# Performance Evaluation of the V-BLAST Coset Detector

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***Abstract*— In this paper we address the analytical performance evaluation of the V-BLAST coset detector (CD). The V-BLAST-CD is a sub-optimal detector for spatial multiplexing MIMO systems using non-binary constellations. In the V-BLAST-CD the decision on the transmitted vector is taken by applying the maximum likelihood detector on a list of candidate vectors containing only a subset of the possible transmitted vectors. The list of candidate vectors is generated by applying ordered V-BLAST detection to the subsets induced by partitioning the multidimensional signal constellation according to the mapping by set partitioning principle. Noting that the detection process of the V-BLAST-CD is similar to that of list decoding of error correcting codes, in order to evaluate the error performance of the V-BLAST-CD we can adopt the same approach used to evaluate the error performance of the generalized minimum distance decoding. The accuracy of the analysis is demonstrated by comparing it to simulation results.**

## I. Introduction

Spatial multiplexing over a multiple-input multiple-output (MIMO) channel is a means to increase the spectral efficiency of wireless communication systems [1]. We consider a scheme where the input information sequence is de-multiplexed into $n_T$ sub-streams which are modulated by an $M^2$-QAM modulator and transmitted in parallel from $n_T$ antennas at the same time and frequency. The detection of the transmitted data symbols is performed by processing the signals received from $n_R \geq n_T$ antennas. For such a scheme the performance, in terms of symbol error probability, is strongly dependent on the technique that is implemented in the receiver to detect the $n_T$ transmitted sub-streams.

The optimum detector, which is the maximum likelihood detector (MLD), has a complexity proportional to $M^{2n_T}$. Due to this exponential dependence, the complexity of the MLD can be prohibitively large also for moderate values of $M$ and $n_T$. The V-BLAST architecture, proposed in [1], is a sub-optimal detection technique that allows us to detect the $n_T$ transmitted sub-streams while keeping the complexity low. In such a scheme, symbols are detected sequentially according to the well-known nulling and successive interference cancellation process. The V-BLAST operations are equivalent to those of a decision-feedback equalizer (DFE) that operates at different stages in the spatial domain. Hence, the overall performance may be limited by the first error probability of the spatial DFE. The performance loss can be mitigated if an appropriate detection ordering is introduced. In [1] the post-detection signal-to-noise ratio (SNR) is taken as the ordering criterion. Despite its detection simplicity, the main drawback of V-BLAST is that the diversity order in the early stages is lower than in the next ones [2]. The diversity order of V-BLAST at the $n$-th processing step is $n_R - n_T + n$. This is the main cause of the performance gap between V-BLAST and MLD (in the latter the diversity order is equal to $n_R$).

Several sub-optimal detection strategies can be devised to reduce the performance gap compared to the MLD. In [3], for example, the MLD is used to increase the diversity order of first stages and only when the detected symbols are reliable enough can the detection be done with the V-BLAST algorithm. Note that for this detector the complexity is proportional to $M^{2n_i}$, where $n_i$ represents the number of initial stages to which the MLD is applied. The detector we focus on is the V-BLAST coset detector (V-BLAST-CD) proposed in [4]. The V-BLAST-CD is obtained by extending the principle of reduced state sequence estimation [5], based on Mapping by Set Partitioning (MSP), to perform the detection in spatial multiplexing MIMO systems using non-binary constellations. Its complexity is proportional to $\mu^{2n_T}$, where $1 \leq \mu \leq M^2$. In [4] it is shown that the V-BLAST-CD greatly outperforms the V-BLAST detector at the cost of a slight increase of complexity ($\mu = 2$ is considered). In particular, from low-to-intermediate SNR the performance of the V-BLAST-CD is the same as the MLD, while at high SNR the V-BLAST-CD still provides a significant performance gain over the V-BLAST detector. In [4] the benefits of the V-BLAST-CD are demonstrated by computer simulations. In this paper we move a step toward a better understanding of the V-BLAST-CD. To this end, the performance evaluation carried out in this paper allows us to achieve further insights into the features of the V-BLAST-CD.

The paper is organized as follows. The system model is introduced in section II. A short description of the V-BLAST-CD is given in section III, while section IV addresses its performance analysis. In section V the results of the analysis are compared to simulation results.

## II. System Model

The block diagram of the system that we consider is shown in Fig. 1. Let $\mathbf{a} = [a_1, a_2, \ldots, a_{n_T}]^T$ denote the vector of

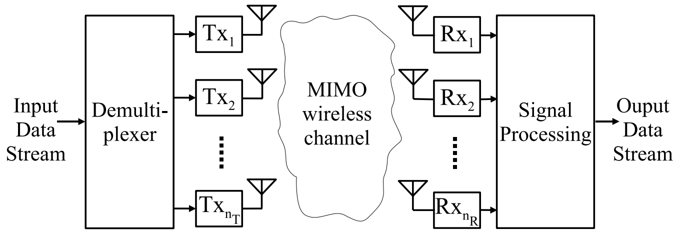


Fig. 1. Block diagram of the spatial multiplexing MIMO system.

transmitted symbols ($(\cdot)^T$ denotes transposition). We assume that symbols transmitted at a particular antenna are equally probable and take a value $s_m$ in the set $\mathcal{S}$ which contains all the $M^2$-QAM constellation points. Let $\mathcal{A}$ denote the set of all $M^{2n_T}$ possible equiprobable transmitted vectors. The received signal vector at a particular time instant is represented in complex baseband form as

$$\mathbf{r} = \mathbf{H}\mathbf{a} + \mathbf{w},$$

where $\mathbf{w} = [w_1, \ldots, w_{n_R}]^T$ is the noise vector of i.i.d. complex Gaussian random variables (RVs) with zero mean and variance $\sigma_w^2$ and $\mathbf{H}$ is the $n_R \times n_T$ channel matrix whose elements are i.i.d. RVs having uniform-distributed phase and Rayleigh-distributed magnitude with average power equal to 1. $\mathbf{H}$ is independent of both $\mathbf{a}$ and $\mathbf{w}$ and it is assumed perfectly known to the receiver. The average radiated power from each antenna is fixed to $1/n_T$. Thus, the total average radiated power is fixed to 1 and it turns out to be independent of the total number of transmitting antennas. The average SNR per transmitted symbol at the receiver is $\bar{\gamma} \triangleq n_R/(n_T\sigma_w^2)$.

## III. The V-BLAST Coset Detector

The principle behind MSP is a geometric partitioning of the signal constellation in subsets of diminishing size, in such a way that minimum Euclidean distance within the subsets increases down the partition chain. A square $M^2$-QAM constellation can be seen as a finite set of points carved out from the two-dimensional integer lattice $\mathbb{Z}^2$. Similarly, the signal transmitted from the $n_T$ antennas can be seen as belonging to a multidimensional constellation carved out from $\mathbb{Z}^{2n_T}$. The partitioning of the constellation in subsets corresponds to the partitioning of the lattice into a sublattice and its cosets. In our proposal, the partition $\mathbb{Z}^{2n_T}/2\mathbb{Z}^{2n_T}$ is considered. Observe that, for each dimension of the $M^2$-QAM signal constellation we have the binary partition $\mathbb{Z}/2\mathbb{Z}$, that generates two subsets each containing $M/2$ points. The partition $\mathbb{Z}^{2n_T}/2\mathbb{Z}^{2n_T}$ divides the multidimensional constellation into $2^{2n_T}$ subsets, each containing $n_T M^2/4$ points. The ordered V-BLAST is used to perform detection for each of the $2^{2n_T}$ subsets [1]. At the end of this procedure a list of $2^{2n_T}$ candidate vectors is generated. The decision is taken by applying the MLD to this reduced set:

$$\hat{\mathbf{a}} = \arg\min_{\hat{\mathbf{a}}_r \in \mathcal{A}_r} \|\mathbf{r} - \mathbf{H}\hat{\mathbf{a}}_r\|^2,$$

where $\hat{\mathbf{a}}_r$ is a vector taken from the reduced set $\mathcal{A}_r$ whose elements are the $2^{2n_T}$ candidate vectors. It is worth noting that the complexity of the MLD on the reduced set of candidate vectors is fixed and independent of the constellation size.

[1]Note that other sub-optimal detectors could be employed to perform the detection in the subsets.

## IV. Performance Analysis

Performance evaluation of the V-BLAST-CD is carried out by following the approach suggested in [6] to evaluate the error performance of the generalized minimum distance (GMD) decoding algorithm. We emphasize that the analysis of [6] can be applied to any list-based sub-optimal detection scheme. The error probability of the $M^2$-QAM symbol is upper bounded as

$$P_s(\mathcal{E}) \leq P_s(\mathcal{E}_{MLD}) + P(\mathcal{F}), \tag{1}$$

where $P_s(\mathcal{E}_{MLD})$ is the symbol error probability of the MLD over the set $\mathcal{A}$ and $P(\mathcal{F})$ is the probability that the transmitted vector is not in the list of candidate vectors. The first term on the RHS of (1) represents an upper bound to the conditional symbol error probability of the MLD in the reduced set of candidate vectors, where the condition is that the transmitted vector belongs to the reduced set $\mathcal{A}_r$. Note that when the partitioning is such that we have just 1 constellation point in each subset, as happens with a 4-QAM constellation, the number of candidate vectors is $2^{2n_T}$ and $P_s(\mathcal{E}) = P(\mathcal{E}_{MLD})$.

### A. Derivation of $P_s(\mathcal{E}_{MLD})$

The term $P_s(\mathcal{E}_{MLD})$ can be approximated as reported in [7], where a tight union bound and an asymptotic bound on the symbol error probability are given for MIMO systems with two-dimensional signal constellations. The approximations on $P_s(\mathcal{E}_{MLD})$ are obtained by calculating the symbol error probability corresponding to an arbitrary $k$-th ($k = 1, \ldots, n_T$) transmitted sub-stream. Let $s_m$ be the symbol transmitted by the $k$-th antenna. We denote as $\mathcal{A}_j$ the subset of $M^{2(n_T-1)}$ vectors in $\mathcal{A}$ which have the symbol $s_m$ in their $k$-th position. We also denote as $\mathcal{A}_i$ the set of $(M^{2n_T} - M^{2(n_T-1)})$ vectors whose $k$-th entry differs from $s_m$. The union bound on $P_s(\mathcal{E}_{MLD})$ is

$$P_s(\mathcal{E}_{MLD}) \leq M^{-2n_T} \sum_{s_m \in \mathcal{S}} \sum_{j \in \mathcal{A}_j} \left( \sum_{i \in \mathcal{A}_i} P_{s_m, ij} \right), \tag{2}$$

where $P_{s_m,ij}$ denotes the pairwise error probability between the vectors $\mathbf{a}_i \in \mathcal{A}_i$ and $\mathbf{a}_j \in \mathcal{A}_j$, given that $s_m$ is transmitted by the $k$-th antenna. As is shown in [7], the closed-form expression of $P_{s_m,ij}$ is given by

$$P_{s_m,ij} = \frac{1}{(1 + r_{s_m,ij})^{2n_R-1}} \sum_{n=0}^{n_R-1} \binom{2n_R - 1}{n} r_{s_m,ij}^n, \tag{3}$$

where

$$r_{s_m,ij} = \frac{\|\mathbf{a}_i - \mathbf{a}_j\|^2}{2\sigma_w^2} + \sqrt{\frac{\|\mathbf{a}_i - \mathbf{a}_j\|^4}{4\sigma_w^4} + \frac{\|\mathbf{a}_i - \mathbf{a}_j\|^2}{\sigma_w^2}} + 1.$$

The computation of (2) would require up to $M^{2n_T}(M^{2n_T} - M^{2(n_T-1)})$ pairwise error probability computations. However, a reduction of this number can be obtained if constellation symmetries are exploited. At high SNR the $P_{s_m,ij}$ given in (3) can be expressed asymptotically as

$$P_{s_m,ij} \approx r_{s_m,ij}^{-n_R} \left( \begin{array}{c} 2n_R - 1 \\ n_R - 1 \end{array} \right),$$

where $r_{s_m,ij} \approx \|\mathbf{a}_i - \mathbf{a}_j\|^2/\sigma_w^2$.

### B. Derivation of $P(\mathcal{F})$

As far as the term $P(\mathcal{F})$ is concerned, the event $\mathcal{F}$ represents the probability that there is at least an error in the subset containing the transmitted vector. An upper bound to $P(\mathcal{F})$ can be obtained by following the analysis derived in [2] for binary modulations. The same approach is here extended to square QAM modulations by using the analysis presented in [8]. In what follows we refer to a generic QAM constellation with $M^2$ signal points. We point out that the probabilities hereafter computed can be used as well when subsets of $M^2$ points are at hand, provided that the SNR is properly scaled.

According to [2] the upper bound on $P(\mathcal{F})$ is

$$P(\mathcal{F}) \leq \sum_{n=1}^{n_T} P_{e,n}, \tag{4}$$

where $P_{e,n}$ denotes the average conditional symbol error probability at the $n$-th step of the detection process, where the condition is that no errors occur in the previous detection steps. The $P_{e,n}$ is computed as [10]

$$P_{e,n} = \int_0^\infty P_e(\gamma) p_n(\gamma) d\gamma, \qquad n = 1, \ldots, n_T, \tag{5}$$

where $P_e(\gamma)$ is the symbol error probability of an $M^2$-QAM signal in an additive white Gaussian noise (AWGN) channel for the instantaneous SNR $\gamma$ and $p_n(\gamma)$ is the probability density function (pdf) of $\gamma$ at the $n$-th detection step. The probability of error for an $M^2$-QAM signal in the AWGN channel can be expressed as [10]

$$P_e(\gamma) = \left(\frac{M-1}{M^2}\right) \left(4MQ\left(\sqrt{\frac{3}{M^2-1}\gamma}\right) - 4\,(M-1)\,Q^2\left(\sqrt{\frac{3}{M^2-1}\gamma}\right)\right), \tag{6}$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{u^2}{2}} du.$$

The main contribution of [2] consists in the derivation of the pdf of $\gamma$ at the different detection steps when the optimal ordering is implemented by the V-BLAST detector. In [2] closed-form analytical expressions are given for the $2 \times n_R$ system, while tight bounds can be obtained for the general $n_T \times n_R$ as shown in [9]. For the sake of simplicity, in the following we derive the analytic expression of $P_{e,n}$ for $n = 1, 2$, in the $2 \times 2$ case. The same approach can be used to obtain the expressions for the other cases but it involves more elaborate mathematics. The pdfs of $\gamma$ at the first and second steps for a $2 \times 2$ system are respectively

$$p_1(\gamma) = \frac{2}{\bar{\gamma}_c} e^{-\gamma/\bar{\gamma}_c} - \frac{3}{2\bar{\gamma}_c} e^{-2\gamma/\bar{\gamma}_c} - \frac{\gamma}{\bar{\gamma}_c^2} e^{-2\gamma/\bar{\gamma}_c} \tag{7}$$

and

$$p_2(\gamma) = 8\frac{\gamma}{\bar{\gamma}_c^2}(1 + 2\frac{\gamma}{\bar{\gamma}_c}) e^{-\gamma/\bar{\gamma}_c}, \tag{8}$$

where $\bar{\gamma}_c = \bar{\gamma}/n_R$ denotes the average SNR per channel. Because the pdf given in (8) is derived by assuming that the symbol detected in the first step is correct, the $P_{e,2}$ will represent the average symbol error probability at the second step given that no error occurs in the interference cancellation step. Observe that in [2] this assumption is made in the derivation of the pdf of $\gamma$ at each step of the V-BLAST detection process. It is important to note that the two expressions in (7) and (8) can be seen as a weighted sum of chi-square pdfs with an even number of degrees of freedom and with different parameters. The expression of the pdf for a chi-square random variable with $2l$ degrees of freedom and average SNR per channel $\gamma_0$ is [10]

$$p(\gamma) = \frac{1}{(l-1)!\gamma_0^l} \gamma^{l-1} e^{-\gamma/\gamma_0}, \qquad \gamma \geq 0. \tag{9}$$

The closed form solution of (5) when (6) is averaged over (9) is given by

$$P_e(M^2, \gamma_0, l) = \frac{M-1}{M^2} \big(M - 1 + 4I_1(M^2, \gamma_0, l) - (M-1)I_2(M^2, \gamma_0, l)\big), \tag{10}$$

where the analytic expressions of $I_1(M^2, \gamma_0, l)$ and $I_2(M^2, \gamma_0, l)$ are given respectively by equation (9) and equation (10) in [8]. Note that the average symbol error probability given in (10) corresponds to that of an $M^2$-QAM modulation with $l$-fold maximal ratio combining (MRC) space diversity in Rayleigh fading channels.

By substituting (7) and (8) in (5) and using (10) the average error probabilities at the first and second step are respectively given by

$$P_{e,1} = 2P_e(M^2, \bar{\gamma}_c, 1) - \frac{3}{4} P_e(M^2, \frac{\bar{\gamma}_c}{2}, 1) - \frac{1}{4} P_e(M^2, \frac{\bar{\gamma}_c}{2}, 2) \tag{11}$$

and

$$P_{e,2} = \frac{1}{2} P_e(M^2, \frac{\bar{\gamma}_c}{4}, 2) + \frac{1}{2} P_e(M^2, \frac{\bar{\gamma}_c}{4}, 3). \tag{12}$$

When (11) and (12) are used to compute the performance of the V-BLAST-CD described in section III, it is necessary to take into account that the number of constellation points in each subset is $M^2/4$, therefore the $\bar{\gamma}_c$ appearing in the above equations must be multiplied by $4(M^2 - 1)/(M^2 - 4)$.

## V. Experimental Results

To demonstrate the accuracy of the bound (1) we consider a system with $n_T = n_R = 2$. Note that in this case the detector of [3] could not provide any complexity reduction. The performance of the V-BLAST-CD is compared to that of

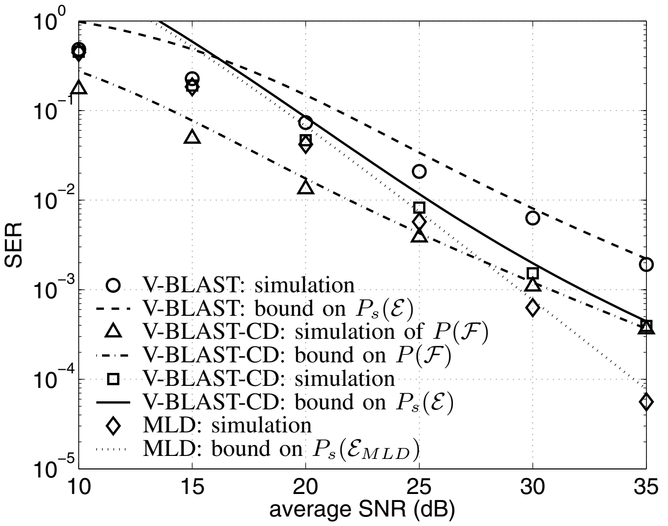


Fig. 2. Performance comparison of V-BLAST, V-BLAST-CD and MLD for a $2 \times 2$ system with 16-QAM modulation.

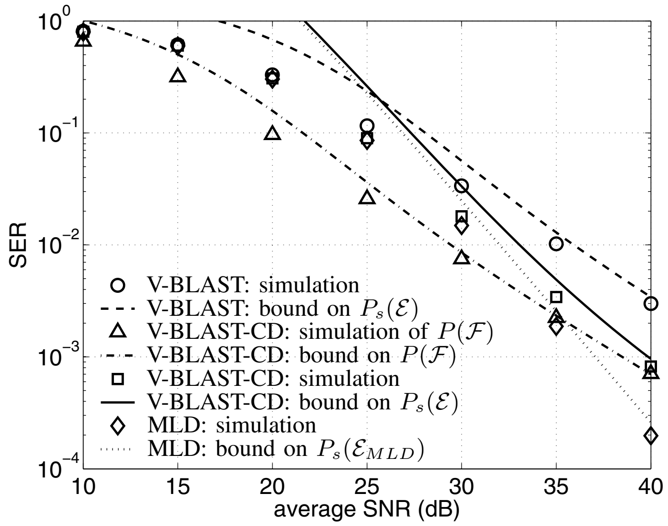


Fig. 3. Performance comparison of V-BLAST, V-BLAST-CD and MLD for a $2 \times 2$ system with 64-QAM modulation.

the V-BLAST and MLD. Figures 2 and 3 report the simulated SER (symbol error rate) versus the average SNR respectively for 16-QAM and 64-QAM modulations. The SER is measured by transmitting a sequence of $10^6$ symbols from each antenna. Together with simulation results, the bounds on the symbol error probability given in (1) for the considered detection techniques are also reported in the figures. We observe that the performance of the V-BLAST-CD algorithm is close to that of the MLD for a SER greater than $10^{-3}$. In this region the bound in (1) is dominated by the error probability of the MLD on the reduced set of candidate vectors. For a lower SER the performance of the V-BLAST-CD is dominated by the symbol error probability of the ordered V-BLAST detector in the subsets. As the figures show, the upper bound (1) is tight at SER lower than $10^{-3}$, where $P_s(\mathcal{E}) \simeq P(\mathcal{F})$.

## VI. Conclusion

The main contribution of this paper consists in the performance evaluation of the V-BLAST-CD proposed in [4]. This is a sub-optimal detector for non-binary modulation formats where the decision is taken by applying the MLD on a reduced set of candidate vectors. The set of candidate vectors is generated by applying the ordered V-BLAST detector to the subsets obtained by a partitioning of the multidimensional transmitted signal constellation according to the MSP principle. By exploiting the conceptual similarity between the detection process implemented by the V-BLAST-CD and that of list decoding of error correcting codes, an upper bound to the symbol error probability of the V-BLAST-CD has been derived by following the same approach proposed in [6] for GMD decoding. The bound on the symbol error probability of the V-BLAST-CD is given by the sum of two terms. The first one represents an upper bound on the symbol error probability of the MLD given that the transmitted vector is among the set of candidate vectors. The second one gives the probability that the transmitted vector is not in the list of candidate vectors. Noticeably, the error performance of the proposed detector turns out to be close to that of MLD at low-to-intermediate SNR. Computer simulations have been used to demonstrate the accuracy of the analysis.

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## References


[1] P. W. Wolniansky, G. J. Foschini, G. D. Golden, and R. A. Valenzuela, "V-BLAST: an architecture for realizing very high data rates over the rich-scattering wireless channel," in *Proc. ISSSE*, Pisa, Italy, pp. 295-300, 29 Sept.-2 Oct. 1998.

[2] S. Loyka and F. Gagnon, "Performance analysis of the V-BLAST algorithm: an analytical approach," *IEEE Trans. Wireless Commun.*, vol. 3, pp. 1326-1337, July 2004.

[3] W. J. Choi, R. Negi, and J. M. Cioffi, "Combined ML and DFE decoding for the V-BLAST system," in *Proc. ICC*, New Orleans, LA, pp. 1243-1248, 18-22 June 2000.

[4] M. Magarini and A. Spalvieri, "A suboptimal detection scheme for MIMO systems with non-binary constellations," in *Proc. PIMRC*, Barcelona, Spain, pp. 1579-1582, 5-8 Sept. 2004.

[5] M. V. Eyuboglu and S. U. H. Qureshi, "Reduced-state sequence estimation with set partitioning and decision feedback," *IEEE Trans. Commun.*, vol. 36, pp. 13-20, Jan. 1988.

[6] D. Agrawal and A. Vardy, "Generalized minimum-distance decoding in Euclidean space: performance analysis," *IEEE Trans. Inform. Theory*, vol. 46, pp. 60-83, Jan. 2000.

[7] X. Zhu and R. D. Murch, "Performance analysis of maximum likelihood detection in a MIMO antenna system," *IEEE Trans. Commun.*, vol. 50, pp. 187-191, Feb. 2002.

[8] C. J. Kim, Y. S. Kim, G. Y. Jeong, J. K. Mun, and H. J. Lee, "SER analysis of QAM with space diversity in Rayleigh fading channels," *ETRI J.*, vol. 17, pp. 25-35, Jan. 1996.

[9] S. Loyka and F. Gagnon, "Analytical framework for outage and BER analysis of the V-BLAST algorithm," in *Proc. IZS*, Zurich, Switzerland, pp. 120-123, Feb. 2004.

[10] J. G. Proakis "Digital communications (4th edition)," McGraw-Hill: New York, 2001.