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On definability of team relations with k-invariant atoms

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ABSTRACT

We study the expressive power of logics whose truth is defined over sets of assignments, called teams, instead of single assignments. Given a team X, any k-tuple of variables in the domain of X defines a corresponding k-ary team relation. Thus the expressive power of a logic $\mathcal{L}$ with team semantics amounts to the set of properties of team relations which $\mathcal{L}$-formulas can define. We introduce a concept of k-invariance which is a natural semantic restriction on any atomic formulae with team semantics. Then we develop a novel proof method to show that, if $\mathcal{L}$ is an extension of FO with any k-invariant atoms, then there are such properties of $(k+1)$-ary team relations which cannot be defined in $\mathcal{L}$. This method can be applied e.g. for arity fragments of various logics with team semantics to prove undefinability results. In particular, we make some interesting observations on the definability of binary team relations with unary inclusion-exclusion logic.

1. Introduction

The origin of *team semantics* goes back to the work of Hodges [15] who presented it to give a compositional semantics as an alternative to *game-theoretic semantics* of IF-logic by Hintikka and Sandu ([13,14]). In the compositional approach it was not sufficient to consider single assignments; instead there was a need to use sets of assignments which are nowadays called *teams*. Väänänen [23] developed this approach further by introducing *dependence atoms* and adding them to first order logic with team semantics. Later various other natural atoms from database theory have been added to this framework – such as *independence atoms* ([8]), *inclusion atoms* and *exclusion atoms* ([4]).

In this paper we present a notion of *k-invariance* of atoms, which is closely related to the study of *arity fragments* of logics with team semantics. We list here some of the most relevant works related to

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the expressive power of arity fragments.[1] In [2] and [7] it is shown that the arity fragments of dependence and independence logic correspond to the *functional* arity fragments of *existential second order logic*, ESO. In [22] it is shown that similarly the arity fragments of exclusion logic correspond to the *relational* arity fragments of ESO. Moreover, in [9] it is shown that inclusion logic has a strict arity hierarchy over graphs. All the results listed here are proven on the level of sentences.

When the expressive power of logics with team semantics is studied on the level of all formulas (not just sentences), the problem is to examine which properties of *team relations* are definable. Galliani [4] has shown that with *inclusion-exclusion logic* one can define exactly those team relations which are ESO-definable. In [20] we show that the relationship between these two logics becomes more delicate when we consider k-ary inclusion-exclusion logic (INEX[k]) and k-ary relational fragment of ESO (ESO[k]). Then all INEX[k]-definable properties are ESO[k]-definable, and conversely all ESO[k]-definable properties *of at most k-ary team relations* are INEX[k]-definable. However, this leaves open what happens to INEX[k]-definability of team relations of higher arity. This question is settled in the current paper as we show, in particular, that the $(k+1)$-*totality* of a team relation cannot be defined in INEX[k].

This undefinability result for INEX[k] can be generalized for quite a rich class $\mathfrak{L}[k]$ of logics with team semantics. The logics in $\mathfrak{L}[k]$ extend FO with any atomic formulas which are either *k-invariant* or *closed downwards*. Being k-invariant intuitively means that such an atom "cannot see" any difference between teams that have the same k-ary team relations; and being closed downwards means that such atoms cannot see when assignments are removed from a team. Our Theorem 7.15 can be used for showing that various properties of certain kind of $(k+1)$-ary team relations are undefinable in $\mathfrak{L}[k]$. In order to prove this theorem we introduce several new definitions and proof techniques which we believe to be useful for the study of team semantics in general. (Indeed, they have already been proven useful for *modal inclusion logic*; see Remark 7.19.)

One particularly interesting case covered by this paper is the expressive power of *unary* inclusion exclusion logic, INEX[1]. This is a rather versatile logic which corresponds to existential *monadic* second order logic ESO[1] on the level of sentences. We show that on one hand various highly nontrivial properties of binary team relations, such 3-*colorability* of the corresponding graph, are definable INEX[1] – but on the other hand some very simple properties, such as *symmetry*, are undefinable in INEX[1].

This paper has the following structure. After some preliminaries in Section 2 we define the notion of k-invariance and the class $\mathfrak{L}[k]$ in Section 3. In Section 4 we present various useful INEX[1]-definable atoms and operators which will be used later in the paper. In Section 5 we discuss the definability of team relations with logics with team semantics in general. In Section 6 we show how certain nontrivial properties of $(k+1)$-ary team relations can be defined in $\mathfrak{L}[k]$ by focusing on the particularly interesting case $k = 1$. The main contributions of this paper are in Section 7 where we develop a novel proof method for showing that various properties of $(k+1)$-ary team relations are undefinable with logics in the class $\mathfrak{L}[k]$. We make some concluding remarks in Section 8.

All the results in this paper are based on PhD Thesis [21] by the author.

2. Preliminaries

In this paper only consider *relational vocabularies* L for simplification, but nothing essential would change if we let L also contain function and constant symbols. Let $\mathcal{M} = (M, \mathcal{I})$ be an L-model. A *team* X for $\mathcal{M}$ is any set of assignments s for $\mathcal{M}$ with a common domain – denoted by $\text{dom}(X)$. For any $\{y_1, \dots, y_k\} \subseteq \text{dom}(X)$ we write

[1] We point out that, in addition to arity fragments, there has also been research on the expressive power of other kinds of fragments of various logics with team semantics. For example, some interesting hierarchy results have been established in [2] and [10] by bounding the number of universal quantifiers.

$$X(y_1 \dots y_k) := \{s(y_1 \dots y_k) \mid s \in X\}.$$

Hence every k-tuple $\vec{y}$ of variables in $\mathrm{dom}(X)$ naturally defines a corresponding k-ary *team relation* $X(\vec{y})$ in the model $\mathcal{M}$. Classes $\mathcal{P}$ of team relations are called *properties* of team relations.

Let X be a team for an L-model $\mathcal{M} = (M, \mathcal{I})$ such that $x \in \mathrm{dom}(X)$. Let $A \subseteq M$ and $F : X \to \mathcal{P}^*(M)$, where $\mathcal{P}^*(M)$ denotes the set of all *nonempty* subsets of M. We use the following notations:

$$X[A/x] := \{s[a/x] \mid a \in A\} \quad \text{and} \quad X[F/x] := \{s[a/x] \mid a \in F(s)\}.$$

The semantics of first order logic (FO) can naturally be generalized from single assignments to sets of assignments. This leads to *team semantics* ([23]) which is defined as follows:

- $\mathcal{M} \vDash_X x_1 = x_2$ iff $s(x_1) = s(x_2)$ for all $s \in X$.
- $\mathcal{M} \vDash_X \neg x_1 = x_2$ iff $s(x_1) \neq s(x_2)$ for all $s \in X$.
- $\mathcal{M} \vDash_X P\,\vec{x}$ iff $s(\vec{x}\,) \in P^{\mathcal{M}}$ for all $s \in X$.
- $\mathcal{M} \vDash_X \neg P\,\vec{x}$ iff $s(\vec{x}\,) \notin P^{\mathcal{M}}$ for all $s \in X$.
- $\mathcal{M} \vDash_X \psi \wedge \theta$ iff $\mathcal{M} \vDash_X \psi$ and $\mathcal{M} \vDash_X \theta$.
- $\mathcal{M} \vDash_X \psi \vee \theta$ iff there are $Y, Y' \subseteq X$ s.t. $Y \cup Y' = X$, $\mathcal{M} \vDash_Y \psi$ and $\mathcal{M} \vDash_{Y'} \theta$.
- $\mathcal{M} \vDash_X \exists x\, \psi$ iff there is $F : X \to \mathcal{P}^*(M)$ such that $\mathcal{M} \vDash_{X[F/x]} \psi$.
- $\mathcal{M} \vDash_X \forall x\, \psi$ iff $\mathcal{M} \vDash_{X[M/x]} \psi$.

(We may write $x_1 \neq x_2$ as a shorthand for $\neg x_1 = x_2$.)

Team semantics is a natural generalization the standard Tarski semantics ($\vDash^T$) for FO as we have

$$\mathcal{M} \vDash_X \varphi \;\text{ iff }\; \mathcal{M} \vDash^T_s \varphi \text{ for all } s \in X,$$

for any formula φ of first order logic.

When FO is extended with new logical atoms (or operators) we obtain more expressive logics with team semantics. Some of the most common atoms that have been studied are dependence atoms ([23]), independence atoms ([8]), inclusion atoms and exclusion atoms ([4]). The addition of these atoms leads to corresponding logics – for example FO extended with dependence atoms is called *dependence logic*.

In most natural extensions of FO the so-called *locality* property is preserved. This means that the truth of a formula φ is determined by only the values of those variables which occur in φ as free variables. That is, a team X satisfies φ if and only if the team $\{s \upharpoonright \mathrm{Fr}(\varphi) \mid s \in X\}$ satisfies φ. As observed e.g. in [17], when FO is extended with new atomic formulas whose semantics local, then it follows that the resulting logic is local as well.[2] Moreover, most of the common extensions of FO also have so-called *empty team property* (ETP), i.e. every formula is true in the empty team $X = \emptyset$. Also this property is preserved when FO is extended with atoms that are true in the empty team.

Remark 2.1. See Section 5.4 of [21] for discussion on ETP and arguments on why we find it natural to require this property for logics with team semantics. We also note that any atom A (resp. operator O) violating ETP has a natural variant that respects ETP but is otherwise essentially equivalent to A (resp. O). Moreover, in [21] we show how to translate a formula φ not having ETP into φ' that has ETP, so that φ and φ' are equivalent in all *nonempty* teams.

[2] This requires the use of so-called "lax semantics" for existential quantifier and disjunction – as in the current paper. An alternative, so called "strict semantics" may violate locality when new atomic formulae are added – even if their semantics is local. (See e.g. [5].)

Most of the logics analyzed in the current paper have empty team property. However, note that the assumption of ETP is not made in Section 7 since it is not needed for proving the undefinability results which we present there.

Definition 2.2. Let φ be a formula with team semantics. We say that φ is *closed downwards* if the following holds for all models $\mathcal{M}$ and teams X:

$$\text{If } \mathcal{M} \vDash_X \varphi \text{ and } Y \subseteq X, \text{ then } \mathcal{M} \vDash_Y \varphi.$$

If every formula of a logic $\mathcal{L}$ with team semantics is closed downwards, then we say that the logic $\mathcal{L}$ is closed downwards. Moreover, we say that a property $\mathcal{P}$ of team relations is closed downwards if it holds that: whenever a team relation $X(v_1 \dots v_k)$ satisfies $\mathcal{P}$, also $Y(v_1 \dots v_k)$ satisfies $\mathcal{P}$ for any subteam $Y \subseteq X$.

If a logic $\mathcal{L}$ is an extension of FO with only downwards closed atoms, then it is easy to show that $\mathcal{L}$ is closed downwards.[3] In particular, FO, dependence logic and exclusion logic are the known to be closed downwards. From the definition above it immediately follows that logics which are closed downwards can only define such properties of teams which are closed downwards.

Example 2.3. Let $\mathcal{M} = (M, \mathcal{I})$ be a model with $M = \{a, b\}$. We consider properties of binary team relations $X(y_1y_2)$. The property of *irreflexivity* of $X(y_1y_2)$ is closed downwards as subsets of an irreflexive relation are also irreflexive. However, the properties of *reflexivity*, *symmetry* and *seriality* are not closed downwards since e.g. the team relation $X(y_1y_2) = M^2$ has all of these properties, but $Y(y_1y_2) = \{(a, a), (a, b)\}$ has none of these properties even though $Y \subseteq X$. We will get back to this example in Section 4.

Finally we present some additional terminology and abbreviations which will be used in the current paper.

Definition 2.4. Let $\mathcal{L}$ be any logic with team semantics. We say that an $\mathcal{L}$-formula φ is *atomic* if it is either a (FO) literal or some (non-FO) atom in $\mathcal{L}$. We also use the following notations for φ:

$$\begin{aligned}
\mathrm{Sf}(\varphi) &:= \text{the set of subformulas of } \varphi; \\
\mathrm{Atom}(\varphi) &:= \{\psi \in \mathrm{Sf}(\varphi) \mid \psi \text{ is atomic}\}; \\
\mathrm{Oper}_{\#}(\varphi) &:= \text{the total number of operators } \wedge, \vee, \exists x \text{ and } \forall x \text{ in } \varphi.
\end{aligned}$$

The notations $\gamma_{=k}$, $\gamma_{\leq k}$ and $\gamma_{\geq k}$ (for $k \in \mathbb{Z}_+$) denote FO-sentences defining that $|M| = k$, $|M| \leq k$ or $|M| \geq k$, respectively.

Remark 2.5. As typically done in this framework, when talking about subformulae of a formula φ, we actually mean different *occurrences* (instances) of the subformulae (note that ψ has two different occurrences in $\psi \vee \psi$). So the elements in $\mathrm{Sf}(\varphi)$ can naturally be considered as *nodes in the syntax tree of* φ.

3. *k*-ary fragments and *k*-invariant atoms

A natural way to restrict the expressive power of logics with team semantics is to put restrictions on the complexity of atoms that can be used. By restricting the *arity* of atoms, shorter tuples of variables are

[3] Also other natural closure properties such as *closure under unions* ([4]) and *closure upwards* ([6]) have been studied. Also closure under unions is preserved when FO is extended with union closed atoms. However, note that the same does not hold for upwards closure since FO itself is not closed upwards (consider e.g. a literal $x = y$).

allowed to be used and thus (typically) it suffices to check team relations of lower arity when evaluating atoms. For example, the truth condition of k-ary inclusion atoms is defined as follows with respect to k-ary team relations:

$$\mathcal{M} \vDash_X x_1 \dots x_k \subseteq y_1 \dots y_k \quad \text{iff} \quad X(x_1 \dots x_k) \subseteq X(y_1 \dots y_k).$$

Similarly, the truth condition of k-ary exclusion atoms is defined with respect to k-ary team relations:

$$\mathcal{M} \vDash_X x_1 \dots x_k \,|\, y_1 \dots y_k \quad \text{iff} \quad X(x_1 \dots x_k) \cap X(y_1 \dots y_k) = \emptyset.$$

The logic extending FO-with both k-ary inclusion and exclusion atoms is called *k-ary inclusion-exclusion logic* and will be denoted by INEX$[k]$.

Most of the atoms introduced for team semantics (such as dependence and independence atoms) have a similar notion of arity which can be related to team relations of the corresponding arity. We next present a natural semantic constraint – so called *k-invariance* – which bounds the expressive power of arbitrary atoms with team semantics.

Definition 3.1. Let X, Y be teams for an L-model $\mathcal{M}$ such that X, Y have a shared domain D. We say that X and Y are *k-equivalent* if the following holds for all $\{y_1, \dots, y_k\} \subseteq D$:

$$X(y_1 \dots y_k) = Y(y_1 \dots y_k).$$

An atom A (with team semantics) is *k-invariant* if we have

$$\mathcal{M} \vDash_X A \quad \text{iff} \quad \mathcal{M} \vDash_Y A$$

for all models $\mathcal{M}$ and k-equivalent teams X and Y for $\mathcal{M}$.

The notion of k-invariance intuitively states that atoms with this property can only "see up to k-ary relations" in a given team. Hence this property could also be called "k-dimensionality". Also note that the definition of k-invariance is very liberal as it allows e.g. atoms which are not local or even invariant under isomorphisms. However, the undefinability results in Section 7 can be proven without any further restrictions on this definition.

Remark 3.2. We say a few words about the notion of arity of an atom A with team semantics. There are *syntactical* ways to define arity, such as simply by counting how many different variables (or terms) are allowed to occur in A (whence e.g. the inclusion atom $x \subseteq y$ would be binary). However, we think that the arity of A should instead be considered as a *semantical* property. Under this assumption we argue that k-invariance is a necessary condition for the atom A to be k-ary, as otherwise the truth condition of A would require evaluating it with respect team relations that have arity higher than k. See Section 3.3.3 in [21] for a more restricted natural subclass of k-invariant atoms, so called "Qs FO-definable k-ary atoms", which nevertheless cover all the common k-ary atoms with team semantics.

Next we define quite a general class of logics with team semantics by setting the k-invariance restriction on certain atoms.

Definition 3.3. A logic $\mathcal{L}$ belongs to the class $\mathfrak{L}[k]$ if (i) $\mathcal{L}$ is an extension of FO with new atomic formulas satisfying locality[4]; and (ii) all atomic formulas in $\mathcal{L}$ belong to either (or both) of the following two classes:

(a) downwards closed atoms;
(b) k-invariant atoms.

Note that, in particular, all downward closed logics and all k-ary fragments[5] of logics with team semantics (studied so far) belong to the class $\mathfrak{L}[k]$.

4. Useful atoms and operators for team semantics

In this section we will present various natural atoms and operators for logics with team semantics. Their semantics are quite simple, but they turn out to be rather useful for this framework; we will also use them later in Section 6. Semantics for all of the atoms and operators below are defined in such a way that empty team property is preserved when they are added to logics with team semantics. This allows us to express them in logics that have the empty team property (in particular, we want to be able to express them in INEX[1]).

We first present semantics for *constancy atom* $=\!(y)$, *inconstancy atom* $\neq\!(y)$ and *totality atom* $T(y_1 \cup \dots \cup y_n)$.

- $\mathcal{M} \vDash_X =\!(y)$ iff $X = \emptyset$ or $|X(y)| = 1$.
- $\mathcal{M} \vDash_X \neq\!(y)$ iff $X = \emptyset$ or $|X(y)| > 2$.
- $\mathcal{M} \vDash_X T(y_1 \cup \dots \cup y_n)$ iff $X = \emptyset$ or $\bigcup_{i \leq n} X(y_i) = M$.

Constancy atoms are actually unary dependence atoms while inconstancy atoms are unary *nondependence atoms* which were introduced by Galliani in [5]. Väänänen ([24]) has called the latter ones *anonymity atoms* and their properties have also been studied by the author in [21]. Totality atoms (for variable tuples, without ETP) were presented in [5].[6]

Next we present semantics for *constant quantifier* $\mathbf{C}(c_1, \dots, c_n)$, *uniform disjunction*[7] $\sqcup$, *relevant disjunctions* $\overset{\bowtie}{\vee}$, $\overset{\triangleright}{\vee}$, $\overset{\triangleleft}{\vee}$, and *possibility operator* $\triangledown$.

- $\mathcal{M} \vDash_X \mathbf{C}(c_1, \dots, c_n)\, \varphi$ iff there are distinct elements $a_1, \dots, a_n$ in M s.t. $\mathcal{M} \vDash_{X[\{a_1\}/c_1, \dots, \{a_n\}/c_n]} \varphi$. (Note that c_i here are variable symbols.)
- $\mathcal{M} \vDash_X \varphi \sqcup \psi$ iff $\mathcal{M} \vDash_X \varphi$ or $\mathcal{M} \vDash_X \varphi$.
- $\mathcal{M} \vDash_X \varphi \overset{\bowtie}{\vee} \psi$ iff $X = \emptyset$ or there are $Y, Y' \subseteq X$ s.t. $Y, Y' \neq \emptyset$, $Y \cup Y' = X$, $\mathcal{M} \vDash_Y \varphi$ and $\mathcal{M} \vDash_{Y'} \psi$.
- $\mathcal{M} \vDash_X \varphi \overset{\triangleright}{\vee} \psi$ iff $X = \emptyset$ or there are $Y, Y' \subseteq X$ s.t. $Y \neq \emptyset$, $Y \cup Y' = X$, $\mathcal{M} \vDash_Y \varphi$ and $\mathcal{M} \vDash_{Y'} \psi$.
- $\mathcal{M} \vDash_X \varphi \overset{\triangleleft}{\vee} \psi$ iff $X = \emptyset$ or there are $Y, Y' \subseteq X$ s.t. $Y' \neq \emptyset$, $Y \cup Y' = X$, $\mathcal{M} \vDash_Y \varphi$ and $\mathcal{M} \vDash_{Y'} \psi$.
- $\mathcal{M} \vDash_X \triangledown\varphi$ iff $X = \emptyset$ or there is $Y \subseteq X$ s.t. $Y \neq \emptyset$ and $\mathcal{M} \vDash_Y \varphi$.

[4] Locality is assumed here mainly because the study of the expressive power via definability of team relations does not make so much sense when a logic is not local (cf. Remark 5.1). However, most of the results in Section 7 can be proven also without assuming locality.

[5] However, notions of arity may differ in the literature. For example, the dependence atom $=\!(y, z)$ states that $X(yz)$ is a unary function. As this is a property of a binary team relation, it is natural to define that $=\!(y, z)$ is a binary atom which is indeed 2-invariant. However, as such binary team relations are unary functions, the atom $=\!(y, z)$ is sometimes called unary.

[6] The atom $\texttt{All}(y_1 \dots y_n)$ presented by Galliani states that $X(y_1 \dots y_n) = M^n$. The totality atom of the current paper is denoted by $T(y_1, \dots, y_n)$ in [21] but we have modified the notation here to avoid confusion with $\texttt{All}(y_1 \dots y_n)$.

[7] This operator has usually been called *intuitionistic disjunction* (or sometimes *classic disjunction*) in the literature. However, we promote the name "uniform disjunction" by using a perspective from *game theoretic semantics*. Unlike the standard disjunction $\vee$ – which allows the verifier to choose a disjunct based on the values of the variables in a team – uniform disjunction $\sqcup$ forces the verifier to make a uniform choice independently of those values.

Constant quantifier was presented in [16] and a slightly different version of possibility operator was presented in [6].[8] Relevant disjunctions were introduced by the author and they have also been studied in [11] and [25].

Note that the semantics of $\overset{\bowtie}{\vee}$ is otherwise identical to the semantics of the standard disjunction, but it states that both of the disjuncts must be "relevant" for the truth of the disjunctive formula as the disjuncts must be satisfied by some nonempty subteams ($\overset{\triangleright}{\vee}$ and $\overset{\triangleleft}{\vee}$ do the same by setting this requirement to only one of the disjuncts). Thus for example $\forall x(P_1 x \overset{\bowtie}{\vee} P_2)$ holds if and only if every element satisfies either P_1 or P_2 and there are indeed some elements which satisfy P_1 and some elements which satisfy P_2. This is conceptually related to the study of *relevance logics* (see e.g. [1]).[9]

The atoms and the operators presented above do not provide much expressive power when they are added to FO (see [5] and [21] for analysis). However, as we will demonstrate in Section 6, they are often quite useful for expressing properties of teams (or models) in the framework of team semantics. See also the example below.

Example 4.1. Recall the properties of binary team relations from Example 2.3. We first note that the property of irreflexivity of $X(y_1y_2)$ can be defined by the simple FO-formula $y_1 \neq y_2$. As reflexivity and seriality do not hold for empty relations, these properties cannot be defined with any formulas which have the empty team property. Thus below we will consider the definability of these properties for nonempty team relations $X(y_1y_2)$ (i.e. for teams $X \neq \emptyset$).

The seriality of $X(y_1y_2)$ is not definable with any FO-formula as it is not closed downwards. However, seriality can be defined with the totality atom $T(y_1)$ as it states that for every $a \in M$ there is some assignment $s_a \in X$ such that $s_a(y_1) = a$, and thus $(a, s_a(y_2)) \in X(y_1y_2)$ for all $a \in M$.

Reflexivity of $X(y_1y_2)$ can be defined with the formula $\triangledown(y_1 = y_2 \wedge T(y_1))$. We sketch a proof for this claim: $X(y_1y_2)$ is reflexive if and only if there is $Y \subseteq X$ for which $Y(y_1y_2)$ is the *identity relation* of M. It is easy to see that $Y(y_1y_2)$ is the identity relation of M if and only if $\mathcal{M} \vDash_Y y_1 = y_2 \wedge T(y_1)$. Thus $X(y_1y_2)$ is reflexive if and only if $\mathcal{M} \vDash_X \triangledown(y_1 = y_2 \wedge T(y_1))$.

The definability of symmetry will be discussed in Sections 6 and 7.

Next we show how all of the atoms and operators presented above can be expressed by using only unary inclusion and exclusion atoms. This will also be useful for our analysis of the expressive power of INEX[1] in Section 6.

- $\mathcal{M} \vDash_X =\!(y)$ iff $\mathcal{M} \vDash_X \forall x\,(x = y \,\vee\, x \,|\, y)$.
- $\mathcal{M} \vDash_X \neq\!(y)$ iff $\mathcal{M} \vDash_X \exists x\,(x \neq y \,\wedge\, x \subseteq y)$.
- $\mathcal{M} \vDash_X T(y_1 \cup \dots \cup y_n)$ iff $\mathcal{M} \vDash_X \forall x \,\big(\bigvee_{i<n} x \subseteq y_i\big)$.
- $\mathcal{M} \vDash_X \mathbf{C}(c_1, \dots, c_n)\,\varphi$ iff $\mathcal{M} \vDash_X \exists c_1 \dots \exists c_n \big(\bigwedge_{i=1}^{n} =\!(c_i) \wedge \bigwedge_{i\neq j} c_i \neq c_j \wedge \varphi\big)$.
- $\mathcal{M} \vDash_X \varphi \sqcup \psi$ iff $\mathcal{M} \vDash_X (\gamma_{=1} \wedge (\varphi \vee \psi)) \vee \mathbf{C}(c_1)\,\mathbf{C}(c_2)\,\big((c_1 = c_2 \wedge \varphi) \vee (c_1 \neq c_2 \wedge \psi)\big)$.
- $\mathcal{M} \vDash_X \varphi \overset{\triangleright}{\vee} \psi$ iff $\mathcal{M} \vDash_X (\gamma_{=1} \wedge \varphi) \vee \mathbf{C}(c)\,\exists y\,\big((\varphi \vee (y = c \wedge \psi)) \,\wedge\, \neq\!(y)\big)$.
- $\mathcal{M} \vDash_X \triangledown\varphi$ iff $\mathcal{M} \vDash_X \varphi \overset{\triangleright}{\vee} \forall x\,(x = x)$.
- $\mathcal{M} \vDash_X \varphi \overset{\triangleleft}{\vee} \psi$ iff $\mathcal{M} \vDash_X \psi \overset{\triangleright}{\vee} \varphi$ (or $\mathcal{M} \vDash_X (\varphi \vee \psi) \wedge \triangledown\psi$).
- $\mathcal{M} \vDash_X \varphi \overset{\bowtie}{\vee} \psi$ iff $\mathcal{M} \vDash_X (\varphi \overset{\triangleright}{\vee} \psi) \wedge (\varphi \overset{\triangleleft}{\vee} \psi)$ (or $\mathcal{M} \vDash_X \triangledown\varphi \wedge (\varphi \vee \psi) \wedge \triangledown\psi$).

Translations for constancy atoms, inconstancy atoms and uniform disjunction have been given and proven in [5]. Complete proofs for all other translations above are given in [21].

[8] Constant quantifier (for a single variable x) is denoted by $\exists! x$ in [16] and possibility operator (not having ETP) is denoted by $\diamond$ in [6].

[9] Also note that (assuming empty team property) $\overset{\bowtie}{\vee}$ and $\sqcup$ can be seen as dual operators as $\sqcup$ states that a team must split into subteams Y, Y' in a trivial way where other side is left empty, while $\overset{\bowtie}{\vee}$ allows any other way of splitting the team.

It is worth noting that all of the operators above can be expressed by adding just constancy and inconstancy atoms to FO (this results in a logic which is expressively weaker than INEX[1] and collapses to FO on the level of sentences). We also point out that the translation for $=(y)$, $\neq(y)$ and $T(y_1 \cup \cdots \cup y_n)$ can be generalized for tuples $\vec{y}$ of variables. Finally, related to the topic of new atoms and operators for team semantics, we mention that Kuusisto ([18]) has introduced the concepts of generalized atoms and generalized quantifiers for team semantics.

5. Definability of team relations

In this section we first make a couple of notes on the definability of team relations with a given logic $\mathcal{L}$ with team semantics. Then we review the known results on the definability of team relations in k-ary inclusion exclusion logic; concentrating on the case $k = 1$ which we argue to be particularly interesting.

5.1. On definability of team relations and expressive power of logics

By saying that a property $\mathcal{P}$ of k-ary team relations is definable in a logic $\mathcal{L}$ with team semantics, we mean that by fixing a tuple $y_1 \dots y_k$ of distinct variables – in the given order – there is an $\mathcal{L}$-formula $\varphi(y_1 \dots y_k)$ such that

$$\mathcal{M} \vDash_X \varphi \;\text{ iff }\; X(y_1 \dots y_k) \in \mathcal{P}.$$

Hence the expressive power of any logic $\mathcal{L}$ can essentially be reduced to the definability of team relations with $\mathcal{L}$. However, two remarks are important to make here.

Remark 5.1. Reducing the expressive power of a formula $\varphi(y_1 \dots y_n)$ to definability of team relations $X(y_1 \dots y_n)$, as done above, requires that φ is local. If φ was not local, we should also consider team relations $X(y_1 \dots y_n \vec{x})$ of higher arity, where the variables in $\vec{x}$ do not occur in φ. This would make the definition much more involved and problematic. We will not discuss this issue further, as all the logics studied in the current paper are assumed to be local (which we argue to be a very natural property for logics in general).

Remark 5.2. In the general case, the expressive power an $\mathcal{L}$-formula $\varphi(y_1 \dots y_k)$ is associated with the class of model-relation pairs $(\mathcal{M}, X(y_1 \dots y_k))$ such that φ satisfies X in $\mathcal{M}$. However, for most of our analysis in the current paper, we interested in such properties of team relations which are independent of the structure of the model (such as the property of $X(y_1 y_2)$ being symmetric). Also note that the analysis here is on the level of all $\mathcal{L}$-formulas – if we would consider only sentences, then the expressive power would simply amount to the class of models with satisfy the given sentence (completely ignoring team relations).

If $\mathcal{L}$ has the empty team property, then we cannot define classes of relations that do not contain the empty relation. Because we are mainly interested in logics which have the empty team property, for simplicity we assume hereafter *by default* that $\emptyset \in \mathcal{P}$ for all classes $\mathcal{P}$ of team relations that we study. For example, if we say that "the relation $X(y_1 y_2)$ is reflexive", we actually mean that "$X(y_1 y_2)$ is either reflexive *or* the empty relation".

It is very important to note the difference between defining relations in a *model* and relations in a *team*. Consider e.g. the property of symmetry of binary relations which is clearly FO-definable as a property of a relation in a model. However, as observed in Example 2.3, the symmetry of team relations is not closed downwards and therefore it is not definable in FO – nor in any other downwards closed logic.

Most of the logics with team semantics that have been studied can only define such properties of $\mathcal{P}$ of team relations which are definable in *existential second order logic*, ESO. For such team relations there exists an ESO-formula $\Phi(R)$, with the *free relation variable* R, such that $\Phi(R)$ is true under those interpretations for R which belong to $\mathcal{P}$. It also makes sense to say that an $\mathcal{L}$-formula $\varphi(y_1 \dots y_n)$ is equivalent[10] to $\Phi(R)$ if the following holds for all admissible models $\mathcal{M}$ and teams X:

$$\mathcal{M} \vDash_X \varphi \quad \text{iff} \quad \mathcal{M}[X(y_1 \dots y_k)] \vDash \Phi(R).$$

In k-ary ESO, denoted by ESO[k], we only allow existential quantification of at most k-ary relation variables, but free relation variables R may have any arity.

Remark 5.3. Because such ESO[0]-formulas which do not contain first order variables are essentially first order sentences, ESO[0]-definable properties of team relations have often been called FO-definable in the literature. However, we have decided to avoid this terminology because it is somewhat ambiguous; by FO-definable properties of team relations one can also mean such properties which are definable by an FO-formula (with team semantics). For example, as observed in Example 4.1 the irreflexivity of a team relation $X(y_1y_2)$ can be defined with an FO-formula, but the reflexivity of $X(y_1y_2)$ cannot be defined with any FO-formula. However, the reflexivity of $X(y_1y_2)$ is easily defined with the ESO[0]-formula $\forall x Rxx$ (when the free relation variable R is interpreted as the team relation $X(y_1y_2)$).

Note that when a property of a team relation is ESO[0]-definable, then the corresponding property of a relation in a model is definable by an FO-sentence. By interpreting team relations with free relation variables R, then an ESO[0]-formula $\Phi(R)$ can get a "direct access" to the complete team, while FO-formulas can only check conditions for assignments one at a time (as an FO-formula is satisfied by a team X if and only if it is satisfied by all assignments in X).

5.2. *Expressive power of* INEX[k] *on the level of all formulas*

Next we focus our attention to the expressive power of k-ary inclusion-exclusion logic INEX[k]. In [20] we showed that all ESO[k]-definable properties of *at most k-ary team relations* can be defined in INEX[k]. Let us inspect the special cases when $k = 1$ and when $k = 2$.

1. ESO[1]-definable properties of *unary* team relations are definable in INEX[1].
2. ESO[2]-definable properties of *binary* team relations are definable in INEX[2].

We first note that, by these results, the expressive power of INEX[2] is rather strong. Indeed, it is hard to think of natural properties of relations which cannot be "simulated" with the properties of binary relations or which are not definable in ESO[2]. Hence we argue that the study of the expressive power of INEX[k], on the level of all formulas, is not so interesting when $k \geq 2$.

Let us then focus on the case of INEX[1]. It is well-known that, for unary relations, ESO[1]-definable properties simply amount to FO-definable properties.[11] Hence, by the result on the upper bound of the expressive power of INEX[1], we only know that INEX[1] can express all ESO[0]-definable properties of unary team relations.[12] Thus our focus is to study which properties of binary (or of higher arity) team relations can be defined in INEX[1].

[10] Note that this equivalence is not completely direct as φ is defined with team semantics, while Φ has the standard Tarski semantics. However, this is the standard way in the literature to form a link between ESO and formulae (with free variables) that have team semantics.
[11] To our best knowledge this is a "folklore result". One way to prove this result is to use an EF-game in a similar manner as e.g. in the proof Proposition 7.2 in [19].
[12] Note, however, that *on the level of sentences* INEX[1] captures exactly the expressive power of ESO[1]. Moreover, as we show in [22], on the level of sentences inclusion atoms can be removed from INEX[1] without lowering the expressive power, but on the level of formulas inclusion atoms are essential in INEX[1] for defining properties that are not closed downwards.

By the results in [20] we also know that all INEX[k]-definable properties of team relations must be ESO[k]-definable. However, these earlier results leave open whether INEX[1] can define some natural ESO[1]-definable properties of *binary* team relations, such as the following two:

(a) $X(y_1y_2)$ is symmetric.
(b) $X(y_1y_2)$ is k-colorable for a given $k \in \mathbb{Z}_+$.

We will see that the rather complex property (b) is indeed definable in INEX[1], for any $k \in \mathbb{Z}_+$. However, interestingly it will turn out that the much more simple looking property (a) cannot be defined in INEX[1].

Example 5.4. Even though symmetry is a very natural property of binary relations in general, it might not be so clear to see "what do the symmetric teams look like" when we interpret teams as e.g. *databases*. For an example, consider a team X – with $y_d, y_a \in \mathrm{dom}(X)$ – in which assignments record information about flights between different cities. The departure city of a given flight is recorded to the variable y_d and arrival city is recorded to y_a. Suppose e.g. that X contains information on every flight operated weekly by the airline company Finnair. Now, if the team relation $X(y_dy_a)$ is symmetric, it means that whenever one takes a direct flight using Finnair, it is possible to get a two-way direct flight (which returns within one week after the arrival to the destination).

6. Defining various nontrivial $(k+1)$-ary team relations in $\mathfrak{L}[k]$

Recall that $\mathfrak{L}[k]$ denotes the class of (local) logics extending FO with any k-invariant and downwards closed atoms. In this section we will analyze some particular cases where logics in $\mathfrak{L}[k]$ can define properties of team relations, whose arity is higher than k. It is important to remember here that even if all the atoms in a logic $\mathcal{L}$ are k-invariant, it may still be possible to define highly nontrivial properties of team relations of higher arity as in $\mathcal{L}$ we also have access to all the logical operators of FO. Our main focus in this section is on which binary team relations can be defined in unary inclusion-exclusion logic (which belongs to $\mathfrak{L}[1]$).

6.1. ESO[0]*-definable binary relations that can be defined in* INEX[1]

In this subsection we consider some elementary ESO[0]-definable properties of binary team relations. For INEX[1], some of these properties turn out be definable while others will be proven undefinable in Section 7.

Let X be a team for which $y_1, y_2 \in \mathrm{dom}(X)$. The table in Fig. 1 contains elementary properties for the team relation $X(y_1y_2)$. We first note that by applying our translation from ESO[k] to INEX[k] in [20] (or [21]), we can express all ESO[2]-definable properties of binary team relations in INEX[2]. Moreover, as this translation is very straightforward, we can find INEX[2]-formulas $\varphi'(y_1y_2)$ which are syntactically almost identical to the (canonical) ESO[0]-formulas $\varphi(R)$ which define these properties for a relation R. The formula $\varphi(R)$ is translated into $\varphi'(y_1y_2)$ simply as follows.[13]

$$\varphi' = \varphi \quad \text{for FO-literals (not containing } R);$$
$$(Rx_1x_2)' := x_1x_2 \subseteq y_1y_2, \quad (\neg Rx_1x_2)' := x_1x_2 \,|\, y_1y_2;$$
$$(\psi \wedge \theta)' := \psi' \wedge \theta', \quad (\psi \vee \theta) := \psi' \underset{y_1y_2}{\vee} \theta';$$
$$(\exists x\, \psi)' := \exists x\, \psi', \quad (\forall x\, \psi)' := \forall x\, \psi'.$$

[13] It is worth noting that this translation stays very simple also when φ contains k-ary second order quantifications $\exists P$ – such quantifiers are simply replaced with repeated quantifications $\exists w_1 \dots \exists w_k$ of *first order variables* w_i, literals with P are translated as: $(P\vec{x})' := \vec{x} \subseteq w_1 \dots w_k$ and $(\neg P\vec{x})' := \vec{x} \,|\, w_1 \dots w_k$ and all disjunctions are additionally required to preserve the values for $w_1 \dots w_k$. See [20] or [21] for details.

Property of the relation $X(y_1y_2)$	Canonical INEX[2]-formula defining the property	An INEX[1]-formula defining the property
Irreflexivity	$\forall x\,(xx \mid y_1y_2)$	$y_1 \neq y_2$
Non-irreflexivity	$\exists x\,(xx \subseteq y_1y_2)$	$\triangledown(y_1 = y_2)$
Reflexivity	$\forall x\,(xx \subseteq y_1y_2)$	$\triangledown(y_1 = y_2 \wedge T(y_1))$
Non-reflexivity	$\exists x\,(xx \mid y_1y_2)$	$\mathbf{C}(x)(x \neq y_1 \vee x \neq y_2)$
Seriality	$\forall x_1 \exists x_2\,(x_1x_2 \subseteq y_1y_2)$	$T(y_1)$
Non-seriality	$\exists x_1 \forall x_2\,(x_1x_2 \mid y_1y_2)$	$\mathbf{C}(x)(x \neq y_1)$
Symmetry	$\forall x_1 \forall x_2\,(x_1x_2 \mid y_1y_2 \underset{y_1y_2}{\vee} x_2x_1 \subseteq y_1y_2)$	Undefinable in INEX[1]
Non-symmetry	$\exists x_1 \exists x_2\,(x_1x_2 \subseteq y_1y_2 \wedge x_2x_1 \mid y_1y_2)$	$\mathbf{C}(x_1,x_2)\big((x_1 = y_1 \wedge x_2 = y_2) \overset{\triangleright}{\vee} (x_1 \neq y_2 \vee x_2 \neq y_1)\big)$
2-totality	$\forall x_1 \forall x_2\,(x_1x_2 \subseteq y_1y_2)$	Undefinable in INEX[1]
Non-2-totality	$\exists x_1 \exists x_2\,(x_1x_2 \mid y_1y_2)$	$\mathbf{C}(x_1)\,\mathbf{C}(x_2)(x_1 \neq y_1 \vee x_2 \neq y_2)$

Fig. 1. Definability/undefinability in INEX[1] and INEX[2] for certain elementary ESO[0]-definable binary team relations and the formulae defining these properties. (2-totality means the property of $X(y_1y_2)$ being the full binary relation M^2.)

The only slightly more complex detail in the translation above is that standard disjunctions are translated into *term value preserving disjunctions* $\underset{y_1y_2}{\vee}$. This operator has otherwise the same semantics as the standard disjunction, but it additionally requires that the team relation for the tuple y_1y_2 is preserved when the team is split – that is, $X(y_1y_2) = Y(y_1y_2) = Y'(y_1y_2)$ when X is split into Y and Y'. Term value preserving disjunction $\underset{\vec{y}}{\vee}$ for k-tuples $\vec{y}$ of variables can be expressed in INEX[k]; see [20] or [21] for details.

For example, reflexivity is canonically defined by the formula $\forall x\, Rxx$ which is translated into $\forall x\,(xx \subseteq y_1y_2)$. Note however, that ESO[0]-formula defining the property must also be translated in negation normal form and operators $\rightarrow$ and $\leftrightarrow$ are assumed to be expressed with other operators (thus e.g. symmetry is defined by $\forall x_1 \forall x_2(\neg Rx_1x_2 \vee Rx_2x_1)$).

Several of the properties in Fig. 1 can also be defined in INEX[1], but it is often nontrivial to write a formula that defines them, and there seems to be no systematic and simple method for finding them like in the case of INEX[2]. Formulas defining the properties often become shorter when we apply INEX[1] definable operators like constant quantification and possibility operator (recall Section 4). The formulas defining irreflexivity, reflexivity and seriality have already been discussed in Example 4.1 – we sketch proofs for all the remaining cases below. (Recall that in all of the cases here we assume that $X \neq \emptyset$.)

- *Non-irreflexivity*: $X(y_1y_2)$ is not irreflexive if and only if there is $s \in X$ such that $s(y_1) = s(y_2)$. This clearly holds if and only if $\mathcal{M} \vDash_X \triangledown(y_1 = y_2)$.
- *Non-reflexivity*: $X(y_1y_2)$ is not reflexive if and only if there is $a \in M$ s.t. for all $s \in X$: $s(y_1) \neq a$ or $s(y_2) \neq a$. It is easy to see that this is true if and only if $\mathcal{M} \vDash_X \mathbf{C}(x)(x \neq y_1 \vee x \neq y_2)$.
- *Non-seriality*: $X(y_1y_2)$ is not serial if and only if there is $a \in M$ such that $s(y_1) \neq a$ for all $s \in X$. This clearly holds if and only if $\mathcal{M} \vDash_X \mathbf{C}(x)(x \neq y_1)$.
- *Non-symmetry*: $X(y_1y_2)$ is not symmetric if and only if there are $a, b \in M$ s.t. $a \neq b$ and the following conditions hold:

  $$\begin{cases} \text{there is } s \in X \text{ for which } s(y_1) = a \text{ and } s(y_2) = b \\ \text{for all } r \in X:\ r(y_1) \neq b \text{ or } r(y_2) \neq a. \end{cases}$$

  This is true iff $\mathcal{M} \vDash_X \mathbf{C}(x_1,x_2)\big((x_1 = y_1 \wedge x_2 = y_2) \overset{\triangleright}{\vee} (x_1 \neq y_2 \vee x_2 \neq y_1)\big)$.
- *Non-2-totality*: $X(y_1y_2)$ is not the full relation M^2 if and only if there are $a, b \in M$ s.t. for all $s \in X$: $s(y_1) \neq a$ or $s(y_2) \neq b$. Clearly this is true if and only if $\mathcal{M} \vDash_X \mathbf{C}(x_1)\,\mathbf{C}(x_2)(x_1 \neq y_1 \vee x_2 \neq y_2)$.
- *Symmetry* and *2-totality* are both undefinable in INEX[1] – as will be shown by Corollaries 7.17 and 7.16, respectively.

6.2. ESO[1]-*definable binary relations that can be defined in* INEX[1]

Above we saw that many natural ESO[0]-definable properties of binary team relations can be defined in INEX[1]. Here we show that there are also some interesting INEX[1]-definable properties of team relations which are beyond ESO[0]-definability. See the following example.

Example 6.1. Let $\mathcal{M} = (M, \mathcal{I})$ be a model and X be a team for $\mathcal{M}$ such that $\mathrm{dom}(X) = \{y_1, y_2\}$. We define the undirected graph $\mathcal{G}_X = (M, E_X)$, where the relation $E_X \subseteq M^2$ is the *symmetric closure* of $X(y_1y_2)$ (that is, $E_X := \{(a,b) \mid (a,b) \in X(y_1y_2) \text{ or } (b,a) \in X(y_1y_2)\}$.) Now it holds that:

(1) $\mathcal{G}_X$ is disconnected iff

$$\mathcal{M} \vDash_X \gamma_{\geq 2} \wedge \exists x_1 \exists x_2 \big(x_1 \,|\, x_2 \wedge \big((y_1 \subseteq x_1 \wedge y_2 \subseteq x_1) \vee (y_1 \subseteq x_2 \wedge y_2 \subseteq x_2)\big)\big).$$

(2) $\mathcal{G}_X$ is k-colorable iff

$$\mathcal{M} \vDash_X \gamma_{\leq k} \vee \exists x_1 \dots \exists x_k \Big(\bigwedge_{i \neq j} x_i \,|\, x_j \wedge T(x_1 \cup \dots \cup x_k) \wedge \bigvee_{x_1,\dots,x_k} \{y_1 \subseteq x_i \wedge y_2 \,|\, x_i \;\mid\; i \leq k\}\Big).$$

In the latter formula $\underset{x_1,\dots,x_k}{\vee}$ is *term value preserving disjunction* which requires that the team is split into subteams in such a way that the values of the variable x_i, for each $i \leq k$, in each subteam are the same as in the initial team before the split. This operator is indeed definable in INEX[1]; see [20] or [21] for details.[14]

We explain briefly why the equivalences (1)–(2) above hold. Supposing that $|M| \geq 2$, the graph $\mathcal{G}_X$ is disconnected if and only if there are nonempty disjoint $A, B \subseteq M$ such that for each assignment $s \in X$ we have either $s(y_1), s(y_2) \in A$ or $s(y_1), s(y_2) \in B$. Moreover, A and B satisfy these conditions with respect to $X(y_1y_2)$ if and only if the quantifier free part of the formula in (1) is true in a team Y that is obtained by extending X in such a way that $Y(x_1) = A$ and $Y(x_2) = B$. It thus follows that the equivalence in (1) holds.

For the equivalence in (2), we first note that $\mathcal{G}_X$ is trivially k-colorable if $|M| \leq k$. When $|M| > k$, the graph $\mathcal{G}_X$ is k-colorable if and only if we can split M into nonempty disjoint subsets $A_1, \dots, A_k$ (covering M) such that for each $s \in X$ we have $s(y_1) \in A_i$ and $s(y_2) \in A_j$ for some $i \neq j$. The sets $A_1, \dots, A_k$ satisfy these conditions if and only if the quantifier free part of the formula in (2) is true in a team Y that is obtained by extending X in such a way that $Y(x_i) = A_i$ for each $i \leq k$. It thus follows that the equivalence in (2) holds.

It is worth noting that the also properties above would have been easy to define in INEX[2] simply by using the canonical ESO[1]-formulas and applying our translation to them. However, it was again much harder to find INEX[1]-formulas defining these properties.

Note that in the example above we were "forcing symmetry" by considering an undirected graph obtained from the symmetric closure of $X(y_1y_2)$. We cannot simply express that $(M, X(y_1y_2))$ forms an undirected graph since the symmetry of $X(y_1y_2)$ is not definable in INEX[1]. This follows from a more general result which we present in the next section.

[14] Note that the operator $\underset{x_1,\dots,x_k}{\vee}$ here only preserves the values for (single) variables x_i and not necessarily preserves the value of the tuple $x_1 \dots x_k$ as $\underset{x_1 \dots x_k}{\vee}$ (without commas) which we discussed earlier. In order to preserve values for k-tuples we would need to use INEX[k].

7. $\mathfrak{L}[k]$-undefinable $(k+1)$-ary team relations

In this Section we will develop a proof method for showing that for any logic $\mathcal{L}$ in $\mathfrak{L}[k]$ (recall Definition 3.3), there exist such properties of $(k+1)$-ary team relations which are not definable in $\mathcal{L}$. In particular, this method can be used to show that the property of $X(y_1 \dots y_{k+1})$ being the full $(k+1)$-ary team relation M^{k+1} cannot be defined with any logic in $\mathfrak{L}[k]$.

Our proof method is intuitively based on analyzing the cardinality of sets of such assignments whose removal alters some k-ary team relations during the "evaluation process" of formulas. We give several auxiliary definitions and lemmas in the following subsections and finally in Subsection 7.4 we present Theorem 7.15 which can be used as a tool for proving various undefinability results for any logics in the class $\mathfrak{L}[k]$.

Unless specified differently, throughout this section $\mathcal{L}$ denotes any logic that is an *extension of* FO *with any atomic formulas which are local* (note that we do not assume any closure properties or empty team property).

7.1. Satisfying evaluations

In this section we define functions called *evaluations* which assign teams to nodes in a syntax tree of a formula. These teams are assigned in the way corresponding to the truth conditions of the operators in FO (with team semantics). *Satisfying evaluations* intuitively correspond to "correct semantic reasoning" for showing that a formula φ is true in a given team X. Similar concepts have been defined and used earlier in the context of *boolean dependence logic* in [3].[15]

In the definition below, remember that by $\mathrm{Sf}(\varphi)$ we mean the set of occurrences of the subformulae of φ (cf. Remark 2.5). This is essential here as different occurrences of formulae may need to be mapped to different teams.

Definition 7.1. Let A be a set. We write $\mathcal{E}_A$ for the class of functions, called *evaluations*, so that for each $E \in \mathcal{E}_A$ we have:

$$E : \mathrm{Sf}(\varphi) \to \{X \mid X \text{ is a team for } A\} \quad \text{for some } \varphi \in \mathcal{L}$$

and the following conditions hold for the (occurrences of) subformulas of φ:

- $E(\psi \wedge \theta) = E(\psi) = E(\theta)$.
- $E(\psi \vee \theta) = E(\psi) \cup E(\theta)$.
- $E(\exists x\, \psi)[F/x] = E(\psi)$ for some $F : E(\exists x \psi) \to \mathcal{P}^*(A)$.
- $E(\forall x\, \psi)[A/x] = E(\psi)$.

In the conditions above it is assumed that ψ is the occurrence following the quantifier Qx in $Qx\,\psi$ for $Q \in \{\exists, \forall\}$, and ψ, θ are the occurrences of the conjuncts/disjuncts in $\psi \circ \theta$ for $\circ \in \{\wedge, \vee\}$.

For each $E \in \mathcal{E}_A$ we define the *set of atomic teams for* E, denoted by $E(\mathrm{Atom})$ as follows.

$$E(\mathrm{Atom}) := \{E(\psi) \mid \psi \in \mathrm{dom}(E) \text{ is atomic}\}.$$

Note that for each evaluation E, there is a unique formula φ and a unique team X such that $\mathrm{dom}(E) = \mathrm{Sf}(\varphi)$ and $E(\varphi) = X$. We then say that *E is an evaluation for φ in X*.

[15] Related definitions here were developed independently by the author. We have adopted here some terminology and notations from [3] for uniformity.

Consider a model $\mathcal{M}$ with universe M. If X is a team for $\mathcal{M}$ and $E \in \mathcal{E}_M$ is an evaluation for φ in X, then E naturally corresponds to "an attempt of proof" for the claim that $\mathcal{M} \vDash_X \varphi$. This attempt is successful if each subformula of φ is assigned to a team which satisfies it. This naturally leads to the following definition.

Definition 7.2. Let $\mathcal{M} = (M, \mathcal{I})$ be an L-model. We call $E \in \mathcal{E}_M$ *a satisfying evaluation in* $\mathcal{M}$ it the following holds:

$$\mathcal{M} \vDash_{E(\psi)} \psi \text{ for each } \psi \in \mathrm{dom}(E).$$

If $E \in \mathcal{E}_M$ is an evaluation for φ in X and E is a satisfying evaluation in $\mathcal{M}$, we say that E *is a satisfying evaluation for* φ *in* $(\mathcal{M}, X)$. We write

$$\mathrm{Sat}(\mathcal{M}, X, \varphi) := \{E \in \mathcal{E}_M \mid E \text{ is a satisfying evaluation for } \varphi \text{ in } (\mathcal{M}, X)\}.$$

Remark 7.3. For E to be a satisfying evaluation in $\mathcal{M}$, it suffices that $\mathcal{M} \vDash_{E(\psi)} \psi$ for each $\psi \in E(\mathrm{Atom})$. This is easy to see by the definition of evaluations.

As one would expect, a formula is true if and only if there is *at least* one satisfying evaluation for it. This is stated in the following lemma which is easy to prove (see Lemma 5.6 in [21] for a complete proof).

Lemma 7.4. *Let* $\mathcal{M}$ *be a* L*-model,* X *be a team and* $\varphi \in \mathcal{L}$*. Then*

$$\mathcal{M} \vDash_X \varphi \quad \textit{iff} \quad \mathrm{Sat}(\mathcal{M}, X, \varphi) \neq \emptyset.$$

7.2. Removal of extension sets

In this subsection we define the *extension set* $Y_{s\prec}$ for an assignment $s \in X$ in a given team Y for which $\mathrm{dom}(X) \subseteq \mathrm{dom}(Y)$. Then we show how the removal of s from X is related to the removal of $Y_{s\prec}$ from Y (Lemma 7.8). This will be one of the key elements for proving our undefinability results later.

Definition 7.5. Let s be an assignment and Y a team s.t. $\mathrm{dom}(s) \subseteq \mathrm{dom}(Y)$. The *extension set of* s *in* Y, denoted by $Y_{s\prec}$, is defined as follows:

$$Y_{s\prec} := \{r \in Y \mid \; r \upharpoonright \mathrm{dom}(s) = s\}.$$

The next example shows how the extension sets in a team "evolve" when we modify the team in different ways related to the truth conditions of the operators in FO. This example also makes it easier to follow the proof for Lemma 7.8 which we present later.

Example 7.6. Let X be a team for a model $\mathcal{M} = (M, \mathcal{I})$ and let $s \in X$.

- We first observe that $X_{s\prec} = \{s\}$.
- Let $Y' \cup Y'' = Y$. Then we have

$$Y'_{s\prec} = \{r \in Y' \mid r \in Y_{s\prec}\} \text{ and } Y''_{s\prec} = \{r \in Y'' \mid r \in Y_{s\prec}\}.$$

- Let $F : Y \to \mathcal{P}^*(M)$ and $x \notin \mathrm{dom}(X)$. Then

$$Y[F/x]_{s\prec} = \{r[a/x] \in Y[F/x] \mid \; r \in Y_{s\prec}, a \in F(r)\}.$$

- Let $x \notin \mathrm{dom}(X)$. Then

$$Y[M/s]_{s\prec} = \{r[a/x] \in Y[M/x] \mid r \in Y_{s\prec},\, a \in M\}.$$

Lemma 7.7. *Let X and Y be teams for which $\mathrm{dom}(X) \subseteq \mathrm{dom}(Y)$ and let $s_1, s_2 \in X$. Then it holds that:*

$$\textit{If } s_1 \neq s_2, \textit{ then } Y_{s_1\prec} \cap Y_{s_2\prec} = \emptyset.$$

Proof. If there is $r \in Y_{s_1\prec} \cap Y_{s_2\prec}$, then $s_1 = r \upharpoonright \mathrm{dom}(X) = s_2$. □

Let Y and Z be teams for which $\mathrm{dom}(Z) \subseteq \mathrm{dom}(Y)$. We use the following abbreviation for the team that is obtained from Y by removing the extension set of s in Y for all $s \in Z$.

$$Y \setminus^{\prec} Z := Y \setminus \bigcup_{s\in Z} Y_{s\prec}.$$

Suppose that $\mathcal{M} \vDash_X \varphi$, for $\varphi \in \mathcal{L}$, and let E be a satisfying evaluation for φ in $(\mathcal{M}, X)$. If φ contains atoms that are not closed downwards, then we do not generally have $\mathcal{M} \vDash_{X\setminus X'} \varphi$ for all $X' \subseteq X$, but for some X' this might be the case. In the following lemma we show that, in order to prove that $\mathcal{M} \vDash_{X\setminus X'} \varphi$ holds, it suffices that we check that all the atomic formulas ψ, which are not closed downwards, remain true when the extension sets for all $s \in X'$ are removed from $E(\psi)$.

Lemma 7.8. *Let $\mathcal{M}$ be a model and X a team for $\mathcal{M}$. Let $\varphi \in \mathcal{L}$ such that none of the variables $x \in \mathrm{dom}(X)$ is quantified in φ. Let $E \in \mathrm{Sat}(\mathcal{M}, X, \varphi)$ and let X' be a subteam of X. Suppose that the following holds:*

$$\mathcal{M} \vDash_{E(\psi)\setminus^{\prec} X'} \psi \;\; \textit{for every } \psi \in \mathrm{Atom}(\varphi) \textit{ which is not closed downwards.}$$

Then we have $\mathcal{M} \vDash_{X\setminus X'} \varphi$.

Proof. Since $E(\varphi) = X$ and $X_{s\prec} = \{s\}$ for each $s \in X'$, we have

$$E(\varphi) \setminus^{\prec} X' = X \setminus \bigcup_{s\in X'} \{s\} = X \setminus X'.$$

Thus, in order to show that $\mathcal{M} \vDash_{X\setminus X'} \varphi$ holds, it suffices that we prove the following claim by structural induction on φ:

$$\mathcal{M} \vDash_{E(\mu)\setminus^{\prec} X'} \mu \text{ for all } \mu \in \mathrm{Sf}(\varphi).$$

- If μ is an atom which is not closed downwards, then the claim follows from the assumptions. Suppose then that μ is a literal or downwards closed atom. Since $E \in \mathrm{Sat}(\mathcal{M}, X, \varphi)$, we have $\mathcal{M} \vDash_{E(\mu)} \mu$. Thus by downwards closure we have $\mathcal{M} \vDash_{E(\mu)\setminus^{\prec} X'} \mu$.
- Let $\mu = \psi \wedge \theta$. Now $E(\psi \wedge \theta) = E(\psi) = E(\theta)$ and thus by the inductive hypothesis $\mathcal{M} \vDash_{E(\psi\wedge\theta)\setminus^{\prec} X'} \psi$ and $\mathcal{M} \vDash_{E(\psi\wedge\theta)\setminus^{\prec} X'} \theta$, i.e. $\mathcal{M} \vDash_{E(\psi\wedge\theta)\setminus^{\prec} X'} (\psi \wedge \theta)$.
- Let $\mu = \psi \vee \theta$. Let $E(\psi \vee \theta) = Y$, whence $E(\psi) = Y'$ and $E(\theta) = Y''$ for some $Y', Y'' \subseteq Y$ for which $Y' \cup Y'' = Y$. By the inductive hypothesis we have $\mathcal{M} \vDash_{Y'\setminus^{\prec} X'} \psi$ and $\mathcal{M} \vDash_{Y''\setminus^{\prec} X'} \theta$. Now for all $s \in X'$ we have

$$Y_{s\prec} = Y'_{s\prec} \cup Y''_{s\prec}, \quad Y'_{s\prec} = Y_{s\prec} \cap Y' \;\text{ and }\; Y''_{s\prec} = Y_{s\prec} \cap Y''.$$

Therefore

$$\begin{aligned}(Y') \setminus^{\prec} X' \cup (Y'') \setminus^{\prec} X' &= (Y' \setminus \bigcup_{s\in X'} Y'_{s\prec}) \cup (Y'' \setminus \bigcup_{s\in X'} Y''_{s\prec}) \\ &= (Y' \setminus \bigcup_{s\in X'} Y_{s\prec}) \cup (Y'' \setminus \bigcup_{s\in X'} Y_{s\prec}) \\ &= (Y' \cup Y'') \setminus \bigcup_{s\in X'} Y_{s\prec} \\ &= Y \setminus \bigcup_{s\in X'} Y_{s\prec} \;=\; Y \setminus^{\prec} X' \;=\; E(\psi \vee \theta) \setminus^{\prec} X'.\end{aligned}$$

Hence $\mathcal{M} \vDash_{E(\psi\vee\theta)\setminus^{\prec} X'} (\psi \vee \theta)$.

- Let $\mu = \exists x\, \psi$. Let $E(\exists x\, \psi) = Y$, whence $E(\psi) = Y[F/x]$ for some function $F : Y \to \mathcal{P}^*(M)$. By the inductive hypothesis, $\mathcal{M} \vDash_{Y[F/x]\setminus^{\prec} X'} \psi$. By our assumptions $x \notin \mathrm{dom}(X)$ and therefore

$$Y[F/x]_{s\prec} = Y_{s\prec}[(F \upharpoonright Y_{s\prec})/x] \quad \text{ for all } s \in X'.$$

We can define $F' := F \upharpoonright (Y \setminus^{\prec} X')$. Now we have

$$\begin{aligned}(Y \setminus^{\prec} X')[F'/x] &= (Y \setminus \bigcup_{s\in X'} Y_{s\prec})[F'/x] = Y[F/x] \setminus \bigcup_{s\in X'} Y_{s\prec}[(F \upharpoonright Y_{s\prec})/x] \\ &= Y[F/x] \setminus \bigcup_{s\in X'} Y[F/x]_{s\prec} \;=\; Y[F/x] \setminus^{\prec} X'.\end{aligned}$$

Hence $\mathcal{M} \vDash_{(E(\exists x\, \psi)\setminus^{\prec} X')[F'/x]} \psi$, i.e. $\mathcal{M} \vDash_{E(\exists x\, \psi)\setminus^{\prec} X'} \exists x\, \psi$.

- Let $\mu = \forall x\, \psi$. Let $E(\forall x\, \psi) = Y$. Now $E(\psi) = Y[M/x]$ and by the inductive hypothesis $\mathcal{M} \vDash_{Y[M/x]\setminus^{\prec} X'} \psi$. By our assumptions $x \notin \mathrm{dom}(X)$ and therefore

$$Y[M/x]_{s\prec} = Y_{s\prec}[M/x] \quad \text{ for all } s \in X'.$$

Thus we have

$$\begin{aligned}(Y \setminus^{\prec} X')[M/x] &= (Y \setminus \bigcup_{s\in X'} Y_{s\prec})[M/x] = Y[M/x] \setminus \bigcup_{s\in X'} Y_{s\prec}[M/x] \\ &= Y[M/x] \setminus \bigcup_{s\in X'} Y[M/x]_{s\prec} \;=\; Y[M/x] \setminus^{\prec} X'.\end{aligned}$$

Hence $\mathcal{M} \vDash_{(E(\forall x\, \psi)\setminus^{\prec} X')[M/x]} \psi$, i.e. $\mathcal{M} \vDash_{E(\forall x\, \psi)\setminus^{\prec} X'} \forall x\, \psi$. □

7.3. Estimates for the cardinality k-separating sets

In the previous subsection we showed that, under certain conditions, some assignments can be removed from a team X without violating the truth of a formula φ. In this section we will analyze when such assignments exist by giving estimates on the cardinality of so-called *k-separating sets*. We begin by defining so-called *k-separators* which are sets of assignments whose removal alters the values of some k-ary relations in a given team.

Definition 7.9. Let Y, Z be teams for $\mathcal{M}$. We say that Z is a *k-separator of* Y, if the following holds:

$$\text{there is } \vec{y} \in (\mathrm{dom}(Y))^k \text{ s.t. } (Y \setminus Z)(\vec{y}) \subset Y(\vec{y}).$$

(Note that we use $\subset$ to denote the *proper* subset relation.)

Observation 7.10. Recall k-equivalence and k-invariant atoms in Definition 3.1. It is easy to see that if $Z \subseteq Y$ is not a k-separator of Y, then Y is k-equivalent to $(Y \setminus Z)$. Therefore, for any k-invariant atom A, the truth of $\mathcal{M} \vDash_Y A$ implies the truth of $\mathcal{M} \vDash_{Y \setminus Z} A$, when Z is not a k-separator of Y.

Next, for given teams X and Y, we define the *k-separating set of X for Y*. This set consists of all those assignments $s \in X$ whose extension set in Y is a k-separator of Y.

Definition 7.11. Let X, Y be teams for which $\mathrm{dom}(X) \subseteq \mathrm{dom}(Y)$. We define the *$k$-separating set of X for Y*, denoted by $\mathrm{Sep}^k_X(Y)$, as follows:

$$\mathrm{Sep}^k_X(Y) := \{s \in X \mid Y_{s\prec} \text{ is a } k\text{-separator of } Y\}.$$

For any evaluation $E \in \mathcal{E}_A$ (for some φ) in X, we use the following abbreviation:

$$\mathrm{Sep}^k_X(E) := \bigcup_{Y \in E(\mathrm{Atom})} \mathrm{Sep}^k_X(Y).$$

In the results later we want to show that, under certain assumptions for a given evaluation E, we have $\mathrm{Sep}^k_X(E) \neq X$. That is, there are assignments $s \in X$ whose extension sets are not k-separators in any $Y \in E(\mathrm{Atom})$. For proving this, we need the next lemma which gives an estimate for the number of assignments in $\mathrm{Sep}^k_X(Y)$ – with respect to some evaluation E.

Lemma 7.12. *Let $\varphi \in \mathcal{L}$, let X be a team and let $E \in \mathcal{E}_A$ be an evaluation for φ in X. Now for each $Y \in E(\mathrm{Atom})$ we have*

$$|\mathrm{Sep}^k_X(Y)| \;\leq\; (\mathrm{Oper}_\#(\varphi) + |\mathrm{dom}(X)|)^k \cdot |A|^k.$$

Proof. Let $Y \in E(\mathrm{Atom})$. All variables in $\mathrm{dom}(Y)$ are either in $\mathrm{dom}(X)$ or quantified in φ. Hence $|\mathrm{dom}(Y)| \leq \mathrm{Oper}_\#(\varphi) + |\mathrm{dom}(X)|$ and thus

$$|(\mathrm{dom}(Y))^k| = |\mathrm{dom}(Y)|^k \leq (\mathrm{Oper}_\#(\varphi) + |\mathrm{dom}(X)|)^k.$$

For every $s \in \mathrm{Sep}^k_X(Y)$ there is (at least one) tuple $\vec{y} \in (\mathrm{dom}(Y))^k$ such that

$$(Y \setminus Y_{s\prec})(\vec{y}) \subset Y(\vec{y}). \tag{$\star$}$$

Since $|(\mathrm{dom}(Y))^k| \leq (\mathrm{Oper}_\#(\varphi) + |\mathrm{dom}(X)|)^k$, in order prove the claim of this lemma, it suffices to show that for each $\vec{y} \in (\mathrm{dom}(Y))^k$, there exist at most $|A|^k$ different assignments s for which the condition $(\star)$ holds.

Let $\vec{y} \in (\mathrm{dom}(Y))^k$. The condition $(\star)$ holds, with respect to $\vec{y}$ and some $s \in X$, if and only if there exists some k-tuple $\vec{a} \in A^k$ such that

$$\vec{a} \in Y(\vec{y}) \;\text{ but }\; \vec{a} \notin (Y \setminus Y_{s\prec})(\vec{y}). \tag{$\star\star$}$$

Since $|A^k| = |A|^k$, it thus suffices to show that for each $\vec{a} \in A^k$ there exists *at most one* $s \in X$ such that the condition $(\star\star)$ holds for the pair $(\vec{a}, s)$.

Let $\vec{a} \in A^k$ and $s_1, s_2 \in X$ such that $(\star\star)$ holds for both $(\vec{a}, s_1)$ and $(\vec{a}, s_2)$. Now $\vec{a} \in Y_{s_1\prec}(\vec{y})$ and thus there is $r \in Y_{s_1\prec}$ for which $r(\vec{y}) = \vec{a}$. Since $r \in Y$, but $r(\vec{y}) = \vec{a} \notin (Y \setminus Y_{s_2\prec})(\vec{y})$, we must have $r \in Y_{s_2\prec}$. Hence $r \in Y_{s_1\prec} \cap Y_{s_2\prec}$ and thus by (the contraposition of) Lemma 7.7 it has to be that $s_1 = s_2$. $\square$

Hereafter we will focus our analysis on logics $\mathcal{L}$ belonging to the class $\mathfrak{L}[k]$ of (local) logics extending FO with k-invariant and downward closed atoms (Definition 3.3). Observe that when we fix $k \geq 1$, $\mathcal{L}$-formula φ and a size of the domain for X, then we can choose large enough set A so that $|A|$ is larger than $(\mathrm{Oper}_{\#}(\varphi) + |\mathrm{dom}(X)|)^k$. Then, by Lemma 7.12, we have $|\mathrm{Sep}^k_X(Y)| < A^{k+1}$ for each $Y \in E(\mathrm{Atom})$, where $E \in \mathcal{E}_A$ is an evaluation for φ in X. By using this observation with the result of Lemma 7.8, we can show that, for sufficiently large models $\mathcal{M}$ and teams X with $|\,\mathrm{dom}(X)| = k + 1$, the following holds: if $\mathcal{M} \vDash_X \varphi$ and X is "of the size $|M|^{k+1}$" (i.e. $|M|^{k+1}$ divided by some constant), then there is $s \in X$ such that $\mathcal{M} \vDash_{X\setminus\{s\}} \varphi$. This claim is presented more formally and generally in the next lemma.

Lemma 7.13. *Suppose that $\mathcal{L}$ belongs to $\mathfrak{L}[k]$. Let $m, k, c \geq 1$ and let $\mathcal{M} = (M, \mathcal{I})$ be an L-model for which $M = \{1, \dots, c \cdot p^{3k}\}$, where $p = \max(m + 1, k, 3)$. Let X and X^* be teams for $\mathcal{M}$ such that $X^* \subseteq X$, $|\mathrm{dom}(X)| = k + 1$ and*

$$|X^*| \geq \frac{|M|^{k+1}}{c}.$$

Then for every $\mathcal{L}$-formula φ, for which $\mathrm{Oper}_{\#}(\varphi) \leq m$, the following implication holds:

$$\text{If } \mathcal{M} \vDash_X \varphi, \text{ then there exists } s \in X^* \text{ s.t. } \mathcal{M} \vDash_{X\setminus\{s\}} \varphi.$$

Proof. Let $\varphi \in \mathcal{L}$ such that $\mathrm{Oper}_{\#}(\varphi) \leq m$. Without loss of generality, we may assume that none of $x \in \mathrm{dom}(X)$ is quantified in φ.[16] Suppose that $\mathcal{M} \vDash_X \varphi$, whence by Lemma 7.4 there is $E \in \mathrm{Sat}(\mathcal{M}, X, \varphi)$. Since $\mathrm{Oper}_{\#}(\varphi) \leq m$ and $|\mathrm{dom}(X)| = k + 1$, by Lemma 7.12 we have

$$|\mathrm{Sep}^k_X(Y)| \;\leq\; (m + (k+1))^k \cdot |M|^k \qquad \text{for each } Y \in E(\mathrm{Atom}).$$

Let $Y_{max} \in E(\mathrm{Atom})$ be a team with the largest k-separating set of X ($|\mathrm{Sep}^k_X(Y_{max})| \geq |\mathrm{Sep}^k_X(Y)|$ for all $Y \in E(\mathrm{Atom})$). Now we have

$$|\mathrm{Sep}^k_X(E)| \;\leq\; |E(\mathrm{Atom})| \cdot |\mathrm{Sep}^k_X(Y_{max})| \;\leq\; |\mathrm{Atom}(\varphi)| \cdot |\mathrm{Sep}^k_X(Y_{max})|.$$

The set $\mathrm{Atom}(\varphi)$ has one more element than the number of the connectives $\wedge$ and $\vee$ in φ. Since $\mathrm{Oper}_{\#}(\varphi) \leq m$, we thus have $|\mathrm{Atom}(\varphi)| \leq m + 1$. Hence

$$|\mathrm{Sep}^k_X(E)| \;\leq\; |\mathrm{Atom}(\varphi)| \cdot |\mathrm{Sep}^k_X(Y_{max})| \;\leq\; (m+1)((m+1)+k)^k |M|^k.$$

- Suppose first that $p = m + 1$. Now $m + 1 \geq k, 3$ and thus

$$\begin{aligned}(m+1)((m+1)+k)^k &\leq (m+1)(2(m+1))^k = 2^k(m+1)^{k+1} \\ &< (m+1)^k(m+1)^{k+1} \leq (m+1)^{3k} = p^{3k}.\end{aligned}$$

- Suppose then that $p = k$. Now $k \geq m + 1, 3$ and thus

$$(m+1)((m+1)+k)^k \leq k\,(2k)^k = 2^k k^{k+1} < k^k k^{k+1} < k^{3k} = p^{3k}.$$

[16] If some of the variables in $\mathrm{dom}(X)$ were quantified in φ, we could instead consider a formula φ' where all of these quantifications (and the variables in their scopes) are replaced with quantifications of fresh variables. (Assuming locality) it is easy to see that then $\mathcal{M} \vDash_X \varphi$ iff $\mathcal{M} \vDash_X \varphi'$, and thus we may prove the claim of Lemma 7.13 for φ' instead.

- Finally suppose that $p = 3$. Now $3 \geq m+1, k$ and thus

$$(m+1)((m+1)+k)^k \leq 3\,(2 \cdot 3)^k < 3^{3k} = p^{3k}.$$

In all cases above $(m+1)((m+1)+k)^k < p^{3k}$. Hence we have

$$\begin{aligned} |\mathrm{Sep}^k_X(E)| &\leq (m+1)((m+1)+k)^k |M|^k \\ &< p^{3k}|M|^k = \frac{(c \cdot p^{3k})|M|^k}{c} = \frac{|M| \cdot |M|^k}{c} = \frac{|M|^{k+1}}{c} \leq |X^*|. \end{aligned}$$

Since $|\mathrm{Sep}^k_X(E)| < |X^*|$ there is (at least one) $s \in X^*$ for which $s \notin \mathrm{Sep}^k_X(E)$. We select and fix one such s. Note that every $\psi \in \mathrm{Atom}(\varphi)$ which is not closed downwards, is a k-invariant atom. Hence, in order to prove that $\mathcal{M} \vDash_{X \setminus \{s\}} \varphi$, by using Lemma 7.8, it suffices that we prove the following:

$$\mathcal{M} \vDash_{E(\psi) \setminus (E(\psi)_{s\prec})} \psi \;\text{ for every } k\text{-invariant atom } \psi \in \mathrm{Atom}(\varphi).$$

Let $\psi \in \mathrm{Atom}(\varphi)$ be a k-invariant atom. Because $E \in \mathrm{Sat}(\mathcal{M}, X, \varphi)$, we have $\mathcal{M} \vDash_{E(\psi)} \psi$. Since $s \notin \mathrm{Sep}^k_X(E)$, in particular $s \notin \mathrm{Sep}^k_X(E(\psi))$ and thus $E(\psi)_{s\prec}$ is not a k-separator of $E(\psi)$. Hence, recalling Observation 7.10, we have $\mathcal{M} \vDash_{E(\psi) \setminus (E(\psi)_{s\prec})} \psi$. □

Remark 7.14. There are several ways for improving the estimates in Lemmas 7.12 and 7.13 – by e.g. separately considering the *quantifier depth* of φ and the *number of connectives* in φ instead of $\mathrm{Oper}_\#(\varphi)$. However, the optimization of these estimates is not necessary for proving our results in the next subsection.

7.4. Theorem for proving undefinability results for $\mathfrak{L}[k]$

By using Lemma 7.13, we can prove the following theorem which can be used for proving undefinability results for various logics in $\mathfrak{L}[k]$ which extend FO with k-invariant atoms and downwards closed atoms. Thus, in particular, it can be applied for k-ary fragments of logics with team semantics, such as INEX[k].

Theorem 7.15. *Let $\mathcal{P}$ be a property of $(k+1)$-ary team relations. Assume that there is a constant c such that for any finite model $\mathcal{M} = (M, \mathcal{I})$, with at least c elements, there are teams X and X^* for $\mathcal{M}$ such that the following conditions hold:*

1. $X^* \subseteq X$.
2. $\mathrm{dom}(X) = \{y_1, \ldots, y_{k+1}\}$.
3. $X(y_1 \ldots y_{k+1})$ *has the property* $\mathcal{P}$.
4. $(X \setminus \{s\})(y_1 \ldots y_{k+1})$ *does not have the property* $\mathcal{P}$ *for any* $s \in X^*$.
5. $|X^*| \geq \frac{|M|^{k+1}}{c}$.

Now the property $\mathcal{P}$ cannot be defined with any logic in the class $\mathfrak{L}[k]$.

Proof. For the sake of contradiction, suppose that $\mathcal{P}$ can be defined with a logic $\mathcal{L}$ in $\mathfrak{L}[k]$. Thus there is $\varphi \in \mathcal{L}$ which defines $\mathcal{P}$; let $m = \mathrm{Oper}_\#(\varphi)$. Let $\mathcal{M}$ be a model, for which $M = \{1, \ldots, c \cdot p^{3k}\}$ where $p = \max(m, k, 3)$, and let X, X^* be teams for $\mathcal{M}$ so that they satisfy the properties given in the assumptions. Since $X(y_1 \ldots y_{k+1})$ has the property $\mathcal{P}$, we have $\mathcal{M} \vDash_X \varphi$. By Lemma 7.13 there is $s \in X^*$, such that $\mathcal{M} \vDash_{X \setminus \{s\}} \varphi$. This is a contradiction, because the team relation $(X \setminus \{s\})(y_1 \ldots, y_{k+1})$ does not have the property $\mathcal{P}$. □

The assumptions of Theorem 7.15 may look quite technical, but the core idea is rather simple: *those properties of teams that are very sensitive to removal of assignments cannot be defined in* $\mathfrak{L}[k]$. By "sensitive" we mean that for any team X with the given property $\mathcal{P}$, there are several assignments (namely those in $X^* \subseteq X$) such that the removal of any single one of them makes X to lose the property $\mathcal{P}$. By "several" we mean that the number of such assignments (namely $|X^*|$) is at least $|M^{k+1}|/c$, for some constant c.

By using Theorem 7.15, we can prove that several simple properties of team relations are undefinable in $\mathfrak{L}[k]$. The first of our undefinability results shows that, for any $k \geq 1$, the $(k+1)$-totality of $X(y_1 \dots y_{k+1})$ – i.e. $X(y_1 \dots y_{k+1})$ being the total $(k+1)$-ary relation M^{k+1} – cannot be defined in $\mathfrak{L}[k]$.

Corollary 7.16. *For any* $k \geq 1$, $(k+1)$*-totality of* $X(y_1 \dots y_{k+1})$ *cannot be defined in* $\mathfrak{L}[k]$.

Proof. Let $c = 1$ and let $\mathcal{M}$ be any finite model. Let X be the team for $\mathcal{M}$ for which $\mathrm{dom}(X) = \{y_1, \dots, y_{k+1}\}$ and $X(y_1 \dots y_{k+1}) = M^{k+1}$. Let $X^* = X$; now $X(y_1 \dots y_{k+1})$ satisfies $(k+1)$-totality, but $(X \setminus \{s\})(y_1 \dots y_{k+1})$ does not satisfy $(k+1)$-totality for any $s \in X^*$. Moreover, we have

$$|X^*| = |X| = |M|^{k+1} = \frac{|M|^{k+1}}{c}.$$

Thus the claim follows from Theorem 7.15. □

The proof for the corollary above uses Theorem 7.15 in the most trivial form since there were $|M|^{k+1}$ assignments in X^* such that removal of any of them sufficed for violating the given property. In the proofs of the next corollaries we will apply Theorem 7.15 for smaller teams X^*.

Corollary 7.17. *Symmetry of* $X(y_1 y_2)$ *cannot be defined in* $\mathfrak{L}[1]$.

Proof. Let $c = 2$ and let $\mathcal{M}$ be any finite model with at least 2 elements. Let X be the team for $\mathcal{M}$ s.t. $\mathrm{dom}(X) = \{y_1, y_2\}$ and $X(y_1 y_2) = M^2$. We define

$$X^* := \{s \in X \mid s(x_1) \neq s(x_2)\}.$$

Now $X(y_1 y_2)$ is clearly symmetric by being a total binary relation. However, $(X \setminus \{s\})(y_1 y_2)$ is not symmetric for any $s \in X^*$ (removal of a *single* edge from any 2-cycle immediately violates the symmetry). We also have

$$|X^*| = |M|^2 - |M| \geq \frac{|M|^2}{2} = \frac{|M|^2}{c}.$$

Thus the claim follows from Theorem 7.15. □

Corollary 7.18. $X(y_1 y_2)$ *being a linear order cannot be defined in* $\mathfrak{L}[1]$.

Proof. Let $c = 2$ and let $\mathcal{M}$ be any finite model with at least 2 elements. Let X be a team for $\mathcal{M}$ such that $\mathrm{dom}(X) = \{y_1, y_2\}$ and $X(y_1 y_2)$ is a linear order. Let $X^* = X$; now $(X \setminus \{s\})(y_1 y_2)$ is not a linear order for any $s \in X^*$ (removal of any edge from a linear order violates the comparability property). Also

$$|X^*| = |X| = \frac{|M|^2 - |M|}{2} + |M| > \frac{|M|^2}{2} = \frac{|M|^2}{c}.$$

Thus the claim follows from Theorem 7.15. □

Note that the properties of k-totality, symmetry and being a linear order are not closed downwards and thus not definable in any downwards closed logic.

Remark 7.19. Relevant disjunction $\overset{\bowtie}{\vee}$ and possibility operator $\triangledown$ have been studied by Hella and Stumpf in [11] in the context of *modal inclusion logic*. They show that, in the modal case, *all* inclusion atoms can be expressed with $\overset{\bowtie}{\vee}$ (or alternatively with $\triangledown$). However, in order to express a k-ary inclusion atom with $\triangledown$, we need a formula whose size is exponential with respect to k. This proof in [11] uses very similar methods as we use in this section.[17]

As shown in [12], a dual result holds for *modal dependence logic*: any dependence atom can be expressed with uniform disjunctions $\sqcup$, but in order to express a k-ary dependence atom with $\sqcup$, we need a formula of size exponential to k. However, this claim is proven by using very different kinds of methods than the ones used here and in [11].

8. Conclusion

On the level of all formulas, the study of the expressive power of logics with team semantics essentially amounts to the definability of team relations. In this paper we have developed various useful concepts and techniques for studying the definability of team relations. In particular, Theorem 7.15 can be used for proving various undefinability results for a rather rich family $\mathfrak{L}[k]$ of logics extending FO with k-invariant and downwards closed atoms. We also believe that the lemmas and definitions leading to this result could be adjusted for other kind of expressivity analysis in the framework of team semantics.

We have also demonstrated that the expressive power of unary inclusion-exclusion logic, INEX[1], is quite interesting as with it one can define various nontrivial properties of binary team relations, such as k-colorability of a corresponding graph. However, some simple properties such as symmetry of a team relation turned out to be undefinable. The full characterization of the expressive power of INEX[1] on the level of formulas remains still open.

As a final remark we note that, for using Theorem 7.15 to prove our undefinability results, it was essential to use finite models. It is thus natural to ask whether these undefinability results would hold if one only considered *infinite models*. However, in the infinite case we would need an essentially different proof method, and thus we leave this question open for future research.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] A.R. Anderson, N.D. Belnap, Entailment: The Logic of Relevance and Neccessity, vol. I, Princeton University Press, 1975.

[17] When this method was initially presented by the author (via private communication), Hella and Stumpf realized that a very similar technique could be used in [11]. Thus we have good reasons to believe that our proof techniques can be applied also in future research for this framework – especially for logics that are not closed downwards.

[2] A. Durand, J. Kontinen, Hierarchies in dependence logic, ACM Trans. Comput. Log. 13 (4) (2012) 31:1–31:21.
[3] J. Ebbing, L. Hella, P. Lohmann, J. Virtema, Boolean dependence logic and partially-ordered connectives, J. Comput. Syst. Sci. 88 (2017) 103–125.
[4] P. Galliani, Inclusion and exclusion dependencies in team semantics - on some logics of imperfect information, Ann. Pure Appl. Log. 163 (1) (2012) 68–84.
[5] P. Galliani, The Dynamics of Imperfect Information, ILLC Dissertation Series, Institute for Logic, Language and Computation, Amsterdam, 2012.
[6] P. Galliani, Upwards closed dependencies in team semantics, Inf. Comput. 245 (2015) 124–135.
[7] P. Galliani, M. Hannula, J. Kontinen, Hierarchies in independence logic, in: Proceedings of CSL 2013, 2013, pp. 263–280.
[8] E. Grädel, J.A. Väänänen, Dependence and independence, Stud. Log. 101 (2) (2013) 399–410.
[9] M. Hannula, Hierarchies in inclusion logic with lax semantics, in: Proceedings of ICLA 2015, 2015, pp. 100–118.
[10] M. Hannula, J. Kontinen, Hierarchies in independence and inclusion logic with strict semantics, J. Log. Comput. 25 (3) (2015) 879–897.
[11] L. Hella, J. Stumpf, The expressive power of modal logic with inclusion atoms, in: Proceedings of GandALF 2015, 2015, pp. 129–143.
[12] L. Hella, K. Luosto, K. Sano, J. Virtema, The expressive power of modal dependence logic, in: Proceedings of AiML 2014, 2014, pp. 294–312.
[13] J. Hintikka, G. Sandu, Informational independence as a semantical phenomenon, in: J.E. Fenstad (Ed.), Logic, Methodology and Philosophy of Science VIII, North-Holland, 1989, pp. 571–589.
[14] J. Hintikka, G. Sandu, Game-theoretical semantics, in: J. van Benthem, A. ter Meulen (Eds.), Handbook of Logic and Language, Elsevier, 1997, pp. 361–410.
[15] W. Hodges, Compositional semantics for a language of imperfect information, Log. J. IGPL 5 (4) (1997) 539–563.
[16] J. Kontinen, J.A. Väänänen, On definability in dependence logic, J. Log. Lang. Inf. 18 (3) (2009) 317–332.
[17] J. Kontinen, A. Kuusisto, J. Virtema, Decidability of predicate logics with team semantics, in: Proceedings of MFCS 2016, 2016, pp. 60:1–60:14.
[18] A. Kuusisto, A double team semantics for generalized quantifiers, J. Log. Lang. Inf. 24 (2) (2015) 149–191.
[19] L. Libkin, Elements of Finite Model Theory, Texts in Theoretical Computer Science. An EATCS Series, Springer, 2004.
[20] R. Rönnholm, Capturing k-ary existential second order logic with k-ary inclusion-exclusion logic, Ann. Pure Appl. Log. 169 (3) (2018) 177–215.
[21] R. Rönnholm, Arity fragments of logics with team semantics, PhD Thesis, Tampere University Press, 2018, http://urn.fi/URN:ISBN:978-952-03-0912-1.
[22] R. Rönnholm, The expressive power of k-ary exclusion logic, Ann. Pure Appl. Logic, Special Issue of WoLLIC 2016 170 (9) (2019) 1070–1099.
[23] J.A. Väänänen, Dependence Logic - a New Approach to Independence Friendly Logic, London Mathematical Society Student Texts, vol. 70, Cambridge University Press, 2007.
[24] J.A. Väänänen, An atom's worth of anonymity, manuscript.
[25] F. Yang, J. Väänänen, Propositional team logics, Ann. Pure Appl. Log. 168 (7) (2017) 1406–1441.