Approximation of the Two-Dimensional Output Process of a Retrial Queue with MMPP Input

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Abstract. In this paper, we review a retrial queue with MMPP input and two-way communication. Incoming requests arriving at the server and finding it busy join the source of retrial calls and try to enter the server again after some exponentially distributed time. While idle, the server makes outgoing calls and serves them with another delay parameter. MMPP (Markov Modulated Poisson Process) is an input process in which control is driven by a continuous Markov chain. Changing its state entails a change in the intensity of the input process. For this model, we present an asymptotic approximation of the two-dimensional characteristic function under the condition of a long delay of requests in the source of retrial calls. For this approximation, we carried out a numerical experiment, where asymptotic results were compared to computations obtained via simulation.

Keywords: Output process · Retrial queue · Two-way communication · Asymptotic analysis method · Simulation · Markov modulated poisson process

1 Introduction

The specific property of RQ systems [10,16] with two-way communication [16] is the presence of different request types, which gives rise to many new service disciplines. For this reason, RQ systems with two-way communication are a powerful tool in design and optimization of real-life systems with multiple random access to a resource. Despite that these systems are well studied, their output process is still a complex and insufficiently explored area to research.

In modern telecommunication networks, there are also point processes with a varying rate of calls incoming. To simulate such jobs within the framework of queuing theory, the Markov Modulated Poisson Process (MMPP) [2,10] is used. It has a mechanism for taking into account the temporal inhomogeneity of the

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arrival rate of requests and also gives analytically processable queuing results [11]. For this reason, MMPP is widely used in Internet research, in particular, using MMPP in [13], a traffic model that accurately approximates the LRD (Long Range Dependence) characteristics of Internet traffic traces, was built. Using the concepts of sessions and streams, the proposed MMPP model simulates the actual hierarchical behaviour of Internet users generating packets. It allows traffic simulation with the desired characteristics, that have a clear physical meaning. The results prove that the queuing traffic behaviour generated by the MMPP model is consistent with the model created by the actual traces of packets collected at the edge router under various scenarios and loads.

Earlier, we presented a similar work, where a retrial queue with Poisson input process is described [3]. In this paper, we take into consideration an improved model with MMPP, which is more suitable for modelling real optimization problems. We find the approximation of the characteristic function of the number of served requests in the considered system using the method of asymptotic analysis. Subsequently, we determine the applicability of the asymptotic results by comparing them to calculations provided with simulation software, which was designed especially for this research.

2 Mathematical Model

MMPP is qualified with two matrices. Matrix of infinitesimal characteristics $\boldsymbol{Q}$ defines the state. Value q_{ij} determines the intensity of the transition of the process from the state i to the state j, and the value $-q_{ii}$ is the intensity of leaving the state i. The matrix $\boldsymbol{Q}$ has property $\sum_j q_{ij} = 0$. The diagonal matrix $\boldsymbol{\Lambda}$ specifies the rate of calls for each of the states of the process.

Let us consider the RQ system with MMPP input. An incoming request takes the server if it is idle. The server, in turn, starts serving it for some exponentially distributed time with parameter μ_1. When an incoming request cannot access the server, it travels to the source of retrial calls, where waits for exponentially distributed time with parameter σ. While free from serving incoming requests, the server produces requests itself with the intensity α and serves them with parameter μ_2.

We denote the following notations: *i(t)* is the number of requests in the orbit at the moment t, *k(t)* is the state of the server: *0*—idle, *1*—busy serving an incoming request, *2*—busy serving an outgoing request; $m_1(t)$ is the number of served input process requests at the moment t, $m_2(t)$ is the number of served outgoing requests at the moment t, $n(t)$ is the state of the input process at the moment t.

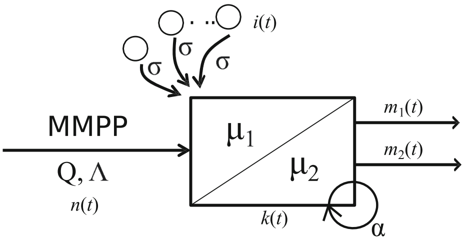

Fig. 1. RQ system with two-way communication

3 Kolmogorov Differential Equations System

We consider the five-dimensional Markov process

$$\{k(t), n(t), i(t), m_1(t), m_2(t)\}$$

Based on the formulated Markov process, we introduce probabilities

$$P\{k(t) = k, n(t) = n, i(t) = i, m_1(t) = m_1, m_2(t) = m_2\}$$

and write down for them the Kolmogorov differential equations system

$$\begin{aligned}
\frac{\partial P_0(n,i,m_1,m_2,t)}{\partial t} &= -(\lambda_n + i\sigma + \alpha)P_0(n,i,m_1,m_2,t) \\
&+ P_1(n,i,m_1-1,m_2,t)\mu_1 + P_2(n,i,m_1,m_2-1,t)\mu_2 \\
&+ \sum_{v=1}^{N} P_0(v,i,m_1,m_2,t)q_{vn}, \\
\frac{\partial P_1(n,i,m_1,m_2,t)}{\partial t} &= -(\lambda_n + \mu_1)P_1(n,i,m_1,m_2,t) \\
&+ (i+1)\sigma P_0(n,i+1,m_1,m_2,t) + \lambda_n P_0(i,m_1,m_2,t) \\
&+ \sum_{v=1}^{N} P_1(v,i,m_1,m_2,t)q_{vn}, \\
\frac{\partial P_2(n,i,m_1,m_2,t)}{\partial t} &= -(\lambda_n + \mu_2)P_2(n,i,m_1,m_2,t) \\
&+ \lambda_n P_2(n,i-1,m_1,m_2,t) + \alpha P_0(n,i,m_1,m_2,t) \\
&+ \sum_{v=1}^{N} P_2(v,i,m_1,m_2,t)q_{vn}.
\end{aligned} \tag{1}$$

Since the obtained system is infinite, we introduce the partial characteristic functions, denoting $j^2 = -1$. Such wise we passed to the system, having only three equations.

$$H_k(n,u,u_1,u_2,t) = \sum_{i=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} e^{jui}e^{ju_1m_1}e^{ju_2m_2}P_k(n,i,m_1,m_2,t).$$

We rewrite system (1) considering introduced partial characteristic functions

$$
\begin{aligned}
\frac{\partial H_0(n,u,u_1,u_2,t)}{\partial t} &= -(\lambda_n+\alpha)H_0(n,u,u_1,u_2,t)\\
&+ j\sigma\frac{\partial H_0(n,u,u_1,u_2,t)}{\partial u}\\
&+ \mu_1 e^{ju_1}H_1(n,u,u_1,u_2,t) + \mu_2 e^{ju_2}H_2(n,u,u_1,u_2,t)\\
&+ \sum_{v=1}^{N} H_0(v,u,u_1,u_2,t)q_{vn},\\
\frac{\partial H_1(n,u,u_1,u_2,t)}{\partial t} &= -(\lambda_n+\mu_1)H_1(n,u,u_1,u_2,t)\\
&- j\sigma e^{-ju}\frac{\partial H_0(n,u,u_1,u_2,t)}{\partial u}\\
&+ \lambda_n H_0(n,u,u_1,u_2,t) + \lambda_n e^{ju}H_1(n,u,u_1,u_2,t)\\
&+ \sum_{v=1}^{N} H_1(v,u,u_1,u_2,t)q_{vn},\\
\frac{\partial H_2(n,u,u_1,u_2,t)}{\partial t} &= -(\lambda_n+\mu_2)H_2(n,u,u_1,u_2,t)\\
&+ \lambda_n e^{ju}H_2(n,u,u_1,u_2,t)\\
&+ \alpha H_0(n,u,u_1,u_2,t) + \sum_{v=1}^{N} H_2(v,u,u_1,u_2,t)q_{vn}.
\end{aligned}
\tag{2}
$$

For further analysis let us denote

$$
\boldsymbol{H}_k(u,u_1,u_2,t) = \{H_k(1,u,u_1,u_2,t), H_k(2,u,u_1,u_2,t), \dots, H_k(N,u,u_1,u_2,t)\},
$$

diagonal unit matrix $\boldsymbol{I}$ with size N. Then (2) will be rewritten in the following form

$$
\begin{aligned}
\frac{\partial \boldsymbol{H}_0(u,u_1,u_2,t)}{\partial t} &= (\boldsymbol{Q}-\boldsymbol{\Lambda}-\alpha\boldsymbol{I})\boldsymbol{H}_0(u,u_1,u_2,t)\\
&+ \mu_1 e^{ju_1}\boldsymbol{H}_1(n,u,u_1,u_2,t)\\
&+ \mu_2 e^{ju_2}\boldsymbol{H}_2(u,u_1,u_2,t) + j\sigma\frac{\partial \boldsymbol{H}_0(u,u_1,u_2,t)}{\partial u},\\
\frac{\partial \boldsymbol{H}_1(u,u_1,u_2,t)}{\partial t} &= \boldsymbol{\Lambda}\boldsymbol{H}_0(u,u_1,u_2,t)\\
&+ (\boldsymbol{Q}+(e^{ju}-1)\boldsymbol{\Lambda}-\boldsymbol{I}\mu_1)\boldsymbol{H}_1(u,u_1,u_2,t)\\
&- j\sigma e^{-ju}\frac{\partial \boldsymbol{H}_0(u,u_1,u_2,t)}{\partial u},\\
\frac{\partial \boldsymbol{H}_2(u,u_1,u_2,t)}{\partial t} &= \alpha\boldsymbol{H}_0(u,u_1,u_2,t)\\
&+ (\boldsymbol{Q}+(e^{ju}-1)\boldsymbol{\Lambda}-\boldsymbol{I}\mu_2)\boldsymbol{H}_2(u,u_1,u_2,t).
\end{aligned}
\tag{3}
$$

4 Asymptotic Analysis Method

We solve the obtained system with the asymptotic analysis method with the limit condition of a long delay of requests in the orbit ($\sigma \to 0$).

Denoting $\epsilon = \sigma, u = \epsilon w, \boldsymbol{F}_k(w, u_1, u_2, t, \epsilon) = \boldsymbol{H}_k(u, u_1, u_2, t)$ we (3) as

$$\begin{aligned}
\frac{\partial \boldsymbol{F}_0(w, u_1, u_2, t, \epsilon)}{\partial t} &= (\boldsymbol{Q} - \boldsymbol{\Lambda} - \alpha \boldsymbol{I})\boldsymbol{F}_0(w, u_1, u_2, t, \epsilon) \\
&+ \mu_1 e^{ju_1}\boldsymbol{F}_1(w, u_1, u_2, t, \epsilon) + \mu_2 e^{ju_2}\boldsymbol{F}_2(w, u_1, u_2, t, \epsilon) \\
&+ j\frac{\partial \boldsymbol{F}_0(w, u_1, u_2, t, \epsilon)}{\partial w}, \\
\frac{\partial \boldsymbol{F}_1(w, u_1, u_2, t, \epsilon)}{\partial t} &= \boldsymbol{\Lambda}\boldsymbol{F}_0(w, u_1, u_2, t, \epsilon) \\
&+ (\boldsymbol{Q} + (e^{j\epsilon w} - 1)\boldsymbol{\Lambda} - \boldsymbol{I}\mu_1)\boldsymbol{F}_1(w, u_1, u_2, t, \epsilon) \\
&- je^{-j\epsilon w}\frac{\partial \boldsymbol{F}_0(w, u_1, u_2, t, \epsilon)}{\partial w}, \\
\frac{\partial \boldsymbol{F}_2(w, u_1, u_2, t, \epsilon)}{\partial t} &= \alpha\boldsymbol{F}_0(w, u_1, u_2, t, \epsilon) \\
&+ (\boldsymbol{Q} + (e^{j\epsilon w} - 1)\boldsymbol{\Lambda} - \boldsymbol{I}\mu_2)\boldsymbol{F}_2(w, u_1, u_2, t, \epsilon).
\end{aligned} \tag{4}$$

The solution for system (4) is formulated in Theorems 1 and 2.

Theorem 1. *Let $i(t)$ is the number of requests in the orbit at the moment t, then in the stationary regime we obtain*

$$\lim_{\epsilon \to 0}\{\sum_{k=0}^{2}\boldsymbol{F}_k(w, 0, 0, t, \epsilon)\} = \lim_{\sigma \to 0} Me^{jw\sigma i(t)} = e^{jw\kappa},$$

where κ is a positive root of the equation

$$\kappa \boldsymbol{R}_0(\kappa)\boldsymbol{e} = [\boldsymbol{R}_1(\kappa) + \boldsymbol{R}_2(\kappa)]\boldsymbol{\Lambda}\boldsymbol{e}.$$

Vectors $\boldsymbol{R}_k$ are defined as

$$\begin{cases}
\boldsymbol{R}_0(\kappa) = \boldsymbol{r}\{\boldsymbol{I} + [\boldsymbol{\Lambda} + \kappa\boldsymbol{I}](\mu_1\boldsymbol{I} - \boldsymbol{Q})^{-1} + \alpha(\mu_2\boldsymbol{I} - \boldsymbol{Q})^{-1}\}^{-1}, \\
\boldsymbol{R}_1(\kappa) = \boldsymbol{R}_0(\kappa)[\boldsymbol{\Lambda} + \kappa\boldsymbol{I}](\mu_1\boldsymbol{I} - \boldsymbol{Q})^{-1}, \\
\boldsymbol{R}_2(\kappa) = \alpha\boldsymbol{R}_0(\kappa)(\mu_2\boldsymbol{I} - \boldsymbol{Q})^{-1}.
\end{cases}$$

The row vector $\boldsymbol{r}$ is the stationary probability distribution of the background process $n(t)$, which is obtained as the unique solution for the system $\boldsymbol{rQ} = 0, \boldsymbol{re} = 1$.

Proof. In (4), we denoted $u_1 = u_2 = 0$, which allows us to remove processes $m_1(t)$ and $m_2(t)$ from consideration. Thus we get a system of equations yet for the three-dimensional process $\{n(t), k(t), i(t)\}$ and consider it in the stationary regime, which spares us from the time derivative t.

Let us denote

$$\boldsymbol{F}_k(w,\epsilon) = \lim_{t\to\infty} \boldsymbol{F}_k(w,0,0,t,\epsilon).$$

Then we obtain

$$\begin{aligned}
&(\boldsymbol{Q} - \boldsymbol{\Lambda} - \alpha\boldsymbol{I})\boldsymbol{F}_0(w,\epsilon) + \mu_1\boldsymbol{F}_1(w,\epsilon) + \mu_2\boldsymbol{F}_2(w,\epsilon) + j\boldsymbol{F}_0'(w,\epsilon) = 0,\\
&\boldsymbol{\Lambda}\boldsymbol{F}_0(w,\epsilon) + (\boldsymbol{Q} + (e^{j\epsilon w} - 1)\boldsymbol{\Lambda} - \boldsymbol{I}\mu_1)\boldsymbol{F}_1(w,\epsilon) - je^{-j\epsilon w}\boldsymbol{F}_0'(w,\epsilon) = 0,\\
&\alpha\boldsymbol{F}_0(w,\epsilon) + (\boldsymbol{Q} + (e^{j\epsilon w} - 1)\boldsymbol{\Lambda} - \boldsymbol{I}\mu_2)\boldsymbol{F}_2(w,\epsilon) = 0.
\end{aligned} \tag{5}$$

Making the passage to the limit $\epsilon \to 0$ in (5) results in

$$\begin{aligned}
&(\boldsymbol{Q} - \boldsymbol{\Lambda} - \alpha\boldsymbol{I})\boldsymbol{F}_0(w) + \mu_1\boldsymbol{F}_1(w) + \mu_2\boldsymbol{F}_2(w) + j\boldsymbol{F}_0'(w) = 0,\\
&\boldsymbol{\Lambda}\boldsymbol{F}_0(w) + (\boldsymbol{Q} - \boldsymbol{I}\mu_1)\boldsymbol{F}_1(w) - j\boldsymbol{F}_0'(w) = 0,\\
&\alpha\boldsymbol{F}_0(w) + (\boldsymbol{Q} - \boldsymbol{I}\mu_2)\boldsymbol{F}_2(w) = 0.
\end{aligned} \tag{6}$$

Solution for the system will be found as

$$\boldsymbol{F}_k(w) = \Phi(w)\boldsymbol{R}_k, \tag{7}$$

where $\boldsymbol{R}_n$ is the server's state stationary probability distribution, and $\Phi(w)$ is the asymptotic approximation of the characteristic function of the number of requests in the orbit under the condition of their long delay. Substituting (7) in (6) and dividing it by $\Phi(w)$, we get

$$\begin{aligned}
&(\boldsymbol{Q} - \boldsymbol{\Lambda} - \alpha\boldsymbol{I})\boldsymbol{R}_0 + \mu_1\boldsymbol{R}_1 + \mu_2\boldsymbol{R}_2 + j\frac{\Phi'(w)}{\Phi(w)}\boldsymbol{R}_0 = 0,\\
&\boldsymbol{\Lambda}\boldsymbol{R}_0 + (\boldsymbol{Q} - \boldsymbol{I}\mu_1)\boldsymbol{R}_1 - j\frac{\Phi'(w)}{\Phi(w)}\boldsymbol{R}_0 = 0,\\
&\alpha\boldsymbol{R}_0 + (\boldsymbol{Q} - \boldsymbol{I}\mu_2)\boldsymbol{R}_2 = 0.
\end{aligned} \tag{8}$$

Since w only appears in $\frac{\Phi'(w)}{\Phi(w)}$, other equation terms do not depend on w. It means that $\Phi(w)$ is exponential. Taking into account that $\Phi(w)$ has the meaning of an asymptotic approximation of the characteristic function of the number of requests in the source of retrial calls, we can clarify the form of this function

$$\frac{\Phi'(w)}{\Phi(w)} = \frac{e^{j\kappa w} j\kappa}{e^{j\kappa w}}, \tag{9}$$

which follows to $j\frac{\Phi(w)'}{\Phi(w)} = -\kappa$. Let us substitute this expression into (8). Then we obtain

$$\begin{aligned}
&(\boldsymbol{Q} - \boldsymbol{\Lambda} - \alpha\boldsymbol{I})\boldsymbol{R}_0 + \mu_1\boldsymbol{R}_1 + \mu_2\boldsymbol{R}_2 - \kappa R_0 = 0,\\
&\boldsymbol{\Lambda}\boldsymbol{R}_0 + (\boldsymbol{Q} - \boldsymbol{I}\mu_1)\boldsymbol{R}_1 + \kappa\boldsymbol{R}_0 = 0,\\
&\alpha\boldsymbol{R}_0 + (\boldsymbol{Q} - \boldsymbol{I}\mu_2)\boldsymbol{R}_2 = 0.
\end{aligned} \tag{10}$$

Let us write down the normality condition for the stationary distribution of the number of served requests

$$\boldsymbol{R}_0 + \boldsymbol{R}_1 + \boldsymbol{R}_2 = \boldsymbol{r}.$$

Based on this equation, as well as on the last two equations of (10), we write the system as

$$\begin{cases} \boldsymbol{R}_1 = \boldsymbol{R}_0[\boldsymbol{\Lambda} + \kappa \boldsymbol{I}](\mu_1 \boldsymbol{I} - \boldsymbol{Q})^{-1}, \\ \boldsymbol{R}_2 = \alpha \boldsymbol{R}_0(\mu_2 \boldsymbol{I} - \boldsymbol{Q})^{-1}, \\ \boldsymbol{R}_0 + \boldsymbol{R}_1 + \boldsymbol{R}_2 = \boldsymbol{r}. \end{cases} \tag{11}$$

Let us sum up the equations of system (5)

$$[\boldsymbol{F}_0(w,\epsilon) + \boldsymbol{F}_1(w,\epsilon) + \boldsymbol{F}_2(w,\epsilon)]\boldsymbol{Q} + \boldsymbol{F}_1(w,\epsilon)(e^{jw\epsilon} - 1)\boldsymbol{\Lambda}$$
$$+ \boldsymbol{F}_2(w,\epsilon)(e^{jw\epsilon} - 1)\boldsymbol{\Lambda} + je^{-jw\epsilon}(e^{jw\epsilon} - 1)\boldsymbol{F}'_0(w,\epsilon) = 0.$$

Multiplying the resulting equations by the unit column vector $\boldsymbol{e}$, we obtain

$$\{\boldsymbol{F}_1(w,\epsilon) + \boldsymbol{F}_2(w,\epsilon)\}\boldsymbol{\Lambda e} + je^{-jw\epsilon}\boldsymbol{F}'_0(w,\epsilon)\boldsymbol{e} = 0.$$

Then we substitute product (7) into the resulting equation

$$[\boldsymbol{R}_1 + \boldsymbol{R}_2]\boldsymbol{\Lambda e} + j\frac{\Phi'(w)}{\Phi(w)}\boldsymbol{R}_0\boldsymbol{e} = 0$$

and make the replacement

$$[\boldsymbol{R}_1 + \boldsymbol{R}_2]\boldsymbol{\Lambda e} - \kappa \boldsymbol{R}_0\boldsymbol{e} = 0. \tag{12}$$

From (12) we can express κ with $\boldsymbol{R}_0$,$\boldsymbol{R}_1$ and $\boldsymbol{R}_2$. In addition, we can rewrite system (11) as follows

$$\begin{cases} \boldsymbol{R}_0(\kappa) = \boldsymbol{r}\{\boldsymbol{I} + [\boldsymbol{\Lambda} + \kappa \boldsymbol{I}](\mu_1 \boldsymbol{I} - \boldsymbol{Q})^{-1} + \alpha(\mu_2 \boldsymbol{I} - \boldsymbol{Q})^{-1}\}^{-1}, \\ \boldsymbol{R}_1(\kappa) = \boldsymbol{R}_0(\kappa)[\boldsymbol{\Lambda} + \kappa \boldsymbol{I}](\mu_1 \boldsymbol{I} - \boldsymbol{Q})^{-1}, \\ \boldsymbol{R}_2(\kappa) = \alpha \boldsymbol{R}_0(\kappa)(\mu_2 \boldsymbol{I} - \boldsymbol{Q})^{-1} \end{cases}$$

Theorem 1 is auxiliary since the general solution for the system is stated in Theorem 2 and needs the results obtained at this stage, namely, the normalized average amount of requests in the source of retrial calls κ and the stationary probability distribution of the server's state $\boldsymbol{R}_k$.

Theorem 2. *The asymptotic approximation of the two-dimensional characteristic function of the number of served requests of the MMPP input process and the number of served outgoing requests for some time t has the form*

$$\lim_{\sigma \to 0} M\{\exp(ju_1 m_1(t))\exp(ju_2 m_2(t))\}$$
$$= \lim_{\epsilon \to 0}\{\sum_{k=0}^{2} \boldsymbol{F}_k(0, u_1, u_2, t, \epsilon)\}\boldsymbol{e} = \boldsymbol{R} \cdot \exp\{\boldsymbol{G}(u_1, u_2)t\}\boldsymbol{ee},$$

where matrix $\boldsymbol{G}(u_1, u_2)$ can be written as

$$\boldsymbol{G}(u_1,u_2) = \begin{bmatrix} \boldsymbol{Q} - \boldsymbol{\Lambda} - (\alpha+\kappa)\boldsymbol{I} & \mu_1 e^{ju_1}\boldsymbol{I} & \mu_2 e^{ju_2}\boldsymbol{I} \\ \boldsymbol{\Lambda} + \kappa\boldsymbol{I} & \boldsymbol{Q} - \mu_1\boldsymbol{I} & 0 \\ \alpha\boldsymbol{I} & 0 & \boldsymbol{Q} - \mu_2\boldsymbol{I} \end{bmatrix}^T,$$

row vector $\boldsymbol{R} = \{\boldsymbol{R}_0, \boldsymbol{R}_1, \boldsymbol{R}_2\}$ is the two-dimensional stationary probability distribution of the process $\{k(t), n(t)\}$, where $\boldsymbol{R}_k$ has dimension N, κ is the normalized average number of requests in the orbit, and e and ee are unit vector columns of dimensions N and $N \cdot K$, where K is the number of server's states.

Proof. After making the passage to the limit $\lim_{\epsilon\to 0} \boldsymbol{F}_k(w, u_1, u_2, t, \epsilon) = \boldsymbol{F}_k(w, u_1, u_2, t)$ in resulting system (4), it will be written as follows

$$\begin{aligned} \frac{\partial \boldsymbol{F}_0(w,u_1,u_2,t)}{\partial t} &= (\boldsymbol{Q} - \boldsymbol{\Lambda} - \alpha\boldsymbol{I})\boldsymbol{F}_0(w,u_1,u_2,t) \\ &+ \mu_1 e^{ju_1}\boldsymbol{F}_1(w,u_1,u_2,t) \\ &+ \mu_2 e^{ju_2}\boldsymbol{F}_2(w,u_1,u_2,t) + j\frac{\partial \boldsymbol{F}_0(w,u_1,u_2,t)}{\partial w}, \\ \frac{\partial \boldsymbol{F}_1(w,u_1,u_2,t)}{\partial t} &= \boldsymbol{\Lambda}\boldsymbol{F}_0(w,u_1,u_2,t) + (\boldsymbol{Q} - \boldsymbol{I}\mu_1)\boldsymbol{F}_1(w,u_1,u_2,t) \\ &- j\frac{\partial \boldsymbol{F}_0(w,u_1,u_2,t)}{\partial w}, \\ \frac{\partial \boldsymbol{F}_2(w,u_1,u_2,t)}{\partial t} &= \alpha\boldsymbol{F}_0(w,u_1,u_2,t) + (\boldsymbol{Q} - \boldsymbol{I}\mu_2)\boldsymbol{F}_2(w,u_1,u_2,t). \end{aligned} \tag{13}$$

Solution for (13) will be found as

$$\boldsymbol{F}_k(w,u_1,u_2,t) = \Phi(w)\boldsymbol{F}_k(u_1,u_2,t). \tag{14}$$

Substituting (14) into (13) and dividing both parts of equations by $\Phi(w)$ we obtain

$$\begin{aligned} \frac{\partial \boldsymbol{F}_0(u_1,u_2,t)}{\partial t} &= (\boldsymbol{Q} - \boldsymbol{\Lambda} - \alpha\boldsymbol{I})\boldsymbol{F}_0(u_1,u_2,t) + \mu_1 e^{ju_1}\boldsymbol{F}_1(u_1,u_2,t) \\ &+ \mu_2 e^{ju_2}\boldsymbol{F}_2(u_1,u_2,t) + j\frac{\Phi'(w)}{\Phi(w)}\boldsymbol{F}_0(u_1,u_2,t), \\ \frac{\partial \boldsymbol{F}_1(u_1,u_2,t)}{\partial t} &= \boldsymbol{\Lambda}\boldsymbol{F}_0(u_1,u_2,t) + (\boldsymbol{Q} - \boldsymbol{I}\mu_1)\boldsymbol{F}_1(u_1,u_2,t) \\ &- j\frac{\Phi'(w)}{\Phi(w)}\boldsymbol{F}_0(u_1,u_2,t), \\ \frac{\partial \boldsymbol{F}_2(u_1,u_2,t)}{\partial t} &= \alpha\boldsymbol{F}_0(u_1,u_2,t) + (\boldsymbol{Q} - \boldsymbol{I}\mu_2)\boldsymbol{F}_2(u_1,u_2,t). \end{aligned} \tag{15}$$

Function $\Phi(w)$ has form (9). After substituting it, (15) will have the form

$$\begin{aligned}\frac{\partial \boldsymbol{F}_0(u_1,u_2,t)}{\partial t} &= (\boldsymbol{Q}-\boldsymbol{\Lambda}-(\alpha+\kappa)\boldsymbol{I})\boldsymbol{F}_0(u_1,u_2,t)\\ &+\mu_1 e^{ju_1}\boldsymbol{F}_1(u_1,u_2,t)\\ &+\mu_2 e^{ju_2}\boldsymbol{F}_2(u_1,u_2,t),\\ \frac{\partial \boldsymbol{F}_1(u_1,u_2,t)}{\partial t} &= (\boldsymbol{\Lambda}+\kappa\boldsymbol{I})\boldsymbol{F}_0(u_1,u_2,t)+(\boldsymbol{Q}-\boldsymbol{I}\mu_1)\boldsymbol{F}_1(u_1,u_2,t)\\ &+0\boldsymbol{F}_2(u_1,u_2,t),\\ \frac{\partial \boldsymbol{F}_2(u_1,u_2,t)}{\partial t} &= \alpha\boldsymbol{F}_0(u_1,u_2,t)+0\boldsymbol{F}_1(u_1,u_2,t)\\ &+(\boldsymbol{Q}-\boldsymbol{I}\mu_2)\boldsymbol{F}_2(u_1,u_2,t).\end{aligned} \tag{16}$$

Let us denote

$$\boldsymbol{FF}(u_1,u_2,t)=\{\boldsymbol{F}_0(u_1,u_2,t),\boldsymbol{F}_1(u_1,u_2,t),\boldsymbol{F}_2(u_1,u_2,t)\},$$

$$\boldsymbol{G}(u_1,u_2)=\begin{bmatrix}\boldsymbol{Q}-\boldsymbol{\Lambda}-(\alpha+\kappa)\boldsymbol{I} & \mu_1 e^{ju_1}\boldsymbol{I} & \mu_2 e^{ju_2}\boldsymbol{I}\\ \boldsymbol{\Lambda}+\kappa\boldsymbol{I} & \boldsymbol{Q}-\mu_1\boldsymbol{I} & 0\\ \alpha\boldsymbol{I} & 0 & \boldsymbol{Q}-\mu_2\boldsymbol{I}\end{bmatrix}^T,$$

$\boldsymbol{G}(u_1,u_2)$ is the transposed matrix of system coefficients (16). Then we obtain the following matrix equation

$$\frac{\partial \boldsymbol{FF}(u_1,u_2,t)}{\partial t}=\boldsymbol{FF}(u_1,u_2,t)\boldsymbol{G}(u_1,u_2),$$

the general solution of which is

$$\boldsymbol{FF}(u_1,u_2,t)=\boldsymbol{C}e^{\boldsymbol{G}(u_1,u_2)t}. \tag{17}$$

Finding a unique solution corresponding to the functioning of the system under consideration requires us to set the initial condition

$$\boldsymbol{FF}(u_1,u_2,0)=\boldsymbol{R}, \tag{18}$$

where row vector $\boldsymbol{R}$ is the two-dimensional stationary probability distribution of server's state $k(t)$, which was found in Theorem 1. With (18) described, we solve can solve the Cauchy problem (17)

$$\boldsymbol{FF}(u_1,u_2,t)=\boldsymbol{R}e^{G(u_1,u_2)}.$$

Since we are focusing on the probability distribution of requests in output processes the marginal distribution is needed. For this, we multiply row vector $\boldsymbol{FF}(u_1,u_2,t)$ by unit vector-column $\boldsymbol{e}$ of size N and the right part of the equation by unit vector-column $\boldsymbol{ee}$ of size $K\cdot N$. We obtain

$$\boldsymbol{FF}(u_1,u_2,t)\boldsymbol{e}=\boldsymbol{R}e^{\boldsymbol{G}(u_1,u_2)t}\boldsymbol{ee}. \tag{19}$$

(19) is the solution for the considered system.

5 Explicit Probability Distribution

Characteristic function (19) allows us to move to an explicit formula for calculating probabilities of the number of served requests in output processes $m_1(t)$ and $m_2(t)$. (19) contains the matrix exponent, for which we apply the similarity transformation [4]

$$\boldsymbol{G}(u_1, u_2) = \boldsymbol{T}(u_1, u_2)\boldsymbol{GJ}(u_1, u_2)\boldsymbol{T}(u_1, u_2)^{-1},$$

where $\boldsymbol{T}(u_1, u_2)$ is an eigenvector matrix of $\boldsymbol{G}(u_1, u_2)$, and $\boldsymbol{GJ}(u_1, u_2)$ is a diagonal eigenvalue matrix of $\boldsymbol{G}(u_1, u_2)$. This conversion is objective for any power m of some matrix $\boldsymbol{A}^m$, which follows it is also valid for the matrix exponent

$$e^{\boldsymbol{G}(u_1,u_2)t} = \boldsymbol{T}(u_1, u_2) \cdot \begin{bmatrix} e^{t\Lambda_1(u_1,u_2)} & 0 & 0 \\ 0 & e^{t\Lambda_2(u_1,u_2)} & 0 \\ 0 & 0 & e^{t\Lambda_3(u_1,u_2)} \end{bmatrix} \cdot \boldsymbol{T}(u_1, u_2)^{-1},$$

where Λ_n is an eigenvalue of $\boldsymbol{G}(u_1, u_2)$. Then the distribution is written as follows

$$\boldsymbol{F}(u_1, u_2, t)\boldsymbol{E} = \boldsymbol{R} \cdot \boldsymbol{T}(u_1, u_2) \cdot \begin{bmatrix} e^{t\Lambda_1(u_1,u_2)} & 0 & 0 \\ 0 & e^{t\Lambda_2(u_1,u_2)} & 0 \\ 0 & 0 & e^{t\Lambda_3(u_1,u_2)} \end{bmatrix} \cdot \boldsymbol{T}(u_1, u_2)^{-1} \cdot \boldsymbol{E}.$$

To restore the distribution, we use the inverse Fourier transform for discrete values

$$P(m_1, m_2, t) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-i \cdot u_1 \cdot m_1} e^{-i \cdot u_2 \cdot m_2} \boldsymbol{F}\boldsymbol{F}(u_1, u_2, t)\boldsymbol{e}\, du_1 du_2. \quad (20)$$

The resulting formula characterizes the probability of servicing m_1 input process requests and m_2 outgoing requests at the moment t in the system under consideration.

6 Numerical Examples

Let us compare simulation output with the calculations based on the obtained asymptotic results. σ affects accuracy, since the solution of the system was obtained under the asymptotic condition of a long delay of requests in the orbit.

We measure the accuracy of the results with the Kolmogorov-Smirnov distance, which is calculated as

$$\Delta = \max_{0 \le i \le \infty} \left| \sum_{v=0}^{i} (P_0(v) - P_1(v)) \right|,$$

where $P_0(v)$ and $P_1(v)$ are comparable probability distributions.

Let us set the following parameters

$$\alpha = 0.6, \mu_1 = 2, \mu_2 = 1.5, t = 15,$$

$$\boldsymbol{Q} = \begin{bmatrix} -0.5 & 0.2 & 0.3 \\ 0.15 & -0.2 & 0.05 \\ 0.3 & 0.4 & -0.7 \end{bmatrix}, \boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}.$$

The input process intensity can be written in form $\boldsymbol{r} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{e}$, after calculation of which, we get the value 0.72. For the parameters set, we obtained the following results.

Let us denote: Δ_S is the KS distance values for the summary distribution, which implies, that served incoming and outgoing requests are homogeneous, and Δ_{TD} is the KS distance values for the two-dimensional distribution of served requests, which are, in the two-dimensional case, of different types.

Table 1. KS distance values for various σ

σ	10	1	0.6	0.4	0.2	0.1	0.05	0.01
Δ_S	0.053	0.045	0.04	0.036	0.028	0.023	0.018	0.016
Δ_{TD}	0.059	0.049	0.042	0.035	0.024	0.015	0.01	0.003

In Table 1, we can notice that for lower values of σ asymptotic results of the two-dimensional distribution are more accurate. Let us raise system load by setting up the new intensity matrix with greater values of diagonal elements

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}.$$

For these parameters, the overall input process intensity is 1.07. Calculations for the new set of parameters are

Table 2. KS distance values for various σ with high system load

σ	10	1	0.6	0.4	0.2	0.1	0.05	0.01
Δ_S	0.037	0.029	0.024	0.02	0.015	0.01	0.008	0.008
Δ_{TD}	0.066	0.048	0.039	0.031	0.019	0.01	0.006	0.002

Based on the performed experiments, we can conclude that a tendency towards an accuracy increase of asymptotic results is always observed when decreasing σ. For a value of σ exceeding the intensity of the input process, the accuracy does not exceed 0.066 (the longest KS distance, which is observed in the case of a two-dimensional probability distribution), which indicates a high degree of accuracy of the obtained approximation. Raising system load with input process requests, as can be seen in Table 2, has a positive effect on the accuracy of the asymptotic results. It is because more events occur within a fixed time interval during simulation.

7 Conclusion

In this paper, we have described the process of finding the asymptotic approximation of the two-dimensional characteristic function of the number of incoming and outgoing requests that have finished serving in retrial queue with two-way communication under the condition of a long delay in the source of retrial calls. This allows retrieving different performance characteristics, including the correlation of the processes in the system output. Moreover, we used it to calculate probability values for further experiments.

Carried out numerical experiments show that obtained approximation gives high accuracy results, and for this reason, it can be used for further research of this type of system.

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