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# Tauberian theorems via the generalized Nörlund mean for sequences in 2-normed spaces

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**Abstract.** In this paper, we will show Tauberian conditions under which ordinary convergence of the sequence $(x_n)$ in 2-normed space $X$, follows from $T_n^{p,q}$-summability. In fact we give a necessary and sufficient Tauberian condition for this method of summability. Also, we prove that Tauberian Theorems for these summability methods are valid with Schmidt-type slowly oscillating condition as well as with Hardy-type "big O" condition.

*Keywords:* $T_n^{p,q}$-summability method; Tauberian theorems; Nörlund summability; Schmidt-type oscillating slowly sequences; Hardy-type condition; 2-normed space

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# 1. Introduction

The Tauberian theory based on the following fact. If the sequence $(x_n)$ converges, i.e.

$$\lim_{n\to\infty} x_n = l$$

exists then it follows that the limit in sense of a regular summability method $(T_n)$ exists and

$$\lim_{n\to\infty} T_n(x_n) = l.$$

The converse of the above fact is not true (or "does not hold") in general. Conditions under which the converse follows are known as Tauberian conditions, and the result with such conditions is known Tauberian theorem. The Tauberian theorems are investigated for many summability methods under different conditions, see for example ([2, 3, 5, 7–11]). In 1976, the well-known Hardy–Littlewood Tauberian theorem was extended to the multidimensional case by Vladimirov [14]. After that paper, the work began on a systematic investigation of the Tauberian theory of generalized functions from the standpoint of both pure mathematics and its application in theoretical and mathematical physics. In [4], some multidimensional Tauberian theorems for generalized functions were established along with their application in mathematical physics. In recent year the Tauberian theorems were proved in 2-normed spaces for the Cesàro summability method (see [12]).

The convolution $(p * q)$ of two non-negative sequences $(p_n)$ and $(q_n)$ is defined by

$$R_n := (p * q)_n = \sum_{k=0}^{n} p_k q_{n-k} = \sum_{k=0}^{n} p_{n-k} q_k.$$

In case $(p * q)_n \neq 0$ for all $n \in \mathbb{N}$, the generalized Nörlund transform $(T_n^{p,q})$ of the sequence $(x_n)$ is given as follows

$$T_n^{p,q} = \frac{1}{(p * q)_n} \sum_{k=0}^{n} p_k q_{n-k} x_k.$$

The sequence $(x_n)$ is generalized Nörlund summable to $L$ (see [1]), if

$$\lim_{n\to\infty} T_n^{p,q} = L. \tag{1.1}$$

Let us define

$$A(n,t) := \{q_{\lambda_n - k} - q_{n-k} : k = 0, 1, 2, \ldots, n; \lambda > 1\}$$

and

$$B(n,t) := \{q_{k-\lambda_n} - q_{n-k} : k = 0, 1, 2, \ldots, \lambda_n; 0 < \lambda < 1\},$$

where $\lambda_n := [\lambda n]$ denotes the integral part of $\lambda n$.

Let us suppose that the sequences $p = (p_n)$ and $q = (q_n)$ satisfies the following conditions:

$$p_n \leq q_n, \quad n \in \mathbb{N},$$
$$q_n \geq 1, \quad n \in \mathbb{N},$$
$$\sup_n A(n, \lambda) < \infty,$$

and

$$\sup_n B(n, \lambda) < \infty.$$

If

$$\lim_{n\to\infty} x_n = L$$

implies (1.1), then the method $(N, p, q)$ is regular. The necessary and sufficient condition for the $(N, p, q)$ method to be regular is (see [6])

$$p_{n-k}q_k = o(R_n) \ \ (n \to \infty, k \in \mathbb{N}),$$

and

$$\sum_{k=0}^{n} |p_{n-k}q_k| = O(R_n) \ \ (n \to \infty).$$

**Remark 1.1.** In case when $p_n \equiv q_n \equiv 1$ for $n \in \mathbb{N}$, the $(N, p, q)$ method coincides the Cesàro $(C, 1)$ summability. For $q_n = 1$ we get the Nörlund method $(\overline{N}, p)$. In case when $p_n = \binom{n+\beta}{\beta}$, $q_n = \binom{n+\alpha-1}{\alpha}$, we get the $(C, \alpha, \beta)$ ([1]) method. Finally, for $p_n = \lambda_n$ and $q_n = 1$, we get the generalized de la Vallée-Poussin method.

In this paper we will prove Tauberian theorems for the $(N, p, q)$ summability method in 2-normed spaces.

**Definition 1.2.** A sequence $(x_n)$ converges to $L$ in a 2-norm $X$, i.e.

$$x_n \overset{\|\cdot,\cdot\|_X}{\longrightarrow} L,$$

if

$$\lim_{n\to\infty} \|x_n - L, y\| = 0,$$

for all $y \in X$.

A sequence $(x_n)$ in a 2-normed space $X$ is $T_n^{p,q}$ summable to $L \in X$ and write, in sign: $x_n \overset{\|\cdot,\cdot\|_X}{\longrightarrow} L(T_n^{p,q})$, if

$$\lim_{n\to\infty} \|T_n^{p,q} - L, y\| = 0,$$

for all $y \in X$.

**Theorem 1.3.** *In a 2-normed space $X$, $\lim_n x_n = L \in X$, implies $\lim_n T_n^{p,q} = L$ in $X$. The converse statement is not true in general.*

***Proof.*** Let us suppose that $\lim_n x_n = L$ in a 2-normed space $X$. It is clear that, for every $\epsilon > 0$, there exists an $n_0$ such that for every $n > n_0$ and any $y \in X$ we have

$$\|x_n - L, y\| < \epsilon :$$

and for any $n < n_0$, $y \in X$ there exists a $M > 0$ such that

$$\|x_n - L, y\| \leq M.$$

Now we can estimate as follows:

$$\begin{aligned}
&\|T_n^{p,q} - L, y\| \\
&= \left\| \frac{1}{(p*q)_n} \sum_{k=0}^{n} p_k q_{n-k} x_k - L, y \right\| = \left\| \frac{1}{(p*q)_n} \sum_{k=0}^{n} p_k q_{n-k}(x_k - L), y \right\| \\
&\leq \left\| \frac{1}{(p*q)_n} \sum_{k=0}^{n_0} p_k q_{n-k}(x_k - L), y \right\| + \left\| \frac{1}{(p*q)_n} \sum_{k\in\{n_0+1,\cdots,n\}} p_k q_{n-k}(x_k - L), y \right\| \\
&\leq M \cdot \frac{A_{n_0}^{p,q}}{(p*q)_n} + \epsilon,
\end{aligned}$$

where $A_{n_0}^{p,q} = \sum_{k=0}^{n_0} p_k q_{n-k}$. Hence, we get desired result. $\square$

**Example 1.4.** Consider $X = \mathbb{R}^3$ and

$$\|x, y\| = \max\{|x_1y_2 - x_2y_1|, |x_1y_3 - x_3y_1|, |x_2y_3 - x_3y_2|\},$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$. Let

$$x_n = \left(1 + (-1)^n, 2 + (-1)^n, 3 + \frac{3(-1)^n}{2}\right),$$

and $y = (y_1, y_2, y_3) \in X$.

If we put $p_n = n$ and $q_n = 1$, then we have

$$\begin{aligned}
T_n^{p,q}(1 + (-1)^n) &= 1 + \frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}, \\
T_n^{p,q}(2 + (-1)^n) &= 2 + \frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}, \\
T_n^{p,q}\left(3 + \frac{3(-1)^n}{2}\right) &= 3 + \frac{3(-1)^n}{2(n+1)} + \frac{3[(-1)^n - 1]}{4n(n+1)}.
\end{aligned}$$

Now we will prove that $T_n^{p,q} \to (1, 2, 3)$ in the 2-normed space $X$.

$$\begin{aligned}
&\lim_{n\to\infty} \|T_n^{p,q} - L, y\| \\
&= \lim_{n\to\infty} \left\| \left( \frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}, \frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}, \frac{3(-1)^n}{2(n+1)} + \frac{3[(-1)^n - 1]}{4n(n+1)} \right), y \right\|
\end{aligned}$$

$$= \lim_{n\to\infty} \max \Bigg\{ \left| y_2\left(\frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}\right) - y_1\left(\frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}\right)\right|, \\ \left| y_3\left(\frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}\right) - y_1\left(\frac{3(-1)^n}{2(n+1)} + \frac{3[(-1)^n - 1]}{4n(n+1)}\right)\right|, \\ \left| y_3\left(\frac{(-1)^n}{n+1} + \frac{(-1)^n - 1}{2n(n+1)}\right) - y_2\left(\frac{3(-1)^n}{2(n+1)} + \frac{3[(-1)^n - 1]}{4n(n+1)}\right)\right| \Bigg\} = 0.$$

So $(x_n)$ is $T_n^{p,q}$-summable to $(1,2,3)$ in 2-normed space $X$. Now we will prove that $(x_n)$ does not converge to $(1,2,3)$ in 2-normed space $X$. Let $y = (1,1,1) \in \mathbb{R}^3$ then

$$\lim_{n\to\infty} \|x_n - L, y\| = \lim_{n\to\infty} \left\| \left((-1)^n, (-1)^n, \frac{3(-1)^n}{2}\right), (y_1, y_2, y_3) \right\| \\ = \lim_{n\to\infty} \max\Bigg\{ |(-1)^n \cdot y_2 - (-1)^n \cdot y_1|, \left|(-1)^n \cdot y_3 - \frac{3(-1)^n}{2} \cdot y_1\right|, \\ \left|(-1)^n \cdot y_3 - \frac{3(-1)^n}{2} \cdot y_2\right| \Bigg\} \neq 0,$$

sequence $(x_n)$ is not convergent.

## 2. Tauberian theorems for $T_n^{p,q}$-summability method

**Theorem 2.1.** *Let $(p_n)$ and $(q_n)$ be two sequences of real numbers defined as above and*

$$\liminf_{n\to\infty} \frac{R_{\lambda_n}}{R_n} > 1, \quad \lambda > 1 \tag{2.1}$$

*where $\lambda_n = [\lambda n]$. Suppose that $\lim_n T_n^{p,q} = L$, in 2-normed space $X$. Then $(x_n)$ is convergent to the same number $L$ in 2-normed space $X$ if and only if*

$$\inf_{\lambda>1} \limsup_{n\to\infty} \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{i=n+1}^{\lambda_n} p_i q_{\lambda_n - i}(x_i - x_n), y \right\| = 0 \tag{2.2}$$

*and*

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{i=\lambda_n+1}^{n} p_i q_{n-i}(x_n - x_i), y \right\| = 0. \tag{2.3}$$

**Definition 2.2.** The sequence $(x_n) \in X$ is slowly oscillating (see [13]) in a 2-normed space if

$$\inf_{\lambda>1} \limsup_{n\to\infty} \max_{n\le k\le\lambda_n} \|x_k - x_n, y\| = 0$$

for all $y \in X$, or equivalently

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \max_{\lambda_n\le k\le n} \|x_n - x_k, y\| = 0$$

for all $y \in X$.

Denoting $\Delta x_n = x_n - x_{n-1}$, we can rewrite the above conditions to the form

$$\inf_{\lambda>1} \lim_{n\to\infty} \sup_{n\leq k\leq \lambda_n} \max \left\| - \sum_{i=k+1}^{n} \Delta x_i, y \right\| = 0$$

and

$$\inf_{0<\lambda<1} \lim_{n\to\infty} \sup_{\lambda_n\leq k\leq n} \max \left\| \sum_{i=k+1}^{n} \Delta x_i, y \right\| = 0,$$

for all $y \in X$.

We will need the following lemmas.

**Lemma 2.3** ([3]). *For the sequences of real numbers $(p_n)$ and $(q_n)$, condition (2.1) is equivalent to*

$$\lim_{n\to\infty} \inf \frac{R_n}{R_{\lambda_n}} > 1, \quad 0 < \lambda < 1.$$

**Lemma 2.4.** *Let $(p_n)$ and $(q_n)$ be the sequences defined as above and relation (2.1) is satisfied. Assume that $x = (x_n)$ is $T_n^{p,q}$-convergent to $L$, in the 2-normed space $X$. Then for every $\lambda > 0$,*

$$\lim_n \|T_{\lambda_n}^{p,q} - L, y\| = 0$$

*for every $y \in X$.*

***Proof.*** Case 1: $\lambda > 1$. Then from the definition of $\lambda = (\lambda_n)$, we get

$$\lim_n (n - \lambda_n) = \lim_n (R_{\lambda_n} - R_n).$$

Now from given conditions, for every $\epsilon > 0$ we have:

$$\|T_{\lambda_n}^{p,q} - L, y\| \leq \|T_{\lambda_n}^{p,q} - T_n^{p,q}, y\| + \|T_n^{p,q} - L, y\| \leq \epsilon.$$

Case 2: $0 < \lambda < 1$. For $\lambda_n = [\lambda \cdot n]$, for any natural number $n$, we can conclude that $(T_{\lambda_n}^{p,q})$ does not appear more than $[1 + \lambda^{-1}]$ times in the sequence $(T_n^{p,q})$. In fact if there exist integers $k, l$ such that

$$n \leq \lambda \cdot k < \lambda(k+1) < \cdots < \lambda(k+l-1) < n+1 \leq \lambda(k+l),$$

then

$$n + \lambda(l-1) \leq \lambda(k+l-1) < n+1 \Rightarrow l < 1 + \frac{1}{\lambda}$$

and

$$\|T_{\lambda_n}^{p,q} - L, y\| \leq \left(1 + \frac{1}{\lambda}\right) \|T_n^{p,q} - L, y\| \leq \epsilon.$$

From this, $\lim_n \|T_{\lambda_n} - L, y\| = 0$ follows. □

**Lemma 2.5.** *Let* $(p_n)$ *and* $(q_n)$ *be the sequences defined as above and relation* (2.1) *be satisfied. Let* $x = (x_n)$ *be* $T_n^{p,q}$*-convergent to* $L$*, in* 2*-normed space* $X$*. Then for every* $\lambda > 0$*,*

$$\lim_n \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} x_k - L, y \right\| = 0 \quad \text{for} \quad \lambda > 1 \tag{2.4}$$

*and*

$$\lim_n \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} x_k - L, y \right\| = 0 \quad \text{for} \quad 0 < \lambda < 1, \tag{2.5}$$

*for every* $y \in X$*.*

***Proof.*** Case 1: $\lambda > 1$. We get

$$\begin{aligned}
&\left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} x_k - L, y \right\| \\
&= \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} x_k - L, y - \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k q_{\lambda_n - k} x_k - L, y \right\| \\
&= \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} x_k - L, \right. \\
&\qquad \left. y - \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k (q_{n-k} + q_{\lambda_n - k} - q_{n-k}) x_k - L, y \right\| \\
&\leq \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} x_k - L, y - \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k q_{n-k} x_k - L, y \right\| \\
&\quad + \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k (q_{\lambda_n - k} - q_{n-k}) x_k - L, y \right\|.
\end{aligned} \tag{2.6}$$

We know that

$$\limsup_n \frac{R_n}{R_{\lambda_n} - R_n} = \left( \liminf_n \frac{R_{\lambda_n}}{R_n} - 1 \right)^{-1} < \infty. \tag{2.7}$$

Now from (2.6) and (2.7), we get (2.4).

Case 2: $0 < \lambda < 1$. Then we have

$$\begin{aligned}
&\left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} x_k - L, y \right\| \\
&= \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{n-k} x_k - L, y - \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k q_{n-k} x_k - L, y \right\|
\end{aligned}$$

$$= \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{n-k} x_k - L, \right. \tag{2.8}$$

$$\left. y - \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k (q_{\lambda_n - k} + q_{n-k} - q_{\lambda_n - k}) x_k - L, y \right\|$$

$$\leq \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{n-k} x_k - L, y - \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} x_k - L, y \right\|$$

$$+ \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k (q_{n-k} - q_{\lambda_n - k}) x_k - L, y \right\|. \tag{2.9}$$

In this case, we know

$$\limsup_{n\to\infty} \frac{R_{\lambda_n}}{R_n - R_{\lambda_n}} = \left( \liminf_{n\to\infty} \frac{R_n}{R_{\lambda_n}} - 1 \right)^{-1} < \infty. \tag{2.10}$$

From (2.9) and (2.10), we get (2.5). □

***Proof of Theorem 2.1.*** Let $\lim_n x_n = L$, and $\lim_n T_n^{p,q} = L$, in 2-normed space $X$. Applying Lemma 2.5, we get relation (2.2) for $\lambda > 1$, and (2.3) for $0 < \lambda < 1$.

Sufficiency. Let $\lim_n T_n^{p,q} = L$ in 2-normed space $X$ and conditions (2.1), (2.2) and (2.3) hold. We will prove that $\lim_n x_n = L$ in $X$. Or equivalently, $\lim_n (T_n^{p,q} - x_n) = 0$ in 2-normed space $X$.

For $\lambda > 1$, we have

$$x_n - T_n^{p,q} = \frac{R_{\lambda_n}}{R_{\lambda_n} - R_n} (T_{\lambda_n}^{p,q} - T_n^{p,q}) - \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} (x_k - x_n).$$

From relation (2.1) and Lemma 2.4, we obtain

$$\left\| \frac{R_{\lambda_n}}{R_{\lambda_n} - R_n} (T_{\lambda_n}^{p,q} - T_n^{p,q}), y \right\| < \epsilon,$$

for every $y \in X$. From (2.2), for every $\epsilon > 0$ we get

$$\left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} (x_k - x_n), y \right\| < \epsilon,$$

for every $y \in X$. From last relations we have proved that $\lim_n (T_n^{p,q} - x_n) = 0$, in 2-normed space $X$.

Now for the case $0 < \lambda < 1$, we get

$$x_n - T_{\lambda_n}^{p,q} = \frac{R_n}{R_n - R_{\lambda_n}} (T_n^{p,q} - T_{\lambda_n}^{p,q}) + \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} (x_n - x_k).$$

From relation (2.1) and Lemma 2.4, we have

$$\left\| \frac{R_n}{R_n - R_{\lambda_n}}(T_n^{p,q} - T_{\lambda_n}^{p,q}), y \right\| < \epsilon,$$

for every $y \in X$. From relation (2.3), for every $\epsilon > 0$ we get

$$\left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k}(x_n - x_k), y \right\| < \epsilon,$$

for every $y \in X$. Hence, we have proved that $\lim_n (T_{\lambda_n}^{p,q} - x_n) = 0$, in 2-normed space $X$. Now proof of the Theorem follows from Lemma 2.4. □

In what follows we will show that under the conditions that $(x_n)$ is a slowly oscillating sequence (see [13]), the $T_n^{p,q}$-summability implies the convergence in the ordinary sense.

**Theorem 2.6.** *Let $X$ be a 2-normed space and $(x_n) \in X$ be $T_n^{p,q}$-limitable to $L$. If $(x_n)$ is slowly oscillating in 2-normed space $X$, then $(x_n)$ converges to $L$ in $X$.*

***Proof.*** In case $\lambda > 1$ let us suppose that $T_n^{p,q}$ converges to $L$ in $X$. To prove that $(x_n) \to L$ in $X$, it is enough to prove that

$$\lim_n \|T_n^{p,q} - x_n, y\| = 0,$$

for every $y \in X$. Let us start with

$$\begin{aligned} \|T_n^{p,q} - x_n, y\| &= \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} x_k - x_n, y \right\| = \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k}(x_k - x_n), y \right\| \\ &= \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \sum_{j=k+1}^{n} \Delta x_j, y \right\| \le \max_{0 \le k \le n} \left\| \sum_{j=k+1}^{n} \Delta x_j, y \right\|. \end{aligned}$$

Taking limit superior in both sides of the above relation and then infimum, we get

$$\inf_{\lambda > 1} \lim_{n \to \infty} \sup \|T_n^{p,q} - x_n, y\| = 0.$$

Hence, it is proved that $(x_n)$ converges to $L$ in $X$.

The case $0 < \lambda < 1$ is similar to the previous one and for this reason we omit it. □

The following result shows that if $(x_n)$ satisfies Hardy ([6]) conditions, and is $T_n^{p,q}$-summable, then it converges in the ordinary sense.

**Theorem 2.7.** *Let $(x_n) \in X$ be $T_n^{p,q}$-summable to $L$ in 2-normed space $X$. If $(x_n)$ satisfies relation*

$$n \Delta x_n = 0(1),$$

*then $(x_n)$ converges to $L$ in $X$.*

***Proof.*** It is enough to prove that

$$\lim_n \|T_n^{p,q} - x_n, y\| = 0$$

for every $y \in X$. First, suppose that $\lambda > 1$. From the condition

$$n\Delta x_n = 0(1),$$

it follows that for every $\epsilon > 0$, there exists an $n_0$ such that for every $n > n_0$ we have

$$|n\Delta x_n| < \epsilon.$$

A routine calculation gives

$$\begin{aligned}\|T_n^{p,q} - x_n, y\| &= \left\|\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k} x_k - x_n, y\right\| = \left\|\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}(x_k - x_n), y\right\| \\ &= \left\|\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\sum_{j=k+1}^{n}\Delta x_j, y\right\| \le \max_{0\le k\le n}\left\|\sum_{j=k+1}^{n}\Delta x_j, y\right\|.\end{aligned}$$

From above relations, we get

$$\|T_n^{p,q} - x_n, y\| \le \max_{0\le k\le n}\left\|\sum_{j=k+1}^{n}\Delta x_j, y\right\| \le \epsilon.$$

Hence, it is proved that $(x_n)$ converges to $L$ in $X$.

The second case, when $0 < \lambda < 1$, can be proved similarly. □

**Competing interests.** The authors declare that they have no competing interests.

**Authors contributions.** All the authors contributed equally and significantly in writing this paper.

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