9-2022

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Available online at www.sciencedirect.com

Manufacturing Letters

Manufacturing Letters 33 (2022) 872–879

50th SME North American Manufacturing Research Conference (NAMRC 50, 2022)

Minimax Registration for Point Cloud Alignment

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Abstract

The alignment, or rigid registration, of three-dimensional (3D) point clouds plays an important role in many applications, such as robotics and computer vision. Recently, with the improvement in high precision and automated 3D scanners, the registration algorithm has become critical in a manufacturing setting for tolerance analysis, quality inspection, or reverse engineering purposes. Most of the currently developed registration algorithms focus on aligning the point clouds by minimizing the average squared deviations. However, in manufacturing practices, especially those involving the assembly of multiple parts, an envelope principle is widely used, which is based on minimax criteria. Our present work models the registration as a minimization problem of the maximum deviation between two point clouds, which can be recast as a second-order cone program. Variants for both pairwise and multiple point clouds registrations are discussed. We compared the performance of the proposed algorithm with other well-known registration algorithms, such as iterative closest point and partial Procrustes registration, on a variety of simulation studies and scanned data. Case studies in both additive manufacturing and reverse engineering applications are presented to demonstrate the usage of the proposed method.

*Keywords:*Rigid registration; 3D point clouds; Metrology; Reverse engineering; Additive Manufacturing; Second-order cone programming.

1. Introduction

The point cloud is a type of data format that is generally created by coordinate measuring machines (CMM) or three-dimensional (3D) scanners to describe the geometric information of a physical object or an environment in the digital world [1]. With the improvement in automation, precision, and speed of the 3D scanner during the past decades, it has attracted more interest in manufacturing and engineering applications [2-4]. On the one hand, the point clouds surveyed by scanners can be utilized for quality inspection purposes, where the point cloud of each produced part is compared with the nominal model and the tolerancing design to determine whether the corresponding part is qualified [5, 6]. On the other hand, reverse engineering (RE) techniques collect the point cloud from a physical object to (re-)construct its digital model for reengineering or redesign purposes [7, 8]. The point clouds surveyed by 3D scanners are recently utilized to bridge the gap between design and manufacturing, known as the digital twin, to build more realistic virtual models for better design and production [9, 10].

The raw point clouds collected by CMM or 3D scanners also incorporate some nuisance factors, other than the geometric information of the physical parts, such as the noises in the scanning process and the posture or position of the scanned parts [11]. These factors prevent the direct alignments between the point clouds and the nominal design or among point clouds. Registration plays an important role in aligning the point clouds with respect to predetermined criteria [12]. To avoid confusion, in this paper, we define the registration as its one major subset, the rigid registration, for manufacturing applications. Rigid registration is an algorithm that removes the translational and rotational factors in the point clouds in order to align them to a target [13]. Therefore, registration algorithms are important for many applications involving point clouds, such as robotics [12] and computer vision [14].

Nomenclature

3D	three-dimension
RE	reverse engineering
AM	additive manufacturing
CMM	coordinate measuring machine
ICP	iterative closest point
PPA	partial Procrustes analysis
CAD	computer-aided design
GD&T	geometric dimensioning and tolerancing
X^n	point cloud n
X_i^n	the ith row (the coordinates of point i) in point cloud n
Γ	rotation matrix
γ	translation vector
GPA	generalized Procrustes analysis
OPR	ordinary Procrustes registration
i.i.d.	independent and identically distributed

Due to the noisy nature of the point clouds, the registration algorithms are designed for aligning point clouds to optimize certain criteria. The two most popular examples are partial Procrustes analysis (PPA) [15], which is generally used in statistical shape analysis, and iterative closest point (ICP) [16], which is widely implemented in computer vision and robotics communities. PPA is defined to be a minimization problem that finds the optimal rotation matrices and translation vectors to minimize the Procrustes distance between one point cloud and another, or the point clouds and their mean configuration, if more than two point clouds are of interest. ICP, on the other hand, minimizes the distance between point clouds by alternating the correspondence estimates and their corresponding transformation factors.

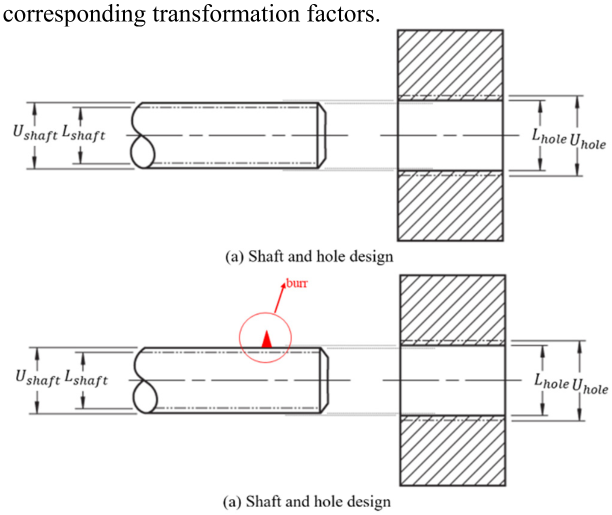


Figure 1. Tolerancing design for shaft and burr design: (a) original tolerance design; (b) one small out-of-tolerance burr on the shaft, while the overall average radius of the shaft is in-tolerance

One potential issue of applying these algorithms in a manufacturing setting is their registration criteria. Both methods minimize the averaged squared distance among all the points in the point clouds (even though we can register the point clouds according to a subset of corresponding pairs in ICP). However, the envelope principle, or the Taylor principle, is the golden standard adopted by geometric dimensioning and tolerance (GD&T) standards, including ASME Y14.5 and ISO tolerancing standards, to define the tolerance regions [17] for manufacturing and production system design and analysis [18]. The tolerance region, or acceptance region in quality inspection, is defined as the area around the nominal design covered by a pair of upper and lower boundaries. In other words, the qualified parts, defined by the tolerance design, are the ones whose maximum deviation from the nominal design should be within the tolerance region. This principle also has real practical insights in assembly analysis. For example, in the classic shaft and hole design (Figure 1), if a small burr exists on the shaft part, which makes this spot out of the tolerance region, these two parts cannot be assembled. However, when analyzing the surveyed point cloud for quality inspection, PPA or ICP algorithms, which register the point clouds by minimizing the average squared deviations, could underestimate such an effect. Another important application involving point cloud processing is RE, whose objective is to build a computer-aided design (CAD) model based on multiple views or multiple scans for accurate model construction. One important component of the CAD model is tolerance design, which is also missing in RE applications. Recently, Geng and Bidanda [17] proposed a tolerance estimation procedure for tolerance inference from the point clouds of multiple parts with the same design. In their paper, even though multiple scans for each part are performed to get the mean scan model to reduce the effect of the scanner, PPA is utilized for registration of multiple scans, which could potentially overestimate the original tolerance designs.

In this paper, we propose a new registration procedure for manufacturing applications. The alignment of the point clouds is based on minimizing the area between the upper and lower envelopes, which cover all the registered points in between. In other words, the rotation matrices and the translation vectors of the rigid registration are sought by minimizing the maximum deviations among point clouds, or a minimax problem. Pairwise registration, which aligns one point cloud to another, is firstly modeled and, later, recast as a second-order cone program for fast optimization. Next, the alignment of multiple point clouds is proposed by increasing the constraints of the pairwise version.

The remainder of this paper is organized as follows. Section 2 briefly surveys the two most important rigid registration techniques, which are ICP and PPA. Algorithms and their major usages are presented. Next, minimax registration models and algorithms are proposed in Section 3 for both the pairwise case and the case involving multiple point clouds. Simulation studies are designed in Section 4 to validate the performance of the proposed algorithm. Comparison results among the three registration algorithms are presented. Case studies regarding reverse engineering and quality inspection for additive manufacturing are presented and discussed in Section 5. Finally, concluding remarks and directions for future research that are motivated by our new algorithm are in Section 6.

2. Registration Algorithms

In this section, we first introduce the basic concepts in rigid registration. The point cloud aligning problem is generalized as an optimization model. Two widely adopted algorithms, ICP and PPA, are briefly summarized. Major steps of both methods are presented, together with their primary applications.

2.1. Basics of Rigid Registration

Let $X^1, X^2, \ldots, X^N \in \mathbb{R}^{K\times 3}$ are N matrices representing one point cloud containing the coordinates of K rows, whose row, X_i^n, contains x-, y-, and z-coordinates of the ith point in nth point cloud, $i=1,\ldots,K$, $n=1,\ldots,N$.

Rigid registration is a way to align two point clouds without changing their size or shape by minimizing a metric measuring the distance between the point clouds through changing the rational and translational factors. There are three major components in the rigid registration: rotation matrix, translation vector, and a target point cloud.

A rotation matrix is a 3×3 matrix defined as

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \end{pmatrix} \quad (1)$$

where the elements Γ_{ij}, $i,j=1,2,3$, are determined by three rotation angles about the x-, y-, and z-axes, respectively. By multiplying the rotation matrix to a point cloud X_i^n, the new matrix $X_i^n\Gamma$ represents the point cloud with the same size and shape but has a different orientation.

Translation vector, on the other hand, is a 3-dimensional vector, defined as

$$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \quad (2)$$

where the elements γ_i, $i=1,2,3$, represent the length of translation along x-, y-, and z-directions, respectively.

Rigid registration, or a registration in general, is an algorithm that aligns the points clouds by removing the effects caused by the above two factors for purposes such as statistical modeling and inference, quality inspection. To achieve this goal, one point cloud is kept fixed as a reference or a target, and other point clouds are transformed to align, or best match, this target. The target can be the nominal point cloud from the original design in quality inspection applications or an arbitrary point cloud in RE projects.

2.2. Iterative Closest Point Algorithm

Iterative closest point (ICP), also named as iterative corresponding point, introduced by Besl and McKay [16], is an algorithm that repetitively updates the rotation matrix and translation vector to minimize a deviation metric between the target and the other point clouds. The deviation metric is generally a measurement of the distance between two point clouds, such as the sum of squared differences between the corresponding points. ICP is one of the most widely utilized registration algorithms in computer vision and robotics.

Essentially, the ICP algorithm can be performed iteratively via the following steps:

- Step 1: Match each point in the point cloud with the closed point in the target point cloud;
- Step 2: Estimate the combination of the rotation matrix and translation vector based on deviation metric, which aligns each point to the corresponding target point;
- Step 3: Transform the points using the estimated transformation coefficients;
- Step 4: Repeat the above procedures until convergence (<0.0001).

Variants of the ICP algorithm can be applied to register different types of geometric data, including general point clouds, line segment sets, implicit or parametric curves and surfaces, and faceted surfaces [16]. Furthermore, the point-wise correspondence relationship can also be explored via this algorithm, which makes it a widely used registration method in different applications, including multi-view registration, computer vision, and medical imaging. However, the ICP algorithm is generally limited to one-to-one registration that one point cloud, also called source, is aligned to the target. It becomes an issue when applying ICP to make inferences in statistical shape analysis, where multiple point clouds are available, and the mean is unknown in RE applications.

2.3. Partial Procrustes Analysis

Partial Procrustes analysis (PPA), or generalized Procrustes analysis (GPA) in general, is a set of registration algorithms developed by the statistical shape analysis (SSA) community to study the statistical behavior regarding shapes. PPA is a specialized version of GPA that remains the size factor in the point clouds during registration. It maps the point cloud X to a new space, called size-and-shape space, where statistical properties have been studied in [19].

We are considering the general case where $n\geq 2$ point clouds are available, which are $X_1,\ldots,X_n$. The point clouds are random samples from a population with a mean μ, which is unknown. Our objective is to estimate the mean by averaging the sample configurations while removing the other variational factors, including translation and rotation. To achieve this objective, we can formulate a least-squares model to find an estimate of μ. Therefore, we minimize

$$G_p = \inf_{\Gamma_i, \gamma_i, \mu} \sum_{i=1}^{n} \|(X_i \Gamma_i + 1_k \gamma_i^T) - \mu\|^2, \tag{3}$$

where Γ_i is a 3-D rotation matrix, 1_k is a vector of k ones, and γ_i is the location parameter, for $i = 1, \ldots, k$.

PPA is a particular case of GPA, developed by Gower and Ten Berge [79], which includes a scaling factor in the above minimization model. Therefore, we can adopt the GPA procedure by removing the steps regarding the size. The PPA algorithm proceeds as follow to solve the above least squares problem:

- Step 1: Center the point clouds to remove the location. The centering process can be done by multiplying each configuration matrix with a Helmert submatrix H, defining

$$X_i^P = H X_i \tag{4}$$

The Helmert submatrix is defined to be the $(k-1) \times k$ matrix whose jth row is

$$(h_j, \ldots, h_j, -jh_j, 0, \ldots, 0), \quad h_j = -[j(j+1)]^{-1/2}, \tag{5}$$

and the jth row consists of h_j repeated j times, one $-jh_j$, and $k-j-1$ zeros.

- Step 2: For the ith configuration matrix, let

$$\overline{X}_{(i)} = \frac{1}{n-1} \sum_{j \neq i} X_j^P \tag{6}$$

- Step 3: Do ordinary Procrustes registration (OPR), involving only rotation, of X_i^P's onto $\overline{X}_{(i)}$ to generate new X_i^P for all i. The OPR is a simplified version of PPA that only register one configuration, X_1, onto another one, X_2, whose locations have been removed, by solving

$$\min_{\Gamma} \|X_2 - X_1 \Gamma\|^2 \tag{7}$$

This problem is well studied by Jackson and Horn et al. [20]. The minimizer is given by $\widehat{\Gamma} = UV^T$, where U and V can be obtained through the following singular value decomposition problem,

$$\frac{X_2^T X_1}{\|X_1\| \, \|X_2\|} = V \Lambda U^T. \tag{8}$$

- Step 4: Repeat Steps 2 and 3 until G_p converge (< 0.0001).

According to the above procedure, it is obvious that PPA could register multiple point clouds simultaneously while minimizing the total squared distance. The algorithm is computationally efficient, which generally converges in 2-5 steps.

PPA is widely used in statistical shape analysis when the size factor is under consideration since it transforms the point clouds from the configuration space to the size-and-shape space. In this space, if the point-wise variance is small, each row of the configuration approximates an independent multivariate Normal random variable [21]. Thus, the coordinates, or the three columns of the point clouds, can be seen as independently and normally distributed in a manufacturing setting, which could simplify future statistical and other analytical procedures.

The primary issue with the PPA algorithm is overlooking the local deviation. Because of the sum of squares loss as the objective function, the total deviations among points are evenly distributed among the point cloud. For example, a small surveyed outlier could shift the alignment between two point clouds, potentially degrading the following quality inspection and analysis.

Many other rigid registration algorithms exist, such as coherent point drift [22], kernel correlation [23], Gaussian mixture models [24], which have the potential to perform pose and correspondence registration simultaneously. However, due to indirect registration, these methods tend to provide a relatively higher point-wise deviation among point clouds.

3. Formulation of Minimax Registration

Classic rigid registration algorithms, such as ICP or PPA presented in the previous section, align the point clouds mostly based on a least-square criterion. The analyses are usually straightforward and simple. However, the least-square method is to minimize the average deviations between/among point clouds. This may not be appropriate when significant local deviations exist or when constructing a tolerance region in a manufacturing setting. We propose to register the point clouds to achieve the smallest deviation band, which is the minimum area covering all the deviated points of the surveyed point cloud, for geometric quality inspection, tolerance specification, and nominal RE-model inference. This smallest deviation band is in line with the geometric dimensioning and tolerancing (GD&T) standards, especially ASME Y14.5, which is based on the minimum envelope principle, also known as the Taylor principle, which represents the maximum (or minimum) allowable size for a part must be within the same range as the maximum size (or minimum size).

In this study, we focus on aligning a pair of point clouds. This case can be widely used for quality inspection, whose objective is to align the point cloud surveyed from an additively manufactured part to the nominal design model. After alignment, the maximum deviation can be compared with the tolerance design to determine whether it is intolerance.

3.1. Model for Pair Registration

The minimum envelope criteria can be modeled as a minimax problem. The envelope is a pair of boundaries covering all the points within them. In other words, it can be mathematically defined as the maximum deviation between each pair of corresponding points of two or more point clouds. Here, without loss of generality, we define this envelope to be symmetric in that the distances between the center line and the upper or lower envelope boundaries are the same. In this case, the width of the envelope is the absolute maximum distance between the corresponding points. Therefore, to construct a minimum envelope is to find the registration algorithm that minimizes the maximum deviation.

Let X_1 and X_2 be a pair of point clouds, each of which has K points. We want to seek the rotation matrix Γ, presented in Equation 1, and a translation vector γ as Equation 2 by solving the following minimax model,

$$\min_{\Gamma,\gamma} \max_{i=1,\ldots,K} \|X_i^1\Gamma + \gamma^T - X_i^2\|^2, \quad (9)$$

where $\|\cdot\|$ is the ℓ^2 norm of a row vector, and X_i^j is the ith row of the point cloud j, $j=1, 2$. The objective function is a minimax problem, which is nonconvex.

To solve this problem, we can reformate the model as

$$\begin{aligned} \min_{\Gamma,\gamma} \quad & d^2 \\ s\,t \quad & \|X_i^1\Gamma + \gamma^T - X_i^2\|^2 \le d^2, \ \forall i=1,\ldots,K \end{aligned} \quad . \quad (10)$$

This model is a second-order cone programming problem with K second-order cone constraints[25, 26]. With this equivalence form, the minimax model becomes a convex optimization problem. Algorithms, such as primal-dual interior point methods [27] or the product-form Cholesky factorization approach [28], can be utilized to solve the second-order cone programming problem. Convex optimization servers, including CVXPY [29] in Python, can be utilized.

3.2. Model for Multiple Registration

Multiple registrations exist in many applications, including RE or 3D scanning applications, where multiple parts or scans are available ($n \ge 3$).

Similar to Equation 10, we formulate this multiple registration problem as

$$\min_{\Gamma^n,\gamma^n} \max_{\substack{i=1,\ldots,K;\\ n,m=1,\ldots,N;\\ n\neq m}} \|X_i^n\Gamma^n + (\gamma^n)^T - X_i^m\|^2. \quad (11)$$

Mimicking the previous procedures, we can have the below equivalent reformulation,

$$\begin{aligned} \min_{\Gamma^n,\gamma^n} \quad & d^2 \\ s\,t \quad & \|X_i^n\Gamma^n + (\gamma^n)^T - X_i^m\|^2 \le d^2, \ \forall i=1,\ldots,K, \\ & \forall n,m=1,\ldots,N, \\ & n\neq m \end{aligned} \quad (12)$$

This formulation has $\binom{N}{2} \times K$second-order cone constraints.

4. Simulation Study

We now illustrate and validate the minimax registration algorithm with a simulated hemispherical design (Figure 2). We simulate the point clouds collected from a AM-printed parts, whose deviational behavior is modeled in [30]. Configurations are simulated from a design with a hemispherical shape, whose radius is 20 millimeters (mm). Assume the standard deviation for each of the points collected by the simulated production process is 0.5 mm. Then, $N(0, 0.5)$ independent and identically distributed (*i.i.d.*) errors in the unit of mm are added to each coordinate of the points to emulate the random error in the production process. There are 100 layers in total, while fifty landmarks are simulated on each layer. A point cloud with an ideal hemispherical design is also generated as the nominal design for quality inspection. Thirty simulated point clouds are generated, and each is compared with the nominal model. The proposed minimax registration algorithm is compared with the classic ICP and PPA algorithms with the least square distance. The results are presented as a box plot and pairwise comparison in Figure 3.

Fig. 2. Geometric design of the simulated hemispherical design.

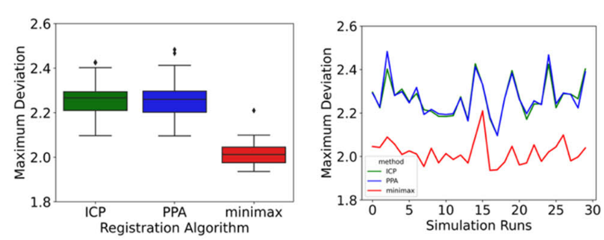


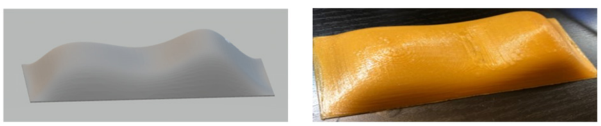
Fig. 3. Comparison results of the simulation experiment among the proposed minimax registration, ICP, and PPA algorithms: (a) box plot; (b) line plot of pairwise comparison. The red line represents the maximum deviation from the proposed minimax registration algorithm; the blue line represents the maximum deviation from the PPA algorithm; the green line represents the maximum deviation from the ICP algorithm.

It can be easily seen that the proposed minimax registration algorithm outperforms ICP or PPA algorithm regarding the maximum deviation, showing that it provides the envelope with the smallest distance covering all the points.

5. Case Studies

5.1. A Reverse Engineering Application

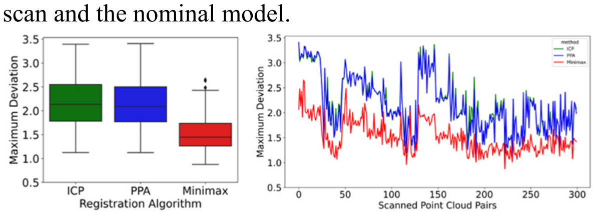
An arbitrary unique freeform object, shown in Figure 4, is printed by the LulzBot TAZ 3-D printer, a fused deposition modeling (FDM) machine, using gold metallic 2.85 mm polymer. The part is unique in the sense that the part created by the printer is of high variability, which causes the part to deviate from the initial design and the true dimensions are unknown. The point cloud samples are collected through 30 independent sequential scans covering the full body of the target part using FARO Platinum 8' Arm Laser Scanner. Pairwise registration is performed, which could be later utilized for estimation of the nominal design or construction of the tolerance region for reproduction.



(a) (b)

Fig. 4. The design and the physical part of the freeform object.: (a) original STL file; (b) photo of the physical part printed by FDM.

Each pair of surveyed point clouds are registered via the above all three registration algorithms, i.e., ICP, PPA, and minimax registration. The results of maximum deviations for 300 pairwise registrations are presented in Figure 5. The observation is close to the simulation result that the minimax registration provides the smallest envelope among all pairs, which is in consistent with the ASME standards. Also, it can be noticed that, with the same objective, the performances of ICP and PPA are similar with much smaller difference. One of the major observations is that, for each scan, the proposed minimax registration algorithm outperforms ICP and PPA algorithms. However, the differences among the three algorithms of interest depend on the true deviation between the scan and the nominal model.



(a) (b)

Fig. 5. Comparison results among the proposed minimax registration, ICP, and PPA algorithms for 300 pairwise registration of 30 surveyed point cloud collected from an object printed by LulzBot FDM printed: (a) box plot; (b) line plot of pairwise comparison. The red line represents the maximum deviation from the proposed minimax registration algorithm; the blue line represents the maximum deviation from the PPA algorithm; the green line represents the maximum deviation from the ICP algorithm.

5.2. Quality Inspection in Additive Manufacturing

AM processes produce parts by adding material layer-by-layer, making it distinct from traditional manufacturing processes. They decompose the 3D design into many thin layers and fabricate each layer sequentially along with the build orientation [31]. In this way, most of the features are "byproducts" of this process, in which the part surfaces are the layers' surrounding profiles[32]. Therefore, conventional geometric inspections based on parametric features are not reliable. 3D scanning could provide a detailed geometric description of the printed objects, which is more appropriate for quality inspection.

A half ball-shaped design is selected for the experiment. The LulzBot TAZ FDM printer is utilized to perform the experiment using a 2.85 mm gold metallic polymer filament, which is presented in Figure 6. The layer thicknesses, 0.18 mm and 0.38 mm, are selected for comparative study, which is similar to the work in [32]. The maximum deviations between the objects printed with these two settings and the nominal design are reported in Figures 7 and 8 with the three registration algorithms presented in this study. The results are similar to the conclusion in [32] that the larger layer thickness $\theta = 0.38$ provides a smaller maximum deviation. However, it can also be observed that the variance of the maximum deviation of the objects printed with layer thickness $\theta = 0.38$ is greater when compared to the ones printed with layer thickness $\theta = 0.18$. Even though fewer layers are printed with larger layer thickness, which causes less deviation in the printed parts, the uncertainty of the actual printed layer thickness is greater due to the thicker layer.

Fig. 6. Photo of the physical half ball-shaped object printed by LulzBot TAZ FDM printer.

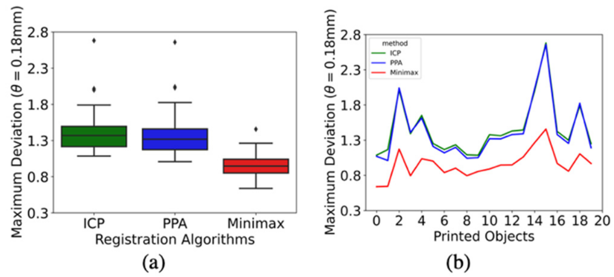


Fig. 7. Comparison results among the proposed minimax registration, ICP, and PPA algorithms for the point clouds collected from 20 objects printed by LulzBot FDM printed with layer thickness $\theta = 0.18$: (a) box plot; (b) line plot. The red line represents the maximum deviation from the proposed minimax registration algorithm; the blue line represents the maximum deviation from the PPA algorithm; the green line represents the maximum deviation from the ICP algorithm.

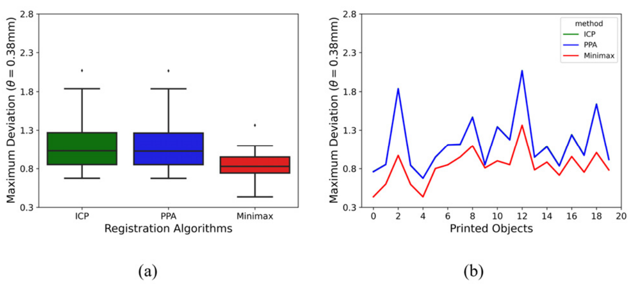


Fig. 8. Comparison results among the proposed minimax registration, ICP, and PPA algorithms for the point clouds collected from 20 objects printed by LulzBot FDM printed with layer thickness $\theta = 0.38$: (a) box plot; (b) line plot. The red line represents the maximum deviation from the proposed minimax registration algorithm; the blue line represents the maximum deviation from the PPA algorithm; the green line represents the maximum deviation from the ICP algorithm.

6. Concluding Remarks and Future Work

This paper introduces a novel registration algorithm that is in line with the GD&T standards and provides the smallest envelope covering the point clouds. This registration algorithm can be utilized for various applications, such as quality inspection, reverse engineering, process analysis for additive manufacturing, and computer vision. The original minimax registration formulation is recast and solved by an equivalent second-order cone programming problem. Both pairwise registration and multiple procedures are presented. A simulation experiment and case studies in RE and AM applications are utilized to validate the performance of the proposed algorithm. The experiments have shown the proposed minimax registration procedure outperforms the conventional rigid registration algorithms, such as ICP and PPA, and provides the smallest envelope.

Even though a multiple registration model has been introduced in this study, the number of second-order cone constraints can increase quadratically with respect to the number of scans. Since the number of points in a point cloud K is generally large, it can be computationally intensive to solve a multiple registration problem. Our future work lies in developing new algorithms to increase the computational speed of the proposed problem based on the methods in discrete and computational geometry.

New directions of research can also be developed for online registration. In applications such as computer vision and reverse engineering, point clouds are generally collected sequentially. Therefore, the registration procedure needs to perform multiple times as the number of point clouds increases. Since the second-order cone programming can be time-consuming, it can be inefficient to perform the registration multiple times. As the number of the constraints increases quadratically with respect to the number of point clouds, the computational time also increases significantly if the point clouds are registered in a sequential fashion. Even though we could set one point cloud, say the first surveyed one, as fixed, and then register the later ones to the fixed point cloud, the generated envelope, or minimax deviation, is suboptimal. Therefore, an “online” minimax registration can be more efficient for these applications.

Acknowledgements

The work of Zhaohui Geng was supported by the National Institute of Standards and Technology under Grant No. 70NANB21H038.

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