Hacettepe Journal of Mathematics and Statistics
Volume 44 (2) (2015), 317 – 322

Representation and characterization of rapidly varying functions

Nebojša Elez* and Dragan Djurčić†

Abstract

In this paper representations and characterizations of the class of rapidly varying functions in the sense of de Haan, for index $+\infty$, will be proved. The statements of this theorems will be given in a form that is used by Karamata. Also, some characterization of normalized rapidly varying functions are proved.

2000 AMS Classification: 26A12.

Keywords: Rapidly varying function; Regular variation.

Received 08 /08 /2013 : Accepted 31 /01 /2014 Doi : 10.15672/HJMS.2015449105

1. Introduction and Results

Karamata's theory of regular variation (see e.g. [6]) was appeared during the thirties of last century as a result of the first serious study of Tauberian type theorems for integral transformations (see e.g. [7] and [8]). The main object in this theory is the class of slowly varying functions in the sense of Karamata which is denoted by SV.

A measurable function $f : [a, \infty) \mapsto (0, \infty)$ $(a > 0)$ is called slowly varying in the sense of Karamata if it satisfies the following condition

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = 1, \tag{1.1}$$

for every $\lambda > 0$,

L. de Haan in [5] introduced the class of rapidly varying functions (denoted by R_∞), with the index of variability $+\infty$. In fact, this notion has already appeared in some Karamata's papers (see e.g. [11]), but in a less distinctly form. In recent years, the Theory of rapid variability and its generalizations have experienced great development in asymptotic analysis and in mathematics in general (see e.g. [1], [2], [3], [4] and [10]),

*University of East Sarajevo Faculty of Philosophy 71420 Pale Bosnia and Hercegovina. Email: `jasnaelez@gmail.com`

†University in Kragujevac Faculty of technical sciences Svetog Save 65, 32000 Čačak Serbia. Email:`dragan.djurcic@ftn.kg.ac.rs`

This paper is supported by the Ministry of Education and Science of Republic of Serbia, Grant No. 174032.

simultaneously with Karamata's theory of regular variability (see [1]). Important properties of the class R_∞ can be seen in [4].

A measurable function $f : [a, \infty) \mapsto (0, \infty)$ $(a > 0)$ is called rapidly varying in the sense of de Haan with the index of variability ∞, if it satisfies the following condition

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \infty, \tag{1.2}$$

for every $\lambda > 1$.

1.1. Remark. In this paper we will consider a function $f \in R_\infty$ defined on the interval $(0, \infty)$. Analogous results can be obtained if the domain of a function f is interval $[a, \infty)$, $a > 0$.

According to results from [1], rapidly varying function f satisfies condition

$$\lim_{x\to\infty} \inf_{\mu\geq\lambda} \frac{f(\mu x)}{f(x)} = \infty \tag{1.3}$$

for every $\lambda > 1$ and it follows that for some $x_0 > 0$ function f is bounded on every interval (x_0, x). Also, $f(x) \to \infty$ for $x \to \infty$ holds.

Now, we give the representation of a functions from functional class R_∞ in Karamata's form.

1.2. Remark. In the following theorem, operator $\underline{D}$ is lower Dini derivative (see [9])

$$\underline{D}g(x) = \underline{\lim}_{y\to x} \frac{g(y) - g(x)}{y - x}, \quad \text{for } g : \mathbb{R} \to \mathbb{R},\ x \in \mathbb{R},$$

and denotation $\sim$ represents strong asymptotic equivalence relation.

1.3. Theorem. *For a function $f : (0, \infty) \mapsto (0, \infty)$ the next assertions are mutually equivalent:*

(a) *function f belongs to the class R_∞;*
(b) *there is a non-decreasing, absolutely continuous function $g : \mathbb{R} \mapsto \mathbb{R}$ such that $\lim_{x\to\infty} \underline{D}g(x) = \infty$ and there is a measurable function $j : (0, \infty) \mapsto (0, \infty)$ such that $j(x) \sim x$ for $x \to \infty$, so that*

$$f(x) = \exp\big(g(\log(j(x)))\big),$$

for all $x > 0$;
(c) *there are a measurable functions $j : (0, \infty) \mapsto (0, \infty)$ and $h : (0, \infty) \mapsto [0, \infty)$, such that $\lim_{x\to\infty} h(x) = \infty$ and $j(x) \sim x$ for $x \to \infty$, for which holds*

$$f(x) = exp\left\{ c + \int_0^{j(x)} h(u) \frac{du}{u} \right\},$$

for all $x > 0$ and for some $c \in \mathbb{R}$.

Now, we give the characterization of a elements from the class R_∞ in Karamata's form.

1.4. Theorem. *Let $f : (0, \infty) \mapsto (0, \infty)$ be a measurable function. Then $f \in R_\infty$ if and only if for all $\alpha > 0$ there is a measurable function $j_\alpha : (0, \infty) \mapsto (0, \infty)$ such that $j(x) \sim x$, for $x \to \infty$, and there is a non-decreasing function $k_\alpha : (0, \infty) \mapsto (0, \infty)$, so that*

$$f(x) = x^\alpha \cdot k_\alpha\big(j_\alpha(x)\big), \qquad \textit{for } x > 0.$$

The following theorem gives a few characterization of elements of one proper subclass of class R_∞, which could be called class of normalized rapidly varying functions (see, e.g., [1]).

1.5. Theorem. *For a measurable function $f:(0,\infty)\mapsto(0,\infty)$ the following assertions are mutually equivalent:*

(a) $\lim\limits_{\substack{x\to\infty\\ \lambda\to 1_+}} \log_\lambda \dfrac{f(\lambda x)}{f(x)}=\infty$;

(b) $\dfrac{f(x)}{x}=o\,(\underline{D}f(x))$, *for $x\to\infty$ (o is Landau symbol [1]);*

(c) *there exists a function $g:\mathbb{R}\mapsto\mathbb{R}$ such that $\lim\limits_{x\to\infty}\underline{D}g(x)=\infty$ so that holds:*

$$f(x)=\exp(g(\log(x)))$$

for all $x>0$;

(d) *for all $\alpha\in\mathbb{R}$ function $\frac{f(x)}{x^\alpha}$ is increasing on some interval $[x_\alpha,\infty)$.*

1.6. Remark. 1) Theorem 1.5 holds even without assumption that the function f is measurable, but this assumption should be included because in Theorem 1.5 one important subclass of class R_∞ is characterized.

2) The fact that for a measurable function $f:(0,\infty)\mapsto(0,\infty)$ exists a measurable function $h:(0,\infty)\mapsto\mathbb{R}$ such that $\lim\limits_{x\to\infty}h(x)=\infty$, and for which is $f(x)=\exp\{c+\int_0^x h(u)\frac{du}{u}\}$ for all $x>0$ and some $c\in\mathbb{R}$, implicates (c) from Theorem 1.5 (and, also implicates (a), (b) and (d) from Theorem 1.5). The proof is analog to the proof (c) $\Rightarrow$ (a) $\Rightarrow$ (b) from Theorem 1.3. The opposite direction need not to be true without additional conditions.

3) If f is absolutely continuous function, opposite direction in 2) is true. That can be proved analogously to the proof (b) $\Rightarrow$ (c) from Theorem 1.3.

2. Proofs

Proof of Theorem 1.3. (a) $\Rightarrow$ (b) Let $f\in R_\infty$. Let construct sequence (x_n) of positive real numbers with the following properties:

1° (x_n) is strictly increasing sequence and $\lim\limits_{n\to\infty}x_n=\infty$,

2° $\lim\limits_{n\to\infty}\dfrac{x_{n+1}}{x_n}=1$, and

3° $\dfrac{f(x)}{f(y)}>2$ for all $x>0$ and all $y>0$, for which $x\geq x_{n+1}>x_n\geq y\geq x_1$, where $n\in\mathbb{N}$.

Let $x_1>0$ so that f is locally bounded on the interval $[x_1,\infty)$ and let $x_{n+1}=(\lambda_n+\frac{1}{n})x_n$ for $n\in\mathbb{N}$, where $\lambda_n=\sup\{\lambda\geq 1 \mid f(\lambda x_n)\leq 2\sup\limits_{x_1\leq t\leq x_n}f(t)\}$. Clearly, for all $n\in\mathbb{N}$ there exists λ_n in $\mathbb{R}$ and $x_n\leq\lambda_n x_n<x_{n+1}$. If $x\geq x_{n+1}>x_n\geq y>0$, then $x>\lambda_n x_n$ for $n\in\mathbb{N}$, and according to the definition of the sequence (λ_n) it is $f(x)>2f(y)$ for $x_1\leq y\leq x_n$. Especially, $f(x_{n+1})>2f(x_n)$ for $n\in\mathbb{N}$, which yields $\lim\limits_{n\to\infty}f(x_n)=\infty$. As f is locally bounded function on the interval $[x_1,\infty)$, it follows $\lim\limits_{n\to\infty}x_n=\infty$.

According to the definition of sequence (λ_n) it can be concluded that sequences (μ_n) and (y_n) are such that, for every $n\in\mathbb{N}$, it follows that $\mu_n\in(\lambda_n-\frac{1}{n},\lambda_n)$ and $y_n\in[x_1,x_n]$, for which it is $\dfrac{f(\mu_n x_n)}{f(y_n)}\leq 2$. Then, according to the theorem of uniform convergence for rapidly varying functions (see (1.3) or [1]), it follows $\limsup_{n\to\infty} \frac{\mu_n x_n}{y_n} \le 1$, i.e., $\limsup_{n\to\infty} \mu_n \le 1$, so it follows that $\lim_{n\to\infty} \lambda_n = 1$. Thus, $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$.

Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a linear function on $[t_n, t_{n+1}]$ such that $g(t_n) = \ln f(x_n)$, where $t_n = \ln x_n$, for every $n \in \mathbb{N}$. Also, $g(t) = e^t - x_1 + g(t_1)$, for $t < t_1$. Now, we have that g is a continuous, piecewise smooth and strictly increasing (hence, absolutely continuous and non-decreasing) function, and (from 1° and 3°) it satisfies

$$g'(t) = \frac{g(t_{n+1}) - g(t_n)}{t_{n+1} - t_n} = \frac{\ln f(x_{n+1}) - \ln f(x_n)}{\ln x_{n+1} - \ln x_n} > 0,$$

for any $t \in (t_n, t_{n+1})$, $n \in \mathbb{N}$. Furthermore, (from 2° and 3°) it satisfies

$$\lim_{t\to\infty} g'(t) = \lim_{n\to\infty} \frac{\ln \frac{f(x_{n+1})}{f(x_n)}}{\ln \frac{x_{n+1}}{x_n}} \ge \lim_{n\to\infty} \frac{\ln 2}{\ln \frac{x_{n+1}}{x_n}} = \infty,$$

for $\mathbb{R} \ni t \ne t_n$, for every $n \in \mathbb{N}$. Thus, $\lim_{x\to\infty} \underline{D}g(x) = \infty$.

Now, let $j(x) = e^{g^{-1}(\ln f(x))}$, for $x > 0$. A function $j(x)$ is measurable, because $f(x)$ is a measurable function, and $\exp\big(g^{-1}(\log(t))\big)$ is a piecewise smooth function (and hence it is absolutely continuous function), for $t > 0$.

Now, we will show that $j(x) \sim x$, for $x \to \infty$. From condition 3° it follows that $\frac{f(x)}{f(x_{n-1})} > 2$ and $\frac{f(x_{n+2})}{f(x)} > 2$, for some $x \in [x_n, x_{n+1})$ and $n \in \mathbb{N}$, $n \ge 2$. For those x and n we obtain

$$f(x_{n-1}) < f(x) < f(x_{n+2}),$$

so, we have

$$g^{-1}\big(\ln f(x_{n-1})\big) < g^{-1}\big(\ln f(x)\big) < g^{-1}\big(\ln f(x_{n+2})\big).$$

Furthermore, for those x and n, we have

$$t_{n-1} = \ln x_{n-1} < \ln j(x) < \ln x_{n+2} = t_{n+2},$$

and finally we obtain that

$$\frac{x_{n-1}}{x_{n+1}} < \frac{x_{n-1}}{x} < \frac{j(x)}{x} < \frac{x_{n+2}}{x} < \frac{x_{n+2}}{x_n}.$$

Hence, from 2° it is satisfied that $j(x) \sim x$, for $x \to \infty$, and $f(x) = \exp(g(\log(j(x))))$ for $x > 0$.

(b) $\Rightarrow$ (c) Let functions g and j have properties given in (b). Let

$$g_0 = \begin{cases} g(x), & \text{for } x > 0, \\ g(0), & \text{for } x \le 0. \end{cases}$$

Let $h(x) = \underline{D}g_0(\ln x)$ for $x > 0$. Then h is measurable, locally integrable, and $\lim_{x\to\infty} h(x) = \infty$ holds. Also, it is

$$\int_0^{j(x)} h(u)\frac{du}{u} = \int_0^{j(x)} \underline{D}g_0(\ln u)\frac{du}{u} = \int_{-\infty}^{\ln j(x)} \underline{D}g_0(t)dt =$$

$$= \int_0^{\ln j(x)} \underline{D}g(t)dt = g(\ln j(x)) - g(0) = \ln f(x) - c,$$

for a constant $c = g(0) \in \mathbb{R}$, and for all $x > 0$.

Thus, $f(x) = \exp\left\{c + \int_0^{j(x)} h(u)\frac{du}{u}\right\}$, for all $x > 0$ and for mentioned $c \in \mathbb{R}$.

(c) $\Rightarrow$ (a) Let $\lambda > 1$ and $M \in \mathbb{R}$. Then, there is $x_0 > 0$ such that $h(x) > \dfrac{2M}{\ln \lambda}$ and $\lambda^{-\frac{1}{4}} < \dfrac{j(x)}{x} < \lambda^{\frac{1}{4}}$, for $x > \dfrac{x_0}{\lambda}$. Hence, it follows

$$\begin{aligned}
\ln \frac{f(\lambda x)}{f(x)} &= \int\limits_0^{j(\lambda x)} h(u)\frac{du}{u} - \int\limits_0^{j(x)} h(u)\frac{du}{u} = \int\limits_{j(x)}^{j(\lambda x)} h(u)\frac{du}{u} > \\
&> \frac{2M}{\ln \lambda} \cdot \big(\ln j(\lambda x) - \ln j(x)\big) = \frac{2M}{\ln \lambda} \cdot \ln\left(\lambda \cdot \frac{j(\lambda x)}{\lambda x} \cdot \frac{x}{j(x)}\right) > \\
&> \frac{2M}{\ln \lambda} \cdot \ln\left(\lambda \cdot \lambda^{-\frac{1}{4}} \cdot \lambda^{-\frac{1}{4}}\right) = M,
\end{aligned}$$

for $x > x_0$. Therefore, it holds that $\lim\limits_{x\to\infty} \dfrac{f(\lambda x)}{f(x)} = \infty$, for every $\lambda > 1$. Also, f is a measurable function, as a composition of three function: a measurable function j, an absolutely continuous function $\int\limits_0^x h(u)\dfrac{du}{u}$ and an exponential function. Hence, $f \in R_\infty$. $\square$

Proof of Theorem 1.4. ($\Rightarrow$) If $f \in R_\infty$, then $\dfrac{f(x)}{x^\alpha} \in R_\infty$, for $x > 0$, and every fixed $\alpha > 0$. From Theorem 1.3, it follows that $\frac{f(x)}{x^\alpha} = \exp\Big\{g_\alpha\big(\log(j_\alpha(x))\big)\Big\}$, for that α and every $x > 0$, where $g_\alpha : \mathbb{R} \mapsto \mathbb{R}$ is a non-decreasing function and $j_\alpha : (0,\infty) \mapsto (0,\infty)$ is a measurable function such that $j_\alpha(x) \sim x$, for $x \to \infty$. If we take that $k_\alpha(t) = \exp\big\{g_\alpha(\log(t))\big\}$, for $t > 0$, we obtain that Theorem holds for this direction.
($\Leftarrow$) For arbitrary $\alpha > 0$, if there is a measurable function $j_\alpha : (0,\infty) \mapsto (0,\infty)$ such that $\lim\limits_{x\to\infty} \dfrac{j_\alpha(x)}{x} = 1$ and a non-decreasing function $k_\alpha : (0,\infty) \mapsto (0,\infty)$, that is satisfied $f(x) = x^\alpha \cdot k_\alpha(j_\alpha(x))$, for $x > 0$ we obtain that

$$\frac{f(\lambda x)}{f(x)} = \lambda^\alpha \cdot \frac{k_\alpha\big(j_\alpha(\lambda x)\big)}{k_\alpha\big(j_\alpha(x)\big)} \geq \lambda^\alpha,$$

for $\lambda > 1$ and sufficiently large x. The previous inequality holds because $j_\alpha(\lambda x) \geq \sqrt{\lambda}x \geq j_\alpha(x)$ for mentioned α, λ and sufficiently large x. Therefore, it follows that $\lim\limits_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \infty$, for $\lambda > 1$. Also, f is a measurable function. Finally, $f \in R_\infty$. $\square$

Proof of Theorem 1.5. (a) $\Leftrightarrow$ (c) Let introduce function f in the following way: $f(x) = \exp(g(\log(x)))$, for $x > 0$. Then, equivalence (a) $\Leftrightarrow$ (c) follows from:

$$\varliminf_{\substack{x\to\infty\\ \lambda\to 1_+}} \log_\lambda \frac{f(\lambda x)}{f(x)} = \varliminf_{\substack{t\to\infty\\ \delta\to 0_+}} \frac{g(t+\delta)-g(t)}{\delta} = \varliminf_{t\to\infty} \underline{D}g(t).$$

(b) $\Leftrightarrow$ (c) Again, let introduce function f as $f(x) = \exp(g(\log(x)))$, for $x > 0$. Then, from the fact that $\underline{D}g(x) = \underline{D}(\ln f(e^x)) = \dfrac{\underline{D}f(e^x)e^x}{f(e^x)}$, for all $x \in \mathbb{R}$, it follows $\varliminf\limits_{x\to\infty} \underline{D}g(x) = \infty$ is equal to the fact that $\frac{f(t)}{t} = o(\underline{D}g(t))$, for $t \to \infty$. This proves the equivalence (b) $\Leftrightarrow$ (c).

(c) $\Leftrightarrow$ (d) Once more, let introduce function f by $f(x) = \exp(g(\log(x)))$, for $x > 0$. Then the function $\frac{f(x)}{x^\alpha}$, $x > 0$ and $\alpha \in \mathbb{R}$, is increasing on an interval $[x_\alpha, \infty)$ if and only if the function $\ln \dfrac{f(e^t)}{e^{\alpha t}} = g(t) - \alpha t$, for $t \in \mathbb{R}$ and the same $\alpha \in \mathbb{R}$, is increasing on an interval $[t_\alpha, \infty)$, and this last condition is equivalent to the fact that $\varliminf_{t\to\infty} Dg(t) \geq \alpha$ for all $\alpha \in \mathbb{R}$. The last fact is equivalent to the fact that $\varliminf_{t\to\infty} Dg(t) = \infty$. □

References

[1] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular variation*, Cambridge Univ. Press, Cambridge, 1987.

[2] D. Djurčić, D. R. Kočinac, M. R. Žižović, Some properties of rapidly varying sequences, *J. Math. Anal. Appl.* **327(2)** (2007), 1297–1306.

[3] D. Djurčić, A. Torgašev, Some asymptotic relations for generalized inverse, *J. Math. Anal. Appl.*, **335(2)** (2007), 1397–1402.

[4] N. Elez, D. Djurčić, Some properties of rapidly varying functions, *J. Math. Anal. Appl.*, **401(2)** (2013), 888–895.

[5] L. de Haan, *On Regular variation and its applications to the weak convergence of sample extremes*, Math. Centre tracts, Vol. **32**, CWI, Amsterdam, 1970.

[6] J. Karamata, Sur un mode de croissance régulière des functions, *Mathematica (Cluj)* **4** (1930), 38–53.

[7] J. Karamata, Neuer Beweis und Verallgemeinerung der Taubersche Sätz, welche die Laplacessche und Stieltjessche Transformationen betreffen, *J. Reine Angew. Math.* **164** (1931), 27–39.

[8] J. Karamata, Sur un mode de croissance régulière. Théorèmes fondamenteaux, *Bull Soc. Math. France* **61** (1933), 55–62.

[9] I.P. Natanson, *Theory of functions of a real variable*, (Translated by Leo F. Boron with the collaboration of Edwin Hewitt), Frederick Ungar Publishing Co., New York, 1955.

[10] S. Matucci, P. Řehák, Rapidly varying decreasing solutions of half-linear difference equations, Math. and Com. Modelling, **49(7-8)**, (2009), 1692–1699.

[11] E. Seneta, Functions of regular variation, *LNM* Vol. **506**, Springer, New York, 1976.