IMK-14 – Istraživanje i razvoj u teškoj mašinogradnji 26(2020)1, SR 19-23
UDC 621 ISSN 0354-6829
doi: 10.5937/IMK2001019Z

Primena metode konačnih razlika kod određivanja ugiba nosača kontinualno promenljivog poprečnog preseka

Nebojša Zdravković[1]*, Mile Savković[1], Goran Marković[1], Goran Pavlović[2]
[1]Fakultet za mašinstvo i građevinarstvo u Kraljevu Univerziteta u Kragujevcu, Srbija
[2]Istraživačko-razvojni centar Alfatec d.o.o., Niš, Srbija

Jedan od kriterijuma za dimenzionisanje čeličnih konstrukcija, neretko i oštriji od kriterijuma napona, jeste njihova deformacija. Prekomerno deformisanje, iako je naponsko stanje u dozvoljenim granicama, može dovesti do pojave dopunskih opterećenja u nosećoj strukturi, i nepravilnog funkcionisanja i kraćeg radnog veka uređaja i opreme. Zbog toga se, pored ostalih, propisuju i uslovi u pogledu dopuštenih deformacija koje noseća čelična konstrukcija mora da ispuni. Naponsko-deformaciona analiza noseće strukture se sprovodi u ranoj fazi projektovanja, najčešće metodom konačnih elemenata (MKE). U radu je prikazan postupak primene metode konačnih razlika (MKR) pri određivanju ugiba na jednostavnom primeru elastično oslonjenog konzolnog nosača, kutijastog poprečnog preseka, sa linearnom promenom njegove visine. Zavisnost ugiba od ulaznih parametara je dobijena korišćenjem programa MATLAB, pri čemu su dati i uporedni rezultati iz konačno-elementnog modela, izrađenog u programu ANSYS. Rezultati iz obe numeričke metode su pokazali veoma visoku podudarnost.

Ključne reči: noseće strukture, promenljiv poprečni presek, ugib, metoda konačnih razlika, MATLAB

1. UVOD

Projektovanje čeličnih nosećih konstrukcija podrazumeva ispunjavanje niza tehničkih zahteva, što se dokazuje sprovođenjem proračunskih procedura kao što su dokaz napona, deformacija, elastične i dinamičke stabilnosti, čvrstoće veza. S obzirom da je uobičajeno da delovi ili cela konstrukcija imaju velike dužine, često se dešava da je kriterijum maksimalno dozvoljenih deformacija merodavan za dimenzionisanje strukture. Ovaj uslov se uglavnom iskazuje kroz maksimalne vrednosti pomeranja karakterističnih tačaka strukture pri delovanju maksimalnog radnog opterećenja.

Nedovoljna krutost noseće strukture, tokom rada, može dovesti do pojave prekomernog deformisanja i pojave dopunskih statičkih i dinamičkih opterećenja. Ovo dalje vodi ka smanjenju operativne pouzdanosti i/ili skraćivanju veka trajanja podsklopova ili cele mašine. Dakle, ograničavanje deformisanja noseće konstrukcije garantuje dobre eksploatacione karakteristike i dug vek mašinske opreme.

Na primer, za određene tipove dizalica, propisane su maksimalne vrednosti ugiba karakterističnih tačaka na nosećoj konstrukciji. Dopušten ugib na sredini nosača mosne dizalice raspona L iznosi $\frac{L}{600}$ [1]. Kod ramnih dizalica, propisane su vrednosti dopuštenog ugiba na sredini raspona i na kraju prepusta: dopusten ugib na sredini raspona je $(1/1000 \div 1/600)L$, dok je na kraju prepusta $(1/400 \div 1/200)L_1$, gde je L – raspon nogu a L_1 dužina prepusta [2].

Izračunavanje ugiba nosećih struktura je relativno jednostavno ukoliko je u pitanju neizmenljiva geometrije i nepromenljivi poprečni preseci nosača. Međutim, to nije uvek slučaj. Tipičan primer strukture sa izmenljivom geometrijom je zglobna strela kod auto-dizalica, kod koje se geometrija strukture menja u zavisnosti od radnog položaja [3]. Ipak, ako su poprečni preseci nosača nepromenljivi, moguće je, uz određene aproksimacije, dobiti analitički model za izračunavanje pomeranja vrha strele [4]. Dobijanje analitičkih modela za izračunavanje pomeranja karakterističnih tačaka strukture omogućava da se ograničenje ugiba koristi kod optimizacije nosećih struktura u cilju redukovanja mase [5-9].

Provera ugiba složenijih nosećih struktura se, u najvećem broju slučajeva, sprovodi primenom MKE [10-12], i to za jedno konstrukciono rešenje strukture koje je konačno ili blisko konačnom. Ovo odgovara završnoj fazi projektovanja, kada je veoma teško ili nemoguće implementirati bilo kakve promene u obliku i dimenzijama elemenata noseće konstrukcije, u slučaju da su pomeranja karakterističnih tačaka pod uticajem radnog opterećenja veća od dozvoljenih. Stoga, potreban je pristup koji se može primeniti u ranoj fazi projektovanja strukture, koji bi omogućio generalizovan pregled mogućih konstrukcionih rešenja koja zadovoljavaju kriterijum dopuštenih deformacija.

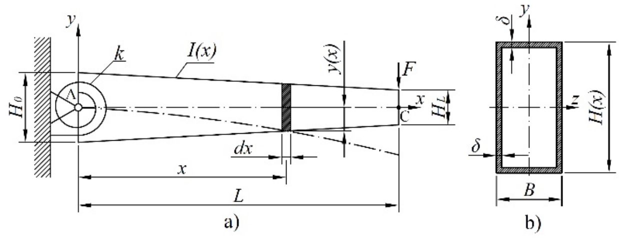
Pored MKE, autori su koristili i druge numeričke pristupe pri rešavanju nelinearnih problema kod određivanja ugiba. Na primer, ugao rotiranja poprečnih preseka grede je predstavljen u obliku stepenog reda i u tom obliku zamenjen u nelinearnoj diferencijalnoj jednačini savijanja, pri različitim opterećenjima i graničnim uslovima [13]. Drugi autori su koristili MKR za formiranje numeričke šeme za rešavanje problema deformisanja greda pri savijanju [14-16].

Rad je fokusiran na prikaz postupka primene MKR pri određivanju ugiba nosećih struktura na jednostavnom primeru konzolnog nosača sa kontinualno promenljivim kutijastim poprečnim presekom. Pored promenljivog poprečnog preseka sa linearno promenljivom visinom, u razmatranje je uzeta i elastičnost oslonca preko njegove zadate krutosti. Postupak je primenljiv i na složenije više-segmentne strukture sa kompleksnijom promenama poprečnih preseka.

*Kontakt adresa autora: Fakultet za mašinstvo i građevinarstvo u Kraljevu Univerziteta u Kragujevcu, Dositejeva 19, 36000 Kraljevo, zdravkovic.n@mfkv.kg.ac.rs

2. MATEMATIČKI MODEL

Model elastično oslonjene čelične konzole, dužine L, promenljivog poprečnog preseka, je dat na slici 1. Nosač se savija u ravni xy pod uticajem koncentrisane sile F koja deluje u krajnjoj tački C, pri čemu se formira elastična linija oblika $y(x)$. Radi jednostavnosti prikaza postupka, ne umanjujući opštost pristupa, zanemaren je uticaj sopstvene težine nosača. Elastično oslanjanje nosača u tački A je predstavljeno oprugom krutosti k. Takođe, u cilju dobijanja preglednijeg numeričkog modela, usvojeno je da visina kutijastog poprečnog preseka $H(x)$ linearno opada sa porastom podužne koordinate x, što je, inače, i najčešći oblik promene poprečnih preseka nosača u praksi. Ipak, treba napomenuti da je ovaj pristup primenljiv na bilo koji drugi vid promene poprečnog preseka. Visine poprečnog preseka na početku ($x=0$) i kraju nosača ($x=L$) iznose respektivno H_0 i H_L. Debljina zidova nosača δ je konstantna po poprečnom preseku i u podužnom pravcu. Širina poprečnog preseka B je konstantna duž koordinate x.



Slika 1: a) model elastično ukliještene konzole, b) poprečni presek.

Diferencijalna jednačina savijanja konzolnog nosača je dobro poznata i glasi:

$$\frac{d^2y(x)}{dx^2} = -\frac{M(x)}{EI(x)} \tag{1}$$

gde je E modul elastičnosti materijala nosača.

Ako se definiše bezdimenzioni dekrement visine kutijastog poprečnog preseka

$$\eta = \frac{H_0 - H_L}{L} = H_L \frac{\psi - 1}{L} \tag{2}$$

gde je $\psi = H_0 / H_L$ odnos visina početnog i krajnjeg preseka, promena visine preseka glasi

$$H(x) = H_0 - \eta x \tag{4}$$

dok moment inercije za osu z, uz zanemarivanje malih veličina višeg reda, može biti napisan u obliku

$$I(x) \approx \frac{\delta}{6}\left[(H_0 - \eta x - 2\delta)^3 + 3B(H_0 - \eta x - \delta)^2\right] \tag{5}$$

Funkcija momenta savijanja, bez uticaja sopstvene težine, je:

$$M(x) = F(L - x) \tag{6}$$

Dakle, polazna jednačina postaje:

$$\frac{d^2y(x)}{dx^2} = \frac{6F(x-L)}{E\delta\left[(H_0 - \eta x - 2\delta)^3 + 3B(H_0 - \eta x - \delta)^2\right]} \tag{7}$$

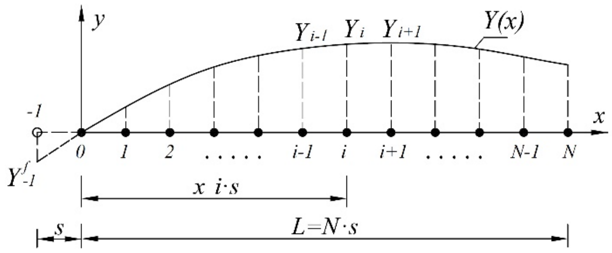
Granični uslovi definišu vrednost ugiba i nagiba u osloncu A, pri čemu nagib zavisi od krutosti elastičnog oslonca k:

$$y(0) = 0 \tag{8}$$

$$\frac{dy}{dx}(x=0) = \frac{M(x=0)}{k} = \frac{EI(x)}{k}\left.\frac{d^2y}{dx^2}\right|_0 \tag{9}$$

3. DISKRETIZACIJA PRIMENOM MKR

Na slici 2 prikazana je mreža tačaka kojom je izdeljen nosač kroz MKR. Na dužini L se nalazi niz od $N=L/s$ tačaka, gde je s usvojeni korak diskretizacije. Transformacijom graničnih uslova su obuhvaćene i tačka oslonca A sa numeracijom 0 i fiktivna tačka na neutralnoj osi levo od oslonca sa numeracijom -1, kojoj odgovara fiktivno pomeranje Y^f_{-1}.



Slika 2: Mreža tačaka iz MKR

Prema MKR, aproksimacioni izrazi za prvi i drugi izvod funkcije elastične linije nosača u tački i mreže su:

$$\left(\frac{dy}{dx}\right)_i \approx \frac{-Y_{i-1} + Y_{i+1}}{2s} \tag{10}$$

$$\left(\frac{d^2y}{dx^2}\right)_i \approx \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{s^2} \tag{11}$$

Ako se stavi za podužnu koordinatu da je $x=is$ i upotrebi aproksimacija (11), dobija se diskretizovan oblik jednačine savijene konzole napisane za arbitrarnu tačku i, koji važi za ceo domen:

$$Y_{i-1} - 2Y_i + Y_{i+1} = = \frac{6s^2F(is - L)}{E\delta\left[(H_0 - \eta is - 2\delta)^3 + 3B(H_0 - \eta is - \delta)^2\right]} \tag{12}$$

Transformacijom graničnih uslova u osloncu A dobijaju se njihovi diskretizovani oblici:

$$Y_0 = 0 \tag{13}$$

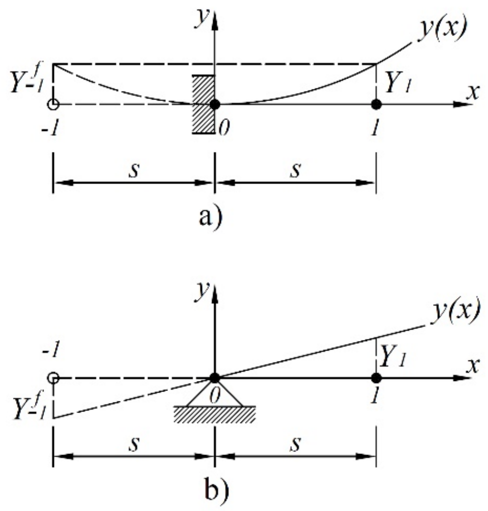
$$Y^f_{-1} = -\frac{2EI_0 - ks}{2EI_0 + ks} Y_1 \tag{14}$$

gde je

$$I_0 = I(x=0) \approx \frac{\delta}{6}\left[(H_0 - 2\delta)^3 + 3B(H_0 - \delta)^2\right] \tag{15}$$

moment inercije poprečnog preseka u osloncu A.

Grafička interpretacija diskretizovanih graničnih uslova (13) i (14) pri ekstremnim vrednostima krutosti oslonačke opruge k je prikazana na slici 3. Ukoliko krutost teži beskonačnoj vrednosti ($k \to \infty$), dobija se da je $Y^f_{-1} = Y_1$, što znači da funkcija $y=y(x)$ ima lokalni ekstrem u tački mreže $i=0$, a to odgovara ukliještenju (slika 3a). Ako imamo da krutost oslonačke opruge teži nuli ($k \to 0$), onda sledi da je $Y^f_{-1} = -Y_1$, što odgovara zglobnom načinu oslanjanja (slika 3b).



Slika 3: Grafička interpretacija diskretizovanih graničnih uslova a) uklještenje pri $k \to \infty$, b) zglob pri $k \to 0$.

Sada se može napisati sistem od N algebarskih jednačina za diskretne tačke $i=0,1,\ldots, N-1$, u kojima figurišu nepoznata pomeranja Y_j, $j=1,2,\ldots,N$. Pri tome, jednačine elastične linije za tačke 0 i 1 obuhvataju pomeranja Y_0 i Y^f_{-1}, koja se eliminišu korišćenjem diskretizovanih graničnih uslova (13) i (14). To ih izdvaja iz niza rekurentnih jednačina, koje se dobijaju za interval tačaka mreže $i=2\div N-1$. Ovakva struktura sistema linearnih algebarskih jednačina, koja je vrlo pogodna za programiranje, ima sledeći oblik:

$$i=0:$$

$$\frac{2ks}{2EI_0+ks}Y_1=-\frac{s^2FL}{EI_0} \tag{16}$$

$$i=1:$$

$$-2Y_1+Y_2=\frac{6s^2F(s-L)}{E\delta\left[(H_0-\eta s-2\delta)^3+3B(H_0-\eta s-\delta)^2\right]} \tag{17}$$

$$i=2\div N-1:$$

$$Y_{i-1}-2Y_i+Y_{i+1}= \tag{18}$$

$$=\frac{6s^2F(is-L)}{E\delta\left[(H_0-\eta is-2\delta)^3+3B(H_0-\eta is-\delta)^2\right]}$$

4. NUMERIČKI PRIMER I KOMPARACIJA REZULTATA SA MKE

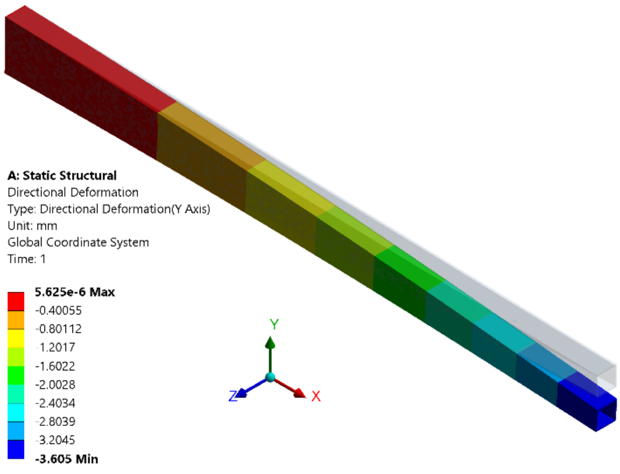
U cilju provere tačnosti primenjenog postupka za izračunavanje ugiba pomoću MKR, sproveden je proračun na primeru konzole sa sledećim numeričkim vrednostima parametara: $L=3000[mm]$, $F=-1000[N]$, $H_L=100[mm]$, $B=100[mm]$, $\delta=5[mm]$, $E=2.1\cdot10^5[N/mm^2]$, pri čemu je variran odnos visina početnog i krajnjeg poprečnog preseka u intervalu $\Psi=1.5\div2.5$ i krutost oslonca u intervalu $k=10^9\div10^{10}[Nmm/rad]$. Za korak diskretizacije je usvojeno $s=1[mm]$, tako da broj jednačina i nepoznatih pomeranja iznosi $N=3000$.

Za rešavanje sistema jednačina i generisanje dijagrama zavisnosti ugiba krajnje tačke konzole (koji odgovara pomeranju Y_N) od varijabilnih parametara Ψ i k korišćen je program MATLAB, dok je za analizu MKE korišćen program ANSYS. Tabela 1 daje komparaciju rezultata iz modela MKE i predstavljenog proračunskog modela na bazi MKR, pri variranju parametara k i Ψ. Može se videti da su razlike dobijenih vrednosti iz ova dva numerička pristupa manje od 1%, što govori o tačnosti navedene metode.

Tabela 1: Uporedni rezultati modela iz MKE i MKR

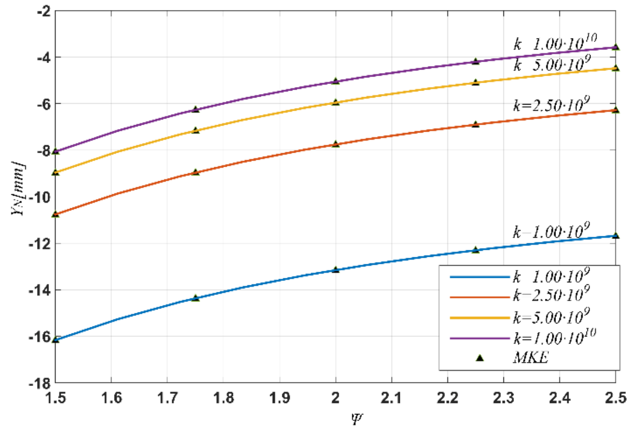
k *[Nmm/rad]*	*Ψ*	*Y_N [mm]*		*Δ*
		MKE	MKR	
10^9	1.50	-16.185	-16.164	0.13%
	1.75	-14.385	-14.361	0.16%
	2.00	-13.176	-13.152	0.18%
	2.25	-12.325	-12.301	0.19%
	2.50	-11.705	-11.682	0.20%
$2.50\cdot10^9$	1.50	-10.785	-10.764	0.20%
	1.75	-8.984	-8.961	0.25%
	2.00	-7.775	-7.752	0.30%
	2.25	-6.925	-6.901	0.34%
	2.50	-6.305	-6.282	0.37%
$5.00\cdot10^9$	1.50	-8.985	-8.964	0.24%
	1.75	-7.184	-7.161	0.32%
	2.00	-5.975	-5.952	0.39%
	2.25	-5.125	-5.101	0.46%
	2.50	-4.505	-4.482	0.52%
10^{10}	1.50	-8.085	-8.064	0.26%
	1.75	-6.284	-6.261	0.36%
	2.00	-5.075	-5.052	0.46%
	2.25	-4.225	-4.201	0.56%
	2.50	**-3.605**	-3.582	0.65%

Na slici 4 prikazan je konačno-elementni model, izgrađen u programu ANSYS, kojem odgovara rezultat poslednjeg testa iz Tabele 1 (podebljana vrednost ugiba vrha nosača).



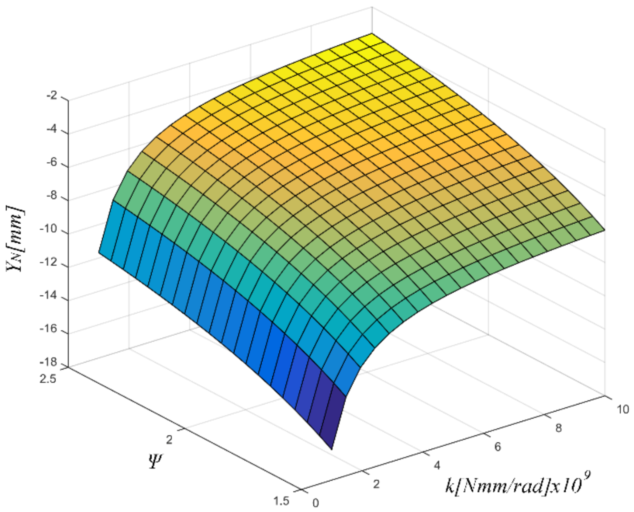
Slika 4: ANSYS MKE model

Grafička interpretacija komparacije rezultata iz MKR i MKE data je na slici 5. Linije predstavljaju dobijenu zavisnost ugiba vrha nosača (pomeranje Y_N iz sistema algebarskih jednačina) od odnosa visina početog i krajnjeg poprečnog preseka, pri različitim vrednostima krutosti oslonca k. Diskretne vrednosti predstavljaju rezultate iz modela MKE.



Slika 5: Grafička interpretacija tabele 1

Slika 6 prikazuje zavisnost ugiba pri kontinualnoj promeni parametara Ψ i k unutar zadatih intervala.



Slika 6: Zavisnost ugiba od parametara Ψ i k unutar zadatih intervala

5. ZAKLJUČAK

Predstavljen postupak primene MKR za određivanje ugiba nosača sa kontinualno promenljivim poprečnim presekom i varijabilnom krutošću oslonca je dao rezultate koji su, praktično, identični sa rezultatima iz MKE. Odstupanje između rezultata obe metode je u okviru 1%. Diferencijalne formulacije jednačine savijanja grede i graničnih uslova su, preko aproksimacija iz metode centralnih konačnih razlika, prevedene na domen linearnih algebarskih jednačina. Iz diskretizovanih graničnih uslova je eliminisano pomeranje fiktivnog čvora, koje se javlja pri pisanju diskretizovane jednačine savijanja za oslonu tačku.

Granični uslovi utiču na oblik samo prve dve jednačine sistema, dok sve ostale jednačine imaju rekurzivnu formu koja je idealna za programiranje. Ova osobina numeričkog modela omogućava efikasno izračunavanje pomeranja diskretnih tačaka mreže i dobijanje zavisnosti ugiba bilo koje tačke od bilo kojeg konstrukcionog parametra. Ovaj pristup se može primeniti i za bilo koji drugi vid promene poprečnog preseka i način oslanjanja nosača.

ZAHVALNICA

Rad je podržan od strane Ministarstva prosvete, nauke i tehnološkog razvoja Republike Srbije kroz Ugovor o realizaciji i finansiranju naučno-istraživačkog rada u 2020. godini, čiji je evidencioni broj 451-03-68/2020-14/200108.

LITERATURA

[1] Eurocode 3: Design of steel structures - Part 6: Crane supporting structures, EN 1993-6.

[2] Petković Z., Ostrić D.: Metal structures in mechanical engineering I (in Serbian), ISBN 86-70803-274-7, Faculty of Mechanical Engineering Belgrade, 1995.

[3] Bošnjak, S., Zrnić, N., Dragović, B. (2009): Dynamic Response of Mobile Elevating Work Platform under Wind Excitation, Strojniški vestnik - Journal of Mechanical Engineering, ISSN 0039-2480, Vol. 55, No. 2, pp 104-113, 2009.

[4] Zdravković N., Petković Z., Gašić M., Savković M.: Research of deflection-payload dependence of the auto crane articulated boom , International Conference on material Handling, Construction and Logistics MHCL'12, Belgrade, Serbia, ISBN 978-86-7083-763-8, pp. 101-106, 2012.

[5] Farkas Jozsef: *Optimum Design of Metal Structures*, Akademiai Kiado, Budapest, 1984.

[6] Gasic M., Savkovic M., Bulatovic R., Petrovic R.: Optimization of a pentagonal cross section of the truck crane boom using Lagrange's multipliers and differential evolution algorithm, Meccanica, Vol. 46, pp. 845–853. doi:10.1007/s1 1012-010-9343-7, 2011.

[7] Gasic M, Savkovic M, Bulatovic R, Petrovic R.: Optimization of trapezoidal cross section of the truck crane boom by Lagrange's multipliers and by differential evolution algorithm (de). Strojniški vestnik – Journal of Mechanical Engineering, 57(2011)4, pp. 304-312, doi: 10.5545/sv-jme.2008.029, 2011.

[8] Pinca BC, Tirian OG, Socalici VA, Ardeleadn DE: Dimensional optimization for the strength structure of a traveling crane, WSEAS Transactions on Applied and heoretical Mechanics 4 (4), ISSN: 1991-8747, pp 147-156, 2009.

[9] Selmic R, Cvetkovic R, Mijailovic R: Optimization of cross-section in structures, monograph, ISBN 86-7395-213-1, Faculty of Transport and Traffic Engineering, Belgrade, 2006.

[10] Sreedhara Rao Gunakala et al: A Finite Element Method for the Beam equation Using MATLAB, International Journal of Applied Science and Technology, ISSN 2221-0997, Vol. 2(8), pp. 80-88, 2012.

[11] Norbaya Omar, Fatimah De'nan: Finite element analysis of deflection and stress in triangular web profiled steel section (triwp) cantilever beam, DIGES PMU, ISSN 2289-6376, Vol. 3, pp. 1-6, 2016.

[12] Sanjay Kumar: Comparison of Deflection and slope of Cantilever Beam with Analytical and Finite Element Method for Different Loading Conditions, International Journal of Engineering Science and Innovative Technology (IJESIT), Vol. 5, Issue 6, pp. 45-51, 2016.

[13] Samir Z. Al-Sadder, Mohammad H. Dado: An Accurate Numerical Scheme for Large Deflection Analysis of Non-Prismatic Inextensible Slender Beams Subjected to General Loading and Boundary Conditions, Advances in Structural Engineering, Vol. 8(6), pp. 585–594. https://doi.org/10.1260/136943305776318392, 2005.

[14] Adel A. Al-Azzawi, Dhiaa M. Theeban: Large Deflection of Deep Beams on Elastic Foundations, Journal of the Serbian Society for Computational Mechanics, ISSN 1820-6530, Vol. 4, No. 1, pp. 88-101, 2010.

[15] K.S. Thankane and T. Stys: Finite Difference Method for Beam Equation with Free Ends Using Mathematica, Southern African Journal of Pure and Applied Mathematics, ISSN 1993-8594, Vol. 4, pp. 61-78, 2009.

[16] AL-Sadder, S., AL-Rawi, R. Finite difference scheme for large-deflection analysis of non-prismatic cantilever beams subjected to different types of continuous and discontinuous loadings, Arch Appl Mech, Vol. 75, pp. 459–473, https://doi.org/10.1007/s00419-005-0422-5, 2006.

IMK-14 – Research & Development in Heavy Machinery 26(2020)1, EN 19-23
UDC 621 ISSN 0354-6829
doi: 10.5937/IMK2001019Z

The determination of the deflection of the beam with continuously varying cross-section by the finite difference method

Nebojša Zdravković[1*], Mile Savković[1], Goran Marković[1], Goran Pavlović[2]
[1]Faculty of Mechanical and Civil Engineering in Kraljevo, University of Kragujevac, Serbia
[2] Alfatec R&D Centre LTD, Niš, Serbia

Amongst others, the deformation criterion is to be fulfilled in the process of steel structure design. Often, it can be more restrictive than the stress criterion. Although the stress state is within the permissible limit, excessive structure deformation can lead the additional load to emerge in the carrying structure and cause improper operation and shorter life of attached devices and equipment. For that reason, besides other requirements, the allowable deformation limits of the steel carrying structure are defined. The stress-strain analysis of the structure is usually conducted by utilization of finite element method (FEM) in the early phase of the design process. Alternatively, this paper shows the application procedure of the finite difference method (FDM) for the deflection determination on the simple elastically restrained cantilever beam with continuously varying box-like cross-section and linear change of section height. Dependence of the deflection on the design parameters is obtained through programming in MATLAB and the comparison is made with the results from the finite element model built in ANSYS. The results from both numerical approaches showed excellent compliance.

Keywords: steel structure, non-uniform beam, deflection, finite difference method, MATLAB

1. INTRODUCTION

The design of steel carrying structures results in the fulfilment of many technical requirements, which are to be proven through calculation procedures for the stress, deformation, elastic and dynamic stability, connections safety, etc. Since the whole structure and/or its segments usually overcome great distances, it is very often the case that the maximum deformation criteria is the key one for its design. This condition is commonly defined through permissible deformation in characteristic points of the structure, with maximum operational load applied.

During operation, insufficient stiffness of the carrying structure can lead to an excessive deformation and cause the occurrence of the additional static and dynamic loads. Further on, this eventually jeopardizes the operational safety and decreases the operational life of the subassemblies or the whole machine. So, the limitation of deformability provides good exploitation features and long life of the equipment.

For example, for each type of cranes, there are prescribed maximum deflection values for the characteristic points on its carrying structure. Permissible static deflection at mid-span point of the bridge crane is $\frac{L}{600}$ [1], where L denotes the span. For the case of the gantry cranes, there is the prescription for the static deflection values for the mid-span point and the endpoint of the overhang: it is $(1/1000\div1/600)L$ for the mid-span, while it is $(1/400\div1/200)L_1$ for the overhang endpoint, where L is the distance between the gantry legs and L_1 denotes the overhang length [2].

It is a relatively easy task to calculate the deflection of the supporting structure if it has an unchangeable geometry and uniform cross-sections of the beams. However, it is a rather rare situation. The articulated booms within truck-cranes are the typical example of the structure with variable geometry configuration, which changes with the working position [3]. Still, if the cross-sections of the segments are uniform, it is possible, with some simplifications, to build an analytical model for the determination of boom tip displacements [4]. Obtaining the analytical model for the calculation of the characteristic points' displacements enables the utilization of deformation conditions in the optimization of the supporting structure to reduce its mass [5-9].

The check of the deflection in more complex structures, in most of the cases, is conducted on FEM models [10-12]. It is usually done for the final design version or the one close to the final. This takes place in the late design phase when it is far more difficult or even impossible to implement any design changes in the structure model, in case it does not meet the regulations about displacements of the characteristic points. Hence, there is a need for an approach that would be applicable in the early design phase to enable some more general perspective on possible design solutions that fulfill the criteria of permissible deformations.

Besides FEM, the authors used other numerical methods to solve nonlinear problems within deflection determination. For example, the angle of rotation is represented by a power series and substituted into the derived governing nonlinear differential bending equation for the prismatic and non-prismatic inextensible beams [13]. Others used the FDM to form the numerical scheme for solving the problem of deformation of the beams in bending [14-16].

The paper is focused on the FDM application procedure for the determination of the supporting structure deflection on the simple example of cantilever non-uniform beam, with continuously varying box-like cross-section. Along with the linearly variable cross-section height, the model also considers the support elasticity. The procedure is also applicable to the multi-segment structure with more complex changes in the cross-sections.

* Corresponding author: Faculty of Mechanical and Civil Engineering in Kraljevo, Dositejeva 19, 36000 Kraljevo, zdravkovic.n@mfkv.kg.ac.rs

2. THE MATHEMATICAL MODEL

The model of an elastically supported cantilever beam with length L, with a continuously variable box-like cross-section, is depicted in Fig. 1. The beam is bending in the *xy* plane under the force F which acts at the endpoint C, forming the elastic curve *y(x)*. To reduce the extent of work, without limiting the generality of the approach, the influence of the beam self-weight is neglected. Elastic restraint of the beam in support A is represented by the flexural spring with stiffness *k*. Also, to make the numerical model more clear and simple, it is adopted that the box-like cross-section height *H(x)* linearly changes along the longitudinal coordinate *x*, which is the most common case in praxis. Nevertheless, it should be stressed that this approach is applicable for any type of the cross-section shape and change. The cross-section height at start point A ($x=0$) is denoted as H_0, while endpoint ($x=L$) cross-section height is denoted as H_L. The wall thickness δ is uniform both over the box-like cross-section and along longitudinal direction. The cross-section width *B* is constant along longitudinal coordinate *x*.

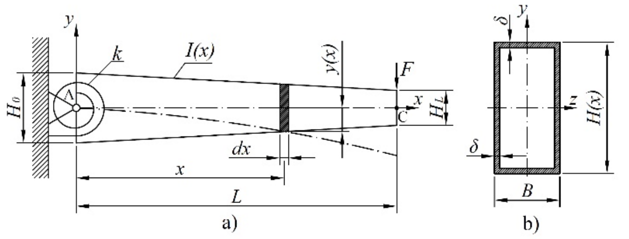


Figure 1: a) Tapered cantilever beam model with elastic support, b) Box-like cross-section with variable height.

The differential equation for the beam in bending is well known:

$$\frac{d^2y(x)}{dx^2} = -\frac{M(x)}{EI(x)} \tag{1}$$

where *E* is the modulus of material elasticity.

The non-dimensional decrement of the cross-section height can be defined as

$$\eta = \frac{H_0 - H_L}{L} = H_L \frac{\psi - 1}{L} \tag{2}$$

where $\psi = H_0 / H_L$ - the ratio between starting and ending cross-section height. Then, the height change can be written as follows

$$H(x) = H_0 - \eta x \tag{4}$$

The moment of inertia for *z* axis, while ignoring small quantities of higher order, can be noted as

$$I(x) \approx \frac{\delta}{6}\left[(H_0 - \eta x - 2\delta)^3 + 3B(H_0 - \eta x - \delta)^2\right] \tag{5}$$

The bending moment, without the self-weight influence, is written as

$$M(x) = F(L - x) \tag{6}$$

Hence, the governing equation becomes

$$\frac{d^2y(x)}{dx^2} = \frac{6F(x-L)}{E\delta\left[(H_0 - \eta x - 2\delta)^3 + 3B(H_0 - \eta x - \delta)^2\right]} \tag{7}$$

The boundary conditions define the values of displacement and the slope in support A, where the slope depends on the spring stiffness *k*:

$$y(0) = 0 \tag{8}$$

$$\frac{dy}{dx}(x=0) = \frac{M(x=0)}{k} = \frac{EI(x)}{k}\left.\frac{d^2y}{dx^2}\right|_0 \tag{9}$$

3. THE FDM DISCRETIZATION

Fig. 2 shows central finite difference grid scheme where the length of the beam *L* is equally divided by *N=L/s* grid points (nodes) into *N* segments with length $s = L / N$. Later transformation of the boundary conditions includes the support point A denoted as node *0* and one fictitious grid point on the left side of the support, denoted as node *-1*. Displacement of this fictitious node is denoted as Y^f_{-1}.

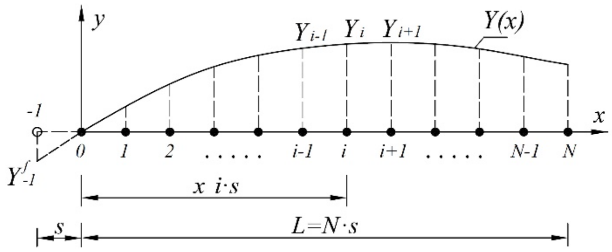


Figure 2: FDM grid scheme

According to the central FDM, approximations for derivatives of shape function *y(x)* in grid point *i* are as follows:

$$\left(\frac{dy}{dx}\right)_i \approx \frac{-Y_{i-1} + Y_{i+1}}{2s} \tag{10}$$

$$\left(\frac{d^2y}{dx^2}\right)_i \approx \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{s^2} \tag{11}$$

By putting for axial coordinate *x=is*, with insertion of approximation (10), the governing differential bending equation gets its algebraic form, written for arbitrary node *i*, which covers the whole length domain of the beam:

$$Y_{i-1} - 2Y_i + Y_{i+1} = \\ = \frac{6s^2F(is - L)}{E\delta\left[(H_0 - \eta is - 2\delta)^3 + 3B(H_0 - \eta is - \delta)^2\right]} \tag{12}$$

Transformed boundary conditions for the support point A are as follows:

$$Y_0 = 0 \tag{13}$$

$$Y^f_{-1} = -\frac{2EI_0 - ks}{2EI_0 + ks} Y_1 \tag{14}$$

where

$$I_0 = I(x=0) \approx \frac{\delta}{6}\left[(H_0 - 2\delta)^3 + 3B(H_0 - \delta)^2\right] \tag{15}$$

represents the moment of inertia of the starting cross-section.

Graphical illustration of transformed boundary conditions (13) and (14), for two extreme values of spring stiffness, is shown in Fig. 3. If stiffness value tends to infinity ($k \to \infty$), then it is $Y^f_{-1} = Y_1$, which means that function $y=y(x)$ has the local extremum in the grid node $i=0$ (this corresponds to fixed support – Fig. 3a). On the other hand, if stiffness value tends to zero, ($k \to 0$), then it is $Y_{-1}^{f} = -Y_1$, which gets close to pinned support (Fig. 3b).

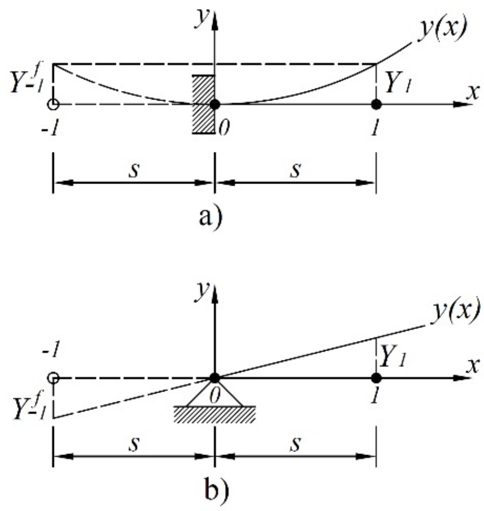


Figure 3: Shape function around the support for two extreme cases a) $k \to \infty$ (clamped support), b) $k \to 0$ (pinned support).

At this phase, a set of N algebraic equations can be written for the grid nodes $i=0,1,\dots, N-1$, with the node displacements Y_i, $i=1,\dots,N$ as unknowns. Thereby, the equations of the shape function written for the nodes 0 and 1 include displacements Y_0 and Y^f_{-1}, which are eliminated by using transformed boundary conditions (13) and (14). This gives them a different form in comparison to all other equations, written for the interval $i=2\div N\text{-}1$. Such system of linear algebraic equations, very convenient for programming, has the following form:

$$i=0:$$

$$\frac{2ks}{2EI_0+ks}Y_1 = -\frac{s^2FL}{EI_0} \tag{16}$$

$$i=1:$$

$$-2Y_1+Y_2 = \frac{6s^2F(s-L)}{E\delta\left[(H_0-\eta s-2\delta)^3+3B(H_0-\eta s-\delta)^2\right]} \tag{17}$$

$$i=2\div N-1:$$

$$Y_{i-1}-2Y_i+Y_{i+1} = \tag{18}$$

$$= \frac{6s^2F(is-L)}{E\delta\left[(H_0-\eta is-2\delta)^3+3B(H_0-\eta is-\delta)^2\right]}$$

4. NUMERICAL EXAMPLE AND COMPARISON WITH FEM MODEL RESULTS

To test the accuracy of presented procedure for determination of deflection, a numerical example with the following parameters is calculated: $L=3000[mm]$, $F=-1000[N]$, $H_L=100[mm]$, $B=100[mm]$, $\delta=5[mm]$, $E=2.1\cdot 10^5[N/mm^2]$. The ratio between starting and ending cross-section height is varied in the interval $\Psi=1.5\div 2.5$, while the spring stiffness is changed in interval $k=10^9\div 10^{10}[Nmm/rad]$. Discretization pitch is adopted as $s=1[mm]$, so the number of equations (unknowns) is $N=3000$.

MATLAB software is utilized for solving the system and for generating the diagrams of dependence of endpoint displacement (which is Y_N) on the variable parameters Ψ and k. FEM analysis are conducted in ANSYS software. Table 1 presents the comparison of the numerical results obtained from both methods. It can be noted that the results differ in less than 1%, which confirms the accuracy of the FDM approach.

Table 1: Comparison of results from FEM and FDM

k [Nmm/rad]	Ψ	Y_N [mm] MKE	Y_N [mm] MKR	Δ
10^9	1.50	-16.185	-16.164	0.13%
	1.75	-14.385	-14.361	0.16%
	2.00	-13.176	-13.152	0.18%
	2.25	-12.325	-12.301	0.19%
	2.50	-11.705	-11.682	0.20%
$2.50\cdot 10^9$	1.50	-10.785	-10.764	0.20%
	1.75	-8.984	-8.961	0.25%
	2.00	-7.775	-7.752	0.30%
	2.25	-6.925	-6.901	0.34%
	2.50	-6.305	-6.282	0.37%
$5.00\cdot 10^9$	1.50	-8.985	-8.964	0.24%
	1.75	-7.184	-7.161	0.32%
	2.00	-5.975	-5.952	0.39%
	2.25	-5.125	-5.101	0.46%
	2.50	-4.505	-4.482	0.52%
10^{10}	1.50	-8.085	-8.064	0.26%
	1.75	-6.284	-6.261	0.36%
	2.00	-5.075	-5.052	0.46%
	2.25	-4.225	-4.201	0.56%
	2.50	**-3.605**	-3.582	0.65%

Figure 4 depicts FEM model built in ANSYS, i.e. the case that corresponds to the result with the bolded value in Table 1.

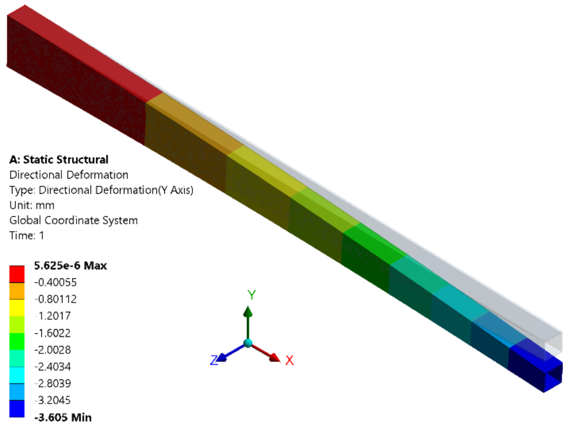


Figure 4: FEM model in ANSYS

Figure 5 presents the graphical interpretation of results comparison between the two methods. The lines are the obtained dependences of the endpoint displacement (Y_N) on the continuously changed parameter Ψ for the fixed values of the parameter k. Point markers are the corresponding results from FEM.

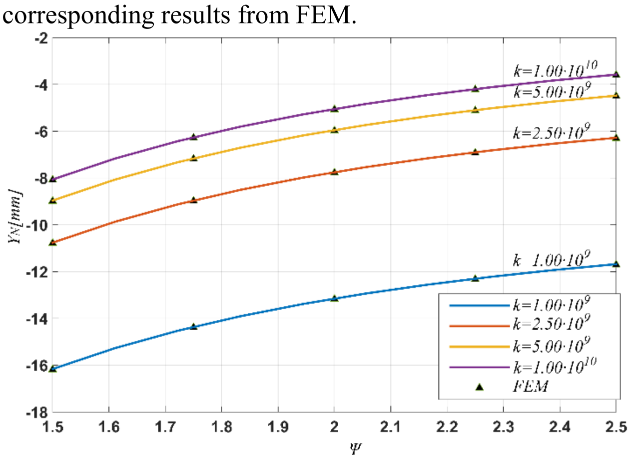


Figure 5: Graphical interpretation of the results in Table1

Figure 6 shows the change of endpoint displacement Y_N with continuous variation of both parameters Ψ and k in the defined domains.

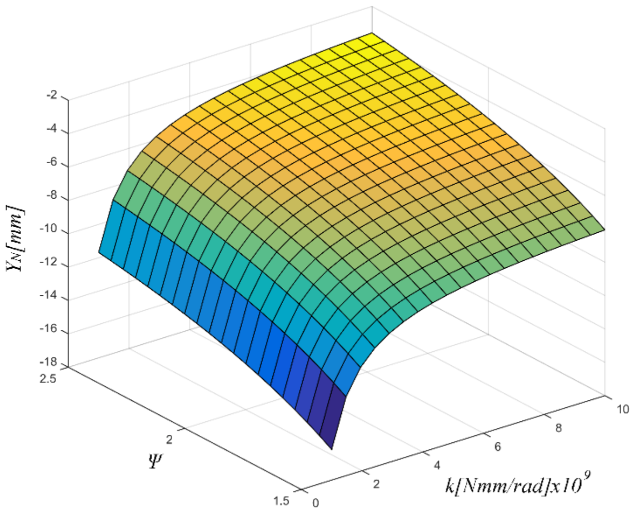


Figure 6: The dependence of endpoint displacement on continually varying parameters Ψ and k

5. CONCLUSION

The presented procedure for the application of FDM in the determination of deflection of the elastically supported non-uniform cantilever beam yielded the results which are practically identical with the ones from FEM. The deviation between the results are within 1%. Differential formulations of the beam bending and the boundary conditions were transformed by FDM approximations into the form linear algebraic equations. The displacement of the fictitious node, which figured in the algebraic form of the slope boundary condition, was eliminated out of the system.

Boundary conditions influenced the form of first two equations of the system only, while all others have the same recursive form, which is suitable for the programming. This feature of the numerical model enables efficient calculation of the nodes' displacements and obtaining the dependence between the displacement of any node and any design parameter. This approach is suitable for application for any other type of change of the cross-section and boundary conditions.

ACKNOWLEDGEMENTS

The paper was supported by the Ministry of education, science and technological development of the Republic of Serbia through the Contract for the scientific-research activity realisation and financing in 2020, ID 451-03-68/2020-14/200108.

REFERENCES

[1] Eurocode 3: Design of steel structures - Part 6: Crane supporting structures, EN 1993-6.

[2] Petković Z., Ostrić D.: Metal structures in mechanical engineering I (in Serbian), ISBN 86-70803-274-7, Faculty of Mechanical Engineering Belgrade, 1995.

[3] Bošnjak, S., Zrnić, N., Dragović, B. (2009): Dynamic Response of Mobile Elevating Work Platform under Wind Excitation, Strojniški vestnik - Journal of Mechanical Engineering, ISSN 0039-2480, Vol. 55, No. 2, pp 104-113, 2009.

[4] Zdravković N., Petković Z., Gašić M., Savković M.: Research of deflection-payload dependence of the auto crane articulated boom , International Conference on material Handling, Construction and Logistics MHCL'12, Belgrade, Serbia, ISBN 978-86-7083-763-8, pp. 101-106, 2012.

[5] Farkas Jozsef: *Optimum Design of Metal Structures*, Akademiai Kiado, Budapest, 1984.

[6] Gasic M., Savkovic M., Bulatovic R., Petrovic R.: Optimization of a pentagonal cross section of the truck crane boom using Lagrange's multipliers and differential evolution algorithm, Meccanica, Vol. 46, pp. 845–853. doi:10.1007/s1 1012-010-9343-7, 2011.

[7] Gasic M, Savkovic M, Bulatovic R, Petrovic R.: Optimization of trapezoidal cross section of the truck crane boom by Lagrange's multipliers and by differential evolution algorithm (de). Strojniški vestnik – Journal of Mechanical Engineering, 57(2011)4, pp. 304-312, doi: 10.5545/sv-jme.2008.029, 2011.

[8] Pinca BC, Tirian OG, Socalici VA, Ardeleadn DE: Dimensional optimization for the strength structure of a traveling crane, WSEAS Transactions on Applied and heoretical Mechanics 4 (4), ISSN: 1991-8747, pp 147-156, 2009.

[9] Selmic R, Cvetkovic R, Mijailovic R: Optimization of cross-section in structures, monograph, ISBN 86-7395-213-1, Faculty of Transport and Traffic Engineering, Belgrade, 2006.

[10] Sreedhara Rao Gunakala et al: A Finite Element Method for the Beam equation Using MATLAB, International Journal of Applied Science and Technology, ISSN 2221-0997, Vol. 2(8), pp. 80-88, 2012.

[11] Norbaya Omar, Fatimah De'nan: Finite element analysis of deflection and stress in triangular web profiled steel section (triwp) cantilever beam, DIGES PMU, ISSN 2289-6376, Vol. 3, pp. 1-6, 2016.

[12] Sanjay Kumar: Comparison of Deflection and slope of Cantilever Beam with Analytical and Finite Element Method for Different Loading Conditions, International Journal of Engineering Science and Innovative Technology (IJESIT), Vol. 5, Issue 6, pp. 45-51, 2016.

[13] Samir Z. Al-Sadder, Mohammad H. Dado: An Accurate Numerical Scheme for Large Deflection Analysis of Non-Prismatic Inextensible Slender Beams Subjected to General Loading and Boundary Conditions, Advances in Structural Engineering, Vol. 8(6), pp. 585–594. https://doi.org/10.1260/136943305776318392, 2005.

[14] Adel A. Al-Azzawi, Dhiaa M. Theeban: Large Deflection of Deep Beams on Elastic Foundations, Journal of the Serbian Society for Computational Mechanics, ISSN 1820-6530, Vol. 4, No. 1, pp. 88-101, 2010.

[15] K.S. Thankane and T. Stys: Finite Difference Method for Beam Equation with Free Ends Using Mathematica, Southern African Journal of Pure and Applied Mathematics, ISSN 1993-8594, Vol. 4, pp. 61-78, 2009.

[16] AL-Sadder, S., AL-Rawi, R. Finite difference scheme for large-deflection analysis of non-prismatic cantilever beams subjected to different types of continuous and discontinuous loadings, Arch Appl Mech, Vol. 75, pp. 459–473, https://doi.org/10.1007/s00419-005-0422-5, 2006.