1}\left(\|x\|_{\tau_2,\beta}\right) \leq \|x\|_{\tau_1,\alpha} \leq \mu_{2,\alpha}\circ\mu_{1,\beta}^{-1}\left(\|x\|_{\tau_2,\beta}\right)\,.$$

This establishes the result. □

In particular, a relationship between $q-$norms and $\tau-$homogeneous $q-$norms is obtained from (3.9) by setting $\tau_1 = \mathbf{1}_n$ or $\tau_2 = \mathbf{1}_n$. Relation (3.7) represents one example of this general relation, when $\tau_2 = \lambda\tau_1$ or vice versa. Note that $\tau-$homogeneous $q-$norms w.r.t. different weights r are usually not equivalent (different from Remark

3.1), since in general $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are not linear functions (e.g. (3.7) in Remark 3.2).

The $\tau-$homogeneous $q-$norm of a vector in $\mathbb{R}^n$ satisfies the triangle inequality in Definition 2.1 when $\tau = \mathbf{1}_n$ and $p \geq 1$. For other cases we have instead the following additive inequality.

Lemma 3.3 (Additive inequality for homogeneous $q-$norm)**.** *The $\tau-$homogeneous $q-$norm satisfies the following inequality for two vectors $x, y \in \mathbb{R}^n$, with $q \geq 1$,*

$$\|x+y\|_{\tau,q} \leq \max\left\{1, 2^{\frac{1}{\min\tau}-\frac{1}{q}}\right\}\left(\|x\|_{\tau,q} + \|y\|_{\tau,q}\right) . \tag{3.10}$$

Proof. Using the definition, we have

$$\begin{aligned}
\|x+y\|_{\tau,q} &= \left(\sum_{i=1}^{n} |x_i + y_i|^{\frac{q}{\tau_i}}\right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^{n} (|x_i| + |y_i|)^{\frac{q}{\tau_i}}\right)^{\frac{1}{q}} \\
&\leq \left(\sum_{i=1}^{n} \max\left\{1, 2^{\frac{q}{\tau_i}-1}\right\} |x_i|^{\frac{q}{\tau_i}} + \sum_{i=1}^{n} \max\left\{1, 2^{\frac{q}{\tau_i}-1}\right\} |y_i|^{\frac{q}{\tau_i}}\right)^{\frac{1}{q}} \\
&\leq \left(\max\left\{1, 2^{\frac{q}{\min\tau}-1}\right\}\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} |x_i|^{\frac{q}{\tau_i}} + \sum_{i=1}^{n} |y_i|^{\frac{q}{\tau_i}}\right)^{\frac{1}{q}} \\
&\leq \max\left\{1, 2^{\frac{1}{\min\tau}-\frac{1}{q}}\right\}\left(\|x\|_{\tau,q} + \|y\|_{\tau,q}\right) .
\end{aligned}$$

The second inequality comes from (A.8), the fourth inequality comes from (A.7) for $q \geq 1$. When $q = \infty$, we have instead

$$\begin{aligned}
&\|x+y\|_{\tau,\infty} = \max_i \left\{|x_i + y_i|^{\frac{1}{\tau_i}}\right\} \leq \max_i \left\{(|x_i| + |y_i|)^{\frac{1}{\tau_i}}\right\} \\
&\leq \max\left\{1, 2^{\frac{1}{\min\tau}-1}\right\}\left(\max_i \left\{|x_i|^{\frac{1}{\tau_i}}\right\} + \max_i \left\{|y_i|^{\frac{1}{\tau_i}}\right\}\right) \\
&\leq \max\left\{1, 2^{\frac{1}{\min\tau}-1}\right\}\left(\|x\|_{\tau,\infty} + \|y\|_{\tau,\infty}\right) .
\end{aligned}$$

□

Remark 3.4 (Additive inequality and triangle inequality for $qh-$norm)**.** *If $\min\tau > 1$, then there exists some $q \in [1, \min\tau]$, s.t. the inequality* (3.10) *is again in the form of triangle inequality as in Definition 2.1, i.e. $\|x+y\|_{\tau,q} \leq \|x\|_{\tau,q} + \|y\|_{\tau,q}$ for all $q \in$*

$[1, \min\tau]$. *This is always possible by scaling the weight vectors by* $L > \max\{1, 1/\min\tau\}$. *Note that the value of the qh−norm is changed with such L−scaling on weight vector* τ *as shown in Remark 3.2.*

3.4 Homogeneous $\mathcal{L}_p-$norm

Similar to the introduction of the $\tau-$homogeneous vector $q-$norm for a vector, we also need to extend the traditional $\mathcal{L}_p-$norm in signal space to be applicable to homogeneous Input-Output Map G of arbitrary weight vectors for input and output.
First, as the Definition 2.3, we introduce the $\tau-$homogeneous $\mathcal{L}_p-$space

Definition 3.4 ($\tau-$homogeneous $\mathcal{L}_p-$space). *For* $p \geq 1$ *the set* $\mathcal{L}_{\tau,p}$ *consists of all signals* $u : [0,\infty) \to \mathbb{R}$*, which are measurable and satisfy* $\int_0^\infty |u(t)|^{\frac{p}{\tau}}\,\mathrm{d}t < \infty$.
For multivariable signals $u : [0,\infty) \to \mathbb{R}^n$ *the signal space* $\mathcal{L}^n_{\tau,p}$ *consists of all measurable signals such that*

$$\int_0^\infty \|u(t)\|^p_{\tau,q}\,\mathrm{d}t < \infty\,,$$

where $\|\cdot\|_{\tau,q}$ *is the* $\tau-$*homogeneous* $q-$*norm in* $\mathbb{R}^n$. *From the Lemma 3.1, all* $\tau-$*homogeneous* $q-$*norms are equivalent, thus when* $u \in \mathcal{L}^n_{\tau,p}$ *for some* q*, it is also true for all* $q \geq 1$.

Compared to Definition 3.4, the weight vector τ is part of the definition of $\tau-$homogeneous $\mathcal{L}_p-$space. Again, we omit the superscript of n in $\mathcal{L}^n_{\tau,p}$ when no ambiguity exist. Similarly, the extended $\tau-$homogeneous $\mathcal{L}_p-$space can be defined as the set of all signals s.t. $u_T \in \mathcal{L}_{\tau,p}$ for any $T \in [0,\infty)$, which is denoted as $u \in \mathcal{L}_{\tau,pe}$.

Definition 3.5 (The $\tau-$homogeneous $\mathcal{L}_p-$norm of a signal). *Define a mapping from a signal* $u \in \mathcal{L}_{\tau,p}$ *to a real non-negative number that provides a continuous measure of size for such a signal with its weight vector* τ_u*, defined by*

$$\|u\|_{\tau_u,\mathcal{L}_p} \triangleq \left(\int_0^\infty \|u(t)\|^p_{\tau_u,q}\ \mathrm{d}t\right)^{\frac{1}{p}}\,. \tag{3.11}$$

Mentioned in Definition 3.4, when $u \in \mathcal{L}_{\tau,p}$ the τ−homogeneous $\mathcal{L}_p$−norm in (3.11) *exists and is finite for all $q \geq 1$. When $p = \infty$, this is*

$$\|u\|_{\tau_u,\mathcal{L}_\infty} = \sup_{t\geq 0} \|u(t)\|_{\tau_u,q} \ , \tag{3.12}$$

that is the supremum of the τ−homogeneous vector q−norm for the vector $u(t)$ for all time. Such definition is a natural extension of the vector homogeneous norm in finite dimension in Definition 3.3 to signals in infinite dimensional space.

Lemma 3.4. *The $\mathcal{L}_{\tau,p}$−space is a linear signal space, i.e. if $x, y \in \mathcal{L}_{\tau,p}$ then $ax + by \in \mathcal{L}_{\tau,p}$ for any $a, b \in \mathbb{R}$.*

Proof. For the proof the generalization of the weak triangle inequality is used [55]

$$f(x+y) \leq f(2x) + f(2y) \ , \quad \forall\, x, y \in [0,\infty) \ , \tag{3.13}$$

where f is a class $\mathcal{K}$ function defined on $[0,\infty)$ (monotonically increasing function). Consider

$$\begin{aligned}
\int_0^\infty \|ax(t) + by(t)\|_{\tau,q}^p \,\mathrm{d}t &= \int_0^\infty \left(\sum_{i=1}^n |ax_i(t) + by_i(t)|^{\frac{q}{\tau_i}} \right)^{\frac{p}{q}} \mathrm{d}t \\
&\leq \int_0^\infty \left(\sum_{i=1}^n (|ax_i(t)| + |by_i(t)|)^{\frac{q}{\tau_i}} \right)^{\frac{p}{q}} \mathrm{d}t \\
&\leq \int_0^\infty \left(\sum_{i=1}^n |2a|^{\frac{q}{\tau_i}} |x_i(t)|^{\frac{q}{\tau_i}} + \sum_{i=1}^n |2b|^{\frac{q}{\tau_i}} |y_i(t)|^{\frac{q}{\tau_i}} \right)^{\frac{p}{q}} \mathrm{d}t \\
&\leq \int_0^\infty \left(A \sum_{i=1}^n |x_i(t)|^{\frac{q}{\tau_i}} + B \sum_{i=1}^n |y_i(t)|^{\frac{q}{\tau_i}} \right)^{\frac{p}{q}} \mathrm{d}t \\
&\leq (2A)^{\frac{p}{q}} \int_0^\infty \left(\sum_{i=1}^n |x_i(t)|^{\frac{q}{\tau_i}} \right)^{\frac{p}{q}} + (2B)^{\frac{p}{q}} \int_0^\infty \left(\sum_{i=1}^n |y_i(t)|^{\frac{q}{\tau_i}} \right)^{\frac{p}{q}} \mathrm{d}t \\
&= (2A)^{\frac{p}{q}} \int_0^\infty \|x(t)\|_{\tau,q}^p \,\mathrm{d}t + (2B)^{\frac{p}{q}} \int_0^\infty \|y(t)\|_{\tau,q}^p \,\mathrm{d}t < \infty \,,
\end{aligned}$$

where $A = \max_i \left\{ |2a|^{\frac{q}{\tau_i}} \right\}$ and $B = \max_i \left\{ |2b|^{\frac{q}{\tau_i}} \right\}$, and we apply several times inequality (3.13) to the monotonically increasing function $|\cdot|^r$, $r > 0$. This concludes the proof for finite p. For $p=\infty$ it is similar from (3.12)

$$\begin{aligned}
\sup_{t\geq 0}\|ax(t)+by(t)\|_{\tau,q} &= \sup_{t\geq 0}\left(\sum_{i=1}^{n}|ax_i(t)+by_i(t)|^{\frac{q}{\tau_i}}\right)^{\frac{1}{q}} \\
&\leq \sup_{t\geq 0}\left(\sum_{i=1}^{n}(|ax_i(t)|+|by_i(t)|)^{\frac{q}{\tau_i}}\right)^{\frac{1}{q}} \\
&\leq \left(\sup_{t\geq 0}\left[\sum_{i=1}^{n}|2a|^{\frac{q}{\tau_i}}\,|x_i(t)|^{\frac{q}{\tau_i}}\right]+\sup_{t\geq 0}\left[\sum_{i=1}^{n}|2b|^{\frac{q}{\tau_i}}\,|y_i(t)|^{\frac{q}{\tau_i}}\right]\right)^{\frac{1}{q}} \\
&\leq \left(A\sup_{t\geq 0}\left[\sum_{i=1}^{n}|x_i(t)|^{\frac{q}{\tau_i}}\right]+B\sup_{t\geq 0}\left[\sum_{i=1}^{n}|y_i(t)|^{\frac{q}{\tau_i}}\right]\right)^{\frac{1}{q}} \\
&= (2A)^{\frac{1}{q}}\sup_{t\geq 0}\|x(t)\|_{\tau,q}+(2B)^{\frac{1}{q}}\sup_{t\geq 0}\|y(t)\|_{\tau,q}<\infty\,.
\end{aligned}$$

□

Remark 3.5 (Equivalence of $\mathcal{L}_{ph}-$norm with the same weight vector)**.** *It is apparent from Lemma 3.1, that the $\tau-$homogeneous $\mathcal{L}_p-$norm* (3.11) *are equivalent for different choice of q.*

Remark 3.6 (Effect of weight scaling on $\mathcal{L}_{ph}-$norms)**.** *Similar to Remark 3.2, scaling the homogeneous degree $d=-\tau_t$, and the weight vectors τ_x, τ_u, τ_y by a positive constant $L\leq\min\{p,q\}$, affects the value of the homogeneous $\mathcal{L}_p-$norm* (3.11) *of the involved signals, i.e. for $u\in\mathcal{L}_{L\tau,p}$*

$$\|u\|_{L\tau_u,\mathcal{L}_p}=\left(\int_0^\infty\|u(t)\|^p_{L\tau_u,q}\,\mathrm{d}t\right)^{\frac{1}{p}}=\left(\int_0^\infty\|u(t)\|^{p/L}_{\tau_u,q/L}\,\mathrm{d}t\right)^{\frac{1/L}{p/L}}=\|u\|^{1/L}_{\tau_u,\mathcal{L}_{p/L}}\,.$$

*Note that the $\|u\|^{1/L}_{\tau_u,\mathcal{L}_{p/L}}$ uses the τ_u-homogeneous $q/L-$norm instead of the (original) homogeneous $q-$norm, which have different values. Nonetheless, it implies that for any $L\leq\min\{p,q\}$ the spaces $\mathcal{L}_{L\tau,p}$ and $\mathcal{L}_{\tau,p/L}$ are identical, i.e. $u\in\mathcal{L}_{L\tau,p}\Leftrightarrow u\in\mathcal{L}_{\tau,p/L}$. Although the two spaces are identical, the power of $1/L$ indicates that the $L\tau-$homogeneous $\mathcal{L}_p-$norm is **not** equivalent to the $\tau-$homogeneous $\mathcal{L}_{p/L}-$norm.*

The triangle inequality in Definition 2.1 is valid for $\tau-$homogeneous $\mathcal{L}_p-$norm when $\tau=\mathbf{1}_n$, i.e. for two signals $x,y\in\mathcal{L}_p$, $\|x+y\|_{\mathbf{1}_n,\mathcal{L}_p}\leq\|x\|_{\mathbf{1}_n,\mathcal{L}_p}+\|y\|_{\mathbf{1}_n,\mathcal{L}_p}$. For other cases, the more general additive inequality is valid.

Lemma 3.5 (Additive inequality for $\tau-$homogeneous $\mathcal{L}_p-$norm)**.** *The $\tau-$homogeneous $\mathcal{L}_p-$norm satisfies the following inequality for two signal $x, y \in \mathcal{L}_{\tau,p}$, with $q \geq 1$, for all $p \geq 1$ (including the case $p = \infty$)*

$$\|x{+}y\|_{\tau,\mathcal{L}_p} \leq \max\left\{1, 2^{\frac{1}{\min\tau}-\frac{1}{q}}\right\}\left(\|x\|_{\tau,\mathcal{L}_p} + \|y\|_{\tau,\mathcal{L}_p}\right) . \tag{3.14}$$

Proof. First of all, signal $x + y \in \mathcal{L}_{\tau,p}$ from Lemma 3.4. Then similar to the classical proof of the triangle inequality for $\mathcal{L}_p-$norm, we have

$$\begin{aligned}
\|x{+}y\|^p_{\tau,\mathcal{L}_p} &= \int_0^T \|x(t){+}y(t)\|^p_{\tau,q}\,\mathrm{d}t = \int_0^T \|x(t){+}y(t)\|_{\tau,q}\,\|x(t){+}y(t)\|^{p-1}_{\tau,q}\,\mathrm{d}t \\
&\leq \max\left\{1, 2^{\frac{1}{\min\tau}-\frac{1}{q}}\right\}\int_0^T \left(\|x(t)\|_{\tau,q} + \|y(t)\|_{\tau,q}\right)\|x(t){+}y(t)\|^{p-1}_{\tau,q}\,\mathrm{d}t \\
&\leq \max\left\{1, 2^{\frac{1}{\min\tau}-\frac{1}{q}}\right\}\left[\left(\int_0^T \|x(t)\|^p_{\tau,q}\,\mathrm{d}t\right)^{\frac{1}{p}} + \left(\int_0^T \|y(t)\|^p_{\tau,q}\,\mathrm{d}t\right)^{\frac{1}{p}}\right] \\
&\cdot\left(\int_0^T \|x(t){+}y(t)\|^p_{\tau,q}\,\mathrm{d}t\right)^{\frac{p-1}{p}} \\
&= \max\left\{1, 2^{\frac{1}{\min\tau}-\frac{1}{q}}\right\}\left(\|x\|_{\tau,\mathcal{L}_p} + \|y\|_{\tau,\mathcal{L}_p}\right)\|x{+}y\|^{p-1}_{\tau,\mathcal{L}_p} .
\end{aligned}$$

From here (3.14) follows immediately. The first inequality derives from Lemma 3.3, the second inequality comes from Hölder's inequality (A.9). From (3.10) as well as definition of homogeneous $\mathcal{L}_\infty-$norm (3.12), it is also easy to derive that

$$\begin{aligned}
&\|x{+}y\|_{\tau,\mathcal{L}_\infty} = \sup_{t\geq 0}\|x(t){+}y(t)\|_{\tau,q} \leq \max\left\{1, 2^{\frac{1}{\min\tau}-\frac{1}{q}}\right\}\left(\sup_{t\geq 0}\|x(t)\|_{\tau,q}{+}\sup_{t\geq 0}\|y(t)\|_{\tau,q}\right) \\
&\leq \max\left\{1, 2^{\frac{1}{\min\tau}-\frac{1}{q}}\right\}\left(\|x\|_{\tau,\mathcal{L}_\infty}{+}\|y\|_{\tau,\mathcal{L}_\infty}\right) .
\end{aligned}$$

Or simply (3.14) is also valid when $p = \infty$. □

Remark 3.7 (Additive inequality and triangle inequality for $\mathcal{L}_{ph}-$norm)**.** *From Remark 3.4, by $L-$scaling on the weight vector τ, there always exists some $q \in [1, \min\tau]$, s.t.* (3.14) *appears in the form of triangle inequality as in Definition 2.1, i.e. $\|x{+}y\|_{\tau,\mathcal{L}_p} \leq \|x\|_{\tau,\mathcal{L}_p} + \|y\|_{\tau,\mathcal{L}_p}$ for all $p \geq 1$, $q \in [1, \min\tau]$, $\forall\, x, y \in \mathcal{L}_{\tau,p}$. Note that, with the $L-$scaling on the weight vector τ, the value of the $\mathcal{L}_{ph}-$norm is also changed as shown in Remark 3.6.*

Remark 3.8 (Relationship between homogeneous $\mathcal{L}_p-$norm and traditional $\mathcal{L}_p-$norm)**.** *Similar to Remark 3.3, introduce the companion signal of* $u(\cdot)$ *as*

$$u^{\frac{1}{\tau_u}}(t) = \left[\lceil u_1(t) \rfloor^{\frac{1}{\tau_{u_1}}}, \cdots, \lceil u_m(t) \rfloor^{\frac{1}{\tau_{u_m}}}\right]^\top . \tag{3.15}$$

It is clear that the $\mathcal{L}_{ph}-$*norm equals the traditional* $\mathcal{L}_p-$*norm for its homeomorphic mapping, namely* $\|u\|_{\tau_y,\mathcal{L}_p} = \left\|u^{\frac{1}{\tau_u}}\right\|_{\mathcal{L}_p}$.

Remark 3.9 (Relationship between $\mathcal{L}_{\tau,p}-$spaces)**.** *As a direct consequence from Remark 2.4, if a signal* $u \in \mathcal{L}_{\tau_u,1} \cap \mathcal{L}_{\tau_u,\infty}$, *then* $u \in \mathcal{L}_{\tau_u,p}$ *for* $p \in [1,\infty]$ *by simply using its companion signal in Remark 3.8.*
Further, in extended τ_u-*homogeneous* $\mathcal{L}_p-$*space with finite* T, *we have* $\mathcal{L}_{\tau_u,\infty e} \subset \mathcal{L}_{\tau_u,pe} \subset \mathcal{L}_{\tau_u,1e}$ *for similar reasons.*

3.5 Finite-gain Homogeneous $\mathcal{L}_2-$stability

Similar to the $\mathcal{L}_2-$stability, the finite-gain homogeneous $\mathcal{L}_2-$stability for homogeneous Input-Output Map can be defined as

Definition 3.6 (Finite-gain homogeneous $\mathcal{L}_2-$stability)**.** *The Input-Output Map* G *is called* $L\tau-$*homogeneous* $\mathcal{L}_2-$*stable if*

$$u \in \mathcal{L}_{L\tau_u,2} \Rightarrow y \in \mathcal{L}_{L\tau_y,2} .$$

The map G *is said to be finite-gain* $L\tau-$*homogeneous* $\mathcal{L}_2-$*stable (finite-gain* $\mathcal{L}_{2h}-$*stable in short), if there exists a finite constant* $\gamma_L > 0$ *and a finite constant* $\beta_L \geq 0$, *s.t.*

$$\|(G(u))_T\|_{L\tau_y,\mathcal{L}_2} \leq \gamma_L \|u_T\|_{L\tau_u,\mathcal{L}_2} + \beta_L , \quad \forall T \in \mathbb{R}_{\geq 0}, u \in \mathcal{L}_{L\tau_y,2e} ,$$

then the $\mathcal{L}_{2h}-$gain *of* G *is defined as*

$$\gamma_L' \triangleq \inf \{\gamma_L \mid \exists \beta_L \text{ such that (3.16) holds}\} .$$

Similar to the classical situation [50], if G has finite $\mathcal{L}_{ph}$−gain then it is automatically $\mathcal{L}_{ph}$−stable. By taking $u \in \mathcal{L}_{L\tau_u,2} \subset \mathcal{L}_{L\tau_u,2e}$ and letting $T \to \infty$, we obtain

$$\|G(u)\|_{L\tau_y,\mathcal{L}_2} \leq \gamma_L \|u\|_{L\tau_u,\mathcal{L}_2} + \beta_L\,, \quad \forall\, u \in \mathcal{L}_{L\tau_u,2}\,, \tag{3.16}$$

implying that $G(u) \in \mathcal{L}_{L\tau_y,2}$.

In the rest of this thesis, the subscript L might not be repeatedly added on other functions for readability purpose. Note that $\mathcal{L}_{2h}$−stability (3.16) agrees with finite-gain $\mathcal{L}_2$−stability for LTI system [29, 50], when $\tau_t = 0, L\tau_u = \mathbf{1}_m, L\tau_y = \mathbf{1}_r$.

Proposition 3.1 (Finite-gain $\mathcal{L}_{2h}$−stability with zero bias)**.** *If a homogeneous Input-Output Map G is finite-gain $\mathcal{L}_{2h}$−stable, then $\beta_L = 0$ in* (3.16) *if $L\tau_t \neq -2$.*

Proof. Since (3.16) is true for all $u \in \mathcal{L}_{L\tau_u,2}$, then (3.16) is also true for the dilated input and dilated output with dilated time as in (3.1) and (3.2), i.e.

$$\|\tilde{y}\|_{L\tau_y,\mathcal{L}_2} \leq \gamma_L \|\tilde{u}\|_{L\tau_u,\mathcal{L}_2} + \beta_L\,, \quad \forall\, u \in \mathcal{L}_{L\tau_u,2}\,. \tag{3.17}$$

The $\mathcal{L}_{2h}$−norm of $\tilde{y}$ is

$$\begin{aligned} \|\tilde{y}\|_{L\tau_y,\mathcal{L}_2} &= \left|\int_0^\infty \left\|\tilde{y}(\tilde{t})\right\|^2_{L\tau_y,q} \mathrm{d}\tilde{t}\right|^{\frac{1}{2}} = \left|\int_0^\infty \kappa^2 \left\|y(t)\right\|^2_{L\tau_y,q} \mathrm{d}\tilde{t}\right|^{\frac{1}{2}} \\ &= \kappa \left|\int_0^\infty \kappa^{L\tau_t} \left\|y(t)\right\|^2_{L\tau_y,q} \mathrm{d}t\right|^{\frac{1}{2}} = \kappa^{1+\frac{L\tau_t}{2}} \|y\|_{L\tau_y,\mathcal{L}_2}\,. \end{aligned} \tag{3.18}$$

That is, the $\mathcal{L}_{2h}$−norm of dilated output $\|\tilde{y}\|_{L\tau_y,\mathcal{L}_2}$ equals the $\mathcal{L}_{2h}$−norm of $\|y\|_{L\tau_y,\mathcal{L}_2}$ scaled by $\kappa^{1+L\tau_t/2}$. Such scaling also applies to input $\|\tilde{u}\|_{L\tau_u,\mathcal{L}_2}$. Then (3.17) is equivalent to

$$\|y\|_{L\tau_y,\mathcal{L}_2} \leq \gamma_L \|u\|_{L\tau_u,\mathcal{L}_2} + \beta_L/\kappa^{1+\frac{L\tau_t}{2}}\,, \quad \forall\, u \in \mathcal{L}_{L\tau_u,2}\,. \tag{3.19}$$

The arbitrariness of κ implies $\beta_L = 0$ when $L\tau_t \neq -2$, more specifically, when $L\tau_t < -2, \kappa \to \infty$ and when $L\tau_t > -2, \kappa \to 0$. For LTI systems, similar observations can be found in Proposition 2.1 with $\tau_t = 0, L\tau_u = \mathbf{1}_m, L\tau_y = \mathbf{1}_r$ [50]. □

3.6 Definition of a Family of Homogeneous $\mathcal{H}_\infty-$norms

From Definition 3.6 we may define the homogeneous $\mathcal{H}_\infty-$norms as

Definition 3.7 (Definition of a family of homogeneous $\mathcal{H}_\infty-$norms)**.** *If the homogeneous Input-Output Map G is finite-gain $\mathcal{L}_{2h}-$stable, we propose a family of homogeneous $\mathcal{H}_\infty-$norm ($\mathcal{H}_{\infty h}-$norm in short) for G when $L\tau_t \neq -2$ as per*

$$\gamma_L' \triangleq \inf \left\{ \gamma \left| \sup_{\|u\|_{L\tau_u,\mathcal{L}_2} \neq 0, u \in \mathcal{L}_{\tau,2}} \frac{\|y\|_{L\tau_y,\mathcal{L}_2}}{\|u\|_{L\tau_u,\mathcal{L}_2}} \leq \gamma \right. \right\}. \tag{3.20}$$

From Proposition 3.1, when G is homogeneous and $L\tau_t \neq -2$, from finite-gain $\mathcal{L}_{2h}-$stability we have $\beta_L = 0$. It is clear that the family of $\mathcal{H}_{\infty h}-$norms equals the $\mathcal{L}_{2h}-$gain, when the $\mathcal{L}_{2h}-$gain is defined in inequality, the $\mathcal{H}_{\infty h}-$norm is defined in fraction. On the other hand, if (3.20) is finite, G also has finite homogeneous $\mathcal{L}_{2h}-$gain γ_L'.

Since the $\mathcal{L}_{2h}-$norm in Definition 3.5 only involves the weight vector of the signal, yet for homogeneous Input-Output Map defined in Definition 3.3, there exists the effect of time scaling described by (3.1) and (3.2). We shall discuss the $\mathcal{H}_{\infty h}-$norm in (3.20) for both cases.
From the signal point of view, i.e. without considering the time scaling, γ_L' in (3.20) is of homogeneous degree 0, since the homogeneous degrees of the $\mathcal{L}_{2h}-$norm in the numerator and the denominator of the right hand side of (3.20) are both 1. When considering the signal u and y as input and output for some homogeneous Input-Output Maps, i.e. with time scaling, the value of (3.20) still remains unchanged, since both the numerator and denominator are scaled by the same coefficient $\kappa^{1+L\tau_t/2}$ as described by (3.18).
Therefore the norm in (3.20) retains constant for any input and its dilation. Again for LTI system with degree $\tau_t = 0$, when choosing $L\tau_u = \mathbf{1}_m, L\tau_y = \mathbf{1}_r$, the homogeneous $\mathcal{H}_\infty-$norm (3.20) is the traditional $\mathcal{H}_\infty-$norm.

In contrast to Section 3.2, where the classical $\mathcal{H}_\infty-$norm is valid only when $\tau_u = \ell\mathbf{1}_m, \tau_y = \ell\mathbf{1}_r$ for some $\ell > 0$, the $\mathcal{H}_{\infty h}-$norm (3.20) is constant for dilated input.

Remark 3.10 (Weight vector for input and output must be positive). *Since* τ_{y_i} *and* τ_{u_i} *are assumed to be all positive real numbers in Definition 3.1, the* $\mathcal{L}_{ph}-$*norm of a signal in Definition 3.5 as well as the* $\mathcal{H}_{\infty h}-$*norm of the Input-Output Map in Definition 3.7 is thus well defined. Note that, in sliding mode with discontinuous vector field the weight vectors for some inputs are allowed to be zero. This is not allowed for Definition 3.7, so is not considered in this thesis.*

Remark 3.11 ($L-$scaling allows different interpretation of $\mathcal{L}_{2h}-$norm). *With different scaling L, the physical meaning as well as the value of such homogeneous* $\mathcal{H}_\infty-$*norm in Definition 3.7 is different, refer to Remark 3.6. We fix* $p = 2$ *and allow the scaling* L *on* $(\tau_u, \tau_y, -\tau_t)$*, since*

- *The traditional* $\mathcal{H}_\infty-$*norm is associated with the worst* $\mathcal{L}_2-$*gain. When there also exists other* $\mathcal{L}_p-$*gain [50], they are not frequently discussed. The two upper estimations for either linear systems or exponentially convergent nonlinear systems are brought up in Section 2.6.*

- *The partial differential inequality (PDI)* (4.6) *used in next section with state space model is mostly associated with the quadratic term of the input and the output. Therefore, in order to assemble the similarity, we adopt the* $\mathcal{L}_{2h}-$*norm in Definition 3.5.*

As reflected in [50], the $\mathcal{L}_p-$*gain of LTI systems has different value for different* p *when it exists, the* $\mathcal{H}_{\infty h}-$*norms* (3.20) *with different* $L-$*scaled weight are also different both in physical meaning and in value.*

3.7 Extension to Homogeneous Finite-gain $\mathcal{L}_p-$stability

The $\mathcal{L}_{2h}-$stability with $L-$scaled weight from Definition 3.6 can be extended to $\mathcal{L}_{ph}-$stability with original weight, i.e.

Definition 3.8 (Finite-gain homogeneous $\mathcal{L}_p-$stability). *The Input-Output Map G is called finite-gain* $\tau-$*homogeneous* $\mathcal{L}_p-$*stable (finite-gain* $\mathcal{L}_{ph}-$*stable in short), if there* *exists a finite constant* γ_p *and a finite constant* β_p*, s.t.*

$$\|y\|_{\tau_y,\mathcal{L}_p} \leq \gamma_p \|u\|_{\tau_u,\mathcal{L}_p} + \beta_p \,, \quad \forall\, u \in \mathcal{L}_{\tau,p} \,, \tag{3.21}$$

then the $\mathcal{L}_{ph}-$gain *of* G *is defined as*

$$\gamma_p' \triangleq \inf \{\gamma_p \mid \exists\, \beta_p \text{ such that (3.21) holds}\} \,.$$

Since the inequality (3.16) with zero bias, $\|y\|_{L\tau_y,\mathcal{L}_2} \leq \gamma_L \|u\|_{L\tau_y,\mathcal{L}_2}$ can be rewritten as

$$\|y\|^{1/L}_{\tau_y,\mathcal{L}_{2/L}} \leq \gamma_L \|u\|^{1/L}_{\tau_u,\mathcal{L}_{2/L}} \,, \quad \forall\, u \in \mathcal{L}_{L\tau,2}$$

from Remark 3.6. Thus the $L\tau-$homogeneous $\mathcal{L}_2-$gain γ_L is correlated to the $\tau-$homogeneous $\mathcal{L}_p-$gain γ_p by $\gamma_p = \gamma_L^L, L = 2/p$. This will be brought up again from the perspective of Partial Differential Inequality in the next Chapter.

3.8 Homogeneous $\mathcal{L}_p-$gain for Continuous Memoryless Input-Output Maps

A function $g : \mathbb{R}^m \to \mathbb{R}^r$ can be viewed as an operator G that assigns to every input signal $u(\cdot)$ the output signal $y(\cdot) = G(u(\cdot))$ given by $y(t) = g(u(t))$, i.e. the output $y(t)$ depends only on the present value of the input $u(t)$, and not on its past or future, and thus it is called memoryless. It is homogeneous if $g(\nu_\kappa^{\tau_u} u) = \nu_\kappa^{\tau_y}(g(u))$ for every $u \in \mathbb{R}^m$ and $\kappa > 0$. This means that each component function $g_i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \cdots, r$, is τ_u-homogeneous of degree τ_{y_i}. According to Definition 3.1 the operator G is of arbitrary homogeneous degree d. We assume that the function g is continuous w.r.t. u. We first introduce the following lemma.

Lemma 3.6 (Simplified from [13, 8])**.** *Suppose* $\phi(u)$ *and* $\chi(u)$ *are continuous real-valued homogeneous functions of degree* $m > 0$ *w.r.t. weight vector* τ_u*, and* $\phi(u)$ *is* *positive definite. Then for all $u \in \mathbb{R}^m$, with the surface $S = \{u : \phi(u) = 1\}$*

$$\left(\min_S \chi(u)\right)\phi(u) \leq \chi(u) \leq \left(\max_S \chi(u)\right)\phi(u)\,.$$

With Lemma 3.6, we can show the following Theorem.

Theorem 3.1 (Existence of $\mathcal{L}_{ph}$−gain for all continuous memoryless Input-Output Maps)**.** *A continuous memoryless Input-Output Map $y = g(u)$ is finite-gain $\mathcal{L}_{ph}$−stable with zero bias, for every $1 \leq p \leq \infty$. Its $\mathcal{L}_{ph}$−gain is the same for all values of $1 \leq p \leq \infty$ and it can be calculated as*

$$\gamma = \max_{\|u\|_{\tau_u,q}=1} \|g(u)\|_{\tau_y,q}\,. \tag{3.22}$$

Proof. As assumed, the function $y = g(u)$ is continuous in the input u. Using the notation from (3.15), $y^{\frac{1}{\tau_y}}$ is also a continuous function of the input u since $\tau_y > 0$. Then setting $\chi(u(t)) = \|g(u(t))\|_{\tau_y,q}$ and $\phi(u(t)) = \|u(t)\|_{\tau_u,q}$, both are continuous homogeneous functions of u with degree 1. And both are positive definite, therefore from Lemma 3.6, we have

$$\|y(t)\|_{\tau_y,q} \leq \left(\max_S \|g\left(u(t)\right)\|_{\tau_y,q}\right)\|u(t)\|_{\tau_u,q}\,, \tag{3.23}$$

where the surface $S = \left\{u : \|u\|_{\tau_u,q} = 1\right\}$ is the homogeneous unit sphere w.r.t. u. Since the Input-Output Map is time invariant and memoryless, $\max_S \|g(u)\|_{\tau_y,q}$ is a constant independent of time t, denote it as $\gamma = \max_S \|g(u)\|_{\tau_y,q}$ as in (3.22). Then we have for all $u \in \mathcal{L}_{\tau_u,p}$

$$\|y\|_{\tau_y,\mathcal{L}_p} = \left(\int_0^\infty \|y(t)\|^p_{\tau_y,q}\,\mathrm{d}t\right)^{\frac{1}{p}} \leq \gamma\left(\int_0^\infty \|u(t)\|^p_{\tau_u,q}\,\mathrm{d}t\right)^{\frac{1}{p}} = \gamma\,\|u\|_{\tau_u,\mathcal{L}_p}\,,$$

which means that the memoryless Input-Output Map G is finite-gain $\mathcal{L}_{ph}$−stable from Definition 3.8 with $\beta = 0$. Such $\mathcal{L}_{ph}$−gain can be achieved by the particular $u^\star$ that maximizes the value $\max_{\|u\|_{r_u,q}=1} \|g(u)\|_{r_y,q}$. Thus the $\mathcal{L}_{ph}$−gain equals γ for all $1 \leq p < \infty$. For the case of $p = \infty$, (3.23) suffices since the operator G is memoryless. □

3.9 A Discussion of Inverse of Homogeneous Input-Output Maps

The inverse of a LTI Input-Output Map is simple to derive, if its transfer function or state space model is available. This is probably not true for homogeneous Input-Output Map with non-zero homogeneous degree.

Proposition 3.2 (Homogeneous degree of inverse of homogeneous Input-Output Map)**.** *If a homogeneous Input-Output Map G has an inverse, its inverse G^{-1} has the same homogeneous degree as G.*

Proof. The cascade of $G^{-1}G$ means that $u(t)$, the input of G, is reconstructed at the output of G^{-1}. Suppose that the homogeneous degree of G is d_1 and that of G^{-1} is d_2. When the input is homogeneously dilated, e.g. in Fig. 3.1, we use the scalar case to simplify the notation, as $\kappa^{\tau_u} u\left(\kappa^{-d_1} t\right)$, from (3.1) and (3.2). The output of G becomes $\kappa^{\tau_y} y\left(\kappa^{-d_1} t\right)$. For the output of G^{-1} to be a reconstruction of the input of G, i.e. again $\kappa^{\tau_u} u\left(\kappa^{-d_1} t\right)$, the homogeneous degree of G^{-1} must be $d_2 = d_1$.
This is also true for the cascade of GG^{-1}, i.e. disregard of left inverse or right inverse. □

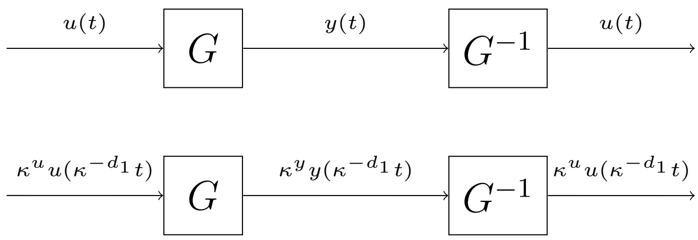

Figure 3.1: Inverse of homogeneous Input-Output Map G

The topic of properties of the inverse of homogeneous Input-Output Maps is seldom studied or discussed. So not a lot is known about it, and left to possible future research.

3.10 Homogeneous $\mathcal{L}_p-$stability for Interconnected Systems

One of the most important applications of the classical $\mathcal{L}_p-$stability concept is in the study of the $\mathcal{L}_p-$stability of interconnected systems. A central result, with many consequences in robust control, is the *small-gain theorem*. It provides a sufficient condition for the finite-gain $\mathcal{L}_p-$stability of the negative feedback interconnection of two finite-gain $\mathcal{L}_p-$stable systems, in terms of their $\mathcal{L}_p-$gains [15, 50, 29]. Moreover, it is well-known that two finite-gain $\mathcal{L}_p-$stable systems connected in cascade or in parallel leads to a finite-gain $\mathcal{L}_p-$stable system. Note that, since the classical $\mathcal{L}_p-$norms are not *equivalent*, there is a small-gain theorem for each p.

In this section we obtain the corresponding small-gain theorems for each p and the result for cascade systems derived from the homogeneous $\mathcal{L}_p-$stability concepts introduced in the previous sections for general homogeneous systems.

3.10.1 Homogeneous Small Gain Theorem for Feedback Interconnected Systems

Since the additive inequality as shown in Lemma 3.5 is different from the traditional triangle inequality in Definition 2.1 for traditional $\mathcal{L}_p-$norm, so the homogeneous small gain theorem with interconnected system is also in a different form.

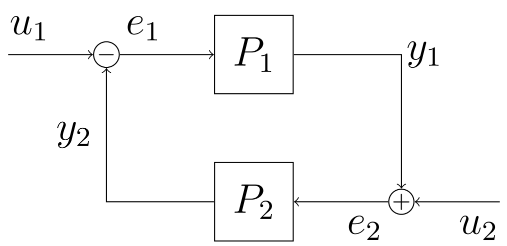

Figure 3.2: Homogeneous small gain theorem

Theorem 3.2 (Homogeneous small gain theorem)**.** *For two homogeneous Input-Output Maps P_1 and P_2, interconnected as shown in Figure 3.2, if the weight vectors of P_1, P_2*

are matched, i.e. the weight vector for output of P_2 and input of P_1 being $\tau_1 \in \mathbb{R}^{n_1}_{>0}$ and the weight vector for output of P_1 and input of P_2 being $\tau_2 \in \mathbb{R}^{n_2}_{>0}$, and further both P_1 and P_2 are $\mathcal{L}_{ph}-$stable for some $p \in [1, \infty]$ with gain γ_1, γ_2 and bias β_1, β_2 as in (3.21) respectively, then the interconnected system is $\mathcal{L}_{ph}-$stable if

$$\gamma_1\gamma_2 < \frac{1}{c_1c_2}\,, \tag{3.24}$$

where

$$c_1 = \max\left\{1, 2^{\frac{1}{\min \tau_1}-\frac{1}{q}}\right\}, \quad c_2 = \max\left\{1, 2^{\frac{1}{\min \tau_2}-\frac{1}{q}}\right\}.$$

Proof. Shown in Figure 3.2, the interconnected system has two inputs $u_1(t) \in \mathbb{R}^{n_1}, u_2(t) \in \mathbb{R}^{n_2}$, whose weight vectors are τ_1, τ_2. We have

$$\begin{aligned} e_1 &= u_1 - y_2\,, \quad y_1 = P_1(e_1)\,, \\ e_2 &= u_2 + y_1\,, \quad y_2 = P_2(e_2)\,. \end{aligned}$$

From the additive inequality (3.14) as well as the assumption of P_1, P_2 being $\mathcal{L}_{ph}-$stable as in (3.21), then assuming $u_1 \in \mathcal{L}_{\tau_1,p}$, $u_2 \in \mathcal{L}_{\tau_2,p}$, the homogeneous $\mathcal{L}_p-$norm of e_1 is

$$\begin{aligned} \|e_1\|_{\tau_1,\mathcal{L}_p} = \|u_1 - y_2\|_{\tau_1,\mathcal{L}_p} &\le \max\left\{1, 2^{\frac{1}{\min \tau_1}-\frac{1}{q}}\right\}\left(\|u_1\|_{\tau_1,\mathcal{L}_p} + \|y_2\|_{\tau_1,\mathcal{L}_p}\right) \\ &\le c_1\left(\|u_1\|_{\tau_1,\mathcal{L}_p} + \gamma_2\|e_2\|_{\tau_2,\mathcal{L}_p} + \beta_2\right)\,. \end{aligned}$$

Similarly

$$\|e_2\|_{\tau_2,\mathcal{L}_p} \le c_2\left(\|u_2\|_{\tau_2,\mathcal{L}_p} + \gamma_1\|e_1\|_{\tau_1,\mathcal{L}_p} + \beta_1\right)\,.$$

Combining both inequalities, we have

$$\begin{aligned} \|e_1\|_{\tau_1,\mathcal{L}_p} &\le c_1\left[\|u_1\|_{\tau_1,\mathcal{L}_p} + \gamma_2c_2\left(\|u_2\|_{\tau_2,\mathcal{L}_p} + \gamma_1\|e_1\|_{\tau_1,\mathcal{L}_p} + \beta_1\right) + \beta_2\right] \\ &= c_1\|u_1\|_{\tau_1,\mathcal{L}_p} + \gamma_2c_1c_2\|u_2\|_{\tau_2,\mathcal{L}_p} + \gamma_1\gamma_2c_1c_2\|e_1\|_{\tau_1,\mathcal{L}_p} + \gamma_2c_1c_2\beta_1 + c_1\beta_2\,. \end{aligned}$$

When (3.24) is true, then we have

$$\|e_1\|_{\tau_1,\mathcal{L}_p} \le \frac{1}{1-\gamma_1\gamma_2c_1c_2}\left(c_1\|u_1\|_{\tau_1,\mathcal{L}_p} + \gamma_2c_1c_2\|u_2\|_{\tau_2,\mathcal{L}_p} + \gamma_2c_1c_2\beta_1 + c_1\beta_2\right)\,.$$

Similarly, the homogeneous $\mathcal{L}_2$−norm of e_2 is bounded by

$$\|e_2\|_{\tau_2,\mathcal{L}_p} \leq \frac{1}{1-\gamma_1\gamma_2 c_1 c_2}\Big(c_2\,\|u_2\|_{\tau_2,\mathcal{L}_p} + \gamma_1 c_1 c_2\,\|u_1\|_{\tau_1,\mathcal{L}_p} + \gamma_1 c_1 c_2 \beta_2 + c_2\beta_1\Big)\;.$$

Thus the closed loop system is $\mathcal{L}_{ph}$−stable. Since the additive inequality (3.14) is also valid for $p = \infty$, so the above bound is also true for $p = \infty$. □

A classical use of the small-gain theorem leads to a robustness interpretation: If P_1 is the nominal system and P_2 an uncertainty, the stability of P_1 is preserved for all P_2 with sufficiently small gain satisfying (3.24). It is important to note that no restriction on the homogeneity degrees of P_1 and P_2 is imposed, i.e. they can be different.

Remark 3.12. *Due to the scaling property of the weight vectors and the degree of homogeneity, it is always possible to achieve* $\min\{\tau_1, \tau_2\} > 1$ *as noted in Remark 3.4. Then choosing* $q \in [1, \min\{\tau_1, \tau_2\}]$*, we recover the well-known small-gain condition* $\gamma_1\gamma_2 < 1$ *in* (3.24).

3.10.2 Homogeneous $\mathcal{L}_p$−gain for Cascaded Homogeneous Systems

In the homogeneous small gain theorem the cascaded system must have matching weight vector. If there exists a cascaded system as in Figure 3.3, we can derive a similar result of existence of $\mathcal{L}_{ph}$−gain. However, if the weight vectors are related as $\tau_{y_1} = \ell\tau_{u_2}$, $\ell > 0$, i.e. the weight of output of G_1 equals the weight of the input of G_2 multiplied by some positive integer, we can conclude the following

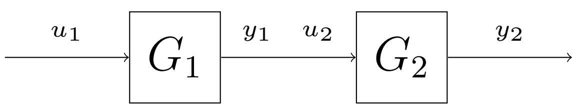

Figure 3.3: Homogeneous $\mathcal{L}_p$−gain for cascaded homogeneous system.

Theorem 3.3 (Homogeneous $\mathcal{L}_p$−gain for cascaded homogeneous system)**.** *Let two homogeneous systems* G_1*,* G_2 *be cascaded as in Figure 3.3 with weight vector correlated as*

$\tau_{y_1} = \ell\tau_{u_2}, \ell > 0$. If G_1 has $\tau-$homogeneous $\mathcal{L}_p-$gain γ_1 and G_2 has $\ell\tau-$homogeneous $\mathcal{L}_p-$gain γ_2 for some $p \in [1, \infty]$, then the cascade system G_2G_1 has $\mathcal{L}_{ph}-$gain $\gamma_1\gamma_2$ with weight vector of the input and the output being τ_{u_1} and $\ell\tau_{y_2}$.

Proof. From definition of the $\mathcal{L}_{ph}-$stability (3.21), we have for $u_1 \in \mathcal{L}_{\tau_{u_1},p}$

$$\begin{aligned} \|y_2\|_{\ell\tau_{y_2},\mathcal{L}_p} &\leq \gamma_2 \|u_2\|_{\ell\tau_{u_2},\mathcal{L}_p} + \beta_2 = \gamma_2 \|y_1\|_{\tau_{y_1},\mathcal{L}_p} + \beta_2 \\ &\leq \gamma_2 \left(\gamma_1 \|u_1\|_{\tau_{u_1},\mathcal{L}_p} + \beta_1\right) + \beta_2 = \gamma_1\gamma_2 \|u_1\|_{\tau_{u_1},\mathcal{L}_p} + \gamma_2\beta_1 + \beta_2 \,. \end{aligned} \tag{3.25}$$

Thus the $\mathcal{L}_{ph}-$gain of the cascaded system G_2G_1 is $\gamma_1\gamma_2$ with input weight vector τ_{u_1} and output weight vector $\ell\tau_{y_2}$.

□

Note that for both interconnection results, the common signals have to have *compatible* weights. For system in Figure 3.2 this means that $\tau_{y_2} = \tau_{e_1} = \tau_2$ and $\tau_{y_1} = \tau_{e_2} = \tau_1$. For the cascaded system in Figure 3.3 this is relaxed to $\tau_{y_1} = \ell\tau_{u_2}, \ell > 0$.

Remark 3.13. *Note that for both Theorem 3.2 and Theorem 3.3, the homogeneous degree for subsystems can be different. This results from the fact that the $\mathcal{L}_{ph}-$gain is an input-output relationship and the behaviour within each sub homogeneous system does not matter from the input-output perspective. When P_1, P_2 have different homogeneous degrees (even of different sign), the closed loop system in Figure 3.2 does not behave homogeneously any longer, yet the conclusion of Theorem 3.2 and Theorem 3.3 remain true. Such behavior is also observed in [5], where the homogeneous small gain theorem does not consider inputs and allows different degrees for each sub-system.*

3.11 Examples

We include some simple examples of the $\mathcal{H}_{\infty h}-$norm of continuous memoryless homogeneous functions.

Example 3.2 ($\mathcal{H}_{\infty h}-$norm of continuous memoryless homogeneous function)**.** *Consider the homogeneous function $y(u) = u_1 u_2$, with weight vector $L\tau_u = (L, 2L), L\tau_y = 3L$. From Theorem 3.1, the Input-Output Map formed by this continuous homogeneous function is finite-gain $\mathcal{L}_{2h}-$stable, and its $\mathcal{H}_{\infty h}-$norm can be found by finding the maximum of the following function when $q = 2$*

$$\gamma_L^2 = \max_{\|u\|=1} \frac{\|y\|^2_{L\tau_y,2}}{\|u\|^2_{L\tau_u,2}} = \max_{\|u\|=1} \frac{|u_1 u_2|^{\frac{2}{3L}}}{|u_1|^{\frac{2}{L}} + |u_2|^{\frac{1}{L}}} . \tag{3.26}$$

This can be easily derived by a search program on the unit sphere w.r.t. u.

Here we use another method. For the value of such homogeneous function of zero degree we may set one variable to $0, \pm 1$, and set the other variable to $\mathbb{R}$. Since we have only two variables, such approach is more convenient. The right hand side of (3.26) becomes

$$\gamma_L^2\Big|_{u_1=\pm 1} = \frac{|u_2|^{\frac{2}{3L}}}{1 + |u_2|^{\frac{1}{L}}} \ \textit{or}\ \gamma_L^2\Big|_{u_1=0, u_2\neq 0} = 0 .$$

The case of $u_1 = 0$ is excluded, since the other case is positive, which is surely bigger than zero. The extremum of $\gamma_L^2\Big|_{u_1=\pm 1}$ along $u_2 \in \mathbb{R}$ can be found by taking its partial derivative against x_2 equal to zero, which gives us

$$\frac{\frac{2}{3L} \lceil u_2 \rfloor^{\frac{2-3L}{3L}} \left(1 + |u_2|^{\frac{1}{L}}\right) - \frac{1}{L} \lceil u_2 \rfloor^{\frac{5-3L}{3L}}}{\left(1 + |u_2|^{\frac{1}{L}}\right)^2} = 0 .$$

The solutions happens to be at $u_2 = \pm 2^L$. Therefore we have the extremum of (3.26) at

$$|u_1| = \kappa^L, |u_2| = 2^L \kappa^{2L}, \forall\, \kappa > 0$$

and the value is

$$\gamma_L = \frac{2^{\frac{1}{3}}}{3^{\frac{1}{2}}} = 0.7274 . \tag{3.27}$$

Figure 3.4 shows the value of

$$\zeta_L = \max_{\|u\|=1} \frac{\|y\|_{L\tau_y,2}}{\|u\|_{L\tau_u,2}}$$

with the coordinate of $u_1 = \sin\theta, u_2 = \cos\theta$. As shown by (3.27), the value of γ_L for this *example is always* 0.7274*, independent of the scaling factor* L*. Further, the function* $y(u) = u_1u_2$ *can be homogeneous with weight* $L\tau_u = (L, L), L\tau_y = 2L$*. It is easily verified that* $\gamma_L = 1/\sqrt{2}$ *in this case is also independent of* L*.* □

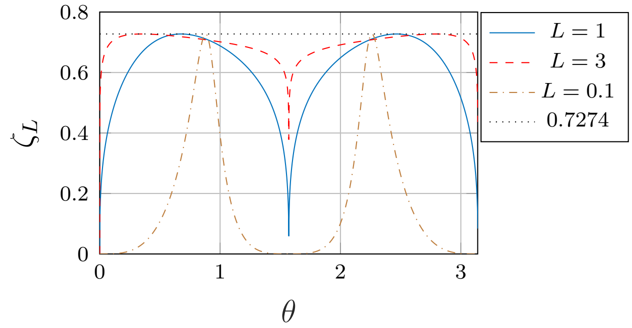

Figure 3.4: ζ_L for homogeneous memoryless Input-Output Map in Example 3.2 along unit sphere.

Example 3.3 ($\mathcal{H}_{\infty h}-$norm of continuous memoryless homogeneous Input-Output Map)**.** *Take another simple example of memoryless homogeneous Input-Output Map*

$$y(t) = \left\lceil c_1 \left\lceil u_1(t) \right\rfloor^3 + c_2 \left\lceil u_2(t) \right\rfloor^2 \right\rfloor^{\frac{1}{3}},$$

where the weight vector is $L\tau_u = (2L, 3L), L\tau_y = 2L$*. From Theorem 3.1, it is finite-gain* $\mathcal{L}_{2h}-$*stable and its* $\mathcal{H}_{\infty h}-$*norm can be found by finding the maximum of the following function*

$$\gamma_L^2 = \max_{\|u\|=1} \frac{\|y\|^2_{L\tau_y,2}}{\|u\|^2_{L\tau_u,2}} = \max_{\|u(t)\|=1} \frac{\left| c_1 \left\lceil u_1 \right\rfloor^3 + c_2 \left\lceil u_2 \right\rfloor^2 \right|^{\frac{1}{3L}}}{|u_1|^{\frac{1}{L}} + |u_2|^{\frac{2}{3L}}}. \tag{3.28}$$

Similar to Example 3.2, we would like to show the analytical results of γ_L^2 *for all* (c_1, c_2) *and* L*. When* $u_1 = 0, u_2 \neq 0$*,*

$$\gamma_L^2\Big|_{u=(0,u_2)} = \frac{\left| c_2 \left\lceil u_2 \right\rfloor^2 \right|^{\frac{1}{3L}}}{|u_2|^{\frac{2}{3L}}} = c_2^{\frac{1}{3L}}.$$

Similarly, when $u_1 \neq 0, u_2 = 0$*,* $\gamma_L^2|_{u=(u_1,0)} = c_1^{\frac{1}{3L}}$*. From homogeneity, any point of* $(\tilde{u}_1 \neq 0, \tilde{u}_2)$ *can be dilated to two cases:* $u_1 = \pm 1$*, and* $u_2 = \tilde{u}_2\kappa^{3L} \in \mathbb{R}$ *by choosing* $\kappa = |\tilde{u}_1|^{-\frac{1}{2L}}$*, and since the homogeneous degree of right hand side of* (3.28) *is zero,* *therefore*

$$\gamma_L^2\Big|_{u=(\tilde{u}_1,\tilde{u}_2)} = \kappa^0\,\gamma_L^2\Big|_{u=\left(\pm 1,\tilde{u}_2|\tilde{u}_1|^{-\frac{3}{2}}\right)} = \frac{\left|\pm c_1 + c_2\lceil u_2\rfloor^2\right|^{\frac{1}{3L}}}{1+|u_2|^{\frac{2}{3L}}}\,,$$

whose extremum, when it exists, can be found by setting its partial derivative against u_2 *to zero, which is*

$$\frac{1}{3L}\left\lceil \pm c_1 + c_2\lceil u_2\rfloor^2\right\rfloor^{\frac{1-3L}{3L}} 2c_2\,|u_2|\left(1+|u_2|^{\frac{2}{3L}}\right) = \frac{2}{3L}\lceil u_2\rfloor^{\frac{2-3L}{3L}}\left|\pm c_1 + c_2\lceil u_2\rfloor^2\right|^{\frac{1}{3L}}\,,$$

simplified into

$$c_2\lceil u_2\rfloor^{\frac{6L-2}{3L}} = \pm c_1\,.$$

This gives us the maximum at

$$\gamma_L^2\Big|_{u=\left(\pm\kappa^{2L},\pm\kappa^{3L}|c_1/c_2|^{\frac{3L}{6L-2}}\right)} = c_1^{\frac{1}{3L}}\left|1+\left|\frac{c_1}{c_2}\right|^{\frac{1}{3L-1}}\right|^{\frac{1-3L}{3L}}\,.$$

In summary, we have

$$\gamma'_L = \max\left\{c_1^{\frac{1}{6L}}, c_2^{\frac{1}{6L}}, c_1^{\frac{1}{6L}}\left|1+\left|\frac{c_1}{c_2}\right|^{\frac{1}{3L-1}}\right|^{\frac{1-3L}{6L}}\right\}\,,$$

which happens respectively at

$$u = \left(\pm\kappa^{2L},0\right)\quad,\left(0,\pm\kappa^{3L}\right),\quad\left(\pm\kappa^{2L},\pm\kappa^{3L}\,|c_1/c_2|^{\frac{3L}{6L-2}}\right),\quad\forall\,\kappa>0\,.$$

Figure 3.5 shows the value of

$$\zeta_L^L = \max_{\|u(t)\|=1}\frac{\|y\|^L_{L\tau_y,2}}{\|u\|^L_{L\tau_u,2}}$$

with the coordination of $u_1 = \sin\theta, u_2 = \cos\theta$ *with* $c_1 = 1, c_2 = 2$.
For $L = 1$, ${\gamma'_L}^L = c_2^{\frac{1}{6}} = 1.1225$ *at* $u = \left(0,\pm\kappa^{3L}\right)$.
For $L = 0.1$, ${\gamma'_L}^L = c_1^{\frac{1}{6}}\left|1+\left|\frac{c_1}{c_2}\right|^{\frac{1}{3L-1}}\right|^{\frac{1-3L}{6}} = 1.1646$ *at* $u = \left(\pm\kappa^{2L},\pm\kappa^{3L}\,|c_1/c_2|^{\frac{3L}{6L-2}}\right)$.
□

Besides Theorem 3.1, a discontinuous (but not singular) memoryless homogeneous Input-Output Map G could also have a finite $\mathcal{H}_{\infty h}-$norm.

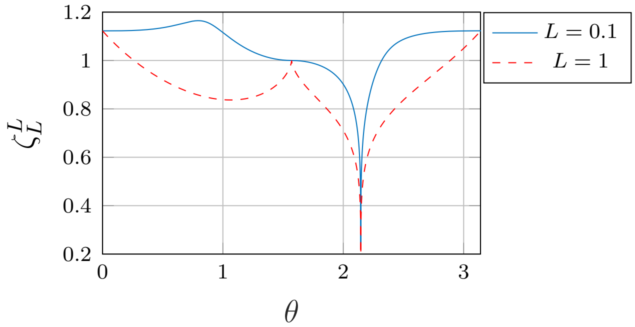

Figure 3.5: ζ_L^L for homogeneous memoryless Input-Output Map in Example 3.3 along unit sphere with $c_1 = 1, c_2 = 2$.

Example 3.4 ($\mathcal{H}_{\infty h}-$norm of discontinuous memoryless homogeneous)**.** *Take another example of memoryless homogeneous Input-Output Map* $y(t) = u_1(t)^2 \text{sign}\,(u_2(t))$*, with weight vector of* $L\tau_u = (L, L)\,, L\tau_y = 2L$*. Note that* $y(t)$ *is discontinuous for* $u_1(t) \neq 0, u_2(t) = 0$*. Yet, with the similar approach shown above, it is easily verified that* $\gamma_L = 1$ *when* $u_1 = \kappa^L, u_2 \to 0$. □

Apparently, the $\mathcal{H}_\infty-$norm in Example 3.2 and Example 3.4 is independent of the choice of L, while the γ_L of Example 3.3 is dependent on L. We show both possibilities in these three examples.

4 Homogeneous $\mathcal{L}_p-$gain for State Space Models

In Chapter 3 we introduce the family of $\mathcal{L}_{ph}-$gain for Input-Output Maps, where the examples that we presented are memoryless and whose homogeneous degree can be freely assigned. On the one hand, finding the $\mathcal{L}_{2h}-$gain γ_L' as per Definition 3.7 for Input-Output Maps is difficult if it has dynamics. On the other hand, most works on homogeneous systems start with a state-space model (SSM) [25, 24, 48]. Thus in this chapter, the concept of the family of $\mathcal{H}_{\infty h}-$norm (or $\mathcal{L}_{ph}-$gain) for Input-Output Map is extendeds to SSM, where the intermediate states allow the use of the Lyapunov function.

We first introduce the continuous homogeneous SSM to be used in the rest of this thesis and extend the definition of weighted homogeneity for an Input-Output Map to be suitable for SSM. Then a partial differential inequality (PDI, or so called dissipation inequality in [50]) is shown to be connected to the $\mathcal{L}_{2h}-$stability for SSM. Thereafter, we show that every continuous homogeneous asymptotically stable SSM has a finite $\mathcal{H}_{\infty h}-$norm. Meanwhile, the previous works in [48, 25, 24] that apply $\mathcal{L}_p-$gain or $\mathcal{H}_\infty-$norm on homogeneous systems are compared with the results in this thesis. Especially, the homogeneous $\mathcal{L}_\infty-$stability and Input-to-State Stability are also included for interested readers.

The materials in this Chapter are published in [60].

4.1 State Space Model for Continuous Homogeneous Dynamics

Consider the following SSM for continuous homogeneous dynamics

$$\Sigma_h : \begin{cases} \dot{x} = f(x, u)\,, \\ y = h(x, u)\,, \end{cases} \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^r$ is the output to be minimized. By continuous we mean that the vector field $f(x, u)$ is continuous in x, u and the function $h(x, u)$ in continuous in x and u. With the SSM, we can extend Definition 3.1 by adding

Definition 4.1 (Weighted homogeneity for SSM [3]). *Fix a set of coordinates $x = (x_1, \cdots, x_n)^\top$ in $\mathbb{R}^n$. Let $\tau_x = (\tau_{x_1}, \cdots, \tau_{x_n})^\top$ be an n-tuple of positive real numbers.*

- *The one-parameter family of dilation $\nu_\kappa^{\tau_x}$ (associated with weight vector τ_x) is defined as*

  $$\nu_\kappa^{\tau_x}(x) \triangleq (\kappa^{\tau_{x1}} x_1, \cdots, \kappa^{\tau_{xn}} x_n)^\top\,, \quad \forall\, x \in \mathbb{R}^n,\, \forall\, \kappa > 0$$

  and denote $\nu_\kappa^\tau(x, u) = \left(\nu_\kappa^{\tau_x}(x)^\top, \nu_\kappa^{\tau_u}(u)^\top\right)^\top$. The numbers τ_{x_i} are called the homogeneous weight of x_i.

- *The homogeneous degree of function $h_i(x, u)$ can be attributed as weight vector of output y_i, i.e. $k_{h_i} = \tau_{y_i}$.*

- *The vector field $f(x, u)$ in Σ_h (4.1) is said to be τ-homogeneous of degree d if each component f_i is τ-homogeneous of degree $k_{f_i} = d + \tau_{x_i}$, that is,*

  $$f_i\left(\nu_\kappa^\tau(x, u)\right) = \kappa^{d+\tau_{x_i}} f_i(x, u), \quad \forall\,(x, u) \in \mathbb{R}^{n+m},\, \forall\,\kappa > 0,\, \forall\, i = 1, \cdots, n\,. \tag{4.2}$$

  or

  $$f\left(\nu_\kappa^\tau(x, u)\right) = \kappa^d \nu_\kappa^\tau\left(f_i(x, u)\right), \quad \forall\,(x, u) \in \mathbb{R}^{n+m},\, \forall\,\kappa > 0\,.$$

For homogeneous dynamics $\dot{x}_i = f_i(x,u)$*, as in* (4.2)*, we have* $d = -\tau_t$ *in accordance with Definition 3.1 [35], since*

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(x,u) \quad \Leftrightarrow \quad \frac{\kappa^{\tau_{x_i}}\,\mathrm{d}x_i}{\kappa^{\tau_t}\,\mathrm{d}t} = \kappa^{d+\tau_{x_i}} f_i(x,u) = f_i\left(\nu_\kappa^\tau(x,u)\right) .$$

The Input-Output Map formed by system Σ_h (4.1) *is homogeneous of degree* d*, if its vector field* $f(x,u)$ *is homogeneous of degree* d *with the same weight vectors.*

From homogeneity, the origin is an equilibrium when $u \equiv 0$. Further from assumption of continuity, we have $\min_i \{\tau_{x_i}\} + d > 0$ as well as $\tau_{y_i} = k_{h_i} > 0$ by Theorem 4.1 in [8]. In the rest of this thesis, $\min_i \{\tau_{x_i}\}$ is denoted as $\min \tau_x$ without ambiguity.

Remark 4.1. *From another aspect, for the system* Σ_h (4.1)*, if the initial value* $x_0 = x(0)$*, input and time are dilated for system* Σ_h (4.1) *along*

$$\begin{array}{lll} x_0 & \to & \tilde{x}_0 = \nu_\kappa^{\tau_x}(x_0)\,, \\ u(t) & \to & \tilde{u}(\tilde{t}) = \nu_\kappa^{\tau_u}(u(t))\,, \\ t & \to & \tilde{t} = \kappa^{\tau_t} t\,, \end{array} \tag{4.3}$$

for any $\kappa > 0$*, then this leads to the trajectory and output for system* Σ_h (4.1) *being dilated as per*

$$\begin{array}{lll} x(t) & \to & \tilde{x}(\tilde{t}) = \nu_\kappa^{\tau_x}(x(t))\,, \\ y(t) & \to & \tilde{y}(\tilde{t}) = \nu_\kappa^{\tau_y}(y(t))\,, \\ t & \to & \tilde{t} = \kappa^{\tau_t} t\,. \end{array} \tag{4.4}$$

That is, for each initial value x_0*, if the input is* u*, let the trajectory and output be* x, y*. Then for all* $\kappa > 0$*, the dilated initial value* $\tilde{x}_0$ *and dilated input* $\tilde{u}$ *with dilated time* $\tilde{t}$ *as in* (4.3) *results in the dilated trajectory* $\tilde{x}$ *and dilated output* $\tilde{y}$ *with dilated time* $\tilde{t}$ *as in* (4.4)*.*

Remark 4.2. *As shown in Proposition 3.1, the Input-Output Map formed by system* Σ_h (4.1) *is homogeneous only when* $x_0 = 0$*, under which case, the dilation of* (4.3) *and* (4.4) *infers* (3.1) *and* (3.2)*.*

Example 4.1 (A linear Input-Output Map with nonlinear vector field and output function)**.** *As brought up in Remark 2.5, it is possible to build an Input-Output Map G* *with a SSM with nonlinear vector field and output function. Take the example of the following continuous homogeneous scalar SSM*

$$\begin{aligned} \dot{x} &= -x - \lceil x \rfloor^{\frac{1}{2}} u + bu^2 \,, \\ y &= \lceil x \rfloor^{\frac{1}{2}} + u \,, \end{aligned}$$

with weight vectors $\tau_y = \tau_u = 1, \tau_x = 2$ and degree $d = 0$. From Remark 4.1, it is clear that this SSM forms a linear Input-Output Map from input-output perspective. When scaling the input by κ, the initial value needs to be scaled by κ^2 in order for the output to be linearly scaled by the same constant κ. Since $\tau_t = -d = 0$, there exists no scaling on time.

4.2 Finite-gain Homogeneous $\mathcal{L}_2-$stability with State Space Model

With the SSM available, Definition 3.6 needs to be extended to include the initial value x_0. Similar to Definition 2.11, we have the following

Definition 4.2 (Finite-gain $\mathcal{L}_{2h}-$stability for SSM)**.** *The system Σ_h (4.1) is called finite-gain $\mathcal{L}_{2h}-$stable if there exists a finite constant γ_L and for each initial value x_0 a finite constant $\beta_L(x_0)$, s.t.*

$$\|y\|_{L\tau_y,\mathcal{L}_2} \leq \gamma_L \|u\|_{L\tau_u,\mathcal{L}_2} + \beta_L(x_0), \quad \forall\, u \in \mathcal{L}_{L\tau_u,2} \,. \tag{4.5}$$

Definition 3.6 covers the Input-Output Map, which contains the SSM system. Since Definition 4.2 requires that one constant γ_L satisfies (4.5) for all x_0 with $\beta_L(x_0)$, thus when $x_0 = 0$, Definition 4.2 infers Definition 3.6 from Remark 4.2.

Remark 4.3 (Homogeneous degree of $\beta_L(x_0)$)**.** *In contrast to Proposition 3.1, where $\beta_L = 0$ if the homogeneous Input-Output Map G is finite-gain $\mathcal{L}_{2h}-$stable, with initial value of state x_0 the homogeneous degree of $\beta_L(x_0)$ as a function of x_0 should be $1 +$*

$L\tau_t/2$ when the system Σ_h (4.1) is finite-gain $\mathcal{L}_{2h}-$stable. The derivation is similar to derivation of (3.19), thus not repeated here.

For the $\mathcal{H}_\infty-$norm in Definition 3.7 to be applicable to $\mathcal{L}_{2h}-$stability in Definition 4.2, we need $\beta_L(x_0) = 0$. This is true from Remark 4.3. Such requirement is also found in LTI systems, i.e. for the Definition 3.7 to be valid, the trajectory should evolve from the origin [65, 50, 29]. We shall define another $\mathcal{H}_{\infty h}-$norm with the help of a Partial Differential Inequality.

Definition 4.3 (Upper estimate of $\mathcal{H}_{\infty h}-$norm)**.** *The continuous homogeneous system Σ_h (4.1) has a finite $\mathcal{L}_{2h}-$gain ($\mathcal{H}_{\infty h}-$norm) less than γ_L, if there exists a positive definite continuously differentiable τ_x-homogeneous storage function $V(x)$ of degree $2 - Ld > 0$ with $V(0) = 0$, such that the following value function J satisfies*

$$J(V_x, x, u) \triangleq V_x f(x,u) + \|y\|^2_{L\tau_y,q} - {\gamma_L}^2 \|u\|^2_{L\tau_u,q} + \epsilon \|x\|^2_{L\tau_x,q} \leq 0\,,\ \forall\,(x,u) \in \mathbb{R}^{n+m} \quad (4.6)$$

for some $\epsilon \geq 0$.

Note that (4.6) consists a partial differential inequality (PDI, or so called dissipation inequality in [50]). The γ_L in Definition 4.3 inferring existence of $\mathcal{H}_{\infty h}-$norm ($\mathcal{L}_{2h}-$gain) in Definition 3.7 is shown in the following theorem. We first need the definition of zero-state detectability.

Definition 4.4 (Zero-state detectability [50])**.** *The system Σ_h (4.1) is called zero-state detectable, if when $u(t) = 0, y(t) = 0, \forall\, t \geq 0$ implies $\lim_{t\to\infty} x(t) = 0$.*

Then we have the following theorem:

Theorem 4.1. *If the condition of Definition 4.3 is satisfied, then the system Σ_h (4.1) is finite-gain $\mathcal{L}_{2h}-$stable and has the $\mathcal{H}_{\infty h}-$norm in Definition 3.7 less or equal to γ_L. When $\epsilon > 0$, the unperturbed system Σ_h (4.1) is globally asymptotically stable at the origin. When $\epsilon = 0$, if system Σ_h (4.1) is further zero-state detectable, then the unperturbed system Σ_h (4.1) is also globally asymptotically stable at the origin.*

Proof. The proof is similar to that in Lemma 2.1. From Definition 4.3, ensure (4.6) for all time leads to

$$\int_0^\infty J(V_x, x, u)\,\mathrm{d}t = V(x(\infty)) - V(x_0) + \|y\|^2_{L\tau_y,\mathcal{L}_2} - {\gamma_L}^2 \|u\|^2_{L\tau_u,\mathcal{L}_2} + \epsilon \|x\|^2_{L\tau_x,\mathcal{L}_2} \leq 0\,.$$

Therefore for all input $u \in \mathcal{L}_{L\tau_u,2}$, we have

$$\begin{aligned}\|y\|^2_{L\tau_y,\mathcal{L}_2} &\leq V(x(\infty)) + \|y\|^2_{L\tau_y,\mathcal{L}_2} \leq {\gamma_L}^2 \|u\|^2_{L\tau_u,\mathcal{L}_2} + V(x_0) - \epsilon \|x\|^2_{L\tau_x,\mathcal{L}_2} \\ &\leq {\gamma_L}^2 \|u\|^2_{L\tau_u,\mathcal{L}_2} + V(x_0)\,.\end{aligned}$$

From Jensen's inequality (A.6), we have

$$\|y\|_{L\tau_y,\mathcal{L}_2} \leq \gamma_L \|u\|_{L\tau_u,\mathcal{L}_2} + \sqrt{V(x_0)}\,, \quad \forall\, u \in \mathcal{L}_{L\tau_u,2}\,. \tag{4.7}$$

Thus, the system Σ_h (4.1) is finite-gain $\mathcal{L}_{2h}-$stable from Definition 4.2. Note that the homogeneous degree of $\beta_L(x_0) = \sqrt{V(x_0)}$ w.r.t to x_0 is $1 - Ld/2 = 1 + L\tau_t/2$, as predicted in Remark 4.3. When $x_0 = 0$, $V(0) = 0$ as assumed in Definition 4.3, then from (4.7) and Definition 3.7

$$\gamma'_L = \sup_{\|u\|_{L\tau_u,\mathcal{L}_2} \neq 0, u \in \mathcal{L}_{L\tau_u,2}} \frac{\|y\|_{L\tau_y,\mathcal{L}_2}}{\|u\|_{L\tau_u,\mathcal{L}_2}} \leq \gamma_L\,,$$

therefore γ_L is an upper bound for the $\mathcal{H}_{\infty h}-$norm γ'_L in Definition 3.7. Further, (4.6) with $u \equiv 0$ reads

$$\frac{\partial V(x)}{\partial x} f(x,0) \leq -\|h(x,0)\|^2_{L\tau_y,q} - \epsilon \|x\|^2_{L\tau_x,q}\,, \quad \forall\, x \in \mathbb{R}^n\,.$$

When $\epsilon = 0$, $V(x)$ is a weak Lyapunov function. Since $V(x)$ is of homogeneous degree $2 - Ld$ and positive definite, it is also radially unbounded from homogeneity. Then from zero-state detectability and LaSalle's Invariance principle, the origin is the globally asymptotically stable equilibrium for the unperturbed system Σ_h (4.1) [30].
When $\epsilon > 0$, $V(x)$ is a strict Lyapunov function, and Lyapunov's theorem implies global asymptotic stability of the origin $x = 0$ for the unperturbed system, without assuming detectability. □

Since LTI systems also belong to continuous homogeneous systems, Theorem 4.1 is also true for LTI systems. Similar conclusions can be found in [50]. Further the converse of Theorem 4.1 is also true for LTI systems. That is, a stable and causal LTI system has finite $\mathcal{H}_\infty-$norm, which can be derived with the help of transfer function in Hardy space [59] or with the help of SSM [65].

However, the converse of Theorem 4.1 with classical $\mathcal{H}_\infty-$norm is not always true for nonlinear systems. Namely, if the system Σ_h (4.1) is globally asymptotically stable, the classical $\mathcal{H}_\infty-$norm might not exist. This is shown in Example 3.1 when each element of the weight vectors of the input and the output are not all equal. On the contrary, we would like to show that such converse theorem is true for the family of $\mathcal{H}_{\infty h}-$norm. In order to do that, we first introduce the following lemma

Lemma 4.1 ([13, 22]). *Let $\psi(x)$ and $\omega(x)$ be two continuous real-valued homogeneous functions of degree $s > 0$ w.r.t. weight vector τ, where $\omega(x)$ is positive definite, and*

$$\{x \in \mathbb{R}^n \backslash \{0\} : \omega(x) = 0\} \subseteq \{x \in \mathbb{R}^n \backslash \{0\} : \psi(x) < 0\} \ .$$

Then there exists positive real numbers $\gamma^\star$ and $c > 0$ such that for all $\gamma \geq \gamma^\star$ and all $x \in \mathbb{R}^n \backslash \{0\}$ it holds $\psi(x) - \gamma\omega(x) < -c \left\|x\right\|_{\tau,q}^{s}$.

With the Lemma we prove the following Theorem.

Theorem 4.2. *If the unperturbed continuous homogeneous system Σ_h (4.1) is locally asymptotically stable, then the condition from Definition 4.3 is met for $L < 2/\left(d + \max \tau_x\right)$. Therefore system Σ_h (4.1) is finite-gain $L\tau-$homogeneous $\mathcal{L}_2-$stable and has a finite $\mathcal{H}_{\infty h}-$norm.*

Proof. By restricting the system Σ_h (4.1) to be continuous and asymptotically stable, a continuously differentiable, strict Lyapunov function $V_l(x)$ of homogeneous degree $2 - Ld > \max\left(L\tau_x\right)$ (effectively $L < 2/\left(d + \max \tau_x\right)$) when $u \equiv 0$ exists. This is a direct consequence of [3, Theorem 5.8] and [47]. Due to this homogeneous, strict Lyapunov function when $u \equiv 0$, there exist $c_1 > 0$ such that $\dot{V}_l(x, 0)$ of homogeneous degree 2 satisfies (Corollary 5.4 in [3])

$$\dot{V}_l(x, 0) \leq -c_1 \left\|x\right\|_{L\tau,q}^{2} \ , \quad \forall\, x \in \mathbb{R}^n \ ,$$

for any $q \geq 1$. Since $\|h(x,0)\|^2_{L\tau_y,q}$ is also a homogeneous positive semi-definite function of x with degree 2, there also exists $c_2 > 0$, such that

$$\|h(x,0)\|^2_{L\tau_y,q} \leq c_2 \|x\|^2_{L\tau,q} , \quad \forall\, x \in \mathbb{R}^n .$$

Thus with this $V_l(x)$, any $V(x) = aV_l(x)$ with $a > \underline{a} = \frac{c_2+\epsilon}{c_1}$ guarantees that for all $x \in \mathbb{R}^n \backslash \{0\}$

$$\frac{\partial V(x)}{\partial x} f(x,0) + \|h(x,0)\|^2_{L\tau_y,q} + \epsilon \|x\|^2_{L\tau_x,q} \leq (-ac_1 + c_2 + \epsilon) \|x\|^2_{L\tau_x,q} < 0 . \tag{4.8}$$

Then define two homogeneous value functions

$$\omega(x,u) \triangleq \|u\|^2_{L\tau_u,q} ,$$
$$\psi(x,u) \triangleq \frac{\partial V}{\partial x} f(x,u) + \|h(x,u)\|^2_{L\tau_y,q} + \epsilon \|x\|^2_{L\tau_x,q} .$$

Both are continuous in x or u and have the homogeneous degree 2 w.r.t. $L-$scaled weight vector. Clearly $\omega(x,u) \geq 0$ and $\omega(x,u) = 0 \Leftrightarrow u = 0$, if we can ensure that $\psi(x,0) < 0$, then according to Lemma 4.1 such finite γ_L exists, when $\gamma > \gamma_L$, (4.6) is satisfied. Such V and γ_L satisfy the Definition 4.3. From Theorem 4.1, the system Σ_h (4.1) is finite-gain $\mathcal{L}_{2h}-$stable and such finite γ_L is an upper bound for the $\mathcal{H}_{\infty h}-$norm γ'_L in Definition 3.7. Therefore the $\mathcal{H}_{\infty h}-$norm for system Σ_h (4.1) is also finite.
Note that here $\psi(x,0) < 0$ is equivalent to (4.8), which is shown above to be valid for every storage function $V_l(x)$ (strict homogeneous Lyapunov function when $u(\cdot) = 0$) and $a > \underline{a}$. □

Remark 4.4 (Relationship between Definition 4.3 and Definition 3.7)**.** *Definition 4.3, which derive the upper bound of $\mathcal{H}_{\infty h}-$norm from storage function and vector field, is a subset of Definition 3.7, since*

- *Definition 4.3 is not applicable to the systems where non-observable unstable states exist, whereas Definition 3.7 for such systems from solely input-output perspective might still be applicable.*

- *In Theorem 4.2, Definition 4.3 predicts the existence of only a subset of Definition 3.7, i.e. with $L < 2/\left(d + \max \tau_x\right)$. This restriction results from the use of* *a continuously differentiable storage function $V(x)$ from the converse Lyapunov Theorem, whereas Definition 3.7 does not impose such restriction from an input-output perspective.*

That is, Definition 4.3 infers Definition 3.7, but not necessarily the other way around.

Remark 4.5 (Construction of the storage function for the Partial Differential Inequality inequality)**.** *The PDI in Definition 4.3 depends on the construction of a storage function. Although there is no general method to obtain storage functions, for certain classes of homogeneous systems the recent work [58] provides a methodology to construct them, using generalized homogeneous forms and the sum of squares (S.O.S.) technique. When we have a smooth $V(x)$ of homogeneity degree $p-d$ satisfying inequality* (4.6)*, it is possible to build another smooth storage function $V^{\frac{p'-d}{p-d}}(x)$ of homogeneous degree $p'-d$ for $p'>p$. On the contrary, for $p'<p$ the differentiability of function $V^{\frac{p'-d}{p-d}}(x)$ at $x=0$ might be lost.*

Remark 4.6 (The role of ϵ)**.** *The $\epsilon \geq 0$ in* (4.6) *serves similarly to the sub-optimal ARE brought up in* (2.18) *for LTI systems.*

4.3 Upper estimate of Homogeneous $\mathcal{L}_2-$gain

In [48, 25, 24], the classical $\mathcal{L}_p-$gain or the $\mathcal{H}_\infty-$norm is shown to exist for some particular continuous homogeneous systems, where $\tau_u = \ell \mathbf{1}_m, \tau_y = \ell \mathbf{1}_r, \ell > 0$. Yet no systematic method is provided to estimate their value.
Contrary to the previous works of [48, 25, 24], the authors of [61] provide a general method and a simplified method to calculate an upper estimate of the $\mathcal{H}_{\infty h}-$norm for the Continuous Super-Twisting-like Algorithm (CSTLA). In [60], such method is generalized to all continuous homogeneous systems Σ_h (4.1).

4.3.1 General Case

Proposition 4.1 (Upper estimate of $\mathcal{L}_2$−gain for system Σ_h (4.1))**.** *When the condition of Definition 4.3 is met for system Σ_h (4.1), for each V the smallest $\gamma_L^\star(V)$ that satisfies (4.6) is*

$$\begin{aligned} \gamma_L^\star(V_x) &= \sqrt{\max_{\|(x,u)\|=1} \zeta(V_x, x, u)}\,, \\ \zeta(V_x, x, u) &= \frac{V_x f(x,u) + \|y\|^2_{L\tau_y,q}}{\|u\|^2_{L\tau_u,q}}\,, \|u\| \neq 0\,. \end{aligned} \tag{4.9}$$

The homogeneous real-valued function $\zeta(V_x, x, u)$ can be written as $\zeta(x,u)$ since V_x is also a homogeneous function of x.

Proof. It is easy to see that for $\gamma_L \geq \gamma_L^\star$ from (4.9) when substituted into (4.6) guarantees that $J(V_x, x, u) \leq 0$. On the other hand, any $\gamma_L < \gamma_L^\star$ leads to the value function being indefinite. Therefore $\gamma_L^\star$ in (4.9) is the smallest γ_L that can maintain $J(V_x, x, u) \leq 0$ for this storage function V at all time.
When $u = 0$, γ_L has no impact on the value of J. When $u \neq 0$, both numerator and denominator of ζ in (4.9) are of homogeneous degree 2. Therefore $\zeta(x,u)$ is a homogeneous function of degree 0, whose value in the whole space $\mathbb{R}^{n+m}\backslash\{0\}$ can be projected onto the unit sphere. This is due to that by definition of homogeneity we have

$$\zeta\left(\nu_\kappa^{L\tau}(x,u)\right) = \kappa^0 \zeta(x,u)$$

for all $\kappa > 0$, then from any point $(x,u) \in \mathbb{R}^{n+m}\backslash\{0\}$ we can find κ with $x_i' = \kappa^{L\tau_{x_i}} x_i, u_j' = \kappa^{L\tau_{u_j}} u_j$, such that (x', u') is on the unit sphere by solving

$$\sum_{i=1}^{n} \kappa^{2L\tau_{x_i}} x_i^2 + \sum_{j=1}^{m} \kappa^{2L\tau_{u_j}} u_j^2 = 1\,. \tag{4.10}$$

When $\kappa \to 0$, the left hand side of (4.10) converges to 0 (vector weight is positive). When $\kappa \to \infty$, it diverges to ∞. By continuity of functions there exists a solution of κ which makes the left hand side equal to 1 such that $\zeta(x', u') = \zeta(x,u)$. Thus, we need to search only on the unit sphere of (x,u) to probe the value of ζ in $\mathbb{R}^{n+m}\backslash\{0\}$. Actually, the value of $\zeta(V_x, x, u)$ can be found on any closed surface that encloses the origin. The proof is straightforward. □

The detailed procedure to search the value of $\max_{\|(x,u)\|=1}\{\zeta(x,u)\}$ is described in A.7.1 for the interested reader.

4.3.2 Affine Case

Now consider the continuous homogeneous system Σ_h (4.1) being affine in u, and the output being devoid of u, i.e.

$$\Sigma_{ha}:\begin{cases}\dot{x}=f(x)+g(x)u\,,\\ y=h(x)\,,\end{cases} \tag{4.11}$$

where g is a homogeneous $n\times m$ matrix-valued function of x.
From homogeneity, the homogeneous degree of $f(x)$ must match that of $g(x)u$, i.e. $\tau_{x_i}-\tau_t=k_{f_i}=k_{g_{ij}}+\tau_{u_j}$ for all $i=1,\cdots,n, j=1,\cdots,m$, where k_{f_i} and $k_{g_{ij}}$ denote the component-wise homogeneous degree of $f_i(x)$ and $g_{ij}(x)$, respectively. With continuity of $g(x)$ in x, we conclude that $k_{g_{ij}}\geq 0$ [8, Theorem 4.1].
For each u_j, $g_{ij}(x)$ cannot be zero for all i, otherwise such input does not affect the system. Therefore $L\tau_{u_j}\leq L\tau_{u_j}+Lk_{g_{ij}}=L\max\tau_x+Ld<2$ from non-negativeness of $k_{g_{ij}}$. Further, we denote $g=(g_1,\dots,g_m)$, where g_i is the i-th column of matrix g.

Proposition 4.2 (Upper estimate of $\mathcal{L}_2-$gain for system Σ_h (4.1) when affine in input)**.** *When Definition 4.3 is met for continuous homogeneous system* Σ_{ha} (4.11)*, with additionally assuming* $L<2/(d+\max\tau_x)$ *and* $q=2$*, for each* V *the smallest* $\gamma_L^\star$ *that satisfies* (4.6) *can be derived by solving the following optimization problem*

$$\begin{aligned}\gamma_L^\star(V)&=\inf\left\{\gamma\,\middle|\,\sup_{\|x\|=1}\mu(V_x,x,\gamma)\leq 0\right\}\,,\\ \mu(V_x,x,\gamma)&=J(V_x,x,0)+\sum_{i=1}^{m}\gamma^{\frac{-2L\tau_{u_i}}{2-L\tau_{u_i}}}C_i\left|V_xg_i(x)\right|^{\frac{2}{2-L\tau_{u_i}}}\,,\\ C_i&=\left|\frac{L\tau_{u_i}}{2}\right|^{\frac{L\tau_{u_i}}{2-L\tau_{u_i}}}-\left|\frac{L\tau_{u_i}}{2}\right|^{\frac{2}{2-L\tau_{u_i}}}\,.\end{aligned} \tag{4.12}$$

Proof. By choosing the storage function V in Definition 4.3, we have

$$J\left(V_x, x, u\right) = V_x\left(f(x) + g(x)u\right) + \|y\|^2_{L\tau_y,2} - \gamma^2 \|u\|^2_{L\tau_u,2}$$
$$= J\left(V_x, x, 0\right) + V_x g(x) u - \gamma^2 \|u\|^2_{L\tau_u,2} \,.$$

Since for any x when $u_i \to \pm\infty, \forall\, i$, we have $J \to -\infty$ by homogeneity. The partial derivative of $J\left(V_x, x, u\right)$ against u reads

$$\frac{\partial J\left(V_x, x, u\right)}{\partial u_i} = V_x g_i(x) - 2\gamma^2 \left|u_i\right|^{\frac{2-L\tau_{u_i}}{L\tau_{u_i}}} \,, \text{ where } \frac{\partial \|u\|^2_{L\tau_u,2}}{\partial u} = \frac{2}{L\tau_{u_i}} \left|u_i\right|^{\frac{2-L\tau_{u_i}}{L\tau_{u_i}}} \,.$$

Setting $\partial J/\partial u = 0$ leads to a unique solution, thus by continuity of the function, the maximum of J w.r.t u lies along

$$u_i^\star\left(V_x, x\right) = \left|\frac{L\tau_{u_i}}{2\gamma^2}\right|^{\frac{L\tau_{u_i}}{2-L\tau_{u_i}}} \left\lceil V_x g_i(x)\right\rfloor^{\frac{L\tau_{u_i}}{2-L\tau_{u_i}}} \,, i = 1, \cdots, m \,. \tag{4.13}$$

Inserting this homogeneous $u^\star$ back to value function J yields

$$J(V_x, x, u^\star) = J\left(V_x, x, 0\right) + \sum_{i=1}^{m} \left|\gamma^2\right|^{\frac{-L\tau_{u_i}}{2-L\tau_{u_i}}} C_i \left|V_x g_i(x)\right|^{\frac{2}{2-L\tau_{u_i}}} \,.$$

With additional assumption of $L\tau_{x_i} + Ld < 2$, we have $\max\left(L\tau_u\right) < L\tau_{x_i} + Ld < 2$, which implies $C_i > 0$ and the power of γ is negative in (4.12). Thus, if $\gamma \to \infty$, then $\mu = J\left(V_x, x, 0\right) \leq 0$, using the storage function $V = aV_l, a > \underline{a}$ in Theorem 4.2. If $\gamma \to 0$ there exist some x s.t. the summation of the remaining terms diverges to $+\infty$. So from continuity of function as well as $\mu(V_x, x, \gamma)$ being strictly decreasing w.r.t γ, there exist one smallest $\gamma > 0$, s.t. $\max_x \mu(V_x, x, \gamma) \leq 0$. Therefore $\gamma > \gamma_L^\star$ from (4.12) guarantees that $J(V_x, x, u^\star) \leq 0$. Then with $\gamma < \gamma_L^\star$, the value function is indefinite. By such properties, we can use convergence search on $\gamma \in \mathbb{R}_+$, based on the sign of $\max_x(\mu)$ on unit sphere w.r.t. x to find the smallest $\gamma_L^\star$ that guarantees negative semi-definiteness of $J(V_x, x, u^\star)$.

Even though $\mu(V_x, x, \gamma)$ is a real-valued homogeneous function of degree 2, we only need to check its sign, not its value. Therefore the search procedure can still be done on the surface of unit sphere with respect to x only. □

The result of Proposition 4.2 suggests the following search procedure to find an estimation of $\gamma_L^\star(V)$: $\max_{\|x\|=1} \mu\left(V(x), x, \eta\right)$. Note that this procedure is analogous to the

Algorithm 3 Search procedure for Proposition 4.2

```
Initialize the upper limit γ_u with large enough real positive number
Initialize the lower limit γ_l = 0
while (γ_u − γ_l)/γ_l ≥ 10^{-7} do
    Choose one γ ∈ (γ_l, γ_u), e.g. γ = (γ_u + γ_l)/2
    Evaluate max_{||x||=1} μ(V_x, x, γ) in (4.12)
    if max_{||x||=1} μ(V_x, x, γ) ≤ 0 then
        γ_u = γ
    else
        γ_l = γ
    end if
end while
γ_L^⋆ = γ_u
```

search performed to estimate the $\mathcal{L}_2$−norm using the Hamiltonian matrix (see Section 2.5.2). For the Hamiltonian matrix, we need to check whether its eigenvalues stay on the imaginary axis with each γ, here we need to check the sign of $\max_{\|x\|=1} \mu\left(V(x), x, \gamma\right)$.

Remark 4.7. *The search of* $\max_{\|x\|=1} \mu\left(V(x), x, \gamma\right)$ *is similar to that described in Algorithm 4. The difference is that we only need to search on the surface of* $\|x\| = 1$ *instead of the surface of* $\|(x, u)\| = 1$. *Besides that, a further iteration on* γ *is necessary. Thus by using the search of* (4.12), *we can reduce the computational complexity from* $O\left(\upsilon^{n+m-1}\right)$ *in* (4.9) *to* $O\left(\upsilon^{n}\right)$. *When* $\tau_u = \ell \mathbf{1}_m$, *the computational complexity can be further reduced to* $O\left(\upsilon^{n-1}\right)$, *with similar method in Proposition 4.1, i.e.*

$$
\begin{aligned}
\gamma^\star(V) &= \left| \max_{\|x\|=1} \frac{\left(\frac{\ell}{2}\right)^{\frac{\ell}{2-\ell}} \left(1-\frac{\ell}{2}\right) \sum_{i=1}^m \left|V_x g_i(x)\right|^{\frac{2}{2-\ell}}}{-V_x f(x) - \|h(x)\|_{\tau_y,q}^2} \right|^{\frac{2-\ell}{2\ell}} \\
&= \left| \max_{\|x\|=1} \frac{\left(\frac{\ell}{2}\right)^{\frac{\ell}{2-\ell}} \left(1-\frac{\ell}{2}\right) \sum_{i=1}^m \left|V_x g_i(x)\right|^{\frac{2}{2-\ell}}}{-J(V_x, x, 0)} \right|^{\frac{2-\ell}{2\ell}} .
\end{aligned}
\tag{4.14}
$$

Refer to the example of the CSTLA in [61].

For each V, we may find the smallest $\gamma_L^\star(V)$ that satisfies Definition 4.3 by applying

Proposition 4.1 or 4.2. In order to find the smallest upper bound, we shall search for

$$\inf_V \gamma_L^\star(V)\,. \tag{4.15}$$

Note that (4.15) might not give a tight upper bound of γ_L' from (3.20). The optimality of (4.15) is not guaranteed, i.e. the actual $\mathcal{L}_{2h}-$gain in time domain might be smaller than (4.15) [4]. Only when we find some actual input that achieves the ratio of $\mathcal{L}_{2h}-$norm of output over input equals to (4.15), we can say $\gamma_L^\star = \gamma_L'$.

Remark 4.8 (Proposition 4.1 and 4.2 for LTI systems)**.** *The method in Proposition 4.1 and 4.2 also applies to LTI systems, where we can set weight vector $\tau_x = \mathbf{1}_n, \tau_y = \mathbf{1}_r, \tau_u = \mathbf{1}_m$, and $d = 0$, then*

- *for $L = 1$, the Proposition 4.2 is equivalent to solving an algebraic Riccati inequality (ARI) for an LTI system. For example, take the LTI system $\dot{x} = Ax+Bu, y = Ex$ and storage function of $V = x^\top Px$, where $P = P^\top > 0$. First of all, we have $C_i = 1/4$, and (4.12) becomes*

$$\begin{aligned}\mu(V_x, x, \gamma) &= J(V_x, x, 0) + \frac{1}{4\gamma^2}\sum_{i=1}^m |V_x g_i(x)|^2 \\ &= x^\top \left(PA + A^\top P + E^\top E\right) x + \frac{1}{4\gamma^2}\sum_{i=1}^m \left|2x^\top PB_i\right|^2 \\ &= x^\top \left(PA + A^\top P + \frac{1}{\gamma^2}PBB^\top P + E^\top E\right) x\,.\end{aligned}$$

  This is of the same form as the ARE (2.8) when $D = 0$ [4]. So for this case we have $\gamma_L^\star = \gamma_L' = \gamma^\dagger$ when $L = 1$.

- *for $L = 2/p, p > 1$, ${\gamma_L^\star}^L$ in Proposition 4.1 and 4.2 provides an upper bound for the $\mathcal{L}_p-$gain of the LTI system. Refer to Section 2.6 for previous results of upper estimates of the $\mathcal{L}_p-$gain. In this thesis we propose another bound for the $\mathcal{L}_p-$gain by solving the PDI of the storage function for each particular $p \geq d+\max\tau_x$. This will be brought up in the next section.*

Therefore, by using Proposition 4.1 or 4.2, a method to calculate the $\mathcal{H}_{\infty h}-$norm defined by (3.20) is proposed. This is an extension to [61], where it is applied only to a special continuous homogeneous system CSTLA with $L = 1$, and it is also an extension to [25, 24], where it applies only to weight vector of the input and the output being equal and affine in the input. For LTI systems, such family of $\mathcal{H}_{\infty h}-$norm allows another estimate of the $\mathcal{L}_p-$gain in state space.

4.4 Extension to Homogeneous $\mathcal{L}_p-$stability

With the state space model, we can also write the PDI (4.6) for the case of homogeneous $\mathcal{L}_p-$gain.

Definition 4.5 ($\mathcal{L}_{ph}-$gain)**.** *The continuous homogeneous system Σ_h (4.1) has a $\mathcal{L}_{ph}-$gain less than γ_p, if there exists a positive definite continuously differentiable storage function $V(x)$ of homogeneous degree $p-d>0$ with $V(0)=0$, such that the following value function satisfies*

$$J(V_x, x, u) \triangleq V_x f(x,u) + \|y\|^p_{\tau_y,q} - {\gamma_p}^p \|u\|^p_{\tau_u,q} + \epsilon \|x\|^p_{\tau_x,q} \leq 0\,,\ \forall\,(x,u) \in \mathbb{R}^{n+m} \quad (4.16)$$

for some $\epsilon \geq 0$.

The proof is similar to that of Theorem 4.1. The relationship between the $L\tau-$homogeneous $\mathcal{L}_2-$gain γ_L and the $\tau-$homogeneous $\mathcal{L}_p-$gain γ_p is shown in the next remark from the perspective of the PDI.

Remark 4.9 (Effect of weight scaling on Partial Differential Inequality)**.** *Recall that, as already observed in Remark 3.2 and Remark 3.6, scaling the homogeneity degree and weight vectors by a positive constant $L>0$ changes the values of the homogeneous vector norms as well as the homogeneous signal norms involved. Performing the scaling, we can also relate the $L\tau-$homogeneous $\mathcal{L}_p-$gain with the $\tau-$homogeneous $\mathcal{L}_{p/L}-$gain.*
For example, from Theorem 4.2, there exists a finite γ_L for $2 > L(d + \max\tau_x)$ if the unperturbed continuous homogeneous system Σ_h (4.1) is asymptotically stable (now with $L-$scaled weight and degree). The dissipation inequality (4.6) is now

$$\frac{\partial V(x)}{\partial x} f(x,u) + \|y\|^2_{L\tau_y,q} - \gamma^2 \|u\|^2_{L\tau_u,q} \leq -\epsilon \|x\|^2_{L\tau_x,q}\,, \quad \forall\,(x,u) \in \mathbb{R}^{n+m}.$$

From (3.7) *in Remark 3.2, the previous inequality can be re-written as*

$$\frac{\partial V(x)}{\partial x} f(x,u) + \|y\|^{2/L}_{\tau_y, q/L} - \tilde{\gamma}^{2/L} \|u\|^{2/L}_{\tau_u, q/L} \leq -\epsilon \|x\|^{2/L}_{\tau_x, q/L} \, , \quad \forall (x,u) \in \mathbb{R}^{n+m} \, ,$$

with $\tilde{\gamma} = \gamma^L$. *Relating to Theorem 4.2, the* $\tau-$*homogeneous* $\mathcal{L}_{2/L}-$*gain equals the* $L\tau-$*homogeneous* $\mathcal{L}_2-$*gain to the power of* L. *However, we must be aware that the* $\tau-$*homogeneous* $\mathcal{L}_{p_1}-$*gain is not related to the* $\tau-$*homogeneous* $\mathcal{L}_{p_2}-$*gain for* $p_1 \neq p_2$ *with such scaling on weight and degree. The previous analysis only says that the* $\tau-$*homogeneous* $\mathcal{L}_{p_2}-$*gain can be derived from the* $L\tau-$*homogeneous* $\mathcal{L}_2-$*gain, with* $L = 2/p_2$. *Yet the latter has nothing to do with the* $\tau-$*homogeneous* $\mathcal{L}_2-$*gain when* $L \neq 1$.

As described in Remark 4.9, the upper bound of the $\tau-$homogeneous $\mathcal{L}_p-$gain γ_p can be related to an upper bound of the $L\tau-$homogeneous $\mathcal{L}_2-$gain γ_L by

$$\gamma_L{}^L = \gamma_p \, , \text{ where } L = \frac{2}{p},$$

i.e. if γ_L is an upper bound for the $L\tau-$homogeneous $\mathcal{L}_2-$gain, then γ_L^L is an upper bound of the $\tau-$homogeneous $\mathcal{L}_p-$gain.

Theorem 4.3. *If the unperturbed continuous homogeneous system* Σ_h (4.1) *is locally asymptotically stable, then the condition from Definition 4.5 is met for* $p > d + \max \tau_x$.

The proof is similar as that for Theorem 4.2.

Remark 4.10. *Suppose the sub-systems satisfy the condition of Theorem 3.2 or Theorem 3.3 for some* $p = p_1$. *Further if they have state space realization* Σ_h (4.1) *and meet the condition from Theorem 4.3, then the conclusion in Theorem 3.2 and Theorem 3.3 is also true for all* $p \in [p_1, \infty]$.

4.5 Homogeneous $\mathcal{L}_\infty-$stability and Input-to-State Stability of Homogeneous Systems

In Definition 4.5, the $\mathcal{L}_{ph}-$gain can be derived from the PDI and storage function. Yet, the case of $p = \infty$, i.e. $L = 0$ is not yet explicitly considered. If for system Σ_h (4.1) the output $y = x$ is selected, then $\mathcal{L}_{\infty h}-$stability is intimately related to Input-to-State Stability (ISS). In fact, note that if inequality (4.16) is satisfied for some $p \geq 1$ and $\epsilon \geq 0$, then function V qualifies as an *ISS Lyapunov function* [50, 26, 29], and therefore the system is ISS. Moreover, Theorem 4.2 assures that if the unperturbed system has $x = 0$ as an asymptotically stable equilibrium point, then the perturbed system is ISS by simply choosing $y = x$. This result is well-known from [1, 7]. It generalizes partial results obtained in [48] for classical homogeneous systems, and in [24] for a particular class of weighted homogeneous systems affine in the input. The results presented in this work make it explicit that the ISS Lyapunov function should satisfy the inequality (4.16) in the homogeneous case, in contrast to the general form presented in [7]. Moreover, the ISS inequality is given explicitly and an estimation of the ISS-gain is provided using homogeneous norms, which are more appropriate in this context.

4.5.1 Homogeneous $\mathcal{L}_\infty-$stability and Input-to-State Stability

A particular definition of ISS for homogeneous systems, tailored to homogeneous systems is proposed, which is an extension of the homogeneous $\mathcal{L}_\infty-$stability with finite-gain.

Definition 4.6. *The homogeneous system* Σ_h (4.1)*, with homogeneity degree* d*, is said to be homogeneous input-to-state stable (ISS) if there exist positive constants* M*,* κ*,* γ_{iss} *such that for any input* $u(\cdot) \in \mathcal{L}_{\infty h} = \mathcal{L}_\infty$ *and any initial value* $x_0 \in \mathbb{R}^n$*, the trajectory* $x(t)$ *of* Σ_h (4.1) *satisfies for all* $t \geq 0$

$$\|x(t)\|_{\tau_x,q} \leq \beta\left(\|x_0\|_{\tau_x,q}, t\right) + \gamma_{iss}\|u(\cdot)\|_{\tau_u,\mathcal{L}_\infty}, \tag{4.17}$$

where q is some number $q \geq 1$ and β is the following $\mathcal{KL}$ function

$$\beta(v,\, t) = \begin{cases} \begin{cases} M\left(v^{-d} - \kappa t\right)^{-\frac{1}{d}} & \text{when } t \leq \frac{1}{\kappa} v^{-d} \\ 0 & \text{when } t \geq \frac{1}{\kappa} v^{-d} \end{cases} & \text{if } d < 0\,, \\ M \exp(-\kappa t) v & \text{if } d = 0\,, \\ M\left(v^{-d} + \kappa t\right)^{-\frac{1}{d}} & \text{if } d > 0\,. \end{cases} \tag{4.18}$$

Note that the *ISS-gain function* is *linear* in the $\mathcal{L}_{\infty h}$−norm $\|u(\cdot)\|_{\tau_u, \mathcal{L}_\infty}$ (cfr. (4.17)). This linearity derives from the homogeneity and the use of homogeneous norms. In the classical norms, the relationship is nonlinear (see e.g. [7]). This illustrates the convenience of adopting homogeneous $\mathcal{L}_\infty$−norms for homogeneous systems.

It is possible to write an ISS inequality similar to (4.17) using classical norms. From the relationship between homogeneous norms given in Lemma 3.2 we can obtain

$$\|x(t)\|_q \leq \tilde{\beta}\left(\|x_0\|_q\,, t\right) + \tilde{\gamma}_{\text{iss}}\left(\|u(\cdot)\|_{\mathcal{L}_\infty}\right)\,,$$

but with $\tilde{\gamma}_{\text{iss}}(\cdot)$ a non–linear function, and $\tilde{\beta}$ different from (4.18). This is the form of the result presented in [7].

Theorem 4.4. *Consider a homogeneous system Σ_h (4.1). If the unperturbed system, with $u = 0$, is locally asymptotically stable at the origin, then system Σ_h (4.1) is ISS with a linear gain function and it has also a finite $\mathcal{L}_{\infty h}$−gain.*

Proof. From Theorem 4.3, we can show that for any $p > d + \max \tau_x$, $p \geq 1$, and considering $y = x$, there exists a homogeneous $V(x)$ such that inequality (4.16) is satisfied for some $\epsilon > 0$, i.e.

$$\frac{\partial V(x)}{\partial x} f(x,u) \leq -\left(\epsilon + 1\right) \|x\|^p_{\tau_x, q} + \gamma^p \|u\|^p_{\tau_u, q}\,, \quad \forall\,(x,u) \in \mathbb{R}^{n+m}\,.$$

Then, for $\|x\|_{\tau_x,q} \geq \frac{\gamma k^{\frac{1}{p}}}{(\epsilon+1)^{\frac{1}{p}}} \|u\|_{\tau_u,q}$, with $k > 1$, the inequality

$$\frac{\partial V(x)}{\partial x} f(x,u) \leq -\frac{k-1}{k}\left(\epsilon + 1\right) \|x\|^p_{\tau_x,q} \tag{4.19}$$

is satisfied. Note that, since $V(x)$ when $u=0$ is a strict Lyapunov function, there exist $0<\alpha_1\le\alpha_2$ such that

$$\alpha_1 \|x\|_{\tau_x,q}^{p-d} \le V(x) \le \alpha_2 \|x\|_{\tau_x,q}^{p-d}\,, \quad \forall\, x\in\mathbb{R}^n\,. \tag{4.20}$$

Define $b=\frac{\gamma k^{\frac{1}{p}}}{(\epsilon+1)^{\frac{1}{p}}}$ and $c=\alpha_2\left(b\|u(\cdot)\|_{\tau_u,\mathcal{L}_\infty}\right)^{p-d}$, where $\|u(\cdot)\|_{\tau_u,\mathcal{L}_\infty}=\sup_{t\ge0}\|u(t)\|_{\tau_u,q}$ is used, then the set

$$\Omega_c=\{x\in\mathbb{R}^n: V(x)\le c\}$$

is such that

$$\mathcal{B}_b=\left\{x\in\mathbb{R}^n: \|x\|_{\tau_x,q}<b\|u(\cdot)\|_{\tau_u,\mathcal{L}_\infty}\right\}\subset\Omega_c\,.$$

As a consequence, for each x on the boundary of Ω_c, we have $\|x\|_{\tau_x,q}\ge b\|u(\cdot)\|_{\tau_u,\mathcal{L}_\infty}$. Therefore, at any $t\ge0$ such that $x(t)$ is on the boundary of Ω_c, we have $\|x(t)\|_{\tau_x,q}\ge b\|u(\cdot)\|_{\tau_u,\mathcal{L}_\infty}\ge\frac{\gamma k^{\frac{1}{p}}}{(\epsilon+1)^{\frac{1}{p}}}\|u(t)\|_{\tau_u,q}$. Then from (4.19) it follows that

$$\frac{\partial V(x(t))}{\partial x}f(x(t),u(t))<0$$

at any $t\ge0$ for $x(t)$ on the boundary of Ω_c, and it can be concluded that for any initial condition $\tilde{x}(0)$ in the interior of Ω_c, the solution $\tilde{x}(t)$ of $\dot{x}=f(x,u)$ is defined for all $t\ge0$ and $\tilde{x}(t)\in\Omega_c$ for all $t\ge0$.

Two cases need to be studied separately. First of all, when $x(0)\in\Omega_c$. For all $t\ge0$, $x(t)$ satisfies

$$\|x(t)\|_{\tau_x,q}^{p-d}\le\frac{1}{\alpha_1}V(x(t))\le\frac{c}{\alpha_1}=\frac{\alpha_2}{\alpha_1}\left(b\|u(\cdot)\|_{\tau_u,\mathcal{L}_\infty}\right)^{p-d}\,,$$

which implies

$$\|x(t)\|_{\tau_x,q}\le\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{p-d}}b\|u(\cdot)\|_{\tau_u,\mathcal{L}_\infty} \tag{4.21}$$

and

$$\|\tilde{x}(\cdot)\|_{\tau_x,\mathcal{L}_\infty}=\sup_{t\ge0}\|\tilde{x}(t)\|_{\tau_x,q}\le\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{p-d}}b\|u(\cdot)\|_{\tau_u,\mathcal{L}_\infty}\,.$$

Secondly, when $x(0)\in\overline{\Omega}_c$, since $\mathcal{B}_b\subset\Omega_c$, $x(0)\in\overline{\mathcal{B}}_b$, i.e. $\|x(0)\|_{\tau_x,q}\ge b\|u(\cdot)\|_{\tau_u,\mathcal{L}_\infty}$. As long as $\|x(t)\|_{\tau_x,q} \geq b\,\|u(\cdot)\|_{\tau_u,\mathcal{L}_\infty} \geq \frac{\gamma k^{\frac{1}{p}}}{(\epsilon+1)^{\frac{1}{p}}}\,\|u(t)\|_{\tau_u,q}$, we have

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} = \frac{\partial V(x(t))}{\partial x} f(x(t),u(t)) \leq -\frac{k-1}{k}(\epsilon+1)\,\|x(t)\|^p_{\tau_x,q} < 0\,.$$

This inequality implies

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} \leq -\frac{k-1}{k}\frac{(\epsilon+1)}{\alpha_2^{\frac{p}{p-d}}} V^{\frac{p}{p-d}}(x(t))\,.$$

Define the function, depending on $v \in \mathbb{R}_{\geq 0}$ and $t \in \mathbb{R}_{\geq 0}$

$$\Phi\left(v,\, t;\, l,\, k\right) = \begin{cases} \begin{cases} \left(v^{-l} + lkt\right)^{-\frac{1}{l}} & \text{when } t \leq -\frac{1}{lk}v^{-l} \\ 0 & \text{when } t \geq -\frac{1}{lk}v^{-l} \end{cases} & \text{if } l < 0\,, \\ \exp\left(-kt\right) v & \text{if } l = 0\,, \\ \left(v^{-l} + lkt\right)^{-\frac{1}{l}} & \text{if } l > 0\,. \end{cases} \tag{4.22}$$

Then Φ is a $\mathcal{KL}$ function, since $\Phi\left(0,\, t;\, l,\, k\right) = 0$, it is monotonically increasing in v, decreasing in t and $\lim_{t\to\infty} \Phi\left(v,\, t;\, l,\, k\right) = 0$. Since the solution to the differential equation $\dot{v} = -\kappa v^{\ell}$ for $\ell \in \mathbb{R}_{>0}$ is given by $v(t) = \Phi\left(v(0),\, t;\, \ell - 1,\, \kappa\right)$, with Φ defined by (4.22), and using the comparison lemma [29], we conclude that $V(x(t))$ satisfies

$$V(x(t)) \leq \Phi\left(V(x_0),\, t;\, \frac{d}{p-d},\, \frac{k-1}{k}\frac{(\epsilon+1)}{\alpha_2^{\frac{p}{p-d}}}\right)\,.$$

Moreover,

$$\begin{aligned} \|x(t)\|^{p-d}_{\tau_x,q} &\leq \frac{1}{\alpha_1} V(x(t)) \leq \frac{1}{\alpha_1}\Phi\left(V(x_0),\, t;\, \frac{d}{p-d},\, \frac{k-1}{k}\frac{(\epsilon+1)}{\alpha_2^{\frac{p}{p-d}}}\right) \\ &\leq \frac{1}{\alpha_1}\Phi\left(\alpha_2\,\|x_0\|^{p-d}_{\tau_x,q},\, t;\, \frac{d}{p-d},\, \frac{k-1}{k}\frac{(\epsilon+1)}{\alpha_2^{\frac{p}{p-d}}}\right)\,, \end{aligned}$$

and therefore

$$\|x(t)\|_{\tau_x,q} \leq \beta\left(\|x_0\|_{\tau_x,q}\,, t\right)\,, \tag{4.23}$$

where

$$\beta\left(\|x_0\|_{\tau_x,q}, t\right) = \left[\frac{1}{\alpha_1}\Phi\left(\alpha_2 \|x_0\|_{\tau_x,q}^{p-d}, t;\ \frac{d}{p-d}, \frac{k-1}{k}\frac{(\epsilon+1)}{\alpha_2^{\frac{p}{p-d}}}\right)\right]^{\frac{1}{p-d}} .$$

It can be shown that β coincides with (4.18) with

$$M = \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{p-d}}, \quad \kappa = \frac{k-1}{k}\frac{(\epsilon+1)}{\alpha_2} \times \begin{cases} \frac{-d}{p-d} & \text{if } d<0\,, \\ \frac{1}{p} & \text{if } d=0\,, \\ \frac{d}{p-d} & \text{if } d>0\,. \end{cases} \tag{4.24}$$

Thus, as long as $V(x(t)) > c$, the function $V(x(t))$ is decreasing, and this shows in particular that $x(t)$ is bounded, and

$$\|x(t)\|_{\tau_x,q}^{p-d} \le \frac{1}{\alpha_1}V(x(t)) \le \frac{1}{\alpha_1}V(x(0))\,.$$

Moreover, there is some finite time T such that $V(x(t)) = c$. Afterwards, $x(t)$ follows (4.21) for $t \ge T$.

From the previous analysis, we conclude that, there exists a finite T, s.t. (4.23) is satisfied for $t \in [0,T]$ and for $t \ge T$ (4.21) is obeyed. When $x(0) \in \Omega_c$, then $T = 0$. Combining both cases, we have for any $k > 1$, that the ISS inequality (4.17) is satisfied with the following constant linear homogeneous ISS-gain

$$\gamma_{\text{iss}} = Mb = \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{p-d}} \frac{k^{\frac{1}{p}}}{(\epsilon+1)^{\frac{1}{p}}}\gamma\,. \tag{4.25}$$

To show that the system is $\mathcal{L}_{\infty h}$−stable with finite gain, recall that $h(x,u)$ is homogeneous and continuous, and therefore using Lemma 3.6 there exist positive constants $c_q > 0$, such that

$$\|h(x,u)\|_{\tau_y,q} \le c_q \|(x,u)\|_{(\tau_x,\tau_u),q}\,, \quad \forall\,(x,u) \in \mathbb{R}^{n+m}\,.$$

From Jensen's inequality (A.6), we have

$$\begin{aligned}\|(x,u)\|_{(\tau_x,\tau_u),q} &= \left(\sum_{i=1}^{n}|x_i|^{\frac{q}{\tau_{x_i}}}+\sum_{j=1}^{m}|u_j|^{\frac{q}{\tau_{u_j}}}\right)^{\frac{1}{q}} = \left(\left(\sum_{i=1}^{n}|x_i|^{\frac{q}{\tau_{x_i}}}\right)^{\frac{q}{q}}+\left(\sum_{j=1}^{m}|u_j|^{\frac{q}{\tau_{u_j}}}\right)^{\frac{q}{q}}\right)^{\frac{1}{q}} \\ &\leq \left(\sum_{i=1}^{n}|x_i|^{\frac{q}{\tau_{x_i}}}\right)^{\frac{1}{q}}+\left(\sum_{j=1}^{m}|u_j|^{\frac{q}{\tau_{u_j}}}\right)^{\frac{1}{q}} = \|x\|_{\tau_x,q}+\|u\|_{\tau_u,q} \,.\end{aligned}$$

Using the above expressions and the ISS inequality (4.17) we arrive at

$$\begin{aligned}\|y(t)\|_{\tau_y,q} &= \|h(x(t),u(t))\|_{\tau_y,q} \leq c_q\,\|x(t)\|_{\tau_x,q}+c_q\,\|u(t)\|_{\tau_u,q} \\ &\leq c_q\left(\beta\left(\|x_0\|_{\tau_x,q},t\right)+\gamma_{\text{iss}}\,\|u(\cdot)\|_{\tau_u,\mathcal{L}_\infty}\right)+c_q\,\|u(\cdot)\|_{\tau_u,\mathcal{L}_\infty} \\ &\leq c_q\beta\left(\|x_0\|_{\tau_x,q},t\right)+c_q\left(\gamma_{\text{iss}}+1\right)\|u(\cdot)\|_{\tau_u,\mathcal{L}_\infty}\,,\end{aligned} \tag{4.26}$$

for all $t \geq 0$. From the previous expression, the finite-gain $\mathcal{L}_{\infty h}-$stability follows immediately. □

Note that given a $V(x)$ of homogeneity degree $p-d$ satisfying inequality (4.16) the ISS finite-gain can be estimated by (4.25), while the parameters of the function β in (4.18) are estimated as in (4.24).

Further note that in (4.25) the value of γ corresponds to the value that satisfies (4.16) for some $p \geq 1$ and with $y = x$. Given some V satisfying (4.16), the best (smallest) value of γ can be calculated using the results of the upper estimate of any $\mathcal{L}_{ph}-$gain. Then we can obtain an upper bound of the homogeneous ISS-gain from γ_{iss} in (4.25). The $\mathcal{L}_{\infty h}-$gain is also upper bounded by $c_q\left(\gamma_{\text{iss}}+1\right)$ as in (4.26).

4.5.2 Upper estimate of γ_{iss} and $\mathcal{L}_{\infty h}-$gain

In contrast to the $\mathcal{L}_{ph}-$gain, which can be derived through the PDI or the HJI by using Proposition 4.1 or 4.2, the upper estimate of the homogeneous ISS-gain depends on the value of some $\mathcal{L}_{ph}-$gain in (4.25). Further, the upper estimate of the $\mathcal{L}_{\infty h}-$gain depends again on the homogeneous ISS-gain γ_{iss} in (4.26).

In order to calculate an upper estimate of the homogeneous ISS-gain γ_{ISS}, α_1 and α_2 in (4.25) need to be derived first. From Lemma 3.6, since both $\|x\|_{\tau_x,q}^{p-d}$ and $V(x)$ are positive definite continuous τ_x-homogeneous functions of degree $p-d$, we have from (4.20) for any $q \geq 1$ that

$$\alpha_1 = \min_{\|x\|_{\tau_x,q}=1} V(x) = \min_{\|x\|=1} \frac{V(x)}{\|x\|_{\tau_x,q}}, \quad \alpha_2 = \max_{\|x\|_{\tau_x,q}=1} V(x) = \max_{\|x\|=1} \frac{V(x)}{\|x\|_{\tau_x,q}}.$$

Then combined with any $\mathcal{L}_p-$gain, (4.25) provides an upper estimate of the ISS-gain, denoted as γ_{ISS}.

Finally, an upper estimate of the $\mathcal{L}_{\infty h}-$gain can be similarly derived. The constant c in (4.26) can be calculated as

$$c_q = \max_{\|(x,u)\|_{(\tau_x,\tau_u),q}=1} \|h(x,u)\|_{\tau_y,q} = \max_{\|(x,u)\|=1} \frac{\|h(x,u)\|_{\tau_y,q}}{\|(x,u)\|_{(\tau_x,\tau_u),q}}.$$

Thus $(\gamma_{\text{ISS}}+1)c_q$ serves as an upper estimate of the $\mathcal{L}_{\infty h}-$gain from (4.26). Note that c_q is independent of the choice of storage function $V(x)$.

Remark 4.11 (Homogeneous small gain theorem with homogeneous ISS-gain)**.** *Theorem 3.2 and Theorem 3.3 are also true with the homogeneous ISS-gain, by using* (4.17) *as well as* (3.10)*. In [5], a homogeneous small gain theorem similar to the Theorem 3.2 is derived by using the homogeneous ISS-gain. Yet the interconnected system considered in [5] excludes the external input, i.e. in Figure 3.2* $u_1 = u_2 = 0$ *and* $y_1 = x_1, y_2 = x_2$ *are assumed.*

4.6 Comparison to Previous Works

There exist some previous works, which apply classical $\mathcal{L}_p-$gain (including $p = \infty$ and Input-to-State stability) on continuous homogeneous systems, where $\tau_u = \ell\mathbf{1}_m, \tau_y = \ell\mathbf{1}_r$, for some $\ell > 0$. In this section, we shall compare the presented results with those to show novelty and improvement.

In [48], the author considers the special class of systems Σ_h (4.1) which are homogeneous in the classical sense, i.e. $\tau_x = \mathbf{1}_n$, $\tau_u = \mathbf{1}_m$, with output $y = x$, and with non-negative homogeneity degree $d \geq 0$. [48] shows that if the origin is asymptotically stable for the unforced system, then for $p \geq 1 + d$ the system (with output $y = x$) is $\mathcal{L}_p$−stable and has finite $\mathcal{L}_p$−gain, with classical norms, i.e. relation (3.21) is satisfied. The presented Theorem 4.3 generalizes these results to an arbitrary homogeneous system, but it shows that to obtain finite $\mathcal{L}_p$−gain it is necessary to consider homogeneous norms. Only in particular cases this is valid for classical signal norms, namely when $\tau_y = \ell \mathbf{1}_r$ and $\tau_u = \ell \mathbf{1}_m$, $\ell > 0$. Note that Theorem 4.3 requires $p > 1 + d$ (from $p > d + \max \tau_x$), which is stricter than the condition in [48]. This is a consequence of the converse Lyapunov theorem for homogeneous systems that assure a smooth Lyapunov function only when $p > 1 + d$. If $p = 1 + d$ a Lyapunov function, which is not differentiable at $x = 0$, can be constructed. The presented Theorem 4.3 together can be extended to cover also the case $p = 1 + d$, but at the expense of a technical issue with the non-differentiability at $x = 0$, as is done in [48]. The details of this extension are not presented here.

In [48] ISS of the system is established (included in the case of $p = \infty$, $y = x$), with a linear gain and using classical norms, so the inequality (4.17) is satisfied from the fact $\tau_x = \mathbf{1}_n$, $\tau_u = \mathbf{1}_m$. The presented Theorem 4.4 extends this result to arbitrary homogeneous systems and shows that the linear gain is valid (only) considering homogeneous norms unless the homogeneous $\mathcal{L}_p$−norms is equivalent to the usual $\mathcal{L}_p$−norms. Moreover, Theorem 4.4 considers also the case of $\mathcal{L}_{\infty h}$−stability for an arbitrary (homogeneous) output $y = h(x, u)$, which is not considered in [48].

In [25], the authors deal with the special class of systems Σ_h (4.1), where $\dot{x} = f(x) + Bu$, with B a constant matrix, which are homogeneous in the classical sense, i.e. $\tau_x = \mathbf{1}_n$, $\tau_u = (d + 1)\, \mathbf{1}_m$, with output $y = h(x)$, $\tau_y = (d + 1)\, \mathbf{1}_r$, and with non-negative homogeneity degree $d \geq 0$. The main result of [25, Theorem 1] states that if the origin is asymptotically stable for the unforced system, then the system is $\mathcal{L}_2$−stable and has finite $\mathcal{L}_2$−gain with classical norms, i.e. relation (3.16) is satisfied. This is characterized using a Hamilton-Jacobi Inequality. The presented Theorem 4.2 generalizes these results to an arbitrary homogeneous system, but it shows that to obtain finite $\mathcal{L}_2$−gain it is necessary to consider homogeneous norms. Only in particular cases this is valid for classical signal norms, namely when $\tau_y = \ell \mathbf{1}_r$ and $\tau_u = \ell \mathbf{1}_m, \ell > 0$. In [25], $\tau_u = (d+1)\,\mathbf{1}_m$, $\tau_y = (d+1)\,\mathbf{1}_r$ are used, but actually $\tau_y = \ell \mathbf{1}_r$ and $\tau_u = \ell \mathbf{1}_m$ could also be considered for each $d > -\min \tau_x$ (from continuity of the vector field). Neither $\mathcal{L}_p-$stability nor ISS stability is considered in [25]. In this thesis all properties are characterized in a unified way using the inequality (4.6) or (4.16), which is more general than the Hamilton-Jacobi Inequality, since the analytical form of $u^\star(x, V_x)$ like (4.13) is not always easily attainable.

In [24], the author generalizes the results of [25] to the special class of weighted homogeneous systems affine in the input, i.e. $\dot{x} = f(x) + G(x)u$, $y = h(x)$, with $f(x)$ a τ_x-homogeneous vector field of degree $d > -\tau_0 \triangleq -\min\{\tau_x\}$, $G(x)$ a matrix with columns being τ_x-homogeneous vector fields of the degree $s \geq -\tau_0$, and homogeneous weight of the input and the output being $\tau_u = (d-s)\,\mathbf{1}_m$, $\tau_y = (d-s)\,\mathbf{1}_r$ (implicitly $d > s$). Since [24] deals with the H_∞ control problem, the following results on $\mathcal{L}_2-$stability are contained implicitly in the paper. In the particular case when $s = -\tau_0$ that strongly restricts matrix $G(x)$, [24] shows that if the origin $x = 0$ is asymptotically stable for the unforced system, then the system is $\mathcal{L}_2-$stable and has finite $\mathcal{L}_2-$gain with classical norms, i.e. relation (3.16) is satisfied. This is characterized using a homogeneous Hamilton-Jacobi Inequality. For the relaxed condition on $G(x)$ that $s \geq -\tau_0$, it is shown that if the origin $x = 0$ is asymptotically stable for the unforced system, then the system is $\mathcal{L}_p-$stable and has finite $\mathcal{L}_p-$gain with classical norms, i.e. relation (3.21) is satisfied for p, when $p \geq {}^{(d+\max\tau_x)}/_{(d-s)}$ (including $p = \infty$). Note that in the proof of [24, Theorem 6.1], the author quote the converse Lyapunov Theorem, but wrongly set the homogeneous degree of $V(x)$ to $p - d \geq \tau_0$ instead of $p - d \geq \max\{\tau_x\}$. [24] does not truly consider ISS stability for **weighted** homogeneous systems. It deals with $\mathcal{L}_\infty-$stability, which is different. Moreover, due to the restrictions imposed on the homogeneity of the output function $y = h(x)$, the particular case $y = h(x) = x$ cannot be considered, because then $\tau_y = (d-s)\,\mathbf{1}_n \leq (d+\tau_0) \neq \tau_x$, unless $\tau_x = \tau_0 \mathbf{1}_n = (d-s)\,\mathbf{1}_n$. In such case, the homogeneous weight of states, input and output are all equal to τ_0, which reduced the conclusion of ISS stability to **classical** homogeneous systems, which is the same as [48]. In contrast to $\mathcal{L}_2-$stability, $\mathcal{L}_p-$stability in [24] is not characterized using a Hamilton-Jacobi

Inequality, despite the fact that the system is affine in the input, and in that case this is still feasible.
The presented Theorem 4.2 generalizes these results to an arbitrary homogeneous system, without the restrictions imposed in paper [24], i.e. the system does not need to be affine in the input, and its homogeneity weights and degree are strongly relaxed. It is further shown that to obtain finite $\mathcal{L}_p-$gain for arbitrary homogeneous system it is necessary to consider homogeneous norms. Only in particular cases this is valid for classical signal norms, namely when $\tau_y = \ell \mathbf{1}_r$ and $\tau_u = \ell \mathbf{1}_m$. But even for the case with classical norms, the presented results are more general than those proposed in [24]. All properties are characterized in a unified way using the inequality (4.6) or (4.16), which is more general than the Hamilton-Jacobi Inequality, that is not always attainable in the non-affine case of the input. Therefore, the result of $\mathcal{H}_\infty$ norm from [25, 24] is generalized to be applicable to all continuous homogeneous systems with arbitrary homogeneous weights for input and output, as well as not necessarily being affine in the input. The conclusion of $\mathcal{L}_p-$stability in [24] is also included in our Theorem 4.3 for $p - Ld > L \max \tau_x$.

In [7] and [1], ISS and other related properties are studied for general weighted homogeneous systems (4.1), and they generalize the results of [48, 24] relating the internal stability of the unforced system and ISS stability. Our results on ISS reproduce those of [1] and [7], although we emphasize the linear relationship between the input and the state (4.17) when using homogeneous norms, which is obscure in [1] and [7]. This linearity issue is clarified in [5] for the more general version of geometric homogeneity by using solely homogeneous norms in contrast to the mixture usage of homogeneous norms and Euclidean norms in [7]. However, [1, 7, 5] do not consider $\mathcal{L}_p-$stability for any value of p. In contrast to [1, 7, 5], we clarify the situation about $\mathcal{L}_p-$stability for general weighted homogeneous systems for arbitrary values of p and the linear homogeneous ISS gain is related to such homogeneous $\mathcal{L}_p-$gain in the proof of Theorem 4.4. Furthermore, the homogeneous small-gain theorem is considered in both [1, 5]. In [1], using the classical norms, the nonlinear ISS-gain is a $\mathcal{K}$ function. In [5] no external inputs are considered. The small-gain theorems derived in both [1, 5] use the (linear or nonlinear) ISS-gain to assure the closed loop stability, and thus are related to

$p = \infty$. In contrast, the homogeneous small-gain theorem obtained in this paper adopts any homogeneous $\mathcal{L}_p-$gain (when it exists, including $\mathcal{L}_{\infty h}-$gain and also homogeneous ISS-gain) to verify closed loop stability.

In [28], the authors adopted a homeomorphic coordinate transformation that can be related to our results. This is done using the companion signals introduced in Remark 3.8: a homogeneous $\mathcal{L}_p-$stable system, according to our Definition 3.7, is also $\mathcal{L}_p-$stable, using the traditional signal norms, from the transformed input $S(u) = u^{\frac{1}{\tau_u}}$ to transformed output $T(y) = y^{\frac{1}{\tau_y}}$, since (3.21) can also be written as

$$\left\| y^{\frac{1}{\tau_y}} \right\|_{\mathcal{L}_p} \leq \gamma_p \left\| u^{\frac{1}{\tau_u}} \right\|_{\mathcal{L}_p} + \beta_p \,, \quad \forall\, u \in \mathcal{L}_{\tau_u, p} \,.$$

In this sense, we generalize this idea, in the context of homogeneous systems, to arbitrary $\mathcal{L}_p-$norms, and not only to the $\mathcal{L}_2-$norm considered in [28].

However, beyond this connection with [28], the objectives and scope of both papers are rather different. [28] is concerned with the relationship between $\mathcal{L}_2-$stability and ISS or iISS for general nonlinear systems, and they show the existence of appropriate input $S(\cdot)$ and output $T(\cdot)$ homeomorphisms, without providing a way to obtain such functions. This dissertation is concerned specifically with general weighted homogeneous systems. It is shown that the homogeneous vector and signal norms are a natural setting for studying input-output and input-to-state stability. One obtains linear finite-gains in all cases, a clear relationship with the internal stability is derived and a method to calculate the gains is provided by means of a dissipation inequality. Moreover, arbitrary values of p are allowed, instead of only $p = 2$ as in [28].

In [28], a feedback interconnected system is also studied. Since the homeomorphic transform of coordinate is only shown to exist without any preference, when treating the feedback interconnected signals, the authors can only assume that constants exist, s.t. $\|S(u_2)\| \leq c \|T(y_1)\|$ (for the case of Figure 3.3) or e.g. $\|S(e_1)\| \leq c_S \|e_1\|$ and $\|y_1\| \leq c_T \|T(y_1)\|$ (for the case of Figure 3.2) in order for the theorems to apply. From homogeneity, it is clear that such constants exist only if both sides are of the same homogeneous degree. For the feedback interconnected system (Figure 3.2), $\left\| e_1^{\frac{1}{\tau_{e_1}}} \right\| \leq c_S \|e_1\|$ and $\|y_1\| \leq c_T \left\| y_1^{\frac{1}{\tau_{y_1}}} \right\|$ imply that the weight vectors for y_1, y_2 are both $\mathbf{1}_n$, thus Theorem 3.2 also serves as improvement as the small gain theorem in [28] by using the additive inequality in Lemma 3.5 .

In [64, 63], the authors apply the classical $\mathcal{L}_2-$gain from disturbance u to $x^{\frac{1}{\tau_x}}$ on the Super-Twisting Algorithm (STA). The STA is homogeneous and has a discontinuous vector field, where the weight vector for the input is $\tau_u = 0$. Apparently, the homogeneous $\mathcal{L}_2-$gain in (3.20) is not well defined for the STA.

Further, the works in [61, 60] consist part of this thesis.

4.7 Applicability of Homogeneous $\mathcal{H}_\infty-$norm to Structured Uncertainty

Now we shall look at whether the above application of classical $\mathcal{H}_\infty-$norm to robustness still apply for the homogeneous $\mathcal{H}_\infty-$norm. Take the example of Figure 2.11. Now we suppose the match of weight vector between G and Δ.

RS From Theorem 3.3, if $\|G_{\iota\to\varpi}\|_{L\tau,\infty} < 1/\|\Delta_U\|_{L\tau,\infty}$, RS of the homogeneous continuous system is guaranteed.

RP In nonlinear systems, the frequency domain performance target ω_p is usually not realizable. The most common requirement is disturbance attenuation, i.e. to minimize some gain from input to output. In another word, to minimize the $\|G_{u\to y}\|_{L\tau,\infty} \leq \gamma_{RP}$ by changing the controller or observer inside the plant G. Under such case, the constant $1/\gamma_{RP}$ should be multiplied to either before input u or after output y. Thereafter $\|G\|_{L\tau,\infty} < 1/\|\Delta_U\|_{L\tau,\infty}$ guarantees RP as well.

D-scale From Proposition 3.2, the D-scaling block and its inverse should share the same homogeneous degree with the plant, in order for the $D^{-1}GD$ to remain homogeneous. Yet, since the inverse of homogeneous D with dynamics is difficult to find, the usefulness of D scaling in LTI system is limited in nonlinear case. This is illustrated in the next example.

Example 4.2 (Take homogeneous function as D scaling). *Take a simple scalar example*

$$\begin{aligned}\dot{x}(t) &= -\left(k+\delta_k(t)\right)\lceil x(t)\rfloor^{1/2} + bu(t)\,,\\ y(t) &= cx(t)\,,\end{aligned}$$

here $L\tau_x = 2L, L\tau_u = L, Ld = -L$. *The stabilizing dynamics* $-\left(k+\delta_k(t)\right)\lceil x\rfloor^{1/2}$ *can be understood as the controller, whose gain varies within a range of* $k+\delta_k(t)$ *and* $|\delta_k(t)| < k$, *thus* $-(k+\delta_k(t)) < 0, \forall\, t > 0$. *We could introduce the extra input and output as*

$$\begin{aligned}\dot{x}(t) &= -k\lceil x(t)\rfloor^{1/2} + bu(t) - \overline{\delta}_k\iota(t)\,,\\ \varpi(t) &= \lceil x(t)\rfloor^{1/2}\,,\quad \iota(t) = \Delta(t)\varpi(t)\,,\end{aligned}$$

where $L\tau_\iota = L\tau_\varpi = L$. *It is simply* $\delta_k\lceil x(t)\rfloor^{1/2} = \overline{\delta}_k\iota(t) = \overline{\delta}_k\Delta(t)\varpi(t)$. *Here we extract the magnitude of* $\delta_k(t)$ *into the input matrix* $\overline{\delta}_k$, *so that* $\|\Delta\|_{L\tau,\infty} = 1$.
Under such setting, the D-scaling can be any diagonal homogeneous function, e.g. $D(\cdot) = \frac{\sqrt{3}}{3}\lceil\cdot\rfloor^{1/2}$, *and* $D^{-1}(\cdot) = 3\lceil\cdot\rfloor^{2}$. *And since the diagonal* Δ *is normalized to 1, the homogeneous* $D\Delta D^{-1}$ *would not change the* $\|D\Delta D^{-1}\|_{L\tau,\infty} = \|\Delta\|_{L\tau,\infty}$, *since the power of any order of* 1 *remains* 1. *As shown in Figure 2.12, the new* $D^{-1}GD$ *reads*

$$\begin{aligned}\dot{x} &= -k\lceil x\rfloor^{1/2} + bu - \overline{\delta}_k\frac{\sqrt{3}}{3}\lceil\hat{\iota}\rfloor^{1/2}\,,\\ \hat{\varpi} &= 3x,\quad \hat{\iota} = \Delta\hat{\varpi}\,.\end{aligned}$$

Yet as shown in the examples, the homogeneous function in D does not change the homogeneous $\mathcal{H}_\infty$−norm since in the value function they are all normalized to $2-d$, yet the numerical scaling has a simple and direct effect.
Therefore, such D-K iteration can be simplified into only one step, i.e. calculating the homogeneous $\mathcal{H}_\infty$−norm $\|G_{\phi\to\varpi}\|_{L\tau,\infty}$ and $\|G_{\iota\to y}\|_{L\tau,\infty}$ and set $D = \alpha\|G_{u\to\varpi}\|_{L\tau,\infty} + (1-\alpha)\,/\,\|G_{\iota\to y}\|_{L\tau,\infty}$ with $\alpha \in (0,1)$.

4.8 Examples

In this section, we introduce some examples of deriving the homogeneous $\mathcal{H}_\infty-$norm.

4.8.1 Scalar SISO system

We would like to show how to derive the $\mathcal{L}_{ph}-$gain analytically for a SISO linear dynamics by using the methods in Theorem 4.2:

$$\dot{x} = -k \lceil x \rfloor^{\frac{1}{z_3}} + b \lceil u \rfloor^{\frac{1}{z_1}} , \quad y = c \lceil x \rfloor^{\frac{z_2}{z_3}} , \tag{4.27}$$

where $z_1, z_2, z_3, k \in \mathbb{R}_{>0}$, with weight vector as $L\tau_x = Lz_3, L\tau_u = Lz_1, Ld = L - Lz_3, L\tau_y = Lz_2$. From Theorem 4.2, system (4.27) has finite $\mathcal{L}_{2h}-$gain for $L < 2/(d + \max \tau_x) = 2$. Then we build a storage function $V(x)$ of homogeneous degree $2 - Ld = 2 - L + Lz_3$ by simply

$$V(x) = \frac{aLz_3}{2 - L + Lz_3} |x|^{\frac{2-L+Lz_3}{Lz_3}} ,$$

whose partial derivative is

$$V_x = \frac{\partial V(x)}{\partial x} = a |x|^{\frac{2-L}{Lz_3}} .$$

Then the value function is

$$\begin{aligned} J(V_x, x, u) &= a \lceil x \rfloor^{\frac{2-L}{Lz_3}} \left(-k \lceil x \rfloor^{\frac{1}{z_3}} + b \lceil u \rfloor^{\frac{1}{z_1}} \right) + \left| c \lceil x \rfloor^{\frac{z_2}{z_3}} \right|^{\frac{2}{Lz_2}} - \gamma^2 |u|^{\frac{2}{Lz_1}} \\ &= \left(|c|^{\frac{2}{Lz_2}} - ak \right) |x|^{\frac{2}{Lz_3}} + ba \lceil x \rfloor^{\frac{2-L}{Lz_3}} \lceil u \rfloor^{\frac{1}{z_1}} - \gamma^2 |u|^{\frac{2}{Lz_1}} . \end{aligned}$$

For $J(V_x, x, 0) < 0$, we must have $a > |c|^{\frac{2}{Lz_2}} / k$. Taking partial derivative of $J(V_x, x, u)$ against u to zero, we have the worst $u^\star(x)$ lies along

$$u^\star(x) = \left| \frac{baL}{2\gamma^2} \right|^{\frac{Lz_1}{2-L}} \lceil x \rfloor^{\frac{z_1}{z_3}} . \tag{4.28}$$

By plugging back $u^\star(x)$ into $J\left(V_x, x, u\right)$, the partial differential inequality (PDI) is turned into a Hamilton-Jacobi Inequality (HJI)

$$J\left(V_x, x, u^\star(x)\right) = \left(|c|^{\frac{2}{Lz_2}} - ak + \left|\frac{ba}{\gamma^L}\right|^{\frac{2}{2-L}} \left|\frac{L}{2}\right|^{\frac{L}{2-L}} \left(1 - \frac{L}{2}\right)\right) |x|^{\frac{2}{Lz_3}} \leq 0\,,$$

which is equivalent to

$$\gamma_L^\star\left(V\right) = \left|\frac{|ba|^{\frac{2}{2-L}}}{ak - |c|^{\frac{2}{Lz_2}}} \left|\frac{L}{2}\right|^{\frac{L}{2-L}} \left(1 - \frac{L}{2}\right)\right|^{\frac{2-L}{2L}}\,. \tag{4.29}$$

The right hand side of (4.29) has only the variable a (in the storage function $V(x)$) and is a continuous function of a, since $L < 2$ from Theorem 4.2. This gives us a chance to find the optimal a, such that the right hand side of (4.29) is smallest. By taking the partial derivative of (4.29) against a, the optimal $a^\star$ (or effectively optimal $V(x)$) lies at

$$a^\star = \frac{2\,|c|^{\frac{2}{Lz_2}}}{Lk} \tag{4.30}$$

and from $L < 2$, $a^\star > |c|^{\frac{2}{Lz_2}}/k$ is also satisfied. Therefore, plugging back (4.30) into (4.29) we have the analytical upper bound of $\mathcal{L}_{ph}-$gain with $p = 2/L$ as

$$\gamma_p \leq {\gamma^\star}_L^L = \frac{b\,|c|^{\frac{1}{z_2}}}{k}\,.$$

Taking this $\gamma_L^\star$ and (4.30) back into (4.28) leads to

$$u^\star(x) = \left|\frac{k}{b}\right|^{z_1} \lceil x \rfloor^{\frac{z_1}{z_3}}\,. \tag{4.31}$$

Actually, with the state starting from x_0, with $u^\star(x)$, we have $\dot{x} = 0$ for system (4.27), whose input and output is

$$u(\cdot) = \left|\frac{k}{b}\right|^{z_1} \lceil x_0 \rfloor^{\frac{z_1}{z_3}}\,, \quad y(\cdot) \ = c\,\lceil x_0 \rfloor^{\frac{z_2}{z_3}}\,.$$

For all time $t \in [0, \infty)$, then the ratio of the $\mathcal{L}_{ph}-$norm of output over input from this constant input to its constant output is

$$\Gamma = \frac{\left\| c \lceil x_0 \rfloor^{\frac{z_2}{z_3}} \right\|_{\tau_y, \mathcal{L}_p}}{\left\| \left| \frac{k}{b} \right|^{z_1} \lceil x_0 \rfloor^{\frac{z_1}{z_3}} \right\|_{\tau_u, \mathcal{L}_p}} = \frac{b \, |c|^{\frac{1}{z_2}}}{k} .$$

Therefore, the γ_p is upper bounded and lower bounded by the same number, thus we conclude that

$$\gamma_p = \frac{b \, |c|^{\frac{1}{z_2}}}{k} , \tag{4.32}$$

for $p = 2/L > 1$. Interestingly, the $\mathcal{L}_{ph}-$gain is itself independent of p, z_1 and z_3, though the worst input $u^\star(x)$ depends on the later two. When $z_1 = z_2 = z_3 = 1$, the system is linear and $V(x) = \frac{a}{2} |x|^2$ is the classical quadratic storage function when $L = 1$. The Bode plot (showing $\mathcal{L}_2-$gain for different frequency of sinusoidal input $u(t) = \sin \omega t$) is plotted in Fig. 4.1. It is clear that for such LTI system, the worst

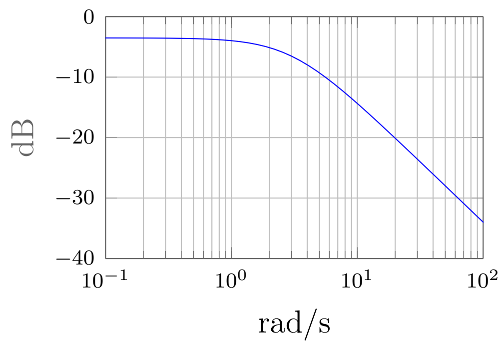

Figure 4.1: Bode plot of system (4.27).
The parameter of system are $z_1 = z_2 = z_3 = 1$ (linear case) and $k = 3, b = 2, c = 1$.

input lies at low frequency, namely a constant input. This also agrees with $u^\star(x)$ in (4.31). For this LTI system, its $\mathcal{L}_p-$gain when $p > 1$ is all equal to (4.32).

4.8.2 Continuous Higher Order Differentiator

Here we show how to find the homogeneous $\mathcal{H}_\infty-$norm for the continuous higher order differentiator in [56]. For a Lebesgue-measurable signal $f(t) = f_0(t) + c\, v(t)$ whose n-th derivative is bounded as $f_0^{(n)}(t) = b\,u(t)$, where $|u(\cdot)| < 1$ and the noise signal $|v(\cdot)| < 1$. A family of continuous differentiators is built according to [56] and reads

$$\begin{aligned}\dot{x}_i &= -k_i \lceil x_1 - f(t) \rfloor^{\frac{\tau_{i+1}}{\tau_1}} + x_{i+1}\,, \quad i = 1, \dots, n-1\,,\\ \dot{x}_n &= -k_n \lceil x_1 - f(t) \rfloor^{\frac{1+d}{\tau_1}}\\ y &= x\end{aligned}$$

where $\tau_i = \tau_{i+1} - d = 1 - (n-i)\,d,\ i = 1, \dots, n$ and $d \in [-1, 0]$ and $\tau_v = 1 - (n-1)\,d, \tau_u = 1 + d$. Then defining the differentiation errors as $e_i \triangleq x_i - f_0^{(i-1)}(t)$ leads to the error dynamics

$$\begin{aligned}\dot{e}_i &= -k_i \lceil e_1 - cv \rfloor^{\frac{\tau_{i+1}}{\tau_1}} + e_{i+1}\,, \quad i = 1, \dots, n-1\,,\\ \dot{e}_n &= -k_n \lceil e_1 - cv \rfloor^{\frac{1+d}{\tau_1}} - b\,u\,.\end{aligned}$$

When $d \in (-1, 0]$, Theorem 4.2 is valid for $L < 2/\left(\max \tau_x + d\right) = 2/\left(1 - n + 2d\right)$. Thus, we can scale the weight vector to $\tau_i = \frac{1-(n-i)d}{1-(n-2)d}$, $i = 1, \dots, n$. Then we have $\tau_u = \frac{1+d}{1-(n-2)d}$, $\tau_v = \frac{1-(n-1)d}{1-(n-2)d}$, $\tau_t = \frac{-d}{1-(n-2)d}$ for $n > 2$. We can use the family of storage functions

$$V(e) = \sum_{i=1}^{n-1} a_i \left|e_i\right|^{\frac{2-d}{\tau_i}} + \sum_{i,j \in \{1, \cdots, n\}}^{i<j} a_{ij} e_i \lceil e_j \rfloor^{\frac{2-d-\tau_j}{\tau_j}}\,. \tag{4.33}$$

In [56] a family of storage functions is proposed with lower bounded homogeneous degree, in which (4.33) also lies. On the other hand, the coefficients in the storage functions from [56] is a subset of that of (4.33). For the case of $v \equiv 0$, we can set $c = 0$.

4.8.3 First Order Integral Sliding Mode Controller

Here we include an example that has discontinuous dynamics, yet the weight vector for input is still non-zero. For a first order scalar system, we can build a first order integral sliding mode controller [34]

$$\begin{aligned}\dot{x}(t) &= u(t) + w(t)\,,\\ u(t) &= -k_1 \lceil x(t) \rfloor^{\frac{1}{2}} - k_2 \int_0^t \lceil x(s) \rfloor^0 \,\mathrm{d}s\,.\end{aligned}$$

Note that this system can be rewritten as the Super Twisting Algorithm (STA) [34]. This system is an exception, since the system can be written in

$$\begin{aligned}\dot{x}_1 &= -k_1 \lceil x_1 \rfloor^{\frac{1}{2}} + x_2 + w\,,\\ \dot{x}_2 &= -k_2 \lceil x_1 \rfloor^{0}\,,\end{aligned}$$

where the vector field is almost everywhere continuous except along $x_1 = 0$. The weight vector is $\tau_x = (2,1)\,, \tau_w = 1, d = -1$. Then with the Lyapunov candidate of degree $2 - d = 3$ as

$$V = a_1 \left(\frac{2}{3} |x_1|^{\frac{3}{2}} - a_{12} x_1 x_2 + \frac{a_2}{3} |x_2|^3 \right)\,,$$

with $a_1, a_{12}, a_2 > 0$, the derivative is

$$\begin{aligned}\dot{V} &= a_1 \left(-k_1 + k_2 a_{12}\right) |x_1| + a_1 \left(-a_{12} - a_2 k_2 \lceil x_1 \rfloor^0 \operatorname{sign}(x_2) \right) |x_2|^2 \\ &\quad + a_1 \left(1 + a_{12} k_1\right) \lceil x_1 \rfloor^{\frac{1}{2}} x_2 + a_1 \left(\lceil x \rfloor^{\frac{1}{2}} - a_{12} x_2 \right) w\,,\end{aligned}$$

From Young's inequality, we need $a_2 \geq a_{12}^3$ for positive definiteness of V, and at least $a_2 k_2 < a_{12} < \frac{k_1}{k_2}$. When V is a strong Lyapunov function, it is clear that γ can be calculated by both methods (4.9) or (4.12) with this Lyapunov function candidate where

$$\begin{aligned}J(V_x, x, w) &= a_1 \left(-k_1 + k_2 a_{12} + E_1/a_1\right) |x_1| + a_1 \left(-a_{12} - a_2 k_2 \lceil x_1 \rfloor^0 \operatorname{sign}(x_2) + E_2/a_1 \right) |x_2|^2 \\ &\quad + a_1 \left(1 + a_{12} k_1\right) \lceil x_1 \rfloor^{\frac{1}{2}} x_2 + a_1 \left(\lceil x_1 \rfloor^{\frac{1}{2}} - a_{12} x_2 \right) w - \gamma^2 w^2\,.\end{aligned}$$

A special treatment need to be taken along $x_1 = 0$, where a set value of $\lceil x_1 \rfloor^0 \in [-1, 1]$ is to be considered. That is, along $x_1 = 0$

$$J(V_x, x_2, w) = a_1 \left(-a_{12} - a_2 k_2 \operatorname{sign}(x_2) + E_2/a_1\right) |x_2|^2 - a_1 a_{12} x_2 w - \gamma^2 w^2 \leq 0$$

as well as

$$J(V_x, x_2, w) = a_1 \left(-a_{12} + a_2 k_2 \operatorname{sign}(x_2) + E_2/a_1\right) |x_2|^2 - a_1 a_{12} x_2 w - \gamma^2 w^2 \leq 0$$

need to be considered. When $x_1 \neq 0$, $J(V_x, x, w)$ is continuous in x, w. The corresponding γ can be derived as shown in Proposition 4.1 or 4.2.

5 Homogeneous $\mathcal{H}_\infty-$controller

As brought up in Section 2.8, when a control input is available for the LTI system, it is possible to build an $\mathcal{H}_\infty-$optimal controller by solving a Partial Differential Inequality with extended input.
In previous works of [25, 24], the authors build a continuous homogeneous $\mathcal{H}_\infty-$controller for some specific homogeneous system. This is discussed in Section 4.6.
In this Chapter, we shall extend such concept to arbitrary continuous homogeneous system, in order to build a continuous homogeneous feedback controller. Note that the situation is more complicated than that in LTI.

5.1 State Space Model

Consider the SSM for continuous homogeneous dynamics with control input,

$$\Sigma_c : \begin{cases} \dot{x} = f(x, u, w)\,, \\ y = h(x, w)\,, \end{cases} \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input and $w \in \mathbb{R}^o$ is the disturbance input and $y \in \mathbb{R}^r$ is the output to be minimized. Here we suppose the full information of the states is available, that is we exclude the necessity of observer design.

The weight vector for x, u, w, y as well as the homogeneous degree of vector field for $f(x, u, w)$ is the same as that in Definition 4.1, thus is not repeated here.

5.2 Two Types of Homogeneous $\mathcal{H}_\infty$−controller

For the closed loop system Σ_c (5.1) to maintain homogeneous (if not, the $\mathcal{H}_{\infty h}$−norm from input to output no longer applies to the closed loop system), we need to build the feedback controller $u(x)$ s.t. the degree of each channel of controller equal to its corresponding weight vector as control input, i.e. $k_{u_i} = \tau_{u_i}$. To that there are two possibilities, namely, one with dynamics or one memoryless:

Example 5.1 (First order homogeneous controller with dynamics). *Take example of a chain integrator, it is shown in [12] that, when the system is in chain integrator form, the homogeneous degree of the closed system can be arbitrarily assigned*

$$\begin{aligned} \dot{x}_1 &= u + w\,, \\ y &= x_1\,. \end{aligned}$$

We can build a linear controller $u(\kappa x) = \kappa u(x)$*, if we fix the weight* $\tau_{x_1} = \tau_u = \tau_w = 1$ *and degree* $d = 0$*. On the other hand, if we set the homogeneous degree of the closed loop system as* $\check{d} \in [-1, 1]$*, and let the controller have the form of*

$$\begin{aligned} u &= -k_1 \lceil x_1 \rfloor^{\frac{1}{1-\check{d}}} + x_2\,, \\ \dot{x}_2 &= -k_2 \lceil x_1 \rfloor^{\frac{1+\check{d}}{1-\check{d}}}\,. \end{aligned}$$

Then the closed loop dynamics is now

$$\begin{aligned} \dot{x}_1 &= -k_1 \lceil x_1 \rfloor^{\frac{1}{1-\check{d}}} + x_2 + w\,, \\ \dot{x}_2 &= -k_2 \lceil x_1 \rfloor^{\frac{1+\check{d}}{1-\check{d}}}\,, \\ y &= x_1\,. \end{aligned}$$

The weight vector and degree of this closed loop dynamics is now $\tau_x = \left(1 - \check{d}, 1\right), \tau_w = 1, d = \check{d}$*. When* $d = -1$*, the* u *is the Super-Twisting Controller.*

Example 5.2 (Higher order memoryless feedback controller). *Take another example,* *for a n-th order chain integrator,*

$$\begin{aligned} \dot{x}_i &= u_i + x_{i+1}\,, \quad i = 1, \cdots, n-1\,, \\ \dot{x}_n &= u_n + w\,. \end{aligned} \tag{5.2}$$

Again we can build linear controller $u(\kappa x) = \kappa u(x)$ *if we set* $\tau_x = \mathbf{1}_n, \tau_u = \mathbf{1}_n, \tau_w = 1, d = 0$. *On the other hand, with the design of* $u_i(x) = -k_i \lceil x_1 \rfloor^{\frac{n-i}{n}}, k_i > 0$, *(5.2) becomes*

$$\begin{aligned} \dot{x}_i &= -k_i \lceil x_1 \rfloor^{\frac{n-i}{n}} + x_{i+1}\,, \quad i = 1, \cdots, n-1\,, \\ \dot{x}_n &= -k_n \lceil x_1 \rfloor^{0} + w\,, \end{aligned}$$

with $\tau_{x_i} = n - i + 1, \tau_{u_i} = n - i, \tau_w = 0, d = -1$.

In the rest of the text, we would only discuss the memoryless homogeneous feedback controller, i.e. the case of Example 5.2. Which is analogous to the design of the $\mathcal{H}_\infty-$controller with full information of the state for LTI system.

5.3 Homogeneous Stablizability and Homogeneous Control Lyapunov Function

For controller design, the concept of stablizability is necessary. When considering homogeneous system, the concept of homogeneous stablizability and the homogeneous control Lyapunov function are introduced [52, 40].

Definition 5.1 ((Homogeneous) Stabilizability [52]). *The homogeneous system* Σ_c *(5.1) is (homogeneously) stabilizable, if there exists a (homogeneous) feedback control policy* $\hat{u}(x)$ *(whose degree* $k_{\hat{u}_i} = \tau_{u_i}, \forall\, i = 1, \cdots, m$*) s.t. the closed loop dynamics* $\dot{x} = f(x, \hat{u}(x), 0)$ *is asymptotically stable.*

For LTI system $\dot{x} = Ax + B_1u + B_2w$, stabilizability is equivalent to existence of a state-feedback control policy, e.g. $\hat{u}(x) = Kx$, s.t. $A + B_1K$ is Hurwitz [65].

Definition 5.2 ((Homogeneous) Control Lyapunov Functions [40, 52]). *A positive-definite (homogeneous) continuously differentiable storage function $V(x)$ with $V(0) = 0$ is called a (homogeneous) control Lyapunov function (CLF) for system Σ_c (5.1) when $w(\cdot) = 0$ if there exists a control strategy $\hat{u}(x)$, s.t.*

$$\frac{\partial V(x)}{\partial x} f(x, \hat{u}(x), 0) < 0\,, \quad \forall\, x \in \mathcal{R}^n \backslash \{0\}\ . \tag{5.3}$$

From the direct Lyapunov theorem, the existence of CLF V and $\hat{u}(x)$ that satisfy (5.3) indicate that such $\hat{u}(x)$ stabilizes system Σ_c (5.1).

For LTI system $\dot{x} = Ax + B_1 u + B_2 w$, with the control Lyapunov function candidate $V = x^\top P x, P = P^\top > 0$ and a state-feedback control policy, e.g. $\hat{u}(x) = Kx$, (5.3) is equivalent to the Lyapunov Inequality. That is, for some matrix $Q = Q^\top > 0$ s.t. $P\,(A + BK) + (A + BK)^\top P = -Q < 0$ [65], or equivalently the closed loop state matrix $A + BK$ is Hurwitz.

Theorem 5.1 (Homogeneous stablizability from control Lyapunov function [40, 47, 17]). *Let ℓ be an integer satisfying $\ell < (2 - Ld)\,/\,(L \max \tau_x)$. The homogeneous system Σ_c (5.1) can be homogeneously stabilized by a continuous homogeneous controller, if and only if there exists an homogeneous control Lyapunov function $V(x)$ of degree $2 - Ld$, which is C^∞ on $\mathbb{R}^n \backslash \{0\}$ and C^ℓ at the origin.*

When $\ell = 1$, i.e. $V(x)$ is continuously differentiable at the origin, $\ell < (2 - Ld)\,/\,(L \max \tau_x)$ is equivalently $L < 2/\,(d + \max \tau_x)$, which agrees with the condition for Theorem 4.2.

Remark 5.1 (Possibility that no stabilizing homogeneous controller exists for some stabilizable homogeneous systems). *In [3], it is noted that some stabilizable homogeneous systems can not be stabilized by any homogeneous feedback control policy $u(x)$, i.e. some homogeneous systems are stabilizable, but not homogeneously stabilizable. Thus the concept of homogeneously stabilizability and the homogeneous CLF brought up in Definition 5.1 and Definition 5.2 is necessary for the design of a stabilizing homogeneous feedback controller.*

5.4 Homogeneous $\mathcal{H}_\infty$−norm for Stabilizing Controller

It is nature that, the system Σ_c (5.1) with a stabilizing controller plugged in, is asymptotically stable when unperturbed. Then from Theorem 4.2 the closed loop system has finite $\mathcal{L}_{2h}$−gain.

5.4.1 The $\mathcal{L}_{2h}$−gain for Stabilizing Controller

As discussed in Section 2.8, for the existence of a preferred controller input that minimizes the value function, we need to introduce the extended output $z\left(y,u\right) = \left(y^\top, u_\theta^\top\right)^\top$, with $\tau_z = \left(\tau_y^\top, \tau_u^\top\right)^\top$ and $u_\theta = \left(\theta^{\tau_{u_1}} u_1, \cdots, \theta^{\tau_{u_m}} u_m\right)^\top$ with $\theta > 0$, then

Definition 5.3 ($\mathcal{L}_{2h}$−gain of closed loop system). *The homogeneous feedback control policy $u(x)$ for system Σ_c (5.1) achieves the $\mathcal{L}_{2h}$−gain from disturbance w to output z less than γ_L, if for each initial value x_0 there exists a finite constant $\beta_L(x_0)$, s.t.*

$$\|z(y,u_\theta)\|_{L\tau_z,\mathcal{L}_2} \leq \gamma_L \|w\|_{L\tau_w,\mathcal{L}_2} + \beta_L(x_0)\,, \quad \forall\, w \in \mathcal{L}_{\tau_w,2}\,. \tag{5.4}$$

This is similar to Definition 4.2, except that the output is the extended output $z(y,u_\theta)$.

Remark 5.2 (Finite-gain $\mathcal{L}_{2h}$−stability with zero bias). *When the condition of Definition 5.3 is met, $\beta_L(x_0)$ in (5.4) is a homogeneous function of initial value x_0 of degree $1+\tau_t/2$, and $\beta_L\left(0\right)=0$, also found in Remark 4.3.*

Definition 5.4 (Partial Differential Inequality for finite $\mathcal{L}_{2h}$−gain). *A homogeneous control strategy $\hat{u}(x)$ achieves the finite $\mathcal{L}_{2h}$−gain from w to $z(y,u_\theta)$ less than γ_L from Definition 5.3, if a $L\tau$−homogeneous storage function $V(x)$ of degree $2-Ld$ exists for system Σ_c (5.1) such that the following partial differential inequality (PDI) is satisfied for all $(x,w) \in \mathbb{R}^{n+o}$*

$$J(V_x,x,\hat{u}(x),w) \triangleq V_x f(x,\hat{u}(x),w) + \|y\|^2_{L\tau_y,q} + \theta^2\|\hat{u}\|^2_{L\tau_u,q} - \gamma_L^2\|w\|^2_{L\tau_w,q} + \epsilon\|x\|^2_{L\tau_x,q} \leq 0 \tag{5.5}$$

for some $\epsilon \geq 0$. *Note that the value function* $J(V_x, x, u, w)$ *is homogeneous of degree 2.*

Note that $\|y\|^2_{L\tau_y,q} + \theta^2 \|\hat{u}\|^2_{L\tau_u,q} = \|z(y, u_\theta)\|^2_{L\tau_z,q}$, thus Definition 5.4 is similar to Definition 4.3.

Theorem 5.2 (Homogeneous stabilizability from Partial Differential Inequality)**.** *If the condition of Definition 5.4 is met for system* Σ_c (5.1) *and some homogeneous controller* $\hat{u}(x)$. *Then the closed loop system has a finite* $L\tau-$*homogeneous* $\mathcal{L}_2-$*gain (*$\mathcal{H}_{\infty h}-$*norm) from disturbance* w *to output* y *(when* $\theta = 0$*) or from disturbance* w *to extended output* $z(y, u_\theta)$. *When* $\epsilon > 0$, *such homogeneous controller* $\hat{u}(x)$ *stabilizes system* Σ_c (5.1). *When* $\epsilon = 0$, *if the system is further zero-state detectable (Definition 4.4) w.r.t. the disturbance* w, *then such homogeneous controller* $\hat{u}(x)$ *stabilizes system* Σ_c (5.1). *For both case, the system is homogeneously stabilizable.*

Proof. The proof is similar as that in Theorem 4.2 by using the Partial Differential Inequality (5.5). And the conclusion of the closed loop system being asymptotically stable is naturally extended to being homogeneously stabilizable. □

5.4.2 Existence of Controller that Minimizes the Value Function

When (5.5) is true for some stabilizing controller $\hat{u}$, the controller that minimizes the value function $J(V_x, x, u, w)$ is of particular interest.

Proposition 5.1 (Existence of controller that minimizes value function)**.** *Suppose the vector field* $f(x, u, w)$ *can be written as addition of two terms* $f(x, u, w) = f_u(x, u) + f_w(x, w)$. *In another word, the control input* u *and disturbance input* w *is uncoupled in the vector field. Then there exists such a homogeneous controller* $u^\star(V_x, x)$, *that achieve the minimum of the value function in* (5.5) *from Definition 5.4, i.e.*

$$J(V_x, x, u^\star(V_x, x), w) \leq J(V_x, x, u, w)\,, \quad \forall\,(x, u, w) \in \mathbb{R}^{n+m+o}\,.$$

Proof. From Theorem 5.1, choosing $\ell = 1$, the homogeneous degree of the storage function V is $2 - Ld > L \max \tau_x$, therefore the degree of each element of $V_x(x)$ w.r.t. x is $2 - Ld - L\tau_{x_i} \geq 2 - Ld - L \max \tau_x > 0$.
Since $V_x f_u(x, u)$ is of homogeneous degree 2 w.r.t. (x, u), **when** x **is fixed** $V_x f_u(x, u)$ might stop being homogeneous w.r.t. u, however, each component in this value function $V_x f_u(x, u)$ w.r.t. u has degree less than 2, thus the term of $\theta^2 \|u\|^2_{L\tau_u,q}$ is the dominating term w.r.t. u in $J(V_x, x, u, w)$ from homogeneity, i.e. $J(V_x, x, u, w) \to \infty$ as $u \to \pm\infty$ for each $(x, w) \in \mathbb{R}^{n+o}$. From continuity of $J(V_x, x, u, w)$ and the Weierstrass theorem that there exists at least a $u^\star(V_x, x)$ that can achieve the minimum of $J(V_x, x, u, w)$ for each fixed (x, w).
The $u^\star(V_x, x)$ can be derived by selecting

$$u^\star(V_x, x) = \arg\min_u J(V_x, x, u, w) = \arg\min_u \left(V_x f_u(x, u) + \theta^2 \|u\|^2_{L\tau_u,q}\right) .$$

Note that $V_x f_u(x, u) + \theta^2 \|u\|^2_{L\tau_u,q}$ is homogeneous of degree 2. Suppose that $\tilde{x} = \nu_\kappa^{\tau_x}(x)$ as well as $\tilde{u}^\star = \nu_\kappa^{\tau_u}(u^\star)$, then we have for each fixed x

$$\kappa^2 \left(V_x f_u(x, u^\star) + \theta^2 \|u^\star\|^2_{L\tau_u,q}\right) \leq \kappa^2 \left(V_x f_u(x, u) + \theta^2 \|u\|^2_{L\tau_u,q}\right) , \quad \forall\, (x, u) \in \mathbb{R}^{n+m} ,$$

which is

$$V_x(\tilde{x})\, f_u(\tilde{x}, \tilde{u}^\star) + \theta^2 \|\tilde{u}^\star\|^2_{L\tau_u,q} \leq V_x(\tilde{x})\, f_u(\tilde{x}, \tilde{u}) + \theta^2 \|\tilde{u}\|^2_{L\tau_u,q} , \quad \forall\ (x, \tilde{u}) \in \mathbb{R}^{n+m} .$$

That is, the $u^\star(V_x(\tilde{x}), \tilde{x})$ collected at $\tilde{x}$ equals the dilated value of the $u^\star(V_x(x), x)$ collected at x w.r.t weight vector τ_u. In another word, $u^\star(V_x, x)$ is a homogeneous controller w.r.t. x, i.e. the homogeneous degree of each element of $u^\star(V_x, x)$ w.r.t. x is τ_{u_i}. □

Remark 5.3. *Extending the output from y to $z(y, u_\theta)$, i.e. including the term $\theta^2 \|u\|^2_{L\tau_u,q}$ in the PDI (5.5), allows the existence of homogeneous $u^\star(V_x, x)$ for the value function $J(V_x, x, u, w)$ for each $(x, w) \in \mathbb{R}^{n+o}$ [4]. Note that, such homogeneous controller $u^\star(V_x, x)$ for Proposition 5.1 might not be unique for each fixed x, and it might not be continuous in x.*
Without the assumption of the control input u and disturbance input w being uncoupled in the vector field, the $u^\star$ that minimize $J(V_x, x, u, w)$ might also depend on disturbance w, which is usually unknown to the controller design. Such cases should be excluded,

since what we wish to design is a continuous homogeneous state-feedback controller.

Theorem 5.3 (Homogeneous controller $u^\star$ for homogeneously stabilizable system)**.** *For a homogeneously stabilizable system Σ_c (5.1), when $L < 2/\left(\max \tau_x + d\right)$, there exists such storage function $V(x)$ that satisfies the condition of Definition 5.4 for some $\epsilon > 0$. With such $V(x)$, the controller $u^\star(V_x, x)$ from Proposition 5.1 stabilizes the system Σ_c.*

Proof. The proof consists of two steps.
First step: Since the system is assumed to be homogeneously stabilizable, from Theorem 5.1, there exists a continuously differentiable CLF V_l of degree $2 - Ld$, with $L < 2/\left(\max \tau_x + d\right)$ by choosing $\ell = 1$. Suppose for such V_l, there exists a $\hat{u}(x)$ and a constant $c_1 > 0$ s.t.

$$\frac{\partial V_l}{\partial x} f(x, \hat{u}(x), 0) \le -c_1 \left\| x \right\|^2_{L\tau_x, q} , \quad \forall\, x \in \mathbb{R}^n ,$$

in another word, such $\hat{u}(x)$ stabilizes system Σ_c (5.1). From homogeneity we also have

$$\left\| h(x, 0) \right\|^2_{L\tau_y, q} + \theta^2 \left\| \hat{u}(x) \right\|^2_{L\tau_u, q} \le c_2 \left\| x \right\|^2_{L\tau_x, q} , \quad \forall\, x \in \mathbb{R}^n ,$$

for some $c_2 > 0$. Let the two homogeneous function of degree 2 w.r.t. (x, u, w)

$$\begin{aligned} \pi(x, u, w) &= \left\| w \right\|^2_{L\tau_w, q} , \\ \psi(x, u, w) &= a \frac{\partial V_l}{\partial x} f\left(x, u, w\right) + \left\| h(x, w) \right\|^2_{L\tau_y, q} + \theta^2 \left\| u \right\|^2_{L\tau_u, q} + \epsilon \left\| x \right\|^2_{L\tau_x, q} . \end{aligned}$$

Then clearly $\pi(x, u, w) \ge 0$ and $w = 0 \Leftrightarrow \pi(x, u, w) = 0$. For any $a > \underline{a} = \left(c_2 + \epsilon\right)/c_1$, the CLF $V = aV_l$ guarantees that when $w = 0$, $\forall\, x \in \mathbb{R}^n \setminus \{0\}$

$$\begin{aligned} \psi(x, \hat{u}(x), 0) &= V_x f(x, \hat{u}(x), 0) + \left\| h(x, 0) \right\|^2_{L\tau_y, q} + \theta^2 \left\| \hat{u} \right\|^2_{L\tau_u, q} + \epsilon \left\| x \right\|^2_{L\tau_x, q} \\ &\le \left(c_2 + \epsilon - ac_1\right) \left\| x \right\|^2_{L\tau_x, q} < 0 . \end{aligned}$$

Therefore from Lemma 4.1, for each such V a finite $\hat{\gamma}$ exist, s.t. $\forall\, (x, w) \in \mathbb{R}^{n+o}$

$$\begin{aligned} &J(V_x, x, \hat{u}(x), w, \hat{\gamma}) = \\ &\frac{\partial V}{\partial x} f(x, \hat{u}(x), w) + \left\| h(x, w) \right\|^2_{L\tau_y, q} + \theta^2 \left\| \hat{u} \right\|^2_{L\tau_u, q} - \hat{\gamma}^2 \left\| w \right\|^2_{L\tau_w, q} + \epsilon \left\| x \right\|^2_{L\tau_x, q} \le -c_3 \left\| x \right\|^2_{L\tau_x, q} \end{aligned} \tag{5.6}$$

for some $c_3 > 0$. Thus such storage function $V(x) = aV_l(x)$ satisfy the condition of Definition 5.4 for some $\epsilon > 0$ when $a > \underline{a} = (c_2 + \epsilon)/c_1$.

Second step: Replace the stabilizing $\hat{u}(x)$ with the $u^\star(V_x, x)$ in Proposition 5.1. Note that in (5.5), the γ has no influence on the $u^\star(V_x, x)$ that minimizes J, since it is multiplied to a term solely of w, which vanishes when taking partial derivative against u. Then we have $J(V_x, x, u^\star(V_x, x), w, \hat{\gamma}) \leq J(V_x, x, \hat{u}(x), w, \hat{\gamma}) \leq -c_3 \|x\|^2_{L\tau_x,q}$ from (5.6), which indicates $\forall\, x \in \mathbb{R}^n \setminus \{0\}$

$$\frac{\partial V}{\partial x} f\left(x, u^\star(V_x, x), 0\right) \leq -\left(c_3 + \epsilon\right) \|x\|^2_{L\tau_x,q} - \|h(x,0)\|^2_{L\tau_y,q} - \theta^2 \|u^\star(V_x, x)\|^2_{L\tau_u,q} < 0\,,$$

since $c_3 > 0$. Thus this $u^\star(V_x, x)$ is also a stabilizing controller for each CLF V_l and $a > \underline{a}$ from first step. When plugging back the $u^\star(V_x, x)$ instead of $\hat{u}(x)$ back in first step, the new $\underline{a}$ should be no bigger than the one derived with $\hat{u}(x)$ for the same reason. From Theorem 5.2, such stabilizing $u^\star(V_x, x)$ achieves finite $\mathcal{L}_{2h}-$gain from w to z. □

For each homogeneously stabilizable system Σ_c (5.1), Theorem 5.1 predicts existence of continuously differentiable CLF V_l and some stabilizing controller $\hat{u}(x)$. Theorem 5.3 shows that, for each CLF V_l, there exist some $\overline{a}$, s.t. the $u^\star\left(a\frac{\partial V_l}{\partial x}, x\right)$, which minimizes the value function $J\left(V_x, x, u^\star(V_x, x), w\right)$, serves also as a homogeneous stabilizing controller. This is already a design method to build such a homogeneous stabilizing controller. In another word, instead of finding some $\hat{u}(x)$, it is preferred to adopt the controller $u^\star\left(a\frac{\partial V_l}{\partial x}, x\right)$ for each V_l.

Similar to Section 4.3, we provide systematic methods to calculate an upper estimate for the $\mathcal{H}_{\infty h}-$norm of system Σ_c (5.1) with some stabilizing controller.

5.5 Upper estimate of $\mathcal{L}_{2h}-$gain

The $u^\star(V_x, x)$ from Proposition 5.1 allows a smaller γ that satisfies $J(V_x, x, u, w, \gamma) \leq 0$.

Proposition 5.2 (Upper estimate of $\mathcal{L}_{2h}$−gain with stabilizing controller)**.** *If the condition of Definition 5.4 is met, with some stabilizing controller* $\hat{u}(x)$ *and storage function* V*, the smallest* $\hat{\gamma}_L(V, \hat{u}(x))$ *that satisfies* (5.5) *is*

$$\begin{aligned} \hat{\gamma}_L(V, \hat{u}(x)) &= \sqrt{\max_{\|(x,w)\|=1} \zeta(V_x, x, \hat{u}(x), w)}\,, \\ \zeta(V_x, x, \hat{u}(x), w) &= \frac{V_x f(x, \hat{u}(x), w) + \|h(x,w)\|^2_{L\tau_y,q} + \theta^2 \|\hat{u}\|^2_{L\tau_u,q}}{\|w\|^2_{L\tau_w,q}}\,, \|w\| \neq 0\,. \end{aligned} \tag{5.7}$$

Proof. The proof is similar to that in Proposition 4.1. □

Proposition 5.3 (Optimality of $u^\star(V_x, x)$ for upper estimate of $\mathcal{L}_{2h}$−gain)**.** *As an extension for Proposition 5.2, for each storage function* V *that satisfies* (5.5)*, the smallest value of* $\hat{\gamma}_L(V, \hat{u}(x))$ *of* (5.7) *with different stabilizing controller* $\hat{u}(x)$ *is achieved when* $u = u^\star(V_x, x)$ *from Proposition 5.1, i.e.* $\gamma_L^\star(V) = \hat{\gamma}_L(V, u^\star(V_x, x)) \leq \hat{\gamma}_L(V, \hat{u}(x))$ *for all stabilizing* $\hat{u}(x)$*.*

Proof. The inequality $J(V_x, x, u^\star(V_x, x), w, \hat{\gamma}) \leq J(V_x, x, u, w, \hat{\gamma})$ in Proposition 5.1 expands into

$$V_x f(x, u^\star(V_x, x), w) + \theta^2 \|u^\star\|^2_{L\tau_u,q} \leq V_x f(x, u, w) + \theta^2 \|u\|^2_{L\tau_u,q}\,, \forall\,(x, u, w) \in \mathbb{R}^{n+m+o}\,,$$

which also includes any stabilizing controller $\hat{u}(x)$. It is clear from the above inequality that

$$\frac{V_x f(x, u^\star(V_x, x), w) + \|h(x,w)\|^2_{L\tau_y,q} + \theta^2 \|u^\star\|^2_{L\tau_u,q} + \epsilon \|x\|^2_{L\tau_x,q}}{\|w\|^2_{L\tau_w,q}} \leq$$
$$\min_{\hat{u}(x)} \frac{V_x f(x, \hat{u}(x), w) + \|h(x,w)\|^2_{L\tau_y,q} + \theta^2 \|\hat{u}\|^2_{L\tau_u,q} + \epsilon \|x\|^2_{L\tau_x,q}}{\|w\|^2_{L\tau_w,q}}\,, \forall\,(x, w) \in \mathbb{R}^{n+o}\,.$$

Thus $\gamma_L^\star(V) \leq \hat{\gamma}_L(V, \hat{u}(x))$ for all stabilizing controller $\hat{u}(x)$ that satisfies (5.5) for each V. □

Theorem 5.2 predicts that a finite γ exists when (5.5) is satisfied in Definition 5.4 for some CLF V and stabilizing homogeneous controller $\hat{u}(x)$. Proposition 5.2 provides an upper estimate of the closed loop $\mathcal{L}_{2h}-$gain $\gamma_L' \leq \hat{\gamma}_L(V, \hat{u}(x))$. Theorem 5.3 shows that the homogeneous controller $u^\star(V_x, x)$ also stabilizes system Σ_c (5.1). Further Proposition 5.3 shows that, with the same V, the closed loop system with the stabilizing homogeneous controller $u^\star(V_x, x)$ achieves a smaller upper estimate of $\mathcal{L}_{2h}-$gain for all stabilizing controller $\hat{u}(x)$.

5.6 Design of $\mathcal{H}_{\infty h}-$controller

Similar to Subsection 4.3.2, it is possible to solve the disturbance input $w^\star(V_x, x)$ analytically that maximizes $J(V_x, x, w)$ w.r.t. w for each fixed $x \in \mathbb{R}^n$ if the vector field of system Σ_h (4.1) is affine in disturbance w.
When restricting the vector field of system Σ_c (5.1) to be affine in control input u, it is also possible to derive the control input $u^\star(V_x, x)$ analytically that maximizes the value function $J\left(V_x, x, u, w\right)$ w.r.t. u for each fixed $(x, w) \in \mathbb{R}^{n+o}$.
When analysing the $w^\star(V_x, x)$ contribute little on the $\mathcal{L}_{ph}-$gain analysis, the analytical solution of $u^\star(V_x, x)$ is naturally interested from the perspective of controller design.

5.6.1 Affine in Control Input

Now consider the continuous homogeneous system Σ_c (5.1) being affine in control input u and that control input u and w is uncoupled in the vector field as brought up in Proposition 5.1, i.e. $f_u(x, u) = g(x)u$, now the SSM is

$$\Sigma_{ca} : \begin{cases} \dot{x} = f_w(x, w) + g(x)u\,, \\ y = h(x, w)\,, \end{cases} \tag{5.8}$$

where g is a homogeneous $n \times m$ matrix-valued function of x, i.e. for all non-zero components g_{ij}.
For each u_j, $g_{ij}(x)$ cannot be zero for all i, otherwise such input does not affect the system. Therefore $L\tau_{u_j} \leq L\tau_{u_j} + Lk_{g_{ij}} = L\tau_{x_j} + Ld < 2$ from non-negativeness of $k_{g_{ij}}$ ($L < 2/\left(\max \tau_x + d\right)$ as in Theorem 5.3). Further, we denote $g = (g_1, \cdots, g_m)$, where g_i is the i-th column of g matrix.

Proposition 5.4 (Stabilizing controller design, extension to [24]). *If the system Σ_{ca} (5.8) is homogeneously stabilizable as in Theorem 5.3 with the control Lyapunov function V_l, then there exists a finite $\underline{\alpha}$, such that for all $\alpha > \underline{\alpha}$, the following continuous homogeneous feedback control law*

$$\hat{u}_i\left(\frac{\partial V_l}{\partial x}, x\right) = -\alpha \left\lceil \frac{\partial V_l}{\partial x} g_i(x) \right\rfloor^{\frac{L\tau_{u_i}}{2-L\tau_{u_i}}} . \tag{5.9}$$

stabilizes system Σ_{ca} (5.8), and the closed loop system is asymptotically stable when $w \equiv 0$.

Proof. From definition of homogeneous stabilizability of control Lyapunov function, there exist some $u(x)$ s.t.

$$\frac{\partial V_l}{\partial x}\left(f_w(x,0) + g(x)u(x)\right) < 0\,, \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}\ . \tag{5.10}$$

First, define three spaces of $x \in \mathbb{R}^n$ for this control Lyapunov function V_l as

$$S_+ \triangleq \left\{\frac{\partial V_l}{\partial x} f_w(x,0) \geq 0\right\}, \quad S_- \triangleq \left\{\frac{\partial V_l}{\partial x} f_w(x,0) < 0\right\}, \quad S_0 \triangleq \left\{\frac{\partial V_l}{\partial x} g(x) = \mathbf{0}_o^\top\right\},$$

as well as a closed surface, namely the homogeneous unit sphere w.r.t x as $S_u \triangleq \left\{\|x\|_{L\tau_x,2} = 1\right\}$.
Apparently, $S_0 \subset S_-$, or else (5.10) can not be true for any $u(x)$. In another word, when $x \in S_+$, the vector $\frac{\partial V_l}{\partial x} g(x)$ can not be a zero vector.
Denoting the vector $\varphi^\top(x) = \frac{\partial V_l}{\partial x} g(x)$, after plugging in the $\hat{u}\left(\frac{\partial V_l}{\partial x}, x\right)$ from (5.9), the Li derivative of the control Lyapunov function (5.10) when $w = 0$ is

$$\frac{\partial V_l}{\partial x} f_w(x,0) - \alpha \sum_{i=1}^{o}\left(|\varphi_i(x)|^{\frac{2}{2-L\tau_{u_i}}}\right), \tag{5.11}$$

now let

$$\pi(x,\varphi) = \sum_{i=1}^{o}\left(|\varphi_i(x)|^{\frac{2}{2-L\tau_{u_i}}}\right), \quad \psi(x,\varphi) \;= \frac{\partial V_l}{\partial x} f(x,0)\,.$$

Apparently, $\pi(x,\varphi) \geq 0$, When $\pi(x,\varphi) = 0$, then $x \in S_0 \subset S_-$, thus $\psi(x,\varphi) < 0$ from (5.10). Therefore from Lemma 4.1, there exist such a $\underline{\alpha}$, s.t. for all $\alpha > \underline{\alpha}$, (5.11) is negative definite. At last, $\max\left(L\tau_u\right) < L\tau_{x_i} + Ld < 2$, thus $\hat{u}(V_x, x)$ in (5.9) is continuous in x. Such $\hat{u}(V_x, x)$ in (5.9) with $\alpha > \underline{\alpha}$ is a continuous homogeneous stabilizing feedback control law. □

Remark 5.4 (Other possible continuous homogeneous controllers)**.** *Another way to design a stabilizing controller like Proposition 5.4 is: For each control Lyapunov function V_l, there exists a finite $\underline{\beta}$, s.t. when $\beta > \underline{\beta}$, the following controller*

$$\hat{u}_i = -\left\lceil \beta \frac{\partial V_l}{\partial x} g_i(x) \right\rfloor^{\frac{L\tau_{u_i}}{2-L\tau_{u_i}}} , \tag{5.12}$$

stabilizes system Σ_{ca} (5.8). The proof is similar. Only when $\tau_u = \ell \mathbf{1}_m$ and $\beta = \alpha^{\frac{2-L\ell}{L\ell}}$ the two approaches give the same stabilizing controller for the same control Lyapunov function V_l .
Other stabilizing controller for control affine system can be found in [40, 2, 17], and discontinuous design in [12].

The design of continuous homogeneous feedback control strategies like (5.9) or (5.12) exclude the case of discontinuous controller, since it would require that $\tau_{u_i} = 0$ for some i. Yet with the extended output z, this would mean that some $\tau_{z_{m+i}} = 0$, then the $\mathcal{L}_{2h}-$norm of the output is not well defined in this case.

Corollary 5.1 ($\mathcal{H}_{\infty h}-$controller)**.** *As direct application of Theorem 5.3 to system Σ_{ca} (5.8). The continuous homogeneous $\mathcal{H}_{\infty h}-$control strategy for any CLF V_l and any $a > \overline{a}$ when choosing $q = 2$ is proposed as*

$$u_i^\star(V_x, x) = -\left| \frac{L\tau_{u_i}}{2\theta^2} \right|^{\frac{L\tau_{u_i}}{2-L\tau_{u_i}}} \left\lceil a \frac{\partial V_l}{\partial x} g_i(x) \right\rfloor^{\frac{L\tau_{u_i}}{2-L\tau_{u_i}}} . \tag{5.13}$$

Proof. This comes directly from Theorem 5.3. Now for system Σ_{ca} (5.8), the value function is turned into

$$\begin{aligned} J(V_x, x, u, w) =& a\frac{\partial V_l}{\partial x}\left(f_w(x,w) + g(x)u\right) + \|y\|^2_{L\tau_y,2} + \theta^2 \|u\|^2_{L\tau_u,2} \\ &- \gamma^2 \|w\|^2_{L\tau_w,2} + \epsilon \|x\|^2_{L\tau_x,2} , \end{aligned}$$

which can be rewritten as

$$J(V_x, x, u, w) = J\left(V_x, x, 0, w\right) + a\frac{\partial V_l}{\partial x}g(x)u + \theta^2 \left\|u\right\|^2_{L\tau_u,2} ,$$

since for any x, w whenever $u_i \to \pm\infty, \forall\, i$, we have $J(V_x, x, u, w) \to \infty$ by homogeneity. The partial derivative of $J(V_x, x, u, w)$ against u reads

$$\frac{\partial J}{\partial u_i} = a\frac{\partial V_l}{\partial x}g_i(x) + \theta^2 \frac{2}{L\tau_{u_i}} \left|u_i\right|^{\frac{2-L\tau_{u_i}}{L\tau_{u_i}}} , \text{ where } \quad \frac{\partial \|u\|^2_{L\tau_u,2}}{\partial u_i} = \frac{2}{L\tau_{u_i}} \left|u_i\right|^{\frac{2-L\tau_{u_i}}{L\tau_{u_i}}} .$$

There exist one unique solution of $\partial J/\partial u = 0$ and by continuity and Weierstrass theorem, the $u^\star(V_x, x)$ that minimizes $J(V_x, x, u, w)$ lies along (5.13). Similar from the proof of Proposition 5.4, $\max\left(L\tau_u\right) < L\tau_{x_i} + Ld < 2$, thus the homogeneous controller (5.13) is continuous in x.
From the proof of Theorem 5.3, there exist such a $\underline{a}$ for each CLF V_l, s.t. (5.13) serves as stabilizing continuous homogeneous control strategy. Further from Proposition 5.3, such $u^\star(V_x, x)$ allows the smallest value of γ that satisfies (5.5) for each CLF $V = aV_l, a > \underline{a}$. □

Remark 5.5 (Simplified step of $\gamma_L^\star$ estimation)**.** *In Theorem 5.3 it is proven that for each CLF $V_l(x)$ with some stabilizing controller $\hat{u}(x)$, there exists a finite number $\underline{a}$, s.t. with $V(x) = aV_l(x)\,, a > \underline{a}$ satisfies the PDI* (5.5) *in Definition 5.4. Thereafter, we can replace the $\hat{u}(x)$ by the $u^\star(V_x, x)$ for this $V(x)$ back in the PDI* (5.5)*, s.t. the $\gamma_L^\star = \hat{\gamma}_L(V, u^\star(V_x, x))$ in* (5.7) *has the smallest value for this storage function $V(x)$ among all stabilizing controller.*
The above procedure involving two stabilizing controllers can be simplified for system Σ_{ca} (5.8)*. With the analytical form of $u^\star(V_x, x)$ in* (5.13)*, which minimizes the PDI* (5.5)*, we can from the beginning skip the homogeneous stabilizing controller $\hat{u}(x)$. For any CLF candidate $V_l(x)$, set $V(x) = aV_l(x)$ and directly plug in the $u^\star(V_x, x)$ from* (5.13) *and verified whether $J(V_x, x, u^\star(V_x, x), 0) < 0$ w.r.t. a (in another word, we search for the $\underline{a}$ for this V_l without the help of any $\hat{u}(x)$). If $J(V_x, x, u^\star(V_x, x), 0) < 0$, then the finite $\gamma_L^\star = \hat{\gamma}_L(V, u^\star(V_x, x))$ in* (5.7) *exists and the controller $u^\star(V_x, x)$ stabilizes the system.*

5.6.2 Affine in Control Input and Disturbance Input

Furthermore, if the system Σ_{ca} (5.8) is further affine in the disturbance w, and the output is devoid of the disturbance w, i.e.

$$\Sigma_{caa}: \begin{cases} \dot{x} = f(x) + g(x)u + e(x)w\,, \\ y = h(x)\,, \end{cases} \tag{5.14}$$

now $e(x)$ is a homogeneous $n \times o$ matrix-valued function of x, denote each component of e as e_{ij}.
For each w_j, $e_{ij}(x)$ cannot be zero for all i, otherwise such input does not affect the system. Therefore $L\tau_{w_j} \leq L\tau_{w_j} + Lk_{e_{ij}} = L\tau_{x_j} + Ld < 2$ from non-negativeness of $k_{e_{ij}}$. Further, we denote $e = (e_1, \cdots, e_m)$, where e_i is the i-th column of e matrix.

Proposition 5.5 (Simplified method)**.** *With system* Σ_{caa} (5.14)*, the upper estimate in Proposition 5.3 when choosing* $q = 2$ *can be found by*

$$\begin{aligned} \gamma_L^\star(V) &= \inf\left\{\gamma \,\middle|\, \sup_{\|x\|=1} \mu(V_x, x, \gamma) \leq 0\right\}, \\ \mu(V_x, x, \gamma) &= J(V_x, x, u^\star(V_x, x), 0) + \sum_{i=1}^{m} |\gamma|^{\frac{-2L\tau_{w_i}}{2-L\tau_{w_i}}} C_i \left|\frac{\partial V}{\partial x} e_i(x)\right|^{\frac{2}{2-L\tau_{w_i}}}, \\ C_i &= \left|\frac{L\tau_{w_i}}{2}\right|^{\frac{L\tau_{w_i}}{2-L\tau_{w_i}}} - \left|\frac{L\tau_{w_i}}{2}\right|^{\frac{2}{2-L\tau_{w_i}}}. \end{aligned}$$

Proof. This proof is again similar to that in Proposition 4.2, thus omitted. □

Similar to that detailed method in Section 4.3.2, the value of the $\gamma_L^\star$ can be found by iteration, during which the sign of $\mu(V(x), x, \gamma)$ is checked.

At last, similar to the result of [24], we can show that if $g(x) = e(x)$, $u^\star\left(a\frac{\partial V_l}{\partial x}, x\right)$ with a large enough a can achieve any $\mathcal{L}_{2h}-$gain $\gamma > \theta$ from input w to output z in the next Proposition.

Proposition 5.6 (Achievable $\mathcal{H}_{\infty h}-$norm under matched controller and disturbance)**.** *As en extension to [24], if system Σ_{caa} (5.14) is homogeneously stabilizable. Further $g(x) = e(x), \tau_u = \tau_w$ (implicitly $m = o$ in dimension). Then for each homogeneous CLF V_l of degree $2 - Ld$ and each $\gamma > \theta$, when choosing $q = 2$ there exists a finite $\underline{a}$, s.t. the optimal controller (5.13) with $V = aV_l, a > \underline{a}$ can achieve the desired $\mathcal{L}_{2h}-$gain γ from w to z.*

Proof. Note that the disturbance that maximizes the value function is in the form of (4.13), i.e.

$$w_i^\star\left(a\frac{\partial V_l}{\partial x}, x\right) = \left|\frac{L\tau_{w_i}}{2\gamma^2}\right|^{\frac{L\tau_{w_i}}{2-L\tau_{w_i}}} \left\lceil a\frac{\partial V_l}{\partial x} e_i(x)\right\rfloor^{\frac{L\tau_{w_i}}{2-L\tau_{w_i}}}, i = 1, \cdots, o\,.$$

With the control $u^\star\left(a\frac{\partial V_l}{\partial x}, x\right)$ from (5.13), which minimizes the value function, and the disturbance $w^\star\left(a\frac{\partial V_l}{\partial x}, x\right)$ from (4.13), which maximizes the value function, the value function can be written as

$$\begin{aligned} J\left(a\frac{\partial V_l}{\partial x}, x, u^\star, w^\star\right) &= a\frac{\partial V_l}{\partial x} f(x) + \|h(x)\|^2_{L\tau_y,2} + \epsilon\, \|x\|^2_{L\tau_x,2} \\ &\quad - \sum_{i=1}^{o} |a|^{\frac{2}{2-L\tau_{u_i}}} \left(|\theta^2|^{\frac{-L\tau_{u_i}}{2-L\tau_{u_i}}} - |\gamma^2|^{\frac{-L\tau_{u_i}}{2-L\tau_{u_i}}}\right) C_i \left|\frac{\partial V_l}{\partial x} g_i(x)\right|^{\frac{2}{2-L\tau_{u_i}}}, \end{aligned} \tag{5.15}$$

where C_i is defined in Proposition 5.5, the three spaces defined in the proof of Proposition 5.4 can be rewritten as

$$S_+ \triangleq \left\{\frac{\partial V_l}{\partial x} f(x) \geq 0\right\}, \quad S_- \triangleq \left\{\frac{\partial V_l}{\partial x} f(x) < 0\right\}, \quad S_0 \triangleq \left\{\frac{\partial V_l}{\partial x} g(x) = \mathbf{0}_o^\top\right\},$$

since $f_w(x, 0) = f(x)$ for system Σ_{caa} (5.14). From homogeneous stabilizability, we have $S_0 \subset S_-$, thus when $x \in S_+$, the vector $\frac{\partial V_l}{\partial x} g(x)$ can not be zero vector. When $x \in S_-$, let

$$\begin{aligned} M_3 &= \min_{x\in S_-\cap S_u} -\frac{\partial V_l}{\partial x} f(x)\,, \\ M_4 &= \max_{x\in S_-\cap S_u} \|h(x)\|^2_{L\tau_y,2} + \epsilon\, \|x\|^2_{L\tau_x,2}\,, \\ M_5 &= \min_{x\in S_-\cap S_u} \sum_{i=1}^{o} \left(|\theta^2|^{\frac{-L\tau_{u_i}}{2-L\tau_{u_i}}} - |\gamma^2|^{\frac{-L\tau_{u_i}}{2-L\tau_{u_i}}}\right) C_i \left|\frac{\partial V_l}{\partial x} g_i(x)\right|^{\frac{2}{2-L\tau_{u_i}}}. \end{aligned}$$

Since $\max\left(L\tau_u\right) < L\tau_{x_i} + Ld < 2$, when $\gamma > \theta$, we have $M_5 \geq 0$, then (5.15) becomes

$$J\left(a\frac{\partial V_l}{\partial x}, x, u^\star, w^\star\right) \leq -aM_3 + M_4 - |a|^{\frac{2}{2-L\min\tau_u}} M_5 \,.$$

If $M_5 = 0$ (i.e. when $\frac{\partial V}{\partial x} g(x) = \mathbf{0}_o^\top$), then $a > a_1 = M_4/M_3$ guarantees that $J\left(a\frac{\partial V_l}{\partial x}, x, u^\star, w^\star\right) \leq 0$ when $x \in S_- \cap S_u$. This is still true when $M_5 > 0$. When $x \in S_+$, let

$$\begin{aligned}
M_6 &= \max_{x\in S_+\cap S_u} \frac{\partial V_l}{\partial x} f(x)\,,\\
M_7 &= \max_{x\in S_+\cap S_u} \|h(x)\|^2_{L\tau_y,2} + \epsilon\,\|x\|^2_{L\tau_x,2}\,,\\
M_8 &= \min_{x\in S_+\cap S_u} \sum_{i=1}^{o} \left(|\theta^2|^{\frac{-L\tau_{u_i}}{2-L\tau_{u_i}}} - |\gamma^2|^{\frac{-L\tau_{u_i}}{2-L\tau_{u_i}}}\right) C_i \left|\frac{\partial V_l}{\partial x} g_i(x)\right|^{\frac{2}{2-L\tau_{u_i}}}\,,
\end{aligned}$$

here $M_8 > 0$ from the analysis above (i.e. $\frac{\partial V_l}{\partial x} g(x,0) \neq \mathbf{0}_o^\top$ when $x \in S_+$ and $\gamma > \theta$), then we have

$$J\left(a\frac{\partial V_l}{\partial x}, x, u^\star, w^\star\right) \leq aM_6 + M_7 - |a|^{\frac{2}{2-L\min\tau_u}} M_8 \,.$$

Since the power on a of the third term is positive and bigger than 1 (simply $\tau_u > 0$ and $\min\left(L\tau_u\right) \leq \max\left(L\tau_u\right) < L\tau_{x_i} + Ld < 2$), thus when $a \to \infty$, $J\left(a\frac{\partial V_l}{\partial x}, x, u^\star, w^\star\right) \to -\infty$. When $a = 0$, the right hand side is positive M_7, from continuity of function, there exists a a_2, s.t. with $a > a_2$, $J\left(a\frac{\partial V_l}{\partial x}, x, u^\star, w^\star\right) \leq 0$ when $x \in S_+ \cap S_u$.
Finally, we can guarantee $J\left(a\frac{\partial V_l}{\partial x}, x, u^\star, w^\star\right) \leq 0$ for any $\gamma > \theta$ with the controller (5.13) when $a > \max\{a_1, a_2\}$ on the homogeneous unit sphere S_u. Since the value function $J\left(a\frac{\partial V_l}{\partial x}, x, u^\star, w^\star\right)$ is homogeneous of degree 2, thus $J\left(a\frac{\partial V_l}{\partial x}, x, u^\star, w^\star\right) \leq 0$ on $x \in \mathbb{R}^n$. □

The condition of Proposition 5.6 is quite strict, which requires $g(x) = e(x)$, under usual case, not all $\gamma > \theta$ is possible with $u^\star\left(a\frac{\partial V_l}{\partial x}, x\right)$. Actually, under such case of $g(x) = e(x)$, the best equivalent controller is $u^\star_{\text{equiv}}(t) = -w(t)$, which is achievable through a high enough gain a in (5.13).

5.7 Examples

Two simple examples are listed to illustrate the procedure of homogeneous $\mathcal{H}_\infty$−controller design.

5.7.1 Scalar System

For a homogeneously stabilizable scalar system

$$\begin{aligned} \dot{x} &= f\lceil x \rfloor^{\frac{1}{\varphi}} + gu + ew\,, \\ y &= cx\,, \end{aligned}$$

where $f > 0, g \neq 0, \varphi > 0$, with weight vectors scaled as $L\tau_x = \varphi L, L\tau_u = L, L\tau_w = L, Ld = L(1-\varphi), L\tau_y = \varphi L$. The extended control output is $z = \begin{bmatrix} y & \theta^L u \end{bmatrix}^\top$. Apparently, $d < 1$. The scalar system is clearly unstable when $u = 0$.

Stabilizing controller

Choose the gain-tunable homogeneous controller candidate as

$$\hat{u}(x) = -\alpha\,\mathrm{sign}(g)\lceil x \rfloor^{\frac{1}{\varphi}}\,,$$

where the closed loop system is

$$\dot{x} = (f - \alpha\,|g|)\lceil x \rfloor^{\frac{1}{\varphi}} + ew\,.$$

Apparently, when the gain of controller $\alpha > f/|g|$, such homogeneous controller Stabilizes the system. Consider a homogeneous storage function of degree $2 - Ld$ as

$$V_l = \frac{\varphi L}{2 - L + \varphi L}\,|x|^{\frac{2-L+\varphi L}{\varphi L}}\,,$$

From the condition of Theorem 5.3, $2 - Ld = 2 - L + \varphi L > \varphi L$, thus $L < 2$. The partial derivative of V_l is $\partial V_l/\partial x = \lceil x \rfloor^{\frac{2-L}{\varphi L}}$. The Li derivative of the storage function is

$$\frac{\partial V_l}{\partial x}\left(f\left\lceil x\right\rfloor^{\frac{1}{\varphi}}+g\hat{u}+ew\right)=\left(f-\alpha\left|g\right|\right)\left|x\right|^{\frac{2}{\varphi L}}+e\left\lceil x\right\rfloor^{\frac{2-L}{\varphi L}}w\,.$$

When $w\equiv 0$, it is negative definite when $\alpha > f/\left|g\right|$. Thus such storage function V_l is a CLF.

The $\mathcal{H}_{\infty h}$ norm for a Stabilizing controller

After replacing the gain of the above Stabilizing controller by $\alpha=\beta f/\left|g\right|,\beta>1$, the closed loop system is now

$$\begin{aligned}\dot{x}&=(1-\beta)\,f\left\lceil x\right\rfloor^{\frac{1}{\varphi}}+ew\,,\\ y&=cx\,.\end{aligned}$$

Then according to Example in Section 4.8, the $\mathcal{L}_{2h}-$gain γ from w to y is

$$\gamma_{w\to y}=\left|\frac{e}{(1-\beta)\,f}\right|^{\frac{1}{L}}\left|c\right|^{\frac{1}{\varphi L}}\;,\tag{5.16}$$

such $\gamma_{w\to y}$ together with the storage function

$$V_{w\to y}=\frac{2\varphi\left|c\right|^{\frac{2}{\varphi L}}}{(2-L+\varphi L)\,(\beta-1)\,f}\left|x\right|^{\frac{2-L+\varphi L}{\varphi L}}$$

satisfies the PDI (5.5). Further, the $\mathcal{L}_{2h}-$gain γ from w to φ is

$$\gamma_{w\to z}=\left|\frac{e}{(1-\beta)\,f}\right|^{\frac{1}{L}}\left(\left|c\right|^{\frac{2}{\varphi L}}+\theta^2\left|\frac{\beta f}{g}\right|^{\frac{2}{L}}\right)^{\frac{1}{2}}\;,\tag{5.17}$$

such $\gamma_{w\to z}$ together with the storage function

$$V_{w\to z}=\frac{2\varphi\left(\left|c\right|^{\frac{2}{\varphi L}}+\theta^2\left|\frac{\beta f}{g}\right|^{\frac{2}{L}}\right)}{(2-L+\varphi L)\,(\beta-1)\,f}\left|x\right|^{\frac{2-L+\varphi L}{\varphi L}}$$

satisfies the PDI (5.5). Apparently, extending the output from y to φ increases the value of the $\mathcal{L}_{2h}-$gain accordingly. Another point to note is that the lower limit of $\gamma_{w\to y}\to 0$ in (5.16), as $\beta\to\infty$. On the other hand, the $\gamma_{w\to z}$ in (5.17) has a non-trivial limit, i.e. when $\beta\to\infty$, $\gamma_{w\to z}\to\theta\left|e/g\right|^{1/L}$.

The $\mathcal{H}_{\infty h}$ controller

From (5.13), the $\mathcal{H}_{\infty h}$ controller for each $V = aV_l$ is

$$u^\star(V_x, x) = -\left|\frac{agL}{2\theta^2}\right|^{\frac{L}{2-L}} \lceil x \rfloor^{\frac{1}{\varphi}} \,. \tag{5.18}$$

Then the closed loop system is

$$\dot{x} = \left(f - |g|^{\frac{2}{2-L}} \left|\frac{aL}{2\theta^2}\right|^{\frac{L}{2-L}}\right) \lceil x \rfloor^{\frac{1}{\varphi}} + ew \,.$$

The Li derivative is

$$\frac{\partial V_l}{\partial x}\left(f \lceil x \rfloor^{\frac{1}{\varphi}} + gu^\star + ew\right) = \left(f - |g|^{\frac{2}{2-L}} \left|\frac{aL}{2\theta^2}\right|^{\frac{L}{2-L}}\right) |x|^{\frac{2}{\varphi L}} + e \lceil x \rfloor^{\frac{2-L}{\varphi L}} w \,,$$

when $w \equiv 0$, apparently for V to be a CLF for this $u^\star(V_x, x)$ (5.18), $f < |g|^{\frac{2}{2-L}} \left|\frac{aL}{2\theta^2}\right|^{\frac{L}{2-L}}$ is needed. That is for $u^\star(V_x, x)$ in (5.18) to be Stabilizing, we need

$$a > \frac{2f^{\frac{2-L}{L}}\theta^2}{|g|^{\frac{2}{L}} L} \,. \tag{5.19}$$

Note that this lower bound is still not the $\underline{a}$ in Theorem 5.3 or Proposition 5.1. Since we need more than V being CLF, but also $J(V_x, x, u^\star, 0) \le 0$ for the γ to exist with this optimal controller (5.18) together with the CLF V.
With $u^\star(V_x, x)$ in (5.18) and $w^\star(V_x, x)$ in (4.13), similar to (5.15) the value function is now

$$\begin{aligned} J(V_x, x, u^\star, w^\star) &= af\, |x|^{\frac{2}{\varphi L}} + |c|^{\frac{2}{\varphi L}} |x|^{\frac{2}{\varphi L}} \\ &- |a|^{\frac{2}{2-L}} \left(\left|\theta^2\right|^{\frac{-L}{2-L}} |g|^{\frac{2}{2-L}} - \left|\gamma^2\right|^{\frac{-L}{2-L}} |e|^{\frac{2}{2-L}}\right) \left(\left|\frac{L}{2}\right|^{\frac{L}{2-L}} - \left|\frac{L}{2}\right|^{\frac{2}{2-L}}\right) |x|^{\frac{2}{L}} \le 0 \,, \end{aligned}$$

which is

$$\left|\gamma^2\right|^{\frac{-L}{2-L}} \le \frac{-af - |c|^{\frac{2}{\varphi L}}}{\left(\left|\frac{L}{2}\right|^{\frac{L}{2-L}} - \left|\frac{L}{2}\right|^{\frac{2}{2-L}}\right) |ae|^{\frac{2}{2-L}}} + \left|\theta^2\right|^{\frac{-L}{2-L}} \left|\frac{g}{e}\right|^{\frac{2}{2-L}} \,.$$

Thus we have

$$\gamma \geq \left(\frac{\left(\left|\frac{L}{2}\right|^{\frac{L}{2-L}} - \left|\frac{L}{2}\right|^{\frac{2}{2-L}}\right)|ae|^{\frac{2}{2-L}}}{-\left(af + |c|^{\frac{2}{\varphi L}}\right)|\theta^2|^{\frac{L}{2-L}} + \left(\left|\frac{L}{2}\right|^{\frac{L}{2-L}} - \left|\frac{L}{2}\right|^{\frac{2}{2-L}}\right)|ag|^{\frac{2}{2-L}}} \right)^{\frac{2-L}{2L}} \theta \,.$$

The smallest γ is when $a \to \infty$, with $\gamma \geq |e/g|^{\frac{1}{L}}\,\theta$, the same as the previous subsection.

5.7.2 Chain Integrator

Consider a chain integrator

$$\dot{x}_1 = x_2 + u_1 + w_1 \,,$$
$$\dot{x}_2 = u_2 + w_2 \,.$$

As described in Section 5.2, the homogeneous degree of the chain integrator can be freely assigned, denoted as d. Let the weight vectors be

$$L\tau_x = (L, L + Ld)\ , L\tau_u = (L + Ld, L + 2Ld)\ , L\tau_w = (L + Ld, L + 2Ld)\ .$$

clearly, for all weight vectors to be positive, $d > -0.5$ is needed. Use the homogeneous storage function of degree $2 - Ld$.

$$V(x) = a_1 \left(\frac{L}{2 - Ld} |x_1|^{\frac{2-Ld}{L}} + a_{12} \lceil x_1 \rfloor^{\frac{2-L-2Ld}{L}} x_2 + a_2 \frac{L + Ld}{2 - Ld} |x_2|^{\frac{2-Ld}{L+Ld}} \right) .$$

From the condition of Theorem 5.3, we need $L < 2/(\max \tau_x + d)$, i.e. when $d \in (-0.5, 0)$, $L < 2/(1 + d)$, when $d \geq 0$, $L < 2/(1 + 2d)$. Yet, probing the partial derivative of the above storage function, we actually need $L < 2/(2 + 2d)$ for the $\mathcal{L}_{ph}-$controller to be continuous.

If we can show that for some a_{12}, a_2 the above storage function is a CLF, then according to (5.13), the $\mathcal{H}_{\infty h}-$controller has the form as

$$u_1^\star = -\left|a_1 \frac{L+Ld}{2\theta^2}\right|^{\frac{L+Ld}{2-L-Ld}} \left\lceil \lceil x_1 \rfloor^{\frac{2-L-Ld}{L}} + a_{12} \frac{2-L-2Ld}{L} |x_1|^{\frac{2-2L-2Ld}{L}} x_2 \right\rfloor^{\frac{L+Ld}{2-L-Ld}} ,$$
$$u_2^\star = -\left|a_1 \frac{L+2Ld}{2\theta^2}\right|^{\frac{L+2Ld}{2-L-2Ld}} \left\lceil a_{12} \lceil x_1 \rfloor^{\frac{2-L-2Ld}{L}} + a_2 \lceil x_2 \rfloor^{\frac{2-L-2Ld}{L+Ld}} \right\rfloor^{\frac{L+2Ld}{2-L-2Ld}} .$$

Such $\mathcal{H}_{\infty h}-$controller stabilizes the system for some $a_1 > \underline{a}_1$. Such $\underline{a}_1$ can be found by verifying whether $J(V_x, x, u^\star, 0) \leq 0$ for all $x \in \mathbb{R}^n$ or simply on the surface of $\|x\| = 1$. And further, by calculating the $\gamma_L^\star$ from Proposition 5.3 or Proposition 5.5 might suggest some preferred set of a_1, a_{12}, a_2 which lead to a smaller $\mathcal{L}_{2h}-$gain for closed loop system.

6 Case Study: Continuous Super-Twisting Like Algorithm

In this chapter, we present the analysis of the continuous super-twisting like algorithm (CSTLA) in detail. Section 6.1 is an extension to the publication [64], in which the traditional $\mathcal{H}_\infty-$norm is applied to the super-twisting algorithm (STA). Yet this is done with a state transformation, which allows the traditional $\mathcal{H}_\infty-$norm from input u to output $y = Ex^{\frac{1}{r_x}}$ being homogeneous with non-zero degree, thus is local. In this chapter as well as in Appendix A.6 we apply a similar method on the CSTLA.
In Section 6.2, the material published in [61], where the homogeneous $\mathcal{H}_\infty-$norm is first introduced and applied to the CSTLA, is also extended to allow the scaling weight vector and degree. Part of the materials in this Chapter are published in [60], where some interesting figures not published are included in the appendix.

6.1 Traditional $\mathcal{H}_\infty-$Norm Analysis with State transformation

In [64, 63], the traditional $\mathcal{H}_\infty-$norm is applied to the super-twisting algorithm (STA) with a state transformation. In this section, a similar approach is presented on the continuous super-twisting like algorithm.

6.1.1 State Space Model of Continuous Super-Twisting Like Algorithm

The state space model of the continuous super-twisting like algorithm is [56, 61]

$$\begin{aligned} \dot{x}_1 &= -k_1 \lceil x_1 \rfloor^{\frac{1}{1-d}} + x_2, \\ \dot{x}_2 &= -k_2 \lceil x_1 \rfloor^{\frac{1+d}{1-d}} + b\,u. \end{aligned} \tag{6.1}$$

This system is homogeneous with the weight vector of $\tau_x = (1-d, 1)\,, \tau_u = 1+d$ and $d \in [-1, 1)$. Depending on d, the following cases can occur, when

$d < -1$, the vector field is unbounded (thus discontinuous) at $x_1 = 0$.

$d = -1$, represents the discontinuous Super-Twisting Algorithm (STA), when $\lceil \cdot \rfloor^0$ is understood in the sense of a differential inclusion, explained in A.2. When $k_2 > b$ and k_1 being greater than some lower bound [51], it provides convergence of the state disregard of the uniformly bounded input $|u| < b/k_2$.

$d \in (-1, 0)$, is bounded input bounded state (BIBS) stable as long as $k_1, k_2 > 0$ [56]. When unperturbed, it provides finite-time convergence of the state from any initial value x_0 [3].

$d = 0$, is an LTI system. It is Hurwitz as long as $k_1, k_2 > 0$, by checking the real part of the eigenvalues of the state matrix

$$A = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}.$$

Suppose its two eigenvalues are $\lambda_1 \lambda_2$, then $\det(A) = \lambda_1 \lambda_2 = k_2 > 0$ and $\mathrm{trace}(A) = \lambda_1 + \lambda_2 = -k_1 < 0$. When $u \equiv 0$, it provides exponential stability from any initial value x_0 [3].

$d \in (0, 1)$, is bounded input bounded state (BIBS) stable as long as $k_1, k_2 > 0$ [56]. When $u \equiv 0$, it provides asymptotic stability from any initial value x_0 [3].

$d \geq 1$, has unbounded (discontinuous) vector field at $x_1 = 0$.

Therefore we are only interested in the case $d \in [-1, 1)$, among which the $\mathcal{L}_{ph}-$norm is well defined for $d \in (-1, 1)$.

6.1.2 Parameter Relationship between k_1, k_2 in the CSTLA

Let us probe the system (6.1) in an different way and multiply the dynamics by a constant $\kappa > 0$, then [35]

$$\kappa\dot{x}_1 = -\kappa k_1 \lceil x_1 \rfloor^{\frac{1}{1-d}} + \kappa x_2,$$
$$\kappa\dot{x}_2 = -\kappa k_2 \lceil x_1 \rfloor^{\frac{1+d}{1-d}} + \kappa b\, u,$$

and after exchanging the states and input (x_1, x_2, u) with $(\hat{x}_1, \hat{x}_2, \hat{u}) = (\kappa x_1, \kappa x_2, \kappa u)$, we have

$$\begin{aligned} \dot{\hat{x}}_1 &= -\kappa^{\frac{-d}{1-d}} k_1 \lceil \hat{x}_1 \rfloor^{\frac{1}{1-d}} + \hat{x}_2, \\ \dot{\hat{x}}_2 &= -\kappa^{\frac{-2d}{1-d}} k_2 \lceil \hat{x}_1 \rfloor^{\frac{1+d}{1-d}} + b\, \hat{u}. \end{aligned} \tag{6.2}$$

If we start at (x_{1o}, x_{2o}) and input $u(t)$ in system (6.1), on the other hand we start at $(\kappa x_{1o}, \kappa x_{2o})$ and input with $\kappa u(t)$ in system (6.2). Consequently in (6.1), we would end up with the same scaled trajectory $(\hat{x}_1, \hat{x}_2, \hat{u})$ when replacing the parameters as per $(k_1, k_2) \mapsto \left(\kappa^{\frac{-d}{1-d}} k_1, \kappa^{\frac{-2d}{1-d}} k_2\right)$. Thus in this homogeneous system, such parameter set (k_1, k_2) is not unique, and can always be scaled in this manner and result in the same trajectory by a linear mapping of the input and state. In view of this, the problem of choosing the parameter set is turned into finding the relationship between the k_1 and k_2 when one is fixed, as shown in [64].
Note that this scaling effect of the parameters has nothing to do with the homogeneous dilation in (4.3) and (4.4). When the initial value and the input are linearly scaled by κ, with such scaled parameter the trajectory is also linearly scaled by κ. Such linear scaling with different parameter does not involve the scaling of the time. On the other hand, when we apply homogeneous dilation to the initial value, input as well as time, then the trajectory of the state is homogeneous dilated (as in (4.3), (4.4)) with the same parameter.

Remark 6.1 (Scaling effect of parameter)**.** *In the linear case, i.e.* $d = 0$*, such pairing is unique, since* $(k_1, k_2) \mapsto (\kappa^0 k_1, \kappa^0 k_2)$*. Note that this scaling effect will also appear*

again later in the simulation. It is interesting to note that the BIBS property exists for $d \in (-1, 1)$ [56], where the STA needs to be excluded. Yet this parameter scaling effect is only valid for a non-zero homogeneous degree, with the linear case singled out.
For the STA the pairing is $\left(L^{\frac{1}{2}}k_1, Lk_2\right)$. That is also in accordance with [64], where the parameter set for the STA observer is $k_1 = 2\sqrt{k_2}$. This means that if k_2 is picked κ times larger, then k_1 should be paired with $\kappa^{\frac{1}{2}}$ times larger, to have the system result in the same linear-scaled trajectory, i.e. the same behaviour. The author of [32] offers the widely used parameter set $k_1 = 1.5\sqrt{b}, k_2 = 1.1b$ for the STA (for system (6.1) *when $d = -1$ and $\|u\|_{\mathcal{L}_\infty} < 1$), where scaling relationship is then mimicked through b.*

6.1.3 Traditional $\mathcal{H}_\infty$−norm with State Transformation

By the state transformation $\xi = x^{\frac{1}{r_x}} = \left(\lceil x_1 \rfloor^{\frac{1}{1-d}}, x_2\right)^\top$, similar to the one first introduced in [39], the transformed system reads

$$\begin{aligned}
\dot{\xi} &= \begin{pmatrix} \frac{1}{1-d} |x_1|^{\frac{d}{1-d}} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1-d} |x_1|^{\frac{d}{1-d}} \left(-k_1 \lceil x_1 \rfloor^{\frac{1}{1-d}} + x_2\right) \\ -k_2 \lceil x_1 \rfloor^{\frac{1+d}{1-d}} + bu \end{pmatrix} \\
&= |x_1|^{\frac{d}{1-d}} \begin{pmatrix} \frac{1}{1-d} \left(-k_1 \lceil x_1 \rfloor^{\frac{1}{1-d}} + x_2\right) \\ -k_2 \lceil x_1 \rfloor^{\frac{1}{1-d}} \end{pmatrix} + \begin{pmatrix} 0 \\ bu \end{pmatrix} \\
&= |x_1|^{\frac{d}{1-d}} \begin{pmatrix} \frac{-k_1}{1-d} & \frac{1}{1-d} \\ -k_2 & 0 \end{pmatrix} \xi + \begin{pmatrix} 0 \\ bu \end{pmatrix} \\
&= |x_1|^{\frac{d}{1-d}} A\xi + Bu\,,
\end{aligned}$$

with

$$A = \begin{pmatrix} \frac{-k_1}{1-d} & \frac{1}{1-d} \\ -k_2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ b \end{pmatrix}.$$

For the preferred choice of the pair (k_1, k_2), now we take an $\mathcal{H}_\infty$−norm perspective. Here we shall define the traditional $\mathcal{H}_\infty$−norms for $d \in [-1, 1)$, see [64, 65],

$$\lambda = \sup_{\|u\|_{\mathcal{L}_2} \neq 0, u \in \mathcal{U}} \frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}}, y = E\xi \tag{6.3}$$

where $E = \operatorname{diag}\left(\sqrt{E_1}, \sqrt{E_2}\right), E_1, E_2 > 0$ for emphasis on which state to minimize.

When $d \in (-1, 1)$, the system (6.1) is BIBS stable [56], so u is simply $\mathcal{U} = \{u \in \mathcal{L}_2\}$. When $d = -1$, the STA can reject non-vanishing input when $|u(\cdot)| < k_2/b$ with k_1 big enough [51], which requires that the input be restricted to the following set [64]

$$\mathcal{U} = \left\{ u \in \mathcal{L}_2 \;\middle|\; \begin{matrix} k_2 < |bu(t)| < M, \; t \in [t_0, t_1] \\ |bu(t)| \leq k_2, \; t \in \mathbb{R}_{\geq 0} \backslash [t_0, t_1] \end{matrix} \right\}$$

for some finite bound M. Then the state is excited (escape the sliding surface) during $t \in [t_0, t_1]$, but still bounded. With $u \in \mathcal{U}$, (6.3) is equivalent to

$$\|y\|_{\mathcal{L}_2}^2 \leq \lambda^2 \|u\|_{\mathcal{L}_2}^2 . \tag{6.4}$$

Now use some storage functions V to define the respective value functions

$$J(V_x, x, u) = \dot{V} + E_1 |x_1|^{\frac{2}{1-d}} + E_2 |x_2|^2 - \lambda^2 |u|^2$$

and assume that the evolution of the state starts at the origin, implying $V(0) = 0$. Then ensuring $J(V_x, x, u) \leq 0$ for all time we have

$$\int_0^T J(V_x, x, u) \, \mathrm{d}t = V(t) - V(0) + \|y\|_{\mathcal{L}_2}^2 - \lambda^2 \|u\|_{\mathcal{L}_2}^2 \leq 0,$$

which leads to (6.4). Consequently, the smallest such λ is the respective norm for the respective k_1, k_2. Thus, we explore the problem by finding the minimum λ that achieves $J(V_x, x, u) \leq 0$ for all time. Finally, we shall investigate the relationship of k_1, k_2 to the λ that leads to a preferable parameter set.

Note that the value-function $J(V_x, x, u)$ is not built homogeneous, if $\tau_u \neq 1$ (implicitly $d \neq 0$). The V used in the next section also disallows the value-function $J(V_x, x, u)$ to be homogeneous unless $d = 0$.

6.1.4 Algebraic Riccati Equation with Bounded States

The following derivation of analytical optimality is similar to the one in [64], however, we extend the case from $d = -1$ to the range $d \in [-1, 1)$.
Using a simple quadratic Lyapunov function $V(x) = \xi^\top P \xi$, with $P = P^\top > 0$, the

value function becomes

$$
\begin{aligned}
&J(V_x,x,u) = 2\xi^\top P\dot{\xi} + \xi^\top E^\top E\xi - \lambda^2 u^2 \\
&= |x_1|^{\frac{d}{1-d}}\,\xi^\top\left(PA + A^\top P + |x_1|^{\frac{-d}{1-d}}\,E^\top E\right)\xi + 2\xi^\top PBu - \lambda^2 u^2 \\
&= -\lambda^2\left|u - \lambda^{-2}B^\top P\xi\right|^2 + |x_1|^{\frac{d}{1-d}}\,\xi^\top\left(PA + A^\top P + |x_1|^{\frac{-d}{1-d}}\,E^\top E + |x_1|^{\frac{-d}{1-d}}\,\lambda^{-2}PBB^\top P\right)\xi\,.
\end{aligned} \tag{6.5}
$$

Since the term $-\lambda^2|u-\lambda^{-2}B^\top P\xi|^2 \le 0$, for $J(V_x,x,u) \le 0$ we need that the following inequality

$$
PA + A^\top P + |x_1|^{\frac{-d}{1-d}}\,E^\top E + |x_1|^{\frac{-d}{1-d}}\,\lambda^{-2}PBB^\top P \le 0 \tag{6.6}
$$

be satisfied. For $d \in [-1,0]$, clearly $\frac{1}{2} \ge \frac{-d}{1-d} \ge 0$. Note that when $d = 0$, which is the linear case, the inequality (6.6) is free of x_1. In this case, λ from (6.3) is of homogeneous degree $d = 0$, which is constant.
For $d \in [-1,0]$ and $u \in \mathcal{U}$, suppose that $|x_1(\cdot)| < x_b$. Then this algebraic Riccati inequality (ARI)

$$
PA + A^\top P + x_b^{\frac{-d}{1-d}}E^\top E + x_b^{\frac{-d}{1-d}}\lambda^{-2}PBB^\top P \le 0 \tag{6.7}
$$

implies (6.6). Further solving the associated algebraic Riccati equation (ARE)

$$
PA + A^\top P + x_b^{\frac{-d}{1-d}}E^\top E + x_b^{\frac{-d}{1-d}}\lambda^{-2}PBB^\top P = 0 \tag{6.8}
$$

leads to (6.7). In [4], for ARE (6.8) it is shown: If A is Hurwitz, $E^\top E$ is positive definite, and the pair (A,E) is observable, then there exists a $\lambda^\star$ such that, for every $\lambda > \lambda^\star$, at least one positive definite solution P exists for (6.8), and for every $\lambda < \lambda^\star$, such solution does not exist. When $d = 0$, this $\lambda^\star$ equals the $\mathcal{H}_\infty$−norm, i.e. the minimum number that keeps $J(V_x,x,u) \le 0$, $\forall\ (x,u) \in \mathbb{R}^{n+m}$ when $d \ne 0$, it serves as an upper bound for the $\mathcal{H}_\infty$−norm.
Note that the value function $J(V_x,x,u)$ (6.5) is homogeneous only when $d = 0$. Therefore most material in Chapter 4 does not apply here.

When $d \in (0,1)$, we have instead of (6.5)

$$
\begin{aligned}
&J(V_x,x,u) = \\
&-\lambda^2\left|u - \lambda^{-2}B^\top P\xi\right|^2 + \xi^\top\left(|x_1|^{\frac{d}{1-d}}\,PA + |x_1|^{\frac{d}{1-d}}\,A^\top P + E^\top E + \lambda^{-2}PBB^\top P\right)\xi\,.
\end{aligned}
$$

Clearly $\frac{d}{1-d} > 0$, the case of

$$|x_1|^{\frac{d}{1-d}} PA + |x_1|^{\frac{d}{1-d}} A^\top P + E^\top E + \lambda^{-2} PBB^\top P \le 0$$

is contrary to the case of $d \in (-1, 0]$, since the matrix $E^\top E + \lambda^{-2} PBB^\top P$ is positive definite and the matrix $PA + A^\top P$ is negative definite. When $|x_1|$ is small, the above inequality can not be true for any $\lambda > 0$.

This is in accordance to the analysis in Example 3.1, i.e. when the $\gamma^\dagger$ in (3.3) (use $\lambda = \gamma^\dagger$ in (6.3)) is of positive homogeneous degree $1 - (1 + d) > 0 \Rightarrow d < 0$, Γ in (3.4) is bigger when the input is bigger. Also when $\gamma^\dagger$ in (3.3) is of negative homogeneous degree $1 - (1 + d) < 0 \Rightarrow d > 0$, Γ in (3.4) is bigger when the input is smaller. Thus though it is strange from its appearance, λ in (6.3) can be finite for case of $d \in (0, 1)$ only when the state does not converge to the equilibrium. Disregard of the fact of BIBS properties, the traditional $\mathcal{L}_2$−gain does not apply unless we restrict the state by $|x_1(\cdot)| > x_b$. This can be guaranteed by a consistently exiting input, e.g. $u(\cdot) > \epsilon$, however such $u \notin \mathcal{L}_2$. Thus such section of $d \in (0, 1)$ should be excluded.

6.1.5 Solution with the Hamilton Matrix

When $d \in [-1, 0]$, the Hamilton matrix can be used to solve the ARE (6.8) as described in Section 2.5.2, we have

$$\begin{aligned} H &= \begin{pmatrix} A & x_b^{\frac{-d}{1-d}} \lambda^{-2} BB^\top \\ -x_b^{\frac{-d}{1-d}} E^\top E & -A^\top \end{pmatrix} \\ &= \begin{pmatrix} \frac{-k_1}{1-d} & \frac{1}{1-d} & 0 & 0 \\ -k_2 & 0 & 0 & b^2 x_b^{\frac{-d}{1-d}} \lambda^{-2} \\ -x_b^{\frac{-d}{1-d}} E_1 & 0 & \frac{k_1}{1-d} & k_2 \\ 0 & -x_b^{\frac{-d}{1-d}} E_2 & \frac{-1}{1-d} & 0 \end{pmatrix}. \end{aligned}$$

If H has no eigenvalue on the imaginary axis, then the matrix $P = P^\top > 0$ can be formed by computing the eigenvectors of H with the method in A.1 [65]. The eigenvalues of H are the zeros of the characteristic polynomial $\det(sI - H)$, which expands into

$$s^4 + \left(b^2 E_2 \lambda^{-2} x_b^{\frac{-2d}{1-d}} - \frac{1}{(1-d)^2} k_1^2 + \frac{2}{1-d} k_2\right) s^2 + \frac{1}{(1-d)^2} k_2^2 - \frac{1}{(1-d)^2} \left(E_2 k_1^2 + E_1\right) b^2 \lambda^{-2} x_b^{\frac{-2d}{1-d}} = 0.$$

Representing any zero on the imaginary axis, set $s = aj$, $a \in \mathbb{R}$, with j the imaginary unit, we have

$$a^4 - \left(b^2 E_2 \lambda^{-2} x_b^{\frac{-2d}{1-d}} - \frac{1}{(1-d)^2} k_1^2 + \frac{2}{1-d} k_2\right) a^2 + \frac{1}{(1-d)^2} k_2^2 - \frac{1}{(1-d)^2} \left(E_2 k_1^2 + E_1\right) b^2 \lambda^{-2} x_b^{\frac{-2d}{1-d}} = 0. \tag{6.9}$$

Now, any parameter of (6.9) is a real number. Let $W \triangleq b^2 E_2 \lambda^{-2} x_b^{\frac{-2d}{1-d}} - \frac{1}{(1-d)^2} k_1^2 + \frac{2}{1-d} k_2$ and $S \triangleq \frac{1}{(1-d)^2} k_2^2 - \frac{1}{(1-d)^2} \left(E_2 k_1^2 + E_1\right) b^2 \lambda^{-2} x_b^{\frac{-2d}{1-d}}$, for brevity. Then the (real) solution of (6.9) satisfies

$$a^2 = \frac{W \pm \sqrt{W^2 - 4S}}{2}.$$

For non-existence of a non-negative real solution for a^2, we can have both roots to be complex, leading to Case I: $W^2 - 4S < 0$, $S > 0$, that is

$$\begin{cases} \left(b^2 E_2 \lambda^{-2} x_b^{\frac{-2d}{1-d}} - \frac{1}{(1-d)^2} k_1^2 + \frac{2}{1-d} k_2\right)^2 < \frac{4}{(1-d)^2} k_2^2 - \frac{4}{(1-d)^2} \left(E_2 k_1^2 + E_1\right) b^2 \lambda^{-2} x_b^{\frac{-2d}{1-d}}, \\ k_2^2 - \left(E_2 k_1^2 + E_1\right) b^2 \lambda^{-2} x_b^{\frac{-2d}{1-d}} > 0 \end{cases}$$

or we may have two real negative roots, defining Case II: $W^2 - 4S \geq 0$, $S > 0$, $W < 0$, that is

$$\begin{cases} \left(b^2 E_2 \lambda^{-2} x_b^{\frac{-2d}{1-d}} - \frac{1}{(1-d)^2} k_1^2 + \frac{2}{1-d} k_2\right)^2 \geq \frac{4}{(1-d)^2} k_2^2 - \frac{4}{(1-d)^2} \left(E_2 k_1^2 + E_1\right) b^2 \lambda^{-2} x_b^{\frac{-2d}{1-d}}, \\ k_2^2 - \left(E_2 k_1^2 + E_1\right) b^2 \lambda^{-2} x_b^{\frac{-2d}{1-d}} > 0\,, \\ b^2 E_2 \lambda^{-2} x_b^{\frac{-2d}{1-d}} - \frac{1}{(1-d)^2} k_1^2 + \frac{2}{1-d} k_2 < 0\,. \end{cases} \tag{6.10}$$

The union of Case I and II is where H has no eigenvalues on the imaginary axis, i.e. also not at the origin. The details of derivation may be found in the Appendix A.6.

Finally, we come to the conclusion that for any fixed k_2, there exists an optimal $\lambda^\star$ which is

$$\lambda^\star = \frac{|b|}{k_2}\sqrt{\left(2\left(1-d\right)k_2E_2+E_1\right)x_b^{\frac{-2d}{1-d}}}\,, \tag{6.11}$$

achieved at

$$k_1^\star = \sqrt{2\left(1-d\right)k_2 - \frac{\left(1-d\right)^2k_2^2E_2}{2\left(1-d\right)k_2E_2+E_1}}\,.$$

Note that $k_1^\star$ lies within the range of

$$k_1^\star = \begin{cases}\sqrt{\frac{3}{2}\left(1-d\right)k_2} & \text{when } E_1/E_2 \to 0\\ \sqrt{2\left(1-d\right)k_2} & \text{when } E_1/E_2 \to \infty\end{cases}\,.$$

Thus the $\mathcal{H}_\infty-$norm optimal range for k_1 with fixed k_2 is

$$\underline{k}_1 \triangleq \sqrt{\frac{3}{2}\left(1-d\right)k_2} \leq k_1^\star \leq \sqrt{2\left(1-d\right)k_2} \triangleq \overline{k}_1\,. \tag{6.12}$$

Note that even though $\lambda^\star$ in (6.11) is local, i.e. with homogeneous degree $-d$, the range of preferred parameter (6.12) is not dependent on state.

6.2 Homogeneous $\mathcal{L}_p-$gain

A detailed analysis of the homogeneous $\mathcal{L}_2-$gain on the continuous super-twisting-like algorithm (CSTLA) is presented in [61]. In this section, we extend the approach to analysis of homogeneous $\mathcal{L}_p-$gain.

6.2.1 Homogeneous Storage Function

We shall only briefly introduce the choice of storage function. Since the CSTLA contains the linear case, where the $\mathcal{H}_\infty-$norm is well studied, we provide, besides the table of collected data, also the comparison of the nonlinear homogeneous system with the linear system.

Since the STA has a discontinuous vector field and $\tau_u = 0$ when $d = -1$, it is considered for the $\mathcal{L}_{ph}-$gain analysis.

With the same SSM (6.1), from Theorem 4.2 as well as Definition 4.5, we must have $p > d + \max \tau_x$. When $d \in (0,1)$, it is $p > 1 + d$, and when $d \in (-1,0)$, it is $p > 1$. Similar to [58, 57, 61], we construct the following homogeneous storage function V of degree $p - d$

$$V(x) = a_1 V_l(x) = a_1 \left(\frac{1-d}{p-d} |x_1|^{\frac{p-d}{1-d}} - a_{12} x_1 \lceil x_2 \rfloor^{p-1} + \frac{a_2}{p-d} |x_2|^{p-d} \right), \qquad (6.13)$$

with $a_1, a_{12}, a_2 > 0$. It is continuously differentiable if $p \geq 2$, and it is positive definite if

$$a_{12} < \left(\frac{a_2}{p-1} \right)^{\frac{p-1}{p-d}},$$

which is obtained by the Young's inequality. Its partial derivative is

$$V_x^\top = a_1 \begin{bmatrix} \lceil x_1 \rfloor^{\frac{p-1}{1-d}} - a_{12} \lceil x_2 \rfloor^{p-1} \\ -(p-1) a_{12} x_1 |x_2|^{p-2} + a_2 \lceil x_2 \rfloor^{p-d-1} \end{bmatrix},$$

the Lie derivative along the trajectories of (6.1) is

$$\begin{aligned} \dot{V} =& a_1 \Big(-k_1 |x_1|^{\frac{p}{1-d}} - a_{12} |x_2|^p + k_1 a_{12} \lceil x_1 \rfloor^{\frac{1}{1-d}} \lceil x_2 \rfloor^{p-1} + \lceil x_1 \rfloor^{\frac{p-1}{1-d}} x_2 \\ & - k_2 a_2 \lceil x_1 \rfloor^{\frac{1+d}{1-d}} \lceil x_2 \rfloor^{p-d-1} + k_2 (p-1) a_{12} |x_1|^{\frac{2}{1-d}} |x_2|^{p-2} + b a_2 \lceil x_2 \rfloor^{p-d-1} u \\ & - b(p-1) a_{12} x_1 |x_2|^{p-2} u \Big). \end{aligned}$$

Clearly, with $p \geq 2$ the condition of Definition 4.5 is met for all $d \in (-1,1)$ (when $d \in (0,1)$, $p > 1 + d$, and when $d \in (-1,0)$, $p > 1$). For negative definiteness of $\dot{V}$ it is necessary that $k_1, a_{12} > 0$, which is satisfied by assumption. If $p = 2$ it is further required that $a_{12} < k_1/k_2$, since the sixth term is combined into the first term. The homogeneous PDI for this storage function is

$$J\left(V_x, x, u\right) = \dot{V} + \|y\|^p_{\tau_y, p} - \gamma^p \|u\|^p_{\tau_u, p} - \epsilon \|x\|^p_{\tau_x, p} \leq 0\,.$$

For $J\left(V_x, x, 0\right) < 0$ when $x \in \mathbb{R}^2 \backslash \{0\}$, we need when $p > 2$ at least

$$a_1 > \frac{|E_1|^{\frac{p}{1-d}} - \epsilon}{k_1}, \; a_{12} > \frac{|E_2|^p - \epsilon}{a_1},$$

and when $p = 2$

$$a_1 > \frac{|E_1|^{\frac{2}{1-d}} - \epsilon}{k_1 - k_2 a_{12}}, \; a_{12} > \frac{|E_2|^2 - \epsilon}{a_1}.$$

For the LTI system, obtained when $d = 0$, we have $\tau_y = \tau_u = 1$, and thus γ is the classical $\mathcal{L}_p$−gain. With the choice of $p = 2$, it provides an upper bound for the traditional $\mathcal{L}_2$−gain. When $p = 3$ or $p = 4$, it provides an upper bound for the $\mathcal{L}_3$−gain or $\mathcal{L}_4$−gain, respectively. When $p = 2$, the storage function (6.13) is in a quadratic form, and we can compare our approach with the traditional ARE solution for linear systems. For the LTI case we also include the results for $\mathcal{L}_3$−gain and $\mathcal{L}_4$−gains.

System (6.1) is affine in u, so (4.13) in Proposition 4.2 using the proposed V, becomes

$$u^\star(x, V_x) = \left|\frac{(1+d)}{p\gamma^p}\right|^{\frac{1+d}{p-1-d}} \left\lceil b a_1 \left(-(p-1) a_{12} x_1 \left|x_2\right|^{p-2} + a_2 \left\lceil x_2 \right\rfloor^{p-d-1}\right) \right\rfloor^{\frac{1+d}{p-1-d}}. \quad (6.14)$$

Since there is only one input, for $\mathcal{L}_{ph}$−gain we can use (4.14) instead, i.e

$$\gamma^\star = \left|\frac{1+d}{p}\right|^{\frac{1}{p}} \left| \max_{\|x\|=1} \left(1 - \frac{1+d}{p}\right) \frac{\left|b a_1 \left(-(p-1) a_{12} x_1 \left|x_2\right|^{p-2} + a_2 \left\lceil x_2 \right\rfloor^{p-d-1}\right)\right|^{\frac{p}{p-1-d}}}{-J\left(V_x, x, 0\right)} \right|^{\frac{p-1-d}{p(1+d)}}$$

The choice of $y = x_1$ or $y = x_2$ depends on the use case of the algorithm. For controller design we shall pick $y = x_1$ and for observer design $y = x_2$ [64]. In contrast to subsection 6.1.4, where the solution of local λ for the ARI with bounded states (by restricting input) is reasonable only when $d \in [-1, 0]$, the $\gamma^\star$ is global for $d \in (-1, 1)$.

6.2.2 The $\mathcal{L}_p$−gain for Linear Case

For $d = 0$ system (6.1) is an LTI system. When choosing $p = 2$, the storage function in (6.13) has homogeneous degree $p = 2 - d = 2$. So in this case, V is a quadratic storage function $x^\top P x$ used in linear $\mathcal{H}_\infty$ analysis [4]. Therefore we can verify our approach of the homogeneous search by comparing with the ARE in (2.8). A simple mapping from

(6.13) to P is

$$P_{11} = \frac{a_1}{2}\,, P_{12} = -\frac{a_1 a_{12}}{2}\,, P_{22} = \frac{a_1 a_2}{2}\,.$$

After simulation (not shown here) the search in Theorem 4.2, Corollary 4.2 and optimal storage function solved using Riccati equation (2.8) lead to the same $\gamma^\dagger$ and optimal storage function.

For the LTI case, the $\lambda^\dagger$ associated to (6.3) has homogeneous degree 0, which equals to γ' in (3.20). From Appendix A.6 the analytical expression for $\lambda^\dagger$ when $d = 0$ is

$$\lambda^\dagger = \begin{cases} b\sqrt{\frac{\left(k_1^2 E_2 + 2k_2 E_2 + 2E_1\right)}{k_1^2\left(4k_2 - k_1^2\right)} + \frac{\sqrt{\left(k_1^2 E_2 + 2k_2 E_2 + 2E_1\right)^2 + k_1^2\left(4k_2 - k_1^2\right)E_2^2}}{k_1^2\left(4k_2 - k_1^2\right)}} & k_1 < k_\mathrm{C} \\ \frac{b}{k_2}\sqrt{k_1^2 E_2 + E_1} & k_1 \geq k_\mathrm{C} \end{cases}$$

where

$$k_\mathrm{C} = 2k_2 + \frac{\sqrt{\left(E_1 + 2k_2 E_2\right)^2 + 4k_2^2 E_2^2} - \left(E_1 + 2k_2 E_2\right)}{2E_2}.$$

For fixed k_2, the optimal k_1 and the corresponding optimal $\mathcal{L}_2-$gain $\lambda^\dagger$ for any E_1, E_2 are

$$k_1^\dagger(k_2) = \sqrt{2k_2 - \frac{k_2^2 E_2}{2k_2 E_2 + E_1}}, \quad \lambda^\dagger(k_1^\dagger, k_2) = \frac{|b|}{k_2}\sqrt{\left(2k_2 E_2 + E_1\right)}\,.$$

By taking the extremes of E_1/E_2, we obtain the $\mathcal{H}_\infty-$norm optimal range as

$$\underline{k}_1^\dagger \triangleq \sqrt{\frac{3}{2}k_2}, \quad \overline{k}_1^\dagger \triangleq \sqrt{2k_2}.$$

Figure 6.1 shows $\lambda^\dagger$ as a function of k_1 for fixed $k_2 = b = 3$. The left sub-figure shows $\lambda^\dagger$ from input u to output $y = x_1$, and the right sub-figure to output $y = x_2$ instead. Figure 6.1 shows that in the linear case, with a fixed k_2, the $\mathcal{H}_\infty-$norm $\lambda^\dagger$ to x_1 stays constant after $k_1 > \overline{k}_1^\dagger$ (cross in left sub-figure), and $\lambda^\dagger$ to x_2 is convex for $k_1 = \underline{k}_1^\dagger$ (represented by a cross in right sub-figure). Since system (6.1) has a single input u, the Bode plot's maximum gain also shows the $\mathcal{H}_\infty-$norm to channel x_1 and x_2 by selecting $y = x$, reflecting Figure 6.1 in the frequency domain. The left sub-figure in Figure 6.2 shows that with $k_1 < \overline{k}_1^\dagger$ the gain has a peak above 0 dB at mid-frequency. With $k_1 \geq \overline{k}_1^\dagger$ the gain is reduced for higher frequency, yet the DC gain is not improved. The right sub-figure in Figure 6.2 shows that the maximum gain gets a minimum at $k_1 = \underline{k}_1^\dagger$, where larger k_1 leads to a larger DC gain, and smaller k_1 leads to a higher gain at higher frequency.

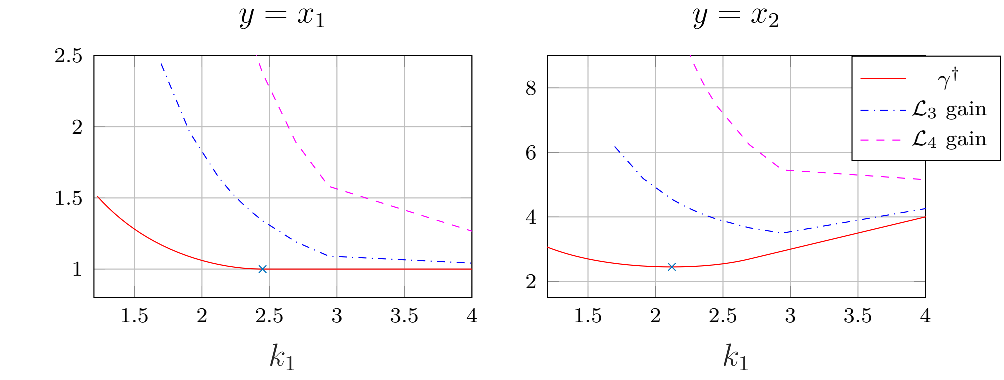

Figure 6.1: Analytical $\mathcal{H}_\infty$−norm $\lambda^\dagger$ for system (6.1).
The parameters of the system are $d = 0$ (linear case) and $k_2 = b = 3$

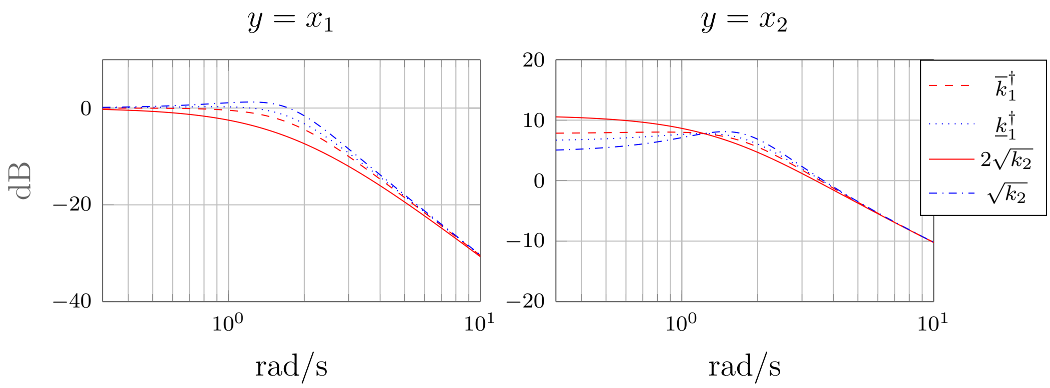

Figure 6.2: Bode plot for system (6.1).
The parameters of the system are $d = 0$ (linear case) and $k_2 = b = 3$

Also in Figure 6.1, the upper estimates for $\mathcal{L}_3$−gain and $\mathcal{L}_4$−gain are plotted. Note that such value is derived from the storage function (6.13), when $d = 0, p = 3$, it is

$$V(x) = a_1 \left(\frac{1}{3} |x_1|^3 - a_{12} x_1 \lceil x_2 \rfloor^2 + \frac{a_2}{3} |x_2|^3 \right) ,$$

when $d = 0, p = 4$, it is

$$V(x) = a_1 \left(\frac{1}{4} |x_1|^4 - a_{12} x_1 \lceil x_2 \rfloor^3 + \frac{a_2}{4} |x_2|^4 \right) ,$$

There are apparently more possible candidates than the above two constructions of storage function of degree 3 or 4, especially the cross term can be designed differently. This highlights the importance of the choice of storage function, which affects how good the upper estimate of $\mathcal{L}_p-$gain, collected from the PDI or the HJI using Proposition 4.1 or 4.2, actually is.

6.2.3 The $\mathcal{L}_{2h}-$gain for Nonlinear Case

For negative d, the upper estimate of $\mathcal{L}_{2h}-$gains $\gamma^\star$ collected from using Proposition 4.1 or 4.2 for the CSTLA are displayed in Figure 6.3, as presented in [61]. It shows the behavior of the $\mathcal{L}_{2h}-$gains from input w to output $y = x_1$ and $y = x_2$, respectively, as k_1 varies, for 4 negative values of $d = \{-0.5, -0.75, -0.9, -0.99\}$, and for each d three fixed values of k_2 are used, given by $k_2 = \{0.99b, b, 1.01b\}$. There is a clear similarity to Figure 6.1. That is, the $\gamma^\star$ to $y = x_1$ (i.e. $E_1 = 1, E_2 = 0$) stops decreasing when k_1 is big enough, and the $\gamma^\star$ to $y = x_2$ (i.e. $E_1 = 0, E_2 = 1$) is convex w.r.t. k_1 for fixed k_2. A more detailed analysis can be found in [61].

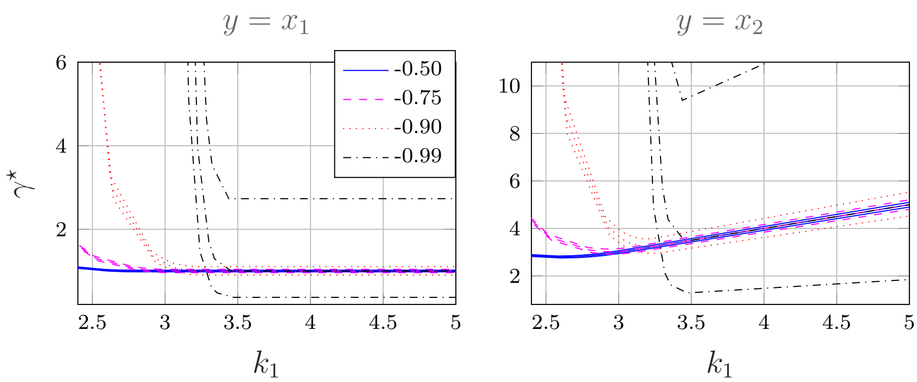

Figure 6.3: Collected $\gamma^\star$ for negative d with varied k_1.
The parameters of the system are $k_2 = b = 3$.

In Appendix A.6, the range of the preferred parameter set, similar to [64], is derived as

$$\underline{k}_1^\star \triangleq \sqrt{\frac{3}{2}\left(1-d\right)k_2}, \quad \overline{k}_1^\star \triangleq \sqrt{2\left(1-d\right)k_2}.$$

Even though the derivation in [64] uses a homogeneous norm of non-zero degree, its parameter range can be verified to be valid by the homogeneous $\mathcal{H}_\infty-$norm as described in [61].

Reflecting on the linear case, when $k_1 \geq \overline{k}_1^\star$, the worst input for both channels is the

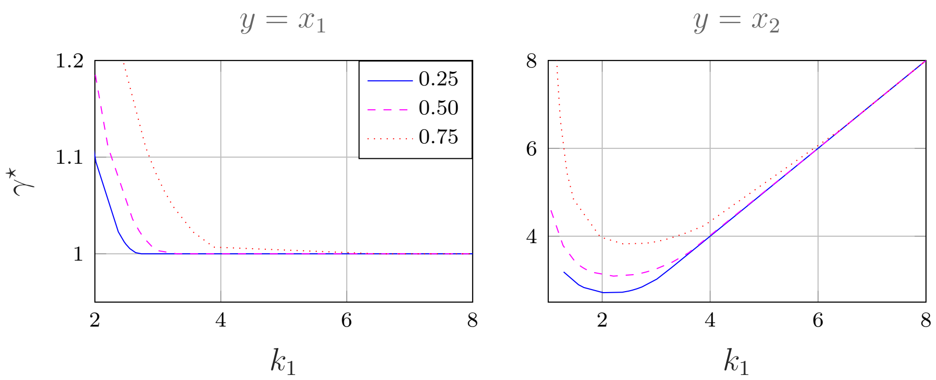

Figure 6.4: Collected $\gamma^\star$ for positive d with varied k_1.
The parameters of the system are $k_2 = b = 3$.

constant input. For (6.1) a constant $u(\cdot) = \overline{u}$ leads to a new equilibrium at $(\bar{x}_1, \bar{x}_2)$ with

$$\bar{x}_1 = \left(\frac{b}{k_2}\,\overline{u}\right)^{\frac{1-d}{1+d}}, \quad \bar{x}_2 = k_1 \left(\frac{b}{k_2}\,\overline{u}\right)^{\frac{1}{1+d}}.$$

Suppose the system start at $x(0) = \bar{x}$, define the ratio of the $\mathcal{L}_{2h}$−norm of output over input as

$$\Gamma(t) = \frac{\|y_T\|_{\tau_y,\mathcal{L}_2}}{\|u_T\|_{\tau_u,\mathcal{L}_2}}, \tag{6.15}$$

especially, denote

$$\begin{aligned} \Gamma_{x_1}(t) &= \left|\frac{\int_0^T \|\bar{x}_1\|^2_{\tau_{x_1},2}\,\mathrm{d}t}{\int_0^T \|\overline{u}\|^2_{\tau_u,2}\,\mathrm{d}t}\right|^{\frac{1}{2}} = \frac{\|\bar{x}_1\|_{\tau_{x_1},2}}{\|\overline{u}\|_{\tau_u,2}} = \frac{|\bar{x}_1|^{\frac{1}{1-d}}}{|\overline{u}|^{\frac{1}{1+d}}} = \left|\frac{b}{k_2}\right|^{\frac{1}{1+d}}, \\ \Gamma_{x_2}(t) &= \left|\frac{\int_0^T \|\bar{x}_2\|^2_{\tau_{x_2},2}\,\mathrm{d}t}{\int_0^T \|\overline{u}\|^2_{\tau_u,2}\,\mathrm{d}t}\right|^{\frac{1}{2}} = \frac{\|\bar{x}_2\|_{\tau_{x_2},2}}{\|\overline{u}\|_{\tau_u,2}} = \frac{|\bar{x}_2|}{|\overline{u}|^{\frac{1}{1+d}}} = k_1\left|\frac{b}{k_2}\right|^{\frac{1}{1+d}}. \end{aligned} \tag{6.16}$$

The value of (6.16) equals the upper bound of $\gamma^\star$ when $k_1 \geq \overline{k}_1^\star$ for negative d collected in Figure 6.3 through Proposition 4.1 or 4.2, i.e.

$$\left.\begin{aligned} \left|\frac{b}{k_2}\right|^{\frac{1}{1+d}} &= \Gamma_{x_1}(t) \leq \gamma'_{x_1} \leq \gamma^\star_{x_1} = \left|\frac{b}{k_2}\right|^{\frac{1}{1+d}} \\ k_1\left|\frac{b}{k_2}\right|^{\frac{1}{1+d}} &= \Gamma_{x_1}(t) \leq \gamma'_{x_2} \leq \gamma^\star_{x_2} = k_1\left|\frac{b}{k_2}\right|^{\frac{1}{1+d}} \end{aligned}\right. \quad \text{when } k_1 \geq \overline{k}_1^\star, d \leq 0.$$

Thus we can conclude that [60]

$$\gamma'_{x_1} = \left|\frac{b}{k_2}\right|^{\frac{1}{1+d}}, \quad \gamma'_{x_2} = k_1 \left|\frac{b}{k_2}\right|^{\frac{1}{1+d}} \ \text{when } k_1 \geq \overline{k}_1^\star, d \leq 0\,. \tag{6.17}$$

Moreover, the behavior of $\gamma^\star$ for positive values of $d = \{0.25,\, 0.5,\, 0.75\}\,, k_2 = b$ are plotted in Figure 6.4, the cases of $k_2 = \{0.99b,\, 1.01b\}$ are omitted, since the differences are too small, i.e. negligible. Note that a similar manner appears as in Figure 6.3, and all three $\gamma^\star$ converge also to (6.17), yet with a bigger k_1 as $\overline{k}_1^\star$ as d grows bigger.
This again shows the great difference between the traditional $\mathcal{L}_2-$gain (in Section 6.1, the case of $d > 0$ must be excluded) and the homogeneous $\mathcal{L}_2-$gain (the $\mathcal{L}_{2h}-$gain is finite and global).

6.3 Simulated Results and Further Observation

In this section, some observations and intuitions are included. Every storage function V mentioned is the optimal storage function that results in the smallest $\gamma^\star$ for each setting of parameters.

6.3.1 Intuition from Figures

Define a function $\zeta_u(x)$ from (4.9), by maximizing its value w.r.t. input u for each fixed state x

$$\zeta_u(x) = \max_{\|u\|\neq 0} \frac{V_x f(x,u) + E_1^2\,|x_1|^{\frac{2}{1-d}} + E_2^2\,|x_2|}{|u|^{\frac{2}{1+d}}}\,, \quad \forall\, x \in \mathbb{R}^n \backslash \{0\}\,.$$

In contrast to Algorithm 4, where the right hand side is maximized on the unit sphere of $\|(x,u)\|_2 = 1$, for each fixed $x \in \mathbb{R}^n$, the function $\zeta_u(x)$ might stop being homogeneous w.r.t. u. In order to find the value of $\zeta_u(x)$, one possibility is presented in the following

- Denoting the dilated state as $\tilde{x}_\kappa = \nu_\kappa^{\tau_x}(x)$, find a $\overline{\kappa}$, s.t. $\|\tilde{x}_{\overline{\kappa}}\|_2 = 1$.
- Divide the section $[0, \overline{\kappa})$ into υ sections, e.g. set $\kappa_i = \frac{i-1}{N}\overline{\kappa}$ for $i = 1, \cdots, \upsilon$

(i.e. $\|\tilde{x}_{\kappa_i}\|_2 < 1$ for $i = 1, \cdots, \upsilon$).

- For each $i = 1, \cdots, \upsilon$, set $u_i = \pm\sqrt{1 - \|\tilde{x}_{\kappa_i}\|_2^2}$ (i.e. $\|(\tilde{x}_{\kappa_i}, u_i)\|_2 = 1$). Evaluate $\zeta(\tilde{x}_{\kappa_i}, u_i)$ and record the maximal value with its corresponding κ_i.

- The refined search to similar to Algorithm 5 around the $\kappa \in [0, \overline{\kappa})$ that results in the biggest value in the previous round.

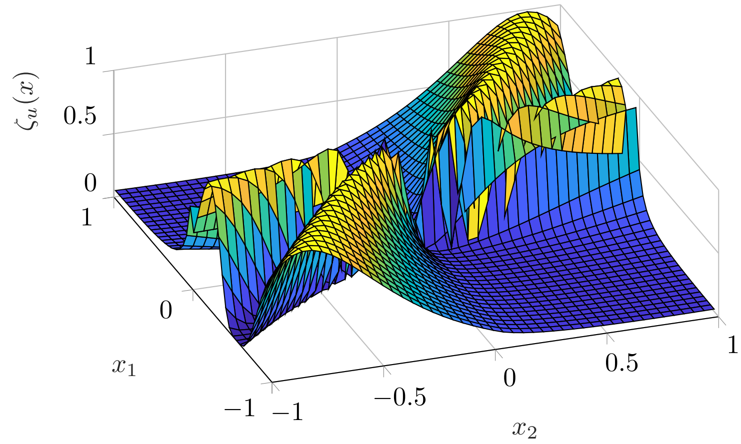

Figure 6.5: Simulation results for $\zeta_u(x)$
The parameters of the system are $d = -0.5$, $E_1 = 0, E_2 = 1$, $k_2 = b = 3$,
$k_1 = \frac{1}{2}\left(\underline{k}_1^\star + \overline{k}_1^\star\right) = 2.799$.

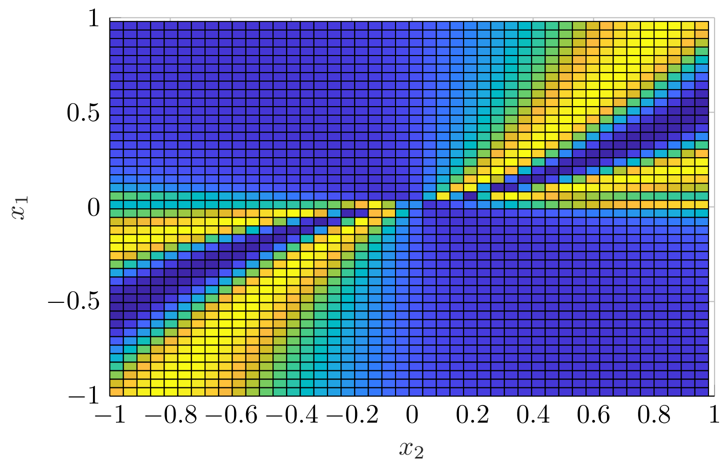

Figure 6.6: Top view of Figure 6.5.
The parameters of the system are $d = -0.5$, $E_1 = 0, E_2 = 1$, $k_2 = b = 3$,
$k_1 = \frac{1}{2}\left(\underline{k}_1^\star + \overline{k}_1^\star\right) = 2.799$.

In the Figure 6.5, the surface of $\zeta_u(x)$ is plotted, the top view of which is shown in Figure

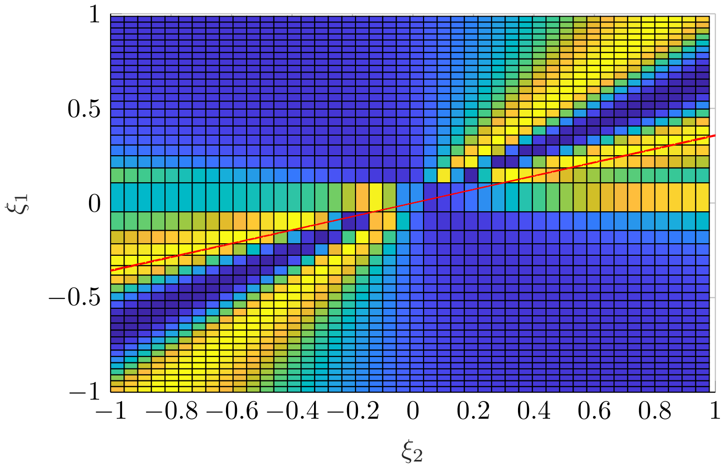

Figure 6.7: Top view of Figure 6.6 with coordinate $\xi = x^{\frac{1}{\tau x}}$.
The parameters of the system are $d = -0.5$, $E_1 = 0, E_2 = 1$, $k_2 = b = 3$, $k_1 = \frac{1}{2}\left(\underline{k}_1^\star + \overline{k}_1^\star\right) = 2.799$. The trajectory of $\zeta(x, u)$ with input $u(t) = \sin(0.01t)$ is also plotted in red.

6.6. After bending the figure with the coordinate of the companion vector $\xi = x^{\frac{1}{\tau x}}$, Figure 6.7 shows the pattern that is linearly shaped.

Figures 6.5, 6.6 and 6.7 reveal that for nonlinear system (6.1) the gain $\gamma^\star$ is not achievable everywhere, and the input should maintain the trajectory of x along the peak of $\zeta_u(x)$ in Figure 6.5, 6.6 and 6.7 as well as use some maximizing $u^\star(x)$ in order to achieve a higher Γ_{x_1} or Γ_{x_2} defined in (6.16). Note that in the linear case, Figure 6.7 with the optimal storage function yields a uniformly achievable $\Gamma = \gamma'$ defined in (6.15) [4].

Figure 6.8 shows that with higher frequency ω in the sinusoidal input $u(t) = \sin(\omega t)$, the trajectory of ξ_1/ξ_2 takes more an “oval shape” ($\xi = x^{\frac{1}{\tau x}}$ as defined in previous subsection). Lower frequency ω leads to a trajectory of ξ_1/ξ_2 more aligned to a line, more specifically, aligned to the peak of ζ_u in Figure 6.7 when $k_1 \geq \overline{k}_1^\star$ (the red line in Figure 6.7 shows the trajectory of ξ with input $u(t) = \sin(0.01t)$). Note that, with the input u that achieves such $\Gamma = \gamma'$ as in (6.17) for some k_1, k_2, the same input also achieves the same value of Γ for smaller k_1 and the same k_2.

More interestingly, as shown in Figure 6.10, even though $\Gamma(t)$ in (6.15) is much smaller than the $\gamma^\star$ for smaller k_1 and the same k_2 as shown in Figure 6.3. Take the setting from Table 6.1 as an example. If we record the function of

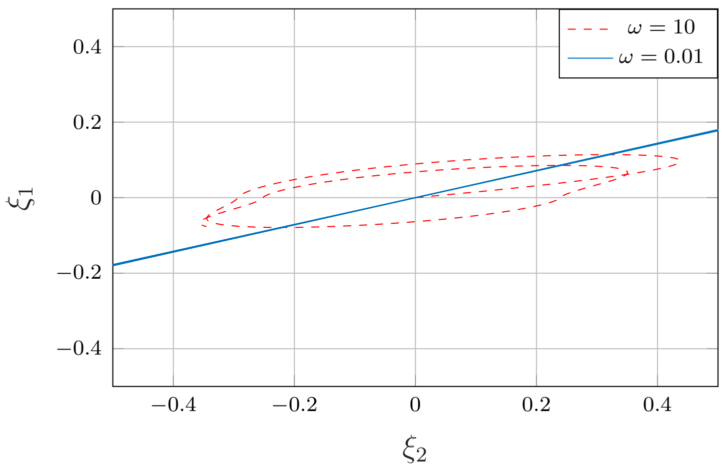

Figure 6.8: State trajectory under input $u(t) = \sin(\omega t)$
The parameters of the system are $d = -0.5$, $k_2 = b = 3$, $k_1 = \frac{1}{2}\left(\underline{k}_1^\star + \overline{k}_1^\star\right) = 2.799$.

Table 6.1: Table of Γ and $\gamma^\star$.

Condition	Γ with constant input	$\gamma^\star$
$E_1 = 1, E_2 = 0$	$\left\|\frac{b}{k_2}\right\|^{\frac{1}{1+d}} = 1$	1.2846
$E_1 = 0, E_2 = 1$	$k_1 \left\|\frac{b}{k_2}\right\|^{\frac{1}{1+d}} = 2.0785$	3.1814

The parameters of the system are $d = -0.5$, $k_2 = b = 3$, $k_1 = 0.8\underline{k}_1^\star = 2.0785$.

$$\zeta(t) = \left| \frac{V_x f\left(x(t), u(t)\right) + E_1^2 \left|x_1(t)\right|^{\frac{2}{1-d}} + E_2^2 \left|x_2(t)\right|}{\left|u(t)\right|^{\frac{2}{1+d}}} \right|^{\frac{1}{2}},$$

as well as the values $\Gamma(t)$ from (6.15) along with the trajectory $x(t)$ and input $u(t) = \sin(0.01t)$, which are shown in Figure 6.10, the value of $\Gamma(t)$ fluctuates closely around 1 when $E_1 = 1, E_2 = 0$ or around 2.0785 when $E_1 = 0, E_2 = 1$. The values of $\zeta(t)$ also fluctuate around such value in Figure 6.10, its magnitude of difference is bigger than that of $\Gamma(t)$, but still quite close (the relationship between $\Gamma(t)$ and $\zeta(t)$ is discussed in Remark 2.6 for LTI systems). This reflects a similar behavior as in Figure 2.6 in Section 2.7 for LTI systems.

Figure 6.9 also shows that such function $\zeta(t)$ though does not travel along the peak of $\zeta_u(x)$ (whose peak is $\gamma_{x_1}^\star = 1.2846$ and $\gamma_{x_2}^\star = 3.1814$ in Table 6.1), the trajectory of ξ_1/ξ_2 is still traveling along some straight line, similar to Figure 6.7 when $\Gamma(t) = \gamma^\star$.

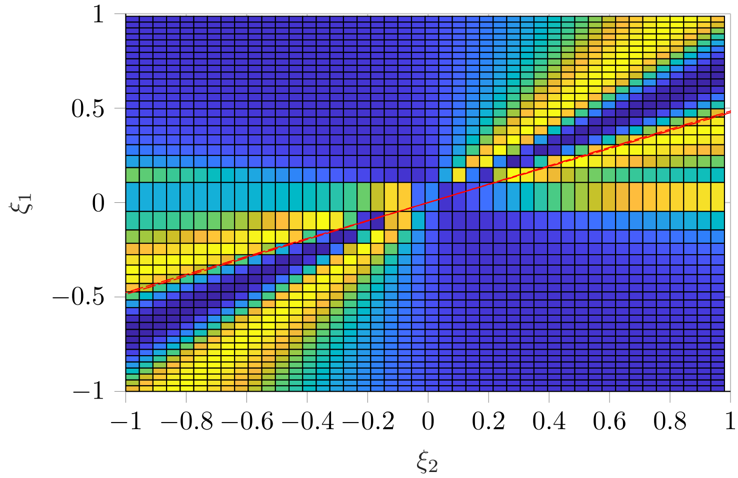

Figure 6.9: Top view of Simulation results for $\zeta_u(\xi)$ for smaller k_1. The parameters of the system are $d = -0.5$, $E_1 = 0, E_2 = 1$, $k_2 = b = 3$, $k_1 = 0.8\underline{k}_1^\star = 2.0785$ and trajectory of $\zeta(x, u)$ with input $u(t) = \sin(0.01t)$.

6.3.2 Worst Input that achieves γ' in Simulation

Other than a constant input, we look for a different worst input that achieves $\Gamma = \gamma'$ as in (6.15). In order to avoid chattering from discretization, we construct such input as

$$u(t) = W\left(D \operatorname{sign}(\sin(\omega t)) + (1 - D)\sin(\omega t)\right), \tag{6.18}$$

where W is the magnitude of u and ω is the frequency in rad/s of the sine component. D proportionates the ratio between the signum function and sine function. The numbers $\Gamma_{x_1} = \Gamma\left(\frac{4\pi}{\omega}\right)$ from (6.15) when $E_1 = 1, E_2 = 0$ and $\Gamma_{x_2} = \Gamma\left(\frac{4\pi}{\omega}\right)$ when $E_1 = 0, E_2 = 1$ are listed in Table 6.2, which again agree with all the $\gamma^\star$ collected for all $k_1 \geq \overline{k}_1^\star$. Since $\gamma^\star$ is the upper bound and the actual achieved ratio of $\mathcal{L}_{2h}-$norm, $\Gamma(t)$ is the lower bound of the $\mathcal{L}_{2h}-$gain. Thus we may say that under the range of $k_1 \geq \overline{k}_1^\star$, we reach the true $\mathcal{H}_{\infty h}-$norm $\gamma' = \gamma^\star = \Gamma\left(\frac{4\pi}{\omega}\right)$ with such non-constant worst input.

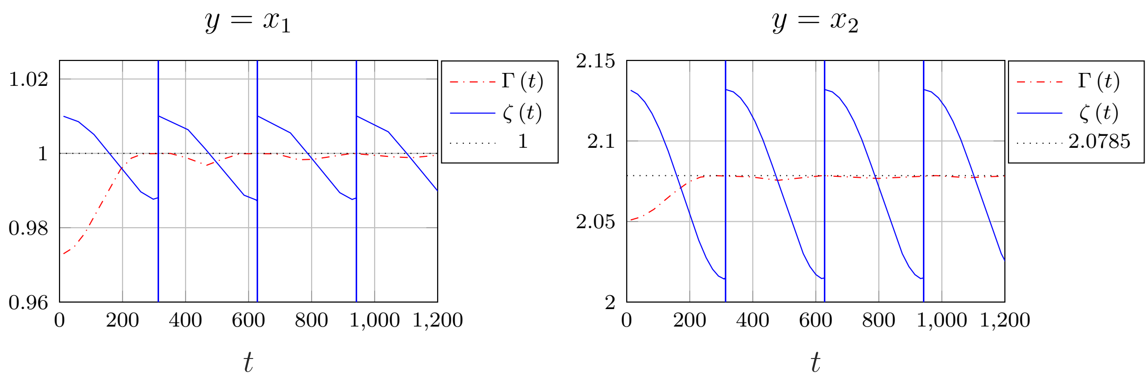

Figure 6.10: The $\zeta(t)$ and $\Gamma(t)$ as in Figure 6.9 with $u(t) = \sin(0.01t)$. The parameters of the system are $d = -0.5$, $E_1 = 0, E_2 = 1$, $k_2 = b = 3$, $k_1 = 0.8\underline{k}_1^\star = 2.0785$.

Table 6.2: Achieved Γ for $k_2 = b = 3$ with u as (6.18).

d	k_1	W	D	ω	Γ_{x_1}	Γ_{x_2}
0.75	$10\overline{k}_1^\star(d) = 12.25$	1	0	0.002	0.9998	12.25
0.50	$10\overline{k}_1^\star(d) = 17.32$	1	0	0.002	0.9998	17.32
0.25	$10\overline{k}_1^\star(d) = 21.21$	1	0	0.002	0.9999	21.21
-0.75	$10\overline{k}_1^\star(d) = 32.40$	0.5	0.45	0.002	0.9999	32.40
-0.90	$10\overline{k}_1^\star(d) = 33.76$	0.7	0.65	0.002	0.9999	33.76
-0.99	$10\overline{k}_1^\star(d) = 34.55$	0.98	0.96	0.0005	0.9960	34.42

6.3.3 Further Discussion of the Worst Input

The surface in Figure 6.12 and 6.13 show the value $\zeta(\theta, u)$ of Proposition 4.1 for the case of $d = -0.5$, $k_2 = b = 3$, $k_1 = \frac{1}{2}\left(\underline{k}_1^\star + \overline{k}_1^\star\right) = 2.799$, i.e.

$$\zeta(\theta, u) = \frac{V_x f(x(\theta, u), u) + E_1^2 \left|x_1(\theta, u)\right|^{\frac{2}{1-d}} + E_2^2 \left|x_2(\theta, u)\right|}{|u|^{\frac{2}{1+d}}},$$

on the unit sphere $\|(x, u)\|_2 = 1$ with coordinates from x to θ by

$$x_1(\theta, u) = \sqrt{1 - u^2}\sin\theta\,, x_2(\theta, u) = \sqrt{1 - u^2}\cos\theta\,,$$

for all $\theta \in [-\pi, \pi]\,, u \in [-1, 1] \setminus \{0\}$. Since $\zeta(x, u)$ is symmetric w.r.t. (x, u), i.e. $\zeta(x, u) = \zeta(-x, -u)$, only the part of $\theta \in [0, \pi]$ is shown in Figure 6.12, 6.13, since $\zeta(\theta, u) = \zeta(\theta + \pi, -u)$.

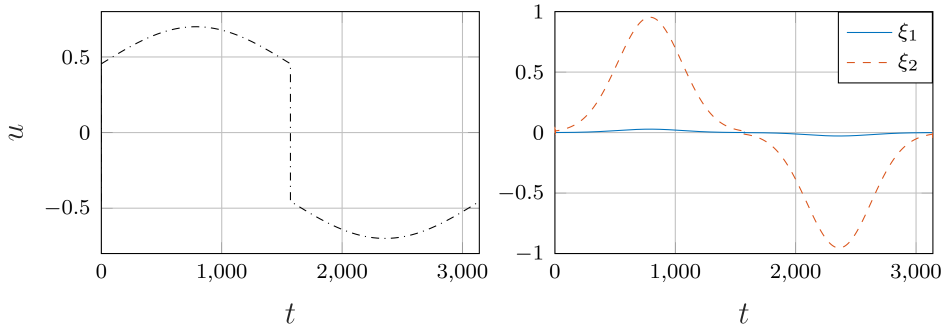

Figure 6.11: Time simulation of the case $d = -0.90$ in Table 6.2.
The parameters of the system are $d = -0.90$, $k_2 = b = 3$,
$k_1 = 10\overline{k}_1^\star = 34.55$, $E_1 = 1$, $E_2 = 0$.

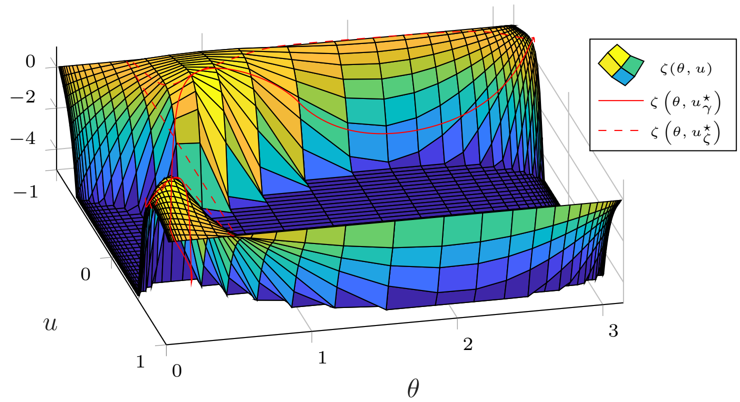

Figure 6.12: $\zeta(\theta, u)$ on the unit sphere for channel x_1
The parameters of the system are $d = -0.5$, $k_2 = b = 3$,
$k_1 = \frac{1}{2}\left(\underline{k}_1^\star + \overline{k}_1^\star\right) = 2.799$, $E_1 = 1$, $E_2 = 0$

Thereafter, use the x coordinate on the unit circle by

$$x_1(\theta) = \sin\theta\,, x_2(\theta) = \cos\theta\,, \quad \forall\,\theta \in [-\pi, \pi]\,. \tag{6.19}$$

and record the input that maximizes $J\left(V_x, x, u\right)$ as

$$u_\gamma^\star(x) = \left|\frac{1+d}{2\gamma^{\star 2}}\right| V_x \begin{bmatrix} 0 \\ b \end{bmatrix}\,. \tag{6.20}$$

Record the value of $\zeta\left(x, u_\gamma^\star(x)\right)$ of each such point. Then project each point $\left(x, u_\gamma^\star(x)\right)$ with dilation ν_κ^τ back onto the unit sphere $\|(\tilde{x}, \tilde{u}^\star)\| = 1$. Since ζ is of homogeneous degree 0, we have $\zeta\left(x, u_\gamma^\star\right) = \zeta\left(\tilde{x}, \tilde{u}_\gamma^\star\right)$. Then rewrite the point $\left(\tilde{x}, \tilde{u}_\gamma^\star\right)$ into coordinate

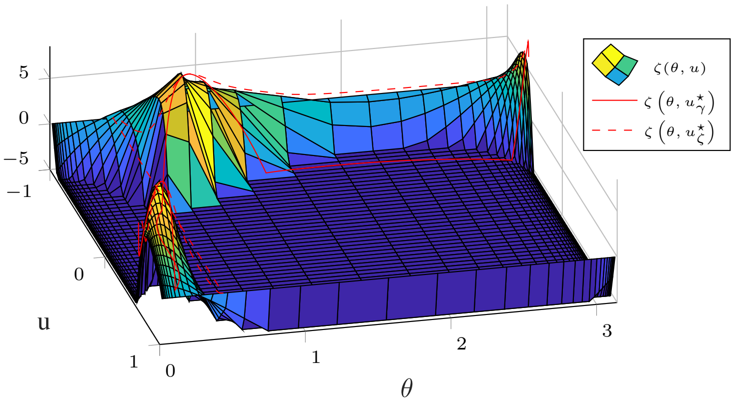

Figure 6.13: $\zeta(\theta, u)$ on the unit sphere for channel x_2
The parameters of the system are $d = -0.5$, $k_2 = b = 3$,
$k_1 = \frac{1}{2}\left(\underline{k}_1^\star + \overline{k}_1^\star\right) = 2.799$, $E_1 = 0$, $E_2 = 1$

$\left(\tilde{\theta}, \tilde{u}^\star_\gamma\right)$ by $\tilde{\theta} = \tan^{-1}(\tilde{x}_1/\tilde{x}_2)$ and plot $\zeta\left(\tilde{\theta}, \tilde{u}^\star_\gamma\right)$in Figure 6.12, 6.13 with red solid line.

At last, another brown dashed line is plotted. First, since the system has only one input u, calculate $\zeta(\theta)$ from (4.14) with the same coordinate (6.19) as

$$\zeta(\theta) = \left| \frac{\left(\frac{1+d}{2}\right)\left(1 - \frac{1+d}{2}\right)\left|V_x[0, b]^\top\right|^{\frac{2}{1+d}}}{-J(V_x, x, 0)} \right|.$$

Then instead of using $u^\star_\gamma(x)$ from (6.20) with fixed $\gamma^\star$, we use

$$u^\star_\zeta(x) = \left|\frac{1+d}{2\zeta(\theta)}\right| V_x \begin{bmatrix} 0 \\ b \end{bmatrix}.$$

The rest is similar to the procedure for the red solid line. The value of $\zeta(\tilde{\theta})$ together with the point of $\left(\tilde{x}, \tilde{u}^\star_\zeta\right)$ on the unit sphere are plotted in brown dashed line.

It is clear that the brown line moves along the peak of positive $\zeta(\theta, u^\star_\zeta)$ along θ, whereas the red line $\zeta(\theta, u^\star_\gamma)$ only touches the peak at three points. Here a sudden change of $u^\star_\zeta$ is observed when the term $-a_{12}x_1 + a_2\lceil x_2 \rfloor^{1-d}$ in (6.14) with $p = 2$ changes sign. When the actual $\zeta(\theta)$ at a certain state is smaller than $\gamma^\star$, then the input u needs to be larger than $u^\star_\gamma$ for the trajectory to travel on the peak of $\zeta(\theta, u)$. Therefore, different from the linear case, u in (4.13) using $u^\star_\gamma$ might not lead to the worst input for all cases.

6.3.4 Shifted Frequency Phenomenon

In the left and right plot of Figure 6.14, for $d = -0.5$ and $k_1 = 3, k_2 = b = 3$, the following function

$$\Gamma(\omega) = \frac{\|y\|_{\tau_y,\mathcal{L}_2}}{\| \left(\kappa \sin(\omega t)\right)_T \|_{\tau_u,\mathcal{L}_2}}, T = \frac{4\pi}{\omega}, \tag{6.21}$$

is plotted for the outputs $y = x_1$ and $y = x_2$, respectively, and the amplitude of the input signal $u = \kappa \sin(\omega t)$ is chosen from the set $\kappa = \{0.5,\ 1,\ 2,\ 3\}$ and its frequency ω ranges from 0.001 rad/s to 100 rad/s. From Figure 6.3 we obtain that the value of $\gamma^\star$ to x_1 is 1, and to x_2 is 3. From Figure 6.14 it is apparent that these gains are attained at a low frequency of the input signal.

Figure 6.14 illustrates another interesting phenomenon. Recall that for homogeneous systems a dilated input signal (3.1) causes a dilated output not only in amplitude, but also in time. Thus, with negative d, if the amplitude of the input u is increased, and if the same value of Γ from (6.21) is to be resulted, then the frequency of the increased-magnitude input u needed to be diminished, as is observed in Figure 6.14. Note that when the amplitude of the input is increased, the frequency plot is shifted to the left, i.e. an input with lower frequency and bigger amplitude achieves the same Γ in (6.21) with negative d.

On the other hand, this is reversed for positive d, since $\tau_t = -d < 0$. Namely, when the amplitude of the input is increased, the frequency plot is shifted to the right, i.e. an input with higher frequency and bigger amplitude achieves the same Γ in (6.21) with positive d. This is shown in Figure 6.15.

The simulations were carried out with forward Euler method with sampling period of $10^{-5}s$.

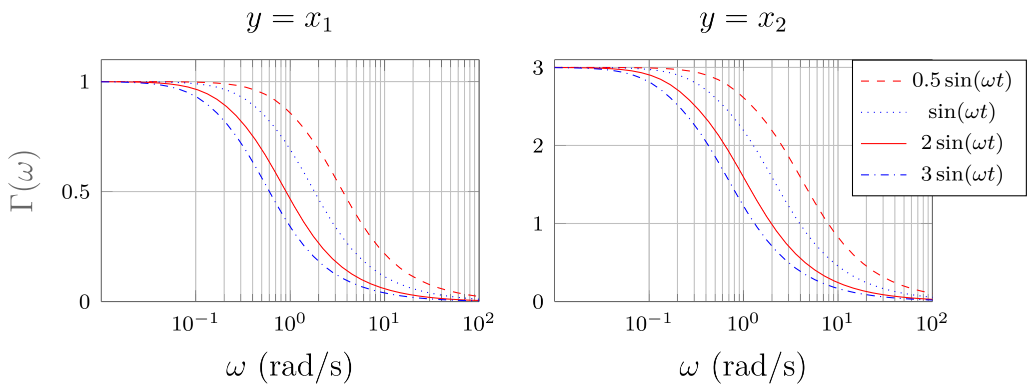

Figure 6.14: $\Gamma(\omega)$ in (6.21) for x_1 (left) and x_2 (right) in response to input $u(t) = \kappa\sin(\omega t)$, for different amplitudes and a range of frequencies. The parameters of the system are $d = -0.5$ and $k_1 = 3, k_2 = b = 3$.

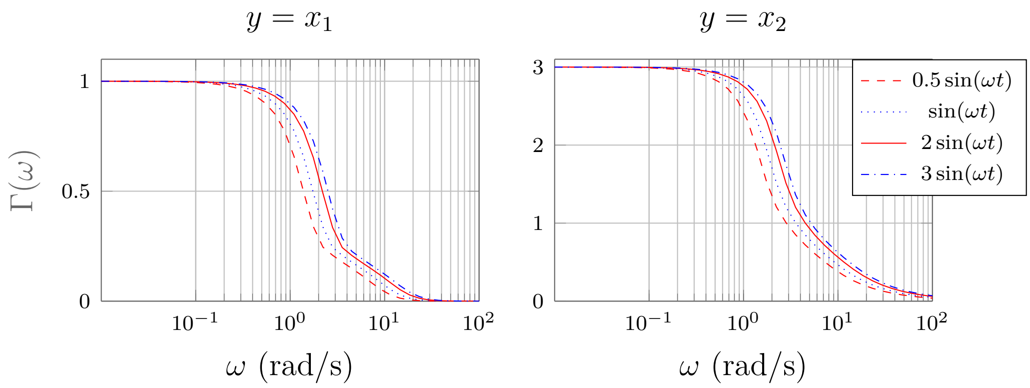

Figure 6.15: $\Gamma(\omega)$ in (6.21) for ξ_1 (left) and ξ_2 (right) in response to input $u(t) = \kappa\sin(\omega t)$, for different amplitudes and a range of frequencies. The parameters of the system are $d = 0.5$ and $k_1 = 3, k_2 = b = 3$.

7 Conclusion and Future Works

7.1 Conclusion

Homogeneous systems and some sliding mode algorithms also built homogeneous with negative homogeneous degree are becoming more popular. When a homogeneous system has negative homogeneous degree, finite-time convergence without input (unperturbed) is guaranteed. Most studies on such systems focus on building homogeneous closed loop systems and show stability for some set of parameter (controller or observer gain) with Lyapunov's direct method. In this case, the range of parameters which guarantees asymptotic stability in the closed loop system is studied.

The $\mathcal{H}_\infty-$analysis represents a systematic method to design a controller or observer that minimizes the worst $\mathcal{L}_2-$gain from input to output. This is also a criterion for gain selection, i.e. instead of providing a range of gain s.t. the closed loop system is asymptotically stable, it provides some optimal gain that minimizes the maximum $\mathcal{L}_2-$gain for the worst possible input. When structured uncertainty exists, $\mu-$synthesis (or called D-K iteration, D for D-scaling and K for $\mathcal{H}_\infty-$controller or observer design) guarantees further robust performance.

However, the traditional $\mathcal{H}_\infty-$norm or $\mathcal{L}_p-$gain applies to homogeneous systems only when the homogeneous systems' input and output have the same weight vectors. Or else, the $\mathcal{L}_p-$gain can be unbounded for arbitrary homogeneous system. In order for the $\mathcal{H}_\infty-$norm or $\mathcal{L}_p-$gain analysis to be applicable for arbitrary continuous homogeneous systems, the homogeneous $\mathcal{L}_p-$norm and the homogeneous $\mathcal{L}_p-$gain (when $p = 2$, homogeneous $\mathcal{H}_\infty-$norm), which remains constant with homogeneous dilation, are introduced in this thesis.

In this thesis,

- the homogeneous $\mathcal{L}_p-$space is shown to be a linear signal space.

- all continuous asymptotically stable homogeneous systems are shown to be homogeneous $\mathcal{L}_p-$stable, including homogeneous $\mathcal{L}_\infty-$stable and Input-to-State stable.

- several systematic methods to calculate such homogeneous $\mathcal{L}_p-$gain are proposed, for both input affine system and non-affine system. The homogeneous $\mathcal{L}_\infty-$gain and homogeneous Input-to-State gain are also included.

- a continuous homogeneous stabilizing controller is proposed for any continuous homogeneously stabilizable system, which can ensure the homogeneous $\mathcal{L}_p-$gain of the closed loop system from input to extended output being less than some finite number.

- the additive inequality of homogeneous $\mathcal{L}_p-$norm of two signals is derived, which allows to derive the homogeneous small gain theorem, i.e. for interconnected homogeneous systems.

With such tools, we can provide guidance for preference of parameters that ensure a smaller homogeneous $\mathcal{L}_p-$gain for the closed loop system, among all parameters that guarantees asymptotic stability. With the homogeneous small gain theorem, such homogeneous $\mathcal{L}_p-$gain can be applied to a cascaded system of homogeneous systems, whose degree are allowed to vary, yet the weight should be matched.

7.2 Future Works

The future works include:

- To include discontinuous vector fields, e.g. differential inclusion described in A.2, whose weight vector of the input and the output are non-zero. One example is the Super-Twisting algorithm with noise, i.e.

$$
\begin{aligned}
\dot{x}_1 &= -k_1 \lceil x_1 + u_2 \rfloor^{\frac{1}{2}} + x_2 + u_1, \\
\dot{x}_2 &= -k_2 \lceil x_1 + u_2 \rfloor^{0},
\end{aligned}
\tag{7.1}
$$

where we can understand $x_1 + u_2$ as the measured x_1 polluted with noise u_2. Here the weight vector and degree are $\tau_x = (2, 1), \tau_u = (1, 2), d = -1$. This allows the definition of $\mathcal{H}_\infty-$norm from Definition 3.7. Yet, in the Definition 4.3, we need to separate the Partial Differential Inequality into two subspaces, one when the vector field is continuous ($x_1 + u_2 \neq 0$ for (7.1)) and the discontinuous case ($x_1 + u_2 = 0$ for (7.1)). Note that $\lceil \cdot \rfloor^0$ is a set-valued function as discussed in A.2 and the treatment need to cover all possible values of such function.

- Possible homogeneous $\mathcal{H}_\infty-$observer designs or coupled with homogeneous $\mathcal{H}_\infty-$controller designs.

- Further studies of candidate Lyapunov functions. For LTI systems, a quadratic Lyapunov function serves well for the $\mathcal{H}_\infty-$controller or observer design. In Proposition 5.1, it is clear that the control Lyapunov function determines the structure of the homogeneous $\mathcal{H}_\infty-$controller. Yet for homogeneous systems, except for the homogeneous degree of the Lyapunov function which is set, the structure is pretty free to choose.

8 Zusammenfassung

Homogene Systeme werden wissenschaftlich zunehmend populärer, u.a. weil unter anderem einige Sliding-Mode-Algorithmen homogen (mit negativen Homogenitätsgrad) sind. Wenn ein homogenes System einen negativen Homogenitätsgrad hat, kann eine endliche Konvergenzzeit im ungestörten Fall garantiert werden. Die meisten Untersuchungen zu solchen Systemen konzentrieren sich auf den Aufbau des homogenen Systems im geschlossenen Regelkreis und weisen Stabilität für einen bestimmten Parametersatz (Regler- oder Beobachterverstärkung) mittels der direkten Methode von Lyapunov nach. In diesem Fall wird der Parameterbereich untersucht, der die asymptotische Stabilität im geschlossenen Regelkreis garantiert.

Die $\mathcal{H}_\infty-$Analyse stellt eine systematische Methode dar, um einen Regler oder Beobachter zu entwerfen, der die $\mathcal{L}_2-$Verstärkung vom Eingang zum Ausgang minimiert. Dadurch wird auch ein Kriterium zur Wahl der Verstärkung gegeben. Anstatt einen Verstärkungsbereich bereitzustellen, in dem das System im geschlossenen Kreis asymptotisch stabil ist, liefert die Methode eine optimale Verstärkung, die die maximale $\mathcal{L}_2-$Verstärkung für den denkbar schlechtesten Eingang minimiert. Wenn es strukturierte Unsicherheit gibt, garantiert die $\mu-$Synthese (auch D-K Iteration genannt) eine weiterhin robuste Performanz.

Allerdings gilt die klassische $\mathcal{H}_\infty-$Norm oder $\mathcal{L}_p-$Verstärkung nur dann für homogene Systeme, wenn Eingang und Ausgang der homogenen Systeme die gleichen Gewichtsvektoren haben. Andernfalls kann die $\mathcal{L}_p-$Verstärkung für beliebige homogene Systeme unbeschränkt sein. Zur Anwendung der $\mathcal{H}_\infty-$Norm- oder $\mathcal{L}_p-$Verstärkungs-Analyse für beliebige kontinuierliche homogene Systeme werden die homogene $\mathcal{L}_p-$Norm sowie die homogene $\mathcal{L}_p-$Verstärkung (bei $p = 2$ homogene $\mathcal{H}_\infty-$Norm), die bei homogener Dilatation konstant bleibt, in dieser Arbeit vorgestellt.

In dieser Doktorarbeit

- wird gezeigt, dass der homogene $\mathcal{L}_p-$Raum ein linearer Signalraum ist.
- wird gezeigt, dass alle stetigen asymptotisch stabilen homogenen Systeme sowohl homogen $\mathcal{L}_p-$stabil, als auch homogen $\mathcal{L}_\infty-$stabil und Eingangs-/Zustands-stabil sind.
- werden mehrere systematische Methoden zur Berechnung einer solchen homogenen $\mathcal{L}_p-$Verstärkung vorgestellt, sowohl für eingangsaffine Systeme als auch für nicht-affine Systeme. Die homogene $\mathcal{L}_\infty-$Verstärkung und die homogene Eingangs-/Zustands-Verstärkung sind ebenfalls berücksichtigt.
- wird für jedes stetige homogen stabilisierbare System ein stabilisierender stetiger homogener Regler vorgeschlagen, der sicherstellen kann, dass die homogene $\mathcal{L}_p-$Verstärkung des geschlossenen Regelkreises vom Eingang zum erweiterten Ausgang kleiner als ein endlicher Wert ist.
- wird die additive Ungleichung der homogenen $\mathcal{L}_p-$Norm zweier Signale hergeleitet, was das homogene Small-Gain Theorem ermöglicht, z.B. für ein vernetztes homogenes System.

Mit diesen Werkzeugen können wir für alle Parameter, die asymptotische Stabilität garantieren, eine Anleitung zur Parameterauswahl geben, die eine kleinere homogene $\mathcal{L}_p-$Verstärkung für das System im geschlossenem Regelkreis gewährleistet. Mit dem homogenen Small-gain Theorem kann eine solche homogene $\mathcal{L}_p-$Verstärkung auf eine Kaskade homogener Systeme angewendet werden, deren Grad variieren darf, wobei deren Gewichtung jedoch passend sein sollte.

A Appendix

A.1 Recovering storage function from the Hamiltonian Matrix

The Hamiltonian matrix H is similar to its negative transpose [65], i.e. with the imaginary matrix

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} ,$$

satisfying $J^2 = -I$, we have

$$J^{-1} H J = -JHJ = -H^\top .$$

Therefore the eigenvalue of H is symmetric w.r.t. the origin (i.e. if λ is an eigenvalue of H, then $-\lambda$ is also an eigenvalue of H). Now collect all the eigenvectors corresponding to eigenvalues with negative real part of H (this rules out the case when H has eigenvalues on imaginary axis, i.e. the real part of some λ is zero). Stack the eigenvalues with negative real part into diagonal matrix Λ and the eigenvectors column-wise in matrix $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$. If P_1 is invertible then $P = P_2 P_1^{-1}$ is the solution of the ARE (2.14), since

$$\begin{bmatrix} A + BR^{-1}D^\top C & BR^{-1}B^\top \\ -C^\top \left(I + DR^{-1}D^\top\right) C & -\left(A + BR^{-1}D^\top C\right)^\top \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \Lambda ,$$

which expands into

$$\begin{aligned} \left(A + BR^{-1}D^{\top}C\right) P_1 + BR^{-1}B^{\top}P_2 &= P_1\Lambda\,, \\ -C^{\top}\left(I + DR^{-1}D^{\top}\right) CP_1 - \left(A + BR^{-1}D^{\top}C\right)^{\top} P_2 &= P_2\Lambda\,. \end{aligned}$$

Premultiplying the first equation by P_1^{-1} and replacing Λ into the second equation, we have

$$\begin{aligned} -C^{\top}\left(I{+}DR^{-1}D^{\top}\right) CP_1 {-} \left(A{+}BR^{-1}D^{\top}C\right)^{\top} P_2 \\ = P_2P_1^{-1}\left(A{+}BR^{-1}D^{\top}C\right) P_1 {+} P_2P_1^{-1}BR^{-1}B^{\top}P_2\,. \end{aligned}$$

Postmultiply the above equation by P_1^{-1} again and we get the ARE (2.14)

$$P\left(A + BR^{-1}D^{\top}C\right) + \left(A + BR^{-1}D^{\top}C\right)^{\top} P + PBR^{-1}P + C^{\top}\left(I + DR^{-1}D^{\top}\right) C = 0\,.$$

with $P = P_2P_1^{-1}$.
For this we need to pay attention to the calculation error. For example, in Matlab© the eigenvalues on the imaginary axis still has real part about 10^{-11} due to calculation error. While in Maple, the error is improved to about 10^{-13}. So in the simulation and examples of this thesis, a bound of 10^{-7} is set as a threshold for determining whether the eigenvalue is on imaginary axis.

A.2 Solution of Discontinuous Vector Filed in Filippov's Sense

Though in this thesis, the homogeneous $\mathcal{L}_p-$gain applies on continuous homogeneous systems, we would also investigate some homogeneous systems with discontinuous vector field, and study some local $\mathcal{H}_\infty-$norm of non-zero degree for such system [64, 63]. Therefore, we need to explain the existence of solution of such discontinuous vector field.
If a vector field is discontinuous, the solution can not be understood in the usual sense.

Take the famous scalar dynamics example in [18]

$$\dot{x}(t) = 1 - 2\ \mathrm{sign}\,(x(t))\ . \tag{A.1}$$

When $x(t) < 0$, the dynamics is $\dot{x}(t) = 3$ and when $x(t) > 0$, the dynamics is $\dot{x}(t) = -1$. In both cases, the trajectory converges to $x = 0$ and can not escape from this equilibrium. Yet the derivative at this equilibrium is $\dot{x}(t)|_{x(t)=0} = 1$, if the vector field is understood as a single-valued function. In order to reconcile this conflict of uniqueness of solution, Filippov generalizes the discontinuous single-valued vector field to differential inclusion, then the solution can be established in the usual sense [18]. (A.1) can be denoted as

$$\begin{aligned} \dot{x}(t) &= 1 - 2\left\lceil x(t) \right\rfloor^0\ , \\ \text{or } \dot{x}(t) &\in 1 - 2\left\lceil x(t) \right\rfloor^0\ . \end{aligned}$$

It has continuous right hand side whenever $x(t) \neq 0$. Here $x(t) = 0$ serves as a hypersurface, in this case a one-dimensional line, between two spaces, where in the two spaces the vector field is continuous. Outside the hypersurface, the existence and uniqueness of solution is guaranteed ([31, Theorem 3.1] or [11, Theorem 1.1]). If the vector field in small neighbourhood outside the hypersurface are both pointing towards to the hypersurface, then vector field can be understood as

$$\begin{aligned} \dot{x}(t) &= \alpha \cdot 3 + (1-\alpha) \cdot -1 = 0\,, \\ \text{where } \alpha &= \frac{3}{3-(-1)} = \frac{3}{4} \end{aligned}$$

and the solution of (A.1) stays on the hypersurface after reaching it. On the other hand, for a non-autonomous system, i.e. with input u, e.g.

$$\dot{x}(t) = bu(t) - 2\left\lceil x(t) \right\rfloor^0\ , \tag{A.2}$$

bound $b|u(t)| < 2$ is necessary for the solution to stay on the hypersurface. If $b|u(t)| > 2$, the solution leaves the hypersurface into the space that $u(t)$ directed.

One example of such differential inclusion dynamics (A.2) that exists in nature is a dry friction force [31], where the state x can be interpreted as the speed of a mass. When the force horizontal is within the limit of dry frictional force, a static mass does not move (i.e. stay in the hypersurface of $x(t) = 0$). A mass with initial speed and horizontal external force less than the limit of frictional force slows down and stops (i.e. converge to and stay in the hypersurface of $x(t) = 0$). When the mass is static and the horizontal external force exceeds the dry frictional force, the mass will start moving in the direction of the force (i.e. exits the hypersurface of $x(t) = 0$). Note that the dry friction force under static or non-static state can vary.

A.3 Jensen's Inequality

Lemma A.1 (Jensen's inequality [10]). *If I is an interval in $\mathbb{R}$ on which $f(x)$ is convex, if $n \geq 2$, w is a positive n-tuple with $\sum_{i=1}^{n} w_i = 1$, x an n-tuple elements in I, then*

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} w_i f(x_i)$$

If f is strictly convex, then the inequality is strict unless $x = \ell \mathbf{1}_n$.

First of all, choose $f(\cdot) = |\cdot|^p, p \geq 1$, which is a convex function, and choose $w = (0.5, 0.5)^\top, x = (2a, 2b)^\top$ and a, b non-negative, we have

$$(a+b)^p \leq 2^{p-1}\left(a^p + b^p\right) . \tag{A.3}$$

Thereafter, from [43, Theorem 5.26] in the section of Jensen-Petrović's Inequality, we have the following lemma

Lemma A.2 ([43]). *Let w, x be two non-negative n-tuples, suppose $x_i \in [0, a]$, $(i = 1, \cdots, n)$ and*

$$\sum_{i=1}^{n} w_i x_i \geq x_j$$

for all $j = 1, \cdots, n$, as well as

$$\sum_{i=1}^{n} w_i x_i \in [0, a] .$$

If $f(x)/x$ is a decreasing function, then

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} w_i f(x_i) . \tag{A.4}$$

If $f(x)/x$ is an increasing function, then the reverse of inequality (A.4) *holds.*

Let $p_1 > p_2 > 0$, and $f(\cdot) = |\cdot|^{\frac{p_1}{p_2}}$, then the function

$$\frac{f(x)}{x} = \lceil x \rfloor^{\frac{p_1 - p_2}{p_2}}$$

is increasing w.r.t. $x \in \mathbb{R}^n$ (since $p_1 > p_2$). Using Lemma A.2 with choice of $w = \mathbf{1}_n$ (the two conditions are met with $a = \infty$), we have

$$\left(\sum_{i=1}^{n} |x_i|^{p_2}\right)^{\frac{p_1}{p_2}} = f\left(\sum_{i=1}^{n} |x_i|^{p_2}\right) \geq \sum_{i=1}^{n} f(|x_i|^{p_2}) = \sum_{i=1}^{n} (|x_i|^{p_2})^{\frac{p_1}{p_2}} = \sum_{i=1}^{n} |x_i|^{p_1} ,$$

i.e.

$$\left(\sum_{i=1}^{n} |x_i|^{p_2}\right)^{\frac{1}{p_2}} \geq \left(\sum_{i=1}^{n} |x_i|^{p_1}\right)^{\frac{1}{p_1}} . \tag{A.5}$$

This indicates that for a vector $x \in \mathbb{R}^n$, $p_1 > p_2 > 0$, we have $\|x\|_{p_2} \geq \|x\|_{p_1}$ where the $p-$norm is defined in (2.1). In another word, for $p \geq 1$, we have for positive a, b

$$(a^p + b^p)^{\frac{1}{p}} \leq a + b \,, \text{ for } p \geq 1 \,. \tag{A.6}$$

Combined with (A.3), we have

$$a^p + b^p \leq (a+b)^p \leq 2^{p-1}\left(a^p + b^p\right) \,, \text{ for } p \geq 1 \,.$$

On the other hand, for $0 < p < 1$ and positive a, b, we have

$$(a^p + b^p)^{\frac{1}{p}} \geq a + b \,, \text{ for } 0 < p < 1 \,,$$

which is

$$(a+b)^p \leq \left(a^p + b^p\right) \,, \text{ for } 0 < p < 1 \,. \tag{A.7}$$

Thus combining the case of $0 < p < 1$ and $p \geq 1$, we have for positive a, b

$$(a+b)^p \leq \max\left\{1, 2^{p-1}\right\}\left(a^p + b^p\right) \,, \text{ for } p > 0 \,. \tag{A.8}$$

A.4 Hölder's Inequality

As a special form of [43, Theorem 4.12] we recall

Lemma A.3 (Hölder's inequality [43])**.** *If f, g are measurable real functions, then the following inequality holds*

$$\int_0^\infty |f(t)g(t)|\,\mathrm{d}t \leq \left(\int_0^\infty |f(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}} \left(\int_0^\infty |f(t)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}, \tag{A.9}$$

for positive real numbers p, q satisfying $p^{-1} + q^{-1} = 1$.

A.5 Young's Inequality

Young's inequality [38], for any positive real numbers a, b, c, and positive real numbers p, q satisfying $p^{-1} + q^{-1} = 1$, the inequality

$$ab \leq \frac{c^p}{p}a^p + \frac{c^{-q}}{q}b^q$$

is satisfied.

A.6 Derivation of $\lambda^\star$ in Subsection 6.1.5

Similar to the derivation in [64], the derivation of $\lambda^\star$ in Section 6.1.5 for the Continuous Super-Twisting Algorithm (CSTLA) is described in this section.

Sectional representation of $\lambda^\star$

From the first inequality in Case II, (6.10), we have

$$k_1^2\left(k_1^2-4\left(1-d\right)k_2\right)\lambda^4+\left(2k_1^2E_2+4k_2(1-d)E_2+4E_1\right)(1-d)^2b^2\lambda^2x_b^{\frac{-2d}{1-d}} + \left(1-d\right)^4b^4E_2^2x_b^{\frac{-4d}{1-d}}\geq 0\,. \tag{A.10}$$

Since $d\in[-1,1)$, thus $1-d\in(0,2]>0$. When $k_1^2\geq 4\left(1-d\right)k_2$, all terms are non-negative. Thus Case II is automatically satisfied, when Case I is infeasible.
When $k_1^2<4\left(1-d\right)k_2$, the left hand side of (A.10) is a downward parabola w.r.t. λ^2, there exists only one positive solution of the equation, which is

$$\lambda_1(k_1,k_2)\triangleq\left(1-d\right)|b|\,x_b^{\frac{-d}{1-d}}\left(\frac{\left(k_1^2E_2+2k_2\left(1-d\right)E_2+2E_1\right)}{k_1^2\left(4\left(1-d\right)k_2-k_1^2\right)} + \frac{\sqrt{\left(k_1^2E_2+2k_2\left(1-d\right)E_2+2E_1\right)^2+k_1^2\left(4\left(1-d\right)k_2-k_1^2\right)E_2^2}}{k_1^2\left(4\left(1-d\right)k_2-k_1^2\right)}\right)^{\frac{1}{2}}\,. \tag{A.11}$$

For (A.10) to hold we need $0\leq\lambda\leq\lambda_1$.
Combing both sections of k_1, both cases for the first inequality are now

$$\text{Case I}:\lambda>\lambda_1 \text{ and } k_1^2<4\left(1-d\right)k_2\,,$$
$$\text{Case II}:\begin{cases}\text{none} & \text{when } k_1^2\geq 4\left(1-d\right)k_2\,,\\ 0\leq\lambda\leq\lambda_1 & \text{when } k_1^2<4\left(1-d\right)k_2\,.\end{cases}$$

For the other two inequalities, first denote two positive number as

$$\lambda_2(k_1,k_2)\triangleq\frac{|b|}{k_2}x_b^{\frac{-d}{1-d}}\sqrt{\left(k_1^2E_2+E_1\right)}\,,$$
$$\lambda_3(k_1,k_2)\triangleq\left(1-d\right)|b|\,x_b^{\frac{-d}{1-d}}\sqrt{\frac{E_2}{k_1^2-2\left(1-d\right)k_2}},\ \text{when } k_1^2>2\left(1-d\right)k_2\,.$$

With the three inequalities, the two cases are equivalent to

$$\text{Case I}:\begin{cases}\lambda>\max\left\{\lambda_1,\lambda_2\right\}\\ k_1^2<4\left(1-d\right)k_2\end{cases}\quad,\quad \text{Case II}:\begin{cases}\lambda\leq\lambda_1\text{ ,when } k_1^2<4\left(1-d\right)k_2\\ \lambda>\max\left\{\lambda_2,\lambda_3\right\}\\ k_1^2>2\left(1-d\right)k_2\end{cases}\,.$$

Among which, when $2(1-d)k_2 < k_1^2 < 4(1-d)k_2$, both case can be rewritten as

$$\begin{aligned} \text{Case I}: &\qquad \lambda > \max\{\lambda_1, \lambda_2\}\ , \\ \text{Case II}: &\quad \lambda_1 \geq \lambda > \max\{\lambda_2, \lambda_3\}\ . \end{aligned}$$

Since the logic for Case I and Case II is "or", thus the smallest $\lambda^\star$ s.t. Case I or Case II is true can be rewritten as

$$\lambda^\star = \begin{cases} \max\{\lambda_1, \lambda_2\} & \text{when } k_1^2 \leq 2(1-d)k_2\,, \\ \max\{\lambda_2, \min\{\lambda_1, \lambda_3\}\} & \text{when } 2(1-d)k_2 < k_1^2 < 4(1-d)k_2\,, \\ \max\{\lambda_2, \lambda_3\} & \text{when } k_1^2 \geq 4(1-d)k_2\,. \end{cases} \tag{A.12}$$

Further $\lambda_2 \propto k_1$ and $\lambda_2 \propto k_2^{-1}$ as well as $\lambda_3 \propto k_1^{-1}$ and $\lambda_3 \propto k_2$. They are both monotonic in k_1 and k_2 in the contrary way. The crossing point between λ_2 and λ_3 w.r.t. k_1 for fixed k_2 lies at

$$\begin{aligned} k_{1,23}^2(k_2) &= 2(1-d)k_2 \\ &\quad + \frac{\sqrt{(E_1 + 2(1-d)k_2E_2)^2 + 4(1-d)^2 k_2^2 E_2^2} - (E_1 + 2(1-d)k_2E_2)}{2E_2}\,. \end{aligned}$$

Clearly, $k_{1,23}^2(k_2) > 2(1-d)k_2$, thus the crossing point between λ_2 and λ_3 w.r.t. k_1 for fixed k_2 is valid (i.e. λ_3 exists).

Convexity of λ_1 on k_1 with fixed k_2

Denote $X(k_1, k_2) = k_1^2E_2 + 2k_2(1-d)E_2 + 2E_1 > 0$ and $Z(k_1, k_2) = k_1^2(4(1-d)k_2 - k_1^2)$. Then (A.11) becomes

$$\lambda_1(k_1, k_2) = (1-d)\,|b|\,x_b^{\frac{-d}{1-d}} \sqrt{\frac{X + \sqrt{X^2 + ZE_2^2}}{Z}}\,.$$

By taking partial derivatives of λ_1^2 w.r.t. k_1 and k_2 we study whether it is convex in k_1 or k_2. To this end, denote

$$X_{k_1} = \frac{\partial X}{\partial k_1} = 2k_1E_2, \quad X_{k_2} = \frac{\partial X}{\partial k_2} = 2(1-d)E_2\,,$$

and

$$Z_{k_1} = \frac{\partial Z}{\partial k_1} = 8\,(1-d)\,k_1 k_2 - 4k_1^3$$
$$Z_{k_2} = \frac{\partial Z}{\partial k_2} = 4\,(1-d)\,k_1^2\,.$$

Then the partial derivatives of λ_1^2 against k_1 and k_2 are

$$\frac{\partial \lambda_1^2}{\partial k_1} = (1-d)^2\, b^2 x_b^{\frac{-2d}{1-d}} \frac{(ZX_{k_1} - Z_{k_1}X)\left(2\sqrt{X^2+ZE_2^2}+2X\right) - E_2^2 Z Z_{k_1}}{2\sqrt{X^2+ZE_2^2}Z^2}$$
$$\frac{\partial \lambda_1^2}{\partial k_2} = (1-d)^2\, b^2 x_b^{\frac{-2d}{1-d}} \frac{(ZX_{k_2} - Z_{k_2}X)\left(2\sqrt{X^2+ZE_2^2}+2X\right) - E_2^2 Z Z_{k_2}}{2\sqrt{X^2+ZE_2^2}Z^2}\,.$$

Expanding $ZX_{k_2} - Z_{k_2}X$ and ZZ_{k_2}, it is easy to verify that $ZX_{k_2} - Z_{k_2}X < 0$ and $ZZ_{k_2} > 0$, thus $\partial\lambda_1^2/\partial k_2 < 0$. λ_1 decreases strictly monotonically with k_2. The only positive solution of $\partial\lambda_1^2/\partial k_1 = 0$ is when

$$k_{1,1}(k_2) = \sqrt{2\,(1-d)\,k_2 - \frac{(1-d)^2\,k_2^2 E_2}{2\,(1-d)\,k_2 E_2 + E_1}}\,.$$

Since $\lambda_1(k_1,k_2) \to \infty$ when $k_1^2 \to 4\,(1-d)\,k_2$ or when $k_1 \to 0$, as a continuous function of k_1, $\lambda_1(k_1,k_2)$ is convex in k_1 with minimum achieved at $k_1 = k_{1,1}(k_2)$. Apparently $k_{1,1}^2(k_2) < 2\,(1-d)\,k_2$. Inserting $k_{1,1}(k_2)$ in (A.11), we have

$$\lambda_1(k_{1,1}(k_2),k_2) = \frac{|b|}{k_2} x_b^{\frac{-d}{1-d}} \sqrt{(2\,(1-d)\,k_2 E_2 + E_1)}\,. \tag{A.13}$$

The only crossing or touching point between $\gamma_1(k_1,k_2)$ and $\gamma_2(k_1,k_2)$ for fixed k_2 is when k_1 equals to

$$k_{1,12}^2\,(k_2) = 2\,(1-d)\,k_2 + \frac{\sqrt{(E_1 + 2\,(1-d)\,k_2E_2)^2 + 4\,(1-d)^2\,k_2^2E_2^2} - (E_1 + 2\,(1-d)\,k_2E_2)}{2E_2}\,.$$

Since $\lambda_1(k_1,k_2) \to \infty$ when $k_1 \to 0$ or when $k_1^2 \to 4\,(1-d)\,k_2$, this point is a touching point, with $\lambda_1(k_1,k_2) \geq \lambda_2(k_1,k_2)$. Note that, the three lower bounds $\lambda_1(k_1,k_2)$, $\lambda_2(k_1,k_2)$, $\lambda_3(k_1,k_2)$ touch or cross at the same point. The touching point corresponds to the Hamilton matrix H having an eigenvalue at the origin for LTI system. Further note that $k_{1,12}^2(k_2) > 2(1-d)k_2$.

$\lambda^\star$ in summary

Inspecting (A.12) when $k_1^2 \leq 2(1-d)k_2$, since $\lambda_1(k_1,k_2) \geq \lambda_2(k_1,k_2)$ from the above analysis, $\lambda^\star = \max\{\lambda_1,\lambda_2\} = \lambda_1$ and for fixed k_2

$$\min_{k_1} \lambda^\star = \lambda_1(k_{1,1}(k_2),k_2)\,, \text{ when } k_1^2 \leq 2(1-d)k_2\,,$$

when $k_1^2 \leq 2(1-d)k_2$ from the convexity analysis above. When $k_1^2 \geq 4(1-d)k_2$, we have

$$\begin{aligned}
\lambda_2(k_1,k_2) &= \frac{|b|}{k_2} x_b^{\frac{-d}{1-d}} \sqrt{(k_1^2 E_2 + E_1)} > |b|\, x_b^{\frac{-d}{1-d}} \sqrt{\frac{4(1-d)E_2}{k_2}} \\
\lambda_3(k_1,k_2) &= (1-d)\,|b|\, x_b^{\frac{-d}{1-d}} \sqrt{\frac{E_2}{k_1^2 - 2(1-d)k_2}} \leq |b|\, x_b^{\frac{-d}{1-d}} \sqrt{\frac{(1-d)E_2}{2k_2}} < \lambda_2(k_1,k_2)\,.
\end{aligned}$$

The inequality is strict, thus the touching point $k_{1,12}^2(k_2) = k_{1,23}^2(k_2) < 4(1-d)k_2$. Therefore, $\lambda^\star = \max\{\lambda_2,\lambda_3\} = \lambda_2$ when $k_1^2 \geq 4(1-d)k_2$, since λ_2 is inversely proportional to k_1. The minimum of $\lambda^\star$ when $k_1^2 \geq 4(1-d)k_2$ is achieved at $k_1^2 = 4(1-d)k_2$. As a consequence, for fixed k_2 when $k_1^2 \geq 4(1-d)k_2$

$$\begin{aligned}
\min_{k_1} \lambda^\star = \lambda_2(4(1-d)k_2,k_2) &= \frac{|b|}{k_2} x_b^{\frac{-d}{1-d}} \sqrt{(4(1-d)k_2E_2 + E_1)} \\
&> \frac{|b|}{k_2} x_b^{\frac{-d}{1-d}} \sqrt{(2(1-d)k_2E_2 + E_1)} = \lambda_1(k_{1,1}(k_2),k_2)\,,
\end{aligned}$$

when compared to (A.13). So this section does not provide the minimal $\lambda^\star$ for fixed k_2.

At last, the section of $2(1-d)k_2 < k_1^2 < 4(1-d)k_2$ is inspected, where

$$\begin{aligned}
& \lambda^\star = \max\{\lambda_2, \min\{\lambda_1,\lambda_3\}\} \\
\Leftrightarrow \quad & \lambda^\star = \begin{cases} \max\{\lambda_1,\lambda_2\} & \text{, when } \lambda_1 < \lambda_3 \\ \max\{\lambda_2,\lambda_3\} & \text{, when } \lambda_1 \geq \lambda_3 \end{cases}
\end{aligned}$$

When $\lambda_1 < \lambda_3$, $\lambda^\star = \max\{\lambda_1, \lambda_2\} = \lambda_1$ ($\lambda_1 \geq \lambda_2$ in all cases). Since λ_1 achieves its minimum at $k_{1,1}^2(k_2) < 2(1-d)k_2$, the value of $\lambda^\star$ can not be smaller than (A.13) under this case.

When $\lambda_1 \geq \lambda_3$, recall that λ_2 and λ_3 are monotonic in k_1, k_2 in a contrary way. From the fact that all lower bounds cross or touch at the same point and again $\lambda_1^\star$ is achieved in $k_1^2 < 2(1-d)k_2$, $\lambda^\star$ in this subsection is not optimal.

Finally, the minimum is $\min_{k_1} \lambda^\star = \lambda_1(k_{1,1}(k_2), k_2)$. Figure A.1 shows the plot of $\lambda_1(k_1, k_2), \lambda_2(k_1, k_2), \lambda_3(k_1, k_2)$ as well as $\lambda^\star(k_1, k_2)$ for fixed k_2 and varying k_1 to each channel (i.e. when $E_1 = 1, E_2 = 0$ to channel $y = \lceil x_1 \rfloor^{\frac{1}{1-d}}$, and when $E_1 = 0, E_2 = 1$ to channel $y = x_2$), with parameterss $d = -0.5$, $k_2 = b = 3$, $e_b = 1$. Figure A.2 shows the same four function, but with fixed k_1 and varying k_2 under the parameters of the system are $d = -0.5, k_1 = 4, b = 3, e_b = 1$. It is clear that $\lambda^\star(k_1, k_2)$ has a minimal value w.r.t. k_1 for each fixed k_2, but is monotonously decreasing w.r.t. k_2 for each fixed k_1.

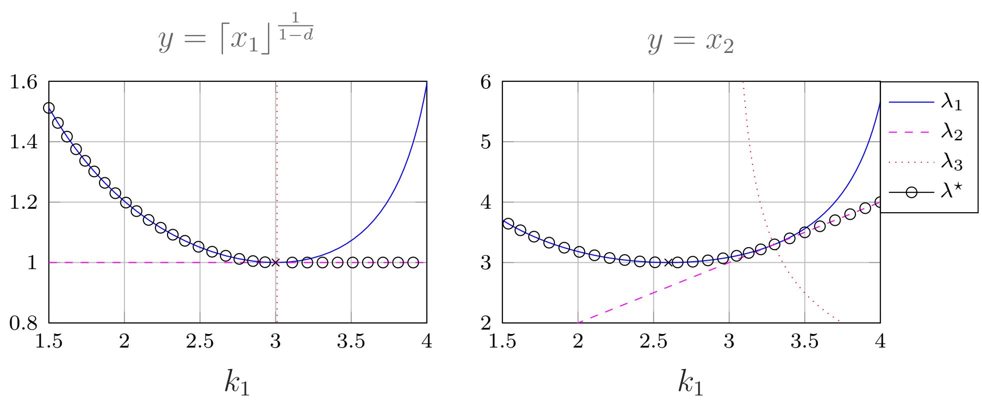

Figure A.1: Calculated λ for fixed k_2.
The parameters of the system are $d = -0.5, k_2 = b = 3, e_b = 1$.

A.7 Algorithms for Search Procedure

The general algorithm for Proposition 4.1 as well as the detailed algorithm for system (6.1) with storage function in (6.13) are listed here.

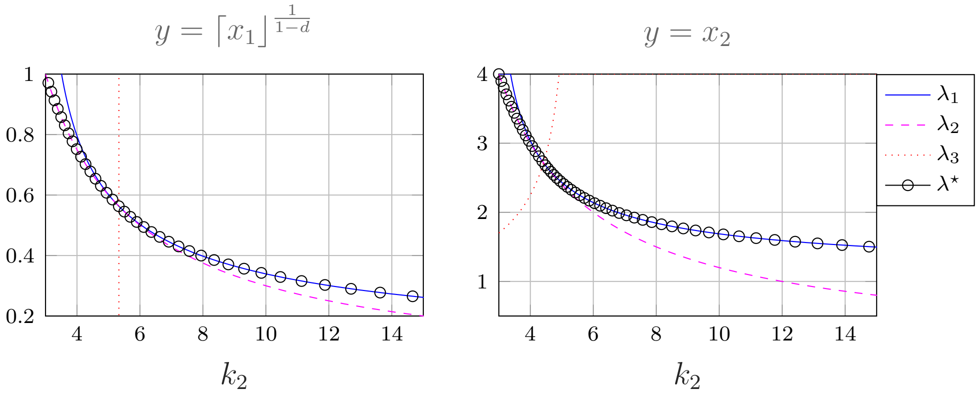

Figure A.2: Calculated λ for fixed k_1.
The parameters of the system are $d = -0.5, k_1 = 4, b = 3, e_b = 1$.

A.7.1 Algorithms for Proposition 4.1

The algorithm of rough search for $\max_{\|(x,u)\|=1} \zeta(V_x, x, u)$ in Section 4.3.1 is listed in Algorithm 4. After the rough search shown in Algorithm 4, several refined search around the point of (x, u) recorded in the last round should be carried out. The maximal value of $\zeta(V_x, x, u)$ on the unit sphere should happen in a neighbourhood of such point of (x, u), since the function $\zeta(V_x, x, u)$ is continuous on the surface of the unit sphere. Note that in practice, several local maxima might occur. In such case refined searches around each local maxima should be carried out.

Therefore, with each fixed V, the rough search for $\max_{\|(x,u)\|=1} \{\zeta(x, u)\} \leq 0$ takes υ^{n+m-1} calculations of $\zeta(x, u)$. The refined search in Algorithm 5 around several local maximal points collected on rough search usually takes less calculations. Thus the computational complexity is of $O\left(\upsilon^{n+m-1}\right)$.

A.7.2 Algorithms for the CSTLA System

The detailed search procedure for the CSTLA system (6.1) with storage function in (6.13), i.e.

$$V_l(x) = \frac{1-d}{p-d} |x_1|^{\frac{p-d}{1-d}} - a_{12} x_1 \lceil x_2 \rfloor^{p-1} + \frac{a_2}{p-d} |x_2|^{p-d} ,$$

Algorithm 4 Procedure of rough search for $\max_{\|(x,u)\|=1} \zeta(V_x, x, u)$ in Proposition 4.1

```
ζ_c = −10^300                    ▷ Record of the maximal ζ(V_x, x, u) in current round
for j_1 = 1, ··· , υ do
    x_1 = −1 + 2j_1/(υ+2)
    for j_2 = 1, ··· , υ do
        x_2 = sqrt(|1 − |x_1|^2|) (−1 + 2j_2/(υ+2))
        ···
        for j_{n+1} = 1, ··· , υ do
            u_1 = sqrt(|1 − ‖x‖_2^2|) (−1 + 2 (j_{n+1}−1)/(υ−1))
            ···
            for j_{n+m−1} = 1, ··· , υ do
                u_{m−1} = sqrt(|1 − ‖x‖_2^2 − Σ_{i=1}^{m−2} |u_i|^2|) (−1 + 2 (j_{n+m−1}−1)/(υ−1))
                u_m = ± sqrt(1 − Σ_{i=1}^{m−1} |u_i|^2 − ‖x‖_2^2)
                Evaluate ζ(V_x, x, u) in (4.9)
                if ζ(V_x, x, u) > ζ_c then
                    ζ_c = ζ(V_x, x, u)
                    (x_r, u_r) = (x, u)          ▷ Record the point for refined search
                end if
            end for
            ···
        end for
        ···
    end for
end for
```

Algorithm 5 Procedure of refined searches after Algorithm 4

```
repeat
    ζ_p = ζ_c            ▷ ζ_p is the record of the maximal ζ(V_x, x, u) in previous round
    for Divide the neighborhood of (x_r, u_r) similarly as Algorithm 4 do
        Evaluate ζ(V_x, x, u)
        if ζ(V_x, x, u) > ζ_c then
            ζ_c = ζ(V_x, x, u)
            (x_r, u_r) = (x, u)          ▷ Record for next round of refined search
        end if
    end for
until (ζ_c − ζ_p)/ζ_p ≤ 10^−7
γ_c = sqrt(ζ_c)
```

is listed in the following. The whole procedure is broken down into several sub-algorithms, i.e. Algorithm 6 and Algorithm 7 are used in the overall Algorithm 8.

First of all, the algorithm to search the region of (a_{12}, a_2) s.t. the storage function V_l (6.13) serves as Lyapunov function when $u = 0$ for system (6.1) with storage function in (6.13) is listed in Algorithm 6 (i.e. checking $\max_{\|x\|_2=1} \dot{V}_l(x, 0) < 0$).
Such region is a convex set, since for any $\alpha \in [0, 1]$ and any two sets of $(\hat{a}_{12}, \hat{a}_2)$, $(\check{a}_{12}, \check{a}_2)$ that belong to the region, the Lyapunov function, from (6.13),

$$V_l\left(\alpha\hat{a}_{12} + (1-\alpha)\check{a}_{12}, \alpha\hat{a}_2 + (1-\alpha)\check{a}_2\right) = \alpha V_l\left(\hat{a}_{12}, \hat{a}_2\right) + (1-\alpha) V_l\left(\check{a}_{12}, \check{a}_2\right)$$

is also a Lyapunov function. Here we abuse the notation $V_l(a_{12}, a_2)$ to emphasize that it represents a different Lyapunov functions with different (a_{12}, a_2), instead of the same Lyapunov function as a function of x. The search is carried out along a_2, and for each a_2, $a_{12} \in \left[0, \left(\frac{a_2}{p-1}\right)^{\frac{p-1}{p-d}}\right]$ or when $p = 2$, $a_{12} \in \left[0, \min\left\{(a_2)^{\frac{1}{2-d}}, k_1/k_2\right\}\right]$. During such search each maximal and minimal value of a_{12}, a_2 of these regions are also recorded.

Thereafter, with the Lyapunov function $V = a_1 V_l$ in (6.13), when $a_1 = 0$, the region of (a_{12}, a_2) s.t. $\max_{\|x\|_2=1} J(V_x, x, 0) \leq 0$ vanishes. This is trivial from

$$J(0, x, 0) = \|y\|^p_{\tau_y, p} \geq 0\,,$$

with $\epsilon = 0$. And when $a_1 \to \infty$, the region of (a_{12}, a_2) s.t. $\max_{\|x\|_2=1} J(V_x, x, 0) \leq 0$ is the same one as the region of (a_{12}, a_2) s.t. V_l is a Lyapunov function. Combining again with that fact that the region of (a_{12}, a_2) is also a convex set, then with all other parameters the same, the region of (a_{12}, a_2) shrinks with a smaller a_1. That allows us to find an a_1 small enough to start with the process.

Finally, the overall process is described in Algorithm 8. With the a_{1s}, the value of γ_c for each $a_1 = 2a_{1s}, 4a_{1s}, 8a_{1s}$ is recorded, when the value of γ_c cease to decrease, we start the process again with a smaller multiplying constant from the point of $a_{1o}/2$, where a_{1o} is the a_1 that achieves the smallest γ_c for all rounds. A repetition of such processes is carried out until the multiplying constant is small enough, a figure of such process is described in upper-left sub-figure of Figure A.3. A clear convexity can be observed in the figure. Other methods for treatment of a_1 can be applied, e.g. additive $a_1 = a_{1s} + i\,, i = 1\,, \cdots$ for first round and $a_1 = a_{1o} + i/2\,, i = 1\,, \cdots$ for second round. Yet, a faster method to find γ_o is always preferred.

Algorithm 6 Procedure of searches of region of (a_{12}, a_2) s.t. the storage function V_l serves as Lyapunov function

Set the step of a_{12} and a_2 as $s_{a_{12}}, s_{a_2}$. ▷ This depends on experience
$flag_m = 0$ ▷ A flag to record whether the region of (a_{12}, a_2) is met with current a_2
$flag_e = 0$ ▷ A flag to record whether the region of (a_{12}, a_2) is existed with current a_2
$j = 1$
while $flag_m + flag_e < 2$ **do**
  $a_2 = js_{a_2}, j = j + 1$
  $record_{a_{12}} = -1$
  $i = 1$
  repeat
    $a_{12} = is_{a_{12}}, i = i + 1$
    Verify whether $\max_{\|x\|_2=1} \dot{V}_l(x, 0) < 0$ for this $V_l(a_{12}, a_2)$
  until $a_{12} > \left(\frac{a_2}{p-1}\right)^{\frac{p-1}{p-d}}$ or $\max_{\|x\|_2=1} \dot{V}_l(x, 0) < 0$
  if $\max_{\|x\|_2=1} \dot{V}_l(x, 0) < 0$ **then**
    $flag_m = 1$
    $record_{a_{12}} = a_{12}$
    if $a_{2l} > a_2$ **then**
      $a_{2l} = a_2$ ▷ Record the minimal of a_2 s.t. V_l serves as Lyapunov function
    end if
    if $a_{2u} < a_2$ **then**
      $a_{2u} = a_2$
    end if
    if $a_{12l} > a_{12}$ **then**
      $a_{12l} = a_{12}$
    end if
    Find the biggest a_{12} s.t. $\max_{\|x\|_2=1} \dot{V}_l(x, 0) < 0$ ▷ Described in Algorithm 3
    if $a_{12u} < a_{12}$ **then**
      $a_{12u} = a_{12}$
    end if
  end if
  if $record_{a_{12}} < 0$ and $flag_m = 1$ **then**
    $flag_e = 1$
  end if
end while

The upper-left subfigure in Figure A.3 shows convergence of

$$\gamma_c{}^2(a_1) = \min_{(a_{12},a_2)} \max_{\|(x,u)\|=1} \zeta\left(V_x(a_1, a_{12}, a_2), x, u\right)$$

formed by the outermost iteration, described in Algorithm 8. The upper-right subfigure

Algorithm 7 Procedure of searches of start value of a_1 (recorded as a_{1s}), s.t. the region of s.t. the region of (a_{12}, a_2) is within a certain percentage of that in Algorithm 6

$d = (a_{2u} - a_{2l}) \min \left\{ 0.7 \frac{(p-1)^{0.1}}{(1+d)^{0.05}} \right\}$ ▷ These a_{2u} and a_{2l} comes from Algorithm 6. And this percentage results from experience
$a_{1u} = 10^5$
$a_{1l} = 0$
while $a_{1u}/a_{1l} \geq 1.001$ **do**
  $a_1 = (a_{1u} + a_{1l})/2$
  Carry out procedure similar to that in Algorithm 6, except that $\max_{\|x\|_2=1} \dot{V}_l(x, 0) < 0$ is replaced by $\max_{\|x\|_2=1} J(V_x, x, 0) \leq 0$. The record of a_{1u}, a_{1l} is also kept
  if $(a_{1u} - a_{1l}) > d$ **then**
    $a_{1u} = a_1$
  else
    $a_{1l} = a_1$
  end if
end while
$a_{1s} = a_{1u}$

Algorithm 8 Overall process for system (6.1) with storage function in (6.13)

Find the Lyapunov region of (a_{12}, a_2) ▷ As described in Algorithm 6
Find a_{1s} small enough ▷ As described in Algorithm 7
$k = 2, a_1 = a_{1s}$
while k > 1.001 **do**
  $\gamma_p = 10^{300}$ ▷ Record of previous round of γ
  $\gamma_c = 10^{300}$ ▷ Record of current round of γ
  $\gamma_o = 10^{300}$ ▷ Record of smallest γ_c of all rounds
  while $\gamma_p > \gamma_c$ **do** ▷ increasing a_1 until the γ_c becomes larger
    $\gamma_p = \gamma_c$
    Rough search with this a_1 ▷ As described in Algorithm 4
    Refined search with this a_1 ▷ As described in Algorithm 5
    if $\gamma_o > \gamma_c$ **then** ▷ Record the smallest value of γ_c and its a_1
      $a_{1o} = a_1$
      $\gamma_o = \gamma_c$
    end if
    $a_1 = k a_1$
  end while
  $a_1 = \max\{a_{1o}/k, a_{1s}\}$ ▷ Roll back from the current minimal value for the next round
  $k = 1 + (k-1)/2$ ▷ Step of $2, 1.5, \cdots, 1.003906, 1.001953$
end while

shows the region of (a_{12}, a_2) (collected similar to that in Algorithm 6) and the rough search as well as the refined search described in Algorithm 4 and 5. The optimal set of (a_{12}, a_2) for γ_o^2 is marked in red cross. The lower subfigure shows

$$\gamma_c{}^2(a_1, a_{12}, a_2) = \max_{\|(x,u)\|=1} \zeta(V_x(a_1, a_{12}, a_2), x, u)\,.$$

To highlight the convexity of $\max_{\|(x,u)\|=1} \zeta(V_x(a_1, a_{12}, a_2), x, u)$ wrt. (a_{12}, a_2), in this figure we set a maximum at 500. Since the values of $\max_{\|(x,u)\|=1} \zeta(V_x(a_1, a_{12}, a_2), x, u)$ with non-optimal (a_{12}, a_2) are too big, the clear convexity is clouded.

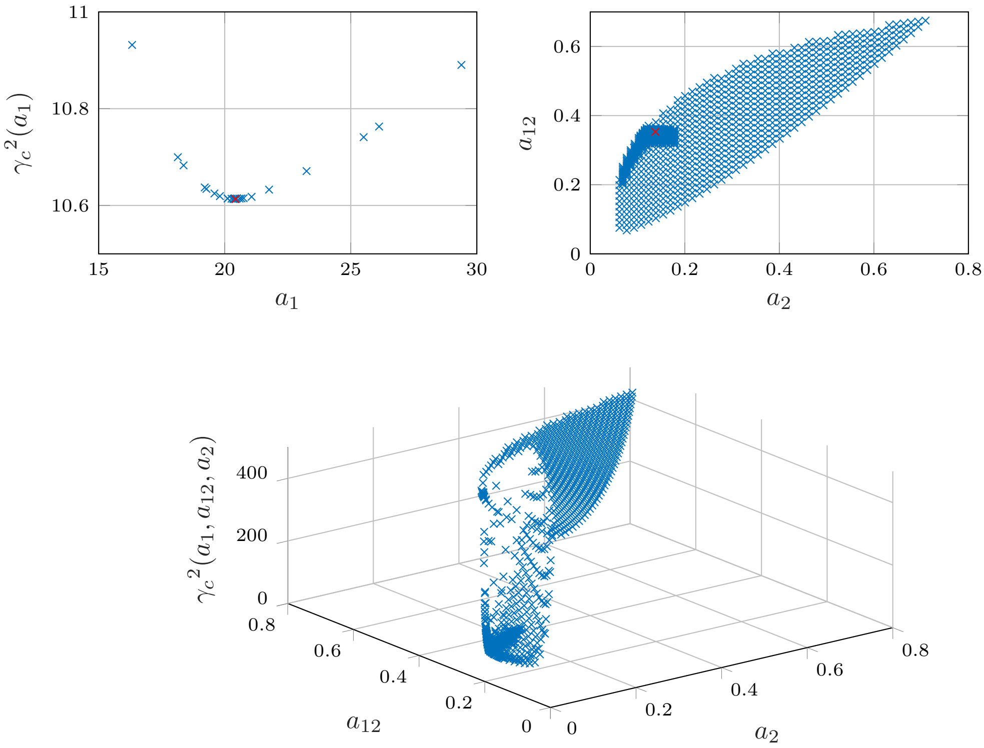

Figure A.3: Search procedure.
The parameters of the system (6.1) are $d = -0.50$ (linear case) and $k_2 = b = 4, k_1 = \left(2\underline{k}_1 + \overline{k}_1\right)/3 = 3.1547$.

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Declaration of primary authorship

I hereby declare and confirm that this thesis

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has been written only by me and without any assistance from third parties. Furthermore, I confirm that no sources have been used in the preparation of this thesis other than those indicated in this thesis itself.

March, 2022

Daipeng Zhang

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