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Rapid Calculation of Compound Interest Processes

By F. C. Belser

Occasion frequently arises for calculating approximately, but quickly, a present value or a sinking fund instalment when neither mathematical tables nor other aids are available.

Although compound interest processes involve lengthy calculations in order to secure accuracy, it is remarkable how closely many of the results can be obtained in a few seconds without the use of either interest tables or logarithmic tables. In fact, the calculations resolve themselves almost into mental arithmetic.

All that is necessary is to remember that, at all ordinary rates of interest, the number of periods in which money doubles itself is, roughly, the same as the number of times the periodical rate per cent. of interest is contained in 70. That is to say: $1.00 will amount to $2.00 in 23⅓ periods at 3% per period; in 17½ periods at 4%; in 14 periods at 5%; and so on. A closer approximation is secured by dividing 69 by the rate and adding .35. By this method it may be determined that $1.00 will amount to $2.00 in 23.35 periods @ 3% $\left(\frac{69}{3} + .35 = 23.35\right)$; in 17.60 periods @ 4%; in 14.15 periods at 5%; and so on.* The first method of approximation is easier to remember, but the second method gives somewhat more accurate results, particularly with the higher rates of interest. The following table shows in comparative form the actual number of periods required for money to double itself and the relative merits of the two methods of approximation.

*This fact is not a coincidence, but is based on sound mathematical reasons. The formula for the accumulation of money at compound interest is $S = (1 + i)^n$, in

Rate of interest	Actual number of periods required	Approximations $\frac{70}{\text{rate}}$	$\frac{69}{\text{rate}} + .35$
1%	69.6607	70.00	69.35
2	35.0028	35.00	34.85
3	23.4498	23.33	23.35
4	17.6730	17.50	17.60
5	14.2067	14.00	14.15
6	11.8956	11.67	11.85
10	7.2725	7.00	7.25
15	4.9595	4.67	4.95
20	3.8018	3.50	3.80

If the formula $\left(\frac{69.32}{\text{rate}} + .34\right)$ is used, the results for all rates of interest up to about 20% are obtained correct to two decimal places.

With this rule in mind, we are thus in a position to calculate rapidly the approximate amount to which $1.00 will accumulate in any given time. Thus, @ 5%, since money doubles itself at this rate in approximately 14.15 periods, we have this result:

$1.00 will become $2.00 in 14.15 periods.

$1.00 will become $4.00 in 14.15 periods more, or 28.30 periods.

$1.00 will become $8.00 in 14.15 periods more, or 42.45 periods.

If the result is desired to, say, 48 periods, correction must be made for 5.55 additional periods, which at 5% is equal to 27.75%. Since it is compound interest that is in question, it is safe to assume an accumulation of a somewhat greater percentage, say 30%. Since 30% of $8.00 is $2.40, the result is:

$1.00 will become $10.40 in 48 periods.

The mathematically correct result is $10.40127.

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which i is the periodical accumulation on $1.00, and n the number of periods. Therefore—

$$n = \frac{\log S}{\log (1 + i)}$$

Where i is less than unity—

$$\log_e (1+i) = i - \tfrac{1}{2} i^2 + 1\text{-}3\, i^3 - \tfrac{1}{4} i^4 \text{ etc.}$$

Since i is very small, the terms of the right hand expression, after the first, rapidly become inconsiderable. Thus if interest is at 5% (that is where i = .05) $\log_e (1 + i) = .05 - .00125 + .00004167$, etc.

So that, roughly speaking,

$$n = \frac{\log_e S}{i}$$

This formula is of general application as an approximation so long as i is as small as it is in the case of ordinary interest rates. When money is doubled S of course equals 2, and $\log_e 2$ equals .693147.

If the result had been required to 40 periods, it would have been necessary to make correction by deducting 2.45 periods @ 5%, or 12.25%. Instead of taking more than the indicated percentage (as was done when adding corrections above), a somewhat smaller percentage should be taken, say 12%. Since 12% of $8.00 is $.96, the result is:

$1.00 will become $7.04 in 40 periods.

The mathematically correct result is $7.03999.

It is also necessary to remember a few simple formulæ. These formulæ are not difficult to remember, as they follow a simple logical sequence. They are as follows:

Present value of $1.00 is equal to	$\dfrac{1}{\text{amount of \$1.00}}$
Amount of an annuity of $1.00 per annum is equal to	$\dfrac{\text{compound interest on \$1.00}}{\text{rate of interest}}$
Present value of an annuity of $1.00 per annum is equal to	$\dfrac{\text{compound discount on \$1.00}}{\text{rate of interest}}$
Sinking fund instalment is the reciprocal of the amount of an annuity or	$\dfrac{\text{rate of interest}}{\text{compound interest on \$1.00}}$
The premium or discount on a bond is equal to	The present value of an annuity of the difference between the coupon rate and the yield rate

Since the compound interest is the amount minus the original principal, and since the compound discount is the original principal minus the present value, all the formulæ resolve themselves into variations of the amount at compound interest.

There are, of course, many other formulæ covering interest calculations, but the foregoing are the most important, and those most frequently used.

To illustrate the ease with which these calculations can be made, examples of a few of the more important applications of the formulæ are given below. For clearness, the processes are set out much more fully than is really necessary in practice. It should be observed that in no case are decimals carried out to more than two or three places, since nothing is claimed for the calculations except approximately correct results in the absence of mathematical tables. The formula $\left(\frac{69}{\text{rate}} + .35\right)$ has been used in these examples for determining the period when \$1.00 amounts to \$2.00.

Ex. 1—What is the amount of \$150.00 for 25 years @ 5%?

The first step is to find the amount of \$1.00 for the given period.

\$1.00 will amount to \$2.00 in 14.15 years.

\$1.00 will amount to \$4.00 in 14.15 years more, or 28.30 years all told.

As the amount for only 25 periods is required, it is necessary to deduct 3.30 years @ 5%, say 15% or \$.60.

\$1.00 therefore becomes \$3.40 in 25 years.

The amount of \$150.00 is equal to \$510.00.

True result—\$507.95.

Ex. 2—What is the present value of \$25,000.00 due 30 years hence, at 6% interest?

The formula for the present value is $\frac{1}{\text{amount of \$1.00}}$

It is therefore necessary to first find the amount of \$1.00 for 30 years @ 6%:

\$1.00 will become \$2.00 in 11.85 years.

\$1.00 will become \$4.00 in 11.85 years more, or 23.70 years.

It is necessary to add 6.30 years @ 6% or 37.18%.

As the interest is compound, 42% or 43% may justifiably be taken, say \$1.72.

\$1.00 will therefore become \$5.72 in 30 years.

This result can be proved by carrying the calculation one stage further, viz.:

\$1.00 will become \$8.00 in 23.70 years plus 11.85 years, or 35.55 years.

Here the correction necessary is a deduction of 5.55 years @ 6%, or 33.30%.

Deducting a somewhat smaller percentage, say 28%, or \$2.24:

\$1.00 will therefore become \$5.76 in 30 years.

This result varies only \$.04 from the first above.

The present value of \$1.00 is then determined thus:

$$\frac{1}{5.76} = \$.1736$$

and the present value of \$25,000.00 is \$4,340.00.

True result—\$4,352.75.

Ex. 3—What is the amount of an annuity of \$1,500.00 for 20 years @ 5%?

$$\text{Formula applicable } \frac{\text{compound interest}}{\text{rate}}$$

In order to determine the compound interest, it is necessary first to determine the amount of \$1.00 for 20 periods @ 5%.

\$1.00 will become \$2.00 in 14.15 years.

To this must be added 5.85 years @ 5%, or 29.2%.

Using a somewhat higher percentage, say 33%, \$1.00 will become \$2.66 in 20 years.

The compound interest is therefore \$1.66, which, divided by the rate of interest, gives \$33.20 as the amount of an annuity of \$1.00.

The amount of an annuity of \$1,500.00 is therefore \$49,800.00.

True result—\$49,598.93.

Ex. 4—What is the value of a leasehold with a rental of \$35,000.00 per annum, having 75 years to run, interest @ 4%?

This problem involves the determination of the present value of an annuity the formula for which is $\frac{\text{compound discount}}{\text{interest rate}}$.

In order to determine the compound discount, it is necessary first to determine the present value of \$1.00, which is equal to

$$\frac{1}{\text{amount of } \$1.00}$$

The amount of \$1.00 for 75 periods at 4% is found as follows:

\$1.00 will amount to \$2.00 in 17.60 years.

\$1.00 will amount to \$4.00 in 17.60 years more, or 35.20 years.

\$1.00 will amount to \$8.00 in 17.60 years more, or 52.80 years.

\$1.00 will amount to \$16.00 in 17.60 years more, or 70.40 years.

To this must be added 4.6 years @ 4%, or 18.4%. Add, say 20%, or \$3.20, then \$1.00 will amount to \$19.20 in 75 years.

The present value is equal to $\frac{1}{19.20}$ or \$.052

and the compound discount is therefore \$1.00 — \$.052, or \$.948.

Using this figure in the original formula, we have $\frac{.948}{.04}$ or \$23.70, for the present value of an annuity of \$1.00 for the period required.

The present value of an annuity of \$35,000.00 therefore equals \$829,500.00

True value—\$828,814.29.

Ex. 5—What is the value of a 5% bond, to yield 4%, maturity 15 years, interest payable semiannually?

The difference between the coupon rate each half year (\$25.00) and the yield rate each half year (\$20.00) is \$5.00, and the premium paid for this bond will represent the present value of an annuity of this difference of \$5.00 payable every interest period for 30 periods, interest being at 2% per period.

The formula for the present value of an annuity is,

$$\frac{\text{compound discount}}{\text{rate}}$$

As in example 4, it is necessary first to determine the amount of \$1.00 for the period, then its present value, and from this its compound discount.

\$1.00 will amount to \$2.00 in 34.85 periods.

From this must be deducted 4.85 periods at 2%, say 9%, or \$.18.

\$1.00 will therefore amount to \$1.82 in 30 periods.

The present value of \$1.00 is equal to $\frac{1}{1.82}$ or \$.549.

The compound discount is therefore \$1.00 — \$.549, or \$.451.

The present value of an annuity of \$1.00 is $\frac{.451}{.02}$ or \$22.55.

Therefore the present value of an annuity of \$5.00 is \$112.75, or the premium on the bond.

This gives as the value of the bond \$1,112.75.

True value—\$1,111.98.

Ex. 6—What is the value of a 5% bond, to yield 5.75%, maturity 40 years, interest semiannually?

The difference between the coupon and the yield rate every half year is \$3.75, as follows:

Coupon		\$25.00
Yield rate @ 2.875%		28.75
	Difference	\$3.75

The discount on the bond will represent the present value of this difference for 80 periods, with interest @ 2.875%.

The calculation is now similar to that in example 5:

\$1.00 will amount to \$2.00 in 24.35 periods.

\$1.00 will amount to \$4.00 in 24.35 periods more, or 48.70 periods.

\$1.00 will amount to \$8.00 in 24.35 periods more, or 73.05 periods.

Add 6.95 periods @ 2.875%, say 21%, or \$1.68.

\$1.00 will amount to \$9.68 in 80 periods.

The present value of \$1.00 is equal to $\frac{1}{9.68}$ or \$.1033.

Therefore the compound discount is .8967.

The present value of an annuity of \$1.00 is therefore

$$\frac{.8967}{.2875} \text{ or } \$31.20.$$

And the present value of an annuity of \$3.75 is \$117.00, which is the discount on the bond.

This makes the value of the bond \$883.00.

True value—\$883.07.

Ex. 7—What sinking fund will it be necessary to set aside annually to extinguish a debt of \$2,000,000 in 20 years @ 5%?

The formula for the sinking fund instalment is

$$\frac{\text{rate}}{\text{compound interest}}$$

The amount of \$1.00 is first determined—

\$1.00 will amount to \$2.00 in 14.15 periods.

Add 5.85 periods @ 5%, or 29.25%.

Using 33% as the increase, or \$.66, \$1.00 will amount to \$2.66 in 20 periods.

The compound interest is therefore \$1.66 and the sinking fund instalment for \$1.00 is $\frac{.05}{1.66}$ or \$.0301.

The sinking fund instalment for \$2,000,000 is then \$60,200.00.
True value—\$60,485.18.

While for all ordinary purposes it is sufficient to remember the formula for the doubling of money, it is interesting to note further extensions of the same principle. Thus: \$1.00 will amount to \$1.50 in as many years as the rate of interest is contained in 41. That is, at 5%, \$1.00 will amount to \$1.50 in 8.2 periods. The formula

$$\left(\frac{40.55}{\text{rate}} + .20\right)$$

gives the exact number of periods to at least two decimal places for all rates up to over 40% per period.

This added data is frequently useful when the interval from one step to another is too great if the period of doubling only is used. Thus, in example 7:

\$1.00 will amount to \$2.00 in 14.15 periods.

\$1.00 will amount to \$3.00 in 8.20 periods more, or 22.35 periods.

The correction is now a deduction of only 2.35 periods at 5%, say 11½% of \$3.00, or \$.345.

\$1.00 will therefore amount to \$2.655 in 20 periods.

This compares with \$2.66 as determined by the first method, and with \$2.65330, the mathematically correct result.

There is no limit to which this principle can be carried; it depends entirely on the efficiency of the individual's memory. It is therefore well to choose for permanent use only those points which one is likely to retain. In the foregoing illustrations the degree of approximation attained depends somewhat on the accuracy with which the allowance for the odd years is estimated, but even if only simple interest is computed for these odd years the results will be sufficiently accurate for many purposes, and experience has fully demonstrated the usefulness of the method herein outlined.