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# Diffusion Limit for Single-Server Retrial Queues with Renewal Input and Outgoing Calls

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**Abstract:** This paper studies a single-server retrial queue with two types of calls (incoming and outgoing calls). Incoming calls arrive at the server according to a renewal process, and outgoing calls of $N-1$ ($N \geq 2$) categories occur according to $N-1$ independent Poisson processes. Upon arrival, if the server is occupied, an incoming call joins a virtual infinite queue called the orbit, and after an exponentially distributed time in orbit enters the server again, while outgoing calls are lost if the server is busy at the time of their arrivals. Although M/G/1 retrial queues and their variants are extensively studied in the literature, the GI/M/1 retrial queues are less studied due to their complexity. This paper aims to obtain a diffusion limit for the number of calls in orbit when the retrial rate is extremely low. Based on the diffusion limit, we built an approximation to the distribution of the number of calls in orbit.

**Keywords:** retrial queue; two-way communication; renewal process; diffusion approximation; incoming call; outgoing call

**MSC:** 60K20; 60K25; 60K30; 68M20

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## 1. Introduction

Retrial queues naturally arise in various applications such as telecommunication and service systems [1,2]. Customers who cannot occupy a server upon arrival join the orbit and repeat their attempt later in these models. Retrial queues have been extensively investigated in the literature. We refer to [1,2] for major results up to the 1990s. For single-server models, most studies focus on models with Poisson or Markovian arrival processes [3–5]. To the best of our knowledge, only a few papers have dealt with models with renewal input. The main difficulty is that the embedded Markov chain of the model has a complicated transition structure that requires the transient analysis of a two-dimensional Markov chain. Furthermore, in [3–5], the total retrial rate of all customers in orbit was assumed to be constant. In contrast to [3–5], in this paper, we considered a GI/M/1 model with outgoing calls and the classical retrial rate where the total retrial rate is proportional to the number of customers in orbit, i.e., each customer in orbit retries independently of other customers. The methodology in [3–6] cannot be applied to GI/M/1 retrial queues with the classical retrial rate. More precisely, if we consider the embedded Markov chain at the pre-arrival epochs, we need the transient solution of a two-dimensional Markov chain representing the dynamics of the number of customers in orbit and the state of the servers. This transient solution is difficult to obtain in explicit form. Retrial queues with outgoing calls (also called two-way communication) have been extensively studied [7–12]. Generating functions for single-server retrial queues with outgoing calls were obtained in [7,8], while asymptotic results were investigated in [9,10,12]. However, in these models, the input is either Poisson or Markovian.

As is introduced in [2], there were some studies by Khomichkov [13,14] for special cases where interarrivals follow hyperexponential, or Erlang, or Coxian distributions of the second order. Furthermore, Yang et al. [15] studied a multiserver model with an exponential service time distribution and Coxian input. However, the retrial model with general renewal input was not yet investigated in the literature to the best of our knowledge, and thus, it is the subject of this paper. We investigated the model in an asymptotic regime when the retrial rate of customers is extremely low. While under this regime, the number of customers in orbit explodes, we show that using an appropriate scaling, the scaled number of customers in orbit converges to a diffusion process. This limiting result was then used to approximate the distribution of the number of customers in orbit. As for related asymptotic results, Sakurai and Phung-Duc [10] studied the asymptotic behavior of an M/G/1 retrial queue with outgoing calls under three regimes: (i) heavy incoming calls, (ii) heavy outgoing calls, and (iii) a low retrial rate. Nazarov et al. [9,12] studied the low retrial and heavy outgoing call asymptotics for Markovian retrial queues under a random environment. However, in [9,10,12], only the stationary distribution was considered.

The rest of this paper is organized as follows. First, Section 2 shows the model in detail. Then, Section 3 presents the basic equations for the distribution of the number of customers in orbit. Sections 4 and 5 show the first- and second-order asymptotics of the distribution, respectively. Section 6 presents an algorithm for the distribution of the number of calls in orbit and some numerical examples. Finally, Section 7 concludes our paper.

## 2. Mathematical Model

We considered a retrial queue with a renewal input process whose probability function of the renewal intervals is given by $A(x)$. We call inbound demands the incoming calls. If the server is idle upon arrival, the incoming call reserves the server for an exponentially distributed time with mean $1/\mu_1$. Otherwise, the incoming call joins the orbit and enters the server after a random time. The delay duration in orbit follows an exponential distribution with mean $1/\sigma$. The behavior of a retrial incoming call from the orbit is the same as that of a fresh one.

During the idle time (of the server), the server generates $N-1$ streams of outgoing calls according to $N-1$ independent Poisson processes. Thus, outgoing calls of type $n$ are generated according to a Poisson process with rate $\alpha_n$ ($n = 2, 3, \ldots, N$). These outgoing calls need to be connected before the server becomes busy. Otherwise, the server is considered to be idle and available for servicing a new incoming call or a call from the retrial orbit. Once an outgoing call of type $n$ is connected, the server becomes busy with this outgoing call for an exponentially distributed time with mean $1/\mu_n$. It is important to point out that when the server is busy with an incoming call (either a new one or from a retrial orbit), outgoing calls are considered lost.

Alternately, we can give the following interpretation (this is due to the memoryless property of the underlying exponential distribution and will not be applicable otherwise) for the server's idle duration. Outgoing calls of type $n$ arrive at the service facility server according to a Poisson process with rate $\alpha_n$, and the service time for outgoing calls of type $n$ is exponentially distributed with mean $1/\mu_n$. Upon arrival of an outgoing call of type $n$, if the server is busy, the outgoing call is lost (does not join the orbit, unlike incoming calls). Otherwise, if the server is idle upon arrival, the outgoing call occupies the server for an exponentially distributed time with mean $1/\mu_n$.

As a result, upon completion of a service (of either an incoming or an outgoing call), the server stays idle for an exponentially distributed time with mean $1/\alpha_o$, where $\alpha_o = \sum_{n=2}^{N} \alpha_n$. During this idle time, an incoming or a retrial call can enter into service. After an idle time, if the server still remains idle, then an outgoing call of type $n$ is connected with probability $\alpha_n/\alpha_o$, and the server is busy serving this outgoing call for an exponentially distributed time with mean $1/\mu_n$.

## 3. Markov Process for the System

Let $i(t)$ denote the number of calls in orbit at time $t$. We studied the transient behavior of $i(t)$ in a heavy traffic regime when the retrial rate $\sigma$ is extremely low, i.e., $\sigma \to 0$. We also obtained an approximation to the stationary probability distribution of $i(t)$ as a by-product.

To analyze the system, we furthermore let $n(t)$ denote the state of the server at time $t$. This process can take the following values: zero if the server is idle; one if the server is busy with an incoming call; $n$ if the server is busy with an outgoing call of type $n$, $n = 2, 3, \ldots, N$.

**Remark 1.** *In the conventional method for the GI/M/1 queue, we considered the pre-arrival epochs as embedded points and constructed a discrete-time Markov chain representing the number of customers in orbit and the state of the server. For the GI/M/1 retrial queue in this paper, if we perform this similarly, we need the transient probabilities of* $\{(n(t),\ i(t)); t \geq 0\}$. *It should be noted that* $\{(n(t),\ i(t)); t \geq 0\}$ *is a level-dependent quasi-birth-and-death process. Furthermore, we need the integral of these transient probabilities with respect to the interarrival time distribution, which is cumbersome and challenging.*

In this paper, we analyzed the model using the supplementary variable approach. To this end, we also denote $z(t)$ as the residual time until the next arrival in the renewal input process. The three-dimensional process $\{n(t),\ i(t),\ z(t)\}$ is Markovian. Denoting:

$$P\{n(t) = n,\ i(t) = i,\ z(t) < z\} = P_n(i, z, t),$$

we have:

$$\begin{aligned}
\frac{\partial P_0(i,z,t)}{\partial t} &= \frac{\partial P_0(i,z,t)}{\partial z} - \frac{\partial P_0(i,0,t)}{\partial z} - \left(i\sigma + \sum_{n=2}^{N} \alpha_n\right) P_0(i,z,t) \\
&\quad + \sum_{n=1}^{N} \mu_n P_n(i,z,t), \\
\frac{\partial P_1(i,z,t)}{\partial t} &= \frac{\partial P_1(i,z,t)}{\partial z} - \frac{\partial P_1(i,0,t)}{\partial z} - \mu_1 P_1(i,z,t) + A(z) \frac{\partial P_1(i-1,0,t)}{\partial z} \\
&\quad + A(z) \frac{\partial P_0(i,0,t)}{\partial z} + (i+1)\sigma P_0(i+1,z,t), \\
\frac{\partial P_n(i,z,t)}{\partial t} &= \frac{\partial P_n(i,z,t)}{\partial z} - \frac{\partial P_n(i,0,t)}{\partial z} - \mu_n P_n(i,z,t) + A(z) \frac{\partial P_n(i-1,0,t)}{\partial z} \\
&\quad + \alpha_n P_0(i,z,t),\ n = 2, \ldots, N,
\end{aligned} \tag{1}$$

where $\dfrac{\partial P_n(i,0,t)}{\partial z} = \dfrac{\partial P_n(i,z,t)}{\partial z}\Big|_{z=0}$.

Then, we rewrite (1) using the partial characteristic functions:

$$H_n(u,z,t) = \sum_{i=0}^{\infty} e^{jui} P_n(i,z,t).$$

$$\begin{aligned}
\frac{\partial H_0(u,z,t)}{\partial t} &= \frac{\partial H_0(u,z,t)}{\partial z} - \frac{\partial H_0(u,0,t)}{\partial z} - H_0(u,z,t) \sum_{n=2}^{N} \alpha_n \\
&\quad + j\sigma \frac{\partial H_0(u,z,t)}{\partial u} + \sum_{n=1}^{N} \mu_n H_n(u,z,t), \\
\frac{\partial H_1(u,z,t)}{\partial t} &= \frac{\partial H_1(u,z,t)}{\partial z} - \frac{\partial H_1(u,0,t)}{\partial z} - \mu_1 H_1(u,z,t) \\
&\quad + A(z)\left[e^{ju} \frac{\partial H_1(u,0,t)}{\partial z} + \frac{\partial H_0(u,0,t)}{\partial z}\right] - j\sigma e^{-ju} \frac{\partial H_0(u,z,t)}{\partial u},
\end{aligned}$$

$$\frac{\partial H_n(u,z,t)}{\partial t} = \frac{\partial H_n(u,z,t)}{\partial z} - \frac{\partial H_n(u,0,t)}{\partial z} - \mu_n H_n(u,z,t)$$
$$+ e^{ju} A(z) \frac{\partial H_n(u,0,t)}{\partial z} + \alpha_n H_0(u,z,t),\ n = 2,\ldots,N. \tag{2}$$

For the compactness of further calculations, we define the matrices $\mathbf{I}_0$, $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{Q}$ of $(N+1)\times(N+1)$ dimension as follows.

$$\mathbf{I}_0 = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{bmatrix},\ \mathbf{I}_1 = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{bmatrix},\ \mathbf{I}_2 = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} -\sum\limits_{n=2}^{N} \alpha_n & 0 & \alpha_2 & \alpha_3 & \ldots & \alpha_N \\ \mu_1 & -\mu_1 & 0 & 0 & \ldots & 0 \\ \mu_2 & 0 & -\mu_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \mu_N & 0 & 0 & 0 & \ldots & -\mu_N \end{bmatrix}.$$

We denote:

$$\mathbf{H}(u,z,t) = \{H_n(u,z,t)\},\ \frac{\partial \mathbf{H}(u,z,t)}{\partial t} = \left\{\frac{\partial H_n(u,z,t)}{\partial t}\right\},$$
$$\frac{\partial \mathbf{H}(u,z,t)}{\partial z} = \left\{\frac{\partial H_n(u,z,t)}{\partial z}\right\},\ \frac{\partial \mathbf{H}(u,z,t)}{\partial u} = \left\{\frac{\partial H_n(u,z,t)}{\partial u}\right\},$$
$$\frac{\partial \mathbf{H}(u,0,t)}{\partial z} = \left\{\frac{\partial H_n(u,0,t)}{\partial z}\right\},\ n = 0,\ldots,N.$$

Using these notations, we obtain (2) in the following form:

$$\frac{\partial \mathbf{H}(u,z,t)}{\partial t} = \frac{\partial \mathbf{H}(u,z,t)}{\partial z} - \frac{\partial \mathbf{H}(u,0,t)}{\partial z} + j\sigma \frac{\partial \mathbf{H}(u,z,t)}{\partial u}[\mathbf{I}_0 - e^{-ju}\mathbf{I}_2]$$
$$+ A(z)\frac{\partial \mathbf{H}(u,0,t)}{\partial z}[\mathbf{I}_2 + e^{ju}\mathbf{I}_1] + \mathbf{H}(u,z,t)\mathbf{Q}. \tag{3}$$

We considered (3) in the limit $z \to \infty$, denoting $\mathbf{H}(u,\infty,t) = \mathbf{H}(u,t)$ and multiplying the equation by unit vector $\mathbf{e}$ (with all elements of one). Since $\mathbf{Qe} = 0$, $\mathbf{I}_0 + \mathbf{I}_1 = \mathbf{I}$, $(\mathbf{I}_0 - \mathbf{I}_2)\mathbf{e} = 0$, $(\mathbf{I}_1 + \mathbf{I}_2)\mathbf{e} = \mathbf{e}$, we obtain another equation:

$$\frac{\partial \mathbf{H}(u,t)}{\partial t}\mathbf{e} = (e^{ju} - 1)\left\{ j\sigma e^{-ju} \frac{\partial \mathbf{H}(u,t)}{\partial u}\mathbf{I}_0 + \frac{\partial \mathbf{H}(u,0,t)}{\partial z}\mathbf{I}_1 \right\}\mathbf{e}. \tag{4}$$

We solved (3) and (4) using the asymptotic-diffusion method under the limit condition $\sigma \to 0$.

## 4. First Step of Asymptotic-Diffusion Analysis

Denoting $\sigma = \varepsilon$ and introducing in (3) and (4) the notations:

$$\tau = t\varepsilon,\ u = w\varepsilon,\ \mathbf{H}(u,t) = \mathbf{F}(w,\tau,\varepsilon),\ \mathbf{H}(u,z,t) = \mathbf{F}(w,z,\tau,\varepsilon),$$

we obtain:

$$\varepsilon \frac{\partial \mathbf{F}(w,z,\tau,\varepsilon)}{\partial \tau} = \frac{\partial \mathbf{F}(w,z,\tau,\varepsilon)}{\partial z} - \frac{\partial \mathbf{F}(w,0,\tau,\varepsilon)}{\partial z} + j\frac{\partial \mathbf{F}(w,z,\tau,\varepsilon)}{\partial w}[\mathbf{I}_0 - e^{-jw\varepsilon}\mathbf{I}_2]$$
$$+ A(z)\frac{\partial \mathbf{F}(w,0,\tau,\varepsilon)}{\partial z}[\mathbf{I}_2 + e^{jw\varepsilon}\mathbf{I}_1] + \mathbf{F}(w,z,\tau,\varepsilon)\mathbf{Q},$$

$$\varepsilon\frac{\partial \mathbf{F}(w,\tau,\varepsilon)}{\partial \tau}\mathbf{e} = (e^{jw\varepsilon}-1)\left\{je^{-jw\varepsilon}\frac{\partial \mathbf{F}(w,\tau,\varepsilon)}{\partial w}\mathbf{I}_0 + \frac{\partial \mathbf{F}(w,0,\tau,\varepsilon)}{\partial z}\mathbf{I}_1\right\}\mathbf{e}. \tag{5}$$

Solving (5) under the limit condition $\varepsilon \to 0$, we obtain Theorem 1.

**Theorem 1.**

$$\lim_{\sigma\to 0}\mathbb{E}e^{jw\sigma i\left(\frac{\tau}{\sigma}\right)} = e^{jwx(\tau)}, \tag{6}$$

*where $x(\tau)$ is the solution of:*

$$x'(\tau) = \mathbf{r}'(0,x)\mathbf{I}_1\mathbf{e} - x(\tau)\mathbf{r}\mathbf{I}_0\mathbf{e}. \tag{7}$$

*Vector $\mathbf{r}(z,x)$ is the joint probability distribution of the server state and the residual time of the next call arrival and $\mathbf{r}(x) = \mathbf{r}(\infty,x)$. Vector $\mathbf{r}'(0,x)$ satisfies:*

$$\mathbf{r}'(0,x)(\mathbf{I} - A^*(s_n)[\mathbf{I}_1 + \mathbf{I}_2])\mathbf{v}_n = 0,$$

$$\mathbf{r}'(0,x)\mathbf{e} = \frac{1}{a_1}. \tag{8}$$

*Here, $s_n$ and $\mathbf{v}_n$ are the eigenvalues and eigenvectors of the matrix $x(\mathbf{I}_0 - \mathbf{I}_2) - \mathbf{Q}$; $a_1$ is the first raw moment of the distribution $A(x)$. Function $A^*(s)$ is the Laplace–Stieltjes transform of the probability function $A(x)$.*

*The components $r(n,z,x)$ of the vector $\mathbf{r}(z,x)$ satisfy the normalization condition:*

$$\sum_{n=0}^{N} r(n,\infty,x) = \mathbf{r}(x)\mathbf{e} = 1, \tag{9}$$

*and:*

$$\mathbf{r}(x)\{\mathbf{Q} - x(\mathbf{I}_0 - \mathbf{I}_2)\} = \mathbf{r}'(0,x)(\mathbf{I}_0 - \mathbf{I}_2). \tag{10}$$

**Proof.** Denoting:

$$\mathbf{F}(w,z,\tau) = \lim_{\varepsilon\to 0}\mathbf{F}(w,z,\tau,\varepsilon),$$

we take the limit by $\varepsilon \to 0$ in (5) to have:

$$\frac{\partial \mathbf{F}(w,z,\tau)}{\partial z} - \frac{\partial \mathbf{F}(w,0,\tau)}{\partial z} + j\frac{\partial \mathbf{F}(w,z,\tau)}{\partial w}(\mathbf{I}_0 - \mathbf{I}_2)$$

$$+A(z)\frac{\partial \mathbf{F}(w,0,\tau)}{\partial z}(\mathbf{I}_2 + \mathbf{I}_1) + \mathbf{F}(w,z,\tau)\mathbf{Q} = 0,$$

$$\frac{\partial \mathbf{F}(w,\tau)}{\partial \tau}\mathbf{e} = jw\left\{j\frac{\partial \mathbf{F}(w,\tau)}{\partial w}\mathbf{I}_0 + \frac{\partial \mathbf{F}(w,0,\tau)}{\partial z}\mathbf{I}_1\right\}\mathbf{e}. \tag{11}$$

We present the solution of (11) in the form:

$$\mathbf{F}(w,z,\tau) = \mathbf{r}(z,x)e^{jwx(\tau)}. \tag{12}$$

Here, $\mathbf{r}(z,x)$ is a row vector with components $r(n,z,x)$ and $x = x(\tau)$ is a scalar function, which is determined later. The components $r(n,z,x)$ of the vector $\mathbf{r}(z,x)$ have the meaning of the limit by $\varepsilon \to 0$ of the joint probabilities that the server is in the $n$-th state and the residual time of arrival is less than $z$. We also denote $\mathbf{r}'(z,x) = \dfrac{\partial \mathbf{r}(z,x)}{\partial z}$. Thereby, we obtain:

$$\mathbf{r}'(z,x) - \mathbf{r}(z,x)[x(\mathbf{I}_0 - \mathbf{I}_2) - \mathbf{Q}] = \mathbf{r}'(0,x)[\mathbf{I} - A(z)(\mathbf{I}_2 + \mathbf{I}_1)], \tag{13}$$

$$x'(\tau) = \mathbf{r}'(0,x)\mathbf{I}_1\mathbf{e} - x(\tau)\mathbf{r}(x)\mathbf{I}_0\mathbf{e}, \tag{14}$$

where (14) is derived by substituting (12) into the second equation of (11). Equation (14) coincides with (7).

In order to obtain (8), we apply the Laplace–Stieltjes transform to (13). Denoting:

$$\mathbf{r}^*(s,x) = \int_0^\infty e^{-sz} d\mathbf{r}_z(z,x), \; A^*(s) = \int_0^\infty e^{-sz} dA(z),$$

we obtain the system of linear equations:

$$\mathbf{r}^*(s,x)\{s\mathbf{I} - [x(\mathbf{I}_0 - \mathbf{I}_2) - \mathbf{Q}]\} = \mathbf{r}'(0,x)[\mathbf{I} - A^*(s)(\mathbf{I}_2 + \mathbf{I}_1)]. \tag{15}$$

Here:

$$\int_0^\infty e^{-sz} d\mathbf{r}'(z,x) = \mathbf{r}'(z,x)e^{-sz}\Big|_0^\infty - \int_0^\infty \mathbf{r}'(z,x) de^{-sz} = -\mathbf{r}'(0,x) + s\mathbf{r}^*(s,x).$$

The solution of (15) depends on values of the vector $\mathbf{r}'(0,x)$. We denote $s_n = s_n(x)$ and $\mathbf{v}_n = \mathbf{v}_n(x)$, $n = 1,2,\ldots,N$ as the eigenvalues and eigenvectors of the matrix $x(\mathbf{I}_0 - \mathbf{I}_2) - \mathbf{Q}$. Since $\det(x(\mathbf{I}_0 - \mathbf{I}_2) - \mathbf{Q}) = 0$, the value $s_0 = 0$ is an eigenvalue of this matrix. Thus, the vector $[\mathbf{I} - A^*(s_0)(\mathbf{I}_2 + \mathbf{I}_1)]\mathbf{v}_0 = 0$ since $A^*(s_0) = A^*(0) = 1$, $(\mathbf{I}_1 + \mathbf{I}_2)\mathbf{e} = \mathbf{e}$, and $\mathbf{v}_0 = C\mathbf{e}$. Considering that:

$$\{s_n\mathbf{I} - [x(\mathbf{I}_0 - \mathbf{I}_2) - \mathbf{Q}]\}\mathbf{v}_n = 0, \tag{16}$$

we obtain the system of linear equations for $\mathbf{r}'(0,x)$:

$$\mathbf{r}'(0,x)[\mathbf{I} - A^*(s_n)(\mathbf{I}_2 + \mathbf{I}_1)]\mathbf{v}_n = 0, \; n = 0,\ldots,N. \tag{17}$$

The determinant of the matrix of the system (17) is zero. In order to obtain a single solution, we take the derivative of (15) with respect to $s$:

$$\mathbf{r}^{*\prime}(s,x)\{s\mathbf{I} - [x(\mathbf{I}_0 - \mathbf{I}_2) - \mathbf{Q}]\} + \mathbf{r}^*(s,x) = -\mathbf{r}'(0,x)A^{*\prime}(s)(\mathbf{I}_2 + \mathbf{I}_1).$$

In the last equation, we set $s = 0$ and multiply the result by unit vector $\mathbf{e}$. Denoting $a_1$ as the mean length of the interval in the renewal input process and taking into account that $\mathbf{Q}\mathbf{e} = 0$, $\mathbf{I}_0\mathbf{e} = \mathbf{I}_2\mathbf{e}$, $(\mathbf{I}_1 + \mathbf{I}_2)\mathbf{e} = \mathbf{e}$, $\mathbf{r}^*(0,x) = \mathbf{r}(x)$, we derive an additional condition:

$$\mathbf{r}'(0,x)\mathbf{e} = \frac{1}{a_1}.$$

Thus, we obtain (8) for the components of the vector $\mathbf{r}'(0,x)$. Substituting $\mathbf{r}'(0,x)$ into (13) and taking into account the normalization condition (9), we obtain (10).

Taking the limit by $z \to \infty$ in (12) and multiplying the result by $\mathbf{e}$, we can write:

$$\mathbf{F}(w,\infty,\tau)\mathbf{e} = \mathbf{r}(\infty,x)\mathbf{e}e^{jwx(\tau)} = e^{jwx(\tau)}.$$

Here, $\mathbf{F}(w,\infty,\tau)\mathbf{e}$ is the limit by $\sigma \to 0$ of the characteristic function $\mathbf{H}(u,\infty,t)\mathbf{e} = \mathbb{E}e^{jui(t)}$, which coincides with (6). The theorem is proven. $\square$

We denote:

$$a(x) = x'(\tau) = \mathbf{r}'(0,x)\mathbf{I}_1\mathbf{e} - x\mathbf{r}(x)\mathbf{I}_0\mathbf{e}. \tag{18}$$

In (18), $\mathbf{r}'(0,x)\mathbf{I}_1\mathbf{e}$ represents the rate of blocked calls that come into the orbit and the second term $x\mathbf{r}(x)\mathbf{I}$ represents the rate of retrial calls from the orbit that successfully enter the idle server.

Function $a(x)$ has the meaning of the drift coefficient of a certain diffusion process that determines the distribution of a scaled number of calls in the orbit.

## 5. Second Step of Asymptotic-Diffusion Analysis

In (3) and (4), we make the following substitutions:

$$\mathbf{H}(u,z,t) = e^{ju\frac{x(\sigma t)}{\sigma}}\mathbf{H}^{(1)}(u,z,t), \tag{19}$$

to obtain the following system of equations.

$$\begin{gathered}
\frac{\partial \mathbf{H}^{(1)}(u,z,t)}{\partial t} + jua(x)\mathbf{H}^{(1)}(u,z,t) = \frac{\partial \mathbf{H}^{(1)}(u,z,t)}{\partial z} - \frac{\partial \mathbf{H}^{(1)}(u,0,t)}{\partial z} \\
+\mathbf{H}^{(1)}(u,z,t)(\mathbf{Q} - x(\mathbf{I}_0 - e^{-ju}\mathbf{I}_2)) \\
+j\sigma\frac{\partial \mathbf{H}^{(1)}(u,z,t)}{\partial u}(\mathbf{I}_0 - e^{-ju}\mathbf{I}_2) + A(z)\frac{\partial \mathbf{H}^{(1)}(u,0,t)}{\partial z}(\mathbf{I}_2 + e^{ju}\mathbf{I}_1), \\
\frac{\partial \mathbf{H}^{(1)}(u,t)}{\partial t}\mathbf{e} + jua(x)\mathbf{H}^{(1)}(u,t)\mathbf{e} = (e^{ju} - 1) \\
\times \left\{ j\sigma e^{-ju}\frac{\partial \mathbf{H}^{(1)}(u,t)}{\partial u}\mathbf{I}_0 + \frac{\partial \mathbf{H}^{(1)}(u,0,t)}{\partial z}\mathbf{I}_1 - xe^{-ju}\mathbf{H}^{(1)}(u,t)\mathbf{I}_0 \right\}\mathbf{e}.
\end{gathered} \tag{20}$$

It should be noted that:

$$\mathbf{H}^{(1)}(u,z,t) = \mathbb{E}e^{ju\left(i(t) - \frac{x(\sigma t)}{\sigma}\right)},$$

which represents the characteristic function of the centered process $i(t) - \frac{x(\sigma t)}{\sigma}$.

Denoting $\sigma = \varepsilon^2$, we introduce the following notations:

$$\tau = t\varepsilon^2,\ u = w\varepsilon,\ \mathbf{H}^{(1)}(u,t) = \mathbf{F}^{(1)}(w,\tau,\varepsilon),\ \mathbf{H}^{(1)}(u,z,t) = \mathbf{F}^{(1)}(w,z,\tau,\varepsilon) \tag{21}$$

and rewrite (20) as:

$$\begin{gathered}
jw\varepsilon a(x)\mathbf{F}^{(1)}(w,z,\tau,\varepsilon) = \frac{\partial \mathbf{F}^{(1)}(w,z,\tau,\varepsilon)}{\partial z} - \frac{\partial \mathbf{F}^{(1)}(w,0,\tau,\varepsilon)}{\partial z} \\
+j\varepsilon\frac{\partial \mathbf{F}^{(1)}(w,z,\tau,\varepsilon)}{\partial w}(\mathbf{I}_0 - \mathbf{I}_2) + \mathbf{F}^{(1)}(w,z,\tau,\varepsilon)\{\mathbf{Q} - x[\mathbf{I}_0 - \mathbf{I}_2 + jw\varepsilon\mathbf{I}_2]\} \\
+A(z)\frac{\partial \mathbf{F}^{(1)}(w,0,\tau,\varepsilon)}{\partial z}(\mathbf{I}_2 + \mathbf{I}_1 + jw\varepsilon\mathbf{I}_1) + O(\varepsilon^2), \\
\varepsilon^2\frac{\partial \mathbf{F}^{(1)}(w,\tau,\varepsilon)}{\partial \tau}\mathbf{e} + jw\varepsilon a(x)\mathbf{F}^{(1)}(w,\tau,\varepsilon)\mathbf{e} \\
= (e^{jw\varepsilon} - 1)\Bigg\{ j\varepsilon e^{-jw\varepsilon}\frac{\partial \mathbf{F}^{(1)}(w,\tau,\varepsilon)}{\partial w}\mathbf{I}_0 \\
+\frac{\partial \mathbf{F}^{(1)}(w,0,\tau,\varepsilon)}{\partial z}\mathbf{I}_1 - xe^{-jw\varepsilon}\mathbf{F}^{(1)}(w,\tau,\varepsilon)\mathbf{I}_0 \Bigg\}\mathbf{e}.
\end{gathered} \tag{22}$$

**Theorem 2.** *$\mathbf{F}^{(1)}(w,\tau) = \lim\limits_{\varepsilon\to 0}\mathbf{F}^{(1)}(w,\tau,\varepsilon)$ is given by:*

$$\mathbf{F}^{(1)}(w,\tau) = \Phi(w,\tau)\mathbf{r}(x), \tag{23}$$

*where $\mathbf{r}(x)$ is given in Theorem 1 and $\Phi(w,\tau)$ is the solution of:*

$$\frac{\partial \Phi(w,\tau)}{\partial \tau} = a'(x)w\frac{\partial \Phi(w,\tau)}{\partial w} + \frac{(jw)^2}{2}b(x)\Phi(w,\tau). \tag{24}$$

*$a(x)$ is given by (18), and $b(x)$ is given by:*

$$b(x) = a(x) + 2\big(\mathbf{g}'(0,x)\mathbf{I}_1 - x\mathbf{g}(x)\mathbf{I}_0 + x\mathbf{r}(x)\mathbf{I}_0\big)\mathbf{e}, \tag{25}$$

*where vector $\mathbf{g}(x)$ with components $\lim_{z\to\infty} g(n,z,x)$ is defined by:*

$$\mathbf{g}(x)(\mathbf{Q} - x(\mathbf{I}_0 - \mathbf{I}_2)) = \mathbf{g}'(0,x)(\mathbf{I}_0 - \mathbf{I}_2) + \mathbf{r}(x)(a(x)\mathbf{I} + x\mathbf{I}_2) - \mathbf{r}'(0,x)\mathbf{I}_1,$$
$$\mathbf{g}(x)\mathbf{e} = 0. \tag{26}$$

*Vector $\mathbf{g}'(0,x)$ is the solution of:*

$$\mathbf{g}'(0,x)(\mathbf{I} - A^*(s_n)[\mathbf{I}_1 + \mathbf{I}_2])\mathbf{v}_n = -A^*(s_n)\mathbf{r}'(0,x)\mathbf{I}_1\mathbf{v}_n - \mathbf{r}^*(s_n)[a(x)\mathbf{I} + x\mathbf{I}_2]\mathbf{v}_n,$$
$$\mathbf{g}'(0,x)\mathbf{e} = -\frac{1}{a_1}\Big\{\mathbf{r}^{*\prime}(0)[a(x)\mathbf{I} + x\mathbf{I}_2] + a_1\mathbf{r}'(0,x)\mathbf{I}_1\Big\}\mathbf{e}. \tag{27}$$

*Here, $s_n$ and $\mathbf{v}_n$ are the eigenvalues and eigenvectors of the matrix $x(\mathbf{I}_0 - \mathbf{I}_2) - \mathbf{Q}$. Vector $\mathbf{r}^*(s_n)$ is defined by:*

$$\mathbf{r}^*(s_n)(s_n\mathbf{I} - (x(\mathbf{I}_0 - \mathbf{I}_2) - \mathbf{Q})) = \mathbf{r}'(0,x)(\mathbf{I} - A^*(s_n)[\mathbf{I}_1 + \mathbf{I}_2]),$$
$$\mathbf{r}^*(s_n)\mathbf{v}_n = -\mathbf{r}'(0,x)A^{*\prime}(s_n)[\mathbf{I}_1 + \mathbf{I}_2]\mathbf{v}_n. \tag{28}$$

*Vector $\mathbf{r}^{*\prime}(0)$ is defined by:*

$$\mathbf{r}^{*\prime}(0)(\mathbf{Q} - x[\mathbf{I}_0 + \mathbf{I}_2]) = a_1\mathbf{r}'(0,x)[\mathbf{I}_1 + \mathbf{I}_2] - \mathbf{r}(x),$$
$$\mathbf{r}^{*\prime}(0)\mathbf{e} = -\frac{a_2}{2a_1}, \tag{29}$$

*where $a_1$ and $a_2$ are first and second raw moments of $A(x)$.*

**Proof.** For the solution of the first equation of the system (22), we propose to seek in the following form:

$$\mathbf{F}^{(1)}(w,\tau,\varepsilon) = \Phi(w,\tau)\{\mathbf{r}(z,x) + jw\varepsilon\mathbf{f}(z,x)\} + O(\varepsilon^2), \tag{30}$$

where $\Phi(w,\tau)$ is a scalar function, which we obtain in further analysis. Substituting (30) into the first equation of the system (22) and taking (13) into account, we obtain the following equation in the limit by $\varepsilon \to 0$:

$$\mathbf{f}'(z,x) + \mathbf{f}(z,x)[\mathbf{Q} - x(\mathbf{I}_0 - \mathbf{I}_2)]$$
$$= \mathbf{r}(z,x)[a(x)\mathbf{I} - x\mathbf{I}_2] + \mathbf{f}'(0,x)[\mathbf{I} - A(z)(\mathbf{I}_1 + \mathbf{I}_2)] + A(z)\mathbf{r}'(0,x)\mathbf{I}_1$$
$$- \frac{\partial\Phi(w,\tau)/\partial w}{w\Phi(w,\tau)}\mathbf{r}(z,x)(\mathbf{I}_0 - \mathbf{I}_2). \tag{31}$$

We express the solution of the equation as:

$$\mathbf{f}(z,x) = C\mathbf{r}(z,x) + \mathbf{g}(z,x) - \boldsymbol{\varphi}(z,x)\frac{\partial\Phi(w,\tau)/\partial w}{w\Phi(w,\tau)}, \tag{32}$$

which we substitute in (31) to have:

$$\boldsymbol{\varphi}'(z,x) + \boldsymbol{\varphi}(z,x)[\mathbf{Q} - x(\mathbf{I}_0 - \mathbf{I}_2)]$$
$$- \boldsymbol{\varphi}(0,x)[\mathbf{I} - A(z)(\mathbf{I}_1 + \mathbf{I}_2)] = \mathbf{r}(z,x)(\mathbf{I}_0 - \mathbf{I}_2), \tag{33}$$

$$\begin{aligned}&\mathbf{g}'(z,x)+\mathbf{g}(z,x)[\mathbf{Q}-x(\mathbf{I}_0-\mathbf{I}_2)]\\&-\mathbf{g}'(0,x)[\mathbf{I}-A(z)(\mathbf{I}_1+\mathbf{I}_2)]=\mathbf{r}(z,x)[a(x)\mathbf{I}+x\mathbf{I}_2]-A(z)\mathbf{r}'(0,x)\mathbf{I}_1.\end{aligned}\tag{34}$$

Because (33) is the derivative of (13), we conclude that:

$$\boldsymbol{\varphi}(z,x)=\frac{\partial\mathbf{r}(z,x)}{\partial x},\tag{35}$$

and for the vector $\boldsymbol{\varphi}(z,x)$, we have the additional condition $\boldsymbol{\varphi}(\infty,x)\mathbf{e}=0$. For $\mathbf{g}(z,x)$, we also define the additional condition $\mathbf{g}(\infty,x)\mathbf{e}=0$.

The next stage of the analysis is devoted to obtaining vectors $\mathbf{g}'(0,x)$ and $\mathbf{g}(x)=\mathbf{g}(\infty,x)$. We apply the Laplace–Stieltjes transform to (34) and write the equation:

$$\begin{aligned}&\mathbf{g}^*(s,x)\{s\mathbf{I}-[x(\mathbf{I}_0-\mathbf{I}_2)-\mathbf{Q}]\}\\&=\mathbf{g}'(0,x)[\mathbf{I}-A^*(s)(\mathbf{I}_2+\mathbf{I}_1)]+\mathbf{r}^*(s,x)[a(x)\mathbf{I}+x\mathbf{I}_2]-\mathbf{r}'(0,x)A^*(s)\mathbf{I}_1.\end{aligned}\tag{36}$$

Since the following equation is true:

$$\{s_n\mathbf{I}-[x(\mathbf{I}_0-\mathbf{I}_2)-\mathbf{Q}]\}\mathbf{v}_n=0,\ n=0,\dots,N,\tag{37}$$

where $s_n$ and $\mathbf{v}_n$ are the eigenvalues and eigenvectors of the matrix $x(\mathbf{I}_0-\mathbf{I}_2)-\mathbf{Q}$, and we derive the equation for the vector $\mathbf{g}'(0,x)$:

$$\begin{aligned}&\mathbf{g}'(0,x)[\mathbf{I}-A^*(s_n)(\mathbf{I}_2+\mathbf{I}_1)]\mathbf{v}_n\\&=\mathbf{r}'(0,x)A^*(s_n)\mathbf{I}_1\mathbf{v}_n-\mathbf{r}^*(s_n,x)[a(x)\mathbf{I}+x\mathbf{I}_2]\mathbf{v}_n,\ n=0,\dots,N.\end{aligned}\tag{38}$$

We also obtain the additional condition for the vector $\mathbf{g}'(0,x)$ as the derivative of (36) multiplied by unit vector $\mathbf{e}$ at the point zero:

$$\mathbf{g}'(0,x)\mathbf{e}=-\frac{1}{a_1}\{\mathbf{r}^{*\prime}(0,x)[a(x)\mathbf{I}+x\mathbf{I}_2]\mathbf{e}-a_1\mathbf{r}'(0,x)\mathbf{I}_1\mathbf{e}\}.\tag{39}$$

To complete the analysis, we need to obtain the vectors $\mathbf{r}^{*\prime}(0,x)$ and $\mathbf{r}^*(s_n,x)$, $n=0,\dots,N$. We differentiate (15) to have:

$$\mathbf{r}^{*\prime}(s,x)\{s_n\mathbf{I}-[x(\mathbf{I}_0-\mathbf{I}_2)-\mathbf{Q}]\}+\mathbf{r}^*(s,x)=-\mathbf{r}'(0,x)A^{*\prime}(s)(\mathbf{I}_1+\mathbf{I}_2).\tag{40}$$

We multiply the equation by eigenvector $\mathbf{v}_n=\mathbf{v}_n(x)$ and substitute the eigenvalues $s_n$:

$$\mathbf{r}^*(s_n,x)\mathbf{v}_n=-\mathbf{r}'(0,x)A^{*\prime}(s_n)(\mathbf{I}_1+\mathbf{I}_2)\mathbf{v}_n,\ n=0,\dots,N.\tag{41}$$

After that, we differentiate (40) by $s$ and set $s=0$ in order to obtain an additional equation:

$$\mathbf{r}^{*\prime}(0,x)\mathbf{e}=-\frac{a_2}{2a_1}.\tag{42}$$

Finally, we obtain the system of equations for the vector $\mathbf{g}(x)$ by setting $s=0$ in (36) and taking into account that $\mathbf{g}^*(0,x)=\mathbf{g}(\infty,x)=\mathbf{g}(x)$, $\mathbf{g}(x)\mathbf{e}=0$:

$$\begin{aligned}&\mathbf{g}(x)[\mathbf{Q}-x(\mathbf{I}_0-\mathbf{I}_2)]\\&=\mathbf{g}'(0,x)(\mathbf{I}_0-\mathbf{I}_2)+\mathbf{r}(x)[a(x)\mathbf{I}-x\mathbf{I}_2]-\mathbf{r}'(0,x)\mathbf{I}_1,\\&\mathbf{g}(x)\mathbf{e}=0.\end{aligned}\tag{43}$$

In order to derive (24), we consider the second equation of (22) with the substitution (30) up to $O(\varepsilon^3)$:

$$\varepsilon^2\frac{\partial\Phi(w,\tau)}{\partial\tau}+jw\varepsilon a(x)\Phi(w,\tau)+(jw\varepsilon)^2a(x)\Phi(w,\tau)\mathbf{fe}$$

$$=jw\varepsilon\Phi(w,\tau)\left\{j\varepsilon\frac{\partial\Phi(w,\tau)/\partial w}{\Phi(w,\tau)}\mathbf{r}(x)\mathbf{I}_0\right.$$

$$\left.+[\mathbf{r}'(0,x)+jw\varepsilon\mathbf{f}'(0,x)]\mathbf{I}_1+jw\varepsilon x\mathbf{r}(x)\mathbf{I}_0-x[\mathbf{r}(x)+jw\varepsilon\mathbf{f}(x)]\mathbf{I}_0\right\}\mathbf{e}$$

$$+\frac{(jw\varepsilon)^2}{2}\Phi(w,\tau)[\mathbf{r}'(0,x)\mathbf{I}_1-x\mathbf{r}(x)\mathbf{I}_0]\mathbf{e}+O(\varepsilon^3).$$

Making simple transformations and taking the limit by $\varepsilon\to0$, we obtain:

$$\frac{\partial\Phi(w,\tau)}{\partial\tau}+(jw)^2a(x)\Phi(w,\tau)\mathbf{f}(x)\mathbf{e}=\frac{(jw)^2}{2}\Phi(w,\tau)a(x)$$

$$+(jw)^2\Phi(w,\tau)\left\{\frac{\partial\Phi(w,\tau)/\partial w}{w\Phi(w,\tau)}\mathbf{r}(x)\mathbf{I}_0+\mathbf{f}'(0,x)\mathbf{I}_1+x\mathbf{r}(x)\mathbf{I}_0-x\mathbf{f}(x)\mathbf{I}_0\right\}\mathbf{e}.$$

Substituting the solution (32) into the obtained equation, we have:

$$\frac{\partial\Phi(w,\tau)}{\partial\tau}=\frac{(jw\varepsilon)^2}{2}\Phi(w,\tau)\{2[\mathbf{g}'(0,x)\mathbf{I}_1-x\mathbf{g}(x)\mathbf{I}_0+x\mathbf{r}(x)\mathbf{I}_0]\mathbf{e}+a(x)\}$$

$$+w\frac{\partial\Phi(w,\tau)}{\partial w}\{-\mathbf{r}(x)\mathbf{I}_0+\boldsymbol{\varphi}'(0,x)\mathbf{I}_1-x\boldsymbol{\varphi}(x)\mathbf{I}_0\}\mathbf{e}. \tag{44}$$

We denote:

$$b(x)=a(x)+2(\mathbf{g}'(0,x)\mathbf{I}_1-x\mathbf{g}(x)\mathbf{I}_0+x\mathbf{r}(x)\mathbf{I}_0)\mathbf{e}. \tag{45}$$

Because the expression $[-\mathbf{r}(x)\mathbf{I}_0+\boldsymbol{\varphi}'(0,x)\mathbf{I}_1-x\boldsymbol{\varphi}(x)\mathbf{I}_0]\mathbf{e}$ is the derivative of $a(x)$ with respect to $x$, we have:

$$\frac{\partial\Phi(w,\tau)}{\partial\tau}=a'(x)w\frac{\partial\Phi(w,\tau)}{\partial w}+b(x)\frac{(jw)^2}{2}\Phi(w,\tau), \tag{46}$$

which coincides with (24). The theorem is proven. □

Function $b(x)$ has the meaning of the diffusion coefficient of a certain diffusion process that determines the distribution of the scaled number of calls in the orbit.

We present a procedure to build an approximation to the stationary distribution of the number of customers in orbit. This general procedure has been used for some related models also [9].

Equation (46) is the Fourier transform of the Fokker–Planck equation for the probability density $p(y,\tau)$ of $y(\tau)$, which is the limit of $\sqrt{\sigma}\left\{i\left(\frac{\tau}{\sigma}\right)-\frac{x(\tau)}{\sigma}\right\}$. We apply the inverse Fourier transform to (44):

$$\frac{\partial p(y,\tau)}{\partial\tau}=-\frac{\partial}{\partial y}\{a'(x)yp(y,\tau)\}+\frac{1}{2}\frac{\partial^2}{\partial y^2}\{b(x)p(y,\tau)\}. \tag{47}$$

The process $y(\tau)$ is the solution of:

$$dy(\tau)=a'(x)yd\tau+\sqrt{b(x)}dw(\tau). \tag{48}$$

We denote:

$$z(\tau) = x(\tau) + \varepsilon y(\tau), \tag{49}$$

where $\varepsilon = \sqrt{\sigma}$. It is noted that $z(\tau) = \lim_{\sigma \to 0} \sigma i(\tau/\sigma)$. Using $dx(\tau) = a(x)d\tau$, we obtain:

$$dz(\tau) = d(x(\tau) + \varepsilon y(\tau)) = (a(x) + \varepsilon y a'(x))d\tau + \varepsilon\sqrt{b(x)}dw(\tau). \tag{50}$$

Using decompositions:

$$a(z) = a(x + \varepsilon y) = a(x) + \varepsilon y a'(x) + o(\varepsilon^2),$$

$$\varepsilon\sqrt{b(z)} = \varepsilon\sqrt{b(x+\varepsilon y)} = \varepsilon\sqrt{b(x) + o(\varepsilon)} = \varepsilon\sqrt{b(x)} + o(\varepsilon^2),$$

we rewrite (50) in the following form up to $o(\varepsilon^2)$:

$$dz(\tau) = a(z)d\tau + \sqrt{\sigma b(z)}dw(\tau). \tag{51}$$

Let $S(z)$ denote the stationary probability density of the process $z(\tau)$.

**Lemma 1.** *The stationary probability density $S(z)$ of $z(\tau)$ has the form:*

$$S(z) = \frac{C}{b(z)} \exp\left\{ \frac{2}{\sigma} \int_0^z \frac{a(x)}{b(x)} dx \right\}, \tag{52}$$

*where $C$ is normalizing constant.*

**Proof.** Since the process $z(\tau)$ is the solution of (51), this process is a diffusion process with drift coefficient $a(z)$ and diffusion coefficient $b(z)$. Hence, the stationary probability density is the solution of the Fokker–Planck equation:

$$-\frac{\partial}{\partial z}\{a(z)S(z)\} + \frac{1}{2}\frac{\partial^2}{\partial z^2}\{\sigma b(z)S(z)\} = 0.$$

The solution of the obtained equation is given by:

$$S(z) = \frac{C}{b(z)} \exp\left\{ \frac{2}{\sigma} \int_0^z \frac{a(x)}{b(x)} dx \right\},$$

which coincides with (52). The lemma is proven. □

To build the diffusion approximation of the process $i(t)$, we use formula:

$$P(i) = \frac{S(\sigma i)}{\sum\limits_{n=0}^{\infty} S(\sigma n)}. \tag{53}$$

Thus, there is no need to specify the value of the normalizing constant $C$.

## 6. The Algorithm for $P(i)$ and Numerical Examples

### *6.1. Algorithm*

Our analysis immediately implies an algorithm to calculate an approximation to the distribution of the number of calls in orbit, i.e., $P(i)$ in (53). The algorithm is implemented as follows:

1. Define $\sigma,\ \mu_1,\ \mu_n,\ \alpha_n,\ n = 2, ..., N$;
2. Define the distribution function $A(x)$ of interarrival times, and calculate its first and second raw moments $a_1$ and $a_2$;
3. Define matrices $\mathbf{I}_0,\ \mathbf{I}_1,\ \mathbf{I}_2$, and form matrix $\mathbf{Q}$ using the initial parameters;
4. Calculate vector $\mathbf{r}'(0, x)$ as the solution of (8) for each $x$;
5. Calculate vector $\mathbf{r}(x)$ as a vector of functions, which is the solution of (9) and (10) for each $x$;
6. Define the drift coefficient $a(x)$ as a function of $x$ using (18);
7. Obtain vector $\mathbf{r}^{*\prime}(0)$ as the solution of (29) for each $x$;
8. Define $\mathbf{g}'(0, x)$ as a vector of functions, which is the solution of (27) for each $x$;
9. Calculate vector $\mathbf{g}(x)$ as the solution of (26) for each $x$;
10. Define the diffusion coefficient $b(x)$ as a function of $x$ using (25);
11. Build the discrete approximation of the probability distribution $P(i)$ of the number of calls in orbit using (53).

### *6.2. Numerical Examples*

For the numerical examples, we set $N = 4$, $\mu_1 = 4.5$, $\mu_2 = 2$, $\mu_3 = 4$, $\mu_4 = 5$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\alpha_4 = 4$. Further, we considered several distributions of interarrival times keeping the mean constant (equal to one):

1. Exponential distribution of interarrival times:
   We assumed that interarrival times of the input flow follow an exponential distribution with mean $\dfrac{1}{\lambda} = 1$, and in this case, the coefficient of variation $(CV)$ is given by $CV = 1$ (the first two graphs in Figure 1);
2. Gamma distribution of interarrival times:
   We assumed that the interarrival times follow a Gamma distribution with shape parameter $k = 0.5$ and rate parameter $\theta = 0.5$. In this case, the mean is given by $\dfrac{k}{\theta} = 1$ and $CV = \sqrt{2}$ (the next two graphs in Figure 1);
3. Hyperexponential distribution of interarrival times:
   We assumed that the interarrival times follow a hyperexponential distribution mixing of two exponential distributions with rates $\lambda_1 = 0.25$ and $\lambda_2 = 4$ and weights $q = 0.2$ and $1 - q = 0.8$. In this case, the mean and $CV$ are given by $\dfrac{q}{\lambda_1} + \dfrac{1-q}{\lambda_2} = 1$ and $CV = \sqrt{5.5}$ (the next two graphs in Figure 1);
4. Deterministic interarrival times:
   We assumed that calls arrive in fixed interarrivals with mean one, and thus, $CV = 0$ (the last two graphs in Figure 1).

In Figure 1, we show the approximation built using (53) for $\sigma = 1$ (on the left-hand side) and $\sigma = 0.1$ (on the right-hand side). The numerical results show the feasibility of our approach. For a relatively large value of $\sigma$ ($\sigma = 1$), the number of calls in orbit is distributed near zero, while it is largely distributed for a small value of $\sigma$ ($\sigma = 0.1$). This observation agrees with the intuition. We also observe that with a relatively small value of $\sigma$ ($\sigma = 0.1$), the distribution is close to a Gaussian distribution. This observation agrees with earlier results on the asymptotic distribution of the number of calls in orbit when $\sigma$ is extremely small for other models such as M/G/1 retrial queues [2].

As we can see, the variance of the number of calls in orbit depends on the variation of the interarrival times of the renewal input process. The larger the coefficient of variance of interarrivals is, the wider the shape of the distribution of the number of calls in orbit. This also agrees with the intuition.

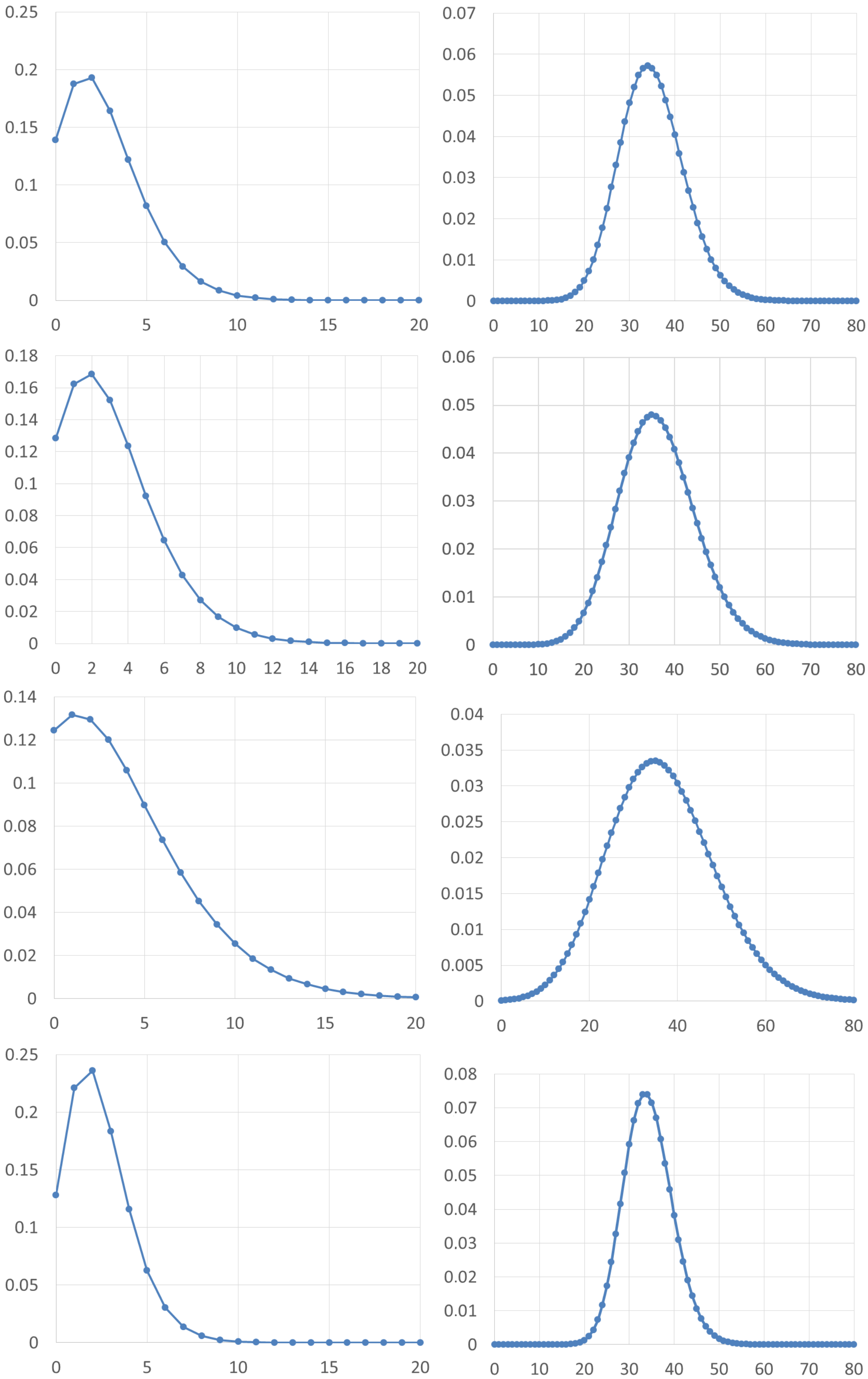


**Figure 1.** Discrete approximation for the probability distribution $P(i)$ of the number of calls in orbit, $\sigma = 1$ (**left**) and $\sigma = 0.1$ (**right**). From top to bottom exponential ($CV = 1$), gamma ($CV = \sqrt{2}$), hyperexponential ($CV = \sqrt{5.5}$), and deterministic ($CV = 0$).

## 7. Conclusions

In this paper, we derived the functional large number theorem and the functional central limit theorem for the number of customers in orbit. Based on these results, we obtained an approximation to the stationary number of customers in orbit.

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