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Solution of Fractional Harmonic Oscillator in a Fractional B-poly Basis

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Abstract - An algorithm for approximating solutions to fractional-order differential equations in fractional polynomial basis is presented. A finite generalized fractional-order basis set is obtained from the modified Bernstein Polynomials, where α is the fractional-order of the modified Bernstein type polynomials (B-polys). The algorithm determines the desired solution in terms of continuous finite number of generalized fractional polynomials in a closed interval and makes use of Galerkin method to calculate the unknown expansion coefficients for constructing the approximate solution to the fractional differential equations. The Caputo's definition for a fractional derivative is used to evaluate derivatives of the polynomials. Each term in a differential equation is converted into matrix form and the final matrix problem is inverted to construct a solution of the fractional differential equations. However, the accuracy and the efficiency of the algorithm rely on the size of the set of B-polys. Furthermore, a recursive definition for generating fractional B-polys and the analytic formulism for calculating fractional derivatives are presented. The current algorithm is applied to solve the fractional harmonic oscillator problem and a number of linear and non-linear fractional differential equations. An excellent agreement is obtained between desired and exact solutions. Furthermore, the current algorithm has great potential to be implemented in other disciplines, when there are no exact solutions available to the fractional differential equations.

Keywords - Fractional Harmonic Oscillator, Generalized Bernstein Polynomials, Galerkin Method, Linear and Non-Linear Fractional Differential Equations, Fractional Basis

1. Introduction

Modified Bernstein polynomials [1] are becoming extremely useful techniques for solving complicating problems in engineering, computer science and physics disciplines. The polynomials are analytically well defined, continuous over an interval, form a finite basis, and represent complicated arbitrary functions to the desired accuracy. Also, because of their analytic nature, they can be differentiated and integrated effortlessly.

In the past years, the fractional-order differential equations have been solved by several authors [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] using numerical as well as analytic techniques. Recently, the authors [1] have solved differential equations using the Galerkin method on the basis of modified Bernstein Polynomials of degree-n over a finite interval. More recently, the authors [13, 14] solved differential equations by means of B-polys and operational matrix methods. Isik et al. [15, 16] have studied linear integral-differential equations as well as the higher order initial and boundary value problems using rational approximation based on B-polys.

Our aim is to present an algorithm to solving fractional-order differential equations by means of generalized Galerkin method and the B-Poly basis of fractional-order. The procedure takes advantage of the continuity and unitary partition property of the generalized fractional-order B-polys on an interval [0, R]. In many applications of B-polys, the matrix formulism obtained by converting a differential equation into matrix form provides greater flexibility to impose initial as well as boundary conditions. The set of B-polys of rational degree (α) on an interval forms a complete basis for continuous (n+1) polynomials. In this paper, example of the fractional harmonic oscillator as well as several examples of fractional differential equation is considered. The solutions to these equations are presented as combination of generalized fractional-order B-polys. In the following sections, we provide briefly Caputo's derivative, define B-poly basis in terms of fractional-order and provide graphs representing absolute errors. Finally we present linear and non-linear examples in which fractional differential equations are solved using the present algorithm.

2. Caputo's fractional differential operator

In this section, we introduce Caputo's fractional operator D^{γ} [17, 18]. The fractional derivative of f(x) in Caputo sense is defined by

$$D^{\gamma} f(x) = J^{m-\gamma} D^{m} f(x) = \frac{1}{\Gamma(m-\beta)} \int_{0}^{x} (x-t)^{m-\gamma-1} f^{(m)}(t)\, dt, \quad (1)$$

$for\, m-1 < \gamma \le m,\ m \in N,\ x > 0,\ f \in C_{-1}^{m}$

The Caputo's derivative of a constant is zero, i.e.

$D^{\gamma} C = 0$ and

$$D^{\gamma} x^{\alpha} = \begin{cases} 0 & \text{for } \alpha \in N_0 \text{ and } \alpha < [\gamma] \\ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\gamma)} x^{\alpha-\gamma} & \text{otherwise} \end{cases} \quad (2)$$

We would like to expand an unknown function in terms of generalized fractional polynomials, for example [1]

$$y(x) = \sum_{i=0}^{n} a_i B_{i,n}(\alpha, x)\, . \quad (3)$$

Where a_i 's are unknown coefficients and α is fractional-order parameter. We make use of the Caputo's derivative property as a linear operator,

$$D^{\gamma}\left(\sum_{i=0}^{n} a_i B_{i,n}(\alpha, x) \right) = \sum_{i=0}^{n} a_i\ D^{\gamma}\left(B_{i,n}(\alpha, x) \right). \quad (4)$$

In the following section we, briefly, define generalized B-Polys and some of its properties.

3. Fractional-order B-Poly basis

The generalized B-polys of n-degree are defined in refs. [1, 19], but here we are presenting a generalized form of the B-polys over an interval [0, R] with α as fractional-order parameter,

$$B_{i,n}(\alpha, x) = \binom{n}{i} \left(\tfrac{x}{R}\right)^{i\alpha} \left(1 - \left(\tfrac{x}{R}\right)^{\alpha}\right)^{n-i}, \quad (5)$$

Using the Binomial expansion in Eq. (5), one may write,

$$B_{i,n}(\alpha, x) = \sum_{k=i}^{n} \beta_{i,k} \left(\tfrac{x}{R}\right)^{\alpha k}, \quad (6)$$

here $\beta_{i,k}$ are defined as

$$\beta_{i,k} = (-1)^{i-k} \binom{n}{k} \binom{k}{i}, \quad (7)$$

One can also generate those B-polys using recursive formula [1] .i.e.

$$B_{i,n}(\alpha, x) = [1 - (\tfrac{x}{R})^{\alpha}]\, B_{i,n-1}(\alpha, x) + (\tfrac{x}{R})^{\alpha}\, B_{i-1,n-1}(\alpha, x).$$

These B-polys represents the basis set and each of the fractional B-polys is positive while the sum of all the positive B-polys of the order α is unity in the entire interval [0, R]. As an example, a graph of the 10 B-polys of the order α = 3/2 in the region [0, 10] is shown in Fig. 1.

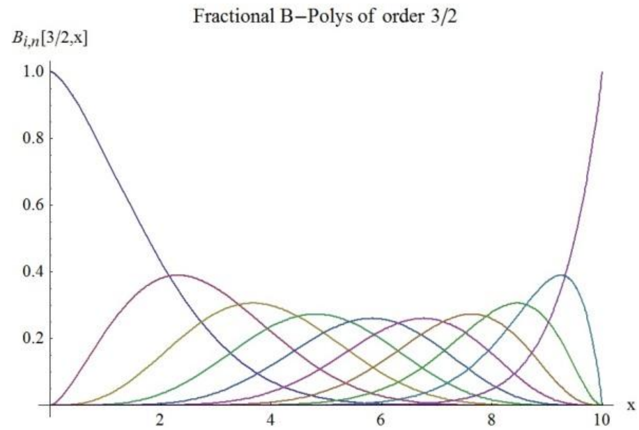

Fig. 1. The set of 10 (n=9) B-Polys of fractional-order α=3/2 are shown in the region [0, 10]. The quantities are dimensionless on both axes.

4. An algorithm for approximating solutions

A generalized B-poly basis set and Galerkin method [1] is employed to approximate the solutions to the fractional-order differential equations. We use expansion formulas given in Eqs.(3-6) and the Caputo's derivative Eq. (2) to approximate the solutions of fractional-order differential equations. A fractional differential equation is converted into matrix form using the techniques described below. The explicit formulas for the matrix formulism are provided in closed expressions concerning inner products of the B-polys and their derivatives in the equations (8-11) employing Eqs. (6-7).

$$b_{i,j} = \left(B_{i,n}(\alpha, x), B_{j,n}(\alpha, x) \right) = \sum_{k=i}^{n} \beta_{i,k} \left(\tfrac{x}{R}\right)^{\alpha k} \sum_{l=j}^{n} \alpha_{i,k} \left(\tfrac{x}{R}\right)^{\alpha l}\ \frac{R}{(k+l)\alpha} \quad (8)$$

The Caputo's derivative of the B-Polys is also given,

$$D^{\gamma} B_{i,n}(\alpha, x) = \sum_{k=i}^{n} \alpha_{i,k} D^{\gamma} \left(\tfrac{x}{R}\right)^{\alpha k} = \sum_{k=i}^{n} \frac{\beta_{i,k}}{R^{\alpha k}} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+1-\gamma)} x^{\alpha k-\gamma} \quad (9)$$

$$d_{i,j}^{(\gamma)} = \left(D^{\gamma} B_{i,n}(\alpha, x), B_{j,n}(\alpha, x) \right) = \sum_{k=i,l=j}^{n} \beta_{i,k} \beta_{j,k} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+1-\gamma)} \frac{R^{1-\gamma}}{((k+l)\alpha+1-\gamma)} \quad (10)$$

$$\text{and } f_i = (f(x), B_{i,n}(\alpha, x)) = \sum_{k=i}^{n} \frac{\beta_{i,k}}{R^{\alpha k}} \int_{0}^{R} f(x)\, x^{\alpha k} dx\, . \quad (11)$$

To show the validity and the capabilities of the algorithm for approximating the solutions of the fractional-order differential equations, we consider several examples below:

Example 1. Consider as a first example of forced Fractional Harmonic Oscillator (FHO) with initial conditions, y (0) = 1 and y' (0) = 0,

$$D^{\gamma} y(t) + \omega^2 y(t) = f(t), \quad t > 0\, . \quad (12)$$

Where, ω is the angular frequency of the oscillator. The second initial condition is only required for $\gamma > 1$, the fractional order of the differential equation. The exact solution to the Eq. (12) can be obtained [20].

$$y(t)=\sum_{k=1}^{m}c_k t^{k-1}E_{\gamma,k}(-\omega^2 t^{\gamma})+\int_0^t f(t-\tau)\tau^{\gamma-1}E_{\gamma,\gamma}(-\omega^2\tau^{\gamma})\,d\tau \cdot \quad (13)$$

Where, $E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ \alpha>0,\ \ \beta>0$, is the Mittag-Leffler function [21]. The initial conditions determine the constants c_k and $m-1<\gamma<m$. Two special cases of interest are (i) for $0<\gamma\le 1$ and (ii) for $1<\gamma\le 2$, the exact solutions to the Eq. (12) for some cases can be obtained from Eq. (13), which are compared with the numerical solutions using the present algorithm as described above. However, the exact solution to Harmonic Oscillator Eq. (12), f (t) = 0, is $y(t)=\sum_{k=0}^{\infty}\frac{(-\omega^2 t^{\gamma})^k}{\Gamma(\gamma k+1)}=E_{\gamma,1}(-\omega^2 t^{\gamma})$ under the initial conditions,i.e. for $\gamma=1$, is $y(t)=e^{-\omega^2 t}$ and for $\gamma=2$, is $y(t)=\cos(\omega t)$. The FHO Eq. (12), f (t) = cos (t), has an exact solution for $\gamma=1$,

$$y(t)\ =\ e^{-\omega^2 t}+\frac{-e^{-\omega^2 t}\omega^2+\omega^2\cos t+\sin t}{\omega^4+1}$$ and for $\gamma=2$ with two initial conditions, y (0) = 1 and y`(0) = 0, is

$$y(t)=\frac{\cos t-\cos\omega t}{\omega^2-1}+\cos\omega t.$$ It is obvious that for $\gamma>1$ and $\alpha>1$, the fractional order of the B-poly, the second initial condition y` (0) =0 is automatically satisfied. We seek numerical solutions to Eq. (12) in the region [0, 2] with initial conditions given above. An approximation to the solution of Eq. (12) may be written,

$$y(t)=\sum_{i=0}^{n}a_i B_{i,n}(\alpha,t)\,. \quad (14)$$

Where the unknown coefficients of the expansion are calculated using Galerkin generalized fractional-order B-poly method by substituting Eq.(14) into Eq. (12) and from the variational property with respect to the coefficients. We get

$$\sum_{i=0}^{n}a_i\{\int_0^R[D^{\gamma}B_{i,n}(\alpha,t)B_{j,n}(\alpha,t)+B_{i,n}(\alpha,t)B_{j,n}(\alpha,t)]dt\}-\int_0^R\cos(t)B_{j,n}(\alpha,t)\,dt=0 \quad (15)$$

Where D^{γ} is the Caputo's derivative Eq. (2) and the Eq. (15) determines (n+1) x (n+1) system of equations, **BA** = **b**, in variables a_0, $a_{1\ldots}$ a_n. The Matrix B has elements

$$B_{i,j}=\int_0^R[D^{\gamma}B_{i,n}(\alpha,t)B_{j,n}(\alpha,t)+B_{i,n}(\alpha,t)B_{j,n}(\alpha,t)]dt\,, \quad (16)$$

and the column matrix **b** has the elements

$$b_j=\int_0^R\cos(t)B_{j,n}(\alpha,t)\,dt\cdot \quad (17)$$

Above integrals are calculated using formulas given in Eqs. (8-11) and the numerical solution of Eq. (12) for the cases $\gamma=1/2,\,1,\,3/2$ and 2 are obtained by solving the Eqs. (14-17) with right hand sidef (t) =cos (t). A typical solution for $\alpha=\gamma=1$ and $\omega=1$, is given

$$y(t)=1.-1.493879\times10^{-7}\,t+0.0000027t^2-0.166683t^3+0.041713t^4$$
$$-0.000073t^5+0.000068t^6-0.000236t^7+0.000036t^8-0.000002t^9.$$

This solution is compared with the exact solution for n=9 B-polys in the interval [0, 2]. The absolute error was determined to be of the order of 10^{-9} between two solutions. Various graphs of the exact solutions of Eq. (12) are depicted in the Fig. 2 and Fig. 3 for f (t) =0, and f (t) =cos (t), respectively. It is obvious that the fractional relaxation appears when $0<\gamma\le 1$ and fractional oscillations show up when $0<\gamma\le 2$. Also, the classical solution for $\gamma=1$ decays exponentially as $t\to\infty$, the fractional solution for $0<\gamma\le 1$ exhibits a faster decay as $t\to 0$ and much slower as $t\to\infty$.

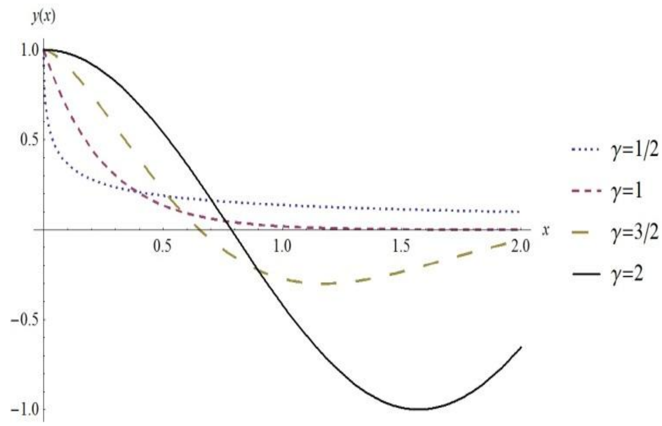

Fig. 2. The Plots of the exact solutions of Eq. (12) with f (t) =0 are shown for $\gamma=1/2,\,1,\,3/2$ and 2. In the plots ω =2 and the range [0, 2] are used.

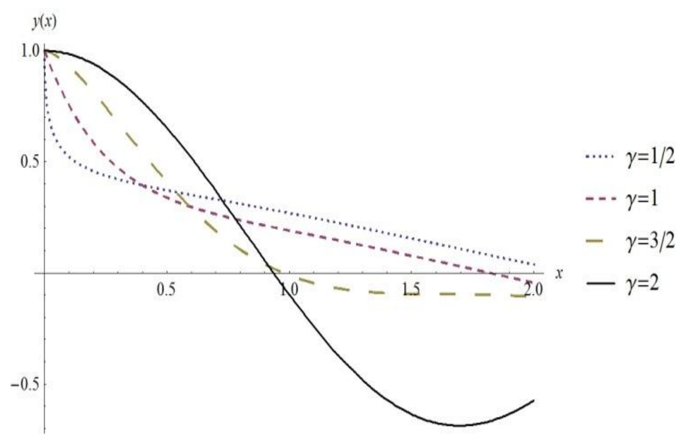

Fig. 3. The Plots of the exact solutions of non-homogeneous Eq. (12) with f (t) = cos (t) are shown for $\gamma=1/2,\,1,\,3/2$ and 2. In these plots ω =2 and the range [0, 2] are used.

In the following, we present eight typical graphs of the absolute error between the exact and the numerical solutions employing the current algorithm. The absolute errors with n=9 and various values of $\alpha=\gamma$ and ω are shown in figures 4 and 5 for homogeneous and nonhomogeneous Eq. (12), respectively.

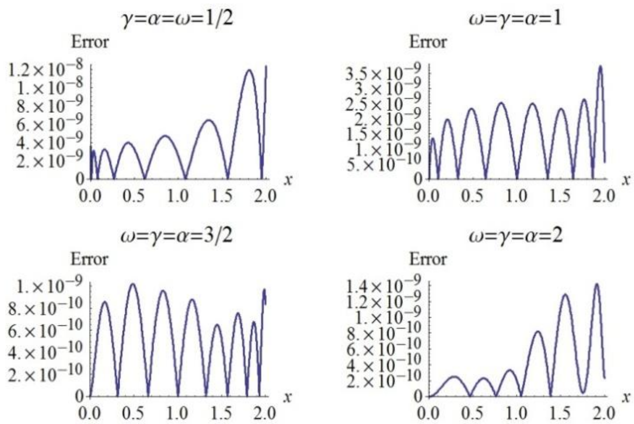

Fig. 4. The absolute error graphs for the right hand side f (t) =0 and $\omega = \gamma = \alpha = 1/2,\ 1,\ 3/2,\ 2$ and n=9.

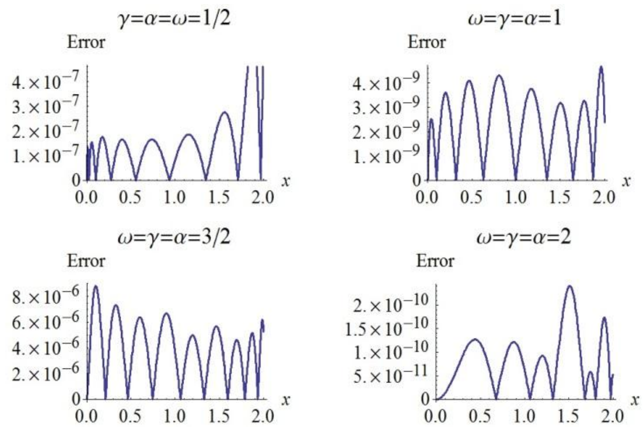

Fig. 5. The absolute error graphs for the right hand side f (t) =cos (t) and $\omega = \gamma = \alpha = 1/2,\ 1,\ 3/2,\ 2$ for n=9.

Example 2. Consider second example of fractional differential equation with two initial conditions, y (0) =1, and y'(0)= -1 [12],

$$D^2 y(t) + D^{3/2} y(t) + y(t) = 1 - x \tag{18}$$

Where $D^{3/2}$ is the Caputo's derivative. This equation has exact solution $y(x) = 1 - x$. Applying the algorithm as described in section 4, we convert the problem into matrix and approximate solution as:

$$\sum_{i=0}^{n} a_i \left\{ \int_0^R [D^2 B_{i,n}(\alpha,x) B_{j,n}(\alpha,x) + D^{3/2} B_{i,n}(\alpha,x) B_{j,n}(\alpha,x) + B_{i,n}(\alpha,x) B_{j,n}(\alpha,x)]\, dx \right\} - \int_0^R (1-x) B_{j,n}(\alpha,x)\, dx = 0 \tag{19}$$

Where Eq. 19 may be written in terms of matrix equation **BA** = **b**, in variables a_0, $a_{1\ldots}$ a_n. The Matrices **B** and **b** have matrix elements given by

$$B_{i,j} = \int_0^R [D^2 B_{i,n}(\alpha,x) B_{j,n}(\alpha,x) + D^{3/2} B_{i,n}(\alpha,x) B_{j,n}(\alpha,x) + B_{i,n}(\alpha,x) B_{j,n}(\alpha,x)]\, dx = 0$$

and

$$f_j = \int_0^R (1-x) B_{j,n}(\alpha,x)\, dx. \tag{20}$$

The integrals are evaluated using the formulas from equations (8-11). The results of the fractional-order differential equation (18) with α= 1 and n=2 in the interval [0, 2] are shown below.

Using the basis set, $\left(1 - x + \frac{x^2}{4},\ x - \frac{x^2}{2},\ \frac{x^2}{4}\right)$, B = $\begin{pmatrix} \frac{11}{15} + \frac{32\sqrt{\frac{2}{\pi}}}{105} & -\frac{7}{15} - \frac{64\sqrt{\frac{2}{\pi}}}{105} & \frac{2}{5} + \frac{32\sqrt{\frac{2}{\pi}}}{105} \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$, and **b**= $\begin{pmatrix} \frac{1}{3} \\ 1 \\ -1 \end{pmatrix}$. We may calculate expansion coefficients (**A**) by multiplying the inverse of matrix **B** with the column matrix **b**. The Matrix of coefficients $A = (1, 0, -1)$ obtained is then multiplied by the basis set in Eq. [14], to obtain the desired solution $y(x) = 1. - 1.\ x$. This example shows the procedure works well in the case of the exact solution as the approximate numerical solution matches with the exact solution.

Example 3. Consider third example of non-homogenous and non-linear fractional differential equation:

$$D^3 y(x) + D^{5/2} y(x) + y^2(x) = x^4, \quad y(0) = 0,\ y'(0) = 0, \text{ and } y''(0) = 2. \tag{21}$$

The exact solution under the boundary conditions of this equation is y(x) = x^2. This is a nonlinear equation it can be solved using iterative scheme described in Ref. [1]. Using the technique described in earlier sections, we may approximate the solution as

$$\sum_{i=0}^{n} a_i \left\{ \int_0^R [D^3 B_{i,n}(\alpha,x) B_{j,n}(\alpha,x) + D^{5/2} B_{i,n}(\alpha,x) B_{j,n}(\alpha,x) + \sum_{k=0}^{n} a_k B_{k,n}(\alpha,x) B_{i,n}(\alpha,x) B_{j,n}(\alpha,x)]\, dx \right\} - \int_0^R x^4 B_{j,n}(\alpha,x)\, dx = 0 \tag{22}$$

The unknown coefficients a_k in the Eq. (22) clearly represent the nonlinearity of the problem in the third term. Integrals are evaluated exactly using the equations (8-11). The initial guess of these coefficients in matrix **A**are obtained omitting third term in the Eq. (22). One can easily evaluate the integrals analytically using the Eqs. (6-7) and the formula,

$$\int_0^R B_{k,n}(\alpha,x) B_{i,n}(\alpha,x) B_{j,n}(\alpha,x)]\, dx = \sum_{k',l,m=0}^{n} \beta_{i,k'} \beta_{j,l} \beta_{k,m} \frac{R}{(k'+l+m)\alpha + 1} \tag{23}$$

The Eq. (22) is converted into a matrix with the help of Eq. (23). The results of the fractional-order differential equation (21) with α= 1 and n=3 in the interval [0, 2] are shown with initial guess of unknown coefficient, $A = (0., 0., 1.3333, 4.1330)$,and initial conditions. The final values of the expansion coefficients after 5 iterations are presented as $A = (0.,\ 0.,\ 1.3333333333,\ 4.0000000000)$ which when multiplied by the basis set Eq. (14) gave the final exact answer $y(x) = 0. + 1.x^2 +\ 4.1 \times 10^{-12} x^3 \cong x^2$. It is clear that the error in the approximate solution is significantly small that may be neglected to provide exact answer.

Example 4. The final example considers non-homogeneous differential equation with an initial condition,

$$D^{\gamma} y(x) + y^2(x) = 1, \quad y(0) = 0. \tag{23}$$

The equation (23) has an exact solution, $y(x) = \frac{e^{2x}-1}{e^{2x}+1}$, for $\gamma = \alpha = 1$. Again using the procedure descried in the above sections, we approximate the solution as

$$\sum_{i=0}^{n} a_i \left\{ \int_0^R [D^{\gamma} B_{i,n}(\alpha,x) B_{j,n}(\alpha,x) + \sum_{k=0}^{n} a_k B_{k,n}(\alpha,x) B_{i,n}(\alpha,x) B_{j,n}(\alpha,x)]\, dx \right\} - \int_0^R B_{j,n}(\alpha,x)\, dx = 0 \tag{24}$$

The Eq. (24) may be written in terms of matrix equation **BA** = **b**, in variables a_0, $a_{1\ldots}$ a_n. The Matrices **B** and **b** have matrix elements, respectively, given by

$$B_{i,j} = \int_0^R [D^{\gamma} B_{i,n}(\alpha,x) B_{j,n}(\alpha,x) + \sum_{k=0}^{n} a_k B_{k,n}(\alpha,x) B_{i,n}(\alpha,x) B_{j,n}(\alpha,x)]\, dx, \tag{25}$$

$$f_j = \int_0^R B_{j,n}(\alpha,x)\, dx$$

The interval over which these integrals are calculated is [0, 1]. The summation of the unknown variables a_k manifest the nonlinearity of the problem in Eq. (24). The initial values of these coefficients are obtained by applying the generalized Galerkin method to the initial data and initial guesses for a_k are determined neglecting the nonlinear terms in Eq. (24). In the approximate solution values for n= 8 B-polys and $\gamma = \alpha = 1$ are used.

Once the initial values of the a_i are obtained they are substituted into Eqs.(24-25) to obtain new estimates for the a_i. This iteration process continues until the converged values of the unknown are obtained. Typical run gave the initial values of the unknown $A = (0., 0.12495, 0.23269, 0.36879, 0.48591, 0.61614, 0.73840, 0.86441, 0.98918)$.

After 8 iterations, values for the coefficients which had converged were used to construct an approximate solution for the nonlinear differential equation. The absolute error is of the order of 10^{-7} between exact and approximate solution obtained for Eq. (23). The final approximate solution with converged coefficients is presented below:

$$y(x) = 0. + 1.00000x - 0.00015x^2 - 0.33148x^3 - 0.01055x^4 + 0.16460x^5 - 0.04805x^6 - 0.02444x^7 + 0.01167x^8$$

In Fig. 6, a plot of the absolute difference between approximate and exact solutions is presented.

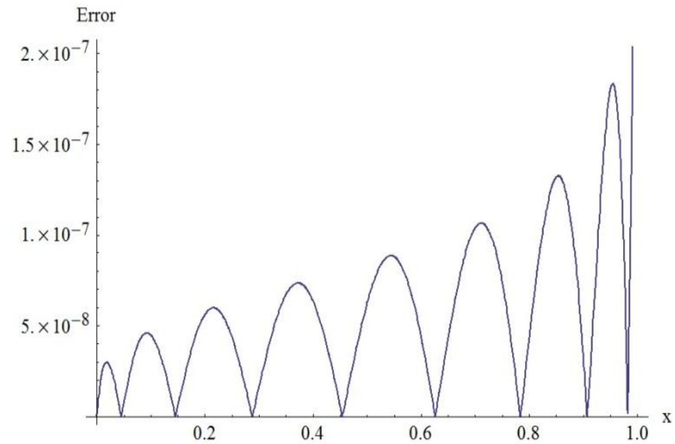

Fig. 6. The absolute error graphs for example 4 with values of $\gamma = \alpha = 1$ and n=8. Only 8 iterations are required getting the converged solution to the level shown.

The absolute difference is obtained using a basis set of 8 B-polys continuous over the interval [0, 1]. It is hoped that this method can be extended to other type of fractional-order nonlinear differential equations as only a small number of B-polys are needed to get a satisfactory solution.

5. Conclusion

In this article, an algorithm based on fractional order generalized B-poly basis and Galerkin method [1] is constructed to solve ordinary, fractional order homogeneous and nonhomogeneous differential equations.

New General explicit formulas are given in Eqs.(3-11) and the Caputo's derivative Eq. (2) are derived to approximate the numerical solutions of fractional-order differential equations.

The explicit formulas for the matrix formulism are constructed in close expressions concerning inner products of the B-polys and their derivatives in the equations (8-11) employing Eqs. (6-7). The method is applied to a variety of examples including fractional harmonic oscillator equation and the differential equations which have exact solutions to compare with.

The solutions obtained using the current algorithm shows that this approach is highly efficient to solve the fractional order problems with small number of B-polys. This method has the ability to be used in variety of disciplines.

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