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Statistical Inference on Desirability Function Optimal Points to Evaluate Multi-Objective Response Surfaces

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DISSERTATION

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STATISTICAL INFERENCE ON DESIRABILITY FUNCTION OPTIMAL POINTS TO EVALUATE MULTI-OBJECTIVE RESPONSE SURFACES

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Abstract

A shortfall of the Derringer and Suich (1980) desirability function is lack of inferential methods to quantify uncertainty. Most articles for addressing uncertainty usually involve robust methods, providing a point estimate that is less affected by variation. Few articles address confidence intervals or bands but not specifically for the Derringer and Suich method. This research provides two valuable contributions to the field of response surface methodology. The first contribution is evaluating the effect of correlation and plane angles on Derringer and Suich optimal solutions. The second contribution proposes and compares 8 inferential methods–both univariate and multivariate–for creating confidence intervals on each desirability function solution for first order and second order models. The effect of the Derringer and Suich method parameters, objective plane angles, and differing correlation between response surfaces are examined through simulation. The 8 proposed methods include a simple best/worst case method, 2 generalized methods, 4 simulated surface methods, and a nonparametric bootstrap method. One of the generalized methods, 2 of the simulated surface methods, and the nonparametric method account for covariance between the response surfaces. Bivariate examples showcase these methods in the first order and second order models. A multivariate real-world case with 3 objectives is also examined. While all 7 novel methods and the best/worst method seem to perform decently on the second order models. The methods which utilize an underlying multivariate-t distribution, Multivariate Generalized (MG) and Multivariate t Simulated Surface (MVtSSig), are recommended methods from this research as they perform well with small samples for both first order and second order models with coverage only becoming unreliable at non-optimal solutions. MG and MVtSSig inference should be

used in conjunction with robust methods such as Pareto Front Optimization to help ascertain which solutions are more likely to be optimal before constructing confidence interval.

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STATISTICAL INFERENCE ON DESIRABILITY FUNCTION OPTIMAL POINTS TO EVALUATE MULTI-OBJECTIVE RESPONSE SURFACES

I. Introduction

A significant element of statistical analysis is to provide decision makers the ability to make well-informed decisions. These decisions occur under uncertainty where information pertaining to the given problem is not guaranteed to be completely accurate due to measurement error, sampling techniques, and other potentially unknown error sources. Response surface methodology (RSM) is decision making through product or process optimization. It is built on design of experiments (DOE), linear models, and optimization techniques to build models that approximate real-world objective functions. When two or more objectives are present, it's seldom rare they can both be fully optimized. To determine optimal values, trade-offs must occur where some subjective element can be incorporated to find a solution that is mutually beneficial for all objectives. Multi-objective RSM (MORSM) extends to optimization when two or more competing, or correlated, objectives are present. MORSM includes some techniques to address competing objectives such as overlaying contour plots, constrained optimization, Pareto fronts, and desirability functions.

Desirability functions (DF) are the core focus of this research. Specifically, the Derringer and Suich method is explored for calculating desirability optimal values as the scaling functions are flexible in shape and used regularly in literature [1]. Current solutions for decision making under uncertainty as it relates to desirability functions and RSM focus on robust solutions rather than inference. There is some literature on inference for desirability functions; however, they use desirability functions that differ from the Derringer and Suich method for scaling and/or require large samples [2, 3]. Due to a lack of an adequate inferential method for the Derringer and Suich method, desirability function optimal solutions are generally reported as a point estimate. Reporting a point estimate value restricts the amount of usable information for a decision maker uncertainty should ideally be quantified in some manner. A review of the literature shows that there are currently no strictly appropriate inferential methods for the Derringer and Suich method. The inferential method that is occasionally used now provides a conservative interval as it uses best and worst case intervals based on confidence or prediction intervals but it also assumes independence of objectives. This assumption can be problematic as it does not appropriately account for the uncertainty between responses. While the best and worst case intervals appear to be empirically conservative, because they are affected by correlation between responses, their coverage is not always guaranteed and it may not provide $100 \cdot (1 - \alpha)\%$ confidence intervals. Decision makers would benefit from an appropriate inferential method for desirability functions to ensure solutions and alternate courses of action are robust to potential error.

This dissertation makes 2 primary contributions to the field of response surface methodology. The first contribution is evaluating the effect of correlation and plane angle on the Derringer and Suich desirability function optimal solution inference to show that the independence assumption is incorrect. The second contribution proposes 8 methods for constructing confidence intervals (CI) around the optimal desirability index for first and second order models. The first method used is already used in literature to construct rudimentary confidence intervals based on the best and worst case values from the linear regression confidence intervals. Seven of the methods are novel for this application and based on the generalized method and bootstrapping techniques as discussed in Section 2.6. The metrics of interest are coverage probability

(CP), average width (AW), and interval symmetry for each of the methods. Both univariate (ignores correlation) and multivariate (characterizes correlation structure) methods are considered in this research to achieve $100 \cdot (1-\alpha)\%$ coverage for CIs on optimal point using Derringer and Suich desirability optimal solutions. Both first order and second order models are utilized. The first order models are used to determine whether the angle between planes has an effect on the metrics of interest. Second order models are used to determine whether the parameter settings for the Derringer and Suich method have an effect on the metrics of interest and to capture use for a real-world data set.

The following chapters are organized as follows. Chapter II provides a review of the literature to address which linear models are used, designed experiments studied, the desirability function structures, some theory of the inferential methods proposed, and the multivariate distributions considered. Chapter III derives the overall methodology of this research including the inference method derivations, problem sets considered, and the simulation study. Chapter IV reviews all of the results across the scenarios and inference methods discussed in Chapter III. Chapter V finishes the dissertation with a succinct explanation of which of the inferential methods are recommended and steps for future research.

II. Background

2.1 Linear Regression Model

The purpose of multiple linear regression (MLR) is to develop a model to approximate a response surface for a continuous random variable. This response surface is composed of the responses, $\boldsymbol{Y}$, and factors (also referred to as predictor or independent variables), $\boldsymbol{X}$. The theoretical linear model is of the form:

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{1}$$

where $\boldsymbol{y}$ is a $n \times 1$ vector of responses, $\boldsymbol{X}$ is a $n \times p$ model matrix of fixed values, $\boldsymbol{\beta}$ is a $p \times 1$ vector of model parameters, and $\boldsymbol{\varepsilon}$ is a $n \times 1$ vector of random error terms that are assumed to be independently and identically distributed (iid) random variables with mean 0 and variance σ^2. The error term is most commonly assumed to be distributed normally for convenience of inference; however, other distributions can be used depending on the needs of the analysis. The two most common estimation techniques for the $\boldsymbol{\beta}$ parameters in this linear model are Ordinary Least Squares (OLS) and Maximum Likelihood. Both OLS and Maximum likelihood have been shown to arrive at the same linear parameters for $\boldsymbol{\beta}$, so for the purposes of this research only OLS estimators are derived [4, 5, 6, 7].

2.1.1 Ordinary Least Squares

The OLS estimators can be derived using Equation 1 in terms of $\boldsymbol{\varepsilon}$ by minimizing the squared error. First, the equation is rearranged and both sides are squared.

$$\begin{aligned}\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} &= (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})'(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}) \\ &= \boldsymbol{y}'\boldsymbol{y} - \boldsymbol{y}'\boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{y} - \boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta} \\ &= \boldsymbol{y}'\boldsymbol{y} - 2\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{y} - \boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta}.\end{aligned}$$

The two middle terms can be added together since their multiplication results in scalar values which are the same. Next, the first and second derivatives are taken with respect to $\boldsymbol{\beta}$. The first derivative is used to solve for the $\hat{\boldsymbol{\beta}}$ estimators and the second derivative shows that the estimators are minimum if the derivative is greater than 0.

$$\begin{aligned}\left.\frac{\partial \boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}}{\partial \boldsymbol{\beta}}\right|_{\hat{\boldsymbol{\beta}}} &= -2\boldsymbol{X}'\boldsymbol{y} + 2\boldsymbol{X}'\boldsymbol{X}\hat{\boldsymbol{\beta}} = 0 \\ &\Rightarrow \boldsymbol{X}'\boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{X}'\boldsymbol{y} \\ &\Rightarrow \hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}. \qquad (2)\end{aligned}$$

Lastly, the second derivatives are

$$\frac{\partial^2 \boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}}{\partial \boldsymbol{\beta}'\boldsymbol{\beta}} = 2\boldsymbol{X}'\boldsymbol{X}. \qquad (3)$$

Since the second derivatives in Equation 3 are greater than 0 as $\boldsymbol{X}'\boldsymbol{X}$ is quadratic, Equation 2 minimize the error and are the OLS estimators. Some of the appeal of the OLS estimators is attributed to the Gauss-Markov Theorem which shows that the OLS estimators are the Best Linear Unbiased Estimators (BLUE) which means that amongst the subset of all unbiased estimators for $\boldsymbol{\beta}$, they have the smallest variance which is shown in Section 2.1.2 [8].

2.1.2 Gauss-Markov Theorem

To show the OLS estimators found in Section 2.1.1 are BLUE, they need to be shown to be unbiased and have the least variance of all unbiased estimators. The expected value is used to show unbiasedness [4, 9, 8].

$$
\begin{aligned}
E[\hat{\boldsymbol{\beta}}] &= E[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}] \\
&= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'E[\boldsymbol{y}] \\
&= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta} \\
&= \boldsymbol{\beta} \qquad (4)
\end{aligned}
$$

$E[\boldsymbol{y}]$ is equal to $\boldsymbol{X\beta}$ since $\boldsymbol{\varepsilon}$ in Equation 1 has mean 0. This shows that the OLS estimators are unbiased. The next step is to find the variance of $\hat{\boldsymbol{\beta}}$ [4, 9, 8].

$$
\begin{aligned}
V[\hat{\boldsymbol{\beta}}] &= V[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}] \\
&= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'V[\boldsymbol{y}]((\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}')' \\
&= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'(\sigma^2)\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1} \\
&= \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1} \\
&= \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1} \qquad (5)
\end{aligned}
$$

Now suppose there was another estimator, $\tilde{\boldsymbol{\beta}} = [(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}' + \boldsymbol{D}]\boldsymbol{y}$, where $\boldsymbol{D}$ is a known matrix.

$$\begin{aligned}
E[\tilde{\boldsymbol{\beta}}] &= E[[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}' + \boldsymbol{D}]\boldsymbol{y}] \\
&= [(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}' + \boldsymbol{D}]E[\boldsymbol{y}] \\
&= [(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}' + \boldsymbol{D}]\boldsymbol{X}\boldsymbol{\beta} \\
&= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{D}\boldsymbol{X}\boldsymbol{\beta} \\
&= \boldsymbol{\beta} + \boldsymbol{D}\boldsymbol{X}\boldsymbol{\beta} \\
&= \boldsymbol{\beta} \qquad (6)
\end{aligned}$$

Equation 6 will only be unbiased as proposed if $\boldsymbol{D}\boldsymbol{X} = \boldsymbol{X}'\boldsymbol{D}' = 0$. Next, the variance of $\tilde{\boldsymbol{\beta}}$ must be checked.

$$\begin{aligned}
V[\tilde{\boldsymbol{\beta}}] &= V[[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}' + \boldsymbol{D}]\boldsymbol{y}] \\
&= [(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}' + \boldsymbol{D}]V[\boldsymbol{y}]([(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}' + \boldsymbol{D}])' \\
&= [(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}' + \boldsymbol{D}]\sigma^2[\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1} + \boldsymbol{D}'] \\
&= \sigma^2\left[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'[\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1} + \boldsymbol{D}'] + \boldsymbol{D}[\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1} + \boldsymbol{D}']\right] \\
&= \sigma^2\left[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1} + (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{D}' + \boldsymbol{D}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1} + \boldsymbol{D}\boldsymbol{D}'\right] \\
&= \sigma^2\left[(\boldsymbol{X}'\boldsymbol{X})^{-1} + \boldsymbol{D}\boldsymbol{D}'\right] \qquad (7)
\end{aligned}$$

It is clear that the variance of $\tilde{\boldsymbol{\beta}}$ is greater than the variance of $\hat{\boldsymbol{\beta}}$ since $\boldsymbol{D}\boldsymbol{D}'$ is a quadratic term and positive [8]. If $\boldsymbol{D}\boldsymbol{D}' = 0$ then $\tilde{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}$. Therefore, the OLS estimators $\hat{\boldsymbol{\beta}}$ minimize the error of the linear model and are BLUE by the Gauss-Markov Theorem since they are unbiased and have the lowest variance of all unbiased estimators.

Additionally, as a result of model assumptions and form, the distribution of predicted values is relatively simple to find. Assuming the error, ε, has mean 0 and constant variance σ^2, the expected value and variance of $\boldsymbol{y}$ are $\boldsymbol{X\beta}$ and $\sigma^2\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$, respectively. $\boldsymbol{H} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ is also known as the hat matrix. Similarly, if the errors are assumed normally distributed, then the predicted values at each location of $\boldsymbol{X\beta}$ are also be normally distributed, i.e., $\hat{\boldsymbol{y}} \sim N(\boldsymbol{X\beta}, \sigma^2\boldsymbol{H})$, where σ^2 can be estimated using the mean square error of the model

$$MSE = \frac{1}{n-p}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})'(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}), \tag{8}$$

and the ratio of SSE and σ^2 is a chi-square distribution

$$\frac{SSE}{\sigma^2} \sim \chi^2_{n-p}. \tag{9}$$

These are all results commonly presented in many linear modeling textbooks [4, 5, 6, 7, 9]

2.1.3 Multivariate Multiple Linear Regression

This multivariate multiple linear regression (MMLR) model can also be generalized to the multivariate case. The multivariate model is detailed in Rencher (2002) and repeated here for readability. The multivariate model is straight forward, using similar notation and multiple linear regression. The theoretical linear model with m correlated responses is of the form:

$$\boldsymbol{Y} = \boldsymbol{XB} + \boldsymbol{\Xi} \tag{10}$$

where $\boldsymbol{Y}$ is now a $n \times m$ matrix of responses, $\boldsymbol{X}$ is a $n \times p$ model matrix of fixed values, $\boldsymbol{B} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_m)$ is a $p \times m$ matrix of model parameters, and $\boldsymbol{\Xi}$ is a $n \times m$ matrix of errors. The multivariate linear model assumes

- $E[\boldsymbol{\Xi}] = \mathbf{0}$
- $cov(\boldsymbol{y}_i) = \boldsymbol{\Sigma}\ \forall\ i = 1, 2, \ldots, n$, where $\boldsymbol{y}_i'$ is the ith row of $\boldsymbol{Y}$
- $cov(y_i, y_j) = \mathbf{0}\ \forall i \neq j$

In other words, the errors have mean vector $\mathbf{0}$, the covariance within rows of $\boldsymbol{Y}$ is constant, and rows of $\boldsymbol{Y}$ are uncorrelated.

$\boldsymbol{B}$ is estimated identically to the univariate case using least squares estimation

$$\hat{\boldsymbol{B}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} \tag{11}$$

which minimizes $\boldsymbol{\Xi}'\boldsymbol{\Xi}$ and has the same BLUE properties as in the univariate model. Finally, the unbiased estimate for covariance can be found by analogy of the univariate case Equation 8

$$\hat{\boldsymbol{\Sigma}} = \boldsymbol{S}_e = \frac{1}{n-p}(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{B}})'(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{B}}) \tag{12}$$

which is a symmetric $m \times m$ matrix. Each diagonal element of $\boldsymbol{S}_e$ is the respective MSE for an individual MLR model of response $r \in 1, \ldots, m$. This covariance estimate is used in proposed methods for inference to incorporate correlation between responses. The MMLR model is explicitly referenced here to remove common confusion between it and its univariate counterpart.

It's important to note that the predicted values and estimated coefficients in this model are identical to individual univariate linear regression models. Key changes include methods of testing and covariance structures to account for correlated responses that is ignored with multiple individual models [10, 11].

The hat matrix, $\boldsymbol{H} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$, for the MMLR is the same one used in MLR as it only relies on the design matrix. $\boldsymbol{H}$ is a projection matrix that maps the observed values $\boldsymbol{Y}$ to the fitted values $\hat{\boldsymbol{Y}}$.

$$\hat{\boldsymbol{Y}} = \boldsymbol{H}\boldsymbol{Y}. \tag{13}$$

The hat matrix helps determine the covariance of $\hat{\boldsymbol{Y}}$ with respect to the design space. The diagonal elements of $\boldsymbol{H}$, h_{ii}, are used when focusing on a univariate row of fitted values.

Rencher (2012) covers many of the derivations for the MMLR model and Helwig (2017) provides a beneficial lecture to succinctly find the expected value and covariance of $\hat{\boldsymbol{Y}}$ by vectorizing $\hat{\boldsymbol{Y}}$, that is, rearranging the $n \times m$ matrix of fitted values to an $n + m \times 1$ vector of the fitted values [10, 12]. A vectorized matrix, $\boldsymbol{M}$, is denoted by $vec(\boldsymbol{M})$ and the Kronecker product is represented by $\otimes$ [12, 13]. As an aside, the Kronecker product of $\boldsymbol{A} \otimes \boldsymbol{B}$ is the process of multiplying each value in $\boldsymbol{A}$ by the entire matrix $\boldsymbol{B}$. For example, let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{B}$ be a $p \times q$ matrix. The resulting matrix from the Kronecker product, $\boldsymbol{C} = \boldsymbol{A} \otimes \boldsymbol{B}$, is a $mp \times nq$ matrix. For example, let $\boldsymbol{A}$ be a 2×2 matrix and $\boldsymbol{B}$ be a $p \times q$ matrix, then $\boldsymbol{C}$ can be written as the following [10, 14, 15].

$$\boldsymbol{C} = \boldsymbol{A} \otimes \boldsymbol{B} = \begin{pmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} \\ a_{21}\boldsymbol{B} & a_{22}\boldsymbol{B} \end{pmatrix} \tag{14}$$

The Kronecker product is helpful to derive the variance calculations when covariance structures are present as it simplifies the calculations as can be seen in the following derivations.

$$
\begin{aligned}
E[vec(\hat{\boldsymbol{Y}})] &= E[vec(\boldsymbol{X}\hat{\boldsymbol{B}})] \\
&= E[vec(\boldsymbol{I}_n\boldsymbol{X}\hat{\boldsymbol{B}})] \\
&= E[(\hat{\boldsymbol{B}}' \otimes \boldsymbol{I}_n)vec(\boldsymbol{X})] \\
&= (\boldsymbol{B}' \otimes \boldsymbol{I}_n)vec(\boldsymbol{X}). \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
V[vec(\hat{\boldsymbol{Y}})] &= V[vec(\boldsymbol{H}\boldsymbol{Y})] \\
&= V[(\boldsymbol{I}_m \otimes \boldsymbol{H})vec(\boldsymbol{Y})] \\
&= (\boldsymbol{I}_m \otimes \boldsymbol{H})V[vec(\boldsymbol{Y})](\boldsymbol{I}_m \otimes \boldsymbol{H})' \\
&= (\boldsymbol{I}_m \otimes \boldsymbol{H})(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_m)(\boldsymbol{I}_m \otimes \boldsymbol{H})' \\
&= (\boldsymbol{I}_m\boldsymbol{\Sigma}\boldsymbol{I}_m \otimes \boldsymbol{H}\boldsymbol{I}_m\boldsymbol{H}') \\
&= \boldsymbol{\Sigma} \otimes \boldsymbol{H} = \boldsymbol{\Sigma} \otimes \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'. \qquad (16)
\end{aligned}
$$

For a specific fitted value, $\hat{\boldsymbol{y}}_i$, the expected value and variance of the fitted value is $E[vec(\hat{\boldsymbol{y}}_{\boldsymbol{i}})] = (\boldsymbol{B}' \otimes \boldsymbol{I}_1)vec(\boldsymbol{x}_i)$ and $V[vec(\hat{\boldsymbol{y}}_{\boldsymbol{i}})] = \Sigma \otimes \boldsymbol{x}_i(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{x}_i'$, respectively. Additionally, when the error matrix, $\boldsymbol{\Xi}$, is assumed to be distributed MVN, the predicted values will have the same distribution [10, 11].

$$
vec(\hat{\boldsymbol{Y}}) \sim N_m((\boldsymbol{B}' \otimes \boldsymbol{I}_n)vec(\boldsymbol{X}), \boldsymbol{\Sigma} \otimes \boldsymbol{H}) \qquad (17)
$$

The expected value of $\hat{\boldsymbol{B}}$ is very straight forward, and the variance of the parameter matrix can be found in a similar fashion to the fitted values.

$$
\begin{aligned}
E[\hat{\boldsymbol{B}}] &= E[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}] \\
&= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'E[\boldsymbol{Y}] \\
&= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}\boldsymbol{B} \\
&= \boldsymbol{B} \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
V[vec(\hat{\boldsymbol{B}})] &= V[vec((\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y})] \\
&= V[vec((\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}\boldsymbol{I}_m)] \\
&= V[(\boldsymbol{I}_m \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}')vec(\boldsymbol{Y})] \\
&= (\boldsymbol{I}_m \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}')V[vec(\boldsymbol{Y})](\boldsymbol{I}_m \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}')' \\
&= (\boldsymbol{I}_m \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}')(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_n)(\boldsymbol{I}_m \otimes \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}) \\
&= (\boldsymbol{I}_m\boldsymbol{\Sigma}\boldsymbol{I}_m \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{I}_n\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}) \\
&= \boldsymbol{\Sigma} \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1} \qquad (19)
\end{aligned}
$$

These derivations show that the multivariate expectation and covariance calculations are simple generalization of the univariate linear regression model [10].

2.2 Response Surface Methodology

A response surface is defined by a dependent variable that can be characterized by two or more independent variables representing a product or process response in the form of the linear model as described in Section 2.1. Response surface methodology (RSM) is the ensemble of statistical techniques used for the optimization involving response surfaces. It is not uncommon to have multiple response surfaces where one is attempting to optimize two or more at the same time, often with conflicting objectives. The general process of RSM is to sequentially fit response surfaces using designed experiments coupled with linear regression for the model and gradient search methods to obtain optimal responses. When the problem involves multi-objective optimization, common techniques such as overlaying contour plots, constrained optimization, Pareto fronts, or desirability functions may be employed. [4, 6]. The first three techniques will be discussed further in the following subsections; However, desirability functions is written in greater detail in Section 2.3 as it is the focus of this research.

2.2.1 Overlaying Contour Plots

Overlaying contour plots is a technique that involves simply placing each of the contour plots on top of each other and visually inspecting the layered plot. One must find where m objectives intersect with one another at solutions that both make sense and are mutually beneficial. While simple, this method becomes increasingly complicated as it requires only looking at two factors at a time while the other $k-2$ factors are held constant. There is also an issue with increasing the number of objectives as it can quickly become incomprehensible with too many layers. Overlaying contour plots is most useful to determine quick, but potentially inefficient, solutions or to find a starting point for more accurate methods [4].

2.2.2 Constrained Optimization

Constrained optimization falls into the realm of mathematical programming. A function, $f(x)$, is to be optimized subject to constraints of the form

$$\begin{gathered} \text{Max } f(x) \\ \text{s.t. } g_i(x) \leq c_i \\ h_j(x) = d_j \end{gathered}$$

where i is the number of inequality constraints and j is the number of equality constraints. c_i, d_j are constants and $g_i(x), h_j(x)$ are functions of the same x in the objective function. This method focuses on optimizing a single objective, $f(x)$, with the other $m-1$ objectives set as constraint functions, $g_i(x)$ and $h_j(x)$. The solution to this problem requires nonlinear programming (NLP) using either direct search or numerical optimization algorithms to find optimum locations. NLP optimization does not guarantee global optimal solutions, but performs well within RSM applications [4].

2.2.3 Pareto Fronts

Pareto fronts can be used to objectively identify an array of points in the solution space that are potentially 'best' in terms of optimizing multiple objectives. A Pareto set is found by searching for solutions that 'Pareto dominate' other solutions and are not dominated themselves. A Pareto dominate solution is one that is at least as good in all of the criteria values and strictly better in at least one of the criteria. For example, if there are two objectives $\boldsymbol{y}_1$ (maximize) and $\boldsymbol{y}_2$ (minimize) then solution 1, $(y_{11}, y_{21}) = (100, 25)$, would Pareto dominate solution 2, $(y_{12}, y_{22}) = (99, 25)$, as the observed y_2 value of solution 1 is at least as good as solution 2 and the observed y_1 value is strictly better [16].

Pareto fronts remove noncontenders from the solution space objectively, but must be further reduced using some optimization formula. Chapman et al. (2014a) compares constrained optimization and desirability functions as a way to optimize the Pareto set of values [16]. Lu et al. (2011a) introduces a utopia point approach to determine a smaller set of promising solutions from the already reduced Pareto set. This approach sets a point that is the absolute ideal for all m objectives and measures the distance of points to the utopia point where a smaller distance indicates a better solution [17]. The Pareto front optimization method using desirability functions has been shown to work well empirically in Myers et al. (2016) Chapman et al. (2014a), Chapman et al. (2014b), and Calhoun (2020) [4, 16, 18, 19]

2.2.4 Design of Experiments for Second Order Models

Optimal points of a response surface are generally found using a second order model. Using a second order model to describe the response surface assumes that the sequential movement along the design space using simple designs such as a factorial design has already been performed. This means that that the region of interest is currently in focus and curvature is found significant, generally through the addition of center points in the experiment design. If curvature is not present, a first order model is fit to the current design. It should be noted that design of experiments, while not required to perform RSM, is exceptionally beneficial in optimizing response surfaces. Depending on the design, there are certain properties such as orthogonality of factors (factors are independent) and rotatability (constant variance for the predicted response at all points of $\boldsymbol{X}$ that are the same distance from the center of the design) [4, 6]. Anderson-Cook (2009) discusses various criteria that are helpful in determining the recommended design using optimality criterion and graphical methods [20].

This research should be agnostic to the design used; However, it is beneficial to describe some of the classical second order design for familiarity. The classical second order designs include the Central Composite Design (CCD), Box-Behnken Design (BBD), and Definitive Screening Design (DSD).

2.2.4.1 Central Composite Design

The CCD was introduced by Box and Wilson in 1951 and is considered to be the most popular second order design [4, 21]. The CCD is a designed experiment that is an extension of a 2^k factorial design which allows quadratic effects to be estimated. There are 2^k corner points, $2k$ axial points, and generally around 3-5 center points where k is the number of factors. The corner points estimate the main effects and interactions, the axial points estimate quadratic effects, and the center points are used to test for curvature [4, 6]. This design has a useful property called rotatability when the axial distance from the center of the design, α, is $\sqrt[4]{2^k}$. Rotatability causes constant variance for the predicted response at all points of $\boldsymbol{X}$ that are the same distance from the center of the design [4, 6]. Figure 1 shows the CCD with two factors.

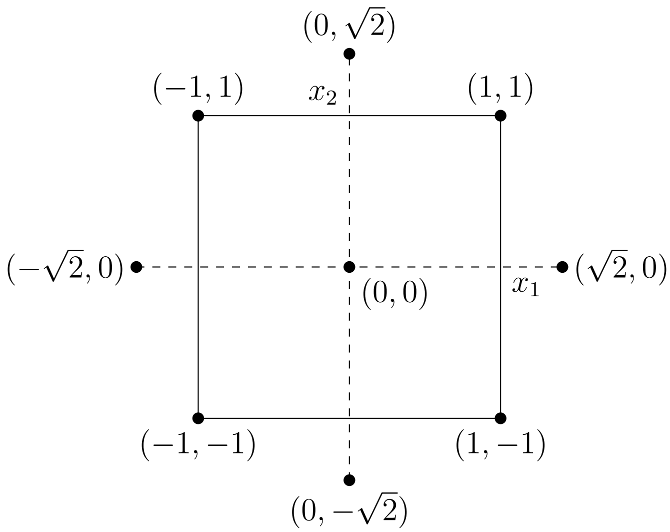


Figure 1. Central Composite Design with 2 Factors and $\alpha = \sqrt{2}$

2.2.4.2 Box-Behnken Design

Introduced by Box and Behnken in 1960, the BBD is a family of three-level designs for fitting second order response surfaces, but it requires a minimum of at least three factors [4, 22]. BBD designs were inspired by balanced incomplete block designs, all combinations of two factors are paired together resembling standard 2^2 factorial designs while the other $k-2$ factors are fixed at 0. The number of runs, N, needed for the k factors is calculated using $N = 2^2 \cdot \binom{k}{2} + n_c = 4 \cdot \binom{k}{2} + n_c$, where n_c is the number of center points. The design is a spherical design and does not provide adequate coverage of the corner points [4, 6]. Figure 2 is the BBD design with three factors where it can be seen that none of the points are in the extreme regions and instead lie in the center of on the edge between each corner. If a three-level design with center points is desired then a modification of the CCD can be used where $\alpha = 1$ which is a cuboidal design called a Face-Centered Cube design.

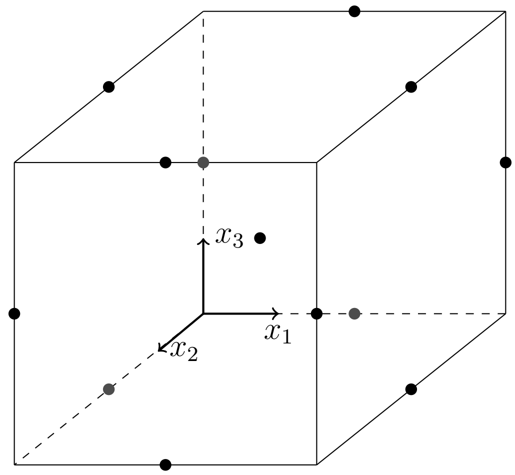


Figure 2. Box-Behnken Design with 3 Factors and center point

2.2.4.3 Definitive Screening Design

Introduced in 2011 by Jones and Nachtsheim, the DSD is typically for first-order models with interactions, but can be adapted to fitting second order models [23]. When using this design for second order models, it heavily relies on the sparsity of effects principle [6]. Aliasing becomes an issues when the sparsity of effects principle does not hold due to too many higher order terms being significant and it becomes difficult to determine which effects are explaining the variance in the model. Jones and Nachtsheim use three levels at each factor with nonregular structures. These structures provide flexible aliasing allowing more terms to be estimated with some correlation between factors. This design is not ideal for actual modeling due to aliasing between factors, but because it can use fewer observations to detect significant effects it is perfect for screening which effects to use in further experiments. This design will generally be used when the current design region contains the optimal region [4].

2.2.4.4 Experimental Design Summary

Each of the aforementioned designs are beneficial in their own right and are by no means an exhaustive list of second order designs that are covered in Myers et al.

(2016) but they are the primary designs [4]. The CCD allows for simple, sequential experimentation. This design may be useful when one is unsure whether the current design region is the optimal region which requires additional searching. The sequential nature of the CCD allows one to introduce additional design points as necessary by starting with a factorial for main effects and screening, adding center points to detect curvature, and finally adding axial points to estimate pure quadratic terms that give the response surface its curved shape. Additionally, center points allow estimation of pure error since there will typically be 3 or more replicates to do so [4].

The characteristics of the BBD make it an important alternative to the CCD. By design, it allows researchers to have three evenly spaced levels of each factor. Similar to the CCD, this design allows degrees of freedom for estimation of pure error with center points as well as some degrees of freedom for lack of fit testing. This design does not attain rotatability properties as easily as the CCD–only specific number of factors such as $k = 4$ and $k = 7$ work–but it does not greatly deviate from rotatability. Myers et al. (2016) explains that the use of the BBD should be confined to situations where prediction of values at the extremes is not needed since there are no factorial points present [4].

Consideration of the DSD requires one to be fairly confident that some second order terms will not be significant. DSD are also attractive when it is suspected that the current design region includes the optimal location, otherwise a simpler design should be used to perform steepest ascent to the optimal region. While this design requires only a moderate number of runs to explore different factors, it is limited in that main effects will be confounded with higher order terms. DSD also have an interesting feature between odd and even number of factors. When k is even, the main effects will all be orthogonal which allows independent estimation of the parameters which should be considered when using the design [4].

There are additional comparisons of the CCD and BBD with respect to design optimality, prediction variance, and axial distance in Chapter 9 of Myers et al. (2016) [4]. Ultimately, the average width of the inference methods considered will be affected by prediction variance of the experimental design but the empirical coverage should be agnostic to the design. The CCD is the design used when considering data. This is primarily due to available data and examples with structures compatible with the CCD as it is flexible and efficient.

2.3 Desirability Functions

Desirability function (DF) is a generic term for a variety of different methods that transform response variables to a scale-free value, d_{ir}, usually between 0 and 1 where $i = 1, \ldots, N$ and $r = 1, \ldots, m$. Data collected with designed experiments allow interpolated solutions to be used as well. If interpolation is used, the number of desirability values N is the sum of the n observations from the designed experiment and all interpolated points considered. Harrington (1965) introduced DFs as a metric for determining optimal points of a response surface [24, 4]. In the trivial case with one objective, the response with the largest desirability is then selected. Multi-objective optimization utilizes a secondary function which combines the m individual DFs into a single desirability index using either an additive or multiplicative form. These functions require the desirability d_{ir} as well as m weights w_r to determine objective importance, usually chosen by the decision maker.

$$D_i^{add} = \sum_{r=1}^{m} w_r d_{ir} \tag{20}$$

Equation 20 is the additive form and represents a weighted arithmetic mean. The additive form allows an individual DF to perform exceptionally well at the expense of others. Equation 21 is the multiplicative form which resembles a weighted geometric mean. The multiplicative form forces each DF to perform moderately well. If one individual DF in the multiplicative form is 0, then the collective desirability index is 0. A limitation of the multiplicative form that is addressed by the Derringer and Suich (DS) method is that it requires non-negative values.

$$D_i^{mult} = \prod_{r=1}^{m} d_{ir}^{w_r} \tag{21}$$

Both the additive and multiplicative version desirabilitys, D_i^{add} and D_i^{mult}, are bounded by $[0, 1]$.

These two desirability functions in Equations 20 and 21 are remarkably similar as they are versions of two of the three Pythagorean means. The additive form is generally used when one or more objectives in an optimization problem are permitted to be zero to allow the others to excel since the arithmetic of one does not affect another. The multiplicative form is used to treat all objectives more equal, thus preventing any single objective from severely outperforming the others using its multiplicative property. Based on the arithmetic and geometric means, the desirability functions behave similarly in that the maximum value that the multiplicative form can achieve is the arithmetic form given the same weights and values.

2.3.1 Harrington Method

The DF proposed by Harrington for one-sided optimization (i.e., maximization or minimization) is defined in Equation 22.

$$d_{ir} = e^{-e^{-y'_{ir}}}, \quad y'_{ir} = -[ln(-ln\ d_{ir})] = b_0 + b_1 y_{ir} \tag{22}$$

where y_{ir} is the observation of interest and b_0, b_1 are the solution of a system of two linear equations from two values of $\boldsymbol{y}_r$, commonly the maximum and minimum [24, 25, 26]. The two-sided DF to obtain a target value is defined in Equation 23.

$$d_{ir} = exp(-|y'_{ir}|^n), \quad y'_{ir} = \frac{2y_{ir} - (U + L)}{U - L} \tag{23}$$

where U and L are some upper and lower specification limit for $\boldsymbol{y}_r$ set by a decision maker [24, 25]. Trautmann and Weihs (2006) derived the underlying distribution for Harrington's DFs, to include the multiplicative desirability index [25]. The authors recommend the application of similar inferential techniques for the DS method due to the flexibility of the function and popularity. An issue with deriving the underlying distribution for the Derringer and Suich method is that it becomes intractable even with the simplest case, which is discussed in Section 2.4.

2.3.2 Derringer and Suich Method

Derringer and Suich (1980) published a modified DF based on Harrington (1965) to provide more flexibility and utility which resulted in it becoming the preferred method [1]. The Derringer and Suich approach is similar in that it was presented as a one-sided and two-sided transformation. Equations 24 and 25 are the transformations for maximization and minimization problems, respectfully. There is no need to actually employ both equations as an minimization problem can be transformed into a maximization problem by applying a negative to all values. After transforming the values, one would simply maximize them using Equation 24. Both equations are included.

$$d_{ir}^{max} = \begin{cases} 0, & y_{ir} < L_r \\ \left(\frac{y_{ir}-L_r}{T_r-L_r}\right)^{l_r}, & L_r \leq y_{ir} \leq T_r \\ 1, & y_{ir} > T_r \end{cases} \tag{24}$$

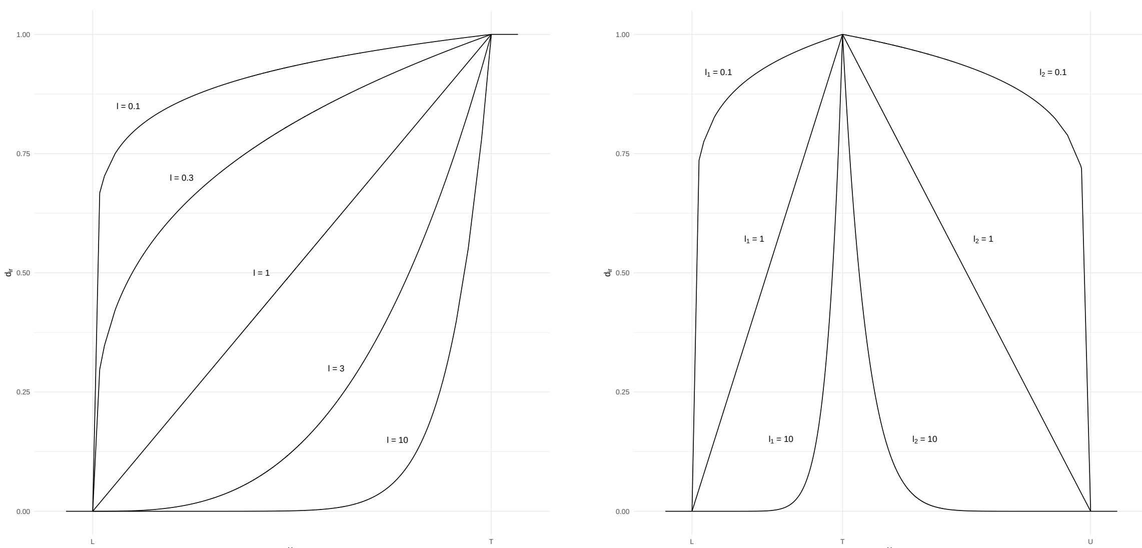
$$d_{ir}^{min} = \begin{cases} 1, & y_{ir} < T_r \\ \left(\frac{U_r-y_{ir}}{U_r-T_r}\right)^{l_r}, & T_r \leq y_{ir} \leq U_r \\ 0, & y_{ir} > U_r \end{cases} \tag{25}$$

Equation 26 is the two-sided transformation when a match target value optimization is desired. This equation is applicable in instances that require precision within certain specifications such as widget size or viscosity of a liquid.

$$d_{ir}^{tgt} = \begin{cases} 0, & y_{ir} < L_r \\ \left(\frac{y_{ir}-L_r}{T_r-L_r}\right)^{l_{1r}}, & L_r \leq y_{ir} \leq T_r \\ \left(\frac{U_r-y_{ir}}{U_r-T_r}\right)^{l_{2r}}, & T_r \leq y_{ir} \leq U_r \\ 0, & y_{ir} > U_r \end{cases} \tag{26}$$

L_r is a specified lower allowable limit, U_r is a specified upper allowable limit, T_r is a specified target to achieve for each response r. L_r is considered an evasion value in maximization, U_r is considered an evasion value in minimization problems, and both are considered evasion values in a match target problem. In all problems T_r is the desirable value that each respective problem is trying to achieve. L_r, U_r, and T_r are all set by the decision maker; However, they can be simply set using values such as the maximum, minimum, and quantile to name a few. l_r and l_{ir} are linearity parameters which affect the slope of the desirability function curve in equations 24-26. As indicated by the subscript, r, each of these values are set for each response independent of each other depending on the requirements for optimizing each response.

The Derringer and Suich method gained popularity due to the simplicity of the equations and flexibility of the emphasis on different responses achieving the target values due to the linearity parameters [4, 27]. Figures 3a and 3b highlight this flexibility using varying levels of linearity.



(a) Maximization/Minimization (Equations 24 and 25)

(b) Match Target (Equation 26)

Figure 3. Desirability functions with differing linearity parameter, l.

These figures are the same as those in Derringer and Suich's 1980 article with slight labeling changes [1]. Figure 3a has a maximization or minimization objective and Figure 3b has a match target objective. Note that the linearity parameter, l_r (or l_{ir} for a match target problem), has a lower bound of 0. When l_r is small, the law of diminishing returns applies and the inverse is true when l_r is large which can be seen in each figure. When l_r is 1, the desirability index remains linear and normality of the values is maintained, given the values were normally distributed from the beginning.

2.3.3 Other Formulations

Additional formulations of the DF have been introduced since Harrington (1965) as well as Derringer and Suich (1980) methods. For example, Castillo et al. (1996) expand on Harrington (1965) and Derringer and Suich (1980) to account for the discontinuity in two-sided transformation using quartic polynomial approximation so the function is differentiable for gradient search based methods [28]. Ch'ng et al. (2005) propose a DF method that transforms the response of interest onto a line with desirability between 0 and 2 based on variable coding formulas that achieves the same purpose as the modified DF but require less computations and are simpler to implement [29]. Wu and Hamada (2000) show a method using the exponential function that removes the need to choose lower and upper values by using the exponential function [30]. Gibb A detailed survey of both simplistic and complicated DFs has been accomplished by Costa (2011) [31]. The methodology for the 7 proposed confidence intervals will be focused on the desirability functions from the Derringer and Suich method as they provide a great deal of flexibility and are prevalent in both literature and software [32].

2.4 Unknown Desirability Distribution

Two primary issues with deriving confidence intervals (CIs) for a desirability index are that there is often only a single observation for each solution in the m response surfaces and D_i is either the sum or product of m random variables. The first issue stems from the fact that a single observation presents no sample variability, so probability distributions must be assumed about the data. There are probability distributions for the sum and product of correlated normally distributed random variables, but d_i is often non-normal [33, 34, 35, 36]. This section provides an example of the simplest case and addresses complex cases.

The MLR model in Section 2.1 set the necessary requirements for modeling response surfaces when assuming a normal distribution for the errors. Normally distributed errors result in the predicted values, $\hat{y}_{ir}$, also being normally distributed for each $i \in 1, \ldots, N, r \in 1, \ldots, m$.

$$\hat{y}_{ir} \sim N(\boldsymbol{x}_i \boldsymbol{\beta}_r, \sigma_r^2 h_{ii}) \tag{27}$$

Equation 27 is used to show the sampling distribution for D_i is difficult, even in an extremely simple case in Section 2.4.1.

2.4.1 Simple Case: Linear Additive Desirability Index

Suppose the additive DF is used with a linearity parameter of $l_1 = l_2 = 1$, this is the simplest form of a DF. If the predicted values $\hat{y}_{ir}$ are within the chosen lower (L_r) and upper (U_r) bounds, then transformation, d_{ir}^{max} or d_{ir}^{min}, remains normally distributed. Recall the DF for maximization in Equation 24.

$$\begin{aligned} d_{ir}^{max} &= \frac{\hat{y}_{ir} - L_r}{T_r - L_r} \\ d_{ir}^{max} &\sim N\left(\frac{\boldsymbol{x}_i \boldsymbol{\beta}_r - L_r}{T_r - L_r}, \frac{\sigma_r^2 h_{ii}}{(T_r - L_r)^2}\right) \end{aligned} \tag{28}$$

Similarly, for minimization in Equation 25

$$\begin{aligned} d_{ir}^{min} &= \frac{U_r - \hat{y}_{ir}}{U_r - T_r} \\ d_{ir}^{min} &\sim N\left(\frac{-(\boldsymbol{x}_i \boldsymbol{\beta}_r - U_r)}{U_r - T_r}, \frac{\sigma_r^2 h_{ii}}{(U_r - T_r)^2}\right) \end{aligned} \tag{29}$$

where $\boldsymbol{x}_i \boldsymbol{\beta}_r$ is the linear combination of a specific solution of $\boldsymbol{x}$ with coefficients from response r. Now consider a simple example of $m = 2$ responses with identical factors, $(\boldsymbol{y}_1, \boldsymbol{X})$ and $(\boldsymbol{y}_2, \boldsymbol{X})$. They can be modeled using MLR and will have an arbitrary objective to maximize $\boldsymbol{y}_1$ and minimize $\boldsymbol{y}_2$. The predicted values can then be plugged into the additive desirability function, the distribution would be as follows.

$$\begin{aligned} D_i^{add} &= \sum_{r=1}^{2} w_r d_{ir} \\ &= w_1 d_{i1} + w_2 d_{i2} \end{aligned} \tag{30}$$

Multiplying d_{i1} by the fixed scalar value w_1 simply carries through to the parameters in Equation 28.

$$w_1 d_{i1} \sim N\left(w_1 \frac{\boldsymbol{x}_i \boldsymbol{\beta}_1 - L_1}{T_1 - L_1}, w_1^2 \frac{\sigma_1^2 h_{ii}}{(T_1 - L_1)^2} \right) \tag{31}$$

The minimization objective distribution is handled similarly,

$$w_2 d_{i2} \sim N\left(-w_2 \frac{(\boldsymbol{x}_i \boldsymbol{\beta}_2 - U_2)}{U_2 - T_2}, w_2^2 \frac{\sigma_2^2 h_{ii}}{(U_2 - T_2)^2} \right) \tag{32}$$

Only by adding the assumption of independence, $\boldsymbol{y}_1 \perp\!\!\!\perp \boldsymbol{y}_2$, does D_i^{max} remain normally distributed,

$$D_i^{max} \sim N\left(w_1 \frac{\boldsymbol{x}_i \boldsymbol{\beta}_1 - L_1}{T_1 - L_1} - w_2 \frac{(\boldsymbol{x}_i \boldsymbol{\beta}_2 - U_2)}{U_2 - T_2}, w_1^2 \frac{\sigma_1^2}{(T_1 - L_1)^2} + w_2^2 \frac{\sigma_2^2}{(U_2 - T_2)^2} \right) \tag{33}$$

The constraint placed on the weights, $\sum_{r=1}^{m} w_r = 1$, ensures this distribution does not result in a degenerate case. This should not be confused with a mixture distribution which uses similar definitions with weights [37, 38]. In this case of DFs, w_r is a fixed scalar value and not the probability of each random variable occurring which is what is assumed for the mixture distribution. Although this example of deriving a distribution is the simplest of cases, the solution only holds if independence of the responses is assumed which is unrealistic in MORSM.

2.4.2 Complex Cases: Non-Linear and/or Multiplicative Desirability Index

The more complex cases of the desirability function will occur when the multiplicative form is used, additional responses are added, the linearity parameter is not equal to 1, or some combination of these complex settings. Using the multiplicative form of the desirability index is beneficial as it prevents individual objectives from severely outperforming each other, but multiplication of two or more distributions becomes a difficult problem. Nadarajah (2016) finds the exact distribution for the product of two correlated normal random variables [36]. Gaunt (2022) further explores the literature for the product of correlated normal random variables which seems to agree that only the bivariate case is available which makes its use limited at this time [39]. When the linearity parameter is not 1, the distribution of d_{ir} becomes unknown except in special cases such as being distributed as a chi-square when $l = 2$ (assuming d_{ir} is properly standardized) [40]. Normality is lost with these issues, preventing closed form inference from being performed at this time which requires more creative solutions to model the possible change in response and desirability at each point. This requires the CI inferential methods to be constructed using approximation techniques to capture dependence between responses.

2.5 Inference using Hypothesis Tests

This section discusses some theory for frequentist-based hypothesis tests. First, introductory theory surrounding the concept is covered. Next, the generalized method from Weerahandi (2003) is discussed for exact hypothesis tests extended from conventional inference methods. Finally, both parametic and nonparametic bootstrap methods are addressed. These hypothesis tests certainly do not cover the entire spectrum of different methods such as likelihood ratios as well as bayesian-based tests;

However, they are included as literature to be possible alternatives to the confidence interval inference methods considered in this research.

2.5.1 Hypothesis Tests

Hypothesis testing is a statistical inference method to evaluate some hypothesis which is a statement about a population parameter. There are two complementary statements within a hypothesis testing problem. These two statements are defined as the null hypothesis ($H_0 : \theta \in \Theta_0$) and the alternative hypothesis ($H_1 : \theta \in \Theta_0^c$) where Θ_0 is a subset from parameter space, Θ, and Θ_0^c is the complement of θ_0 [33].

Hypothesis testing is used as a decision rule to either accept or reject H_0 as the truth where the latter results in deciding H_1 is true. As the decision relies on sample data, which contains uncertainty, incorrect decisions based on the sample data may be made, requiring the evaluation of tests. The two types of errors that can be made with a hypothesis test are the Type I Error and Type II Error. Type I Error occurs when the null hypothesis, H_0, is true but the hypothesis test rejects it and concludes that H_1 is true. Type II Error occurs when the alternative hypothesis, H_1, is true, but the hypothesis test accepts H_0. The probability of Type I and Type II errors are defined as $P(\text{Type I Error}) = \alpha$ and $P(\text{Type II Error}) = \beta$, respectively. The ideal test minimizes both α and β. When determining the test that minimizes both Type I and Type II errors, it is common to control the Type I Error at a specified level and then search the subclass of tests which minimize Type II Error [33, 41].

2.5.2 Generalized Method

Weerahandi (2003) describes the implementation of generalized p-Values and generalized gonfidence intervals, the latter of which is covered in Section 2.6.2. Generalized methods are extensions of conventional inference methods to provide exact fixed-level tests that do not depend on nuisance parameters nor use asymptotic inference which tends to perform poorly for small samples [42]. The following is Weerahandi's definition of a generalized test variable. Let $\boldsymbol{\zeta} = (\theta, \boldsymbol{\delta})$ be a vector of unknown parameters, where θ is the parameter of interest and $\boldsymbol{\delta}$ is a vector of nuisance parameters. Additionally, $\boldsymbol{X}$ is an observable random vector with cdf $F(\boldsymbol{x}; \boldsymbol{\zeta})$ and sample space Ξ. $\boldsymbol{x}$ is a vector of observed values of $\boldsymbol{X}$ such that $\boldsymbol{x} \in \Xi$.

Definition 1 (Generalized Test Variable [42]) *A random variable of the form $T = T(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\zeta})$ is said to be a generalized test variable if it has the following three properties:*

Property 1: *$t_{obs} = t(\boldsymbol{x}; \boldsymbol{x}, \boldsymbol{\zeta})$ does not depend on unknown parameters.*

Property 2: *When θ is specified, T has a probability distribution that is free of nuisance parameters.*

Property 3: *For fixed $\boldsymbol{x}$ and $\boldsymbol{\delta}$, $Pr(T \leq t; \theta)$ is a monotonic function of θ for any given t.*

The search for a generalized test variable can be restricted to functions of complete sufficient statistics. Depending on whether T is stochastically increasing or decreasing, the generalized p-value, $P(x)$, for testing a null hypothesis can be defined as equations 34 and 35, respectively [42].

$$\begin{aligned} &H_0 : \theta \geq \theta_0 \text{ vs } H_1 : \theta > \theta_0 \\ &p(\boldsymbol{x}) = P(T \geq t_{obs} | \theta = \theta_0) \end{aligned} \tag{34}$$

$$\begin{aligned} &H_0 : \theta \leq \theta_0 \text{ vs } H_1 : \theta > \theta_0 \\ &p(\boldsymbol{x}) = P(T \leq t_{obs} | \theta = \theta_0) \end{aligned} \tag{35}$$

These equations are for the simple case of testing a single hypothesized value but the method is not limited to that. Weerahandi (1995) discusses situations that cannot be tested easily, or at all in some cases, using fixed-level testing such as comparing the means of two exponential distributions with possibly censored data and the Behrens-Fisher problem without using nonparametric statistics which tend to be useful but less powerful. The introduction of the generalized method as an extension of conventional methods allow solutions to these simple, yet obfuscated, problems to be found [42].

2.5.3 Bootstrap Method

The bootstrap method, introduced by Efron in the 1970s, provides an alternative means to calculate standard errors of parameters for statistical inference [33]. This method is based on the concept that the characteristics of a population are represented by their analogous counterparts from a sample drawn from that population. A nonparametric bootstrap sample, $\boldsymbol{X}^* = (X_1^*, \ldots, X_n^*)$, is created by resampling n observations from a random sample, $\boldsymbol{X} = (X_1, \ldots, X_n)$, with replacement. Let $\hat{F}$ be the empirical cumulative distribution of $\boldsymbol{x}$, the observed values of the random sample $\boldsymbol{X}$. The empirical distribution places probability $1/n$ on each of the observed values x_i which is used to generate the bootstrap sample, i.e., the bootstrap data points, $\boldsymbol{x}^* = (x_1^*, \ldots, x_n^*)$, are a random sample of size n drawn with replacement from the population of n objects in the observed data, $\boldsymbol{x}$. [43, 44, 45].

The bootstrap method works by drawing many independent B bootstrap samples to use for inference on the population. Each bootstrap sample will contain members of the original data set where some data points will have duplicates and some will not appear at all within a specific bootstrap sample. A potential issue to be aware of with bootstrap sampling is that there will not be any bootstrapped observations that are more extreme than the minimum and maximum values of the observed data set. This characteristic of the bootstrap sample may be unrealistic depending on the true underlying distribution of the data, F. Conversely, it will ensure that the bootstrap samples do not exceed unrealistic values such as when the true random variable is bounded by some values. [45]

There is also the parametric bootstrap which uses $\hat{F}_{par}$ that is an estimate of F, the true distribution, derived from some assumed parametric model for the data. The methodology is nearly identical to that of the nonparametric bootstrap with the only change being that instead of sampling B samples of size n from the data, the B bootstrap samples are drawn from the parametric estimate of the population $\hat{F}_{par}$. From there, some function is applied to each bootstrap sample to represent the parameter of interest, θ. Then the approximated value is found by taking the average across the B samples.

Nonparametric hypothesis testing on $t(\boldsymbol{x})$ for is performed by finding the achieved significance level, ASL.

$$ASL = Prob_{H_0}(t(\boldsymbol{x}^*) \geq t(\boldsymbol{x})) \tag{36}$$

where $t(\boldsymbol{x}^*)$ is test statistic calculated from n bootstrapped values from the original data $\boldsymbol{x}$ and $t(\boldsymbol{x})$ is the observed value from the $\boldsymbol{x}$. The bootstrapped test statistic is calculated B times sampling from $\boldsymbol{x}$ with replacement, compared to the observed value for a value of 0 or 1, and then all B calculations are summed and divided by B to give a significance level as seen in Equation 37.

$$AS\hat{L}_{boot} = \frac{\sum\limits_{i=1}^{B} I_{(t(\boldsymbol{x}_i^{*b}) \geq t_{obs\ i})}}{B} \tag{37}$$

Nonparametric bootstrap hypothesis tests tend to be more conservative than their parametric counterparts, making the test less likely to fail to reject the null hypothesis [45].

2.6 Inference using Confidence Intervals

This section covers some inferential methods based on confidence intervals, bounds around a true parameter of interest. Similar to the hypothesis tests in Secton 2.5, first the general concept behind confidence intervals is discussed. Second, the generalized method is revisited using similar theory to construct confidence bounds using a generalized pivotal quantity. Third, bootstrap method is introduced to consider non-parametric resampling methods. Fourth, a robust estimation method for pareto front optimization adapted for use as an inferential method is discussed. Lastly, a rudimentary method for constructing confidence intervals for desirability functions is covered. All of these methods, with the exception of bootstrapped pairs, are considered within this research for confidence intervals of the desirability optimal points.

2.6.1 Confidence Intervals

A two-sided confidence interval (CI) for some function of parameter θ, $\tau(\theta)$, based on a random sample, $\boldsymbol{X} = (X_1, \ldots, X_n)$, is the pair of functions, $L(\boldsymbol{X})$ and $U(\boldsymbol{X})$, such that $L(\boldsymbol{X}) \leq U(\boldsymbol{X}) \; \forall \; \boldsymbol{x} \in \mathcal{X}$. $\mathcal{X}$ is the set of all possible values of $\boldsymbol{X}$ and $\boldsymbol{x}$ are observed values of $\boldsymbol{X}$. More formally, this interval is a $100(1-\alpha)\%$ CI of the form $P[L(\boldsymbol{X} \leq \tau(\theta) \leq U(\boldsymbol{X})] = 1 - \alpha$ [33, 46, 47, 48].

Bain and Engelhardt (1987) and Casella and Berger (2002) imply coverage probability can be interpreted as the proportion of trials (repeated random samples) wherein the true parameter of interest lies within the CI bounds, $(L(\hat{\theta}), U(\hat{\theta}))$. The size of the interval is generally considered the width in a one-dimensional set. The CI with the smallest width that guarantees $1 - \alpha$ coverage probability is considered the best interval. Width will often be sacrificed in favor of maintaining coverage probability [46, 33]. with the extreme case being a width of $(-\infty, \infty)$ having coverage probability of 1.

2.6.2 Generalized Method

Definition 2 (Generalized Pivotal Quantity) *Let $R = r(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\zeta})$ be a function of $\boldsymbol{X}$ and possibly $\boldsymbol{x}$, $\boldsymbol{\zeta}$ as well. The random quantity R is said to be a generalized pivotal quantity if it has the following two properties:*

Property A*: R has a probability distribution that is free of unknown parameters.*

Property B*: r_{obs} defined as $r_{obs} = r(\boldsymbol{x}; \boldsymbol{x}, \boldsymbol{\zeta})$ ... does not depend on nuisance parameters, $\boldsymbol{\delta}$ [42].*

Definition 3 (Generalized Confidence Interval) *If the subset C_γ of the sample space ρ of R satisfies $(Pr(R \in C_\gamma) = \gamma)$, then the subset Θ_c of the parameter space given by $\Theta_c(r) = \{\theta \in \Theta | r(\boldsymbol{x}; \boldsymbol{x}, \boldsymbol{\zeta}) \in C_\gamma\}$ is said to be a $100\gamma\%$ generalized confidence interval for θ [42].*

A generalized pivotal quantity is formally defined in Definitions 2 and 3. The generalized pivotal quantity is constructed, generally with some sufficient statistic, by rearranging the terms of random variables within a statistic so that the generalized pivot is only defined by observed values, such as the sample mean, and standardized random variables, such as the standard normal distribution. This generalized pivot can be used to generate several new values that are within appropriate uncertainty bounds of the data to be used for confidence intervals where the applicable $\alpha/2$ and $1 - \alpha/2$ quantiles are used for a $100(1 - \alpha)\%$ confidence interval. This is the general form of the univariate generalized method. The generalized method for regression for both the univariate and multivariate cases are constructed in Chapter III.

2.6.3 Bootstrap Method

Similar to hypothesis tests in Section 2.5.3, a bootstrap confidence interval can be calculated. Efron (1994) describes two ways to bootstrap for predicted values of regression models: bootstrapping in pairs and bootstrapping the residuals. Bootstrapping pairs randomly samples n observations, $(\boldsymbol{y}_i', \boldsymbol{x}_i')$, B times from the original data set. Then B models are fit to the bootstrapped samples which can be used to calculate standard errors and mean values. Bootstrapping the residuals assumes that $\boldsymbol{X}\hat{\boldsymbol{\beta}}$ is fixed and randomly samples n rows of residuals $\boldsymbol{e}_i = \boldsymbol{y}_i - \hat{\boldsymbol{y}}_i$ for $i \in 1, \ldots, n$, with replacement, from a linear model constructed with the sample data set. The bootstrapped residuals $\boldsymbol{e}_{i_b}^*$, are then added to the predicted values from the linear model to generate a 'new' bootstrap sample.

$$\boldsymbol{y}_b^* = \boldsymbol{X}\hat{\boldsymbol{\beta}} + \boldsymbol{e}_b^*, \ b \in 1, \ldots, B \tag{38}$$

This bootstrapped sample, $\boldsymbol{c}_b^* = \{(\boldsymbol{x}_1\hat{\boldsymbol{\beta}} + e_{1_b}^*, \boldsymbol{x}_1), \ldots, (\boldsymbol{x}_n\hat{\boldsymbol{\beta}} + e_{n_b}^*, \boldsymbol{x}_n)\}$, is used to fit a 'new' regression model for $b \in B$ [45]. Eck (2018) provides a multivariate generalization of the residual bootstrap method for fixed $X\boldsymbol{B}$ that is used in this research [49]. In the multivariate generalization, an MMLR model is built using the sample data and the residual vectors of each observation, $\boldsymbol{e}_i$, are randomly sampled n times, with replacement to form an $n \times m$ matrix of bootstrapped errors, $\boldsymbol{E}_b^*$. In exactly the same way as the univariate bootstrapped residual method, these bootstrapped residuals are then added to the MMLR model predicted values and used to build a bootstrapped MMLR model.

$$\boldsymbol{Y}_b^* = \boldsymbol{X}\hat{\boldsymbol{B}} + \boldsymbol{E}_b^*, \, b \in 1, \ldots, B \tag{39}$$

2.6.4 Simulated Surface Method

To incorporate uncertainty into response surfaces for Pareto front optimization, Chapman et al. (2014) samples regression parameters of response surfaces from a multivariate normal distribution as

$$\hat{\boldsymbol{\beta}} \sim N_m(\boldsymbol{\beta}, \sigma^2(\boldsymbol{X'X})^{-1}) \tag{40}$$

to simulate B 'new' response surfaces. Calhoun (2020) proposed the multivariate t-distribution where

$$\hat{\boldsymbol{\beta}} \sim t_m(\boldsymbol{\beta}, \sigma^2(\boldsymbol{X'X})^{-1} \cdot \frac{\nu}{\nu - 2}) \tag{41}$$

to avoid understating uncertainty in the original work. These simulated response surfaces are within stochastically appropriate intervals of the original estimates and may be used to approximate additional response surfaces. This method can be adapted by using the B simulated surfaces to predict observations, $\boldsymbol{Y}^*$, would be similar to a parametric bootstrap where the bootstrapped values are the $\boldsymbol{\beta}$ coefficients. The predicted values are calculated as

$$\boldsymbol{Y}_b^* = \boldsymbol{X}\hat{\boldsymbol{\beta}}_b^*. \tag{42}$$

After calculating B sets of predicted values, the $\alpha/2$ and $(1-\alpha)/2$ quantiles for each solution can be used to construct confidence intervals. The method in Chapman et al. (2014b) is currently the only method presented in Myers (2016) for incorporating uncertainty in multi-objective optimization. Although methodology introduces uncertainty to choose more robust solutions, it still does not capture the correlation between surfaces. By using individual linear models there is no covariance structure between surfaces when drawing from the sampling distribution for $\boldsymbol{\beta}$ which may not perform as well as it should [18, 19].

2.6.5 Best/Worst Case Method

Other than the method described in Chapman et al. (2014b), which does not explicitly solve for confidence intervals, one may utilize a best/worst (BW) case scenario for CIs. Chapman et al. (2014a) and He et al. (2012) introduce a form of the BW method. In both articles, only the worst case scenarios are considered. In Chapman's article they are based on the prediction intervals of each individual regression model whereas He's article utilizes the worst case confidence interval for robust desirability optimal values [16, 50]. The method used here is based on the CIs of each regression model and consider both the upper and lower bounds for each response. Individual regression models are fit to each of the r responses.

$$\hat{\boldsymbol{y}}_r = \boldsymbol{X}\hat{\boldsymbol{\beta}}_r \tag{43}$$

The standard CIs for each observation in the candidate set of N locations are constructed using

$$y_{ir} \in (\hat{y}_{ir} \pm t_{1-\alpha/2,n-p}\sqrt{MSE_r \cdot h_{ii}}) \tag{44}$$

If $\boldsymbol{y}_r$ is to be maximized, the worst case is when y_{ir} is at the lower bound and the best case is when y_{ir} is at the upper bound, the inverse is true when $\boldsymbol{y}_r$ is minimized. If $\boldsymbol{y}_r$ has a target objective, the worst case is the largest deviation from the target value and the best case is exactly the target value. The best (worst) case values for both responses at each observation are used to calculate an upper (lower) bound for the desirability, D_i. This method can be extended to multiple objectives relatively easily and is simple to calculate.

2.7 Multivariate Distributions

The Multivariate Normal (MVN), Multivariate t (MVt), Dirichlet, and Wishart are multivariate generalizations of the Normal, t, Beta, and Gamma distributions, respectively. The MVN has one commonly utilized pdf across multiple sources defined in this section whereas the MVt defined here is one of many different derivations that are covered in Kotz (2004) [51]; However, the one listed here is often the suggested derivation as it's a direct multivariate generalization from the univariate t-distribution in the same manner that the multivariate normal distribution is a direct generalization of the univariate normal distribution [51]. These distributions are essential as they are used as sampling distribution to help characterize correlation structures for the inferential methods. The Dirichlet and Wishart distribution are also included for completeness and potential interest in future research.

2.7.1 Multivariate Normal Distribution

Let $\boldsymbol{X}$ be a vector of m random variables with mean vector, $\boldsymbol{\mu}$, and covariance matrix, $\boldsymbol{\Sigma}$, then the pdf of the MVN is defined as

$$f(\boldsymbol{x}|\boldsymbol{\mu}, m, \boldsymbol{\Sigma}) = \frac{1}{(\sqrt{2\pi})^m |\boldsymbol{\Sigma}|^{1/2}} e^{-(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})/2} \tag{45}$$

where $\boldsymbol{x} \in \mathbb{R}$, $\boldsymbol{\mu} \in \mathbb{R}$, $m \in \mathbb{N}$, and $\boldsymbol{\Sigma}$ is positive definite [10]. $N_m(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is used to denote an multivariate normal distribution with m vectors. The bivariate normal distribution is a special case of the MVN when $m = 2$. If $\boldsymbol{\Sigma}$ is a diagonal matrix, the MVN can be factored such that

$$f(\boldsymbol{x}|\boldsymbol{\mu}, m, \boldsymbol{\Sigma}) = f(x_1|\mu_1, \sigma_1^2) \cdot f(x_2|\mu_2, \sigma_2^2) \cdot \cdots \cdot f(x_n|\mu_n, \sigma_n^2)$$

which shows that the MVN becomes the product of independent normal distributions when there is no covariance between variables [33, 52].

2.7.2 Multivariate t Distribution

Let $\boldsymbol{X}$ be a vector of m random variables with shift vector, $\boldsymbol{\mu}$, and correlation matrix, $\boldsymbol{R}$, then the pdf of the MVt used in this research is defined as

$$f(\boldsymbol{x}|\boldsymbol{\mu},\nu,m,\boldsymbol{R}) = \frac{\Gamma(\frac{\nu+m}{2})}{\Gamma(\frac{\nu}{2})}\frac{1}{(\sqrt{\nu\pi})^m|\boldsymbol{R}|^{1/2}}(1+\frac{1}{\nu}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{R}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}))^{-(\nu+m)/2},$$

or more commonly written as

$$f(\boldsymbol{x}|\boldsymbol{\mu},\nu,m,\boldsymbol{R}) = \frac{\Gamma((\nu+m)/2)}{(\nu\pi)^{m/2}\Gamma(\nu/2)|\boldsymbol{R}|^{1/2}}\left[1+\frac{1}{\nu}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{R}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]^{-(\nu+m)/2} \tag{46}$$

where $\boldsymbol{x} \in \mathbb{R}$, $\boldsymbol{\mu} \in \mathbb{R}$, $\nu \in \mathbb{N}$, $m \in \mathbb{N}$, and r_{ij} are entries of $\boldsymbol{R}$ where $-1 \leq r_{ij} \leq 1$. $\boldsymbol{R}$ has corresponding $\boldsymbol{\Sigma}$ as the covariance matrix [51]. There are several derivations of the MVt in Kotz' Multivariate t book. The distribution seen in Equation 46 is used for this research as it is the most common form of the MVt distributions. It is a direct generalization of the univariate t-Distribution. Furthermore, if $\boldsymbol{Y} \sim N_m(\boldsymbol{0},\boldsymbol{\Sigma})$ and if $U \sim \chi^2_\nu$, independent of $\boldsymbol{Y}$, then

$$\boldsymbol{X} = \boldsymbol{\mu} + \boldsymbol{Y}\sqrt{\nu/U} \sim t_m(\boldsymbol{\mu},\boldsymbol{\Sigma}_t) \tag{47}$$

is an MVt distribution with dispersion matrix $\boldsymbol{\Sigma}_t = \boldsymbol{\Sigma}\frac{\nu}{\nu-2}$ [51, 53]. It is trivial to see that the $\boldsymbol{\Sigma}_t$ is the original covariance matrix multiplied by a larger than 1 constant which causes the dispersion matrix to be larger than that of the MVN. This is desired to ensure there is enough variability when covariance is unknown and an estimator for covariance is used. [10].

2.7.3 Dirichlet Distribution

Let $\boldsymbol{Y} = (Y_0, \ldots, Y_m)$ be a vector of m independent random variables where Y_j is distributed $\chi^2_{\nu_j}$ for $i = 0, 1, \ldots, m$, $\nu_j > 0$. The transformation for the Dirichlet distribution, the joint distribution of $Y_1, \ldots, Y_m$ is defined by

$$X_j = \frac{Y_j}{\sum\limits_{i=0}^{m} Y_i}, \qquad j = 1, \ldots, m \tag{48}$$

performing the appropriate transformation of variables results in the pdf of the (standard) Dirichlet Distribution

$$f(\boldsymbol{x}|\boldsymbol{\alpha}) = \frac{\Gamma\left(\sum\limits_{j=0}^{m} \alpha_j\right)}{\prod\limits_{j=0}^{m} \Gamma\left(\alpha_j\right)} \left(1 - \sum_{j=1}^{m} x_j\right)^{\alpha_0 - 1} \prod_{j=1}^{m} x_j^{\alpha_j - 1} \tag{49}$$

where $x_j \geq 0$, $j = 1, \ldots, m$ and $\sum\limits_{j=1}^{m} x_j \leq 1$. This version is the standard Dirichlet as it replaces the degrees of freedom with $\alpha_j = \frac{1}{2}\nu_j$. Letting $\Theta = \sum\limits_{j=0}^{m} \alpha_j$, the expected value and covariance of this distribution are

$$E[X_i] = \frac{\alpha_i}{\Theta},\ V[X_i] = \frac{\alpha_i(\Theta - \alpha_i)}{\Theta^2(\theta + 1)},\ Cov[X_i, X_j] = -\sqrt{\frac{\alpha_i \alpha_j}{(\Theta - \alpha_i)(\Theta - \theta_j)}} \tag{50}$$

This distribution is commonly known as a multivariate beta distribution. The random variables all have values between 0 and 1, and their sum is less than or equal to 1. [52, 54]

The Dirichlet distribution might be beneficial for future research to derive the sampling distribution for the weights in multi-objective decision analysis. The requirement of deriving a sampling distribution for $\boldsymbol{w} = (w_1, \ldots, w_m)$ stems from the fact that weight schemes are generally reliant on subjective priorities and only reflect a snapshot in time. Three or fewer objectives are fairly straight forward to perform sensitivity analysis with simplex plots but as m increases, so does the complexity. The Dirichlet distribution is the only multivariate distribution with random variables that sum to 1 with each individual random variate bounded by (0,1) and may be an ideal candidate as it meets the needs for a vector of weights such as $\boldsymbol{w}$. The initial concept involves the assumption that $\boldsymbol{w} \sim Dirichlet(r, \boldsymbol{\alpha})$ where r is the number of responses and $\boldsymbol{\alpha}$ are concentration parameters that need to be estimated.

2.7.4 Wishart Distribution

Let $\boldsymbol{Z}$ be an $n \times m$, matrix distributed as $N_m(0, I_n \otimes \boldsymbol{\Sigma})$, a multivariate standard normal distribution. The product, $\boldsymbol{X} = \boldsymbol{Z}'\boldsymbol{Z}$ is distributed as a Wishart distribution with n degrees of freedom and covariance matrix $\boldsymbol{\Sigma}$, $\boldsymbol{X} \sim W_m(n, \boldsymbol{\Sigma})$. If $n < m$ then $\boldsymbol{X}$ is singular and will not have a pdf; However, if $n \geq m$ then $\boldsymbol{X}$ is positive definite with probability 1 and has following pdf.

$$f(\boldsymbol{x}|m, n, \boldsymbol{\Sigma}) = \frac{1}{2^{mn/2}\Gamma_m\left(\frac{1}{2}n\right)\left(det(\boldsymbol{\Sigma})\right)^{n/2}} exp(tr\left(-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\right))\left(det(\boldsymbol{x})\right)^{(n-m-1)/2} \tag{51}$$

where $\boldsymbol{X} > 0$, $\Gamma_m(a) = \int\limits_{A>0} exp(tr(-\boldsymbol{X}))det(\boldsymbol{X}^{a-(m+1)/2})d\boldsymbol{X}$ is the multivariate gamma function, $tr(.)$ is the trace of a matrix, and det is the determinant. The expected value and covariance of this distribution are

$$E[\boldsymbol{X}] = n\boldsymbol{\Sigma},\ Cov[x_{ij}, x_{kl}] = n(\sigma_{ik}\sigma_{jk} + \sigma_{il}\sigma_{jk}) \tag{52}$$

for $i, j, k, l = 1, \ldots, m$. The Wishart distribution is convenient for normally distributed data as it has analogous properties to the Chi-square since it is considered a multivariate Chi-square distribution. If $\boldsymbol{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ are independent

$N_m(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $n > m$, then the sample covariance matrix, $\boldsymbol{S} = \frac{1}{n-1}\sum_{i=1}^{n}(\boldsymbol{X}_i - \bar{\boldsymbol{X}})(\boldsymbol{X}_i - \bar{\boldsymbol{X}})'$, is distributed as $W_m(n, \frac{1}{n}\boldsymbol{\Sigma})$. If $A_i \sim W_m(n_i, \boldsymbol{\Sigma}),\ i = 1, \ldots, r,\ A_i$ is $m \times m$ the $\sum_{i=1}^{r} A_i$ is $W_m(n, \boldsymbol{\Sigma})$ where $n = \sum_{i=1}^{r} n_i$ [52, 11]. This distribution could be beneficial in further exploring the covariance structure as the sample covariance in Equation 12. $\boldsymbol{S}_e$ multiplied by $n - p$ is distributed as a Wishart.

$$(n - p) \cdot \boldsymbol{S}_e \sim W_m(n - p, \boldsymbol{\Sigma}) \tag{53}$$

Using the Wishart distribution as the sampling distribution for the covariance matrix in MMLR may be a beneficial addition to this topic in future research. [10, 11, 8]

2.8 Conclusion

A review of the literature indicates a gap for an adequate inference method for Derringer and Suich Desirability Functions commonly used in RSM. A rudimentary method by taking the best and worst case using confidence or prediction intervals from each response variable will sometimes be used to construct simple intervals around optimal desirability values. Unfortunately, this method does not include a covariance component as the intervals are based on the individual linear regression models. The primary contribution of this research is to address the gap in literature by proposing 7 novel methods and comparing them to the best/worst case method. The inference methods assume independence of response variables in multi-objective optimization problems are antithetical to the solution; However, 3 of the proposed novel methods do assume independence to show what a lack of a covariance component can do to the confidence level.

The 7 novel methods include the following. First, a univariate generalized method based on the multiple linear regression model and normality assumptions. This univariate generalized method is then extended it to a multivariate case by implementing the multivariate multiple linear regression model and multivariate normal assumptions. Both the MVN and MVt simulated surface methods are included as well as corrections to incorporate a covariance structure into the sampling distribution of each. Additionally, the multivariate generalization of the residual bootstrap method is included. These 7 novel methods will be compared to the best/worst method bringing the total number of methods considered to 8. Each of these methods consider both first order and second order models that use 2^k factorial design and a CCD, respectively.

III. Methodology

This chapter outlines the methodology for implementing proposed inferential methods with a simulated data set. Capturing the change in coverage, width, and symmetry is main goal with each differing setting. The inferential methods are outlined and derived first to showcase which methods are being used as well as to how they are used. The problem sets for the first and second order models are then introduced to ensure an understanding of how the simulated data is generated. Finally, the simulation design for each type of model is discussed to show the inferential methods and models capture coverage probability, average width, and symmetry in addition to the formulation for these metrics.

3.1 Inferential Methods

This section reiterates some of the inferential methods information presented in Section 2.6. This serves both to show which methods are used for comparison as well as expands upon some of them. The best/worst case method and multivariate bootstrapp residuals method are utilized as they are originally presented; however, the generalized methods and simulated surface methods must be derived and adapted to the problem at hand. In total, there are 8 methods being compared which are abbreviated as follows.

- BW – Best/Worst Case Method (Section 3.1.1)
- UG – Univariate Generalized Method (Section 3.1.2.1)
- MG – Multivariate Generalized Method (Section 3.1.2.2)
- MVNS – MVN Surfaces Method (Section 3.1.3.1)
- MVtS – MVt Surfaces Method (Section 3.1.3.1)
- MVNSSig – MVN Surfaces Method with Covariance (Section 3.1.3.1)

- MVtSSig – MVt Surfaces Method with Covariance (Section 3.1.3.1)
- BSR – Multivariate Bootstrap Residuals Method (Section 3.1.3.2)

3.1.1 Best/Worst Case Method

The best/worst case method is based on the confidence intervals of each regression model. Individual regression models are fit to each response.

$$\hat{\boldsymbol{y}}_r = \boldsymbol{X}\hat{\boldsymbol{\beta}},\ r \in 1, \ldots, m \tag{54}$$

The standard confidence intervals for each observation are constructed using

$$y_{ir} \in (\hat{y}_{ir} \pm t_{1-\alpha/2,n-p}\sqrt{MSE_r \cdot h_{ii}}) \tag{55}$$

If $\boldsymbol{y}_r$ is to be maximized, the worst case that could occur is when y_{ir} is at it's lower bound and the best case is when y_{ir} is at it's upper bound. The inverse is true for $\boldsymbol{y}_r$ is to be minimized.

The worst (best) case values for both responses are used to calculate a lower (upper) bound for the desirability D_i. This method can be extended to multiple objectives relatively easily and is simple to calculate.

3.1.2 Generalized Method

Weerahandi (2003) forms the methodology for generalized pivotal quantities to find exact CIs using monte carlo simulation [42]. The general idea of the generalized method is replace unknown parameters with random variables and estimators using pivotal quantities. Nunnally (2018) provides generalized pivotal quantities for both the univariate and multivariate cases if the underlying distribution of the data can be assumed as normal or MVN [55]. Both pivotal quantities from Nunnally (2018) have been adjusted for the linear regression model. The generalized pivotal quantity for the univariate method is discussed first in Section 3.1.2.1. The multivariate extension of the generalized pivotal quantity is then derived in Section 3.1.2.2.

3.1.2.1 Univariate Pivotal Quantity

The UG method is based on the general pivotal quantity as defined in Section 2.6.2. The linear regression version can be derived from the distribution of $\hat{\boldsymbol{y}}_r$. As previously shown, the distribution for $\hat{y}_{ir}$ at observation i in response r is normal, assuming the errors of the linear model are normally distributed [55].

$$\hat{y}_{ir} \sim N(\boldsymbol{x}_i \boldsymbol{\beta}_r, \sigma_r^2 h_{ii}) \tag{56}$$

A pivot of $\hat{y}_{ir}$ is found by subtracting the mean and dividing by the standard deviation.

$$Z = \frac{\hat{y}_{ir} - \boldsymbol{x}_i \boldsymbol{\beta}_r}{\sqrt{\sigma_r^2 h_{ii}}} \sim N(0,1) \tag{57}$$

The next step is rearranging in terms of $E[y_{ir}] = \boldsymbol{x}_i \boldsymbol{\beta}_r$.

$$E[y_{ir}] = \hat{y}_{ir} - Z\sqrt{\sigma_r^2 h_{ii}} \tag{58}$$

Then σ_r^2 must be replaced to remove nuisance parameters. Let

$$U = \frac{SSE_r}{\sigma_r^2} \sim \chi^2_{n-p} \tag{59}$$

which can be rearranged in terms of σ_r^2

$$\sigma_r^2 = \frac{SSE_r}{U} \tag{60}$$

and used in Equation 58.

$$
\begin{aligned}
E\left[y_{ir}\right] &= \hat{y}_{ir} - Z\sqrt{\sigma_r^2} \cdot \sqrt{h_{ii}} \\
&= \hat{y}_{ir} - Z\sqrt{\frac{SSE_r}{U}} \cdot \sqrt{h_{ii}} \\
R_{E[y_{ir}]} &= \hat{y}_{ir} - \frac{Z}{\sqrt{U/(n-p)}} \cdot \sqrt{MSE_r h_{ii}} \\
&= \hat{y}_{ir} - t \cdot \sqrt{MSE_r h_{ii}}
\end{aligned} \tag{61}
$$

where, t is a t-distributed random variable with $n-p$ degrees of freedom. This change follows since a standard normal distribution Z divided by the square root of a chi-square distribution, U, divided by its degrees of freedom, and $Z \perp\!\!\!\perp U$, is the t-distribution, t, with the same degrees of freedom and U. The notation changes from $E\left[y_{ir}\right]$ to $R_{E[y_{ir}]}$ once the equation represents a generalized pivotal quantity for the parameter of interest as defined in Weerahandi (2003). To construct a generalized CI around D_i, a large number (B) of Monte Carlo draws are sampled from t and used to calculate $R_{E[y_{ir}]}$. $R_{E[y_{ir}]}$ is then used in equations 24-26 for y_{ir} to calculate $R_{d_{ir}}$ and subsequently R_{D_i} using equations 20 or 21. Similar to $R_{E[y_{ir}]}$, $R_{d_{ir}}$ and R_{D_i} are the generalized pivots for d_{ir} and D_i, respectively. They follow naturally from calculating d_{ir} and D_i using $R_{E[y_{ir}]}$ in Equations 20-26 because there are no additional random variables in the desirability function calculations.

3.1.2.2 Multivariate Pivotal Quantity

For the MG method, the standard form of this pivot can be constructed using $\boldsymbol{Y} \sim N_m(\boldsymbol{0}, \boldsymbol{\Sigma})$ and $U \sim \chi^2_\nu$, where $\boldsymbol{Y} \perp\!\!\!\perp U$, which results in the following

$$
\boldsymbol{\mu} + \frac{\boldsymbol{Y}}{\sqrt{U/\nu}} \sim t_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}_t) \tag{62}
$$

which is a multivariate t (MVt) distribution with dispersion matrix $\boldsymbol{\Sigma}_t = \boldsymbol{\Sigma} \cdot \frac{\nu}{\nu-2}$ shown in Kotz (2004) [51]. As shown in Section 2.1.3, the multivariate distribution for $vec(\hat{\boldsymbol{Y}})$ is

$$vec(\hat{\boldsymbol{Y}}) \sim N_m(vec(\boldsymbol{XB}), \boldsymbol{\Sigma} \otimes \boldsymbol{H}) \tag{63}$$

First, a typical pivotal quantity can be made where $E[vec(\boldsymbol{Y})] = vec(\boldsymbol{XB})$

$$\boldsymbol{P} = vec(\hat{\boldsymbol{Y}}) - E[vec(\boldsymbol{Y})] \sim N_m(\boldsymbol{0}, \boldsymbol{\Sigma} \otimes \boldsymbol{H}) \tag{64}$$

Let $U \sim \chi^2_{n-p}$ then

$$t = \boldsymbol{vec(\hat{Y})} + \frac{\boldsymbol{P}}{\sqrt{U/(n-p)}} \sim t_m(vec(\hat{\boldsymbol{Y}}), \boldsymbol{\Sigma}_t \otimes \boldsymbol{H}, n-p) \tag{65}$$

where t_m is distributed MVt with $n-p$ degrees of freedom and $\boldsymbol{\Sigma}_t = \boldsymbol{\Sigma} \cdot \frac{n-p}{n-p-2}$. Thus, the multivariate pivotal quantity is

$$R_{vec(\hat{\boldsymbol{Y}})} = t_m(vec(\hat{\boldsymbol{Y}}), \boldsymbol{\Sigma}_t \otimes \boldsymbol{H}, n-p) \tag{66}$$

where the shift vector $vec(\hat{\boldsymbol{Y}})$ is the predicted value for $vec(\boldsymbol{Y})$. $\boldsymbol{\Sigma}_t$ is the dispersion matrix, $\boldsymbol{H}$ is the hat matrix, and $n-p$ are the degrees of freedom from the original regression model drawn from t_m. In this case, $\boldsymbol{Sigma}$ is estimated by the MMLR covariance estimate, $\boldsymbol{S}_e$.

3.1.3 Bootstrap Methods

This section encompasses 5 of the novel methods which align with bootstrapping sampled data. The simulated surface methods in this section are not strictly defined as parametric bootstrap methods, but they behave in a similar way. Each of the simulated surface methods are described in Section 3.1.3.1 to include MVNS, MVtS, MVNSSig, and MVtSSig. The last method considered is the nonparametric bootstrapped residuals method. BSR is covered in Section 3.1.3.2.

3.1.3.1 Simulated Surfaces

The methods adapted from Chapman et al. (2014b) and Calhoun (2020) are explained here. They rely on resampling from the MVN and MVt distributions. Chapman et al. (2014b) proposed using the distribution of $\hat{\boldsymbol{\beta}}_r$ to simulate B (500 in original article) surfaces which can be used to infer about the true surface [18]. If the errors from a MLR model are assumed normal then

$$\hat{\boldsymbol{\beta}}_r \sim N_m(\boldsymbol{\beta}_r, \sigma_r^2(\boldsymbol{X}'\boldsymbol{X})^{-1}), \tag{67}$$

Chapman et al. (2014b) used the MVN distribution and Calhoun (2020) expanded the research by implementing a MVt distribution to simulate several response surfaces using MSE as an estimate for variance in Pareto front optimization [18, 19]. This methodology has been adapted slightly where B parameter vectors for each response are resampled as $\hat{\boldsymbol{\beta}}_r^*$. Then simulated predicted values, $\hat{\boldsymbol{Y}}_{\boldsymbol{r}}^*$, are calculated using the design matrix $\boldsymbol{X}$,

$$\hat{\boldsymbol{Y}}_{\boldsymbol{r}}^* = \boldsymbol{X}\hat{\boldsymbol{\beta}}_r^* \tag{68}$$

The desirability is calculated for each of the predicted values for each simulated response. These are the MVNS and MVtS methods, respectively.

The MVNS and MVtS methods calculate the surfaces independent of one another. By vectorizing $\hat{\boldsymbol{B}}$ from the MMLR model, a covariance structure can be included. The notation for vectorized $\hat{\boldsymbol{B}}$ is $vec(\hat{\boldsymbol{B}})$, a $pm \times 1$ vector. Once again, if the error matrix is assumed MVN, then the expected value and variance in Section 2.1.3 can be used to find the sampling distribution of $\hat{\boldsymbol{B}}$.

$$vec(\hat{\boldsymbol{B}}) \sim N_m(vec(\boldsymbol{B}), \boldsymbol{\Sigma} \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}) \tag{69}$$

The MVNSSig method uses the sampling distribution in Equation 69 with the covariance estimate $\boldsymbol{S}_e$ from Equation 12. The MVtSSig uses the same estimate for covariance but uses the same changes as implemented in Calhoun (2020) to transform it into the MVt distribution [19].

$$vec(\hat{\boldsymbol{B}}) = t_m(vec(\boldsymbol{B}), (\boldsymbol{S}_e \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}) \cdot \frac{n-p}{n-p-2}) \tag{70}$$

$\hat{\beta}_r$ and $\hat{\boldsymbol{B}}$ can be used as estimates for β_r and $\boldsymbol{B}$ in each of the sampling distributions such that

$$vec(\hat{\boldsymbol{B}}^*) \sim N_m(vec(\hat{\boldsymbol{B}}), \boldsymbol{S}_e \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}) \tag{71}$$

and

$$vec(\hat{\boldsymbol{B}}^*) = t_m(vec(\hat{\boldsymbol{B}}), (\boldsymbol{S}_e \otimes (\boldsymbol{X}'\boldsymbol{X})^{-1}) \cdot \frac{n-p}{n-p-2}) \tag{72}$$

3.1.3.2 Residual Resampling

The BSR method randomly samples n rows of residuals $\boldsymbol{e}_i = \boldsymbol{y}_i - \hat{\boldsymbol{y}}_i$ for $i \in 1, \ldots, n$, with replacement, from a linear model constructed with the sample data set. The bootstrapped residuals $\boldsymbol{e}_i^*$, are then added to the predicted values from the linear model to generate a 'new' bootstrap sample. This bootstrapped sample, $\boldsymbol{c}^* = \{(\boldsymbol{x}_1\hat{\boldsymbol{\beta}} + e_{i_1}^*, \boldsymbol{x}_1), \ldots, (\boldsymbol{x}_n\hat{\boldsymbol{\beta}} + e_{i_n}^*, \boldsymbol{x}_n)\}$, is used to fit a 'new' multivariate regression model. Each of the B multivariate regression models calculated from the bootstrapped samples are used to predicted $\boldsymbol{y}_r$ and subsequently used in the desirability functions to find d_{ir} and D_i.

3.2 Problem Sets

There are a total of 8 problem sets used in this research, 7 intersecting planes and 1 chemical process optimization problem. The first order models use the first 7 data sets. The second order models use the chemical process problem data set as it is commonly used within RSM, borrowed directly from Myers et al. (2016) [4].

3.2.1 First Order Models

The first order models consist of $m = 2$ responses, $(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{X})$, with 7 different levels of correlation. Two optimization patterns are considered, the first is to maximize $\boldsymbol{y}_1$ and minimize $\boldsymbol{y}_2$ (Max/Min) and the second is to maximize $\boldsymbol{y}_1$ and match a target value for $\boldsymbol{y}_2$ (Max/Tgt). The correlation considered between the bivariate responses are $\boldsymbol{\rho} = (-0.8, -0.5, -0.3, 0, 0.3, 0.5, 0.8)$, providing a symmetry of correlations to discover how differing correlation affect the inference methods. The correlations values are also used to construct the plane surfaces to known angles. According to Bretscher (2009) $cos(\theta)$ is equal to the correlation between two planes [56]. The angles that match with each of the respective entries of $\boldsymbol{\rho}$ are $\boldsymbol{\theta} = (143, 120, 107, 90, 73, 60, 37)$. To create data with known correlation and plane angles, first order models of the following form are created with the 7 intersecting plane problem sets.

$$y_{ir} = \beta_{0r} + \beta_{1r}x_{1i} + \beta_{2r}x_{2i} + \epsilon_{ir} \tag{73}$$

The true coefficients used for this simulation design are designated as follows $\boldsymbol{\beta}_{kr} = (\beta_0, \beta_1, \beta_2)$. These correspond to the design matrix $\boldsymbol{X} = (\mathbf{1}, \boldsymbol{x}_1, \boldsymbol{x}_2)$.

$$\rho_1 = -0.8, \quad \theta_1 = 143, \quad \boldsymbol{\beta}_{11} = (10, 9.5, 9.5), \quad \boldsymbol{\beta}_{12} = (10, -7, 1) \tag{74}$$

$$\rho_2 = -0.5, \quad \theta_2 = 120, \quad \boldsymbol{\beta}_{21} = (10, 2.5, 9.5), \quad \boldsymbol{\beta}_{22} = (10, -5.5, -1.5) \tag{75}$$

$$\rho_3 = -0.3, \quad \theta_3 = 107, \quad \boldsymbol{\beta}_{31} = (10, 6.5, 9.5), \quad \boldsymbol{\beta}_{32} = (10, 5.5, -7) \tag{76}$$

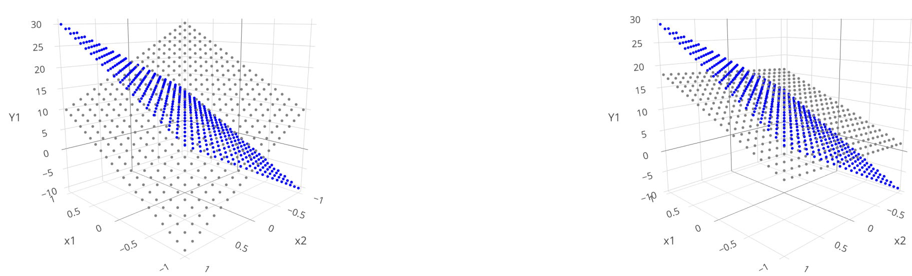
$$\rho_4 = 0, \quad \theta_4 = 90, \quad \boldsymbol{\beta}_{41} = (10, 10, 10), \quad \boldsymbol{\beta}_{42} = (10, 10, -10) \tag{77}$$

$$\rho_5 = 0.3, \quad \theta_5 = 73, \quad \boldsymbol{\beta}_{51} = (10, 6.5, 9.5), \quad \boldsymbol{\beta}_{52} = (10, -5.5, 7) \tag{78}$$

$$\rho_6 = 0.5, \quad \theta_6 = 60, \quad \boldsymbol{\beta}_{61} = (10, 2.5, 9.5), \quad \boldsymbol{\beta}_{62} = (10, 5.5, 1.5) \tag{79}$$

$$\rho_7 = 0.8, \quad \theta_7 = 37, \quad \boldsymbol{\beta}_{71} = (10, 9.5, 9.5), \quad \boldsymbol{\beta}_{72} = (10, 7, 1) \tag{80}$$

These true β_{kr} vectors control the angle between the planes. For example, the planes $y_{i1} = 10 + 9.5x_{1i} + 9.5x_{2i}$ and $y_{ir} = 10 - 7x_{1i} + 1x_{2i}$ correspond to a correlation of $\rho_1 = -0.8$. This means that for ρ_1 in Equation 74, the set of two planes has angle $\theta = cos^{-1}(-0.8) \approx 143°$. Similarly the planes associated with $\rho_4 = 0$ are orthogonal as they have an angle of $90°$. Figure 4 shows examples of the true surface for the 0



(a) 0 Correlation Planes (b) 0.8 Correlation Planes

Figure 4. Examples of Response Surface Planes Associated with Correlation of 0 and Correlation of 0.8

correlation and 0.8 correlation response surface planes. Although the perspective is slightly skewed, the planes in figure 4a are orthogonal.

For the initial design matrix, the first order model sets use a 2^{p-1} factorial designed experiment with $n_c = 5$ center points, where $p-1$ here is the number of factors. There are only 2 factors in this experiment resulting in a total of $2^2 + 5 = 9$ observations. The first order models have center points as it is reasonable to expect that one would conduct a 2^k factorial design with added center points to check for curvature.

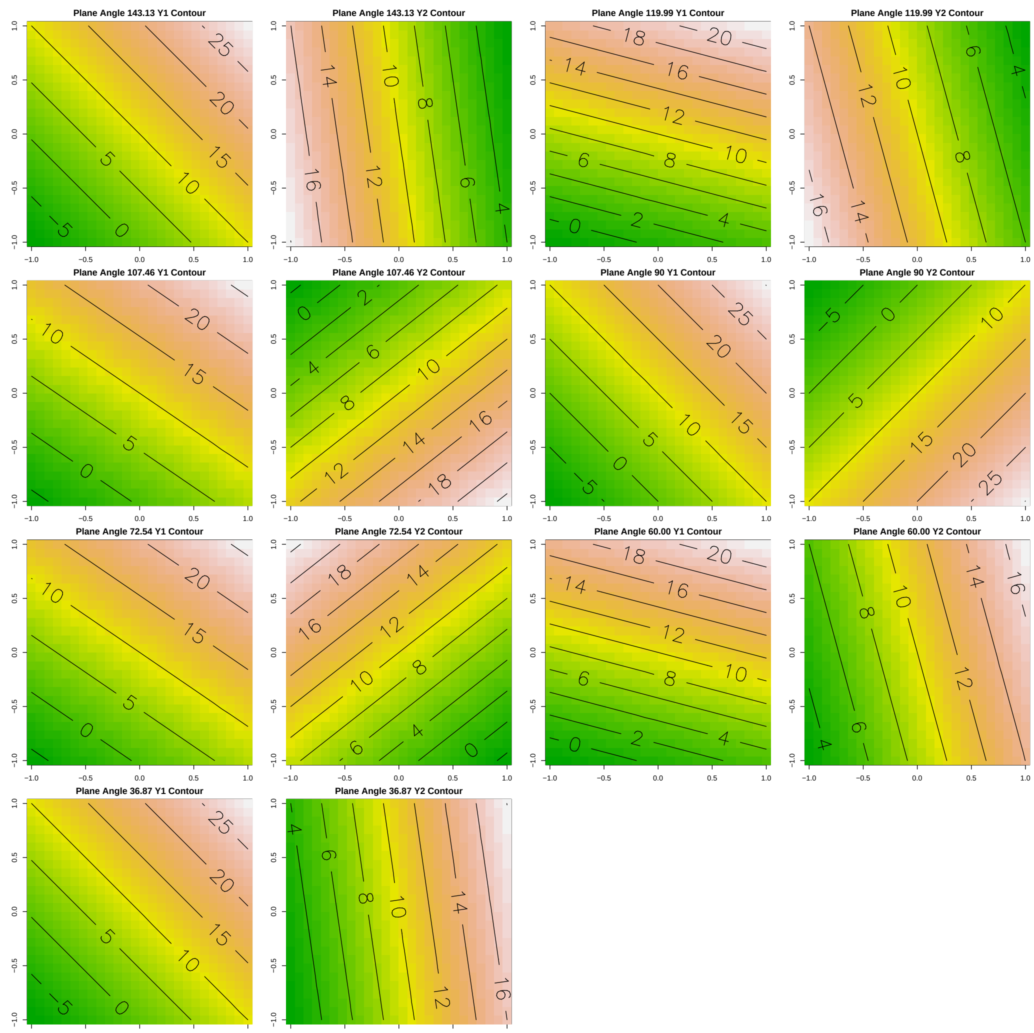


Figure 5. First Order Contour Plots

Figure 5 shows the contours of each of the true response plane surfaces at the respective plane angles. At each plane angle there is a Y_1 and Y_2 contour to display where some of the optimal points might be. For example, consider a Max/Min problem where the angle between planes is 143. The maximum value for Y_1 is in the top right region at $(x_1, x_2) = (1, 1)$ and the minimum value for Y_2 is at $x_1 = 1$ for all values of x_2. This would indicate that the mutually optimal value will likely be at $(x_1, x_2) = (1, 1)$. A more effective way of doing this is to overlay the contour plots; however, that is not the focus here. Something important to notice is that the angles that are opposite of each other (143 and 37, 120 and 60, 107 and 73) have opposite correlation between Y_1 and Y_2 in the contour plots. The plane angles associated with negative correlation (143, 120, 107) have response values negatively correlated response values with respect to the $\boldsymbol{X}$ and the plane angles associated with positive correlation (73, 60, 37) have response values that agree with respect to $\boldsymbol{X}$. The perpendicular planes with a 90° angle has no correlation between responses. Note, the correlation of the planes is different than the correlation between specific response values at observation i.

Simulated random samples are generated by adding a bivariate normal distribution to the true surfaces for each simulation. The bivariate normal distribution is $N_2(\mathbf{0}, \mathbf{\Sigma}_k)$ where

$$\mathbf{\Sigma}_k = \begin{pmatrix} \sigma_{k1}^2 & \rho_k \sigma_{k1} \sigma_{k2} \\ \rho_k \sigma_{k1} \sigma_{k2} & \sigma_{k2}^2 \end{pmatrix} \tag{81}$$

Each $\mathbf{\Sigma}_k$ uses unit variance with their corresponding correlation which effectively makes it a correlation matrix for simplicity in the first order models.

$$\mathbf{\Sigma}_k = \begin{pmatrix} 1 & \rho_k \\ \rho_k & 1 \end{pmatrix} \tag{82}$$

3.2.2 Second Order Models

The problem set used for the second order models is a chemical process optimization problem from Myers et al. (2016) [4]. Table 1 shows the data where there are three response variables, yield, viscosity, and number-average molecular weight with two factors, time and temperature. In the original problem, the three responses are to be simultaneously optimized where yield is maximized, viscosity is to match a target of 65, and molecular weight is minimized.

Table 1. Chemical Process Optimization Problem

	Natural Variables		Coded Variables		Responses		
i	ξ_1(**Time**)	ξ_2(**Temp**)	x_1	x_2	y_1(**Yield**)	y_2(**Viscosity**)	y_3(**Molecular Weight**)
1	80	170	-1	-1	76.5	62	2940
2	80	180	-1	1	77.0	60	3470
‘ 3	90	170	1	-1	78.0	66	3680
4	90	180	1	1	79.5	59	3890
5	85	175	0	0	79.9	72	3480
6	85	175	0	0	80.3	69	3200
7	85	175	0	0	80.0	68	3410
8	85	175	0	0	79.7	70	3290
9	85	175	0	0	79.8	71	3500
10	92.07	175	1.414	0	78.4	68	3360
11	77.93	175	-1.414	0	75.6	71	3020
12	85	182.07	0	1.414	78.5	58	3630
13	85	167.93	0	-1.414	77.0	57	3150

The chemical process experiment is a 13-run central composite design (CCD) with 4 corner points, 4 axial points, and 5 center points for building quadratic multiple linear regression models. The axial distance for the CCD is $\alpha = \sqrt{2}$ which provides rotatability in the design. The 5 center points in this design provide uniform precision in the design space which greatly reduces variation at the center. Using the data from the chemical process optimization problem, the three response surface models in equations 83-85 were fit.

$$\hat{y}_1 = 79.9400 + 0.9951x_1 + 0.5152x_2 + 0.2500x_1x_2 - 1.3764x_1^2 - 1.0013x_2^2 \qquad (83)$$

$$\hat{y}_2 = 70.0002 - 0.1553x_1 - 0.9484x_2 - 1.2500x_1x_2 - 0.6873x_1^2 - 6.6891x_2^2 \qquad (84)$$

$$\hat{y}_3 = 3375.975 + 205.126x_1 + 177.367x_2 - 80x_1x_2 - 41.744x_1^2 + 58.286x_2^2 \qquad (85)$$

True response surfaces, such as the β_{kr} vectors from the first order models, generated for these problems were based on rounded values from the MLR models of $\boldsymbol{Y}$ in Equations 83-85 for each problem.

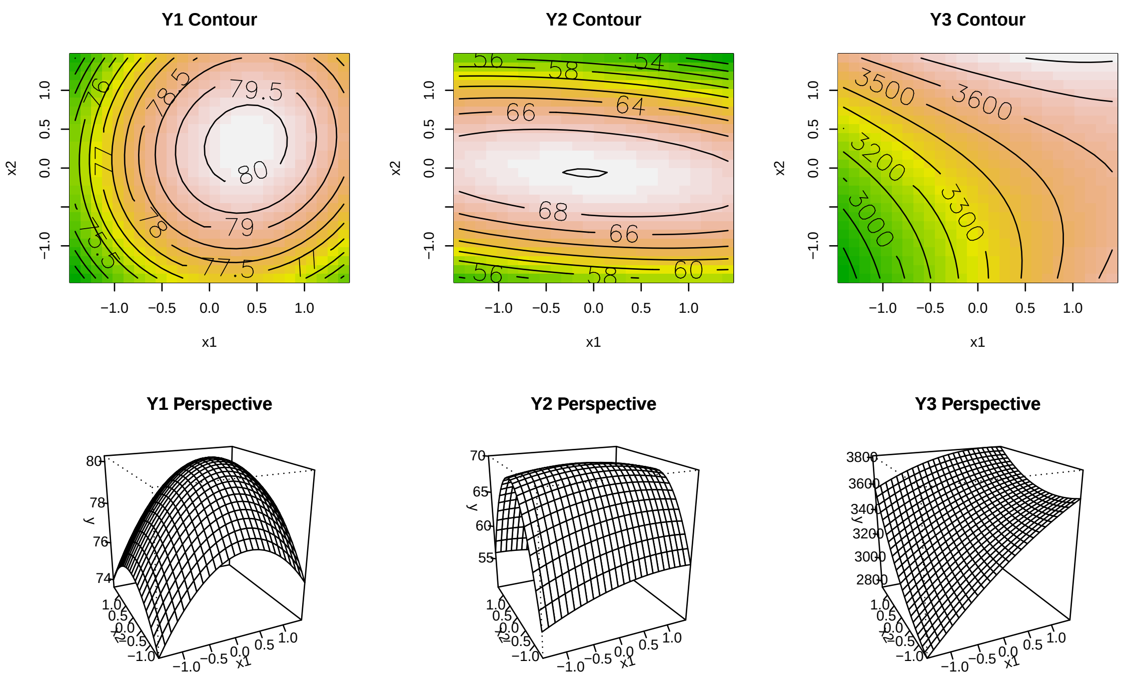


Figure 6. Chemical Process Optimization Problem Contour and Perspective Plots

Figure 6 shows the contours and perspective plots for the true response surface of the chemical process optimization problem. Similar to the first order models, these contours are convenient for showing potential optimal solutions. The issue is they become more convoluted as the number of responses increases and they are essentially useless when the number of factors is greater than two. With these contours, consider a Max/Tgt/Min problem where the target for Y_2 is around 65. The optimal region for Y_1 is somewhere around $(x_1, x_2) = (0.5, 0.5)$, the optimal regions for Y_2 are around $x_2 = 0.5$ or $x_2 = -0.5$ for roughly all values of x_1, and the optimal region for Y_3 is when $(x_1, x_2) = (-1.5, -1.5)$. It should be easy to see that these agree with the perspective plots below the contours as Y_1 is a paraboloid, Y_2 is a saddle shape, and Y_3 is a slightly curved surface. Finding a combined optimal value for these three objectives together requires much more consideration and trade-off.

These problems use the $\boldsymbol{S}_e$ estimate covariance matrix from the original regression models as the true covariance when sampling from a multivariate normal distribution to generate simulated samples. Simulated samples are drawn from distributions of the form $\boldsymbol{Y}_{sim} = \boldsymbol{X}\boldsymbol{B} + N_p(\boldsymbol{0}, \boldsymbol{\Sigma})$ where $\boldsymbol{X}$ is a collection of N candidate points, $\boldsymbol{B}$ is the parameter matrix comprised of the coefficients from Equations 83-85, and $\boldsymbol{\Sigma}$ is replaced by the covariance matrix in Equations 86.

$$\boldsymbol{\Sigma}_{chem} = \boldsymbol{S}_e = \begin{pmatrix} 0.07091 & -0.25988 & 13.46764 \\ -0.25988 & 5.17458 & 36.95738 \\ 13.46764 & 36.95738 & 29695.65911 \end{pmatrix} \tag{86}$$

3.3 Simulation Design

The factors of interest that are considered as possible changes for each model are desirability index used (additive/multiplicative), correlation of response surfaces, $\boldsymbol{\rho} = (-0.8, -0.5, -0.3, 0, 0.3, 0.5, 0.8)$, angle between planes $\boldsymbol{\theta} = (143, 120, 107, 90, 73, 60, 37)$, inferential method (BW, UG, MG, MVNS, MVtS, MVNSSig, MVtSSig, BSR). The first order models do not consider any changes in linearity or weight schemes and only use the simplest parameters, $l = 1$ and $\boldsymbol{w} = (0.5, 0.5)$. The second order models do not have changing correlations or angles between planes, however, they do take into account changing linearity of the Derringer and Suich desirability function and a few different weight schemes. The responses are the CI coverage probability (CP), average width (AW) for each desirability index D_i. Coverage symmetry is also considered for the first order max/min with covariance paired problem as well as the second order max/tgt/min problem which will be discussed further in Section 3.3.1.

3.3.1 First Order Models

There are four optimization scenarios considered for the first order models. The first is a Max/Min problem at three different plane angles, $\boldsymbol{\theta} = (120, 90, 73)$, with all 7 correlation levels in $\boldsymbol{\rho}$ to determine how changing correlation between responses affects CP and AW. The second does the opposite of the first where all 7 different plane angles in $\boldsymbol{\theta}$ and three correlation levels in $\boldsymbol{\rho} = (-0.5, 0, 0.5)$ are considered to determine how changing covariance between responses affects CP and AW. The third is a Max/Min problem that matches each respective plane angle in $\boldsymbol{\theta}$ with the correlation levels in $\boldsymbol{\rho}$ for the covariance matrix. Finally, the last scenario is a Max/Tgt problem that also matches each respective plane angle in $\boldsymbol{\theta}$ with the correlation levels in $\boldsymbol{\rho}$ for the covariance matrix.

3.3.2 Second Order Models

There are three optimization scenarios considered for the second order models. The first is a Max/Min scenario which is a reduced version of the chemical process problem with only Y_1 and Y_3. The second is a Max/Tgt scenario also as a reduced version of the chemical process problem with Y_1 and Y_2. The third is a Max/Tgt/Min scenario with all three objectives. Each of these scenarios considered three different weight schemes and four different linearity parameters for a total of 12 combinations.

The two scenarios with only two objectives use the following weights.

$$\boldsymbol{w}_1 = (w_1, w_2) = (0.5, 0.5) \tag{87}$$

$$\boldsymbol{w}_2 = (0.8, 0.2) \tag{88}$$

$$\boldsymbol{w}_3 = (0.2, 0.8) \tag{89}$$

That is, equal weighting, Y_1 preferred, and Y_2 preferred, respectively. The linearity parameters are slightly different for each one; The two objective Max/Min scenario uses

$$\boldsymbol{l}_1 = (l_1, l_2) = (1, 1) \tag{90}$$

$$\boldsymbol{l}_2 = (0.1, 1) \tag{91}$$

$$\boldsymbol{l}_3 = (1, 10) \tag{92}$$

$$\boldsymbol{l}_4 = (0.1, 10) \tag{93}$$

and the two objective Max/Tgt scenario uses

$$\boldsymbol{l}_1 = (l_1, l_{12}, l_{22}) = (1, 1, 1) \tag{94}$$

$$\boldsymbol{l}_2 = (0.1, 1, 1) \tag{95}$$

$$\boldsymbol{l}_3 = (1, 10, 1) \tag{96}$$

$$\boldsymbol{l}_4 = (0.1, 10, 1) \tag{97}$$

The three objective Max/Tgt/Min scenario has the following weight schemes

$$\boldsymbol{w}_1 = (w_1, w_2, w_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \tag{98}$$

$$\boldsymbol{w}_2 = (0.6, 0.2, 0.2) \tag{99}$$

$$\boldsymbol{w}_3 = (0.2, 0.6, 0.2) \tag{100}$$

which is equal weighting, Y_1 preferred, and Y_2 preferred, respectively. Finally, the linearity parameters used in the three objective scenario are

$$\boldsymbol{l}_1 = (l_1, l_{12}, l_{22}, l_3) = (1, 1, 1, 1) \tag{101}$$

$$\boldsymbol{l}_2 = (0.1, 1, 1, 1) \tag{102}$$

$$\boldsymbol{l}_3 = (1, 10, 1, 1) \tag{103}$$

$$\boldsymbol{l}_4 = (0.1, 10, 1, 1) \tag{104}$$

Recall Figure 3 which indicates that smaller values of l place less emphasis on the value being close to the objective and larger values place more emphasis on being close to the objective. $\boldsymbol{l}$ in each scenario is more or less arbitrarily chosen simply to assess whether there is an effect or not.

3.3.3 Simulation

While the models are based on 9 and 13 run designed experiments, calculating desirability often uses interpolated points within the design region. A grid of candidate points, $\boldsymbol{X}^*$, was constructed with $x_1, x_2 \in (-1, 1)$ using a increments of 0.5 for both the first and second order models. This gives a total of $N = 25$ points for interpolation. The grid of points for the second order models can be extended to the design boundary of $(x_1, x_2) \in (-\sqrt{2}, \sqrt{2})$ and smaller increments can be used to as-

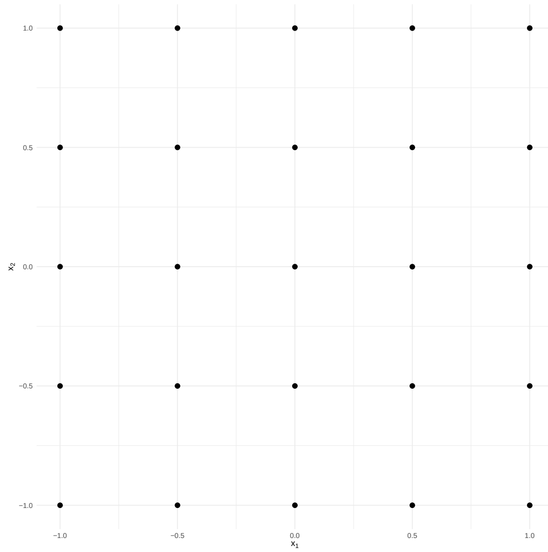


Figure 7. Interpolation Grid of 25 Points, $\boldsymbol{X}^*$ bounded by (-1,1)

sess additional points. This was not considered for this research as it would make the results a bit unmanageable for simulation purposes. This should not present an issue due to the rotatability of the design space prediction variance and optimal regions of the chemical process problem found using previous literature such as Chapman et al. (2014b) and Calhoun (2020). Figure 7 depicts the grid $\boldsymbol{X}$-space of interpolated points and Table 2 is a reference table for the observation number at (x_1, x_2). The solutions that will be covered the most in Chapter IV are observation 1 and 13. Observation 1 corresponds to the lower left quadrant of Figure 7 where $(x_1, x_2) = (-1, -1)$. Observation 13 is the center point where $(x_1, x_2) = (0, 0)$.

		x_1				
		-1	-0.5	0	0.5	1
	-1	1	2	3	4	5
	-0.5	6	7	8	9	10
x_2	0	11	12	13	14	15
	0.5	16	17	18	19	20
	1	21	22	23	24	25

Table 2. Reference of observation number to $\boldsymbol{X}$-space for (x_1, x_2)

Simulations for each method were conducted in the same manner for proper comparison. The primary goal of the simulations are to capture CP and AW using a large number, G, of simulations. $G = 10,000$ simulations were generated for each inference method. Each of the G simulations checked whether a given CI contained the true desirability index, calculated using the true underlying surface from the coefficients in Equations 74-80 and 83-85.

Coverage probability was estimated by summing the number of times a CI contained the true parameter and dividing it by the total number of simulations G.

$$CP = \frac{\sum_{g=1}^{G} I_{\theta \in (L(\hat{\theta}_g)), U(\hat{\theta}_g))}}{G} \tag{105}$$

where I is an indicator function mapping to 1 if the true parameter is in the interval or 0 otherwise, θ is the true parameter, $L(\hat{\theta}_g)$ and $U(\hat{\theta}_g)$ are the lower and upper bounds of the CI for simulation g, respectively. a 95% CI with $G = 10,000$ simulations should have a CP of approximately 9,500/10,000 = 0.95. In this research, the true parameter is the desirability index D_i with the lower and upper bounds being determined by the inferential methods.

Resampling techniques are used in all but the BW method. To help ensure the sample space is adequately explored, $B = 2,000$ samples are taken with each method. CIs from methods using resampling techniques are calculated using the $\alpha/2$ and $(1-\alpha)/2$ quantiles from the B samples within a given simulation. The AW of a CI was calculated by subtracting the lower bound from the upper bound and dividing by G.

$$AW = \frac{\sum_{g=1}^{G} U(\hat{\theta}_g) - L(\hat{\theta}_g)}{G} \tag{106}$$

Also, the average symmetry, ASYM of the confidence interval for each inferential method is included for some of the scenarios considered. The symmetry in this case is the comparison of the sum of the number of times that the true parameter was above a confidence interval minus the number of times the true parameter was below a confidence interval divided by G.

$$ASYM = \frac{\sum_{g=1}^{G} I_{\theta > U(\hat{\theta}_g)} - I_{\theta < L(\hat{\theta}_g)}}{G} \tag{107}$$

Recall the desirability formulas in equations 20-25. The first two equations require w_r and d_{ir} to be defined. d_{ir} is defined by the second two equations which require y_{ir} ($\hat{y}_{ir}$), target value (T_r), evasion value (L_r or U_r), and a linearity parameter (l_r). The set of weights and linearity parameters are detailed in previous sections and the predicted values, $\hat{y}_{ir}$, are calculated using the MMLR models and resampling methods. The target and evasion values must be set necessarily far from the data to prevent truncating the desirability function values as they would result in excess number of 0s and 1s. The $100 \cdot (1 - \alpha/2)\%$ 'prediction' interval on the true value y_{ir} was an easy choice for simple bounds as they would be far enough to prevent truncation. In practice, the bounds could be chosen using the prediction interval around the predicted values from the linear model fit to the data set.

$$y_{ir} \pm t_{1-\alpha/2,n-p} \cdot \sqrt{\sigma_i^2(1 + h_{ii})} \tag{108}$$

For the Max/Min first order model problem, the $\boldsymbol{y}_1$ target (T_1) is the upper bound and the evasion (L_1) is the lower bound of the "true" prediction interval. For $\boldsymbol{y}_2$, T_2 is the lower bound and the evasion (U_2) is the upper bound of the "true" prediction interval. For the Max/Tgt first order model problem, $\boldsymbol{y}_1$ values remain the same as the Max/Max. $\boldsymbol{y}_2$ target is the arbitrarily chosen as the 60th percentile of the true response for each scenario, the upper evasion (U_2) is the upper bound of the "true" prediction interval problem and lower evasion (L_2) is the lower bound of the "true" prediction interval. Similarly, the second order models use the upper bound of the "true" prediction interval for the target and the lower "true" prediction interval bound for the evasion value for maximization problems and the inverse for minimization problems. The second order chemical process problem target for $\boldsymbol{Y}_2$ is a value of 65. The truncation of values may not have much of an effect on the additive desirability index as the desirability D_i may still be larger than 0 if one d_{ir} is 0; However, having a d_{ir} as 0 would result in the entire desirability for the multiplicative desirability index to be 0. Setting the target and evasion values far enough from the response values is important because excess 0s and 1s in the B simulated samples for creating confidence intervals will potentially bias the interval away from the true value.

- Fit response surface model for m responses using linear regression,
- Calculate sample covariance $\boldsymbol{S}_e$ from MMLR,
- Determine, $\boldsymbol{X}^*$, the set of N candidate locations,
- For each r, determine target and evasion values (T_r, L_r, U_r),
- For each r, determine linearity parameters l_r, weights w_r, and DF type
- Calculate initial response surface, $\hat{\boldsymbol{Y}} = \boldsymbol{X}^*\hat{\boldsymbol{B}}$,
- For each observation in $\boldsymbol{X}^*$ using inference method of choice, sample B $\hat{\boldsymbol{y}}_i$ values, $\hat{\boldsymbol{y}}_{ib}$,
- For each observation in $\boldsymbol{X}^*$, calculate $\alpha/2$ and $1-\alpha/2$ quantiles from simulated random sample, $\hat{\boldsymbol{y}}_{ib}$,
- For each observation in $\boldsymbol{X}^*$, calculate applicable desirability function: d_{ir}^{max}, d_{ir}^{min}, or d_{ir}^{tgt},
- For each observation in $\boldsymbol{X}^*$, calculate applicable desirability index: D_i^{add} or D_i^{mult}

Figure 8. Algorithmic steps to compute CP and AW

Figure 8 details the general process for calculating inference for the desirability index, D_i for N candidate locations using the resampling-based inferential methods. In the event that a univariate method is used, the process is the same, but with MSE and individual multiple linear regression models.

3.4 Conclusion

There are 8 inferential methods for confidence bounds on the Derringer and Suich desirability functions proposed in this chapter. UG, MG, MVNS, MVtS, MVNSSig, MVtSSig, and BSR are all novel techniques for this application and are compared to the previously used BW method in Chapter IV. Of the proposed methods, 4 are characterized as univarate (BW, UG, MVNS, MVtS) meaning they assume independence between multiple objectives in MORSM. The remaining 4 are multivariate (MG, MVNSSig, MVtSSig, BSR) techniques which incorporate a covariance struture to capture variation between multiple objectives. The MG and MVtSSig methods are expected to perform the best as they both incorporate a covariance structure and they rely on the multivariate-t distribution which is most appropriate when true covariance is unknown. The univariate methods are expected to perform poorly; However, they provide important information by showing what occurs to confidence when covariance is ignored.

Each of the methods will be use first order and second order models in different scenarios to determine the potential effects of each desirability function setting. The first order models consist of 4 scenarios that each use 7 correlation levels and plane angles where the weights and linearity parameter are simple. The first scenario accounts for the effect due to angles between response planes, the second accounts for the effect due to correlation between responses, the third changes the plane angle with the correlation levels, and the fourth does the same as the third but with

Max/Tgt optimization objectives. The second order models use reduced versions of the chemical processing optimization problem from Myers et al. (2016) in the first two scenarios and a full version in the third scenario. Each of the second order models consider 3 different weight combinations and 4 different linearity combinations for a total of 12 combination each to account for the effect due to the desirability function settings.

Finally, the inference methods all utilize the same simulation plan for calculating the coverage probability, average width, and symmetry in each model. $G = 10,000$ simulations are conducted using $B = 2000$ for the resampling techniques. A collection of $N = 25$ candidate points are considered within the boundaries of the designed experiments for the first and second order models; However, the number of points used in practice are only limited by computation power.

IV. Analysis

This chapter presents analysis on inferential methods used to quantify uncertainty pertaining to Derringer and Suich method desirability functions. The goal of this analysis is to show that the inferential methods that utilize covariance structures maintain empirical coverage when correlation between response surfaces is present. Additionally, the analysis reveals which methods are preferred as they will also maintain the specified empirical coverage. This analysis covers first and second order models with 7 scenarios to capture the change of empirical coverage probability and average width due to Derringer and Suich method parameter settings, correlation between responses, and angles between response surfaces. The inferential methods considered are those covered in Chapter III (BW, UG, MG, MVNS, MVtS, MVNSSig, MVtSSig, BSR). Only MG, MVNSSig, MVtSSig and BSR are multivariate methods which account for covariance. The remaining three methods are considered univariate and assume independence between responses. Although there is some deviation, the primary solutions covered in each scenario are the corner points and center point. All of the analysis in this chapter aim $\alpha = 0.05$ for creating 95% confidence intervals.

4.1 First Order Models

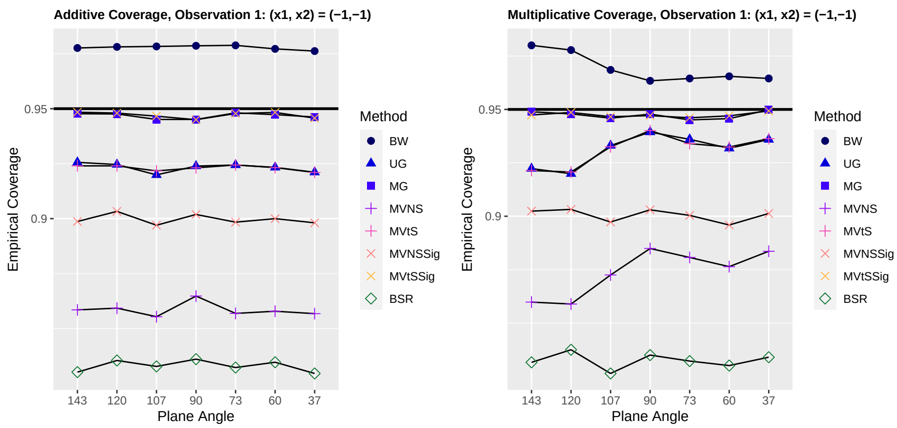
The first order models results are relatively straight forward because the structure of each surface and correlation is simple to control. There are four scenarios that are addressed in this section. First is a Max/Min problem where the correlation between responses is held constant and the angle between the response surface planes is changed. The second is a Max/Min problem where the angle between response surface planes is held constant while correlation is changed. The third and fourth are Max/Min and Max/Tgt problems, respectively, where the angle between response surface planes and correlation change together.

4.1.1 Max/Min Same Correlation with Differing Plane Angles

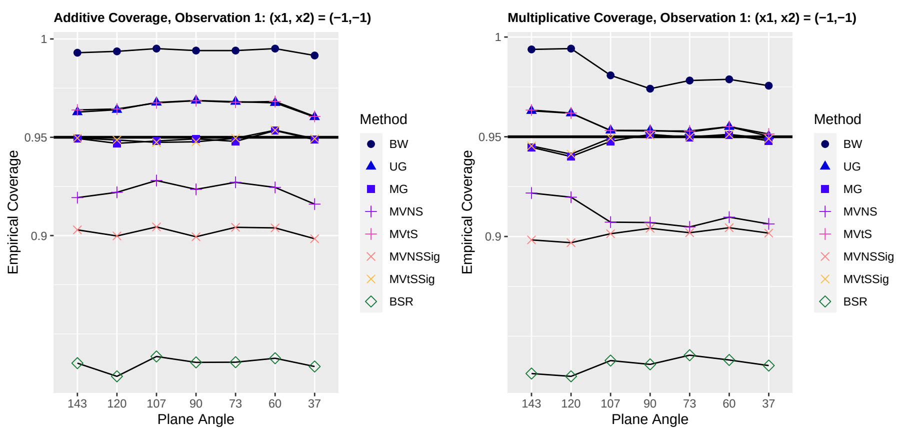
The correlation levels that are considered are when $\boldsymbol{\rho} = (-0.5, 0, 0.5)$ paired with each of the 7 plane angles. The empirical coverage of each correlation level are studied first. Then the widths are covered. Both of these serve to observe how empirical coverage and average width changes with different plane angles.

Figure 9 shows the empirical coverage for the additive and multiplicative desirability index at observation 1 $(x_1, x_2) = (-1, -1)$ for each inferential method. Each of the methods behave incredibly similar across all of the angles for the additive form within correlation level. Notice that as the univariate methods do differ across correlation levels. In both the additive and multiplicative forms, when correlation is negative, the empirical coverage of three inferential methods that do not account for correlation (UG, MVN, MVt) appear to be lower than the specified α level. As the correlation increases, those same inferential methods increase to approximately at or above the specified level, and then they become strictly conservative when correlation is positive. In all cases, the BW method maintains conservative empirical coverage.

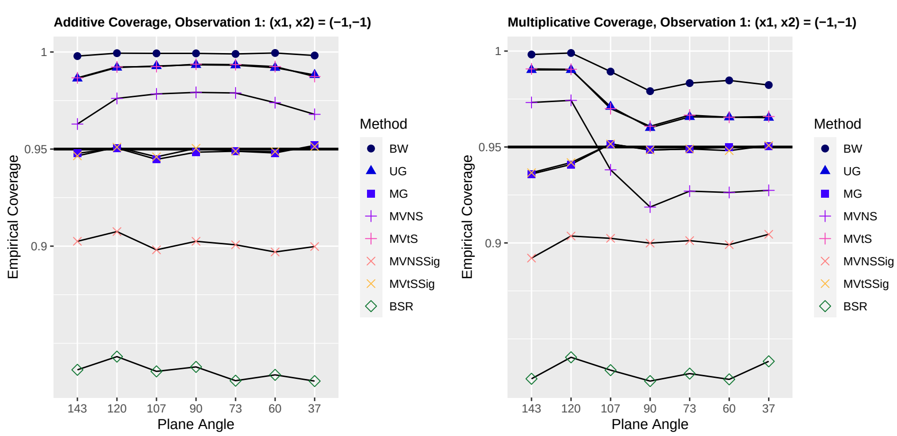
In general, empirical coverage will increase as the average width increases. Width can only increase due to standard error which is affected by the variance (or covariance estimate) and sample size. Since sample size is constant in these problems, it can be conjectured that, given the monotonic nature of the desirability function when a target problem it not considered, increased correlation is increasing the covariance estimate, similarly a negative covariance would decrease that estimate. If correlation is accounted for, this effect does not occur as the average width of the interval is being appropriately sized. The MG, MVNSSig, MVtSSig, and BSR methods are appropriately accounting for the correlation structure and maintain the same empir-



(a) $\rho = -0.5$

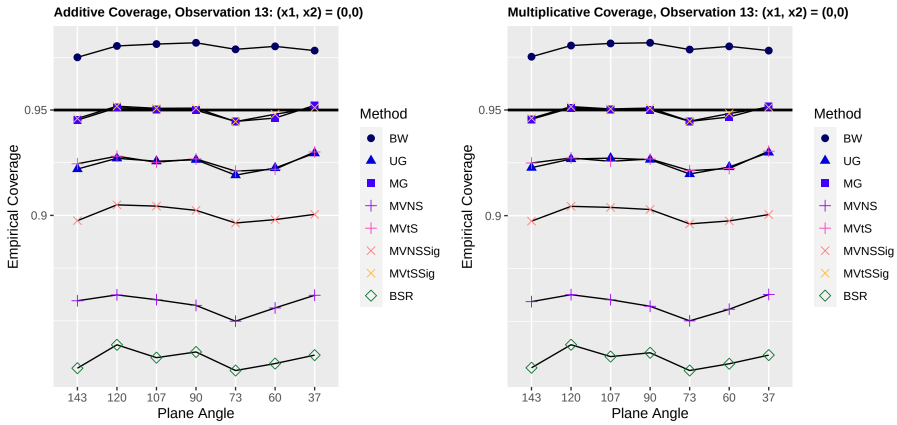


(b) $\rho = 0$

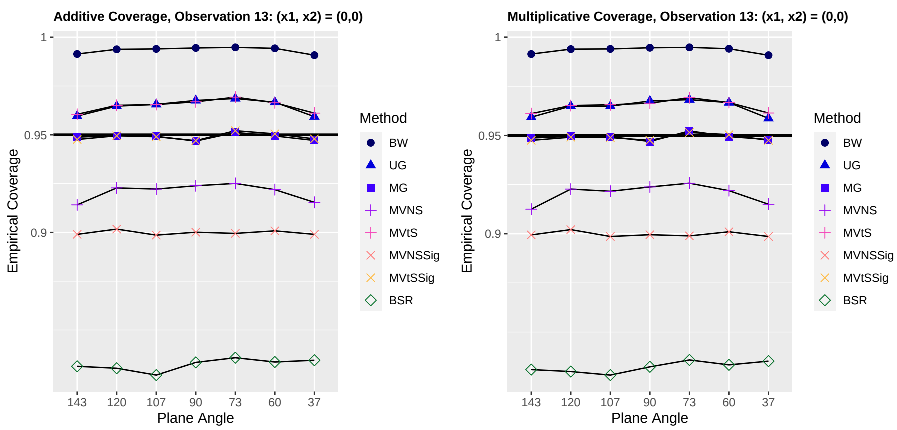


(c) $\rho = 0.5$

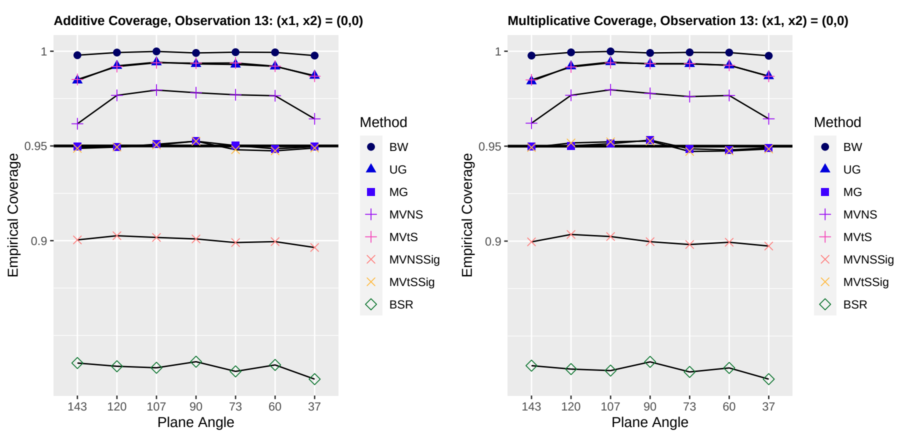
Figure 9. Additive and Multiplicative Plots for Empirical Coverage at Observation 1 (Max/Min Constant Correlation)



(a) $\rho = -0.5$



(b) $\rho = 0$



(c) $\rho = 0.5$

Figure 10. Additive and Multiplicative Plots for Empirical Coverage at Observation 13 (Max/Min Constant Correlation)

ical coverage across all plane angles unlike the univariate counterparts; However, the MVNSSig and BSR methods do not obtain the specified empirical coverage.

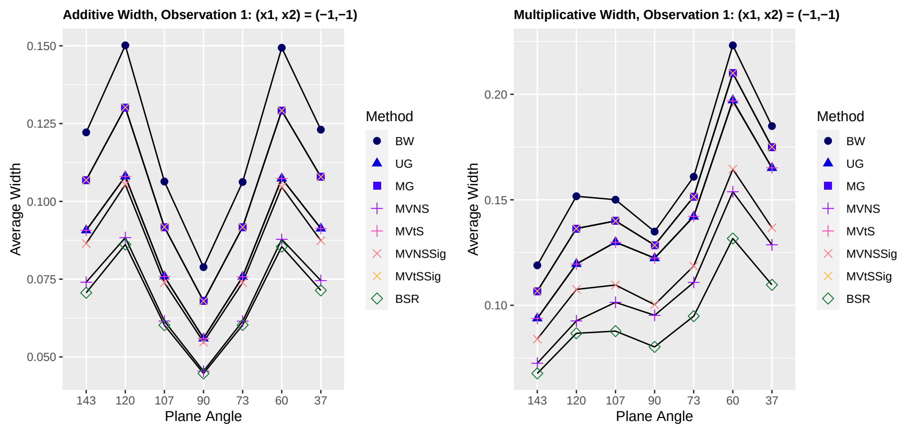
Figure 10 is similar to Figure 9 but is at observation 13 which it is the center of the design, $(x_1, x_2) = (0, 0)$. This figure shows that the center point and corner point have similar results in that the empirical coverage of the univariate methods increase as correlation increases, however, they are less affected by the change in the plane angles.

Figure 11 is the average width at each correlation level for all the plane angles. The information presented here reveals that average width is changing as the plane angle changes for each correlation level. It would seem that the hat matrix $\boldsymbol{H}$ changes the average width appropriately based on the amount of leverage exerted on a specific observation at each plane angle. Also, note that in opposite to the empirical coverage plots, the multivariate methods are adjusting average width to accommodate their constant empirical coverage across each of the correlation levels. This will be seen in greater detail in Section 4.1.2.

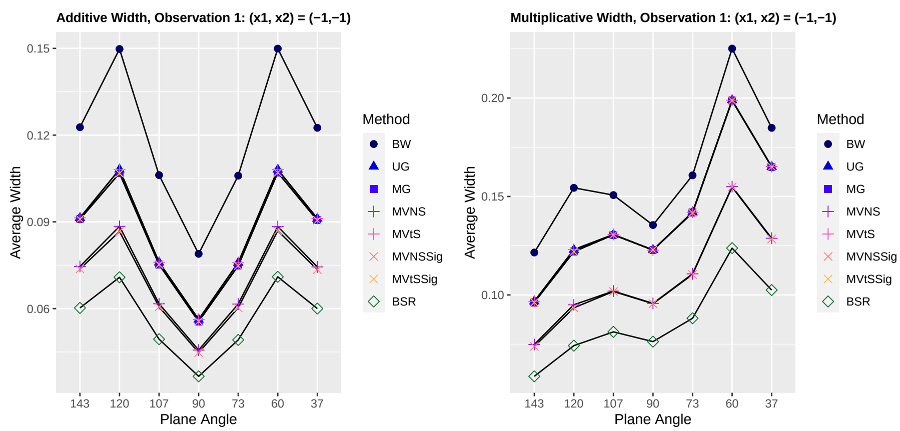
4.1.2 Max/Min Same Plane Angles with Differing Correlation

The plane angles that are considered here are $\boldsymbol{\theta} = (120, 90, 60)$ paired with each of the correlation levels. Once again, the empirical coverage is covered first followed by width.

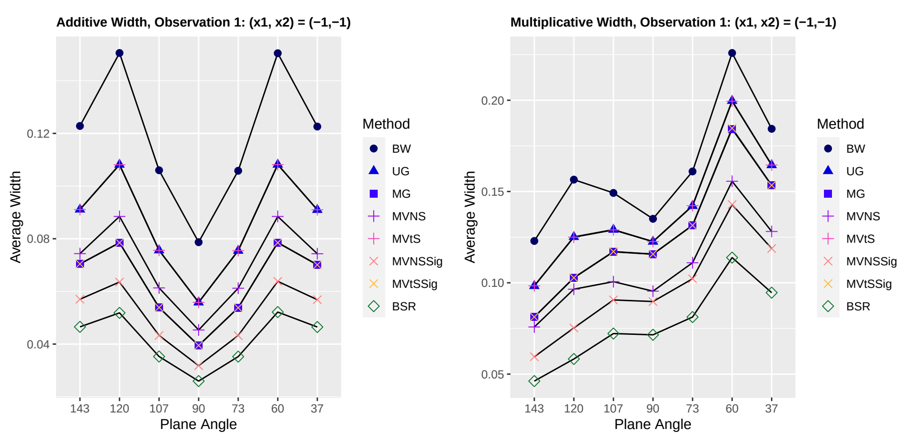
Figure 12 shows the empirical coverage plots for the additive and multiplicative desirability index for observation 1. These examples agree completely with the results in Section 4.1.1. The empirical coverage for each level of correlation between responses changes drastically within each plane angle but is remarkably similar across different plane angles. Figure 13 shows the results for the center point which also agree. For both observations, the empirical coverage for the univariate methods is below the



(a) $\rho = -0.5$



(b) $\rho = 0$



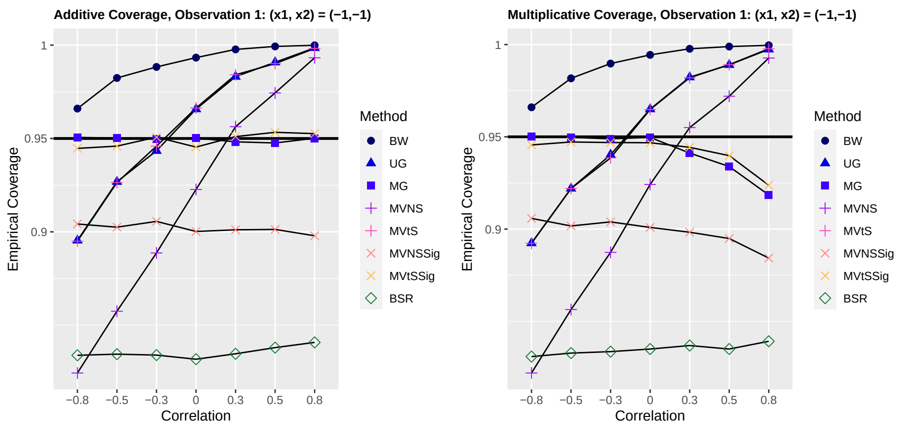
(c) $\rho = 0.5$

Figure 11. Additive and Multiplicative Plots for Average Width at Observation 1 (Max/Min Constant Correlation)

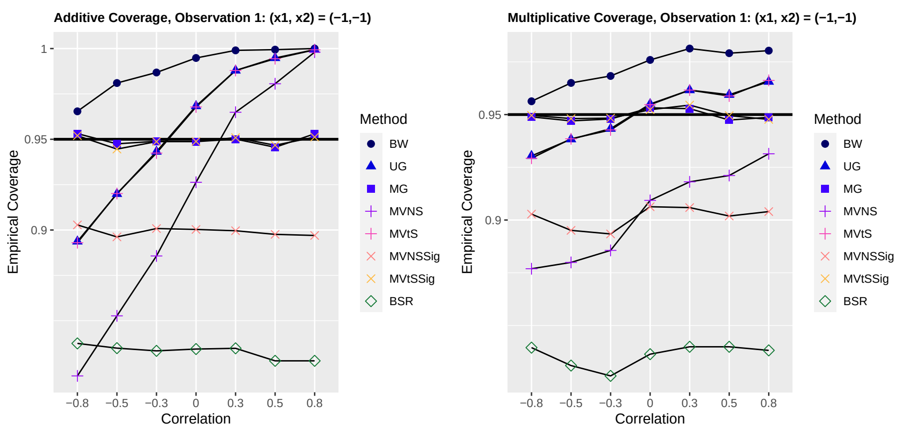
specified α, except for the conservative BW method, but increases as the correlation increases, eventually having conservative empirical coverage. By showing that the plane angle does not necessarily impact the confidence probability of a given desirability index, the angle can be ignored in first order models and more importantly, the methods should generalize to second order models. Additionally, notice that when the plane angle is 120 in Figure 14a the empirical coverage of MG and MVtSSig lessens as correlation increases for the multiplicative form. This directly relates to the likelihood of a point being optimal given the location of said solution in X-space. At the solution $\boldsymbol{x}_1 = (-1, -1)$ for $\theta_2 = 120$, this is the least optimal location. As the correlation increases, the solution becomes even less desirable because as one value increases, the other increases which is the exact opposite of what is desirable in a Max/Min optimization problem.

Figure 13 is the empirical coverage at the center point, observation 13. At this location when prediction variance is lower and the solution space is much more neutral in terms of extreme optimal points for either Y_1 or Y_2, the empirical coverage is well maintained for the multivariate methods but not for the univariate methods. This shows a clear pattern that the univariate methods are severely affected by covariance of the response surfaces that should be accounted for when conducting analysis. Again, the BW method does provide a conservative estimate here.

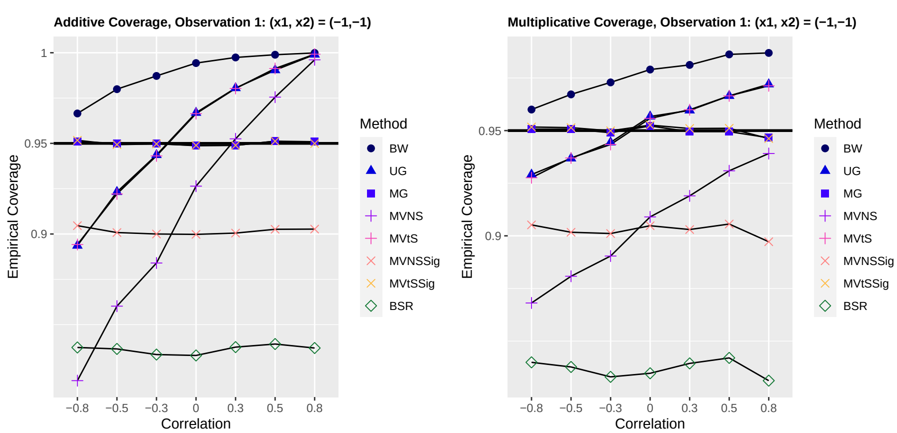
Figure 14 has the average width at observation 1. The results in this figure show an inverse pattern to that of the empirical coverage. It's important to note that the multivariate methods, are all decreasing in average width as correlation increases whereas the univariate methods remain the same. As mentioned in Section 2.6, empirical coverage and average width are trade-off values. Because the multivariate methods maintain empirical coverage, their average width changes with correlation. Since the univariate methods do not account for correlation, they do not change



(a) $\theta = 120$

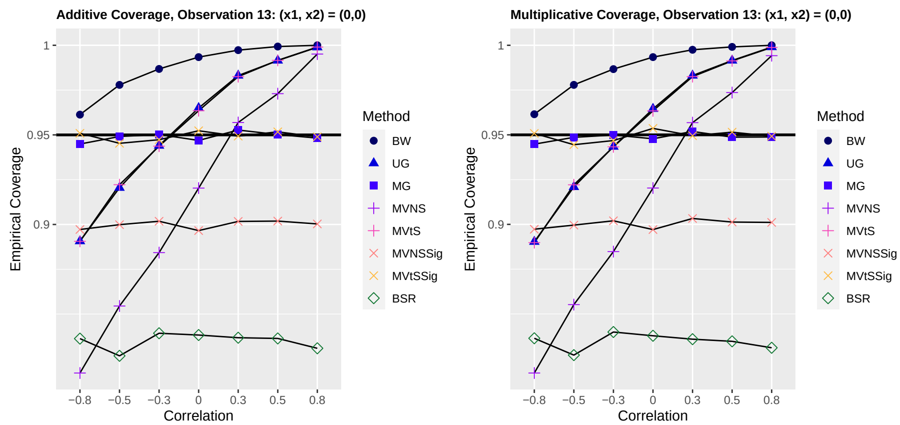


(b) $\theta = 90$

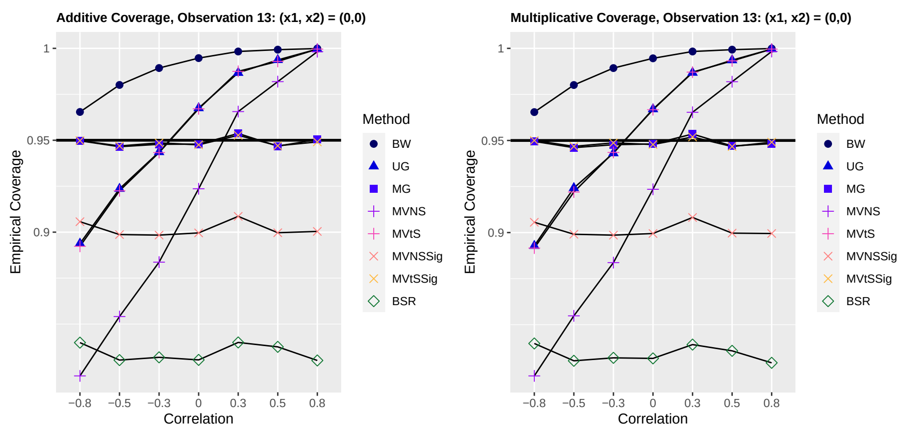


(c) $\theta = 60$

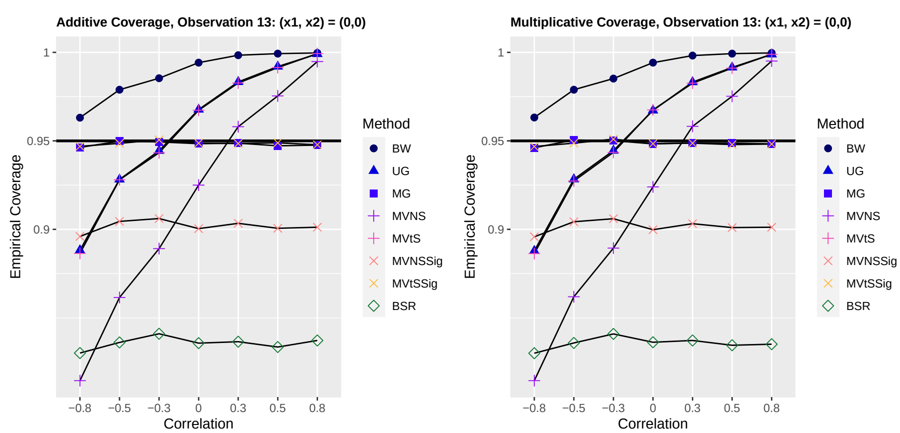
Figure 12. Additive and Multiplicative Plots for Empirical Coverage at Observation 1 (Max/Min Constant Plane Angles)



(a) $\theta = 120$

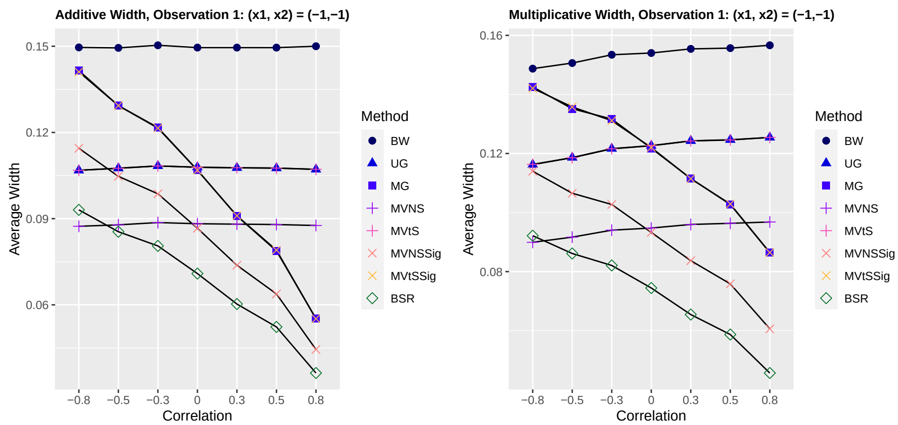


(b) $\theta = 90$

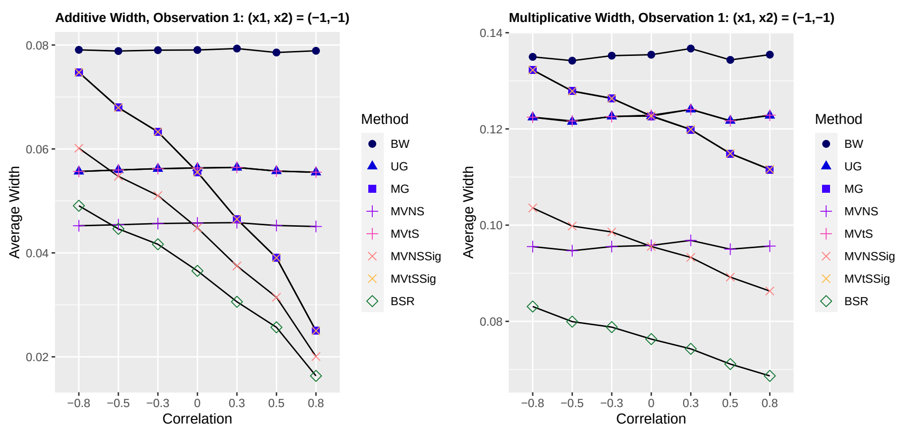


(c) $\theta = 60$

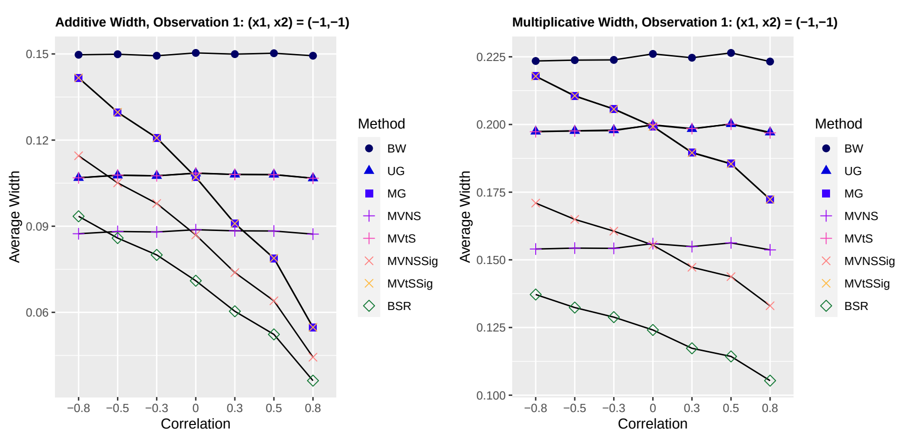
Figure 13. Additive and Multiplicative Plots for Empirical Coverage at Observation 13 (Max/Min Constant Plane Angles)



(a) $\theta = 120$



(b) $\theta = 90$



(c) $\theta = 60$

Figure 14. Additive and Multiplicative Plots for Average Width at Observation 1 (Max/Min Constant Plane Angles)

average width size to accommodate a necessary change in empirical coverage.

4.1.3 Max/Min Matching Plane Angle and Correlation

In this scenario, the plane angle simply matches with the correlation level since the plane angle is less of a concern. For example, at correlation $\rho = -0.8$, the plane angle is $\theta = 143$, at correlation $\rho = 0$, the plane angle is $\theta = 90$. Figure 15 shows the empirical coverage and average width for each correlation level for the additive and multiplicative forms at observation 1. Additional tables and plots for all 25 observations for this scenario are available in Appendices A and C, respectively. All of the observations behaved similarly, requiring only the analysis of a single observation here.

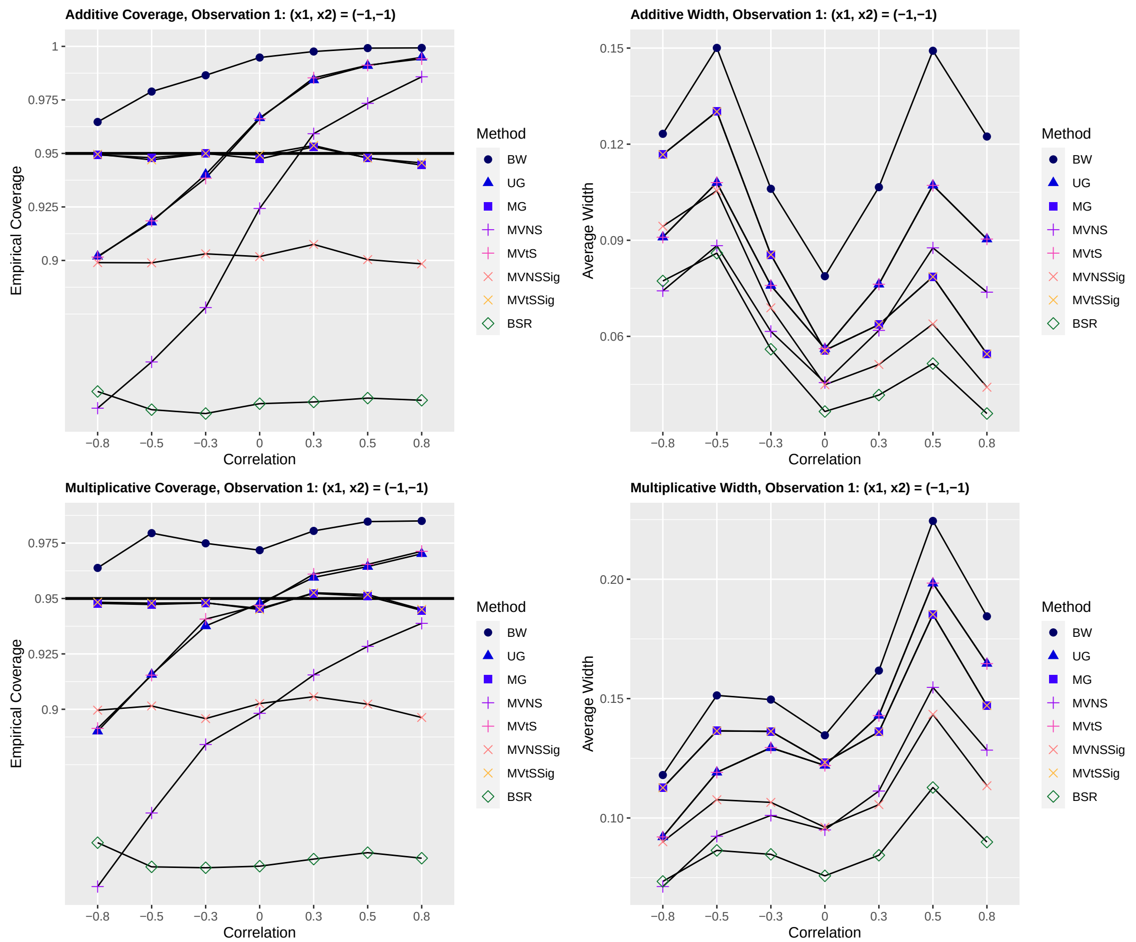


Figure 15. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 1 (Max/Min Matching)

In Figure 15, the empirical coverage shown on the left follows the same pattern as when the plane angles were held constant with changing correlation. Notice that as the correlation increases, the univariate methods empirical coverage gradually grows while the multivariate methods remain the same. At the same time, the average width on the right side of these figures is changing. When the correlation is negative, the multivariate methods MG and MVtSSig have the second largest average width just slightly smaller than the very conservative BW method. As the correlation increases to 0, the univariate and multivariate methods that correspond to each other–UG and MG, MVN and MVNSSig, MVt and MVtSSig–obtain the same width. Then as the correlation further increases to positive correlation, the multivariate methods continue to adjust their average width size and become smaller than the univariate methods to maintain empirical coverage which further agrees with what has been mentioned in previous sections. This can also be seen for the additive form in Table 3 which has empirical coverage and average width with respect to the inferential methods and $\boldsymbol{\rho}$.

				Empirical Coverage				
$\boldsymbol{\rho}$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	**0.9647**	0.9019	**0.9493**	0.831	0.9016	0.899	**0.9496**	0.8388
-0.5	**0.9789**	0.9179	**0.948**	0.8526	0.9185	0.8989	**0.947**	0.8303
-0.3	**0.9865**	0.9402	**0.95**	0.878	0.9384	0.9031	**0.95**	0.8285
0.3	**0.9976**	**0.9843**	**0.9531**	**0.9592**	0.9853	0.9075	**0.9537**	0.8339
0.5	**0.9992**	**0.991**	**0.9479**	**0.9734**	**0.9912**	0.9004	**0.9478**	0.8357
0.8	**0.9993**	**0.9949**	**0.9446**	**0.9858**	**0.9942**	0.8984	**0.9456**	0.8347
				Average Width				
$\boldsymbol{\rho}$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.12323	0.09096	0.11676	0.07421	0.09094	0.09437	0.11676	0.07729
-0.5	0.15006	0.10801	0.13024	0.08831	0.108	0.10548	0.13015	0.08598
-0.3	0.10607	0.07587	0.0855	0.06155	0.0759	0.06891	0.08551	0.05597
0	0.07877	0.05614	0.05561	0.04556	0.05616	0.04489	0.05562	0.03654
0.3	0.10658	0.07625	0.06365	0.06187	0.07627	0.05122	0.06365	0.04168
0.5	0.14916	0.10716	0.07863	0.08764	0.10714	0.06387	0.07861	0.0515
0.8	0.12238	0.09036	0.05448	0.0738	0.09038	0.04412	0.05449	0.03591

Table 3. Additive Desirability Inference Observation 1, (x_1, x_2) = (-1,-1) Empirical Coverage and Average Width

Figure 16 shows the average symmetry of the confidence interval for the additive and multiplicative forms where a positive average symmetry means the true parameter was above the confidence interval more than it was below. Once again, across all observations the additive form had excellent average symmetry of all inferential methods containing the true parameter. The multiplicative form was also very symmetric; However, the true parameter was slightly more often above the true parameter for UG, MVNS and MVtS. In the case of desirability function, having a positive average symmetry is a good thing because it means that the upper bound of the confidence interval is conservative. In other words, a positive average symmetry will help ensure that the chosen solution is at or above a confidence interval making it more robust to poor decisions due to uncertainty. With that said, note that that value on the y-axis is the percentage of times, in decimal format, that the true parameter was above the interval versus below the interval the out of all $G = 10,000$. The value of 0.0225 means that the true parameter was above the interval 225 simulations more than below the interval. Ideal average symmetry for 95% confidence intervals in $G = 10,000$ simulations would be when the true parameter is above the interval 250 times and below the interval 250 times, resulting in a average symmetry of 0.

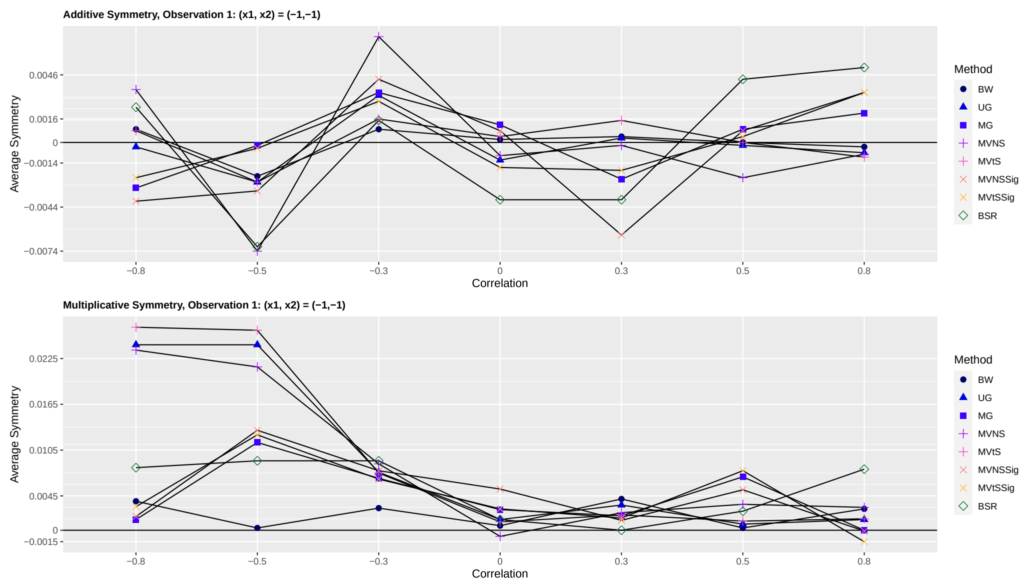


Figure 16. Additive and Multiplicative Plots for Average Symmetry at Observation 1 (Max/Min Matching)

4.1.4 Max/Tgt Matching Plane Angle and Correlation

This last first order scenario changes the response objective from a simple Max/Min problem to a Max/Tgt problem where the correlation and plane angle are still matching. This scenario is meant to increase complexity as the Max/Min scenarios result in a monotonic desirability function. In a Max/Tgt problem, there is a discontinuity at the target value for the target objective. More importantly, the optimal value for Y_2 is achievable whereas the desirability bounds for Y_1 are set necessarily far from the upper and lower bound of the data itself. Figure 17 shows the empirical coverage and average width of the additive and multiplicative forms of the desirability for observation 1. The immediate concern with this scenario is that there is a drop in empirical coverage for all methods when $\rho = -0.3$ except for BW.

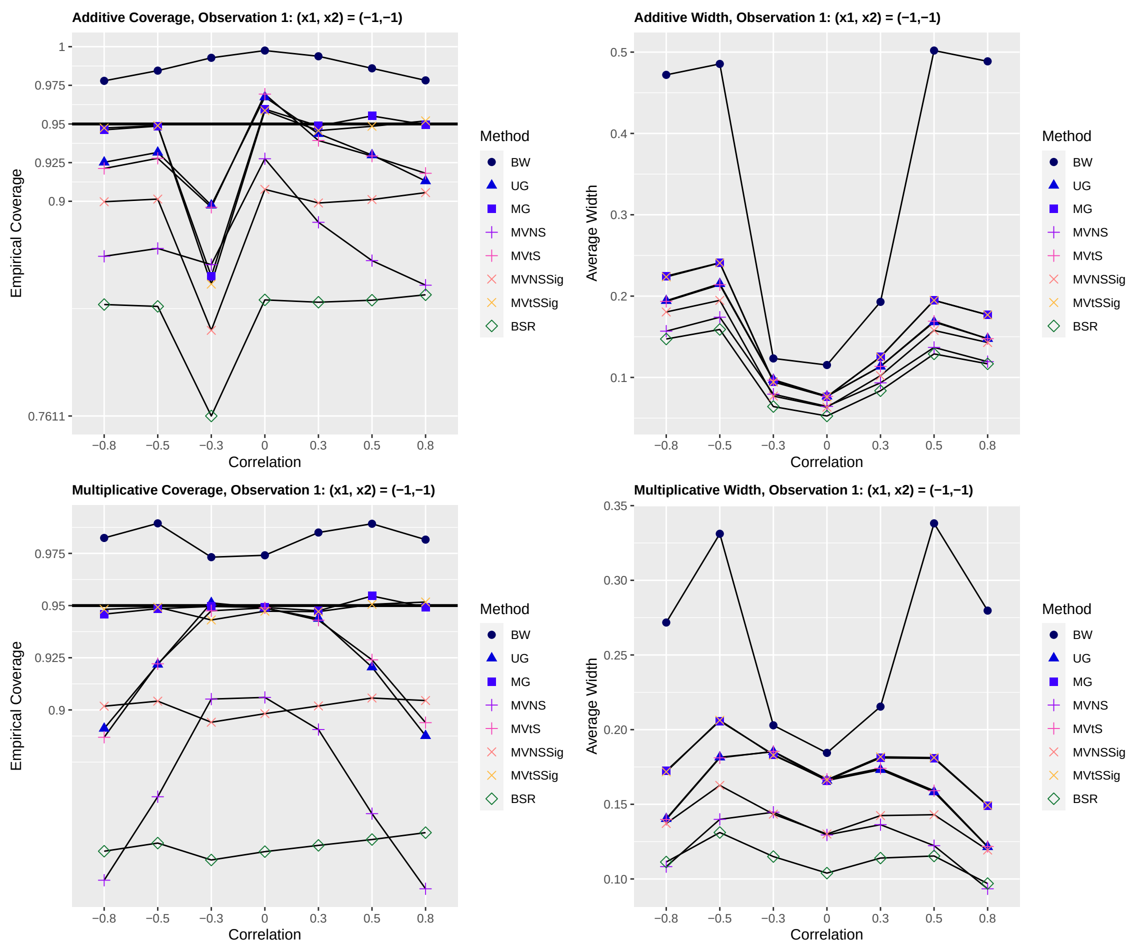


Figure 17. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 1 (Max/Tgt Matching)

Recall Section 3.3.3, the target for each of the correlation levels are chosen arbitrarily as the 60th quantile of the true response plane. This corresponds to (11.6, 11.35, 11.7, 12, 11.7, 11.35, 11.6) for each correlation level in $\boldsymbol{\rho}$, respectively. Referencing the first order contour plots Figure 5, it should be noted that the sudden drop in empirical coverage when $\rho = -0.3$ for observation 1 corresponds to the contour plots with angle 107. This observation at $(x_1, x_2) = (-1, -1)$ is approximately on the contour where the target value is 11.7. This issue disappears in observation 2 which can be seen in Figure 18. The response values for Y_2 at observation 2 at $(x_1, x_2) = (-0.5, -1)$ does not match the target value for any of the correlation levels.

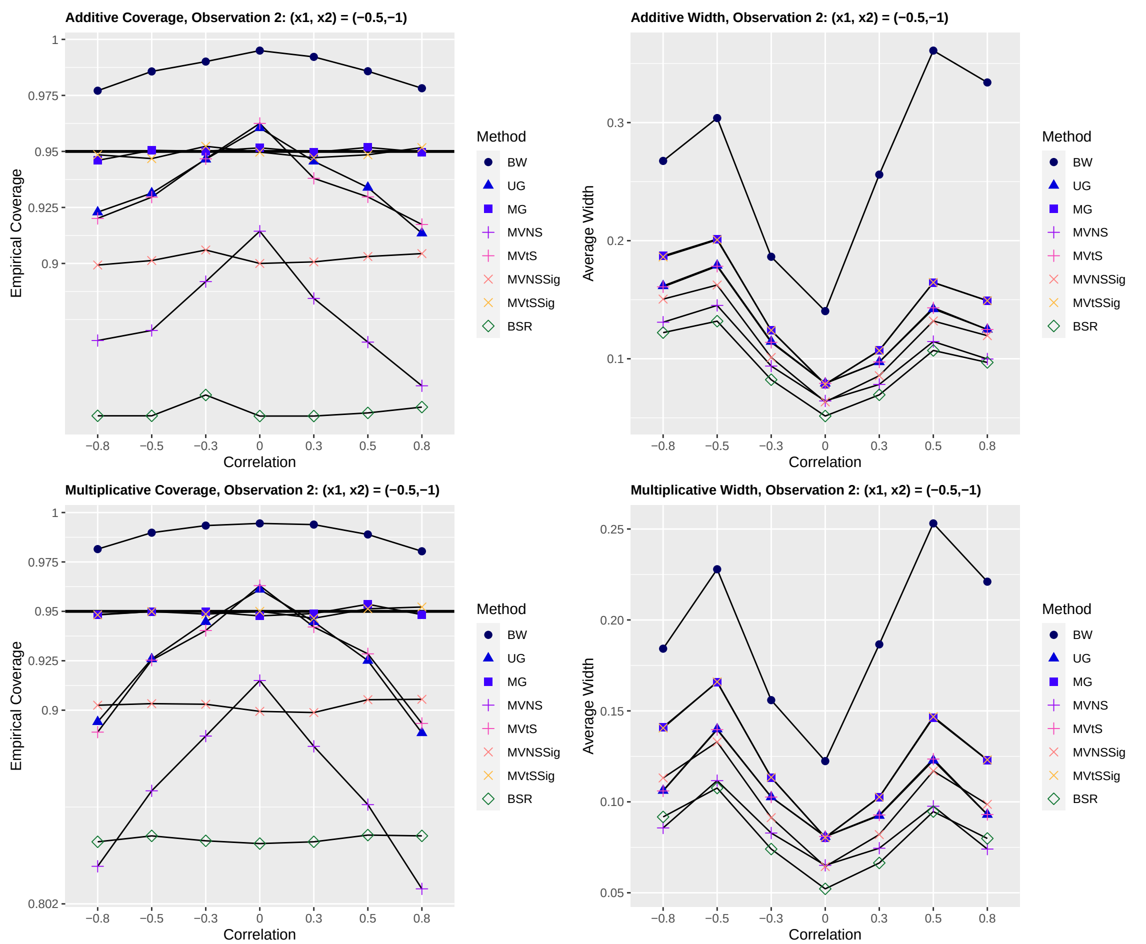


Figure 18. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 2 (Max/Tgt Matching)

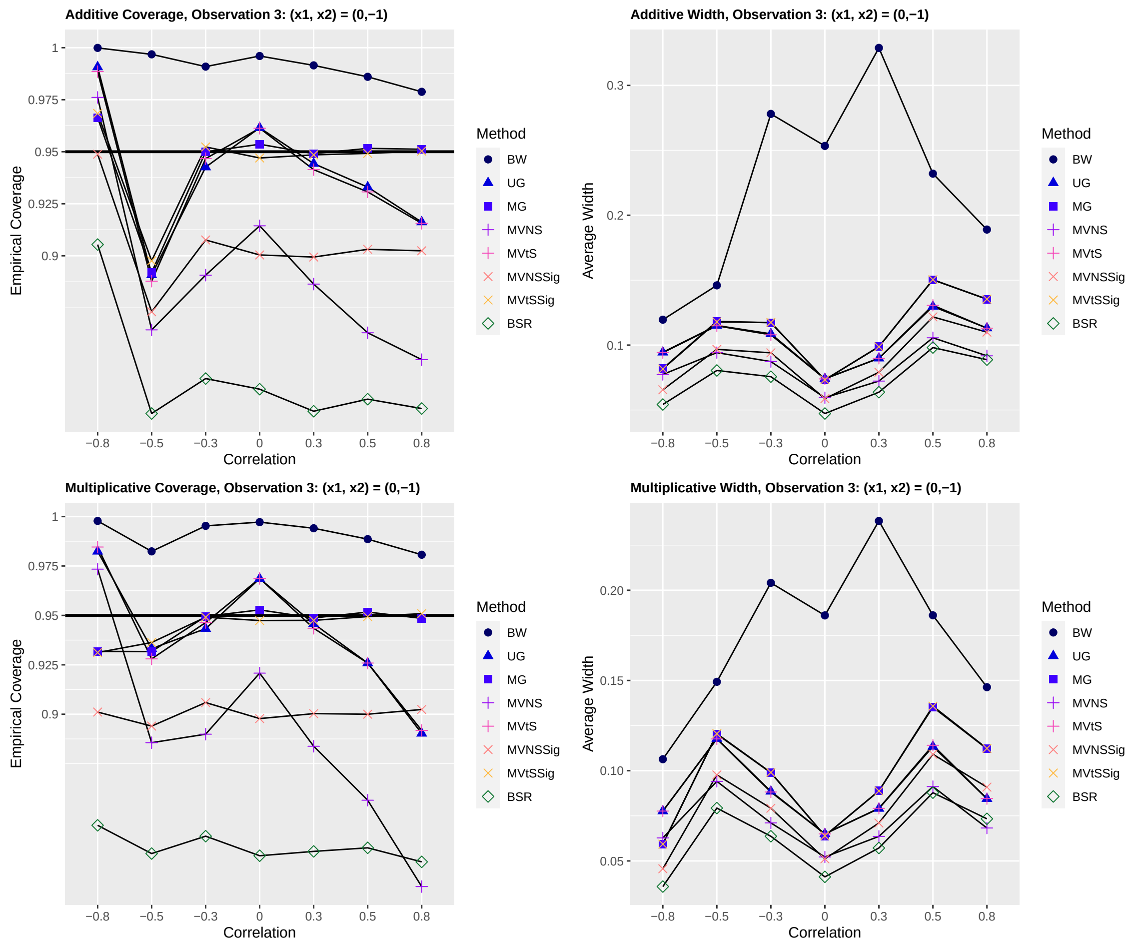


Figure 19. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 3 (Max/Tgt Matching)

The issue returns at observation 3, $(x_1, x_2) = (0, -1)$ when $\rho = -0.5$ as seen in Figure 19. This correlation corresponds with the plane angle 120 and target value of 11.35. Comparing with the contour plots once more finds that $\boldsymbol{Y}_2$ on plane angle 120 at observation 3 once again intersects with the target value of 11.35. This is confirmed for other observations where the drop in empirical coverage occurs. While this limitation is not ideal, upon initial inspection, the average symmetry plots help showcase what is happening in Figures 20-22.

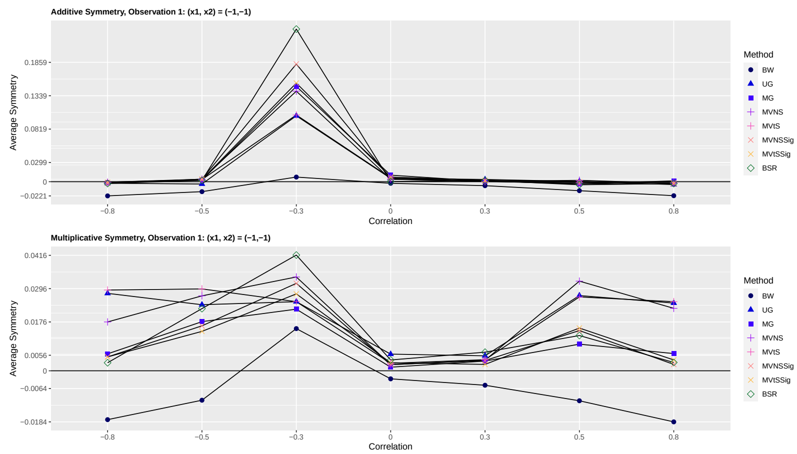


Figure 20. Additive and Multiplicative Plots for Symmetry at Observation 1, Max/Tgt

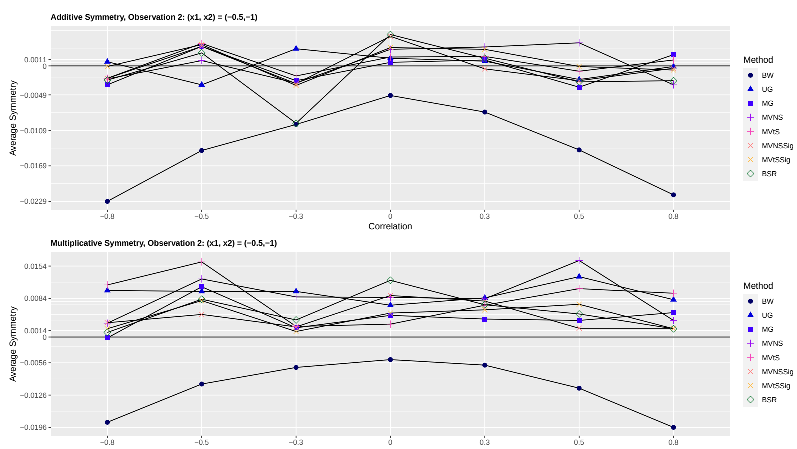


Figure 21. Additive and Multiplicative Plots for Symmetry at Observation 2, Max/Tgt

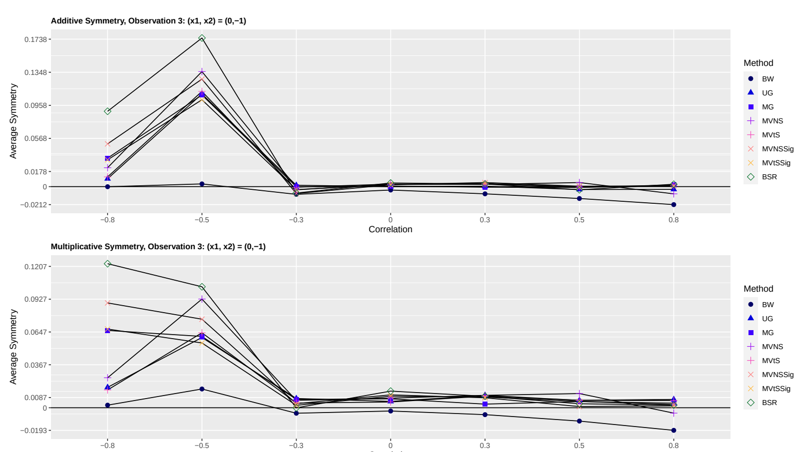


Figure 22. Additive and Multiplicative Plots for Symmetry at Observation 3, Max/Tgt

Figure 20 shows that when the solution corresponds with the most optimal match target objective, the average symmetry skews positive significantly. This means that the true parameter is above the interval much more often than not. Essentially, since the empirical optimal value for Y_2 is near the true optimal, the confidence interval is capturing the value less as it's near the most upper bound of possible values. This is also seen by the different in average symmetry for additive vs multiplicative in this figure. When the additive form is use, the average symmetry is skewed much more than the multiplicative form. This can be explained by the fact that the additive form allows one objective to overshadow the others if it is near optimality, so the true optimal value is often very large and outside of the confidence bounds. The multiplicative form sacrifices the optimality of a single objective in favor of all objectives being more equally optimized so the average symmetry skew is much less severe. Once again, as long as the average symmetry is skewed positively, this is a benefit to the decision maker as the optimal value will be at least as good as the confidence bound indicates if not better. Figure 21 indicates no major issues with average symmetry which agrees with the empirical coverage and average width plots where there are no severe decreases in empirical coverage. Figure 22 helps support the case made prior. Once again, when $\rho - 0.5$, the empirical coverage decreases for the novel methods because that solution corresponds to a true optimal point for Y_2 when the plane angle is $\theta = 120$. In Figure 22, the average symmetry once again skews positively for at $\rho = -0.5$ accounting for the loss in empirical coverage. The BW method is less affected by this issue since the empirical coverage tends to be quite conservative so it still captures the true parameter in most simulations.

4.2 Second Order Models

The first order models results provided excellent insights into the response surface structure; However, the goal is to construct adequate confidence intervals since they are more commonly used in RSM. There are three scenarios that are addressed in this section. First is a Max/Min problem where Y_1 and Y_3 are respectively the yield and molecular weight of the chemical process problem. The second is a Max/Tgt problem where Y_1 and Y_2 are respectively the yield and visocity of the chemical process problem. The third scenario considered is the full chemical process Max/Tgt/Min problem. Recall that the design matrix for these scenarios is a CCD with 5 center points providing uniform precision, therefore prediction variance is highest in the corner points and lowest in the center. These scenarios also incorporate the weights and linearity settings discussed in Section 3.3.2. Each solution discussed mentions which objective is preferred based on the contour plots in Figure 6.

4.2.1 Max/Min Surface Models

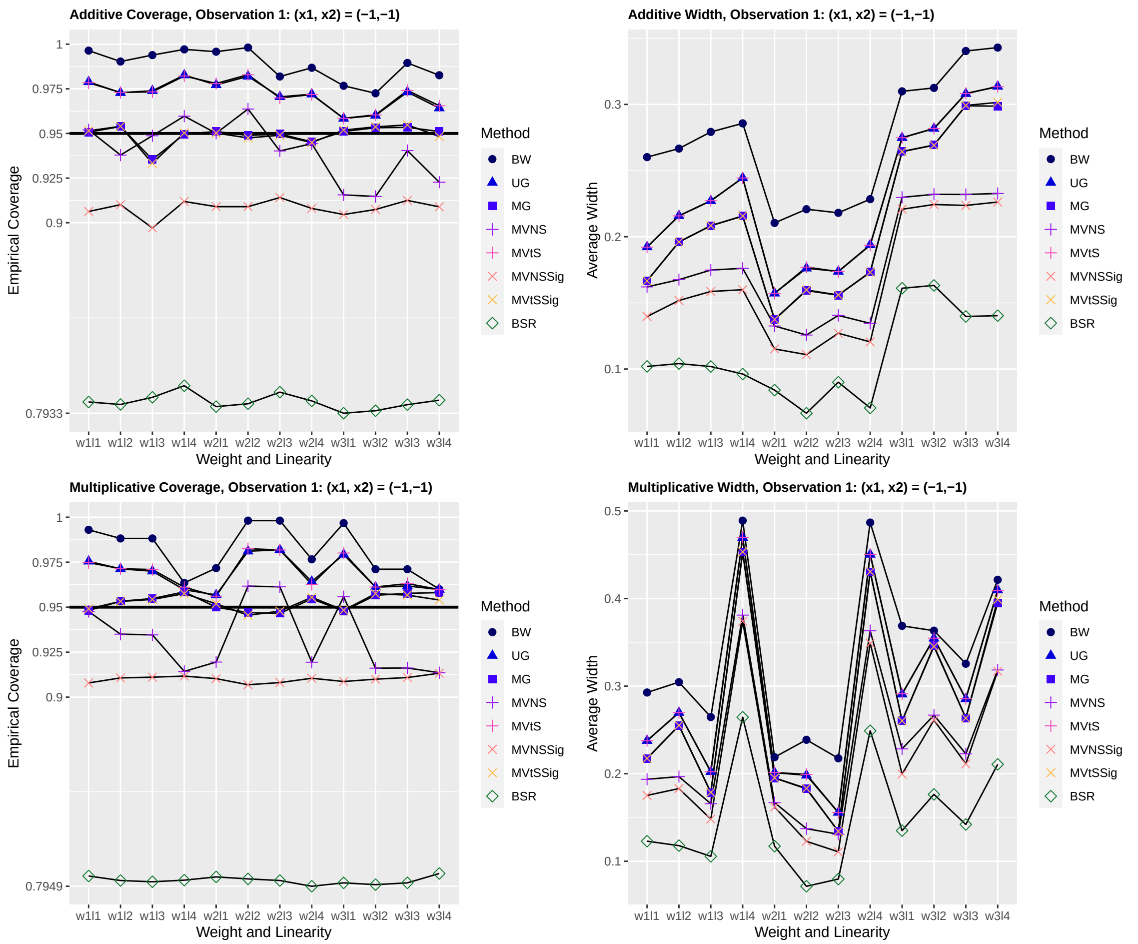


Figure 23. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 1 (Max/Min Second Order)

Akin to the first order Max/Min scenarios, this scenario provides the simplest framework to support the inferential methods generalizing to second order models. Figure 23 shows the empirical coverage and average width for the additive and multiplicative form for observation 1 which favors Y_3 because that's the optimal region for molecular weight. In the additive form, the BW, UG, MVtS methods all appear to be conservative for all weight and linearity combinations. The MG and MVtSSig appear to meet the approximate 95% empirical coverage very well. The MVNSSig and BSR methods both fail to adequately capture the desirability index, however their empirical coverage remains consistent. Lastly, the MVNS method performs well at the simplest $\boldsymbol{w}_i, \boldsymbol{l_j}$ combination, but is very erratic otherwise. The same is true for the multiplicative form.

MG and MVtSSig slightly outperform UG and MVtS with respect to average width. While these changes in average width appear to be small though, it's important to note that the Derringer and Suich desirability index is bound by [0,1]. With such a small interval, any small change is greatly desired. There is a concern for the multiplicative average width at $\boldsymbol{w}_1, \boldsymbol{l}_4$, $\boldsymbol{w}_2, \boldsymbol{l}_4$, $\boldsymbol{w}_3, \boldsymbol{l}_4$ as the average width goes to 0.5 for the BW method and to 0.45 for the UG, MG, MVtS, and MVtSSig methods for $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ and up to 0.4 for $\boldsymbol{w}_3$. This is problematic in practice because it spans roughly half of the range of values. With that said, the issue corresponds with the linearity setting of $\boldsymbol{l}_4$ which is when the emphasis is changed the most on both responses at the same time. At this linearity, $\boldsymbol{l}_4 = (0.1, 10)$, which means that there's little emphasis for Y_1 to achieve the target and high emphasis for Y_3 to achieve the target. The emphasis settings are likely why there is larger average width at $\boldsymbol{w}_1 = (0.5, 0.5)$ and $\boldsymbol{w}_2 = (0.8, 0.2)$ when Y_1 is more important. The average width at $\boldsymbol{w}_3 = (0.2, 0.8)$ is still large but at least slightly smaller than the other two. Avoiding large emphasis for multiple objectives should be avoided if implementing these inferential methods.

Figure 24 shows the empirical coverage and average width for the additive and multiplicative form for observation 13 which favors Y_1 as that's near the optimal region for yield. Recall the that prediction variance is the lowest at the center point for this design due to it's number of center points. The prediction variance appears to greatly affect empirical coverage and average width of all the inference methods. The empirical coverage of MG and MVtSSig becomes more consistent and maintains near exact empirical coverage for all $\boldsymbol{w}_i, \boldsymbol{l}_j$. In addition to the empirical coverage

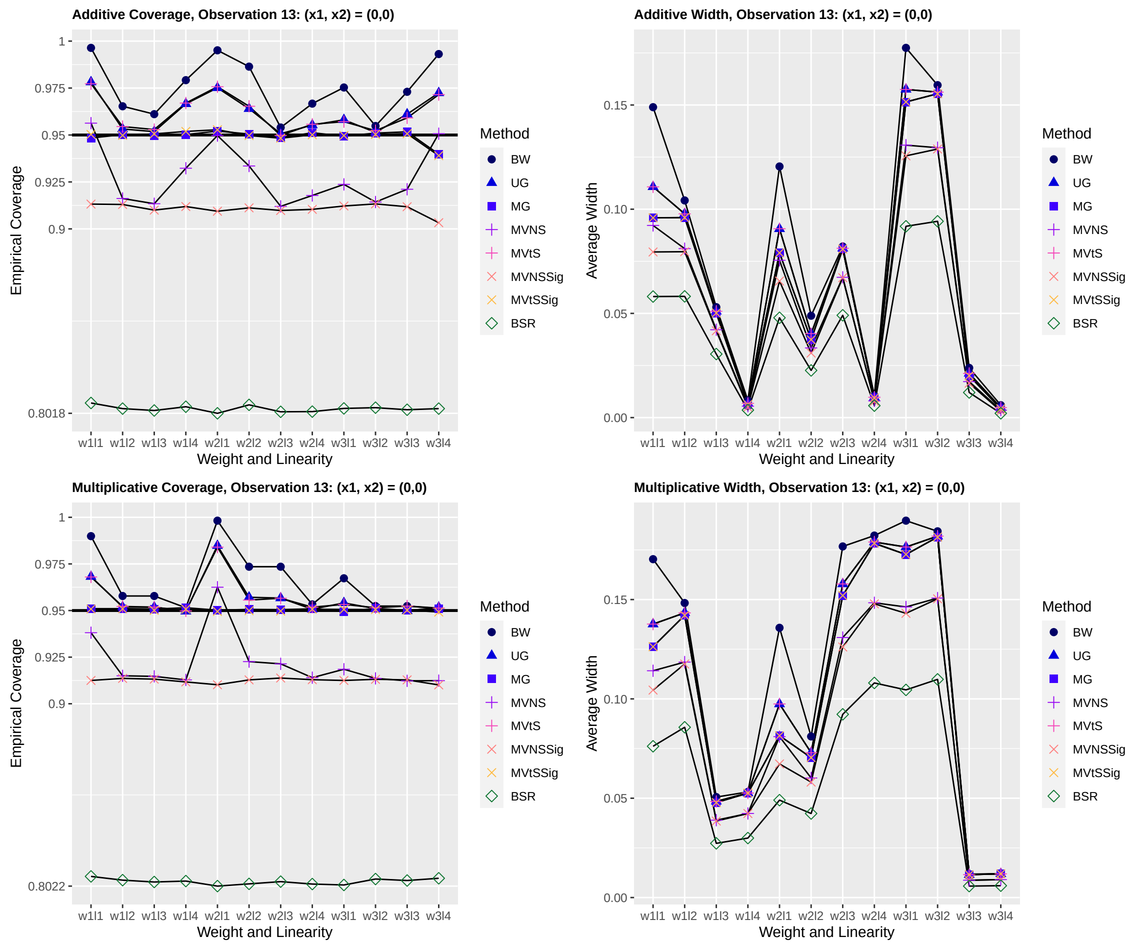


Figure 24. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 13 (Max/Min Second Order)

being maintained, the average width is considerably smaller than the corner point observation. A pattern with this observation is more apparent with the BW, UG, MVNs, and MVtS methods. $\boldsymbol{w}_i$ has a smaller effect on the empirical coverage and average width than $\boldsymbol{l}_j$ as the lines appear slightly periodic in all four of the plots in Figure 24.

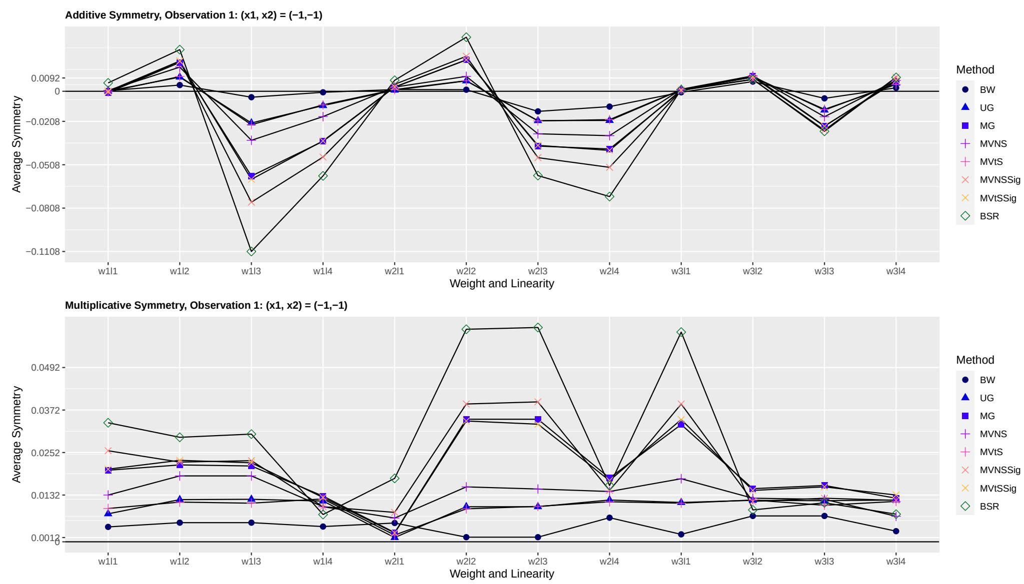


Figure 25. Additive and Multiplicative Plots for Average Symmetry at Observation 1 (Max/Min Second Order)

Figure 25 is the average symmetry plot for observation 1. This plot supports that the weights do not have much of an impact as they attain similar values across the different linearity parameters. The linearity parameter impacts the average symmetry much more, but the values are still very small for both additive and multiplicative forms so the intervals are still adequately capturing true parameter.

4.2.2 Max/Tgt Surface Models

This scenario is serves as a stepping stone to the three objective problem to try to dissect any potential issues in a simpler case. The importance of prediction variance is showcased here as the solutions are less chaotic with solutions near the center of the design. Figure 26 shows the empirical coverage and average width for observation 1. At this solution Y_2 is preferred as it is near the optimal region for viscosity. All inferential methods at this observation struggle to maintain exact 95% confidence, but they all remain very close with MG and MVtSSig appearing to be the most reliable when considering empirical coverage and average width. The empirical coverage is

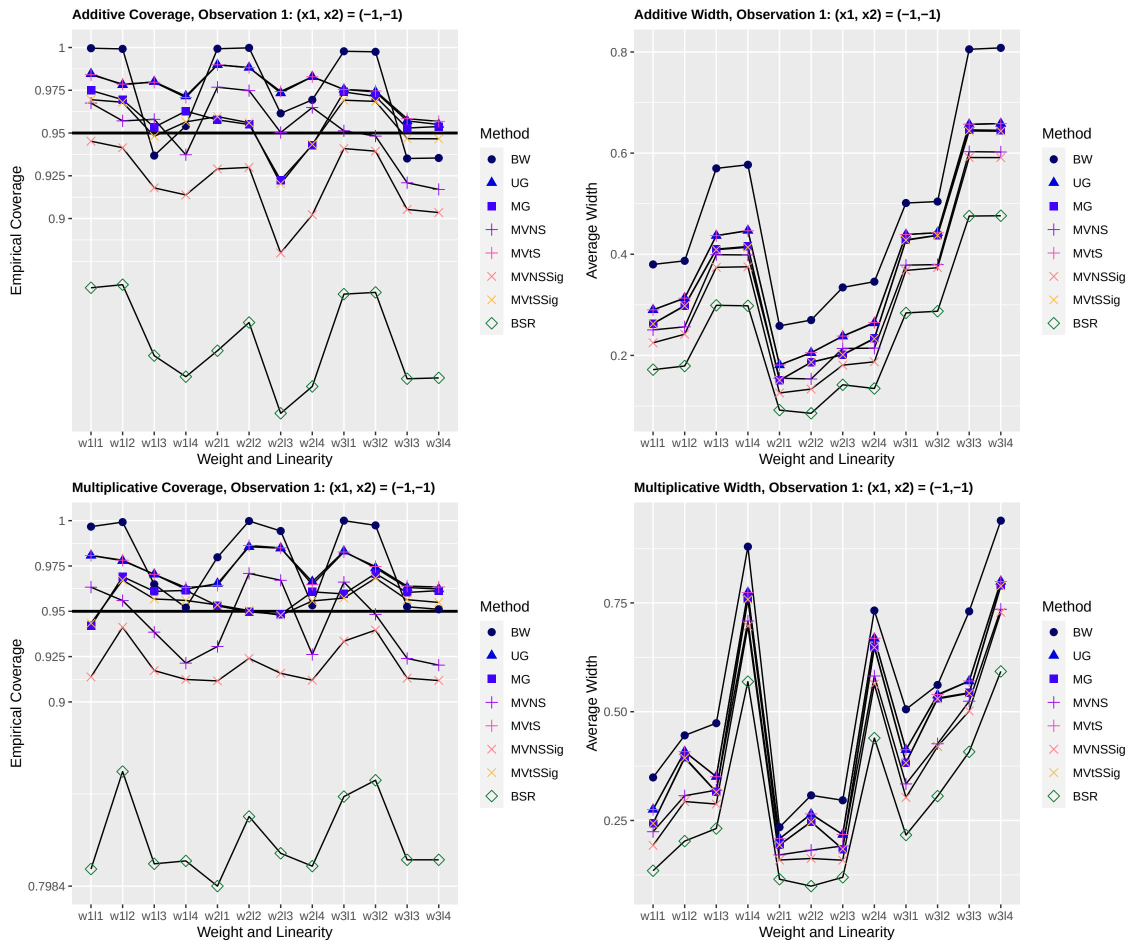


Figure 26. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 1 (Max/Tgt Second Order)

not necessarily the concern with this Max/Tgt problem. The bigger concern is the average width on each of the intervals, especially in the case of the BW method. Based on the fact that when the average width is smallest when $\boldsymbol{w}_2 = (0.8, 0.2)$ is active and when $\boldsymbol{l}_1 = (1, 1, 1)$ or $\boldsymbol{l}_2 = (0.1, 1, 1)$ it is evident that the match target objective is causing the wider intervals. The problem is not necessarily due to the match target problem, but rather the amount of emphasis placed on the objective. The emphasis is set to 10 for Y_2 which means that the solution does not become desirable unless it is very close to the target value, at which point it increases rapidly as seen in Figure 3b. The solution here is near optimal so the desirability, d_2, has a much larger range of values it can take on with each simulation resulting in larger average widths. The match target problem does appear to be affected by it more due to the fact that it's target value is within the bounds of the objective whereas the Y_1 target is larger than the largest value in the data due to the prediction intervals used.

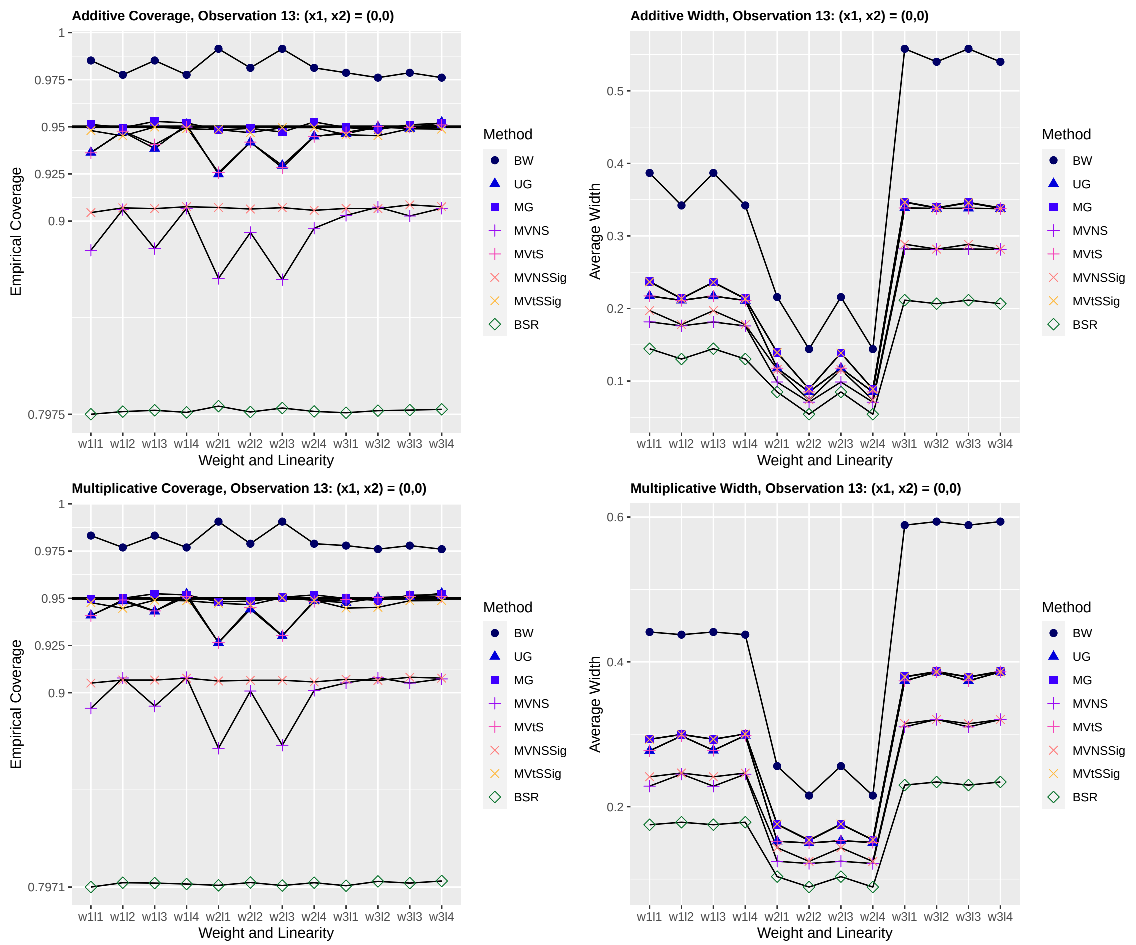


Figure 27. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 13 (Max/Tgt Second Order)

With that said, consider observation 13 in Figure 27 which favors both Y_1 and Y_2. This figure shows that when prediction variance is low, the inferential methods perform exceptionally well. The average width is reduced by nearly half across the $\boldsymbol{w}_i\boldsymbol{l}_j$ combinations and empirical coverage is easily maintained by the MG and MVtSSig methods, and to a lesser extend by the UG, and MVtS methods. The issue when using excessive emphasis in match target problems can be mitigated by adding more data points when conducting the original designed experiment to maintain empirical coverage. Using less emphasis in the DF settings would help combat issues present with large widths and low empirical coverage.

4.2.3 Max/Tgt/Min Surface Models

This scenario serves as the standard that should be met for these inferential methods to be used in real-world scenarios. All tables and plots for this scenario are in Appendix B and D. The section covers observations 1, 4, and 19 specifically as they are the optimal solutions for Y_3, Y_2, Y_1, respectively. This can be seen in Table 4. Additionally, the empirical coverage and average width are examined at observation 13 because it is the center point and observation 5 since the empirical coverage here changes drastically with linearity and weights. The optimal solution for Y_2 actually alternates between 4 and 20 depending on if $\boldsymbol{l}_3$ or $\boldsymbol{l}_4$ is active, which place extra emphasis on Y_2 being closer to optimality.

Figure 28 shows the empirical coverage and average width for the additive and multiplicative plots at observation 1 which favors Y_3, and Y_2 to a slightly lesser degree. Across the 12 settings, UG and MVtS maintain conservative empirical coverage for both the additive and multiplicative forms. The MG and MVtSSig methods maintain

Table 4. Optimal and Worst Solutions for each $\boldsymbol{w}_i\boldsymbol{l}_j$ Combination

w_i	l_j	Add Max	Mult Max	$d_1 Max$	$d_2 Max$	$d_3 Max$	Add Min	Mult Min	$d_1 Min$	$d_2 Min$	$d_3 Min$
1	1	19	3	19	4	1	21	1	1	13	25
1	2	3	2	19	4	1	25	25	1	13	25
1	3	19	19	19	20	1	21	25	1	25	25
1	4	6	6	19	20	1	25	25	1	25	25
2	1	3	3	19	4	1	1	1	1	13	25
2	2	4	3	19	4	1	23	23	1	13	25
2	3	19	19	19	20	1	21	24	1	25	25
2	4	19	7	19	20	1	24	24	1	25	25
3	1	4	4	19	4	1	21	21	1	13	25
3	2	4	2	19	4	1	24	24	1	13	25
3	3	19	19	19	20	1	25	25	1	25	25
3	4	20	8	19	20	1	25	25	1	25	25

empirical coverage across most of the $\boldsymbol{w}_i\boldsymbol{l}_j$ combinations. The additive form empirical coverage from MG and MVtSSig is below 92.5% when $\boldsymbol{w}_2 = (0.6, 0.2, 0.2)$ and when $\boldsymbol{l}_3 = (1, 10, 1, 1)$ which is a combination that favors Y_1 due to the weight and Y_2 due to the linearity parameter. A similar drop in empirical coverage was previously seen in the Max/Tgt scenario because the emphasis on Y_2 is high resulting in a wide range of values it can be as it nears the optimal value. Also, the empirical coverage for the multiplicative form using the MG and MVtSSig methods is around 92.5% when $\boldsymbol{w}_1 = (1/3, 1/3, 1/3)$ and $\boldsymbol{l}_1 = (1, 1, 1, 1)$ which favors Y_3 simply due to the optimal region of the surface indicating suboptimal values do not necessarily have good empirical coverage either. The BW, MVNSSig, and BSR methods do not perform as well as the MG and MVtSSig methods as they appear to be largely affected by the change in linearity and in the case of the latter two, do not meet the specified empirical coverage. All of the methods suffer issues with average width at observation for most of the linearity parameters which is expected based on the results from previous scenarios. Similar to the Max/Tgt second order scenario, the average width when $\boldsymbol{w}_2 = (0.6, 0.2, 0.2)$ is generally smaller than the other two weight schemes in both the additive and multiplicative forms with the exception when $\boldsymbol{l}_4 = (0.1, 10, 1, 1)$. The reason that average width increases for both the additive and multiplicative forms when $\boldsymbol{l}_3$ or $\boldsymbol{l}_4$ is active is due to the shape of the desirability function for that objective. These linearity settings place additional emphasis on Y_2 to be close to the optimal value before the desirability increases. Recall Figure 3b that when emphasis is high, the optimal value is small and stable until it gets close to optimal where it drastically increases. This solution is nearly optimal for Y_2 so the desirability function, d_{12} is at a point where the range of possible desirability values drastically increases, resulting in larger widths. This is seen to an extreme case in the multiplicative form when with $\boldsymbol{l}_4$ because in this case, both the d_{12} slope is large and the slope for d_{11} is large for the opposite reason. The emphasis here is low so since Y_1 is far from optimal, the desirability is less stable and can range from many values with small changes in the response value.

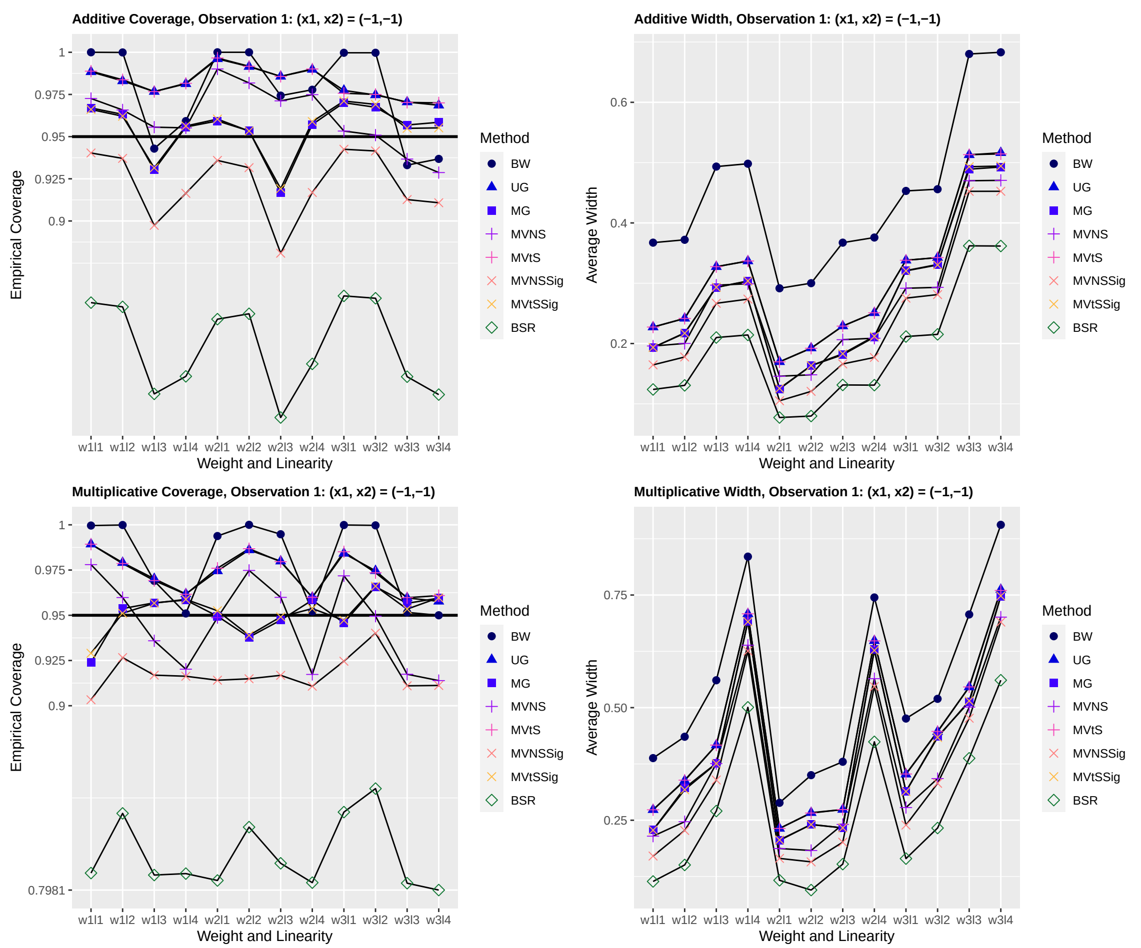


Figure 28. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 1 (Max/Tgt/Min Second Order)

Figure 29 is the empirical coverage and average width for both DF forms at observation 4. This is when Y_2, the target objective, is optimal. Similar issues to observation 1 can be seen here. When $\boldsymbol{l}_3$ or $\boldsymbol{l}_4$ are active, the emphasis placed on Y_2 being close to the target causes wider confidence bounds. Maintaining empirical coverage of multi-objective optimization when a match target objective is rather difficult because the optimal value is within the bounds of the data itself unlike maximization or minimization objectives where the target can be well outside of the possible data values.

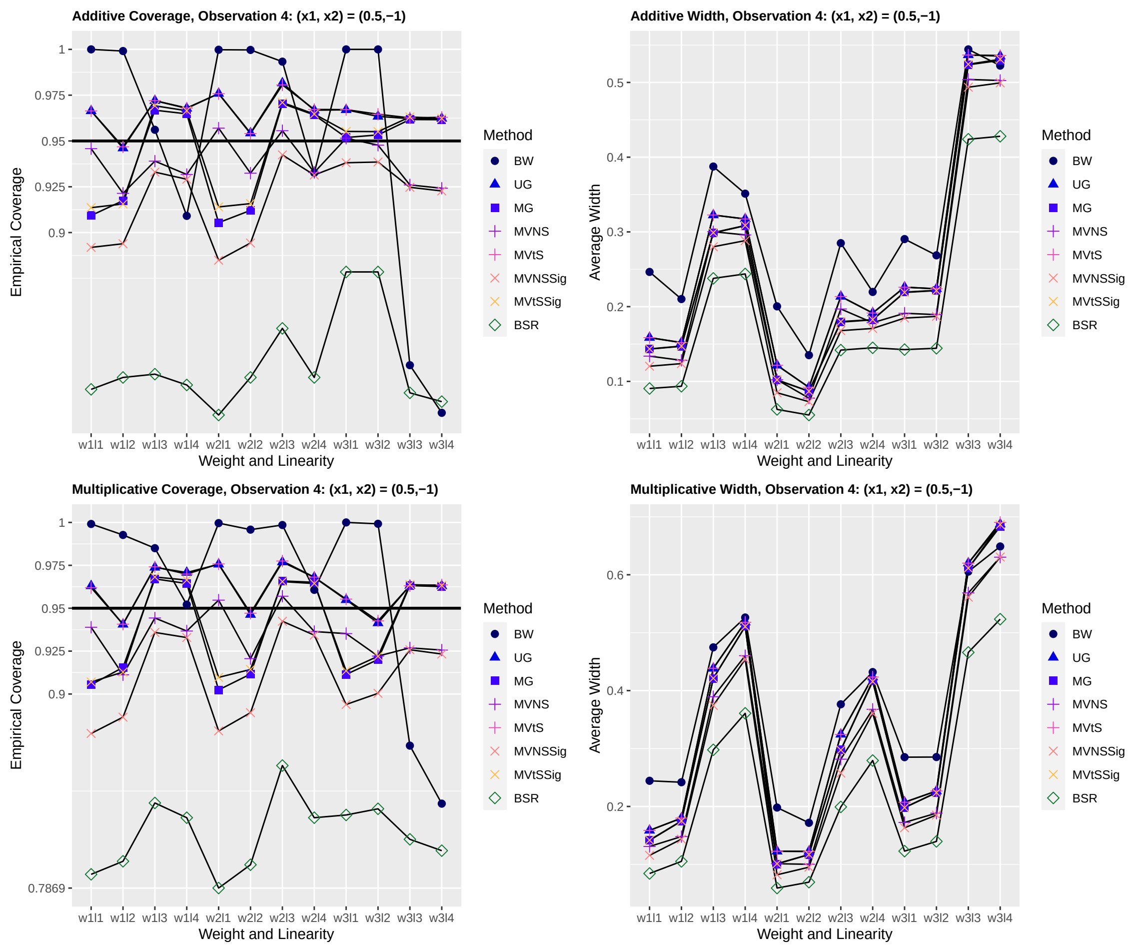


Figure 29. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 4 (Max/Tgt/Min Second Order)

With that said, looking at Figure 30 shows that in the places that empirical coverage decreases for each method, the average symmetry skews positive which means the true parameter is above the bound. If a match target objective is present, a one-sided confidence bound may be more beneficial to utilize.

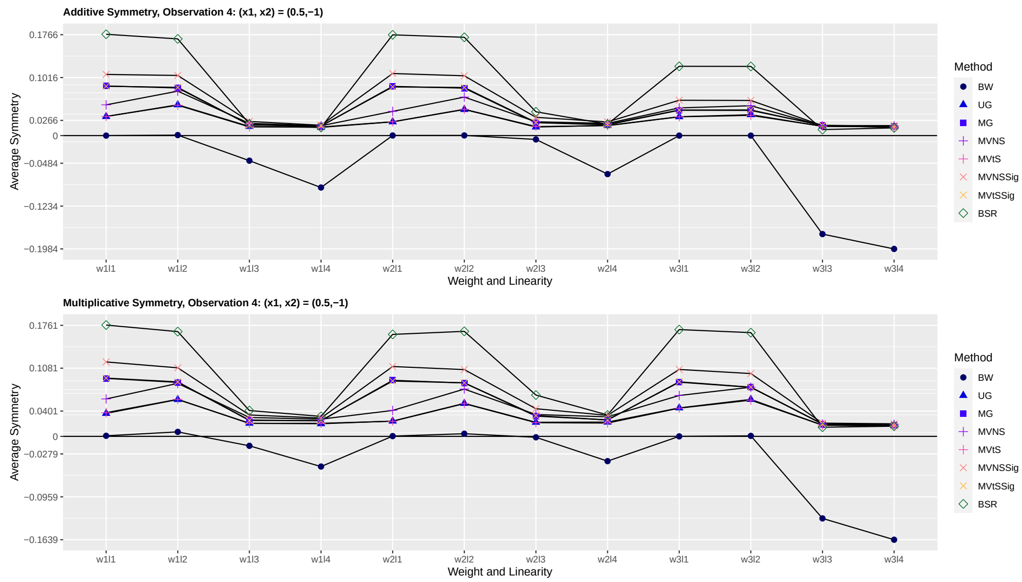


Figure 30. Additive and Multiplicative Plots for Average Symmetry at Observation 4 (Max/Tgt/Min Second Order)

Figure 31 is the empirical coverage and average width for observation 19. This is when Y_1 is optimal and when the additive and multiplicative forms are most likely to be optimal. The prediction variance at this observation is lower than that of the corner points since it is closer to the center. The empirical coverage is better maintained by most of the methods with UG, MG, MVtS, and MVtSsig all performing nearly identical. This point is also near optimal for Y_2 again which appears in when $\boldsymbol{l}_3$ and $\boldsymbol{l}_4$ are active once again but it does not perturb anything too much. The widths of the methods are all fairly respectable with the exception when $\boldsymbol{w}_3$ is active with the two latter linearity parameters but they do all remain below 0.3.

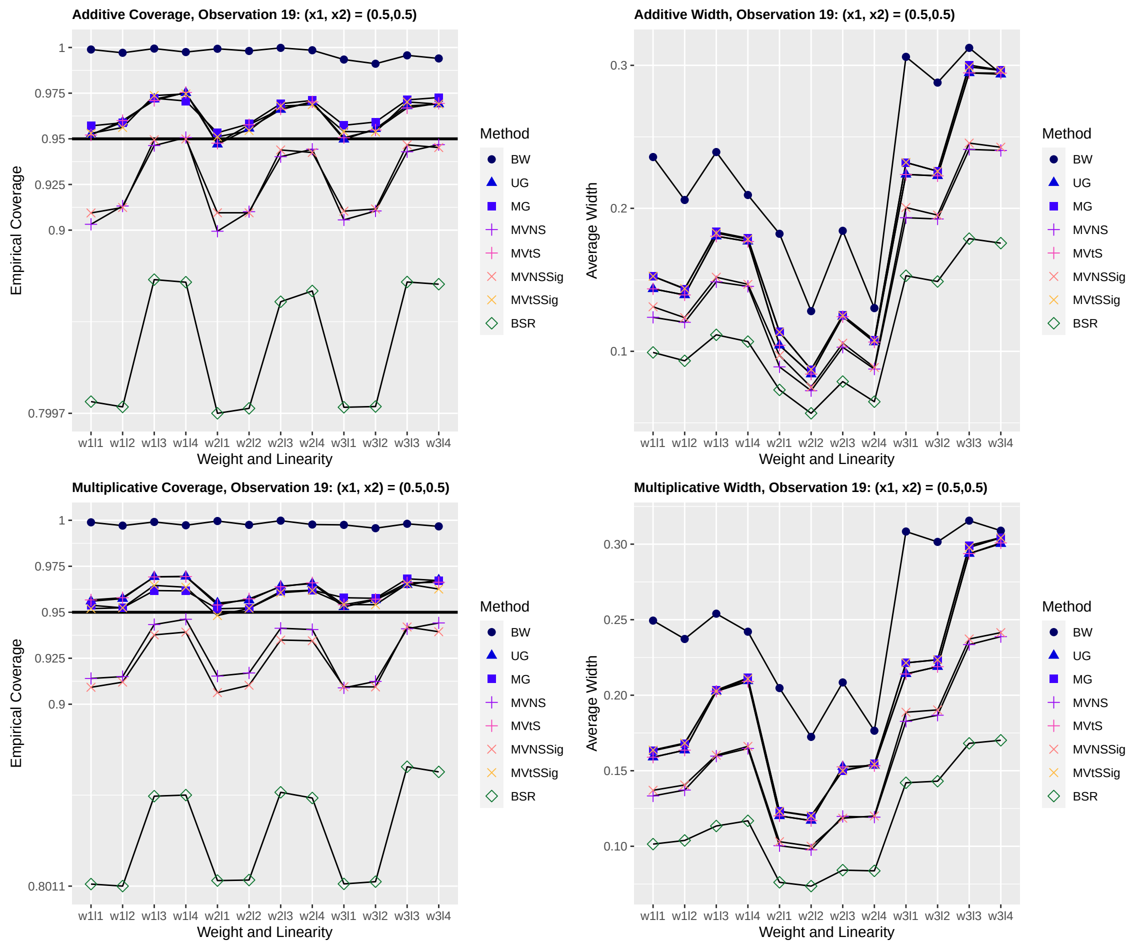


Figure 31. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 19 (Max/Tgt/Min Second Order)

Observation 5 is shown in Figure 32. This observation slightly favors Y_2 but is far from the optimal regions for Y_1 and Y_3. At this solution only the BW method is able to maintain empirical coverage due to its conservative nature. Every other method fails to attain 95% confidence except when $\boldsymbol{l}_3$ or $\boldsymbol{l}_4$ which both making the more likely to be an optimal solution. This observation is showcased primarily to bring attention to the fact that empirical coverage of the best two methods, MG and MVtSSig, drops below 90% but a considerable amount. While initially alarming that they may perform poorly for any of the observations, this is unfortunately a case of the linearity once again which can be seen in the average width of the observation 5 plots and by the average symmetry plots in Figure 33. In this figure, all of the instances when empirical coverage is below the specified threshold, the average symmetry is skewed positive. This means that once again, the linearity is greatly affecting empirical coverage and average width but in a way that is at least beneficial.

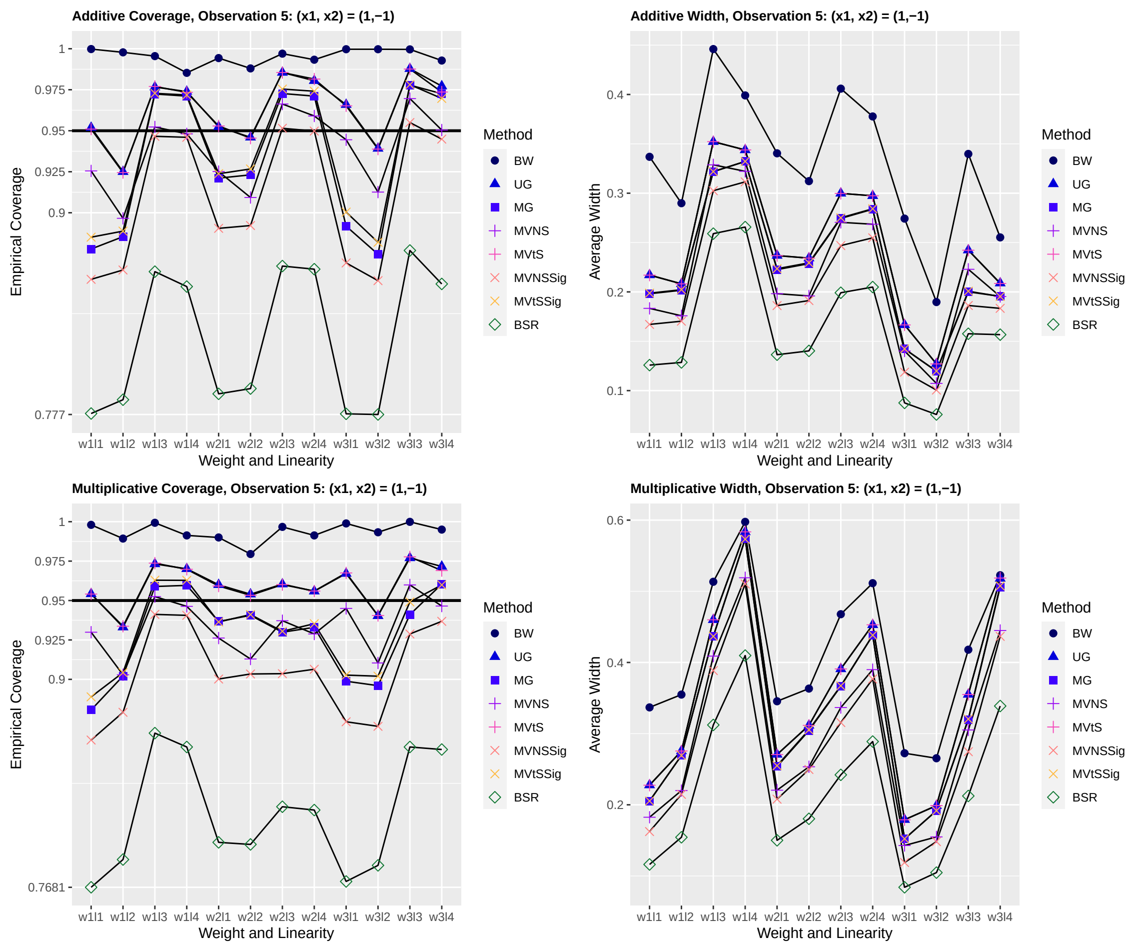


Figure 32. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 5 (Max/Tgt/Min Second Order)

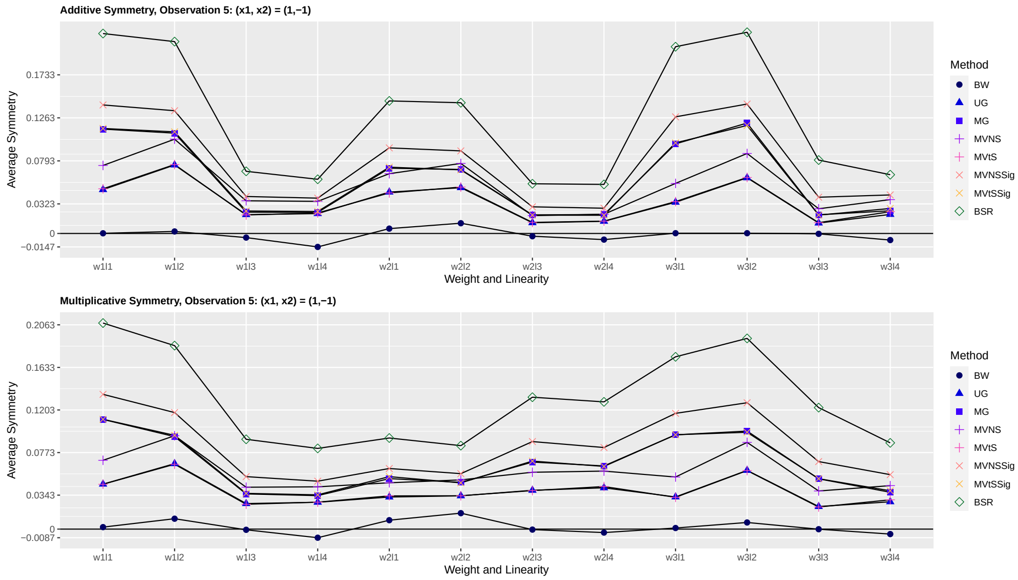


Figure 33. Additive and Multiplicative Plots for Average Symmetry at Observation 5 (Max/Tgt/Min Second Order)

Moving from the region of observation 5 to observation 13 severely decreases the variation once again as seen in Figure 34. At observation 13, Y_1 is favored and is far from the optimal region for Y_3; However, there is not much effect as the prediction variance is small enough at this observation and the emphasis placed on each response is no issue that each method is stable at all desirability function settings. Although there is not much effect, it can still be seen when looking at the average width plots when $\boldsymbol{w}_3$ is active which favors Y_2 resulting in wider confidence bounds, on average. The BW method provides a very conservative confidence interval here with a larger corresponding width. UG, MG, MVtS, and MVtSSig all provide approximately 95% confidence intervals across all desirability settings. On average, the widths for UG, MG, MVtS, and MVtSSig are all in agreement as well hovering around 0.2 when the weight combinations are equal, 0.15 when the weight combination favors Y_1, and around 0.275 when the weight combination favors Y_2. Finally, the methods that perform poorly are the MVN, MVNSSig, and BSR which do not meet the minimum empirical coverage but they do have smaller widths, on average, which may prove to be beneficial depending on an analysts specific needs.

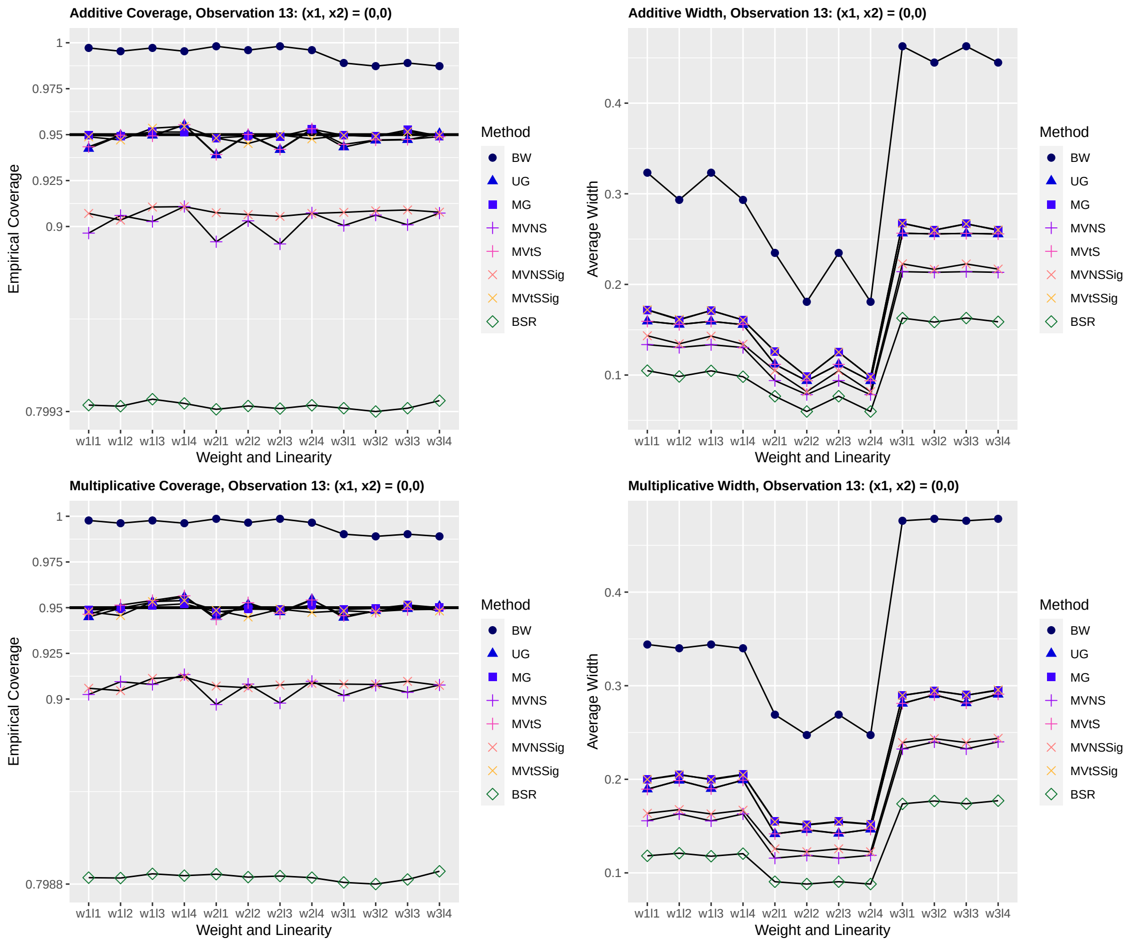


Figure 34. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 13 (Max/Tgt/Min Second Order)

4.3 Discussion & Preferred Inference Methods

Across each of the 7 scenarios, two of the novel methods, MG and MVtSSig, outperform the others in most situations. Most of the time, the first two scenarios show that while the plane angle does not have any significant effect on empirical coverage, it does slightly affect width. The correlation between responses greatly affects empirical coverage and average width if not accounted for with some covariance structure. Each of the multivariate-based methods adjust their empirical coverage and average width to stay within reasonable bounds with MG and MVtSSig both accounting for correlation and maintaining the specified 95% empirical coverage. This is further evidenced in the third and fourth scenarios, where the univariate method empirical coverage change drastically depending on the amount of correlation present between response surfaces, except for BW, but the multivariate methods stay relatively constant. In the third and fourth scenarios, MG and MVtSSig maintain empirical coverage for both the additive and multiplicative methods with adjusted average width as correlation changes. The first time that MG and MVtSSig perform poorly are for specific correlation levels in scenario 4 where the contours of the response surface suggest that empirical coverage decreases when at or near the match target objective but this can be explained by the positive skew in the average symmetry plot as the true parameter is simply more optimal than expected.

In the 3 scenarios focused on second order models, a few common themes arose. An obvious one is that all of the methods perform better as prediction variance decreases, which is the center of the design because of the uniform precision CCD. The empirical coverage decreased and the average width increased when at points where linearity emphasis causes large ranges in possible desirability values. The multiplicative form tends to have more erratic changes in average width, but empirical coverage was more or less similar between forms. It is important to notice that the changes occurring across the $x - axis$ of for the second order plots is not the same as the first order plots which is why there is not a drastic change as seen in the first order plots. In the second order models, the correlation is constant based on the covariance estimate in Equation 86. This covariance corresponds to the correlation matrix

$$\boldsymbol{C} = \begin{pmatrix} 1 & -0.42893 & 0.29331 \\ -0.42893 & 1 & 0.094275 \\ 0.29331 & 0.094275 & 1 \end{pmatrix} \tag{109}$$

Because the correlation is constant, the trend of each inference method looks very similar across each of the weight combinations and emphasis linearity parameters. In addition to the constant correlation for SO models, the correlation is fairly low between each response. Larger correlations should lead to worse performance of the univariate methods.

4.4 Conclusion

The analysis conducted using each of the 8 inferential methods for Derringer and Suich desirability functons were presented in this chapter. The methods were tested for empirical coverage probability, average width, and average symmetry using large Monte Carlo simulation with first and second order models. The first order models considered 4 scenarios to evaluate the effect that plane angles between response surfaces, correlation between responses, and a combination of the two had on CP, AW, and SYM using Max/Min and Max/Tgt problem sets. The second order models used 3 scenarios based on the chemical process optimization problem to examine the effect on CP, AW, and SYM from desirability function parameter settings. The second order models also provided key insights into the reliability of the inferential methods based on the topology of the response surfaces.

The first order model scenarios show that the angle between response planes is irrelevant, it is crucial to incorporate a covariance structure, that the MG and MVtSSig methods perform best for first order models while the BW method was very conservative, and that when a match target is considered, there may be coverage issues at the target value itself. By showing that the angle between planes does not affect empirical coverage, the implementation to second order models becomes straight forward. The covariance structure causes empirical coverage to be adequately met by increasing or decreasing the average width of the confidence interval with the amount of correlation. In general, the MG and MVtSSig methods performed the best as they incorporated a covariance structure and they utilize the multivariate-t distribution which is ideal when true covariance is unknown. It is also important to have the a method like BW that is overall conservative and simple to implement. The decrease in empirical coverage when the match target problem is an example of when the BW method would possibly be a better choice than the MG and MVtSSig methods, however, the decrease in empirical coverage for MG and MVtSSig is when match target problems is addressed by positive skew in average symmetry.

The second order model scenarios show that the weight combinations and linear parameters play a rather large role in the applicability of the inference method. The methods performed poorly in both the additive and multiplicative form when constructing confidence intervals near optimal values when emphasis is high and far from optimal values with emphasis is low. Similar to the first order models, the MG and MVtSSig methods developed in this research outperformed all other methods and the BW methods were conservative, resulting in rather large widths at times. Prediction variance also contributed a lot to the effectiveness of each inference method. While the MG and MVtSSig methods work fairly well for these small samples, to ensure adequate empirical coverage in all scenarios, one should consider adding additional runs to their designed experiment.

Overall, the results of this research suggest that when one is doubtful of which method to use, BW can provide a conservative estimate. More adequate methods that are suggested for use include MG and MVtSSig as they maintain specified empirical coverage while keeping average width relatively small. If using the MG and MVtSSig method, consider using the Pareto Front Optimization techniques from Chapman et al (2014b) and Calhoun (2020) as they will help remove the Pareto dominated points to reduce solution space considered.

V. Conclusion

5.1 Conclusion

The Derringer and Suich desirability function is an incredibly useful tool for determining the optimal point between multiple correlated response surfaces. It is commonly used as a decision making tool as it captures subjectivity of the decision maker and emphasis for each individual response surface in a simple and flexible way. Many methods for reporting the desirability function usually involve some method of making the desirability optimal value robust to variation rather than reporting inference about the optimal values. In the BW method that is sometimes used, it assumes independence between the multiple objectives being optimized and tends to be rather conservative with larger widths. Larger widths can be problematic with the desirability function as it's bounded by [0,1] meaning that every small increment matter significantly.

This research proposed 7 novel methods (UG, MG, MVNS, MVtS, MVNSSig, MVtSSig, BSR) for this application to compare to the conservative BW method to provide adequate confidence intervals around the desirability function optimal values. 3 univariate methods and 4 multivariate methods were proposed that utilize either the generalized method, simulated surface method, or nonparametric bootstrap method. Each of the methods utilized Monte Carlo Simulation to generate $G = 10,000$ simulated regression models with true underlying surfaces that were used to calculate the coverage probability, the average width, and the symmetry of each method. This was performed for both first order and second order models across 7 scenarios to assess the effect of angles between response planes, correlation between response variables, weight combinations, and linearity parameter combinations.

The univariate methods should be avoided as their coverage is not consistent as correlation between response surfaces change with the exception of the BW method which maintained conservative coverage. The MVN, MVNSSig, and BSR methods did maintain constant coverage in most situations; However; coverage was consistently below the specifed limit and should be avoided. The methods that perform best are the MG and MVtSSig methods as they maintained the specified confidence in most situations. The situations these methods performed poorly was in first order models with Max/Tgt objectives when the observation was in an area where the response value for the Tgt problem was the at or near the Tgt value itself; However, this scenario does not cause any issues as the average symmetry shows that the true parameter is above the MG and MVtSSig intervals which is a benefit to the decision maker. The same phenomena would likely be seen if the target and evasion bounds for the Max and Min objectives was within the range of possible response values, leading to more 0s and 1s for the desirability. Additionally, the coverage decreased and average width increased if a MG or MVtSSig confidence interval was constructed around a non-optimal observation which can be combatted by incorporating the research by Chapman et al. (2014b) and Calhoun (2020) to remove Pareto dominated points. If one is concerned about coverage, the BW method is still a good alternative with the primary trade-off being that the average width is larger than each of the other methods. Alternatively, additional observations in the original designed experiment to decrease the prediction is also beneficial in maintaining coverage with small average width.

5.2 Future Research

This final section addresses a few additional research topics that could be explored following this research. Each of the research topics build on this dissertation. The first would address better covariance estimates for randomly sampling using the MG and MVtSSig methods. The second would look at conducting a two-step approach for confidence intervals based on robust optimal points. The third would attempt to incorporate probability distributions into sensitivity analysis to make searching the change in optimal decision spaces simpler.

A potential improvement would be to utilize the Wishart distribution to capture additional variation within the multivariate inference methods to make them more robust. The sample covariance of the multivariate multiple linear regression model multiplied by the degrees of freedom of the model is distributed as a Wishart distribution. Since the sampling distribution of the sample covariance is known, finding an appropriate pivotal quantity or simply using the sample covariance as the 'true' covariance structure may benefit the MG and MVtSSig methods maintain appropriate confidence levels.

The MG and MVtSSig methods should perform well on their own when considering optimal points from the single estimated regression model, however, a more robust method that should be explored is to perform the Pareto Front Optimization with simulated surfaces to remove noncontender points and then apply the MG or MVtSSig method to those points that are stochastically Pareto dominant. The contribution here that extends past the current literature would be to include the covariance structure for the parameter vectors to better characterize the uncertainty between the simulated surfaces. After robust optimal points are found, confidence intervals can be constructed around the candidate solutions with assurance that the confidence is maintained because the MG and MVtSSig methods perform better when the solutions are already around 'true' optimal solutions.

The definition of the Dirichlet Distribution lends itself well to the structure of weight combination from multi-objective decision analysis. When a decision maker chooses weights, it may not necessarily be fully informed, it may only capture a specific point in time, or perhaps the decision maker is simply interested in how robust their optimal decision space is to variation in their choice of weights. This sensitivity analysis becomes more difficult as the number of objectives grows which would benefit from a method to capture it using probability distributions.

Appendix A. First Order Max/Min Model Desirability Inference Results

A.1 Additive Empirical Coverage and Average Width Tables

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9625	0.8982	0.95	0.8257	0.8992	0.9041	0.9502	0.8349
-0.5	0.9786	0.9178	0.9475	0.8515	0.9191	0.8974	0.9454	0.8272
-0.3	0.9845	0.9388	0.9534	0.8823	0.9406	0.9023	0.9535	0.8273
0	0.9946	0.9667	0.9473	0.9225	0.9665	0.903	0.9466	0.8333
0.3	0.9977	0.9844	0.953	0.9578	0.984	0.9065	0.9513	0.8319
0.5	0.9992	0.9907	0.948	0.9725	0.9912	0.9025	0.9477	0.8281
0.8	0.9993	0.9944	0.9445	0.985	0.9941	0.8974	0.9443	0.8355
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.10377	0.0771	0.09856	0.06208	0.07714	0.07896	0.09855	0.06415
-0.5	0.12642	0.09174	0.11019	0.07389	0.09174	0.08827	0.11009	0.07138
-0.3	0.08879	0.0638	0.07175	0.05137	0.06383	0.0575	0.07176	0.04644
0	0.06595	0.04726	0.04675	0.03804	0.04727	0.03747	0.04675	0.0303
0.3	0.08918	0.06412	0.0533	0.05161	0.06413	0.0427	0.05329	0.03456
0.5	0.12554	0.09103	0.06667	0.07331	0.09103	0.05343	0.06667	0.04273
0.8	0.10307	0.07668	0.04606	0.06175	0.07673	0.0369	0.04607	0.0298

Table 5. Additive Desirability Inference Observation 2, (x_1, x_2) = (-0.5,-1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9629	0.8967	0.9514	0.8276	0.8967	0.9041	0.9519	0.8263
-0.5	0.9779	0.9211	0.9471	0.8544	0.9222	0.8972	0.9479	0.8274
-0.3	0.9851	0.9391	0.9508	0.8809	0.9379	0.9041	0.9514	0.8293
0	0.9941	0.9653	0.9502	0.9234	0.9656	0.9031	0.9499	0.8263
0.3	0.9973	0.9835	0.9533	0.9588	0.9836	0.9056	0.9533	0.8319
0.5	0.9989	0.9913	0.9473	0.9729	0.9912	0.8979	0.948	0.8264
0.8	0.9993	0.9948	0.9476	0.9846	0.9952	0.8989	0.9489	0.8318
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.09581	0.07124	0.09103	0.05732	0.07124	0.07291	0.09101	0.05885
-0.5	0.11673	0.08478	0.10173	0.06824	0.08475	0.0815	0.10165	0.06548
-0.3	0.08198	0.05894	0.06626	0.04744	0.05893	0.05308	0.06627	0.04261
0	0.06089	0.04361	0.04317	0.03512	0.04363	0.0346	0.04316	0.02779
0.3	0.08234	0.05917	0.0492	0.04763	0.0592	0.03944	0.04921	0.03171
0.5	0.11591	0.08412	0.06155	0.06768	0.08408	0.04935	0.06158	0.03921
0.8	0.09516	0.07082	0.04253	0.057	0.07084	0.03407	0.04254	0.02734

Table 6. Additive Desirability Inference Observation 3, (x_1, x_2) = (0,-1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9634	0.8956	0.9561	0.8257	0.8963	0.9087	0.9544	0.8289
-0.5	0.9777	0.9229	0.9484	0.8635	0.9238	0.8977	0.9477	0.8384
-0.3	0.986	0.9388	0.9511	0.8836	0.9398	0.9039	0.9514	0.8311
0	0.9941	0.9651	0.9509	0.9232	0.9651	0.9037	0.9507	0.8292
0.3	0.9972	0.9845	0.9529	0.9602	0.9858	0.9038	0.9528	0.8346
0.5	0.9988	0.9913	0.9476	0.9744	0.991	0.8984	0.9477	0.8279
0.8	0.9992	0.9955	0.9506	0.9858	0.9957	0.9031	0.9514	0.831
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.10377	0.07712	0.09859	0.06207	0.07714	0.07896	0.09855	0.06416
-0.5	0.12642	0.09182	0.11013	0.0739	0.09181	0.08826	0.11015	0.0714
-0.3	0.08879	0.06381	0.07176	0.05138	0.06383	0.0575	0.07178	0.04645
0	0.06595	0.04729	0.04677	0.03805	0.04727	0.03747	0.04677	0.0303
0.3	0.08918	0.06411	0.05331	0.05158	0.06412	0.04271	0.0533	0.03457
0.5	0.12554	0.09106	0.0667	0.07329	0.09109	0.05345	0.0667	0.04273
0.8	0.10307	0.07668	0.04607	0.06175	0.0767	0.03691	0.04609	0.0298

Table 7. Additive Desirability Inference Observation 4, (x_1, x_2) = (0.5,-1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9634	0.8972	0.9542	0.8261	0.8976	0.9071	0.9534	0.8304
-0.5	0.9795	0.9238	0.9468	0.8646	0.925	0.8988	0.9472	0.8418
-0.3	0.9864	0.9431	0.9497	0.8833	0.9425	0.9054	0.9492	0.8295
0	0.9936	0.9639	0.9509	0.9207	0.9655	0.904	0.949	0.8327
0.3	0.998	0.9862	0.9514	0.9598	0.9866	0.9049	0.9516	0.8402
0.5	0.9985	0.9915	0.9476	0.9752	0.9917	0.8992	0.949	0.8277
0.8	0.9992	0.9958	0.9515	0.9865	0.9956	0.9035	0.9515	0.836
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.12464	0.09261	0.11838	0.07456	0.09263	0.09486	0.11835	0.07745
-0.5	0.15185	0.11027	0.13232	0.08875	0.11028	0.10598	0.13226	0.08619
-0.3	0.10595	0.07571	0.08536	0.06151	0.07571	0.06888	0.08537	0.05599
0	0.07874	0.05612	0.05563	0.04555	0.05612	0.04489	0.05566	0.03654
0.3	0.10644	0.07606	0.06328	0.06178	0.07606	0.05115	0.06331	0.04167
0.5	0.15078	0.10941	0.08007	0.08803	0.10941	0.06418	0.0801	0.05157
0.8	0.1238	0.09209	0.05528	0.07417	0.09206	0.04433	0.0553	0.03598

Table 8. Additive Desirability Inference Observation 5, (x_1, x_2) = (1,-1) Empirical Coverage and Average Width

				Empirical Coverage				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9638	0.9052	0.9499	0.834	0.9047	0.8999	0.9503	0.839
-0.5	0.9788	0.9216	0.9478	0.8561	0.9212	0.9011	0.9483	0.8309
-0.3	0.9867	0.9382	0.9498	0.8788	0.9397	0.9024	0.951	0.827
0	0.9952	0.9691	0.9484	0.922	0.968	0.9021	0.9475	0.8286
0.3	0.9971	0.9845	0.9494	0.9596	0.9843	0.9045	0.949	0.8336
0.5	0.9988	0.9905	0.9511	0.9742	0.9903	0.9061	0.9493	0.8328
0.8	0.9997	0.9952	0.9474	0.9857	0.9954	0.8984	0.9464	0.8345
				Average Width				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.10374	0.07704	0.09853	0.06207	0.07702	0.07894	0.09852	0.06415
-0.5	0.12641	0.09175	0.11011	0.0739	0.09176	0.08825	0.11009	0.07135
-0.3	0.08879	0.06382	0.0718	0.05137	0.06383	0.05752	0.07177	0.04642
0	0.06595	0.04727	0.04678	0.03802	0.04727	0.03748	0.04675	0.03029
0.3	0.08918	0.06411	0.0533	0.05162	0.06417	0.0427	0.05332	0.03456
0.5	0.12553	0.09103	0.06655	0.07331	0.09105	0.0534	0.06657	0.04274
0.8	0.10304	0.07661	0.0459	0.06171	0.0766	0.0369	0.04588	0.02981

Table 9. Additive Desirability Inference Observation 6, (x_1, x_2) = (-1,-0.5) Empirical Coverage and Average Width

				Empirical Coverage				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9637	0.9003	0.9491	0.8297	0.901	0.9006	0.9487	0.8317
-0.5	0.978	0.9209	0.9455	0.8546	0.9198	0.8985	0.946	0.8257
-0.3	0.9857	0.9401	0.9511	0.8816	0.9398	0.9005	0.9497	0.8222
0	0.995	0.9674	0.9488	0.9237	0.9686	0.9029	0.9474	0.8287
0.3	0.997	0.9831	0.9516	0.9587	0.9831	0.908	0.9516	0.8307
0.5	0.9994	0.9909	0.9486	0.9739	0.9908	0.9037	0.9499	0.8294
0.8	0.9994	0.9946	0.9466	0.9865	0.9947	0.8996	0.9467	0.8339
				Average Width				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.07747	0.05761	0.07354	0.04635	0.05759	0.05894	0.07358	0.04761
-0.5	0.09439	0.06855	0.08222	0.05517	0.06855	0.0659	0.08223	0.05296
-0.3	0.06629	0.04767	0.05358	0.03835	0.04766	0.04294	0.05358	0.03446
0	0.04923	0.03527	0.03492	0.02839	0.03529	0.02798	0.03491	0.02249
0.3	0.06658	0.04784	0.03981	0.03854	0.0479	0.03189	0.0398	0.02565
0.5	0.09373	0.06801	0.04978	0.05474	0.068	0.0399	0.04977	0.03172
0.8	0.07695	0.05726	0.03439	0.0461	0.05728	0.02756	0.03438	0.02212

Table 10. Additive Desirability Inference Observation 7, (x_1, x_2) = (-0.5,-0.5) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9624	0.8982	0.9514	0.8243	0.8981	0.9046	0.9517	0.8245
-0.5	0.9772	0.9199	0.9472	0.8552	0.9203	0.8964	0.9469	0.8254
-0.3	0.9855	0.9393	0.9514	0.8824	0.9381	0.9048	0.9512	0.8274
0	0.9938	0.9676	0.949	0.9242	0.9668	0.9052	0.9497	0.8298
0.3	0.9968	0.9847	0.9544	0.9593	0.9854	0.9079	0.9548	0.8298
0.5	0.9987	0.9911	0.9498	0.9731	0.991	0.9014	0.9499	0.8235
0.8	0.9993	0.996	0.9511	0.9854	0.9959	0.9024	0.9496	0.8278
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.06643	0.04939	0.06307	0.03975	0.04939	0.05056	0.0631	0.0408
-0.5	0.08093	0.05878	0.07048	0.0473	0.05878	0.05651	0.07049	0.04539
-0.3	0.05684	0.04085	0.04594	0.03289	0.04085	0.03681	0.04594	0.02954
0	0.04222	0.03025	0.02993	0.02435	0.03025	0.02399	0.02993	0.01927
0.3	0.05709	0.04104	0.03412	0.03303	0.04103	0.02734	0.03412	0.02199
0.5	0.08037	0.05835	0.04271	0.04694	0.0583	0.03422	0.04268	0.02718
0.8	0.06598	0.0491	0.02948	0.03953	0.04911	0.02363	0.02949	0.01895

Table 11. Additive Desirability Inference Observation 8, (x_1, x_2) = (0,-0.5) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9626	0.8992	0.9533	0.8255	0.8992	0.9063	0.9537	0.8291
-0.5	0.9779	0.9263	0.948	0.8619	0.9261	0.8971	0.9487	0.8324
-0.3	0.987	0.9435	0.9527	0.884	0.9411	0.9065	0.951	0.8251
0	0.9943	0.9649	0.9509	0.9208	0.966	0.9067	0.951	0.8276
0.3	0.9976	0.9858	0.9562	0.9616	0.9861	0.9064	0.9544	0.8351
0.5	0.9986	0.9904	0.95	0.9741	0.9906	0.9007	0.9501	0.8255
0.8	0.9991	0.9954	0.9506	0.9863	0.9955	0.9037	0.951	0.8283
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.07747	0.05758	0.07353	0.04634	0.05761	0.05895	0.07357	0.04762
-0.5	0.09439	0.06854	0.08222	0.05515	0.06858	0.06589	0.0822	0.053
-0.3	0.06629	0.04764	0.05357	0.03835	0.04764	0.04292	0.05359	0.03448
0	0.04923	0.03529	0.03492	0.0284	0.03528	0.02798	0.03491	0.02249
0.3	0.06658	0.04786	0.03981	0.03851	0.04786	0.03188	0.03981	0.02566
0.5	0.09373	0.06801	0.04977	0.05473	0.06802	0.0399	0.04978	0.03171
0.8	0.07695	0.05725	0.0344	0.04609	0.05725	0.02755	0.03441	0.02212

Table 12. Additive Desirability Inference Observation 9, (x_1, x_2) = (0.5,-0.5) Empirical Coverage and Average Width

	Empirical Coverage							
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9643	0.8986	0.9543	0.8281	0.8985	0.9102	0.9545	0.834
-0.5	0.9789	0.9281	0.9457	0.863	0.9262	0.8983	0.9455	0.8367
-0.3	0.9859	0.9425	0.9508	0.8841	0.9429	0.9058	0.9492	0.8317
0	0.9935	0.9654	0.9502	0.921	0.9651	0.903	0.9516	0.8302
0.3	0.9982	0.9863	0.9534	0.9626	0.9864	0.9082	0.9519	0.8378
0.5	0.9983	0.9919	0.9469	0.9757	0.9915	0.898	0.9473	0.8273
0.8	0.9991	0.9951	0.951	0.9865	0.9953	0.9041	0.9497	0.8322
	Average Width							
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.10377	0.0771	0.09857	0.06207	0.07716	0.07896	0.09855	0.06416
-0.5	0.12642	0.09181	0.11013	0.0739	0.09186	0.08825	0.11013	0.0714
-0.3	0.08879	0.06382	0.07175	0.05136	0.06382	0.05752	0.07178	0.04645
0	0.06595	0.04725	0.04677	0.03803	0.04724	0.03747	0.04676	0.0303
0.3	0.08918	0.06411	0.05332	0.05159	0.06412	0.0427	0.05332	0.03457
0.5	0.12554	0.09107	0.06664	0.0733	0.09111	0.05344	0.06668	0.04273
0.8	0.10307	0.07667	0.04603	0.06173	0.07667	0.0369	0.04608	0.02981

Table 13. Additive Desirability Inference Observation 10, (x_1, x_2) = (1,-0.5) Empirical Coverage and Average Width

	Empirical Coverage							
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9649	0.9035	0.9485	0.8355	0.9048	0.901	0.9482	0.8378
-0.5	0.9794	0.9229	0.949	0.8605	0.9215	0.902	0.948	0.8329
-0.3	0.9857	0.9425	0.9527	0.8808	0.9419	0.9025	0.9516	0.8206
0	0.9947	0.9653	0.9498	0.9219	0.9662	0.901	0.9497	0.8254
0.3	0.9976	0.9842	0.9475	0.9603	0.9843	0.9004	0.9477	0.8308
0.5	0.9991	0.9909	0.9529	0.974	0.9914	0.9059	0.9543	0.8304
0.8	0.9996	0.9963	0.9489	0.988	0.9959	0.9017	0.9474	0.8307
	Average Width							
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.09581	0.07123	0.09099	0.05732	0.07122	0.0729	0.09096	0.05884
-0.5	0.11673	0.0848	0.10164	0.06822	0.08481	0.08148	0.1017	0.06545
-0.3	0.08198	0.05893	0.06626	0.04743	0.05892	0.05311	0.06624	0.04259
0	0.06089	0.04362	0.04318	0.03511	0.04364	0.03459	0.04316	0.02778
0.3	0.08234	0.05919	0.04922	0.04765	0.05923	0.03943	0.04925	0.0317
0.5	0.11591	0.08412	0.06155	0.06766	0.08413	0.04933	0.06155	0.03921
0.8	0.09516	0.0708	0.04249	0.05698	0.07079	0.03409	0.0425	0.02735

Table 14. Additive Desirability Inference Observation 11, (x_1, x_2) = (-1,0) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9628	0.9033	0.9485	0.8356	0.9017	0.9011	0.949	0.8342
-0.5	0.9793	0.9256	0.9479	0.8595	0.9252	0.8992	0.9482	0.8311
-0.3	0.9854	0.9408	0.9497	0.8797	0.9407	0.8997	0.9503	0.8222
0	0.9954	0.9666	0.9513	0.921	0.9667	0.9032	0.9508	0.8269
0.3	0.9981	0.983	0.9503	0.9573	0.9831	0.9039	0.9509	0.8273
0.5	0.9987	0.9913	0.9526	0.9731	0.9909	0.9046	0.9529	0.8267
0.8	0.9995	0.9969	0.9499	0.9884	0.997	0.9009	0.951	0.8275
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.06643	0.04936	0.06306	0.03975	0.04938	0.05053	0.06308	0.04079
-0.5	0.08093	0.05874	0.07052	0.04729	0.05881	0.0565	0.0705	0.04537
-0.3	0.05684	0.04087	0.04594	0.03289	0.04085	0.03682	0.04594	0.02953
0	0.04222	0.03026	0.02994	0.02434	0.03025	0.02399	0.02993	0.01926
0.3	0.05709	0.04104	0.03412	0.03304	0.04107	0.02734	0.03414	0.02199
0.5	0.08037	0.0583	0.04269	0.04691	0.05834	0.03421	0.04267	0.02718
0.8	0.06598	0.0491	0.02947	0.03951	0.04908	0.02363	0.02949	0.01895

Table 15. Additive Desirability Inference Observation 12, (x_1, x_2) = (-0.5,0) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9637	0.8955	0.9502	0.8276	0.8957	0.9029	0.9488	0.8336
-0.5	0.979	0.9226	0.9477	0.8619	0.9226	0.8994	0.9479	0.8352
-0.3	0.9859	0.9401	0.9522	0.8857	0.9412	0.9051	0.9514	0.8321
0	0.9943	0.9671	0.9501	0.9233	0.9674	0.9026	0.9514	0.8362
0.3	0.9971	0.9832	0.9576	0.9581	0.9838	0.9103	0.956	0.8377
0.5	0.9986	0.9897	0.9499	0.9729	0.9892	0.9049	0.9505	0.8295
0.8	0.9994	0.9964	0.9523	0.9876	0.9963	0.904	0.9515	0.8296
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.05315	0.03952	0.05049	0.03179	0.03951	0.04044	0.05046	0.03289
-0.5	0.06475	0.04703	0.05641	0.03784	0.04703	0.04519	0.05639	0.03658
-0.3	0.04547	0.03269	0.03675	0.02631	0.03269	0.02945	0.03676	0.02381
0	0.03377	0.0242	0.02396	0.01947	0.0242	0.01919	0.02395	0.01552
0.3	0.04567	0.03283	0.0273	0.02642	0.03283	0.02187	0.02731	0.01772
0.5	0.0643	0.04664	0.03415	0.03754	0.04666	0.02738	0.03414	0.0219
0.8	0.05279	0.03927	0.02358	0.03161	0.03929	0.0189	0.02359	0.01528

Table 16. Additive Desirability Inference Observation 13, (x_1, x_2) = (0,0) Empirical Coverage and Average Width

Empirical Coverage								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.9624	0.9014	0.9513	0.8296	0.9009	0.904	0.9515	0.8283
-0.5	0.9791	0.9256	0.9481	0.86	0.9247	0.8996	0.9465	0.8294
-0.3	0.9864	0.9412	0.952	0.882	0.9402	0.9081	0.9516	0.8259
0	0.9938	0.9677	0.951	0.9288	0.9684	0.9023	0.9508	0.8317
0.3	0.998	0.9867	0.9568	0.9602	0.9865	0.9106	0.9563	0.8321
0.5	0.9986	0.9899	0.9512	0.9743	0.9898	0.9025	0.9502	0.8249
0.8	0.9994	0.9961	0.9497	0.9868	0.9963	0.9032	0.9499	0.8302
Average Width								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.06643	0.04938	0.0631	0.03973	0.04939	0.05055	0.06309	0.0408
-0.5	0.08093	0.05879	0.07054	0.04731	0.05879	0.05649	0.07048	0.04541
-0.3	0.05684	0.04086	0.04596	0.03288	0.04086	0.03682	0.04597	0.02954
0	0.04222	0.03025	0.02992	0.02434	0.03024	0.02398	0.02993	0.01926
0.3	0.05709	0.04106	0.03412	0.03303	0.04103	0.02734	0.03413	0.02198
0.5	0.08037	0.0583	0.04268	0.04692	0.05832	0.0342	0.04268	0.02718
0.8	0.06598	0.04908	0.02949	0.03952	0.0491	0.02362	0.0295	0.01896

Table 17. Additive Desirability Inference Observation 14, (x_1, x_2) = (0.5,0) Empirical Coverage and Average Width

Empirical Coverage								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.9653	0.8959	0.954	0.832	0.8982	0.9087	0.9541	0.8335
-0.5	0.9802	0.9224	0.9473	0.8584	0.9224	0.9003	0.9477	0.8288
-0.3	0.9873	0.9412	0.9502	0.8832	0.9411	0.9037	0.9502	0.8276
0	0.994	0.967	0.9502	0.9228	0.9657	0.9025	0.95	0.8297
0.3	0.9989	0.9872	0.9557	0.9624	0.987	0.9047	0.9552	0.8327
0.5	0.9985	0.9914	0.9482	0.9761	0.9916	0.8997	0.9489	0.8282
0.8	0.9993	0.996	0.9467	0.9862	0.9964	0.9021	0.9467	0.8325
Average Width								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.09581	0.0712	0.09101	0.05729	0.07122	0.0729	0.09101	0.05886
-0.5	0.11673	0.08477	0.10162	0.06824	0.08481	0.08148	0.10166	0.06549
-0.3	0.08198	0.05893	0.06624	0.04742	0.05892	0.05311	0.06628	0.04261
0	0.06089	0.04363	0.04317	0.0351	0.04363	0.0346	0.04317	0.02779
0.3	0.08234	0.05917	0.04924	0.04765	0.0592	0.03942	0.04924	0.03171
0.5	0.11591	0.08414	0.06155	0.06767	0.08411	0.04932	0.06155	0.03921
0.8	0.09516	0.07079	0.04248	0.057	0.07079	0.03408	0.04251	0.02734

Table 18. Additive Desirability Inference Observation 15, (x_1, x_2) = (1,0) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9648	0.9025	0.949	0.8347	0.9017	0.9014	0.95	0.8374
-0.5	0.9805	0.9265	0.9506	0.8619	0.9275	0.901	0.9506	0.8353
-0.3	0.9859	0.9427	0.9506	0.8821	0.9427	0.9027	0.948	0.8298
0	0.9945	0.9658	0.9518	0.9203	0.9649	0.9034	0.953	0.8272
0.3	0.9982	0.9841	0.9461	0.9603	0.9842	0.9023	0.9455	0.8401
0.5	0.9989	0.9918	0.9516	0.9749	0.9919	0.906	0.9513	0.8276
0.8	0.9995	0.9959	0.9491	0.9875	0.9957	0.8995	0.9481	0.8339
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.10377	0.07714	0.09857	0.06208	0.07711	0.07896	0.0985	0.06416
-0.5	0.12642	0.0918	0.11017	0.0739	0.09186	0.08827	0.11015	0.07138
-0.3	0.08879	0.06382	0.07177	0.05136	0.06382	0.05751	0.07172	0.04641
0	0.06595	0.04724	0.04674	0.03803	0.04726	0.03747	0.04675	0.03029
0.3	0.08918	0.06414	0.05332	0.0516	0.06415	0.04272	0.05334	0.03457
0.5	0.12554	0.0911	0.06666	0.07327	0.09113	0.05342	0.06668	0.04272
0.8	0.10307	0.07669	0.04606	0.06171	0.07665	0.03691	0.04604	0.02981

Table 19. Additive Desirability Inference Observation 16, (x_1, x_2) = (-1,0.5) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9626	0.8992	0.9506	0.8326	0.8979	0.9009	0.9508	0.832
-0.5	0.9825	0.9278	0.9486	0.8628	0.928	0.9024	0.9477	0.8307
-0.3	0.9869	0.9406	0.9495	0.8832	0.9402	0.8998	0.9487	0.8231
0	0.9951	0.9655	0.9527	0.9199	0.9666	0.9051	0.9542	0.8261
0.3	0.9983	0.9845	0.9496	0.9581	0.985	0.9015	0.9479	0.8375
0.5	0.9979	0.9914	0.9521	0.9749	0.9912	0.904	0.9523	0.8244
0.8	0.9995	0.996	0.9495	0.9884	0.9959	0.8983	0.9482	0.8294
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.07747	0.05763	0.07354	0.04636	0.05758	0.05894	0.07353	0.04763
-0.5	0.09439	0.06852	0.08222	0.05515	0.06855	0.06588	0.08221	0.05298
-0.3	0.06629	0.04765	0.05359	0.03835	0.04764	0.04294	0.05356	0.03445
0	0.04923	0.03527	0.0349	0.02839	0.03527	0.02797	0.0349	0.02248
0.3	0.06658	0.04787	0.03979	0.03852	0.04788	0.03189	0.03982	0.02566
0.5	0.09373	0.06801	0.04976	0.05471	0.06804	0.03989	0.0498	0.03172
0.8	0.07695	0.05723	0.03439	0.04607	0.05725	0.02755	0.03439	0.02212

Table 20. Additive Desirability Inference Observation 17, (x_1, x_2) = (-0.5,0.5) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.964	0.9028	0.9484	0.8262	0.9023	0.9017	0.9485	0.8292
-0.5	0.9824	0.926	0.9499	0.8634	0.9262	0.9034	0.9493	0.8337
-0.3	0.9853	0.9391	0.9514	0.8809	0.9392	0.9004	0.9509	0.8257
0	0.9944	0.9647	0.9501	0.9231	0.9652	0.8991	0.9519	0.8298
0.3	0.9978	0.9828	0.9545	0.9563	0.9839	0.907	0.9536	0.837
0.5	0.9983	0.9903	0.9537	0.9726	0.9902	0.9051	0.9531	0.8244
0.8	0.999	0.9955	0.9485	0.9877	0.9952	0.8992	0.9487	0.8306
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.06643	0.04938	0.06312	0.03974	0.04937	0.05055	0.06307	0.0408
-0.5	0.08093	0.05879	0.07049	0.04731	0.05879	0.0565	0.07048	0.04539
-0.3	0.05684	0.04086	0.04596	0.03287	0.04086	0.03681	0.04594	0.02952
0	0.04222	0.03024	0.02993	0.02434	0.03024	0.02398	0.02994	0.01926
0.3	0.05709	0.04105	0.03413	0.03303	0.04105	0.02735	0.03414	0.02199
0.5	0.08037	0.0583	0.04269	0.04693	0.05833	0.03421	0.04271	0.02717
0.8	0.06598	0.04908	0.02947	0.03952	0.0491	0.02362	0.02948	0.01895

Table 21. Additive Desirability Inference Observation 18, (x_1, x_2) = (0,0.5) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.963	0.8983	0.9511	0.8311	0.8996	0.8985	0.95	0.8349
-0.5	0.9809	0.9242	0.9478	0.8613	0.9251	0.903	0.9474	0.8293
-0.3	0.986	0.9417	0.9498	0.8787	0.9413	0.9045	0.9495	0.8231
0	0.9945	0.9689	0.9481	0.9267	0.9674	0.898	0.9493	0.8308
0.3	0.9979	0.9844	0.9553	0.9608	0.9848	0.9081	0.9542	0.8346
0.5	0.999	0.9888	0.9486	0.9734	0.9891	0.9048	0.9513	0.828
0.8	0.9989	0.9957	0.9471	0.9864	0.9958	0.9028	0.9472	0.8333
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.07747	0.05757	0.07359	0.04633	0.05759	0.05898	0.07357	0.04763
-0.5	0.09439	0.06853	0.08228	0.05518	0.06854	0.06587	0.08221	0.05299
-0.3	0.06629	0.04764	0.05357	0.03834	0.04766	0.04294	0.05359	0.03447
0	0.04923	0.0353	0.03492	0.02839	0.03528	0.02797	0.03491	0.02248
0.3	0.06658	0.04784	0.03981	0.03852	0.04788	0.03189	0.03981	0.02567
0.5	0.09373	0.06799	0.04978	0.05473	0.06802	0.03989	0.0498	0.03171
0.8	0.07695	0.05724	0.03438	0.04608	0.05727	0.02755	0.03438	0.02212

Table 22. Additive Desirability Inference Observation 19, (x_1, x_2) = (0.5,0.5) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9634	0.8997	0.9506	0.8344	0.9021	0.8994	0.9514	0.8403
-0.5	0.9804	0.9263	0.9484	0.86	0.9245	0.8984	0.9484	0.8322
-0.3	0.988	0.9402	0.9507	0.8805	0.9403	0.9049	0.9507	0.8288
0	0.9946	0.9671	0.9468	0.9226	0.9678	0.9017	0.9475	0.8313
0.3	0.9986	0.9858	0.9539	0.9611	0.9857	0.9067	0.9531	0.837
0.5	0.9986	0.9909	0.9487	0.974	0.9912	0.9019	0.9485	0.8306
0.8	0.9992	0.996	0.9469	0.9877	0.9956	0.9004	0.9476	0.8339
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.10374	0.07705	0.09854	0.06203	0.07704	0.07898	0.09852	0.06418
-0.5	0.12641	0.09175	0.11015	0.0739	0.09176	0.08823	0.11007	0.0714
-0.3	0.08879	0.06381	0.0718	0.05136	0.06384	0.05753	0.07179	0.04644
0	0.06595	0.04724	0.04675	0.03803	0.04726	0.03747	0.04676	0.03029
0.3	0.08918	0.06412	0.05331	0.0516	0.0641	0.04271	0.05334	0.03457
0.5	0.12553	0.09109	0.06656	0.07329	0.09103	0.05341	0.06656	0.04273
0.8	0.10304	0.07661	0.0459	0.06171	0.07661	0.0369	0.04589	0.0298

Table 23. Additive Desirability Inference Observation 20, (x_1, x_2) = (1,0.5) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.962	0.9006	0.9496	0.8306	0.8989	0.8997	0.9504	0.8356
-0.5	0.9815	0.9295	0.9488	0.8674	0.9277	0.9036	0.949	0.8381
-0.3	0.987	0.9415	0.9494	0.886	0.9415	0.9039	0.9491	0.8325
0	0.9937	0.9653	0.9533	0.9191	0.9662	0.9045	0.9541	0.8259
0.3	0.998	0.9845	0.9479	0.961	0.9853	0.9024	0.9486	0.8429
0.5	0.999	0.9903	0.948	0.9753	0.9909	0.9039	0.9483	0.8272
0.8	0.9994	0.9957	0.9486	0.9862	0.9954	0.8979	0.9496	0.8378
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.12464	0.0926	0.11842	0.07457	0.09264	0.09485	0.11832	0.07747
-0.5	0.15185	0.11029	0.13232	0.08873	0.11031	0.10601	0.13223	0.0862
-0.3	0.10596	0.0757	0.08536	0.0615	0.0757	0.06888	0.08533	0.05597
0	0.07876	0.05612	0.05563	0.04555	0.05612	0.04488	0.05563	0.03653
0.3	0.10645	0.07608	0.06333	0.06179	0.07609	0.05115	0.06335	0.04167
0.5	0.15079	0.10944	0.08003	0.08802	0.10947	0.06417	0.0801	0.05156
0.8	0.12379	0.09214	0.05526	0.07411	0.09207	0.04432	0.05528	0.03598

Table 24. Additive Desirability Inference Observation 21, (x_1, x_2) = (-1,1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9623	0.8984	0.95	0.8293	0.8993	0.9019	0.9502	0.8318
-0.5	0.9833	0.928	0.95	0.8678	0.9269	0.9043	0.9491	0.8414
-0.3	0.9879	0.9409	0.949	0.8825	0.9414	0.9032	0.949	0.8308
0	0.9938	0.965	0.9517	0.9215	0.9647	0.9016	0.9516	0.8262
0.3	0.9977	0.9852	0.9507	0.9605	0.9841	0.9033	0.9495	0.8414
0.5	0.9976	0.99	0.9495	0.9757	0.9906	0.9003	0.9498	0.8266
0.8	0.9992	0.9954	0.9483	0.9869	0.9955	0.899	0.9482	0.8333
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.10377	0.07711	0.09855	0.06209	0.07713	0.07898	0.09852	0.06418
-0.5	0.12642	0.09176	0.11015	0.07387	0.09182	0.08825	0.1101	0.07141
-0.3	0.08879	0.06384	0.07177	0.05136	0.0638	0.0575	0.07176	0.04643
0	0.06595	0.04726	0.04676	0.03802	0.04727	0.03746	0.04676	0.03029
0.3	0.08918	0.0641	0.05329	0.05161	0.06415	0.04271	0.05332	0.03457
0.5	0.12554	0.09109	0.06666	0.0733	0.09113	0.05342	0.0667	0.04272
0.8	0.10307	0.07665	0.04607	0.06171	0.07667	0.03691	0.04607	0.0298

Table 25. Additive Desirability Inference Observation 22, (x_1, x_2) = (-0.5,1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9636	0.8996	0.9474	0.8286	0.8997	0.9013	0.9477	0.8304
-0.5	0.9814	0.9267	0.9496	0.864	0.9254	0.9016	0.9488	0.8325
-0.3	0.9864	0.9407	0.95	0.8823	0.9407	0.9027	0.9494	0.8254
0	0.9944	0.9649	0.9502	0.9244	0.9653	0.8964	0.9498	0.8336
0.3	0.9974	0.9847	0.9531	0.9582	0.985	0.9076	0.9523	0.8393
0.5	0.9981	0.9903	0.9497	0.9733	0.9901	0.9034	0.9503	0.8269
0.8	0.9993	0.9956	0.9476	0.9853	0.9952	0.8985	0.9465	0.8322
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.09581	0.07122	0.09098	0.05733	0.0712	0.07292	0.09096	0.05885
-0.5	0.11673	0.08475	0.10178	0.06823	0.08478	0.08147	0.10164	0.0655
-0.3	0.08198	0.05894	0.06627	0.04741	0.05893	0.05309	0.06627	0.04258
0	0.06089	0.04364	0.04319	0.03512	0.04363	0.03459	0.04317	0.02778
0.3	0.08234	0.05918	0.04924	0.04765	0.05921	0.03945	0.04923	0.03172
0.5	0.11591	0.08411	0.06157	0.06768	0.08411	0.04934	0.06158	0.0392
0.8	0.09516	0.07083	0.04253	0.05699	0.07082	0.03407	0.04253	0.02733

Table 26. Additive Desirability Inference Observation 23, (x_1, x_2) = (0,1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9649	0.9013	0.949	0.8271	0.9008	0.8992	0.949	0.835
-0.5	0.9816	0.9248	0.9449	0.8581	0.9255	0.9017	0.9462	0.8326
-0.3	0.9854	0.9407	0.9495	0.8796	0.9404	0.9014	0.9498	0.8254
0	0.9937	0.9689	0.9486	0.9252	0.969	0.8966	0.9472	0.8346
0.3	0.9984	0.9829	0.9532	0.9594	0.9836	0.9096	0.9523	0.8416
0.5	0.9989	0.9892	0.9495	0.9751	0.9895	0.9024	0.9507	0.829
0.8	0.9993	0.9953	0.948	0.9874	0.9952	0.8969	0.9473	0.8352
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.10377	0.07712	0.09855	0.06208	0.07714	0.07899	0.09853	0.06417
-0.5	0.12642	0.09171	0.11006	0.07391	0.09176	0.08826	0.11009	0.07141
-0.3	0.08879	0.06383	0.07177	0.05136	0.06385	0.05751	0.07182	0.04643
0	0.06595	0.04725	0.04674	0.03803	0.04725	0.03745	0.04678	0.03029
0.3	0.08918	0.06413	0.0533	0.05161	0.06413	0.04272	0.05329	0.03457
0.5	0.12554	0.09101	0.06668	0.0733	0.09101	0.05342	0.06669	0.04272
0.8	0.10307	0.07671	0.04604	0.06172	0.07672	0.0369	0.04605	0.0298

Table 27. Additive Desirability Inference Observation 24, (x_1, x_2) = (0.5,1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9624	0.9027	0.9505	0.8352	0.9048	0.8988	0.9503	0.8413
-0.5	0.9802	0.926	0.9478	0.8591	0.9246	0.9008	0.9493	0.8335
-0.3	0.9864	0.9413	0.9489	0.8806	0.9413	0.9014	0.9488	0.8294
0	0.9947	0.9676	0.948	0.9257	0.9667	0.9001	0.9482	0.8336
0.3	0.9989	0.9853	0.9537	0.9596	0.9844	0.9076	0.9522	0.8374
0.5	0.9989	0.9904	0.9493	0.9744	0.9901	0.901	0.9504	0.8336
0.8	0.9994	0.9944	0.9462	0.9876	0.9947	0.8996	0.9457	0.8375
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.12325	0.09089	0.1168	0.07419	0.09098	0.09442	0.11676	0.07733
-0.5	0.15013	0.10806	0.13025	0.08834	0.10806	0.10548	0.1302	0.08601
-0.3	0.10612	0.07589	0.08552	0.06157	0.07593	0.06892	0.08556	0.056
0	0.07879	0.05613	0.05561	0.04557	0.05615	0.04486	0.05563	0.03654
0.3	0.10658	0.07625	0.06366	0.06186	0.07624	0.05122	0.06364	0.04171
0.5	0.14907	0.10712	0.07863	0.08759	0.10706	0.06386	0.07864	0.05149
0.8	0.12239	0.09033	0.05453	0.07376	0.0904	0.04413	0.0545	0.03591

Table 28. Additive Desirability Inference Observation 25, (x_1, x_2) = (1,1) Empirical Coverage and Average Width

A.2 Multiplicative Empirical Coverage and Average Width Tables

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9638	0.8901	0.9478	0.82	0.8914	0.8996	0.9483	0.8398
-0.5	0.9795	0.9157	0.9473	0.8532	0.9154	0.9015	0.9479	0.8289
-0.3	0.9749	0.9376	0.948	0.8841	0.9407	0.8958	0.9481	0.8285
0	0.9718	0.9478	0.9455	0.8982	0.9469	0.9026	0.945	0.8292
0.3	0.9805	0.9595	0.9523	0.9155	0.961	0.9057	0.9526	0.8324
0.5	0.9847	0.9644	0.951	0.9284	0.9654	0.9023	0.9518	0.8353
0.8	0.985	0.9702	0.9446	0.9388	0.9713	0.8963	0.9451	0.8328
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.11799	0.09206	0.11266	0.07127	0.09206	0.09005	0.11267	0.07343
-0.5	0.15134	0.11913	0.13654	0.09234	0.11913	0.10764	0.13644	0.08642
-0.3	0.1496	0.12938	0.13626	0.10109	0.12949	0.10654	0.13628	0.08477
0	0.13463	0.122	0.12322	0.09501	0.12199	0.09612	0.12329	0.07577
0.3	0.16173	0.14295	0.13608	0.11128	0.14287	0.1056	0.13603	0.08439
0.5	0.22442	0.19831	0.18516	0.15471	0.19838	0.1434	0.18524	0.11277
0.8	0.18445	0.16471	0.1471	0.12842	0.16465	0.11351	0.1471	0.08992

Table 29. Multiplicative Desirability Inference Observation 1, (x_1, x_2) = (-1,-1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9636	0.8928	0.9479	0.8151	0.8931	0.9034	0.9484	0.8344
-0.5	0.9773	0.9169	0.9471	0.8513	0.9172	0.8982	0.9477	0.8263
-0.3	0.9839	0.9389	0.9481	0.8833	0.9401	0.9014	0.948	0.8317
0	0.9906	0.9619	0.9453	0.9132	0.9611	0.8996	0.9456	0.8321
0.3	0.9904	0.967	0.9537	0.9317	0.9672	0.9062	0.9556	0.8327
0.5	0.9932	0.976	0.9498	0.9453	0.9754	0.9	0.949	0.8299
0.8	0.9987	0.9939	0.9458	0.9869	0.9941	0.8962	0.9473	0.8322
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.09844	0.07051	0.09301	0.0565	0.07055	0.07444	0.093	0.06052
-0.5	0.12602	0.09262	0.1105	0.07374	0.09265	0.08805	0.11045	0.07093
-0.3	0.0925	0.06893	0.07654	0.0553	0.06899	0.06118	0.07654	0.04929
0	0.07155	0.05466	0.05465	0.04388	0.05466	0.04371	0.05465	0.03515
0.3	0.1044	0.08365	0.07585	0.06701	0.08366	0.06051	0.07582	0.04899
0.5	0.14874	0.12077	0.10453	0.09621	0.12082	0.08274	0.1045	0.06577
0.8	0.10338	0.07725	0.04688	0.06204	0.07728	0.03736	0.04689	0.03009

Table 30. Multiplicative Desirability Inference Observation 2, (x_1, x_2) = (-0.5,-1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.9639	0.8883	0.9525	0.8141	0.888	0.9044	0.9514	0.8283
-0.5	0.9758	0.921	0.9479	0.8554	0.9213	0.8967	0.9476	0.8278
-0.3	0.9852	0.9388	0.9512	0.8794	0.9377	0.9021	0.9501	0.8295
0	0.9941	0.9653	0.9514	0.9234	0.9657	0.9029	0.9492	0.8274
0.3	0.9943	0.9748	0.9558	0.9424	0.9745	0.9061	0.9542	0.8308
0.5	0.9979	0.9875	0.9512	0.9647	0.9868	0.9032	0.9515	0.8283
0.8	1	0.9995	0.9467	0.9981	0.9996	0.8997	0.9452	0.8272
Average Width								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.09103	0.06499	0.086	0.05221	0.06498	0.06886	0.08598	0.05562
-0.5	0.11734	0.08609	0.10288	0.06899	0.08608	0.0822	0.10277	0.06586
-0.3	0.0818	0.05891	0.0662	0.04731	0.05889	0.05294	0.06621	0.04247
0	0.06093	0.04375	0.04333	0.03519	0.04377	0.03468	0.04333	0.02785
0.3	0.08979	0.0689	0.0607	0.05537	0.06895	0.04856	0.06068	0.03914
0.5	0.12218	0.09267	0.07323	0.07425	0.09264	0.05833	0.07327	0.04616
0.8	0.08988	0.06389	0.02932	0.05135	0.06386	0.02328	0.02935	0.01861

Table 31. Multiplicative Desirability Inference Observation 3, (x_1, x_2) = (0,-1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.9645	0.8856	0.956	0.8142	0.8859	0.9073	0.9548	0.8278
-0.5	0.9778	0.9244	0.9487	0.8619	0.9225	0.8965	0.9478	0.8335
-0.3	0.9825	0.9403	0.9486	0.8842	0.9415	0.9026	0.9507	0.8331
0	0.9898	0.959	0.9496	0.9113	0.9583	0.9083	0.9498	0.8312
0.3	0.9965	0.9814	0.9518	0.9493	0.9811	0.9053	0.9536	0.8334
0.5	0.9992	0.9923	0.949	0.9758	0.9921	0.9013	0.9499	0.8249
0.8	0.9992	0.9954	0.9499	0.9849	0.9955	0.9026	0.9515	0.8327
Average Width								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.09871	0.07043	0.09325	0.05665	0.07048	0.07468	0.09324	0.06072
-0.5	0.12813	0.0945	0.11244	0.07578	0.09447	0.08996	0.11241	0.07259
-0.3	0.09597	0.07341	0.08055	0.0588	0.07341	0.06428	0.08054	0.05194
0	0.07165	0.05482	0.05448	0.04397	0.05479	0.04356	0.05449	0.03527
0.3	0.09406	0.07058	0.06098	0.05671	0.07059	0.04881	0.06097	0.03958
0.5	0.12464	0.09037	0.06609	0.0725	0.09045	0.05243	0.06611	0.04166
0.8	0.10363	0.07755	0.04736	0.06235	0.07759	0.03781	0.0474	0.03048

Table 32. Multiplicative Desirability Inference Observation 4, (x_1, x_2) = (0.5,-1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9636	0.8882	0.953	0.8128	0.8878	0.906	0.9545	0.8318
-0.5	0.9775	0.9263	0.9474	0.8612	0.9257	0.8999	0.947	0.841
-0.3	0.9745	0.9452	0.9486	0.8928	0.9466	0.9001	0.9506	0.8355
0	0.9723	0.9475	0.9511	0.9007	0.948	0.9001	0.9515	0.8342
0.3	0.9972	0.9825	0.9517	0.957	0.9826	0.902	0.9509	0.8388
0.5	0.9987	0.9914	0.9479	0.9772	0.992	0.9003	0.949	0.8279
0.8	0.9918	0.9776	0.9512	0.9494	0.9792	0.9064	0.951	0.8331
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.11864	0.0847	0.11208	0.06813	0.08473	0.08978	0.11205	0.07335
-0.5	0.15495	0.11479	0.13625	0.0921	0.11479	0.10896	0.13622	0.08837
-0.3	0.15721	0.13804	0.14374	0.10764	0.13796	0.11217	0.14388	0.08988
0	0.13475	0.12214	0.12239	0.09521	0.12216	0.09522	0.12245	0.07612
0.3	0.11044	0.08156	0.06992	0.06589	0.08157	0.05604	0.06994	0.04566
0.5	0.15049	0.11014	0.0818	0.08798	0.11013	0.06435	0.08182	0.05124
0.8	0.15333	0.12888	0.10564	0.10254	0.12893	0.08348	0.10568	0.06732

Table 33. Multiplicative Desirability Inference Observation 5, (x_1, x_2) = (1,-1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9638	0.9068	0.9501	0.8368	0.9063	0.8994	0.9506	0.8387
-0.5	0.9775	0.9239	0.9498	0.8605	0.925	0.8995	0.9492	0.8298
-0.3	0.9828	0.9395	0.9495	0.88	0.942	0.9011	0.9479	0.8252
0	0.9853	0.9516	0.9461	0.9047	0.9531	0.9011	0.9465	0.8275
0.3	0.996	0.979	0.9481	0.9498	0.979	0.9049	0.9489	0.8285
0.5	0.9968	0.9856	0.9507	0.9638	0.9858	0.9016	0.9502	0.832
0.8	0.998	0.9918	0.9448	0.9792	0.9918	0.899	0.9423	0.8307
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.10756	0.08296	0.10268	0.06569	0.08293	0.08155	0.10266	0.06604
-0.5	0.13425	0.10301	0.11933	0.08176	0.10295	0.09476	0.11928	0.07638
-0.3	0.09597	0.07332	0.08066	0.05883	0.07331	0.06448	0.08059	0.05198
0	0.07942	0.0644	0.06469	0.05161	0.06439	0.05173	0.06467	0.04156
0.3	0.09242	0.06859	0.05864	0.05508	0.06862	0.0468	0.05865	0.03792
0.5	0.13203	0.09975	0.07868	0.08011	0.09972	0.06276	0.07869	0.05011
0.8	0.10682	0.08164	0.054	0.06559	0.08169	0.04307	0.05402	0.03471

Table 34. Multiplicative Desirability Inference Observation 6, (x_1, x_2) = (-1,-0.5) Empirical Coverage and Average Width

				Empirical Coverage				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9643	0.898	0.9491	0.825	0.8985	0.9011	0.9486	0.8304
-0.5	0.9789	0.9199	0.9452	0.8541	0.9191	0.8979	0.945	0.826
-0.3	0.9852	0.939	0.9498	0.8818	0.9393	0.8994	0.949	0.8264
0	0.9932	0.964	0.9484	0.9188	0.9638	0.9019	0.9477	0.8284
0.3	0.9969	0.9806	0.95	0.9525	0.9806	0.9055	0.9501	0.8292
0.5	0.999	0.9908	0.9491	0.9737	0.9909	0.9011	0.95	0.8263
0.8	1	0.999	0.948	0.997	0.999	0.8978	0.9475	0.8295
				Average Width				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.07624	0.05608	0.07226	0.04506	0.05608	0.05788	0.0723	0.04675
-0.5	0.09382	0.06803	0.08168	0.05462	0.06802	0.0654	0.08169	0.05253
-0.3	0.06668	0.04824	0.05412	0.03878	0.04822	0.04335	0.05412	0.03477
0	0.05123	0.03791	0.03779	0.0305	0.03794	0.03026	0.03779	0.02425
0.3	0.06755	0.0492	0.04136	0.03958	0.04924	0.03311	0.04137	0.02668
0.5	0.0939	0.06835	0.05034	0.05494	0.06833	0.04023	0.05034	0.03191
0.8	0.07292	0.05191	0.02423	0.04175	0.05194	0.01933	0.02423	0.0155

Table 35. Multiplicative Desirability Inference Observation 7, (x_1, x_2) = (-0.5,-0.5) Empirical Coverage and Average Width

				Empirical Coverage				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9633	0.8931	0.9513	0.8178	0.8926	0.9049	0.9519	0.8253
-0.5	0.9788	0.9194	0.9477	0.855	0.9184	0.8978	0.9477	0.827
-0.3	0.9858	0.9388	0.9501	0.883	0.9373	0.9042	0.9518	0.8274
0	0.9938	0.9672	0.9491	0.924	0.9662	0.9059	0.949	0.8289
0.3	0.9968	0.9832	0.9553	0.9587	0.9837	0.9096	0.9546	0.8285
0.5	0.9993	0.9931	0.9501	0.9778	0.9932	0.9043	0.9491	0.8218
0.8	0.9998	0.9981	0.9514	0.9946	0.9983	0.9056	0.9499	0.8302
				Average Width				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.06463	0.04703	0.06118	0.03783	0.04703	0.04903	0.0612	0.03958
-0.5	0.07943	0.05685	0.06884	0.04571	0.05683	0.05517	0.06884	0.04428
-0.3	0.05678	0.04079	0.04588	0.03283	0.0408	0.03675	0.04588	0.02949
0	0.04223	0.03027	0.02996	0.02436	0.03028	0.02401	0.02996	0.01928
0.3	0.05741	0.04148	0.03463	0.03338	0.0415	0.02772	0.0346	0.02233
0.5	0.07888	0.05641	0.03999	0.04536	0.05637	0.03195	0.03997	0.02531
0.8	0.06355	0.04588	0.02359	0.03693	0.04589	0.01887	0.02361	0.01513

Table 36. Multiplicative Desirability Inference Observation 8, (x_1, x_2) = (0,-0.5) Empirical Coverage and Average Width

				Empirical Coverage				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9637	0.8944	0.9534	0.8191	0.8941	0.9068	0.9537	0.8278
-0.5	0.9788	0.9237	0.9472	0.8602	0.9242	0.8977	0.9481	0.8352
-0.3	0.9845	0.944	0.9514	0.8851	0.9432	0.9044	0.951	0.8304
0	0.9921	0.9628	0.9513	0.9163	0.961	0.9037	0.9513	0.8299
0.3	0.9977	0.9853	0.9536	0.9624	0.9854	0.9042	0.9549	0.8347
0.5	0.9986	0.9909	0.9497	0.9747	0.9908	0.8997	0.95	0.8261
0.8	0.998	0.9894	0.9511	0.9713	0.9895	0.9057	0.9498	0.8244
				Average Width				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.07499	0.0543	0.07092	0.0437	0.05433	0.05685	0.07096	0.04594
-0.5	0.09251	0.06612	0.0802	0.05316	0.06614	0.06423	0.08016	0.0516
-0.3	0.06937	0.05175	0.05733	0.04159	0.05174	0.04592	0.05737	0.0369
0	0.05128	0.03802	0.03772	0.03056	0.03801	0.03019	0.03771	0.02432
0.3	0.06666	0.048	0.03991	0.03861	0.048	0.03194	0.0399	0.02575
0.5	0.09349	0.06783	0.04961	0.0545	0.06783	0.03963	0.04962	0.03143
0.8	0.08215	0.06401	0.04484	0.05144	0.06401	0.03585	0.04483	0.02876

Table 37. Multiplicative Desirability Inference Observation 9, (x_1, x_2) = (0.5,-0.5) Empirical Coverage and Average Width

				Empirical Coverage				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9647	0.8924	0.9539	0.8183	0.8925	0.9093	0.9552	0.8327
-0.5	0.9804	0.9246	0.9472	0.8604	0.9248	0.8997	0.9477	0.835
-0.3	0.9788	0.9441	0.9495	0.8886	0.9452	0.902	0.9492	0.8337
0	0.9851	0.9531	0.9506	0.9055	0.9534	0.9039	0.9498	0.8325
0.3	0.9981	0.9864	0.9533	0.9631	0.9866	0.9052	0.9539	0.8362
0.5	0.9966	0.984	0.9484	0.9578	0.9842	0.8999	0.9496	0.8261
0.8	0.9873	0.9729	0.9524	0.9396	0.973	0.9023	0.9525	0.8306
				Average Width				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.10016	0.07233	0.09476	0.05822	0.0724	0.0759	0.09476	0.0617
-0.5	0.12421	0.08905	0.10782	0.07157	0.08905	0.08634	0.10778	0.06975
-0.3	0.10415	0.08352	0.08997	0.06687	0.0835	0.07191	0.09	0.05812
0	0.07959	0.06457	0.06443	0.05178	0.06457	0.05152	0.06444	0.04174
0.3	0.08908	0.06403	0.05313	0.0515	0.06405	0.04253	0.05313	0.03446
0.5	0.13633	0.10536	0.08574	0.08433	0.10546	0.06819	0.08577	0.0544
0.8	0.14065	0.12235	0.10605	0.09743	0.12236	0.08409	0.10608	0.0676

Table 38. Multiplicative Desirability Inference Observation 10, (x_1, x_2) = (1,-0.5) Empirical Coverage and Average Width

				Empirical Coverage				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9618	0.9141	0.9496	0.8535	0.915	0.9009	0.9497	0.8375
-0.5	0.9745	0.9297	0.9484	0.8722	0.9292	0.9009	0.9479	0.8335
-0.3	0.9846	0.9399	0.9511	0.8836	0.9404	0.9028	0.9504	0.8188
0	0.9883	0.9548	0.949	0.9078	0.9556	0.9001	0.9496	0.8273
0.3	0.9975	0.984	0.9468	0.9602	0.9836	0.9021	0.9472	0.8308
0.5	0.9992	0.9918	0.951	0.9781	0.9916	0.9	0.9503	0.8282
0.8	0.9999	0.9984	0.9479	0.9941	0.9984	0.902	0.9472	0.8291
				Average Width				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.10659	0.08527	0.10231	0.06801	0.08528	0.08153	0.10227	0.06558
-0.5	0.13088	0.10332	0.11753	0.08253	0.1033	0.09377	0.11757	0.07526
-0.3	0.08435	0.06212	0.06924	0.04996	0.06213	0.05545	0.06919	0.04446
0	0.06757	0.05228	0.05236	0.04204	0.05232	0.04192	0.05237	0.03353
0.3	0.08253	0.05957	0.04979	0.04789	0.05962	0.03977	0.04981	0.03192
0.5	0.11442	0.08224	0.059	0.0661	0.08226	0.04711	0.05898	0.03735
0.8	0.09131	0.06573	0.03322	0.05288	0.06576	0.02652	0.03322	0.02126

Table 39. Multiplicative Desirability Inference Observation 11, (x_1, x_2) = (-1,0) Empirical Coverage and Average Width

				Empirical Coverage				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9622	0.908	0.9493	0.8409	0.9048	0.9003	0.9494	0.8341
-0.5	0.9777	0.9276	0.9485	0.8616	0.9277	0.9006	0.9482	0.8302
-0.3	0.9852	0.9403	0.9497	0.8801	0.9413	0.9	0.9489	0.8227
0	0.9932	0.9643	0.9503	0.919	0.9643	0.9027	0.9507	0.8253
0.3	0.9979	0.9834	0.9506	0.9568	0.9827	0.9028	0.9512	0.8288
0.5	0.9992	0.993	0.9508	0.9789	0.9925	0.9035	0.951	0.8278
0.8	0.9999	0.9994	0.9485	0.9979	0.9995	0.902	0.9501	0.8298
				Average Width				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.06818	0.05168	0.06489	0.04157	0.0517	0.05198	0.06491	0.04194
-0.5	0.08348	0.06216	0.07342	0.04999	0.06223	0.05877	0.07339	0.04721
-0.3	0.05671	0.04071	0.04581	0.03275	0.0407	0.0367	0.04579	0.02943
0	0.04323	0.03161	0.03143	0.02542	0.0316	0.02517	0.03142	0.02016
0.3	0.05713	0.04112	0.03424	0.03309	0.04115	0.02741	0.03426	0.02203
0.5	0.07888	0.05638	0.03992	0.04534	0.05639	0.03193	0.03992	0.02531
0.8	0.0625	0.04445	0.0206	0.03576	0.04445	0.01647	0.0206	0.01321

Table 40. Multiplicative Desirability Inference Observation 12, (x_1, x_2) = (-0.5,0) Empirical Coverage and Average Width

Empirical Coverage								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.9636	0.895	0.9502	0.8282	0.8954	0.9028	0.9484	0.8339
-0.5	0.979	0.9223	0.9476	0.8616	0.923	0.8997	0.9481	0.8358
-0.3	0.9858	0.9401	0.9519	0.8855	0.9406	0.9042	0.9503	0.8315
0	0.9943	0.9666	0.9507	0.9226	0.9679	0.9015	0.9515	0.835
0.3	0.997	0.9832	0.9573	0.9584	0.9835	0.9097	0.9555	0.8377
0.5	0.9985	0.9893	0.9495	0.9732	0.9887	0.9044	0.951	0.8292
0.8	0.9996	0.9968	0.9516	0.9875	0.9962	0.9039	0.9514	0.8307
Average Width								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.05317	0.03955	0.0505	0.03182	0.03955	0.04045	0.05048	0.03289
-0.5	0.06477	0.04708	0.05645	0.03787	0.04708	0.04521	0.05643	0.0366
-0.3	0.04548	0.03271	0.03676	0.02632	0.03271	0.02946	0.03677	0.02381
0	0.03378	0.02421	0.02397	0.01947	0.0242	0.0192	0.02396	0.01552
0.3	0.04568	0.03285	0.02733	0.02643	0.03285	0.02189	0.02734	0.01773
0.5	0.06433	0.04671	0.03427	0.03758	0.04672	0.02744	0.03426	0.02194
0.8	0.05281	0.03932	0.02365	0.03164	0.03933	0.01894	0.02365	0.01531

Table 41. Multiplicative Desirability Inference Observation 13, (x_1, x_2) = (0,0) Empirical Coverage and Average Width

Empirical Coverage								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.9634	0.8991	0.9513	0.8259	0.8988	0.9038	0.9516	0.8285
-0.5	0.9801	0.9244	0.9483	0.8583	0.9231	0.9	0.9472	0.8312
-0.3	0.9842	0.9408	0.9528	0.8828	0.9408	0.9062	0.9515	0.8284
0	0.9924	0.9677	0.9496	0.922	0.9669	0.8992	0.9482	0.829
0.3	0.998	0.9866	0.9569	0.9602	0.986	0.9097	0.9569	0.8325
0.5	0.9962	0.9847	0.9494	0.9626	0.9846	0.9014	0.9498	0.8236
0.8	0.9954	0.9826	0.9496	0.9603	0.9828	0.9028	0.9477	0.8307
Average Width								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.06564	0.04834	0.06226	0.03889	0.04835	0.04987	0.06225	0.04025
-0.5	0.0799	0.05745	0.06939	0.04622	0.05746	0.05556	0.06935	0.04464
-0.3	0.05876	0.04341	0.04831	0.03492	0.04342	0.0387	0.04832	0.03105
0	0.04327	0.03165	0.03135	0.02546	0.03164	0.02512	0.03136	0.02021
0.3	0.05709	0.04106	0.03414	0.03303	0.04104	0.02734	0.03415	0.02198
0.5	0.08527	0.06475	0.05146	0.05205	0.0648	0.04119	0.05148	0.03274
0.8	0.07526	0.06087	0.04672	0.0489	0.06087	0.03736	0.04671	0.02995

Table 42. Multiplicative Desirability Inference Observation 14, (x_1, x_2) = (0.5,0) Empirical Coverage and Average Width

Empirical Coverage								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.9657	0.8943	0.9548	0.8274	0.8953	0.9087	0.9545	0.8338
-0.5	0.98	0.9211	0.9468	0.8571	0.92	0.9014	0.9468	0.8304
-0.3	0.9819	0.9403	0.9507	0.8853	0.9404	0.9022	0.9503	0.8297
0	0.988	0.9589	0.9488	0.9116	0.9585	0.9007	0.9481	0.8321
0.3	0.9987	0.9874	0.9552	0.9629	0.9866	0.9058	0.9547	0.8327
0.5	0.9911	0.9726	0.9473	0.9391	0.9723	0.8985	0.9469	0.8269
0.8	0.9812	0.9655	0.951	0.9318	0.9665	0.9057	0.9511	0.8317
Average Width								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.09398	0.0688	0.08909	0.05535	0.06883	0.07135	0.08907	0.05761
-0.5	0.11461	0.08203	0.09929	0.06599	0.08206	0.07958	0.09933	0.06392
-0.3	0.09075	0.07035	0.07679	0.05649	0.07035	0.0615	0.07684	0.04938
0	0.06769	0.05245	0.0522	0.04213	0.05245	0.0418	0.0522	0.03366
0.3	0.08233	0.05919	0.04926	0.04764	0.05922	0.03941	0.04928	0.03171
0.5	0.14214	0.1173	0.1031	0.09354	0.11726	0.08195	0.10312	0.06505
0.8	0.14448	0.12937	0.11661	0.10288	0.12941	0.0924	0.11667	0.07376

Table 43. Multiplicative Desirability Inference Observation 15, (x_1, x_2) = (1,0) Empirical Coverage and Average Width

Empirical Coverage								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.9606	0.919	0.95	0.8586	0.9188	0.9017	0.9488	0.837
-0.5	0.9736	0.9335	0.9492	0.8752	0.9339	0.8998	0.9485	0.8379
-0.3	0.9853	0.9434	0.9519	0.8825	0.9434	0.9021	0.9516	0.8244
0	0.9899	0.9579	0.9519	0.9116	0.9586	0.903	0.9523	0.8272
0.3	0.9937	0.9746	0.9513	0.9395	0.9747	0.9017	0.9495	0.8343
0.5	0.9989	0.9935	0.9497	0.9787	0.9941	0.9039	0.9494	0.8266
0.8	0.9999	0.9993	0.9506	0.9977	0.9993	0.8994	0.9508	0.8296
Average Width								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.12216	0.10054	0.1178	0.08011	0.10047	0.09387	0.11769	0.07596
-0.5	0.14782	0.11923	0.13398	0.0953	0.11926	0.10688	0.13392	0.08645
-0.3	0.08972	0.06508	0.07296	0.05237	0.06509	0.05845	0.07289	0.04715
0	0.07056	0.0533	0.05324	0.04288	0.05331	0.04263	0.05324	0.03435
0.3	0.0999	0.07811	0.07015	0.06258	0.07811	0.05598	0.07016	0.045
0.5	0.12355	0.08856	0.0631	0.07118	0.0886	0.05041	0.06311	0.04024
0.8	0.09728	0.069	0.03136	0.05549	0.06899	0.02498	0.03136	0.02013

Table 44. Multiplicative Desirability Inference Observation 16, (x_1, x_2) = (-1,0.5) Empirical Coverage and Average Width

Empirical Coverage								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.9616	0.9064	0.9513	0.844	0.9071	0.9013	0.9512	0.8324
-0.5	0.9797	0.9315	0.9478	0.8698	0.9307	0.9007	0.9473	0.8333
-0.3	0.9877	0.9403	0.9498	0.8814	0.9412	0.8989	0.9488	0.8228
0	0.9935	0.9633	0.9518	0.9201	0.9629	0.9072	0.9521	0.8262
0.3	0.9958	0.977	0.9503	0.9418	0.9769	0.9054	0.9513	0.8322
0.5	0.9979	0.9906	0.9512	0.9756	0.9907	0.9036	0.9528	0.8224
0.8	0.9997	0.9983	0.9486	0.994	0.9986	0.9006	0.9488	0.8337
Average Width								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.08263	0.06435	0.07894	0.05167	0.0643	0.06323	0.07893	0.05105
-0.5	0.10087	0.07707	0.08953	0.06191	0.07708	0.07167	0.0895	0.05768
-0.3	0.06599	0.04726	0.05323	0.03803	0.04725	0.04265	0.0532	0.03421
0	0.05002	0.03633	0.03605	0.02923	0.03633	0.0289	0.03608	0.02318
0.3	0.07155	0.0544	0.04783	0.04372	0.05442	0.03828	0.04784	0.03068
0.5	0.09407	0.06851	0.05052	0.05508	0.06855	0.04044	0.05057	0.03214
0.8	0.07448	0.05396	0.02847	0.04344	0.05399	0.02278	0.02847	0.01828

Table 45. Multiplicative Desirability Inference Observation 17, (x_1, x_2) = (-0.5,0.5) Empirical Coverage and Average Width

Empirical Coverage								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.9629	0.9066	0.9482	0.8327	0.906	0.9018	0.9495	0.8292
-0.5	0.9813	0.9276	0.9504	0.8671	0.9288	0.904	0.9502	0.8349
-0.3	0.9852	0.9393	0.951	0.8807	0.9389	0.9	0.9505	0.825
0	0.9942	0.9644	0.9511	0.9229	0.9653	0.8994	0.9523	0.8293
0.3	0.9965	0.9782	0.9535	0.9476	0.9788	0.9084	0.9541	0.8331
0.5	0.9964	0.9848	0.9527	0.9609	0.9849	0.9058	0.953	0.8241
0.8	0.9985	0.9919	0.9467	0.9775	0.9916	0.9002	0.9476	0.8294
Average Width								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.06831	0.05186	0.06509	0.04171	0.05186	0.05213	0.06503	0.04206
-0.5	0.08305	0.06162	0.07289	0.04953	0.06161	0.0584	0.07286	0.04694
-0.3	0.05691	0.04095	0.04605	0.03295	0.04096	0.03688	0.04603	0.02957
0	0.04222	0.03025	0.02995	0.02435	0.03025	0.02399	0.02995	0.01926
0.3	0.06012	0.04503	0.03909	0.03622	0.04504	0.03131	0.03909	0.02509
0.5	0.08449	0.06373	0.05015	0.05126	0.06378	0.04014	0.05018	0.03191
0.8	0.06918	0.05326	0.03607	0.04284	0.05327	0.02889	0.03608	0.02316

Table 46. Multiplicative Desirability Inference Observation 18, (x_1, x_2) = (0,0.5) Empirical Coverage and Average Width

				Empirical Coverage				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9632	0.8999	0.9512	0.8333	0.9013	0.8987	0.9503	0.8346
-0.5	0.9806	0.9253	0.9481	0.8623	0.925	0.903	0.9479	0.83
-0.3	0.9853	0.941	0.9515	0.8776	0.9413	0.9033	0.9512	0.82
0	0.9942	0.9664	0.9485	0.9225	0.9661	0.9007	0.9491	0.8325
0.3	0.9965	0.9808	0.9531	0.9514	0.9813	0.9066	0.9522	0.8344
0.5	0.9931	0.9741	0.9501	0.944	0.9745	0.9025	0.9505	0.8281
0.8	0.9909	0.9781	0.9485	0.9509	0.9778	0.9046	0.949	0.8355
				Average Width				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.07817	0.0585	0.07432	0.04707	0.05854	0.05956	0.0743	0.04809
-0.5	0.09479	0.06911	0.08276	0.05562	0.06912	0.06624	0.08269	0.05328
-0.3	0.0681	0.05005	0.05579	0.04026	0.05007	0.0447	0.05581	0.03589
0	0.05006	0.0364	0.03604	0.02927	0.03638	0.02885	0.03602	0.02323
0.3	0.06928	0.05142	0.04426	0.04138	0.05145	0.03545	0.04427	0.02847
0.5	0.10817	0.08646	0.07368	0.06941	0.08655	0.05892	0.07373	0.0469
0.8	0.09422	0.07875	0.06434	0.06318	0.07882	0.05145	0.06432	0.04124

Table 47. Multiplicative Desirability Inference Observation 19, (x_1, x_2) = (0.5,0.5) Empirical Coverage and Average Width

				Empirical Coverage				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9633	0.8996	0.9505	0.8334	0.9013	0.8996	0.9513	0.8406
-0.5	0.9808	0.9255	0.9489	0.8588	0.9236	0.8983	0.9486	0.8312
-0.3	0.983	0.9411	0.9496	0.8808	0.9396	0.9007	0.9489	0.8258
0	0.9895	0.9608	0.95	0.9145	0.9612	0.9028	0.9508	0.8343
0.3	0.9979	0.9842	0.9523	0.955	0.9835	0.9077	0.9536	0.8385
0.5	0.983	0.9639	0.9477	0.9249	0.9638	0.899	0.9471	0.8323
0.8	0.9759	0.9639	0.9504	0.9242	0.964	0.905	0.9509	0.8391
				Average Width				
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.10348	0.07673	0.09827	0.06176	0.07673	0.07875	0.09824	0.06399
-0.5	0.12545	0.09054	0.10912	0.0729	0.09057	0.08737	0.10904	0.07068
-0.3	0.09568	0.07285	0.08012	0.05856	0.07287	0.06416	0.08012	0.05184
0	0.07066	0.05343	0.05309	0.04297	0.05345	0.04252	0.05311	0.03446
0.3	0.09208	0.06798	0.05817	0.05468	0.06797	0.04656	0.05817	0.03762
0.5	0.17954	0.15675	0.14499	0.12396	0.15666	0.11437	0.14504	0.09104
0.8	0.17798	0.16369	0.15227	0.12902	0.16371	0.11946	0.15226	0.09555

Table 48. Multiplicative Desirability Inference Observation 20, (x_1, x_2) = (1,0.5) Empirical Coverage and Average Width

Empirical Coverage								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.9582	0.9202	0.9483	0.8587	0.9223	0.9026	0.9486	0.8348
-0.5	0.9753	0.9356	0.9481	0.8799	0.9359	0.9008	0.9477	0.8423
-0.3	0.9864	0.9423	0.9509	0.8841	0.9425	0.902	0.9492	0.8301
0	0.9915	0.9573	0.9512	0.9148	0.9585	0.9027	0.954	0.8286
0.3	0.9791	0.9594	0.9513	0.9185	0.9595	0.9019	0.9504	0.8328
0.5	0.9988	0.9899	0.9475	0.9739	0.9905	0.9028	0.9476	0.8271
0.8	0.9998	0.9988	0.9481	0.9962	0.999	0.9021	0.9484	0.8337
Average Width								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.15379	0.12909	0.1488	0.10272	0.12916	0.11837	0.14866	0.09617
-0.5	0.18352	0.15043	0.16739	0.12011	0.15048	0.13347	0.16725	0.10849
-0.3	0.10635	0.07633	0.08596	0.06187	0.07635	0.0692	0.0859	0.05618
0	0.08276	0.06149	0.06142	0.0497	0.06149	0.04933	0.06143	0.04001
0.3	0.17799	0.16079	0.15667	0.12527	0.16075	0.12152	0.15659	0.09607
0.5	0.15178	0.1109	0.08221	0.0891	0.11095	0.06575	0.08228	0.05277
0.8	0.11786	0.08428	0.04062	0.06776	0.08422	0.0324	0.04065	0.02622

Table 49. Multiplicative Desirability Inference Observation 21, (x_1, x_2) = (-1,1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.9592	0.9119	0.9502	0.8469	0.9126	0.9013	0.9501	0.833
-0.5	0.9787	0.9309	0.9481	0.8763	0.9305	0.9033	0.95	0.8435
-0.3	0.9879	0.9409	0.9488	0.8816	0.9415	0.9033	0.9487	0.8274
0	0.9937	0.9632	0.9514	0.9194	0.9631	0.9029	0.9528	0.829
0.3	0.9865	0.9658	0.9501	0.9248	0.966	0.9089	0.9492	0.8314
0.5	0.9963	0.9855	0.9499	0.9647	0.9856	0.903	0.9496	0.8251
0.8	0.9993	0.9954	0.9491	0.987	0.9955	0.9002	0.9498	0.8342
Average Width								
ρ	BW	UG	MG	MVNS	MVtS	MVNSSig	MVtSSig	BSR
-0.8	0.11457	0.09102	0.10984	0.07308	0.09104	0.08793	0.10978	0.07136
-0.5	0.13937	0.1086	0.12465	0.0872	0.10865	0.09975	0.12458	0.08077
-0.3	0.08835	0.06329	0.07125	0.0509	0.06326	0.05708	0.07126	0.04607
0	0.0667	0.04828	0.04791	0.03884	0.04828	0.03837	0.0479	0.03097
0.3	0.11483	0.09614	0.09018	0.07686	0.09616	0.07192	0.09018	0.05778
0.5	0.13137	0.09882	0.07729	0.07942	0.09889	0.06186	0.07737	0.04947
0.8	0.10307	0.07671	0.04608	0.06171	0.07674	0.03687	0.04608	0.02975

Table 50. Multiplicative Desirability Inference Observation 22, (x_1, x_2) = (-0.5,1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9624	0.9062	0.948	0.8377	0.9066	0.902	0.9471	0.8308
-0.5	0.9789	0.9288	0.9491	0.8682	0.9291	0.9003	0.9484	0.837
-0.3	0.9863	0.9407	0.9504	0.8814	0.9412	0.9027	0.9487	0.8256
0	0.9943	0.9649	0.95	0.9243	0.9654	0.8966	0.9499	0.833
0.3	0.9909	0.9693	0.9513	0.9318	0.9689	0.9045	0.9503	0.8362
0.5	0.994	0.9769	0.9511	0.9477	0.9774	0.9045	0.9511	0.8258
0.8	0.9961	0.9859	0.9488	0.9641	0.9862	0.9013	0.9492	0.8323
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.10112	0.07814	0.09655	0.06282	0.07815	0.07735	0.09652	0.06239
-0.5	0.12281	0.0928	0.10863	0.07461	0.09279	0.08693	0.1085	0.06992
-0.3	0.08214	0.05918	0.06649	0.04759	0.05917	0.05326	0.0665	0.04271
0	0.06089	0.04366	0.04321	0.03513	0.04365	0.0346	0.0432	0.02779
0.3	0.09744	0.07845	0.07194	0.06295	0.07848	0.05754	0.0719	0.04602
0.5	0.13005	0.10239	0.08556	0.08218	0.10243	0.06841	0.08559	0.05441
0.8	0.1051	0.0836	0.06165	0.06712	0.08357	0.0493	0.06163	0.0395

Table 51. Multiplicative Desirability Inference Observation 23, (x_1, x_2) = (0,1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9646	0.9039	0.9487	0.8343	0.904	0.8997	0.949	0.835
-0.5	0.9796	0.9267	0.9463	0.8624	0.9272	0.9023	0.9467	0.8324
-0.3	0.9847	0.9389	0.9495	0.8815	0.9396	0.9021	0.9488	0.8253
0	0.9935	0.9665	0.9483	0.9247	0.9672	0.8988	0.9491	0.8326
0.3	0.9939	0.9724	0.9503	0.9379	0.9714	0.9042	0.9506	0.8415
0.5	0.9881	0.9663	0.9503	0.9311	0.9664	0.9041	0.9526	0.8256
0.8	0.9874	0.9727	0.9473	0.9401	0.9717	0.9018	0.9475	0.8357
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.10688	0.08123	0.1018	0.06532	0.08123	0.08158	0.10179	0.06624
-0.5	0.12946	0.09579	0.11352	0.07713	0.09585	0.09099	0.11354	0.07363
-0.3	0.09086	0.06659	0.07431	0.05356	0.06659	0.05953	0.07437	0.04807
0	0.06674	0.04832	0.04782	0.03888	0.04834	0.03831	0.04785	0.03103
0.3	0.10097	0.07934	0.07152	0.0637	0.07934	0.05724	0.0715	0.04615
0.5	0.16026	0.13433	0.12037	0.10744	0.13442	0.0959	0.12041	0.0767
0.8	0.13621	0.11725	0.10023	0.09382	0.1173	0.07996	0.10019	0.06439

Table 52. Multiplicative Desirability Inference Observation 24, (x_1, x_2) = (0.5,1) Empirical Coverage and Average Width

Empirical Coverage								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.9622	0.9044	0.9509	0.838	0.9053	0.8986	0.9496	0.8417
-0.5	0.9794	0.9271	0.9479	0.8615	0.9255	0.9003	0.9495	0.8321
-0.3	0.9842	0.9399	0.9501	0.883	0.9402	0.9003	0.95	0.8247
0	0.9915	0.961	0.9488	0.9197	0.962	0.9009	0.9478	0.8334
0.3	0.9953	0.9754	0.95	0.9423	0.9753	0.9054	0.9504	0.8405
0.5	0.9756	0.9581	0.9473	0.9149	0.9569	0.8992	0.9478	0.8326
0.8	0.971	0.9611	0.9499	0.9192	0.9597	0.9008	0.95	0.8377
Average Width								
ρ	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
-0.8	0.12504	0.09339	0.1187	0.07607	0.09348	0.09591	0.11866	0.07853
-0.5	0.15123	0.10974	0.13161	0.08953	0.10977	0.10646	0.13157	0.08679
-0.3	0.11275	0.08478	0.09367	0.06847	0.0848	0.07528	0.09372	0.06116
0	0.08288	0.06166	0.06127	0.04982	0.06166	0.04921	0.06128	0.04013
0.3	0.11764	0.09081	0.08118	0.07322	0.09085	0.06497	0.08117	0.05269
0.5	0.26757	0.24558	0.23679	0.19143	0.24577	0.18355	0.23695	0.14472
0.8	0.25428	0.23948	0.22863	0.18658	0.23996	0.17746	0.22834	0.14064

Table 53. Multiplicative Desirability Inference Observation 25, (x_1, x_2) = (1,1) Empirical Coverage and Average Width

Appendix B. Second Order Max/Tgt/Min Model Desirability Inference Results

B.1 Additive Empirical Coverage and Average Width Tables

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	1	0.9884	0.967	0.9726	0.9888	0.9403	0.9661	0.8516
$w_1 l_2$	0.9999	0.983	0.9632	0.9658	0.9837	0.9371	0.9621	0.8491
$w_1 l_3$	0.9429	0.9767	0.9303	0.9556	0.9767	0.8974	0.9315	0.7975
$w_1 l_4$	0.9592	0.9813	0.9557	0.9552	0.9816	0.9164	0.9563	0.8079
$w_2 l_1$	1	0.9961	0.9593	0.9901	0.9966	0.9359	0.9604	0.8418
$w_2 l_2$	1	0.9915	0.9534	0.9818	0.9917	0.9316	0.9533	0.845
$w_2 l_3$	0.9743	0.9857	0.9168	0.9712	0.9857	0.8809	0.9192	0.7835
$w_2 l_4$	0.9778	0.9898	0.9572	0.9749	0.9901	0.9169	0.9589	0.8153
$w_3 l_1$	0.9997	0.9774	0.9701	0.9533	0.9757	0.9425	0.9711	0.8556
$w_3 l_2$	0.9997	0.9747	0.9674	0.9509	0.975	0.9415	0.9691	0.8542
$w_3 l_3$	0.9331	0.9705	0.9569	0.9367	0.9703	0.9127	0.9549	0.8077
$w_3 l_4$	0.9368	0.9686	0.9585	0.9287	0.97	0.9108	0.9552	0.7971
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.36729	0.22726	0.19338	0.19591	0.22734	0.16462	0.19353	0.12384
$w_1 l_2$	0.37202	0.24173	0.2177	0.20013	0.2416	0.17769	0.21701	0.13062
$w_1 l_3$	0.49343	0.32732	0.29303	0.29681	0.32746	0.26672	0.29335	0.20997
$w_1 l_4$	0.49816	0.33678	0.30341	0.29908	0.33688	0.27363	0.30369	0.21424
$w_2 l_1$	0.2916	0.16969	0.12479	0.1458	0.16957	0.10508	0.12556	0.07731
$w_2 l_2$	0.30011	0.19248	0.16327	0.1482	0.19215	0.12065	0.1634	0.07976
$w_2 l_3$	0.36728	0.229	0.18143	0.20639	0.22889	0.16617	0.18304	0.13132
$w_2 l_4$	0.3758	0.25093	0.2106	0.20884	0.251	0.17694	0.21199	0.13101
$w_3 l_1$	0.45307	0.33806	0.32093	0.29169	0.33814	0.27529	0.32047	0.2116
$w_3 l_2$	0.45591	0.34245	0.33112	0.29312	0.34248	0.28117	0.33026	0.2152
$w_3 l_3$	0.68013	0.51291	0.48909	0.46993	0.51322	0.45239	0.49318	0.36201
$w_3 l_4$	0.68297	0.51635	0.49272	0.4706	0.51444	0.45247	0.49428	0.36168

Table 54. Additive Desirability Inference Observation 1, (x_1, x_2) = (-1,-1) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	1	0.9895	0.9647	0.9747	0.9891	0.945	0.9661	0.8672
$w_1 l_2$	1	0.9799	0.9665	0.961	0.9796	0.9428	0.9669	0.8597
$w_1 l_3$	0.9033	0.9695	0.9521	0.9345	0.969	0.9072	0.9524	0.796
$w_1 l_4$	0.8993	0.961	0.9486	0.9197	0.961	0.9054	0.9519	0.7959
$w_2 l_1$	0.9985	0.9791	0.9528	0.9546	0.9785	0.922	0.9544	0.8234
$w_2 l_2$	0.9961	0.9705	0.9531	0.9439	0.9713	0.9222	0.9528	0.8203
$w_2 l_3$	0.9581	0.9763	0.9487	0.9497	0.9754	0.9132	0.9525	0.8062
$w_2 l_4$	0.9448	0.9705	0.9487	0.9402	0.9707	0.9128	0.9528	0.806
$w_3 l_1$	1	0.9953	0.9546	0.9894	0.9955	0.935	0.9595	0.8483
$w_3 l_2$	1	0.9881	0.9641	0.9728	0.9887	0.9433	0.9665	0.8568
$w_3 l_3$	0.9352	0.985	0.9551	0.9618	0.9858	0.919	0.9571	0.8146
$w_3 l_4$	0.9014	0.9665	0.95	0.9296	0.9673	0.9125	0.9526	0.8012
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.26217	0.16431	0.14086	0.14021	0.16432	0.11888	0.14094	0.08941
$w_1 l_2$	0.22818	0.15816	0.14905	0.13508	0.15807	0.12624	0.14893	0.09529
$w_1 l_3$	0.43352	0.3164	0.2905	0.29132	0.31649	0.26835	0.2904	0.21921
$w_1 l_4$	0.39954	0.31108	0.30066	0.287	0.31113	0.27736	0.30082	0.2262
$w_2 l_1$	0.25798	0.17471	0.15832	0.14668	0.17464	0.13212	0.15907	0.09666
$w_2 l_2$	0.23759	0.17271	0.16476	0.14495	0.17276	0.13768	0.16535	0.10092
$w_2 l_3$	0.36079	0.24924	0.22615	0.22491	0.24938	0.20455	0.22737	0.16173
$w_2 l_4$	0.3404	0.24774	0.23418	0.22357	0.24771	0.21124	0.23525	0.16676
$w_3 l_1$	0.20981	0.12413	0.09342	0.10523	0.12415	0.07777	0.09398	0.05699
$w_3 l_2$	0.14864	0.09632	0.08632	0.08222	0.09637	0.07294	0.08636	0.05502
$w_3 l_3$	0.31262	0.21097	0.17333	0.19299	0.21093	0.16111	0.17475	0.13091
$w_3 l_4$	0.25145	0.18914	0.17654	0.17479	0.18858	0.16341	0.17686	0.13381

Table 55. Additive Desirability Inference Observation 2, (x_1, x_2) = (-0.5,-1) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	1	0.9861	0.9487	0.9735	0.9871	0.9345	0.9542	0.8656
$w_1 l_2$	0.9998	0.9746	0.951	0.9562	0.9733	0.9343	0.9549	0.8683
$w_1 l_3$	0.8461	0.9584	0.9503	0.9164	0.9575	0.9043	0.9509	0.7948
$w_1 l_4$	0.8217	0.9518	0.9475	0.9095	0.9532	0.9036	0.9499	0.7936
$w_2 l_1$	1	0.9783	0.9706	0.9624	0.9783	0.9528	0.9704	0.8951
$w_2 l_2$	1	0.9758	0.9695	0.9568	0.9757	0.9517	0.9704	0.8944
$w_2 l_3$	0.8056	0.9535	0.9467	0.9142	0.953	0.9099	0.9523	0.7999
$w_2 l_4$	0.8034	0.9527	0.9458	0.9129	0.9522	0.9098	0.9532	0.8
$w_3 l_1$	0.9976	0.9707	0.9448	0.946	0.9711	0.9154	0.945	0.8175
$w_3 l_2$	0.9924	0.9618	0.9444	0.9317	0.9617	0.9146	0.9452	0.8213
$w_3 l_3$	0.9583	0.9793	0.9659	0.9485	0.9797	0.928	0.9686	0.819
$w_3 l_4$	0.9394	0.975	0.9643	0.9416	0.9756	0.9223	0.9658	0.8142
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.22677	0.14447	0.12718	0.12235	0.14441	0.10672	0.12723	0.08053
$w_1 l_2$	0.19412	0.13853	0.13279	0.11742	0.13848	0.11182	0.13271	0.08473
$w_1 l_3$	0.39398	0.31557	0.29252	0.29312	0.31548	0.27358	0.29292	0.23037
$w_1 l_4$	0.36132	0.31092	0.30226	0.28952	0.31087	0.28196	0.30262	0.23669
$w_2 l_1$	0.27012	0.20763	0.19955	0.17657	0.20768	0.16876	0.19911	0.13062
$w_2 l_2$	0.25053	0.20564	0.20333	0.17503	0.20587	0.17199	0.20256	0.13321
$w_2 l_3$	0.57109	0.53154	0.51729	0.49849	0.53154	0.48797	0.51924	0.41597
$w_2 l_4$	0.5515	0.52999	0.52333	0.49721	0.52998	0.49313	0.52496	0.41978
$w_3 l_1$	0.2268	0.15629	0.14399	0.13076	0.15625	0.11993	0.1445	0.08809
$w_3 l_2$	0.20721	0.15451	0.14922	0.12919	0.15443	0.12453	0.14965	0.09158
$w_3 l_3$	0.32713	0.23983	0.21901	0.21836	0.23983	0.19974	0.22	0.16193
$w_3 l_4$	0.30754	0.23841	0.22665	0.21716	0.23809	0.20568	0.22699	0.16633

Table 56. Additive Desirability Inference Observation 3, (x_1, x_2) = (0,-1) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	1	0.9664	0.9093	0.9458	0.9664	0.8919	0.9136	0.8144
$w_1 l_2$	0.9991	0.9461	0.9174	0.9214	0.9468	0.8939	0.9156	0.8209
$w_1 l_3$	0.9561	0.972	0.9668	0.939	0.972	0.933	0.9692	0.8227
$w_1 l_4$	0.9091	0.9678	0.9648	0.9317	0.968	0.9291	0.9665	0.8168
$w_2 l_1$	0.9998	0.9759	0.9054	0.957	0.9758	0.8849	0.914	0.8004
$w_2 l_2$	0.9997	0.9543	0.9121	0.9324	0.954	0.8943	0.9158	0.8209
$w_2 l_3$	0.9933	0.9816	0.9704	0.9556	0.9806	0.9424	0.971	0.8477
$w_2 l_4$	0.9326	0.9668	0.9642	0.9328	0.9672	0.9314	0.9647	0.8209
$w_3 l_1$	1	0.967	0.9519	0.9515	0.9671	0.9381	0.9552	0.8785
$w_3 l_2$	1	0.9634	0.9533	0.9477	0.9646	0.9385	0.9551	0.8785
$w_3 l_3$	0.8276	0.9622	0.9617	0.926	0.9626	0.9247	0.9631	0.8125
$w_3 l_4$	0.8016	0.9623	0.9616	0.9242	0.9629	0.9227	0.9623	0.8077
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.24644	0.15859	0.14323	0.1338	0.15849	0.12026	0.14345	0.09048
$w_1 l_2$	0.21017	0.15193	0.14695	0.12819	0.15186	0.12377	0.14725	0.09347
$w_1 l_3$	0.38746	0.32254	0.29865	0.30021	0.32263	0.28013	0.29902	0.23781
$w_1 l_4$	0.35118	0.31738	0.30831	0.29596	0.31729	0.28832	0.30856	0.24376
$w_2 l_1$	0.20037	0.12153	0.10169	0.10212	0.12154	0.0848	0.10207	0.06249
$w_2 l_2$	0.13507	0.09203	0.08681	0.07763	0.09206	0.07283	0.08679	0.055
$w_2 l_3$	0.28498	0.21366	0.17936	0.19684	0.21356	0.1679	0.18006	0.14186
$w_2 l_4$	0.21968	0.19143	0.18228	0.17851	0.19147	0.1708	0.18259	0.14495
$w_3 l_1$	0.29041	0.22606	0.21944	0.19111	0.22599	0.18471	0.21899	0.14252
$w_3 l_2$	0.26865	0.22385	0.22183	0.18934	0.22398	0.18693	0.22143	0.14432
$w_3 l_3$	0.54423	0.53683	0.52358	0.50396	0.53662	0.49361	0.52429	0.42397
$w_3 l_4$	0.52247	0.53517	0.5295	0.50261	0.53608	0.49939	0.53101	0.42795

Table 57. Additive Desirability Inference Observation 4, (x_1, x_2) = (0.5,-1) Empirical Coverage and Average Width

Empirical Coverage								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.9998	0.9519	0.8779	0.9255	0.951	0.8594	0.8851	0.7776
w_1l_2	0.9978	0.925	0.8854	0.8966	0.9245	0.8651	0.8888	0.786
w_1l_3	0.9955	0.9766	0.9724	0.9523	0.977	0.9466	0.9728	0.8641
w_1l_4	0.9853	0.9739	0.9711	0.9481	0.9734	0.946	0.9718	0.855
w_2l_1	0.9943	0.9524	0.9211	0.9251	0.9529	0.8905	0.9238	0.7896
w_2l_2	0.988	0.946	0.9231	0.9093	0.9455	0.8922	0.9267	0.7928
w_2l_3	0.997	0.9855	0.9727	0.9663	0.9855	0.9515	0.9754	0.8675
w_2l_4	0.9933	0.9805	0.9711	0.959	0.9815	0.9498	0.9741	0.8656
w_3l_1	0.9997	0.9659	0.8918	0.9445	0.9652	0.8694	0.9004	0.7774
w_3l_2	0.9997	0.9392	0.8747	0.9126	0.9388	0.8586	0.882	0.777
w_3l_3	0.9996	0.9878	0.9778	0.9696	0.9876	0.955	0.9781	0.877
w_3l_4	0.9928	0.9772	0.9726	0.9504	0.9736	0.9449	0.9697	0.8566
Average Width								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.33685	0.21716	0.19815	0.18333	0.21698	0.16708	0.19874	0.12583
w_1l_2	0.28987	0.20819	0.20164	0.17581	0.20817	0.17032	0.20223	0.12855
w_1l_3	0.44603	0.35217	0.32181	0.32875	0.35224	0.30254	0.32187	0.25902
w_1l_4	0.39906	0.34381	0.33215	0.32218	0.34383	0.3115	0.33221	0.26569
w_2l_1	0.34041	0.23687	0.22258	0.19821	0.23675	0.18612	0.22349	0.13638
w_2l_2	0.31223	0.23422	0.22865	0.1959	0.23421	0.19123	0.22944	0.14025
w_2l_3	0.40592	0.29987	0.27443	0.27048	0.29978	0.24691	0.27509	0.19915
w_2l_4	0.37774	0.29738	0.28375	0.26856	0.2974	0.25479	0.28423	0.20492
w_3l_1	0.27427	0.1666	0.14217	0.14015	0.16656	0.11867	0.14232	0.08756
w_3l_2	0.18972	0.12707	0.11966	0.1073	0.12707	0.10057	0.11975	0.0758
w_3l_3	0.33978	0.24233	0.20004	0.22289	0.24222	0.18623	0.20072	0.1576
w_3l_4	0.25523	0.20889	0.19551	0.19535	0.20854	0.18334	0.19534	0.15669

Table 58. Additive Desirability Inference Observation 5, (x_1, x_2) = (1,-1) Empirical Coverage and Average Width

Empirical Coverage								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.9994	0.9537	0.9577	0.9061	0.9529	0.9097	0.9529	0.7966
w_1l_2	0.9984	0.9578	0.9594	0.916	0.9575	0.9155	0.9571	0.8026
w_1l_3	0.9997	0.9719	0.9734	0.9468	0.97	0.9483	0.9728	0.8713
w_1l_4	0.9992	0.974	0.9717	0.9509	0.9737	0.949	0.9727	0.8724
w_2l_1	0.9955	0.9543	0.9609	0.908	0.955	0.9127	0.9582	0.8006
w_2l_2	0.9936	0.9582	0.9617	0.9132	0.9588	0.9143	0.9587	0.8026
w_2l_3	0.998	0.9688	0.9753	0.9438	0.9687	0.9473	0.9728	0.8766
w_2l_4	0.9968	0.9713	0.9746	0.9465	0.9707	0.948	0.9721	0.8778
w_3l_1	0.9984	0.9613	0.9512	0.924	0.9596	0.9125	0.9565	0.8018
w_3l_2	0.9969	0.9601	0.9518	0.9213	0.959	0.9162	0.9563	0.7989
w_3l_3	0.9987	0.971	0.9638	0.9478	0.9703	0.9373	0.9637	0.843
w_3l_4	0.9973	0.969	0.9625	0.9446	0.9653	0.9335	0.9598	0.8415
Average Width								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.29981	0.18395	0.19417	0.15853	0.18385	0.1674	0.1944	0.12629
w_1l_2	0.2688	0.17875	0.18489	0.15413	0.17875	0.15926	0.18538	0.12007
w_1l_3	0.3048	0.225	0.2261	0.18884	0.22492	0.19189	0.22586	0.14261
w_1l_4	0.27379	0.2205	0.2221	0.18488	0.22053	0.18697	0.22195	0.13788
w_2l_1	0.38648	0.28489	0.29478	0.24654	0.28493	0.25542	0.29527	0.19428
w_2l_2	0.36787	0.28342	0.28871	0.24532	0.2836	0.25008	0.28912	0.19008
w_2l_3	0.39546	0.36376	0.37014	0.30378	0.36375	0.30914	0.36854	0.22844
w_2l_4	0.37685	0.36284	0.36688	0.3028	0.36271	0.30562	0.36555	0.22495
w_3l_1	0.28057	0.18169	0.18425	0.15374	0.1818	0.15606	0.18493	0.11523
w_3l_2	0.26196	0.17997	0.1836	0.1522	0.18003	0.15535	0.18427	0.11451
w_3l_3	0.28356	0.19923	0.19595	0.16717	0.19911	0.16601	0.19648	0.12247
w_3l_4	0.26495	0.19761	0.1974	0.16578	0.19664	0.16572	0.1969	0.12171

Table 59. Additive Desirability Inference Observation 6, (x_1, x_2) = (-1,-0.5) Empirical Coverage and Average Width

Empirical Coverage								
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	0.997	0.9449	0.9484	0.8996	0.9454	0.9104	0.9506	0.8027
w$_1 l_2$	0.9947	0.953	0.9483	0.911	0.9527	0.9116	0.9517	0.8037
w$_1 l_3$	0.9973	0.963	0.9643	0.9204	0.963	0.9219	0.9639	0.8115
w$_1 l_4$	0.995	0.9657	0.964	0.927	0.9668	0.9248	0.9664	0.8167
w$_2 l_1$	0.9977	0.9435	0.9486	0.8926	0.9426	0.911	0.9506	0.8017
w$_2 l_2$	0.9961	0.9504	0.9479	0.904	0.9501	0.9086	0.9493	0.8069
w$_2 l_3$	0.9979	0.9596	0.963	0.9113	0.959	0.9234	0.9636	0.8103
w$_2 l_4$	0.9964	0.9647	0.9651	0.924	0.9651	0.9268	0.9634	0.8185
w$_3 l_1$	0.9894	0.9442	0.9483	0.8996	0.9435	0.9067	0.9493	0.8035
w$_3 l_2$	0.9865	0.9478	0.9491	0.9042	0.9475	0.9068	0.9507	0.8058
w$_3 l_3$	0.9899	0.9599	0.9651	0.9191	0.9597	0.9249	0.9639	0.8167
w$_3 l_4$	0.987	0.962	0.9662	0.9226	0.9639	0.9238	0.9629	0.8173
Average Width								
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	0.2726	0.15384	0.16549	0.13102	0.15388	0.1404	0.16581	0.10383
w$_1 l_2$	0.24418	0.15013	0.15549	0.1278	0.15014	0.13207	0.15602	0.09766
w$_1 l_3$	0.27273	0.16057	0.17001	0.1338	0.16055	0.14192	0.16972	0.10443
w$_1 l_4$	0.24431	0.15687	0.1615	0.13065	0.15695	0.13413	0.16133	0.09847
w$_2 l_1$	0.20422	0.10899	0.12212	0.09255	0.109	0.10293	0.12202	0.07593
w$_2 l_2$	0.15306	0.09057	0.09538	0.07709	0.09058	0.08078	0.09538	0.05972
w$_2 l_3$	0.2043	0.11262	0.12266	0.09405	0.11256	0.10336	0.12252	0.07627
w$_2 l_4$	0.15314	0.09453	0.09859	0.07875	0.09452	0.08205	0.09858	0.06027
w$_3 l_1$	0.37205	0.2459	0.25629	0.2091	0.24606	0.21761	0.25636	0.16109
w$_3 l_2$	0.355	0.24518	0.24935	0.20843	0.24519	0.21192	0.24953	0.15698
w$_3 l_3$	0.3723	0.25855	0.26799	0.21425	0.2584	0.22215	0.2672	0.16302
w$_3 l_4$	0.35524	0.25758	0.262	0.2135	0.25786	0.21713	0.26192	0.15893

Table 60. Additive Desirability Inference Observation 7, (x_1, x_2) = (-0.5,-0.5) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9968	0.9445	0.9501	0.8997	0.9458	0.9082	0.9505	0.8016
$w_1 l_2$	0.9951	0.9525	0.9514	0.9105	0.9519	0.9081	0.9509	0.8015
$w_1 l_3$	0.9969	0.9575	0.9584	0.9117	0.9577	0.9181	0.9621	0.8059
$w_1 l_4$	0.9952	0.9621	0.9596	0.9186	0.9619	0.9172	0.9596	0.8042
$w_2 l_1$	0.9972	0.9575	0.9475	0.9166	0.9565	0.9099	0.9481	0.8069
$w_2 l_2$	0.9955	0.9548	0.947	0.9148	0.9575	0.9094	0.9475	0.8057
$w_2 l_3$	0.9972	0.9628	0.9535	0.922	0.9621	0.914	0.9551	0.8087
$w_2 l_4$	0.9955	0.962	0.9527	0.9216	0.9611	0.9152	0.9541	0.81
$w_3 l_1$	0.998	0.943	0.9478	0.8921	0.9444	0.9096	0.951	0.8017
$w_3 l_2$	0.9962	0.9499	0.9499	0.9082	0.9505	0.9102	0.9495	0.8046
$w_3 l_3$	0.998	0.954	0.9555	0.9027	0.9525	0.9146	0.9567	0.8069
$w_3 l_4$	0.9963	0.9602	0.9594	0.9153	0.9617	0.9181	0.9602	0.8079
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.2809	0.15283	0.1646	0.12934	0.15279	0.13876	0.16491	0.10208
$w_1 l_2$	0.2521	0.14928	0.15437	0.12626	0.1492	0.13029	0.15486	0.09583
$w_1 l_3$	0.28095	0.15564	0.166	0.13023	0.15566	0.13886	0.16588	0.10196
$w_1 l_4$	0.25214	0.15206	0.15677	0.1271	0.1521	0.13063	0.15662	0.0958
$w_2 l_1$	0.24488	0.14248	0.14679	0.11989	0.1425	0.12301	0.14725	0.09016
$w_2 l_2$	0.2276	0.14119	0.1455	0.11865	0.1411	0.12178	0.14581	0.08933
$w_2 l_3$	0.24491	0.14397	0.1469	0.12033	0.14394	0.12308	0.14734	0.09022
$w_2 l_4$	0.22762	0.14263	0.14606	0.11913	0.14262	0.12206	0.14646	0.0894
$w_3 l_1$	0.20835	0.10777	0.12129	0.09114	0.10771	0.1016	0.12107	0.0746
$w_3 l_2$	0.15651	0.08981	0.09422	0.07596	0.0898	0.07932	0.09423	0.05831
$w_3 l_3$	0.20838	0.10933	0.1208	0.09161	0.10929	0.1015	0.12058	0.07465
$w_3 l_4$	0.15653	0.09148	0.09539	0.07648	0.0915	0.07972	0.09551	0.05847

Table 61. Additive Desirability Inference Observation 8, (x_1, x_2) = (0,-0.5) Empirical Coverage and Average Width

Empirical Coverage								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.9971	0.9468	0.9508	0.9008	0.9467	0.9085	0.948	0.8013
w_1l_2	0.9951	0.9532	0.9491	0.91	0.9532	0.9088	0.9489	0.8015
w_1l_3	0.9972	0.9587	0.9599	0.9109	0.9592	0.9177	0.9622	0.8075
w_1l_4	0.9952	0.9622	0.9592	0.9189	0.9628	0.9189	0.9613	0.8024
w_2l_1	0.99	0.9441	0.9482	0.9006	0.9444	0.9072	0.9479	0.8022
w_2l_2	0.9881	0.9489	0.9484	0.9055	0.9488	0.9076	0.9489	0.8027
w_2l_3	0.9901	0.9548	0.9589	0.9103	0.9559	0.9167	0.9582	0.8063
w_2l_4	0.9882	0.9591	0.959	0.9147	0.9586	0.9167	0.9588	0.8074
w_3l_1	0.9981	0.9587	0.9492	0.9176	0.9584	0.9107	0.9503	0.8089
w_3l_2	0.9959	0.9579	0.9471	0.9164	0.96	0.9101	0.9501	0.8082
w_3l_3	0.9981	0.9656	0.956	0.9226	0.9651	0.9146	0.9569	0.8098
w_3l_4	0.9959	0.9655	0.9561	0.9216	0.9644	0.917	0.9598	0.81
Average Width								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.28555	0.15581	0.16796	0.13202	0.15586	0.14165	0.16821	0.10419
w_1l_2	0.25605	0.15215	0.15746	0.12887	0.15214	0.13293	0.15792	0.09787
w_1l_3	0.28558	0.15892	0.16948	0.13292	0.15895	0.14171	0.16928	0.10413
w_1l_4	0.25607	0.15524	0.16003	0.12974	0.15527	0.1333	0.15988	0.0978
w_2l_1	0.39537	0.25004	0.26084	0.21131	0.25014	0.21982	0.26077	0.16177
w_2l_2	0.37767	0.24942	0.25358	0.21049	0.24928	0.21388	0.25363	0.15755
w_2l_3	0.39541	0.25571	0.26554	0.21305	0.25561	0.22128	0.2653	0.16236
w_2l_4	0.37771	0.2549	0.25922	0.21235	0.25497	0.21557	0.25888	0.15812
w_3l_1	0.2493	0.14548	0.14982	0.12245	0.14551	0.12564	0.15024	0.0921
w_3l_2	0.23159	0.14408	0.14849	0.12122	0.14408	0.12439	0.14882	0.09121
w_3l_3	0.24931	0.14706	0.15004	0.12289	0.14711	0.12568	0.15039	0.0922
w_3l_4	0.23161	0.14569	0.14908	0.12168	0.14525	0.12436	0.14927	0.09113

Table 62. Additive Desirability Inference Observation 9, (x_1, x_2) = (0.5,-0.5) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9983	0.9495	0.9511	0.9021	0.9498	0.9071	0.949	0.7993
$w_1 l_2$	0.9964	0.9522	0.9522	0.9107	0.9532	0.9084	0.9499	0.7971
$w_1 l_3$	0.9985	0.9677	0.9701	0.9315	0.9681	0.9361	0.9697	0.8318
$w_1 l_4$	0.9969	0.97	0.969	0.9384	0.9696	0.9361	0.9683	0.832
$w_2 l_1$	0.9988	0.9421	0.9509	0.8934	0.9401	0.9052	0.9487	0.7965
$w_2 l_2$	0.9968	0.9505	0.9524	0.9096	0.9524	0.909	0.9516	0.7984
$w_2 l_3$	0.9988	0.9631	0.9679	0.9284	0.9634	0.9314	0.9659	0.8261
$w_2 l_4$	0.9973	0.9698	0.9694	0.9401	0.9708	0.9394	0.9688	0.8351
$w_3 l_1$	0.9913	0.9468	0.951	0.9002	0.9474	0.9082	0.9519	0.7935
$w_3 l_2$	0.9887	0.9511	0.9513	0.9073	0.951	0.9096	0.9522	0.7942
$w_3 l_3$	0.9927	0.9675	0.9689	0.9351	0.967	0.9388	0.9699	0.8371
$w_3 l_4$	0.9904	0.9693	0.969	0.9395	0.9718	0.9433	0.9721	0.8447
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.3254	0.1922	0.2056	0.16517	0.19219	0.17621	0.20599	0.13113
$w_1 l_2$	0.28822	0.1871	0.19343	0.16082	0.18706	0.16586	0.194	0.12344
$w_1 l_3$	0.32625	0.21006	0.21761	0.17563	0.20997	0.18362	0.21738	0.13523
$w_1 l_4$	0.28907	0.20514	0.20914	0.17146	0.20521	0.17505	0.20898	0.12805
$w_2 l_1$	0.24774	0.13755	0.15262	0.11759	0.13756	0.12984	0.15235	0.09618
$w_2 l_2$	0.18082	0.11286	0.11829	0.09696	0.11284	0.10125	0.11842	0.07529
$w_2 l_3$	0.24826	0.14695	0.15477	0.12327	0.14692	0.13193	0.15457	0.0979
$w_2 l_4$	0.18133	0.12364	0.12697	0.10339	0.12362	0.10664	0.12689	0.07822
$w_3 l_1$	0.43253	0.3037	0.31622	0.2614	0.30378	0.27162	0.31607	0.20301
$w_3 l_2$	0.41022	0.30249	0.30793	0.26036	0.3028	0.26446	0.30775	0.19772
$w_3 l_3$	0.43407	0.33853	0.3477	0.28183	0.33854	0.29022	0.3471	0.21278
$w_3 l_4$	0.41176	0.33755	0.34145	0.28076	0.33884	0.28564	0.34267	0.2088

Table 63. Additive Desirability Inference Observation 10, (x_1, x_2) = (1,-0.5) Empirical Coverage and Average Width

				Empirical Coverage				
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9982	0.9394	0.9458	0.8945	0.9398	0.9028	0.9461	0.7976
$w_1 l_2$	0.9968	0.9464	0.9461	0.9036	0.9467	0.9049	0.9472	0.7991
$w_1 l_3$	0.9982	0.9544	0.9552	0.9073	0.9539	0.9106	0.9557	0.7965
$w_1 l_4$	0.9968	0.958	0.9541	0.9149	0.9592	0.9116	0.9579	0.8001
$w_2 l_1$	0.9992	0.9577	0.9427	0.9179	0.957	0.9129	0.9499	0.805
$w_2 l_2$	0.9974	0.9565	0.9439	0.9181	0.957	0.9119	0.9525	0.8045
$w_2 l_3$	0.9992	0.9615	0.9491	0.9219	0.9615	0.9146	0.9576	0.8059
$w_2 l_4$	0.9974	0.9613	0.9504	0.9227	0.9612	0.9166	0.9576	0.806
$w_3 l_1$	0.9989	0.9435	0.9489	0.8923	0.9427	0.9082	0.9518	0.798
$w_3 l_2$	0.9977	0.9497	0.9447	0.9034	0.9491	0.9092	0.9507	0.8008
$w_3 l_3$	0.999	0.9523	0.9562	0.9001	0.9525	0.9134	0.9581	0.799
$w_3 l_4$	0.9977	0.9573	0.9547	0.9112	0.9557	0.9165	0.9567	0.8061
				Average Width				
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.33501	0.18163	0.19582	0.15381	0.1816	0.16498	0.19591	0.12109
$w_1 l_2$	0.30503	0.17763	0.18482	0.15022	0.1775	0.15593	0.18517	0.11444
$w_1 l_3$	0.33505	0.18408	0.19624	0.15471	0.184	0.16499	0.1962	0.12106
$w_1 l_4$	0.30508	0.18014	0.18639	0.15118	0.18001	0.15616	0.18631	0.11448
$w_2 l_1$	0.29175	0.16925	0.17449	0.1426	0.16926	0.14628	0.1749	0.10693
$w_2 l_2$	0.27376	0.16771	0.17298	0.14125	0.16774	0.14496	0.17338	0.10596
$w_2 l_3$	0.29177	0.17028	0.17432	0.14285	0.17026	0.14622	0.17475	0.10698
$w_2 l_4$	0.27379	0.16878	0.17315	0.14153	0.16871	0.145	0.17362	0.10604
$w_3 l_1$	0.24833	0.128	0.14416	0.10836	0.12801	0.12082	0.14402	0.08854
$w_3 l_2$	0.19438	0.10774	0.11479	0.09119	0.10777	0.0966	0.11484	0.07085
$w_3 l_3$	0.24835	0.12918	0.14308	0.10867	0.12916	0.12057	0.14283	0.08853
$w_3 l_4$	0.1944	0.10908	0.11533	0.09152	0.10903	0.09682	0.1153	0.071

Table 64. Additive Desirability Inference Observation 11, (x_1, x_2) = (-1,0) Empirical Coverage and Average Width

Empirical Coverage								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.9975	0.9408	0.9471	0.897	0.94	0.9033	0.9468	0.803
w_1l_2	0.9947	0.9461	0.9474	0.9072	0.9469	0.9039	0.9457	0.8042
w_1l_3	0.9976	0.9497	0.949	0.9016	0.9494	0.912	0.9506	0.7998
w_1l_4	0.9948	0.9535	0.9485	0.9129	0.9548	0.9109	0.9526	0.8023
w_2l_1	0.988	0.9438	0.9473	0.8981	0.9442	0.907	0.9513	0.8015
w_2l_2	0.986	0.9472	0.9482	0.9045	0.949	0.9076	0.9517	0.8013
w_2l_3	0.9881	0.9474	0.9494	0.9009	0.9475	0.9073	0.9532	0.8026
w_2l_4	0.9861	0.9513	0.9496	0.9052	0.9499	0.9091	0.9537	0.8035
w_3l_1	0.9983	0.9568	0.9455	0.9174	0.9548	0.9103	0.9484	0.808
w_3l_2	0.9965	0.9553	0.9447	0.9146	0.954	0.9094	0.9494	0.8103
w_3l_3	0.9983	0.9566	0.9475	0.9182	0.958	0.9108	0.9515	0.8085
w_3l_4	0.9965	0.9572	0.9471	0.9172	0.962	0.9069	0.9526	0.7997
Average Width								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.31675	0.1554	0.16765	0.13029	0.15549	0.13978	0.16786	0.10231
w_1l_2	0.28833	0.15198	0.15726	0.1272	0.152	0.13138	0.15777	0.09616
w_1l_3	0.31678	0.15548	0.16711	0.13025	0.15539	0.13944	0.167	0.10204
w_1l_4	0.28836	0.15186	0.15712	0.1272	0.15192	0.13097	0.15698	0.09589
w_2l_1	0.45401	0.25017	0.26098	0.20886	0.25031	0.2173	0.261	0.15883
w_2l_2	0.43696	0.24949	0.25401	0.20819	0.24956	0.21146	0.25393	0.15472
w_2l_3	0.45405	0.24996	0.26059	0.20881	0.24998	0.21716	0.26036	0.15885
w_2l_4	0.437	0.24942	0.25372	0.20802	0.24921	0.21135	0.25359	0.15466
w_3l_1	0.26639	0.14369	0.14816	0.12028	0.14365	0.12348	0.14845	0.09029
w_3l_2	0.24934	0.14215	0.14659	0.1191	0.14227	0.12221	0.14694	0.08939
w_3l_3	0.2664	0.14366	0.14774	0.12024	0.14364	0.12342	0.14814	0.09031
w_3l_4	0.24935	0.14226	0.14639	0.11908	0.14183	0.12194	0.1464	0.08921

Table 65. Additive Desirability Inference Observation 12, (x_1, x_2) = (-0.5,0) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9972	0.9424	0.9496	0.8964	0.9434	0.9071	0.9488	0.8028
$w_1 l_2$	0.9954	0.9496	0.9492	0.906	0.9497	0.9035	0.9471	0.8022
$w_1 l_3$	0.9972	0.9496	0.9515	0.9027	0.9493	0.9106	0.9535	0.806
$w_1 l_4$	0.9954	0.9552	0.9514	0.9109	0.9554	0.9108	0.9545	0.8037
$w_2 l_1$	0.9981	0.9389	0.9482	0.8917	0.9392	0.9075	0.9481	0.8005
$w_2 l_2$	0.996	0.9498	0.9492	0.9031	0.9501	0.9065	0.9452	0.8023
$w_2 l_3$	0.9981	0.9418	0.949	0.8905	0.942	0.9055	0.9498	0.8009
$w_2 l_4$	0.996	0.9528	0.953	0.9074	0.9532	0.9071	0.9477	0.8027
$w_3 l_1$	0.989	0.9432	0.9496	0.9005	0.9447	0.9077	0.9496	0.8011
$w_3 l_2$	0.9873	0.9469	0.9491	0.9062	0.9471	0.9085	0.949	0.7993
$w_3 l_3$	0.989	0.9472	0.9526	0.9009	0.9474	0.909	0.9516	0.8011
$w_3 l_4$	0.9873	0.9506	0.9492	0.9073	0.9489	0.9078	0.9496	0.8052
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.32325	0.15937	0.17174	0.13359	0.15932	0.14332	0.17203	0.10484
$w_1 l_2$	0.29326	0.15578	0.16089	0.1304	0.15573	0.1345	0.16143	0.09841
$w_1 l_3$	0.32325	0.15924	0.17123	0.13351	0.15927	0.14292	0.17113	0.10461
$w_1 l_4$	0.29326	0.15563	0.16077	0.13036	0.15565	0.13403	0.16067	0.09813
$w_2 l_1$	0.23477	0.11179	0.12612	0.09389	0.1118	0.1048	0.12591	0.07659
$w_2 l_2$	0.18079	0.09364	0.09808	0.07841	0.0936	0.08171	0.09807	0.05978
$w_2 l_3$	0.23477	0.11178	0.12542	0.09389	0.11177	0.1047	0.12522	0.07662
$w_2 l_4$	0.18079	0.09357	0.09782	0.07838	0.09358	0.08167	0.09787	0.05975
$w_3 l_1$	0.46276	0.25646	0.26752	0.21407	0.25647	0.2227	0.26727	0.16278
$w_3 l_2$	0.44477	0.25573	0.26001	0.21333	0.25565	0.21663	0.25985	0.15845
$w_3 l_3$	0.46276	0.25625	0.26693	0.21405	0.25631	0.22257	0.2667	0.16287
$w_3 l_4$	0.44477	0.25552	0.25962	0.21333	0.25572	0.21689	0.26011	0.15873

Table 66. Additive Desirability Inference Observation 13, (x_1, x_2) = (0,0) Empirical Coverage and Average Width

Empirical Coverage								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.9972	0.9448	0.9498	0.8994	0.9439	0.9058	0.9491	0.8008
w_1l_2	0.9955	0.9494	0.9501	0.9086	0.949	0.9054	0.9475	0.7998
w_1l_3	0.9972	0.9492	0.9531	0.9032	0.9494	0.9124	0.9522	0.8094
w_1l_4	0.9955	0.9549	0.9536	0.9139	0.9538	0.9129	0.9531	0.807
w_2l_1	0.9979	0.9563	0.9489	0.9164	0.9576	0.9096	0.9505	0.8032
w_2l_2	0.9968	0.957	0.9473	0.9183	0.9547	0.9095	0.9514	0.8041
w_2l_3	0.9979	0.9574	0.9486	0.9169	0.9582	0.911	0.9515	0.804
w_2l_4	0.9968	0.9584	0.9497	0.9188	0.9567	0.9108	0.9517	0.805
w_3l_1	0.9977	0.9404	0.9486	0.8927	0.942	0.9067	0.9496	0.8007
w_3l_2	0.9961	0.9486	0.9506	0.9081	0.9501	0.9063	0.946	0.8032
w_3l_3	0.9977	0.9459	0.9522	0.8933	0.9459	0.9067	0.9503	0.8023
w_3l_4	0.9961	0.9524	0.9533	0.9081	0.9558	0.9098	0.9522	0.8004
Average Width								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.3106	0.15523	0.16736	0.13023	0.15517	0.13977	0.16756	0.10224
w_1l_2	0.28124	0.15173	0.15672	0.12715	0.15171	0.13113	0.15723	0.09599
w_1l_3	0.3106	0.15533	0.16686	0.13018	0.1553	0.13928	0.16675	0.10203
w_1l_4	0.28124	0.15178	0.15672	0.12706	0.15177	0.13064	0.15665	0.09571
w_2l_1	0.2627	0.14348	0.14797	0.12021	0.14346	0.12341	0.14836	0.09025
w_2l_2	0.24508	0.14209	0.14649	0.11904	0.14202	0.12219	0.14686	0.08941
w_2l_3	0.2627	0.1436	0.14757	0.1202	0.14357	0.12333	0.14794	0.09028
w_2l_4	0.24508	0.14224	0.14624	0.11901	0.14222	0.12216	0.14669	0.08938
w_3l_1	0.22617	0.10894	0.12286	0.09154	0.10891	0.1022	0.12267	0.07474
w_3l_2	0.17332	0.0912	0.09547	0.0764	0.09118	0.07961	0.09541	0.05825
w_3l_3	0.22617	0.10899	0.12221	0.09154	0.10904	0.10207	0.12189	0.07475
w_3l_4	0.17332	0.09122	0.09523	0.0764	0.09125	0.07962	0.09539	0.05831

Table 67. Additive Desirability Inference Observation 14, (x_1, x_2) = (0.5,0) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9969	0.9464	0.9508	0.8998	0.9463	0.906	0.9485	0.7975
$w_1 l_2$	0.9956	0.9518	0.9509	0.9075	0.9511	0.9079	0.9491	0.7934
$w_1 l_3$	0.9969	0.9619	0.9612	0.9143	0.9614	0.9235	0.9629	0.8048
$w_1 l_4$	0.9956	0.9668	0.9615	0.9232	0.9678	0.9227	0.9646	0.8058
$w_2 l_1$	0.9896	0.9431	0.9477	0.8972	0.944	0.9054	0.9491	0.8014
$w_2 l_2$	0.9877	0.9478	0.9485	0.9041	0.9475	0.9081	0.948	0.7997
$w_2 l_3$	0.9897	0.9586	0.9603	0.9132	0.9589	0.9207	0.9621	0.8102
$w_2 l_4$	0.9878	0.9619	0.9612	0.9189	0.9627	0.9217	0.9613	0.8093
$w_3 l_1$	0.9984	0.9563	0.9487	0.9187	0.9566	0.9104	0.9534	0.8073
$w_3 l_2$	0.9971	0.9569	0.9476	0.9195	0.9581	0.9136	0.9522	0.8062
$w_3 l_3$	0.9984	0.9648	0.9561	0.9261	0.9644	0.9172	0.9608	0.8109
$w_3 l_4$	0.9971	0.9645	0.9557	0.9242	0.9635	0.9149	0.9576	0.8055
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.32274	0.17985	0.19377	0.1529	0.17981	0.16399	0.19392	0.12073
$w_1 l_2$	0.28839	0.17553	0.18156	0.14922	0.17553	0.15393	0.18213	0.11344
$w_1 l_3$	0.32276	0.18455	0.196	0.15476	0.18463	0.16461	0.19566	0.12098
$w_1 l_4$	0.28841	0.18045	0.18551	0.15102	0.18041	0.15507	0.18532	0.1137
$w_2 l_1$	0.44287	0.28794	0.30043	0.24432	0.28796	0.25419	0.30009	0.18752
$w_2 l_2$	0.42226	0.28719	0.2921	0.24344	0.28703	0.24725	0.29194	0.18258
$w_2 l_3$	0.44291	0.29699	0.30813	0.24781	0.29697	0.25712	0.30754	0.18859
$w_2 l_4$	0.4223	0.29608	0.30085	0.24686	0.29614	0.25054	0.30039	0.18364
$w_3 l_1$	0.28439	0.1686	0.17369	0.14221	0.16866	0.14582	0.17406	0.10685
$w_3 l_2$	0.26378	0.16696	0.17218	0.14083	0.16693	0.14443	0.17251	0.10585
$w_3 l_3$	0.2844	0.17085	0.17398	0.14304	0.17085	0.14602	0.17432	0.10696
$w_3 l_4$	0.26379	0.16926	0.17308	0.14162	0.16882	0.14468	0.17331	0.10583

Table 68. Additive Desirability Inference Observation 15, (x_1, x_2) = (1,0) Empirical Coverage and Average Width

				Empirical Coverage				
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	0.9992	0.9522	0.9515	0.9045	0.9523	0.9105	0.951	0.8008
w$_1 l_2$	0.9977	0.9567	0.9538	0.916	0.9571	0.9133	0.9531	0.801
w$_1 l_3$	0.9994	0.9671	0.9714	0.9407	0.9674	0.9431	0.9693	0.8533
w$_1 l_4$	0.9985	0.9693	0.9706	0.9446	0.9693	0.9442	0.9674	0.8561
w$_2 l_1$	0.9995	0.9485	0.9515	0.8952	0.9478	0.9088	0.9521	0.797
w$_2 l_2$	0.9988	0.9544	0.9527	0.9089	0.9551	0.9117	0.9552	0.8024
w$_2 l_3$	0.9997	0.9649	0.9712	0.933	0.9654	0.9425	0.9679	0.8502
w$_2 l_4$	0.999	0.9685	0.97	0.9437	0.9684	0.9429	0.9691	0.86
w$_3 l_1$	0.993	0.9539	0.9531	0.9054	0.9542	0.9134	0.9569	0.797
w$_3 l_2$	0.991	0.9577	0.9528	0.9115	0.9573	0.9168	0.9572	0.7989
w$_3 l_3$	0.9958	0.9703	0.9716	0.9434	0.9703	0.9488	0.9712	0.8624
w$_3 l_4$	0.9944	0.9714	0.9705	0.9462	0.973	0.9507	0.9737	0.8694
				Average Width				
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	0.30708	0.18667	0.19797	0.16082	0.18669	0.17052	0.19825	0.12799
w$_1 l_2$	0.27448	0.18151	0.18782	0.15648	0.18159	0.16166	0.18834	0.1213
w$_1 l_3$	0.30986	0.21975	0.22255	0.18408	0.21977	0.18876	0.22265	0.14002
w$_1 l_4$	0.27727	0.21518	0.21719	0.18011	0.21519	0.18277	0.21714	0.13448
w$_2 l_1$	0.23675	0.13506	0.14733	0.11553	0.13507	0.12626	0.14753	0.09435
w$_2 l_2$	0.17808	0.11073	0.11677	0.09542	0.11072	0.1005	0.117	0.07542
w$_2 l_3$	0.23842	0.15195	0.15443	0.12747	0.15188	0.13227	0.15421	0.09909
w$_2 l_4$	0.17975	0.13006	0.13256	0.10878	0.13001	0.11164	0.1324	0.08239
w$_3 l_1$	0.39955	0.29116	0.30163	0.25171	0.29119	0.26131	0.30227	0.19771
w$_3 l_2$	0.37999	0.28981	0.2948	0.25065	0.29003	0.25541	0.29541	0.19322
w$_3 l_3$	0.40457	0.35395	0.36127	0.29456	0.35388	0.30134	0.36017	0.22146
w$_3 l_4$	0.38501	0.35312	0.35739	0.29375	0.35349	0.29785	0.35668	0.21799

Table 69. Additive Desirability Inference Observation 16, (x_1, x_2) = (-1,0.5) Empirical Coverage and Average Width

Empirical Coverage								
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	0.9971	0.943	0.9455	0.8952	0.9436	0.907	0.9485	0.7993
w$_1 l_2$	0.9947	0.9502	0.9454	0.9062	0.9508	0.9052	0.9493	0.8018
w$_1 l_3$	0.9975	0.9652	0.9637	0.9258	0.9648	0.9303	0.9649	0.8285
w$_1 l_4$	0.9953	0.9684	0.9611	0.934	0.9693	0.9329	0.9665	0.8301
w$_2 l_1$	0.9983	0.9572	0.9473	0.9151	0.955	0.9082	0.9468	0.809
w$_2 l_2$	0.9968	0.9556	0.9462	0.9151	0.9563	0.9092	0.9502	0.8074
w$_2 l_3$	0.9986	0.9672	0.9582	0.93	0.9676	0.9234	0.9583	0.822
w$_2 l_4$	0.9971	0.9663	0.9576	0.9285	0.9651	0.9232	0.9609	0.8177
w$_3 l_1$	0.9985	0.9434	0.9497	0.8909	0.9417	0.9044	0.9476	0.7991
w$_3 l_2$	0.9956	0.9514	0.9453	0.9065	0.9514	0.9095	0.9505	0.8029
w$_3 l_3$	0.9989	0.9615	0.9647	0.9187	0.9624	0.9253	0.9618	0.8188
w$_3 l_4$	0.9962	0.9675	0.9621	0.9308	0.9672	0.9312	0.9639	0.8326
Average Width								
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	0.26093	0.15133	0.16243	0.1295	0.1513	0.13847	0.1625	0.10306
w$_1 l_2$	0.23198	0.14759	0.15268	0.1262	0.14754	0.13022	0.15299	0.09691
w$_1 l_3$	0.26124	0.16412	0.17173	0.13587	0.16409	0.14298	0.17154	0.10505
w$_1 l_4$	0.23229	0.16043	0.16398	0.13272	0.16033	0.13579	0.1639	0.0992
w$_2 l_1$	0.23452	0.14388	0.14753	0.12155	0.14389	0.12438	0.1481	0.09163
w$_2 l_2$	0.21715	0.14245	0.14655	0.12035	0.14243	0.12334	0.14699	0.09083
w$_2 l_3$	0.23471	0.14965	0.15072	0.12433	0.14963	0.12613	0.15109	0.09257
w$_2 l_4$	0.21734	0.14823	0.15057	0.12308	0.14824	0.12554	0.15096	0.09187
w$_3 l_1$	0.19722	0.10789	0.12019	0.09187	0.10782	0.10191	0.12009	0.07558
w$_3 l_2$	0.14511	0.08898	0.09329	0.07612	0.08897	0.07954	0.09337	0.05915
w$_3 l_3$	0.19741	0.11441	0.1224	0.09507	0.1144	0.10315	0.12222	0.07637
w$_3 l_4$	0.14529	0.09628	0.09965	0.07967	0.09634	0.08276	0.09964	0.0606

Table 70. Additive Desirability Inference Observation 17, (x_1, x_2) = (-0.5,0.5) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9984	0.944	0.9497	0.8963	0.9431	0.9067	0.949	0.8027
$w_1 l_2$	0.9961	0.9515	0.9497	0.9055	0.9519	0.9056	0.9491	0.7965
$w_1 l_3$	0.9989	0.9698	0.9666	0.9352	0.9695	0.9359	0.9702	0.8331
$w_1 l_4$	0.9967	0.9718	0.9665	0.9415	0.9718	0.9345	0.9706	0.8344
$w_2 l_1$	0.9889	0.9436	0.9482	0.8997	0.9438	0.9076	0.9462	0.798
$w_2 l_2$	0.9866	0.9477	0.949	0.9057	0.9477	0.9083	0.9476	0.7989
$w_2 l_3$	0.9906	0.9637	0.9687	0.9279	0.9639	0.9351	0.9674	0.8354
$w_2 l_4$	0.9884	0.9671	0.9682	0.9338	0.9673	0.9348	0.9672	0.8361
$w_3 l_1$	0.9984	0.9556	0.9484	0.9112	0.9566	0.906	0.9495	0.8005
$w_3 l_2$	0.9967	0.9547	0.9482	0.9113	0.9557	0.9061	0.9502	0.802
$w_3 l_3$	0.9985	0.9707	0.9597	0.929	0.9697	0.9233	0.9626	0.8185
$w_3 l_4$	0.9968	0.9679	0.9601	0.9276	0.9663	0.9221	0.9626	0.8155
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.25014	0.14676	0.15725	0.12593	0.14674	0.13446	0.15738	0.10042
$w_1 l_2$	0.22089	0.14297	0.14768	0.12262	0.14296	0.12627	0.14805	0.09427
$w_1 l_3$	0.25084	0.16312	0.1699	0.13465	0.16302	0.14093	0.16945	0.10344
$w_1 l_4$	0.22159	0.15934	0.16279	0.13146	0.15935	0.13404	0.16252	0.09774
$w_2 l_1$	0.33411	0.23265	0.24208	0.19971	0.23267	0.20756	0.24229	0.15547
$w_2 l_2$	0.31656	0.23186	0.23544	0.19897	0.23178	0.20189	0.23571	0.15131
$w_2 l_3$	0.33537	0.26354	0.27134	0.21617	0.2633	0.22272	0.27065	0.1625
$w_2 l_4$	0.31782	0.26284	0.26624	0.21539	0.2626	0.21793	0.26558	0.15851
$w_3 l_1$	0.22642	0.14046	0.1438	0.11867	0.1404	0.1213	0.14422	0.08952
$w_3 l_2$	0.20887	0.13899	0.14278	0.11747	0.139	0.12028	0.14314	0.08867
$w_3 l_3$	0.22684	0.1481	0.14832	0.12272	0.14803	0.12389	0.14868	0.09093
$w_3 l_4$	0.20929	0.1467	0.14839	0.12151	0.14623	0.12341	0.14856	0.09018

Table 71. Additive Desirability Inference Observation 18, (x_1, x_2) = (0,0.5) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9989	0.9523	0.9571	0.9032	0.9521	0.9094	0.953	0.8061
$w_1 l_2$	0.9971	0.9591	0.9588	0.9132	0.96	0.9124	0.9562	0.8032
$w_1 l_3$	0.9994	0.9717	0.9722	0.9463	0.971	0.9495	0.9737	0.8729
$w_1 l_4$	0.9975	0.9752	0.9707	0.9507	0.9754	0.9499	0.9746	0.8715
$w_2 l_1$	0.9993	0.947	0.9533	0.8994	0.9476	0.9095	0.9511	0.7997
$w_2 l_2$	0.9981	0.9557	0.9582	0.9101	0.9577	0.9095	0.9545	0.8024
$w_2 l_3$	0.9998	0.966	0.9692	0.9402	0.9663	0.9439	0.9679	0.8608
$w_2 l_4$	0.9985	0.9702	0.971	0.9443	0.9702	0.9425	0.9687	0.8667
$w_3 l_1$	0.9934	0.9497	0.9574	0.9056	0.9506	0.9104	0.954	0.803
$w_3 l_2$	0.9911	0.9555	0.9592	0.9105	0.9548	0.9116	0.9538	0.8034
$w_3 l_3$	0.9957	0.968	0.9713	0.9429	0.9669	0.9467	0.9702	0.8716
$w_3 l_4$	0.994	0.969	0.9726	0.9468	0.9694	0.9452	0.969	0.8704
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.23587	0.14365	0.15238	0.12378	0.14367	0.13114	0.15248	0.09922
$w_1 l_2$	0.20584	0.1394	0.14358	0.12027	0.13943	0.1235	0.14388	0.09335
$w_1 l_3$	0.23936	0.18051	0.18348	0.14861	0.18052	0.15181	0.18267	0.11154
$w_1 l_4$	0.20933	0.17693	0.17904	0.14543	0.17678	0.14688	0.17833	0.10679
$w_2 l_1$	0.18218	0.10419	0.11359	0.08912	0.1042	0.09711	0.11339	0.07302
$w_2 l_2$	0.12811	0.08403	0.0872	0.07245	0.08402	0.07495	0.08726	0.05664
$w_2 l_3$	0.18428	0.1243	0.12525	0.10287	0.12421	0.10565	0.12485	0.07884
$w_2 l_4$	0.13021	0.10646	0.1078	0.08759	0.10648	0.08874	0.10745	0.06474
$w_3 l_1$	0.30595	0.22375	0.23204	0.19344	0.22365	0.20058	0.23211	0.15286
$w_3 l_2$	0.28793	0.22272	0.22599	0.19255	0.22272	0.19526	0.22591	0.14882
$w_3 l_3$	0.31223	0.29469	0.30019	0.24121	0.29474	0.24562	0.29886	0.1788
$w_3 l_4$	0.29421	0.29376	0.29661	0.24046	0.29452	0.24275	0.29648	0.17566

Table 72. Additive Desirability Inference Observation 19, (x_1, x_2) = (0.5,0.5) Empirical Coverage and Average Width

Empirical Coverage								
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	1	0.9616	0.9684	0.9401	0.9622	0.943	0.9638	0.8589
w$_1 l_2$	0.9996	0.964	0.9661	0.9436	0.9646	0.9403	0.9615	0.8584
w$_1 l_3$	1	0.9609	0.9606	0.9425	0.9615	0.9451	0.9616	0.8861
w$_1 l_4$	0.9998	0.9639	0.9543	0.9466	0.9636	0.9429	0.9595	0.8791
w$_2 l_1$	0.9986	0.9624	0.953	0.9312	0.9629	0.919	0.9535	0.825
w$_2 l_2$	0.9953	0.9602	0.9518	0.9252	0.9592	0.9177	0.9544	0.821
w$_2 l_3$	0.9987	0.9585	0.9391	0.9354	0.9588	0.9202	0.9399	0.8547
w$_2 l_4$	0.9961	0.9535	0.9385	0.9284	0.9545	0.9195	0.9415	0.8486
w$_3 l_1$	1	0.9595	0.9642	0.9317	0.9598	0.9321	0.9624	0.8369
w$_3 l_2$	0.9998	0.9661	0.9674	0.9445	0.965	0.9442	0.9658	0.8674
w$_3 l_3$	1	0.9558	0.9524	0.9327	0.9566	0.9375	0.9543	0.8786
w$_3 l_4$	0.9999	0.9576	0.9572	0.9396	0.954	0.933	0.9512	0.8698
Average Width								
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	0.26228	0.1673	0.16902	0.14262	0.1673	0.14475	0.16908	0.11024
w$_1 l_2$	0.22402	0.16092	0.16299	0.13736	0.16092	0.13924	0.16326	0.10588
w$_1 l_3$	0.30176	0.27005	0.25743	0.23639	0.26992	0.228	0.25751	0.17932
w$_1 l_4$	0.2635	0.26498	0.26041	0.23236	0.26501	0.22956	0.26076	0.17899
w$_2 l_1$	0.25804	0.17603	0.17384	0.14782	0.1761	0.14642	0.17406	0.10829
w$_2 l_2$	0.23508	0.17406	0.17527	0.14611	0.17399	0.14732	0.17566	0.10881
w$_2 l_3$	0.28173	0.22254	0.21011	0.19275	0.22261	0.18312	0.21074	0.14058
w$_2 l_4$	0.25877	0.22095	0.2152	0.19126	0.22087	0.18688	0.21575	0.1428
w$_3 l_1$	0.20987	0.12585	0.12693	0.10663	0.1259	0.10809	0.12655	0.08156
w$_3 l_2$	0.141	0.09706	0.09834	0.08284	0.0971	0.08389	0.09819	0.06382
w$_3 l_3$	0.23357	0.1819	0.16401	0.15875	0.18191	0.14601	0.16419	0.11583
w$_3 l_4$	0.16469	0.15951	0.15536	0.13984	0.15914	0.13713	0.15532	0.10738

Table 73. Additive Desirability Inference Observation 20, (x_1, x_2) = (1,0.5) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	1	0.9912	0.9638	0.9777	0.9912	0.9451	0.967	0.8685
$w_1 l_2$	1	0.9847	0.9645	0.9674	0.9844	0.9446	0.9679	0.8628
$w_1 l_3$	0.9204	0.9773	0.953	0.9511	0.9763	0.9106	0.9461	0.804
$w_1 l_4$	0.9117	0.9686	0.955	0.9382	0.9701	0.9101	0.9509	0.8041
$w_2 l_1$	1	0.9778	0.9709	0.9594	0.9785	0.9506	0.9728	0.8782
$w_2 l_2$	1	0.9743	0.9715	0.9558	0.9755	0.9494	0.9708	0.8755
$w_2 l_3$	0.8973	0.9595	0.9543	0.919	0.9599	0.9074	0.9519	0.7996
$w_2 l_4$	0.8971	0.9545	0.9526	0.9132	0.9552	0.9061	0.9512	0.7997
$w_3 l_1$	0.999	0.9779	0.9513	0.9568	0.9787	0.9221	0.9568	0.8169
$w_3 l_2$	0.9959	0.9719	0.9544	0.9442	0.9728	0.9212	0.9559	0.8159
$w_3 l_3$	0.9788	0.9683	0.9459	0.9398	0.9671	0.901	0.9417	0.7975
$w_3 l_4$	0.9714	0.9626	0.9503	0.9294	0.9648	0.903	0.9477	0.795
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.35884	0.22453	0.19275	0.19196	0.22454	0.16311	0.19273	0.12273
$w_1 l_2$	0.3222	0.21813	0.20226	0.18572	0.21823	0.1707	0.20204	0.12886
$w_1 l_3$	0.48889	0.34037	0.30624	0.31238	0.3405	0.28241	0.30602	0.23018
$w_1 l_4$	0.45224	0.33351	0.31557	0.30625	0.33353	0.29056	0.31555	0.23648
$w_2 l_1$	0.43541	0.32905	0.31282	0.28224	0.32874	0.26714	0.31241	0.20598
$w_2 l_2$	0.41342	0.32737	0.31952	0.28048	0.32727	0.27251	0.31891	0.21013
$w_2 l_3$	0.6695	0.53649	0.51463	0.49938	0.53644	0.48219	0.51704	0.39988
$w_2 l_4$	0.64751	0.53333	0.52095	0.49687	0.53375	0.48762	0.52353	0.40389
$w_3 l_1$	0.35365	0.23928	0.21718	0.2014	0.23951	0.18159	0.21836	0.13283
$w_3 l_2$	0.33166	0.23761	0.22433	0.19934	0.23791	0.18756	0.2253	0.13747
$w_3 l_3$	0.43168	0.29513	0.26669	0.26341	0.29508	0.23804	0.26778	0.18704
$w_3 l_4$	0.40969	0.29368	0.27444	0.26175	0.29263	0.24388	0.27428	0.19129

Table 74. Additive Desirability Inference Observation 21, (x_1, x_2) = (-1,1) Empirical Coverage and Average Width

Empirical Coverage								
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	1	0.9902	0.9683	0.9738	0.9897	0.938	0.9683	0.8334
w$_1 l_2$	0.9995	0.9798	0.9681	0.9533	0.9806	0.9368	0.9671	0.8291
w$_1 l_3$	0.9681	0.9793	0.9483	0.9565	0.9794	0.9099	0.9444	0.8084
w$_1 l_4$	0.9674	0.967	0.9504	0.9347	0.9667	0.911	0.949	0.805
w$_2 l_1$	1	0.9973	0.9625	0.9905	0.9974	0.9306	0.9638	0.8325
w$_2 l_2$	0.9999	0.9845	0.9688	0.9613	0.9847	0.9297	0.966	0.8247
w$_2 l_3$	0.9729	0.9849	0.9322	0.9705	0.9842	0.8944	0.9248	0.8021
w$_2 l_4$	0.9678	0.9709	0.9508	0.9436	0.9706	0.9079	0.944	0.8041
w$_3 l_1$	0.9992	0.974	0.9682	0.9449	0.973	0.9343	0.9677	0.8206
w$_3 l_2$	0.9977	0.9697	0.9671	0.935	0.9697	0.9313	0.9661	0.8189
w$_3 l_3$	0.9658	0.9579	0.9516	0.9195	0.9586	0.9047	0.9483	0.796
w$_3 l_4$	0.9652	0.9536	0.9504	0.9104	0.954	0.9066	0.9479	0.8051
Average Width								
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	0.28028	0.17083	0.1435	0.14681	0.17091	0.12209	0.14371	0.09156
w$_1 l_2$	0.24399	0.16501	0.15496	0.14189	0.16494	0.13228	0.15508	0.09944
w$_1 l_3$	0.4508	0.29244	0.26606	0.26072	0.29234	0.23749	0.26562	0.18231
w$_1 l_4$	0.41451	0.28681	0.2774	0.25616	0.28667	0.24733	0.2772	0.18999
w$_2 l_1$	0.22067	0.12711	0.09066	0.10832	0.12708	0.07571	0.09113	0.05557
w$_2 l_2$	0.15536	0.09969	0.08972	0.08567	0.0997	0.07627	0.08966	0.05734
w$_2 l_3$	0.32298	0.19693	0.15972	0.17484	0.19699	0.14327	0.16044	0.10981
w$_2 l_4$	0.25767	0.17348	0.16318	0.1551	0.17363	0.14614	0.16356	0.11238
w$_3 l_1$	0.35132	0.25636	0.2423	0.22129	0.25643	0.20809	0.24194	0.15885
w$_3 l_2$	0.32954	0.25482	0.24987	0.21993	0.25492	0.21463	0.24903	0.16386
w$_3 l_3$	0.65825	0.47705	0.46058	0.42822	0.47695	0.41453	0.46164	0.32121
w$_3 l_4$	0.63648	0.4747	0.46791	0.42647	0.47374	0.42029	0.46814	0.32503

Table 75. Additive Desirability Inference Observation 22, (x_1, x_2) = (-0.5,1) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	1	0.9864	0.961	0.9682	0.9859	0.9203	0.9599	0.8087
$w_1 l_2$	0.9986	0.9737	0.9579	0.9423	0.9745	0.9173	0.9599	0.8073
$w_1 l_3$	0.994	0.9844	0.9346	0.9652	0.9836	0.9017	0.9328	0.8031
$w_1 l_4$	0.992	0.9671	0.9399	0.9358	0.9664	0.9071	0.9392	0.8036
$w_2 l_1$	0.9995	0.9776	0.9517	0.9527	0.9775	0.9129	0.9514	0.8053
$w_2 l_2$	0.9982	0.9693	0.9522	0.9375	0.9692	0.911	0.9496	0.8038
$w_2 l_3$	0.9911	0.9668	0.9354	0.9395	0.9676	0.8948	0.932	0.7918
$w_2 l_4$	0.9855	0.9557	0.9371	0.9245	0.9575	0.8984	0.9357	0.7958
$w_3 l_1$	1	0.9964	0.9602	0.9884	0.9968	0.9203	0.9621	0.8172
$w_3 l_2$	0.9998	0.9773	0.9594	0.9447	0.977	0.9138	0.9559	0.7996
$w_3 l_3$	0.994	0.9899	0.9223	0.9763	0.9894	0.8885	0.9173	0.7901
$w_3 l_4$	0.9935	0.969	0.9402	0.9403	0.9695	0.9056	0.9395	0.8012
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.2736	0.16074	0.13363	0.13799	0.16071	0.11386	0.13394	0.08489
$w_1 l_2$	0.23937	0.15571	0.14644	0.13379	0.15563	0.12483	0.14651	0.09318
$w_1 l_3$	0.44469	0.24496	0.22076	0.20737	0.24497	0.1862	0.2204	0.13168
$w_1 l_4$	0.41045	0.23966	0.23182	0.20283	0.23978	0.19568	0.23161	0.13871
$w_2 l_1$	0.2549	0.16151	0.14266	0.13631	0.16159	0.11938	0.14314	0.0876
$w_2 l_2$	0.23436	0.15979	0.15084	0.13483	0.15976	0.12629	0.15135	0.09277
$w_2 l_3$	0.35756	0.2058	0.1855	0.17448	0.20587	0.15827	0.18664	0.11461
$w_2 l_4$	0.33701	0.20426	0.19327	0.17311	0.20437	0.16481	0.19457	0.11935
$w_3 l_1$	0.21149	0.11762	0.08117	0.10033	0.11765	0.06816	0.08154	0.05017
$w_3 l_2$	0.14986	0.09366	0.0851	0.08042	0.0937	0.07246	0.08504	0.05414
$w_3 l_3$	0.31414	0.16764	0.13405	0.14237	0.16752	0.11471	0.13505	0.08185
$w_3 l_4$	0.25251	0.14514	0.13656	0.12313	0.14467	0.11566	0.13681	0.08198

Table 76. Additive Desirability Inference Observation 23, (x_1, x_2) = (0,1) Empirical Coverage and Average Width

Empirical Coverage								
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	1	0.9848	0.9548	0.9633	0.984	0.9139	0.9557	0.8044
w$_1 l_2$	0.9977	0.9697	0.953	0.9401	0.9697	0.9112	0.954	0.8036
w$_1 l_3$	0.9974	0.9832	0.9196	0.963	0.9839	0.8901	0.9239	0.7937
w$_1 l_4$	0.9905	0.9594	0.9293	0.9297	0.9604	0.9029	0.934	0.7992
w$_2 l_1$	0.9947	0.9631	0.9507	0.9264	0.9619	0.9095	0.9509	0.7989
w$_2 l_2$	0.9904	0.9558	0.9515	0.9152	0.9559	0.9089	0.9517	0.7993
w$_2 l_3$	0.9969	0.975	0.926	0.958	0.9761	0.8994	0.9279	0.8065
w$_2 l_4$	0.993	0.9585	0.9358	0.9312	0.9582	0.9032	0.9335	0.8017
w$_3 l_1$	0.9995	0.9778	0.9479	0.9553	0.9779	0.91	0.949	0.804
w$_3 l_2$	0.9983	0.9687	0.9481	0.9387	0.9683	0.9083	0.9479	0.8011
w$_3 l_3$	0.993	0.9649	0.9406	0.9399	0.9658	0.906	0.9406	0.8033
w$_3 l_4$	0.9818	0.9566	0.9428	0.9272	0.9617	0.9047	0.9456	0.7972
Average Width								
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	0.31925	0.18202	0.15096	0.15583	0.18203	0.12847	0.15138	0.09479
w$_1 l_2$	0.28092	0.17666	0.16606	0.15132	0.17665	0.14121	0.1662	0.10424
w$_1 l_3$	0.45977	0.21346	0.18875	0.1726	0.21336	0.15079	0.18739	0.09933
w$_1 l_4$	0.42145	0.20642	0.19933	0.16636	0.20656	0.15963	0.19827	0.10538
w$_2 l_1$	0.42149	0.28331	0.26648	0.2424	0.28335	0.22787	0.26661	0.16908
w$_2 l_2$	0.3985	0.282	0.27601	0.24143	0.28204	0.23592	0.27602	0.17502
w$_2 l_3$	0.67443	0.32048	0.30323	0.25555	0.3203	0.24173	0.30405	0.15507
w$_2 l_4$	0.65144	0.31674	0.31077	0.25301	0.31696	0.24823	0.31182	0.15951
w$_3 l_1$	0.29219	0.18047	0.1586	0.15236	0.18032	0.13293	0.15929	0.09701
w$_3 l_2$	0.2692	0.17843	0.16801	0.15066	0.17856	0.14093	0.16858	0.10293
w$_3 l_3$	0.37651	0.20178	0.18224	0.16648	0.20177	0.15146	0.18302	0.10665
w$_3 l_4$	0.35352	0.2001	0.19017	0.16473	0.19923	0.15734	0.19018	0.11103

Table 77. Additive Desirability Inference Observation 24, (x_1, x_2) = (0.5,1) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9998	0.9843	0.9539	0.9631	0.9842	0.9096	0.9552	0.797
$w_1 l_2$	0.9983	0.9718	0.9516	0.9356	0.9693	0.9085	0.9535	0.7959
$w_1 l_3$	0.9959	0.9835	0.9327	0.9637	0.9837	0.9073	0.9359	0.8099
$w_1 l_4$	0.9835	0.9591	0.9408	0.9281	0.9614	0.9087	0.9413	0.808
$w_2 l_1$	1	0.9968	0.9546	0.9888	0.9969	0.9156	0.9575	0.806
$w_2 l_2$	0.9993	0.9757	0.9511	0.947	0.9764	0.9127	0.9552	0.8001
$w_2 l_3$	0.9978	0.9896	0.9381	0.9744	0.9883	0.9135	0.9436	0.8205
$w_2 l_4$	0.9913	0.9683	0.9403	0.9432	0.9678	0.9076	0.937	0.8123
$w_3 l_1$	0.9948	0.9645	0.9515	0.9286	0.9639	0.9108	0.9548	0.801
$w_3 l_2$	0.9896	0.9581	0.9522	0.9186	0.9588	0.9114	0.9538	0.8007
$w_3 l_3$	0.9963	0.9773	0.9134	0.9585	0.977	0.8866	0.9163	0.7909
$w_3 l_4$	0.9858	0.9509	0.9233	0.9212	0.9495	0.8892	0.924	0.7849
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.42894	0.24435	0.20264	0.21193	0.24434	0.17457	0.20287	0.12977
$w_1 l_2$	0.37706	0.23698	0.22257	0.2056	0.23689	0.19164	0.22246	0.14243
$w_1 l_3$	0.50675	0.23423	0.2034	0.19033	0.23436	0.16444	0.20237	0.10932
$w_1 l_4$	0.45487	0.22375	0.21495	0.18075	0.22402	0.1733	0.21398	0.11485
$w_2 l_1$	0.32953	0.17874	0.1223	0.15386	0.17882	0.10388	0.12295	0.07634
$w_2 l_2$	0.23614	0.14288	0.12905	0.12395	0.14288	0.11135	0.12926	0.08282
$w_2 l_3$	0.37622	0.17973	0.14039	0.14822	0.17978	0.11675	0.14085	0.08112
$w_2 l_4$	0.28283	0.13576	0.12561	0.1099	0.13592	0.10181	0.12572	0.06746
$w_3 l_1$	0.56316	0.38087	0.35793	0.32942	0.38081	0.30947	0.35838	0.23128
$w_3 l_2$	0.53203	0.37932	0.37072	0.32785	0.3794	0.32029	0.3707	0.23929
$w_3 l_3$	0.70323	0.31779	0.29823	0.25138	0.31719	0.23423	0.29759	0.14559
$w_3 l_4$	0.6721	0.31278	0.30694	0.24627	0.31254	0.2415	0.30602	0.15175

Table 78. Additive Desirability Inference Observation 25, (x_1, x_2) = (1,1) Empirical Coverage and Average Width

B.2 Multiplicative Empirical Coverage and Average Width Tables

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9996	0.9893	0.9239	0.978	0.9893	0.9033	0.929	0.8075
$w_1 l_2$	0.9999	0.9791	0.9537	0.9598	0.9786	0.9266	0.9511	0.8405
$w_1 l_3$	0.969	0.9702	0.9569	0.9359	0.9692	0.9169	0.9567	0.8064
$w_1 l_4$	0.951	0.9618	0.9584	0.9202	0.9616	0.9163	0.9588	0.8072
$w_2 l_1$	0.9938	0.9745	0.9495	0.9487	0.976	0.9141	0.9525	0.8034
$w_2 l_2$	1	0.9861	0.9379	0.9748	0.9868	0.9149	0.9389	0.8329
$w_2 l_3$	0.9948	0.9799	0.9473	0.9599	0.9796	0.9168	0.9491	0.8129
$w_2 l_4$	0.9521	0.9599	0.9584	0.9173	0.9599	0.9108	0.9537	0.8022
$w_3 l_1$	0.9999	0.9842	0.9458	0.9718	0.9851	0.9246	0.9472	0.8412
$w_3 l_2$	0.9997	0.9746	0.9658	0.9495	0.9735	0.9401	0.9661	0.8542
$w_3 l_3$	0.9516	0.9596	0.9566	0.9174	0.9597	0.911	0.9534	0.8019
$w_3 l_4$	0.95	0.9577	0.9595	0.9139	0.9609	0.9112	0.9597	0.7981
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.38796	0.27327	0.22845	0.21463	0.27317	0.17047	0.22798	0.11408
$w_1 l_2$	0.43536	0.33788	0.32164	0.24673	0.33893	0.22727	0.31766	0.15069
$w_1 l_3$	0.56074	0.4168	0.37634	0.37568	0.4168	0.33936	0.37587	0.27039
$w_1 l_4$	0.83527	0.70821	0.69161	0.63852	0.70834	0.62559	0.69078	0.50067
$w_2 l_1$	0.28866	0.23165	0.20511	0.18723	0.23182	0.16524	0.20626	0.11662
$w_2 l_2$	0.35008	0.26668	0.2407	0.18279	0.26598	0.15756	0.24033	0.09493
$w_2 l_3$	0.37981	0.27373	0.23293	0.24067	0.27374	0.20116	0.23306	0.1525
$w_2 l_4$	0.74461	0.64845	0.62898	0.56428	0.64817	0.54665	0.62707	0.42411
$w_3 l_1$	0.47575	0.35211	0.31377	0.27812	0.35271	0.23924	0.31383	0.16478
$w_3 l_2$	0.51954	0.44656	0.43569	0.34247	0.44727	0.332	0.43378	0.23268
$w_3 l_3$	0.70687	0.54567	0.51258	0.50114	0.54599	0.4765	0.51498	0.38752
$w_3 l_4$	0.90589	0.7621	0.74703	0.70099	0.76018	0.69002	0.74789	0.56087

Table 79. Multiplicative Desirability Inference Observation 1, (x_1, x_2) = (-1,-1) Empirical Coverage and Average Width

Empirical Coverage								
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	1	0.9914	0.9415	0.9822	0.9917	0.9244	0.9448	0.8413
w$_1 l_2$	0.9997	0.9736	0.9547	0.9532	0.9736	0.9314	0.9549	0.8403
w$_1 l_3$	0.8995	0.9546	0.9464	0.915	0.9533	0.9046	0.9484	0.7939
w$_1 l_4$	0.8981	0.9515	0.945	0.9086	0.9509	0.904	0.9482	0.7925
w$_2 l_1$	0.9976	0.9786	0.9471	0.9549	0.9779	0.9135	0.9484	0.8155
w$_2 l_2$	0.9908	0.9667	0.9478	0.933	0.9661	0.9163	0.948	0.8121
w$_2 l_3$	0.9394	0.9631	0.9497	0.9247	0.9625	0.9091	0.9525	0.8004
w$_2 l_4$	0.9223	0.9592	0.9488	0.9194	0.9585	0.909	0.9506	0.8001
w$_3 l_1$	0.9997	0.9878	0.9362	0.9771	0.9881	0.9129	0.9438	0.814
w$_3 l_2$	0.9999	0.9838	0.9533	0.9656	0.9832	0.9294	0.9526	0.8407
w$_3 l_3$	0.9337	0.9656	0.9491	0.9293	0.9649	0.9078	0.9516	0.7963
w$_3 l_4$	0.8985	0.9495	0.9455	0.911	0.9549	0.9113	0.9512	0.8011
Average Width								
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	0.24456	0.1494	0.12071	0.12471	0.14939	0.09812	0.12084	0.07124
w$_1 l_2$	0.24625	0.17478	0.16586	0.14585	0.17467	0.13687	0.16572	0.10081
w$_1 l_3$	0.56347	0.47289	0.45237	0.4262	0.47314	0.40844	0.45288	0.32709
w$_1 l_4$	0.68255	0.60949	0.60162	0.54944	0.6094	0.54318	0.60235	0.43504
w$_2 l_1$	0.24786	0.16915	0.14881	0.1403	0.16908	0.12306	0.14968	0.08922
w$_2 l_2$	0.25072	0.19267	0.18468	0.15961	0.19273	0.15319	0.18543	0.11147
w$_2 l_3$	0.50279	0.39449	0.37414	0.34807	0.39457	0.3288	0.37386	0.25569
w$_2 l_4$	0.55837	0.46035	0.44986	0.40611	0.46016	0.39511	0.44932	0.307
w$_3 l_1$	0.19291	0.12182	0.09397	0.10116	0.12179	0.07645	0.09449	0.05468
w$_3 l_2$	0.17055	0.11449	0.10478	0.0951	0.11455	0.08572	0.10476	0.0628
w$_3 l_3$	0.41667	0.33028	0.30109	0.29172	0.33032	0.2651	0.30099	0.20664
w$_3 l_4$	0.56322	0.50484	0.49744	0.44663	0.5064	0.43994	0.49784	0.34488

Table 80. Multiplicative Desirability Inference Observation 2, (x_1, x_2) = (-0.5,-1) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9999	0.9833	0.9278	0.9679	0.983	0.9107	0.9354	0.8283
$w_1 l_2$	0.9977	0.9646	0.9388	0.9425	0.964	0.9144	0.9398	0.835
$w_1 l_3$	0.8664	0.957	0.9503	0.9134	0.9563	0.9054	0.9527	0.7946
$w_1 l_4$	0.8301	0.951	0.9467	0.9077	0.9524	0.9041	0.9501	0.7926
$w_2 l_1$	1	0.9789	0.9571	0.9632	0.979	0.9415	0.9586	0.8817
$w_2 l_2$	1	0.9729	0.9596	0.9542	0.973	0.9415	0.9604	0.8808
$w_2 l_3$	0.8056	0.9532	0.9465	0.9138	0.9524	0.9099	0.9528	0.7992
$w_2 l_4$	0.8036	0.9525	0.9462	0.9125	0.9517	0.9099	0.9536	0.8004
$w_3 l_1$	0.9948	0.9672	0.9452	0.9363	0.9675	0.9094	0.9449	0.8077
$w_3 l_2$	0.9851	0.9576	0.9449	0.9238	0.957	0.9115	0.9454	0.8068
$w_3 l_3$	0.9686	0.9698	0.9596	0.9371	0.971	0.9231	0.9597	0.8198
$w_3 l_4$	0.9507	0.9685	0.958	0.9327	0.9686	0.9226	0.9595	0.8172
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.21957	0.13868	0.11938	0.11524	0.13859	0.09764	0.11956	0.07163
$w_1 l_2$	0.21482	0.156	0.15002	0.12945	0.15592	0.12366	0.15001	0.09135
$w_1 l_3$	0.50322	0.45054	0.43318	0.4025	0.45043	0.38786	0.43399	0.31051
$w_1 l_4$	0.56502	0.53424	0.52707	0.47772	0.53423	0.47178	0.52761	0.37752
$w_2 l_1$	0.26137	0.18909	0.17832	0.15874	0.18913	0.14807	0.17801	0.11229
$w_2 l_2$	0.26102	0.2066	0.20387	0.1736	0.20685	0.16969	0.20312	0.12902
$w_2 l_3$	0.65168	0.63309	0.62342	0.58235	0.63297	0.57486	0.62464	0.47742
$w_2 l_4$	0.70403	0.703	0.69855	0.64686	0.70301	0.64393	0.69946	0.53464
$w_3 l_1$	0.22384	0.16421	0.15163	0.13616	0.16417	0.12584	0.1523	0.09187
$w_3 l_2$	0.22421	0.18064	0.17526	0.14956	0.18051	0.14577	0.17595	0.10651
$w_3 l_3$	0.41999	0.34264	0.32569	0.29916	0.34251	0.28302	0.32529	0.2191
$w_3 l_4$	0.4468	0.38018	0.37034	0.33205	0.38235	0.32361	0.37184	0.25058

Table 81. Multiplicative Desirability Inference Observation 3, (x_1, x_2) = (0,-1) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9991	0.963	0.9054	0.9389	0.962	0.877	0.907	0.7949
$w_1 l_2$	0.9927	0.9406	0.9155	0.9112	0.9407	0.8865	0.9128	0.8025
$w_1 l_3$	0.985	0.9738	0.9671	0.9443	0.9741	0.9359	0.9682	0.8365
$w_1 l_4$	0.9521	0.9708	0.9644	0.9367	0.9699	0.9329	0.9664	0.8279
$w_2 l_1$	0.9996	0.9756	0.9024	0.9547	0.9758	0.8785	0.9097	0.7869
$w_2 l_2$	0.9958	0.9463	0.9116	0.9206	0.9471	0.8891	0.9143	0.8005
$w_2 l_3$	0.9985	0.9769	0.9659	0.957	0.9775	0.9424	0.9655	0.8583
$w_2 l_4$	0.9607	0.968	0.9651	0.9364	0.9681	0.9343	0.9644	0.8279
$w_3 l_1$	1	0.9549	0.9114	0.9352	0.9553	0.8939	0.9135	0.8295
$w_3 l_2$	0.9992	0.9414	0.9201	0.9221	0.9426	0.9004	0.922	0.8331
$w_3 l_3$	0.8699	0.9632	0.9632	0.9269	0.9631	0.9258	0.9637	0.8153
$w_3 l_4$	0.8361	0.963	0.9626	0.9256	0.9636	0.9233	0.9633	0.8087
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.24432	0.15869	0.14142	0.13091	0.15864	0.11558	0.14184	0.08428
$w_1 l_2$	0.24184	0.17985	0.17464	0.14801	0.17981	0.14355	0.17506	0.10515
$w_1 l_3$	0.47469	0.43829	0.42106	0.38944	0.43857	0.37431	0.42128	0.29778
$w_1 l_4$	0.52623	0.51817	0.51102	0.4603	0.51797	0.45392	0.5106	0.36081
$w_2 l_1$	0.1981	0.12261	0.1007	0.10111	0.12266	0.08202	0.10124	0.05911
$w_2 l_2$	0.17176	0.12239	0.11696	0.10016	0.12238	0.09516	0.11703	0.06928
$w_2 l_3$	0.37639	0.32452	0.29772	0.28161	0.32427	0.25759	0.29738	0.19921
$w_2 l_4$	0.43202	0.42344	0.41703	0.36773	0.4234	0.36073	0.41567	0.27933
$w_3 l_1$	0.2851	0.20737	0.19811	0.17236	0.20728	0.16323	0.19786	0.12288
$w_3 l_2$	0.2853	0.2266	0.22409	0.18843	0.22672	0.18515	0.22376	0.13962
$w_3 l_3$	0.60545	0.62008	0.61121	0.56865	0.62007	0.56136	0.61144	0.4659
$w_3 l_4$	0.64887	0.68733	0.68329	0.63039	0.69036	0.63063	0.68662	0.52334

Table 82. Multiplicative Desirability Inference Observation 4, (x_1, x_2) = (0.5,-1) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.998	0.9542	0.8806	0.9299	0.9541	0.8615	0.8889	0.7681
$w_1 l_2$	0.9893	0.9331	0.9021	0.9029	0.9337	0.8791	0.9043	0.7857
$w_1 l_3$	0.9993	0.9732	0.9589	0.9525	0.9737	0.9412	0.9629	0.8659
$w_1 l_4$	0.9913	0.9701	0.9596	0.9463	0.9698	0.9406	0.9628	0.8571
$w_2 l_1$	0.99	0.96	0.9367	0.9262	0.9592	0.9002	0.9364	0.7966
$w_2 l_2$	0.9795	0.954	0.9406	0.9129	0.9533	0.9034	0.9411	0.7953
$w_2 l_3$	0.9967	0.9604	0.9299	0.9372	0.9601	0.9036	0.9305	0.8192
$w_2 l_4$	0.9913	0.9559	0.933	0.9288	0.956	0.9064	0.9353	0.8171
$w_3 l_1$	0.9989	0.9671	0.8988	0.945	0.9675	0.8731	0.9027	0.7718
$w_3 l_2$	0.9933	0.9403	0.896	0.9104	0.9405	0.8702	0.902	0.782
$w_3 l_3$	0.9999	0.977	0.9411	0.9599	0.9777	0.9288	0.9491	0.8571
$w_3 l_4$	0.995	0.9716	0.9604	0.9465	0.969	0.9368	0.9599	0.8555
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.33686	0.22776	0.20532	0.18251	0.22751	0.16238	0.20534	0.11618
$w_1 l_2$	0.35488	0.27566	0.26929	0.21999	0.2756	0.21468	0.26935	0.15435
$w_1 l_3$	0.51338	0.45996	0.43683	0.40892	0.45995	0.3886	0.43671	0.31219
$w_1 l_4$	0.59757	0.58341	0.5739	0.51903	0.58354	0.51033	0.57318	0.40972
$w_2 l_1$	0.34549	0.27074	0.25388	0.2205	0.2706	0.20791	0.25498	0.15008
$w_2 l_2$	0.3633	0.31151	0.30425	0.25344	0.31159	0.2496	0.30534	0.18035
$w_2 l_3$	0.4678	0.3909	0.36657	0.33665	0.39091	0.3156	0.36723	0.24211
$w_2 l_4$	0.51141	0.45288	0.43802	0.38993	0.45292	0.37681	0.43827	0.28889
$w_3 l_1$	0.27247	0.17913	0.1521	0.14278	0.17891	0.11888	0.15228	0.08408
$w_3 l_2$	0.2654	0.19868	0.19178	0.15491	0.19903	0.14828	0.19165	0.10471
$w_3 l_3$	0.41792	0.35503	0.31982	0.30517	0.35501	0.27465	0.3205	0.21241
$w_3 l_4$	0.52257	0.51799	0.50598	0.44491	0.51858	0.43666	0.50756	0.3388

Table 83. Multiplicative Desirability Inference Observation 5, (x_1, x_2) = (1,-1) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9996	0.9485	0.9529	0.903	0.9488	0.9099	0.9509	0.7951
$w_1 l_2$	0.9985	0.957	0.9561	0.9153	0.9575	0.9147	0.9563	0.8055
$w_1 l_3$	0.9998	0.9616	0.9658	0.9385	0.9618	0.9438	0.9659	0.8602
$w_1 l_4$	0.9992	0.9701	0.9651	0.9484	0.9706	0.9447	0.9683	0.8661
$w_2 l_1$	0.9975	0.9487	0.9578	0.899	0.9473	0.9105	0.9557	0.8007
$w_2 l_2$	0.995	0.9594	0.9615	0.914	0.959	0.9139	0.958	0.8038
$w_2 l_3$	0.999	0.9629	0.9705	0.9369	0.962	0.9443	0.9683	0.8729
$w_2 l_4$	0.9977	0.9703	0.9725	0.946	0.9693	0.9466	0.9701	0.8757
$w_3 l_1$	0.9992	0.9619	0.9477	0.9235	0.9608	0.912	0.9512	0.7989
$w_3 l_2$	0.9955	0.9575	0.9504	0.9195	0.9571	0.914	0.953	0.7997
$w_3 l_3$	0.9993	0.969	0.9566	0.9432	0.9683	0.9327	0.9576	0.8364
$w_3 l_4$	0.9959	0.9654	0.9568	0.9388	0.9615	0.9293	0.9546	0.8361
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.2719	0.1632	0.17333	0.1366	0.16318	0.14611	0.17361	0.10817
$w_1 l_2$	0.29235	0.20301	0.20852	0.17021	0.20296	0.17508	0.20923	0.12919
$w_1 l_3$	0.27853	0.2181	0.22018	0.17163	0.21791	0.17505	0.21996	0.1239
$w_1 l_4$	0.30214	0.28408	0.28707	0.2218	0.28405	0.22415	0.28665	0.15523
$w_2 l_1$	0.35757	0.25158	0.26524	0.21426	0.25171	0.22674	0.26567	0.17037
$w_2 l_2$	0.3861	0.30239	0.307	0.25764	0.30255	0.26199	0.30756	0.1964
$w_2 l_3$	0.36779	0.34372	0.35436	0.27605	0.3437	0.28449	0.35201	0.20381
$w_2 l_4$	0.399	0.41693	0.42238	0.33419	0.41682	0.33701	0.42018	0.23922
$w_3 l_1$	0.26754	0.1685	0.16815	0.13973	0.16854	0.13995	0.16863	0.10219
$w_3 l_2$	0.27645	0.19807	0.20144	0.1642	0.19821	0.1674	0.20211	0.12198
$w_3 l_3$	0.27262	0.20202	0.19612	0.15983	0.20188	0.15619	0.19576	0.11102
$w_3 l_4$	0.28286	0.24039	0.24042	0.1895	0.23943	0.18956	0.23903	0.1334

Table 84. Multiplicative Desirability Inference Observation 6, (x_1, x_2) = (-1,-0.5) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9978	0.9461	0.9492	0.9015	0.9461	0.9086	0.9514	0.8022
$w_1 l_2$	0.9955	0.9532	0.9491	0.9122	0.9546	0.9096	0.9514	0.8068
$w_1 l_3$	0.9981	0.9617	0.9629	0.9209	0.962	0.921	0.9614	0.8125
$w_1 l_4$	0.9958	0.965	0.9616	0.9267	0.9652	0.923	0.9629	0.8162
$w_2 l_1$	0.9986	0.9446	0.9499	0.8968	0.9453	0.9091	0.9508	0.8016
$w_2 l_2$	0.9962	0.9537	0.9483	0.9097	0.9535	0.908	0.9487	0.8061
$w_2 l_3$	0.9987	0.9575	0.9607	0.9123	0.9571	0.9211	0.9615	0.8103
$w_2 l_4$	0.9965	0.9651	0.9618	0.9278	0.9646	0.9242	0.9588	0.8189
$w_3 l_1$	0.9916	0.9442	0.9484	0.899	0.9439	0.907	0.9503	0.8033
$w_3 l_2$	0.9888	0.9486	0.9501	0.9049	0.9473	0.9072	0.9513	0.8053
$w_3 l_3$	0.9921	0.9588	0.9646	0.9188	0.9591	0.9232	0.9628	0.8169
$w_3 l_4$	0.9893	0.9613	0.9651	0.9222	0.9639	0.9234	0.9619	0.8157
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.26074	0.15026	0.1607	0.1261	0.15032	0.13462	0.16113	0.09867
$w_1 l_2$	0.26805	0.17092	0.17647	0.14329	0.17096	0.14766	0.17715	0.10808
$w_1 l_3$	0.26093	0.15954	0.16836	0.12944	0.1595	0.13673	0.16805	0.09927
$w_1 l_4$	0.26828	0.18173	0.18695	0.14732	0.18186	0.15072	0.18661	0.10899
$w_2 l_1$	0.19564	0.10793	0.11905	0.09014	0.10792	0.09912	0.11901	0.0725
$w_2 l_2$	0.18536	0.11607	0.12091	0.09689	0.11606	0.10062	0.12106	0.07343
$w_2 l_3$	0.19577	0.11341	0.12231	0.09209	0.11338	0.10013	0.12218	0.0729
$w_2 l_4$	0.18553	0.12393	0.12827	0.09975	0.12396	0.10303	0.12836	0.07426
$w_3 l_1$	0.35818	0.23737	0.24779	0.20029	0.23754	0.20903	0.24798	0.15394
$w_3 l_2$	0.37158	0.25999	0.26412	0.21932	0.26001	0.22287	0.2645	0.16413
$w_3 l_3$	0.35847	0.2528	0.26248	0.20614	0.25264	0.21428	0.26168	0.15595
$w_3 l_4$	0.37191	0.27664	0.2813	0.22561	0.27692	0.22932	0.28127	0.16631

Table 85. Multiplicative Desirability Inference Observation 7, (x_1, x_2) = (-0.5,-0.5) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9978	0.949	0.9503	0.908	0.9499	0.9093	0.9505	0.8021
$w_1 l_2$	0.9962	0.9542	0.9508	0.9144	0.9545	0.9077	0.9494	0.8011
$w_1 l_3$	0.9979	0.9591	0.9574	0.9154	0.9602	0.9156	0.9607	0.8023
$w_1 l_4$	0.9963	0.9617	0.9589	0.9196	0.9617	0.9175	0.959	0.8028
$w_2 l_1$	0.9964	0.9544	0.9476	0.9135	0.9542	0.9083	0.9478	0.8047
$w_2 l_2$	0.9949	0.9533	0.9458	0.9124	0.9537	0.9094	0.9466	0.8063
$w_2 l_3$	0.9964	0.9592	0.9515	0.919	0.958	0.9112	0.9513	0.8072
$w_2 l_4$	0.9949	0.9587	0.95	0.9191	0.9567	0.9129	0.9523	0.8098
$w_3 l_1$	0.9984	0.947	0.9471	0.8998	0.9474	0.9099	0.9506	0.8021
$w_3 l_2$	0.9967	0.9532	0.9505	0.9124	0.9532	0.9109	0.9476	0.806
$w_3 l_3$	0.9984	0.9566	0.955	0.9078	0.9559	0.9144	0.9574	0.8046
$w_3 l_4$	0.9968	0.9613	0.9573	0.9189	0.9638	0.9176	0.9581	0.8058
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.28102	0.16019	0.16983	0.13369	0.16018	0.14141	0.17022	0.10315
$w_1 l_2$	0.27995	0.17286	0.17831	0.14413	0.17276	0.14836	0.17896	0.10812
$w_1 l_3$	0.28108	0.16461	0.17322	0.13484	0.16464	0.14168	0.17301	0.10297
$w_1 l_4$	0.28002	0.17761	0.1829	0.14537	0.17761	0.14895	0.18262	0.10802
$w_2 l_1$	0.24198	0.15357	0.15672	0.1275	0.15366	0.12996	0.15735	0.09471
$w_2 l_2$	0.24083	0.16157	0.16545	0.13391	0.16144	0.13708	0.16596	0.09999
$w_2 l_3$	0.24201	0.15599	0.15791	0.12802	0.15595	0.13019	0.15846	0.0948
$w_2 l_4$	0.24087	0.16399	0.16705	0.13449	0.16395	0.13749	0.16759	0.10007
$w_3 l_1$	0.21618	0.11628	0.12743	0.09681	0.11621	0.10553	0.12736	0.07682
$w_3 l_2$	0.19743	0.12084	0.12545	0.10038	0.12079	0.10388	0.12567	0.07546
$w_3 l_3$	0.21623	0.11932	0.12897	0.09758	0.11921	0.1058	0.12883	0.07692
$w_3 l_4$	0.19749	0.12449	0.12875	0.10131	0.12426	0.1045	0.12884	0.07557

Table 86. Multiplicative Desirability Inference Observation 8, (x_1, x_2) = (0,-0.5) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9981	0.9518	0.9488	0.9104	0.9512	0.9086	0.9482	0.8048
$w_1 l_2$	0.9963	0.9559	0.9482	0.9136	0.9563	0.9092	0.9482	0.8036
$w_1 l_3$	0.9982	0.9593	0.9581	0.9183	0.9598	0.916	0.9588	0.8034
$w_1 l_4$	0.9964	0.9613	0.9576	0.9194	0.9611	0.9151	0.9573	0.8033
$w_2 l_1$	0.9933	0.9456	0.9504	0.9034	0.9463	0.9067	0.95	0.8014
$w_2 l_2$	0.9918	0.95	0.9491	0.9069	0.9504	0.9063	0.9489	0.8033
$w_2 l_3$	0.9934	0.9575	0.9591	0.913	0.9571	0.9158	0.9581	0.809
$w_2 l_4$	0.9919	0.959	0.9584	0.9165	0.9592	0.9162	0.958	0.8096
$w_3 l_1$	0.9972	0.9577	0.9489	0.9171	0.958	0.9097	0.9511	0.8068
$w_3 l_2$	0.9942	0.9572	0.9475	0.915	0.9577	0.9117	0.9518	0.807
$w_3 l_3$	0.9972	0.9623	0.9535	0.9208	0.9618	0.9131	0.9559	0.8087
$w_3 l_4$	0.9942	0.9615	0.9535	0.918	0.9624	0.9167	0.9585	0.8087
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.28686	0.16459	0.17357	0.13719	0.1646	0.14438	0.17416	0.10523
$w_1 l_2$	0.28554	0.17667	0.18229	0.1472	0.17668	0.15152	0.18313	0.11047
$w_1 l_3$	0.28689	0.16924	0.17719	0.13827	0.16923	0.14463	0.17703	0.10508
$w_1 l_4$	0.28558	0.18168	0.18702	0.14845	0.18166	0.15206	0.18683	0.11025
$w_2 l_1$	0.38995	0.2479	0.2574	0.20792	0.24799	0.21553	0.25756	0.15779
$w_2 l_2$	0.39398	0.26042	0.26528	0.21811	0.26031	0.22204	0.26539	0.16263
$w_2 l_3$	0.39	0.25535	0.26413	0.20998	0.25525	0.21736	0.26393	0.15842
$w_2 l_4$	0.39404	0.26806	0.27311	0.22043	0.26811	0.22416	0.27279	0.16329
$w_3 l_1$	0.24648	0.16128	0.16391	0.13368	0.16132	0.13598	0.16458	0.09907
$w_3 l_2$	0.24569	0.16909	0.17274	0.14009	0.16906	0.14322	0.1734	0.1044
$w_3 l_3$	0.2465	0.16364	0.16507	0.13419	0.16367	0.13614	0.16573	0.09917
$w_3 l_4$	0.24572	0.17155	0.17419	0.14056	0.17072	0.14291	0.17431	0.10408

Table 87. Multiplicative Desirability Inference Observation 9, (x_1, x_2) = (0.5,-0.5) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9989	0.9542	0.9513	0.9146	0.9539	0.9109	0.949	0.8019
$w_1 l_2$	0.9973	0.9562	0.9517	0.9173	0.957	0.9115	0.9508	0.8013
$w_1 l_3$	0.999	0.9622	0.9613	0.9314	0.962	0.927	0.9604	0.8221
$w_1 l_4$	0.9975	0.9641	0.962	0.9335	0.964	0.9269	0.9608	0.82
$w_2 l_1$	0.9995	0.951	0.9482	0.9078	0.9501	0.9059	0.9469	0.799
$w_2 l_2$	0.9978	0.9558	0.9523	0.9151	0.9552	0.9111	0.9524	0.8008
$w_2 l_3$	0.9995	0.9617	0.959	0.9304	0.9621	0.9264	0.959	0.8253
$w_2 l_4$	0.998	0.9682	0.9627	0.9358	0.9673	0.9329	0.963	0.8282
$w_3 l_1$	0.9964	0.9485	0.9523	0.9046	0.9485	0.912	0.9517	0.7944
$w_3 l_2$	0.9948	0.9534	0.9529	0.912	0.9536	0.9114	0.9526	0.7965
$w_3 l_3$	0.997	0.9661	0.9665	0.9354	0.966	0.9389	0.9673	0.8346
$w_3 l_4$	0.9957	0.9682	0.9659	0.9408	0.9706	0.9409	0.9689	0.8419
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.3263	0.20313	0.2109	0.16783	0.20313	0.1749	0.21225	0.12703
$w_1 l_2$	0.33513	0.22793	0.23371	0.18821	0.22795	0.19335	0.23526	0.14011
$w_1 l_3$	0.3275	0.22838	0.23292	0.17986	0.22829	0.18467	0.2328	0.13098
$w_1 l_4$	0.33653	0.25677	0.26145	0.20217	0.25685	0.20569	0.26129	0.14496
$w_2 l_1$	0.25841	0.15033	0.15843	0.12324	0.15018	0.13027	0.15859	0.09423
$w_2 l_2$	0.24819	0.16606	0.17123	0.13556	0.16586	0.13973	0.17172	0.10048
$w_2 l_3$	0.25935	0.16785	0.17119	0.13149	0.16791	0.13572	0.17107	0.0966
$w_2 l_4$	0.24943	0.1899	0.19393	0.14678	0.18997	0.14994	0.19358	0.10464
$w_3 l_1$	0.42436	0.29284	0.30388	0.24709	0.29293	0.25624	0.30388	0.18893
$w_3 l_2$	0.43643	0.31622	0.32262	0.26672	0.31649	0.27141	0.32243	0.19987
$w_3 l_3$	0.42617	0.33602	0.34492	0.26986	0.33606	0.27754	0.34422	0.1988
$w_3 l_4$	0.43841	0.36302	0.36824	0.29127	0.36504	0.29716	0.36977	0.21196

Table 88. Multiplicative Desirability Inference Observation 10, (x_1, x_2) = (1,-0.5) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9989	0.9359	0.9462	0.8911	0.9379	0.9024	0.9442	0.7991
$w_1 l_2$	0.9968	0.9458	0.947	0.9039	0.9467	0.9044	0.9464	0.7969
$w_1 l_3$	0.9989	0.9501	0.9548	0.9018	0.9495	0.9089	0.9514	0.7978
$w_1 l_4$	0.9968	0.9568	0.9539	0.917	0.9566	0.9104	0.955	0.8009
$w_2 l_1$	0.9994	0.9548	0.9443	0.9185	0.9555	0.9107	0.9495	0.8029
$w_2 l_2$	0.9976	0.9546	0.9433	0.9172	0.9542	0.9118	0.9518	0.8034
$w_2 l_3$	0.9994	0.9584	0.9487	0.9217	0.9586	0.914	0.9559	0.8035
$w_2 l_4$	0.9976	0.9579	0.9487	0.9207	0.9593	0.9145	0.9556	0.8062
$w_3 l_1$	0.9987	0.9374	0.9472	0.8888	0.9363	0.9083	0.9497	0.7971
$w_3 l_2$	0.9972	0.9507	0.9454	0.9065	0.9498	0.9096	0.9494	0.8016
$w_3 l_3$	0.9988	0.9454	0.9543	0.8947	0.9441	0.912	0.9542	0.7982
$w_3 l_4$	0.9972	0.9573	0.9543	0.9135	0.9562	0.9153	0.9543	0.8083
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.31075	0.17671	0.19123	0.14551	0.17666	0.15773	0.19173	0.11396
$w_1 l_2$	0.35152	0.22455	0.23088	0.18438	0.22439	0.19008	0.23215	0.13688
$w_1 l_3$	0.31082	0.18084	0.1938	0.14669	0.18086	0.15798	0.19407	0.11389
$w_1 l_4$	0.35161	0.23016	0.23629	0.18595	0.23007	0.19083	0.23647	0.13688
$w_2 l_1$	0.28275	0.16802	0.17301	0.13769	0.16818	0.14131	0.17337	0.10186
$w_2 l_2$	0.30039	0.19568	0.20136	0.16003	0.19575	0.16418	0.20175	0.1183
$w_2 l_3$	0.2828	0.17064	0.17455	0.13816	0.17063	0.14145	0.17466	0.10194
$w_2 l_4$	0.30045	0.19877	0.2036	0.16068	0.19869	0.16447	0.2038	0.11841
$w_3 l_1$	0.22342	0.12693	0.14167	0.1039	0.12694	0.11574	0.14141	0.08359
$w_3 l_2$	0.2489	0.15596	0.16301	0.12672	0.15589	0.1322	0.16323	0.09461
$w_3 l_3$	0.22347	0.12914	0.14192	0.10429	0.1291	0.11566	0.14177	0.0836
$w_3 l_4$	0.24897	0.16006	0.1668	0.12754	0.16106	0.13364	0.16814	0.09515

Table 89. Multiplicative Desirability Inference Observation 11, (x_1, x_2) = (-1,0) Empirical Coverage and Average Width

			Empirical Coverage					
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9974	0.9422	0.9452	0.8993	0.9407	0.9023	0.9465	0.8046
$w_1 l_2$	0.9952	0.9464	0.9472	0.907	0.9474	0.9043	0.9444	0.8047
$w_1 l_3$	0.9975	0.9506	0.9475	0.9053	0.9507	0.9121	0.9505	0.8012
$w_1 l_4$	0.9953	0.955	0.9494	0.913	0.9553	0.9093	0.9523	0.8037
$w_2 l_1$	0.9893	0.9461	0.9478	0.8993	0.9454	0.9075	0.9512	0.8013
$w_2 l_2$	0.9871	0.949	0.9484	0.9058	0.9491	0.9073	0.9513	0.8014
$w_2 l_3$	0.9894	0.9486	0.9498	0.9009	0.9491	0.9072	0.9536	0.8021
$w_2 l_4$	0.9872	0.9515	0.9499	0.9054	0.9503	0.9089	0.9535	0.8022
$w_3 l_1$	0.9983	0.9548	0.9446	0.9187	0.9552	0.9091	0.9497	0.8084
$w_3 l_2$	0.9969	0.9547	0.9438	0.9168	0.954	0.9085	0.9491	0.8074
$w_3 l_3$	0.9983	0.9566	0.947	0.92	0.9567	0.9092	0.951	0.81
$w_3 l_4$	0.9969	0.9543	0.9453	0.9162	0.9593	0.9086	0.9512	0.7987
				Average Width				
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.32107	0.17369	0.18459	0.14314	0.17373	0.15175	0.1849	0.10982
$w_1 l_2$	0.332	0.193	0.19873	0.15883	0.19307	0.16331	0.19954	0.11809
$w_1 l_3$	0.3211	0.17413	0.18478	0.14299	0.17406	0.15122	0.18452	0.10941
$w_1 l_4$	0.33203	0.19331	0.1993	0.15877	0.19351	0.16267	0.19889	0.11761
$w_2 l_1$	0.45452	0.26635	0.27558	0.22058	0.26661	0.22802	0.27582	0.16567
$w_2 l_2$	0.47081	0.28572	0.28993	0.2365	0.28585	0.2397	0.28997	0.17427
$w_2 l_3$	0.45457	0.26673	0.27587	0.22058	0.26682	0.22795	0.27586	0.16571
$w_2 l_4$	0.47086	0.28627	0.2903	0.23639	0.28607	0.23967	0.29039	0.17421
$w_3 l_1$	0.26638	0.15615	0.16072	0.12843	0.15612	0.13171	0.16088	0.09549
$w_3 l_2$	0.26999	0.16651	0.17156	0.1371	0.16671	0.14053	0.17175	0.10194
$w_3 l_3$	0.2664	0.15646	0.16066	0.12837	0.15647	0.13163	0.16093	0.0955
$w_3 l_4$	0.27001	0.16705	0.17168	0.13713	0.16684	0.14045	0.17185	0.10181

Table 90. Multiplicative Desirability Inference Observation 12, (x_1, x_2) = (-0.5,0) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9977	0.9449	0.9488	0.9025	0.9464	0.9059	0.9477	0.8023
$w_1 l_2$	0.9962	0.9497	0.9493	0.9095	0.9514	0.9046	0.9457	0.8021
$w_1 l_3$	0.9977	0.953	0.9511	0.908	0.9539	0.9113	0.9533	0.8044
$w_1 l_4$	0.9962	0.9558	0.9521	0.9134	0.9565	0.912	0.9539	0.8034
$w_2 l_1$	0.9986	0.945	0.9477	0.897	0.9437	0.9071	0.9485	0.8042
$w_2 l_2$	0.9965	0.9517	0.9492	0.9082	0.9527	0.9062	0.9449	0.8026
$w_2 l_3$	0.9986	0.9476	0.949	0.8978	0.9469	0.9077	0.9492	0.8032
$w_2 l_4$	0.9965	0.9545	0.9513	0.9097	0.9542	0.9086	0.9474	0.8023
$w_3 l_1$	0.9902	0.9446	0.949	0.9019	0.9456	0.9082	0.9483	0.7997
$w_3 l_2$	0.989	0.9483	0.9496	0.9075	0.9479	0.908	0.9475	0.7988
$w_3 l_3$	0.9902	0.9494	0.9515	0.9037	0.9489	0.9097	0.9514	0.8013
$w_3 l_4$	0.989	0.9507	0.95	0.9077	0.9491	0.9075	0.9482	0.8058
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.34403	0.18947	0.19969	0.15569	0.18943	0.16365	0.1999	0.11823
$w_1 l_2$	0.34	0.19877	0.20468	0.16308	0.19865	0.1676	0.20517	0.12102
$w_1 l_3$	0.34403	0.18999	0.20014	0.15554	0.18992	0.16297	0.1995	0.11781
$w_1 l_4$	0.34	0.19916	0.20533	0.16302	0.19906	0.16694	0.20475	0.12054
$w_2 l_1$	0.26906	0.14166	0.15498	0.11573	0.14172	0.12569	0.15449	0.09054
$w_2 l_2$	0.2473	0.14615	0.1516	0.11876	0.14583	0.12255	0.15118	0.08808
$w_2 l_3$	0.26906	0.14214	0.15505	0.11577	0.14211	0.12569	0.1545	0.09058
$w_2 l_4$	0.2473	0.14656	0.15214	0.11877	0.14661	0.12252	0.15183	0.08805
$w_3 l_1$	0.47623	0.28103	0.28975	0.23235	0.28107	0.23929	0.2896	0.17377
$w_3 l_2$	0.47832	0.29031	0.29468	0.23997	0.29021	0.24341	0.29458	0.17681
$w_3 l_3$	0.47623	0.28155	0.28996	0.2324	0.28156	0.23924	0.28974	0.17386
$w_3 l_4$	0.47832	0.29072	0.29499	0.24002	0.29101	0.2438	0.29559	0.17711

Table 91. Multiplicative Desirability Inference Observation 13, (x_1, x_2) = (0,0) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9978	0.9485	0.9504	0.9087	0.9491	0.9043	0.9464	0.8025
$w_1 l_2$	0.9968	0.9533	0.9485	0.9123	0.9516	0.906	0.9456	0.804
$w_1 l_3$	0.9978	0.9533	0.9523	0.9132	0.9548	0.9119	0.9534	0.8027
$w_1 l_4$	0.9968	0.9566	0.9513	0.919	0.9565	0.9137	0.9527	0.802
$w_2 l_1$	0.9971	0.9543	0.9456	0.9176	0.9546	0.9106	0.9503	0.8055
$w_2 l_2$	0.996	0.9545	0.9461	0.918	0.9539	0.9107	0.9499	0.8054
$w_2 l_3$	0.9971	0.9552	0.947	0.9171	0.957	0.9109	0.9512	0.805
$w_2 l_4$	0.996	0.9565	0.9458	0.9169	0.9552	0.9119	0.9518	0.8054
$w_3 l_1$	0.9988	0.9447	0.9511	0.9024	0.9468	0.9089	0.9458	0.801
$w_3 l_2$	0.9971	0.952	0.9488	0.9117	0.9527	0.9068	0.9464	0.8039
$w_3 l_3$	0.9988	0.95	0.9525	0.904	0.9507	0.9095	0.9493	0.8027
$w_3 l_4$	0.9971	0.9545	0.9507	0.914	0.9568	0.9074	0.9515	0.8001

Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.32681	0.17978	0.18902	0.14827	0.17963	0.15546	0.18928	0.11259
$w_1 l_2$	0.32021	0.18675	0.19246	0.15385	0.18661	0.15824	0.19311	0.11467
$w_1 l_3$	0.32681	0.1805	0.18944	0.14827	0.18051	0.15488	0.18913	0.11229
$w_1 l_4$	0.32021	0.1874	0.19333	0.15393	0.1873	0.15776	0.19299	0.11428
$w_2 l_1$	0.26515	0.16636	0.16981	0.13707	0.16635	0.13978	0.17038	0.10158
$w_2 l_2$	0.26143	0.17055	0.17463	0.1405	0.17051	0.14377	0.17519	0.10453
$w_2 l_3$	0.26515	0.16688	0.16989	0.13706	0.16682	0.13977	0.17035	0.10159
$w_2 l_4$	0.26143	0.17106	0.17479	0.14046	0.17096	0.14379	0.17534	0.10448
$w_3 l_1$	0.25941	0.13577	0.14725	0.11167	0.13582	0.12042	0.14708	0.087
$w_3 l_2$	0.2349	0.13785	0.14293	0.11291	0.13787	0.11656	0.14295	0.08406
$w_3 l_3$	0.25941	0.13634	0.14732	0.11176	0.13648	0.12034	0.14717	0.08702
$w_3 l_4$	0.2349	0.1386	0.14345	0.11296	0.13832	0.11646	0.14327	0.084

Table 92. Multiplicative Desirability Inference Observation 14, (x_1, x_2) = (0.5,0) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9982	0.9542	0.9472	0.9112	0.9537	0.9072	0.9491	0.7981
$w_1 l_2$	0.9975	0.9568	0.9477	0.914	0.9559	0.907	0.9495	0.7947
$w_1 l_3$	0.9982	0.963	0.9569	0.92	0.9628	0.9185	0.9602	0.8059
$w_1 l_4$	0.9975	0.965	0.9563	0.9239	0.9644	0.9174	0.9604	0.8059
$w_2 l_1$	0.9937	0.9458	0.9481	0.9012	0.9479	0.9062	0.9494	0.8011
$w_2 l_2$	0.9926	0.9504	0.9499	0.9059	0.9506	0.9068	0.9478	0.8018
$w_2 l_3$	0.9937	0.961	0.9597	0.9162	0.9601	0.9208	0.9604	0.8097
$w_2 l_4$	0.9926	0.963	0.9604	0.9214	0.9636	0.9204	0.9588	0.8102
$w_3 l_1$	0.9971	0.9565	0.9492	0.9207	0.9573	0.9147	0.9531	0.8069
$w_3 l_2$	0.9942	0.9553	0.9475	0.9184	0.955	0.9159	0.9529	0.8072
$w_3 l_3$	0.9971	0.9608	0.9536	0.9241	0.9617	0.9192	0.957	0.808
$w_3 l_4$	0.9942	0.9599	0.9522	0.9215	0.9593	0.9133	0.9542	0.8039
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.33335	0.19974	0.20897	0.16465	0.19966	0.17207	0.20947	0.12459
$w_1 l_2$	0.33404	0.21475	0.22115	0.17704	0.21477	0.18205	0.22226	0.13173
$w_1 l_3$	0.33337	0.20701	0.21475	0.16695	0.20708	0.17324	0.21451	0.12474
$w_1 l_4$	0.33406	0.22295	0.22878	0.17946	0.22287	0.18359	0.22853	0.13188
$w_2 l_1$	0.4422	0.28829	0.29899	0.24106	0.28825	0.24954	0.2987	0.18237
$w_2 l_2$	0.44743	0.30273	0.30887	0.25294	0.30257	0.25747	0.30856	0.18822
$w_2 l_3$	0.44224	0.30045	0.31012	0.24516	0.3005	0.25318	0.30954	0.18349
$w_2 l_4$	0.44747	0.31538	0.32148	0.25718	0.31546	0.26159	0.32075	0.18938
$w_3 l_1$	0.28715	0.19796	0.20054	0.16264	0.19795	0.16496	0.20129	0.11957
$w_3 l_2$	0.28793	0.20751	0.21153	0.17047	0.20763	0.17396	0.21214	0.12609
$w_3 l_3$	0.28716	0.20139	0.20204	0.16333	0.2013	0.16525	0.20288	0.11961
$w_3 l_4$	0.28794	0.21111	0.21352	0.1713	0.21006	0.17366	0.21345	0.12567

Table 93. Multiplicative Desirability Inference Observation 15, (x_1, x_2) = (1,0) Empirical Coverage and Average Width

Empirical Coverage								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.9997	0.9523	0.9495	0.9081	0.9518	0.9097	0.9497	0.7998
w_1l_2	0.9984	0.9572	0.9515	0.9204	0.9582	0.9123	0.9519	0.8002
w_1l_3	0.9998	0.9587	0.9631	0.9335	0.9574	0.9328	0.9615	0.8426
w_1l_4	0.999	0.965	0.9611	0.9405	0.9652	0.9368	0.9607	0.8493
w_2l_1	0.9992	0.944	0.9459	0.8976	0.944	0.9053	0.9485	0.7967
w_2l_2	0.9992	0.957	0.952	0.9161	0.9574	0.9142	0.9549	0.805
w_2l_3	0.9994	0.956	0.9602	0.9233	0.9552	0.9306	0.9579	0.8323
w_2l_4	0.9994	0.9655	0.9626	0.9419	0.9652	0.9376	0.9628	0.8531
w_3l_1	0.9975	0.9504	0.9525	0.9008	0.9497	0.9111	0.9552	0.7971
w_3l_2	0.9946	0.9583	0.9538	0.9136	0.9585	0.9155	0.9574	0.8013
w_3l_3	0.9985	0.9651	0.9684	0.9367	0.9649	0.944	0.9682	0.8589
w_3l_4	0.9967	0.9704	0.9678	0.945	0.9711	0.9472	0.9715	0.864
Average Width								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.27763	0.1661	0.17464	0.13875	0.16615	0.14675	0.17524	0.10797
w_1l_2	0.30314	0.20791	0.2135	0.17369	0.20796	0.17853	0.21446	0.13086
w_1l_3	0.28119	0.20853	0.21002	0.16434	0.20858	0.16779	0.21083	0.11913
w_1l_4	0.30813	0.26721	0.26908	0.20948	0.26723	0.21191	0.26987	0.14801
w_2l_1	0.21037	0.12854	0.13544	0.1063	0.12853	0.11313	0.13586	0.08285
w_2l_2	0.21635	0.14249	0.14755	0.11801	0.14247	0.12246	0.14828	0.08928
w_2l_3	0.21263	0.14932	0.14894	0.11802	0.14925	0.11992	0.14868	0.08651
w_2l_4	0.2205	0.18618	0.1896	0.14294	0.18633	0.14548	0.18917	0.10047
w_3l_1	0.36809	0.25553	0.26785	0.21716	0.25558	0.22889	0.26874	0.17102
w_3l_2	0.39763	0.30229	0.30686	0.25704	0.30256	0.26172	0.30795	0.19516
w_3l_3	0.37366	0.32725	0.33715	0.26225	0.32721	0.27097	0.33594	0.19364
w_3l_4	0.40445	0.38918	0.39428	0.31135	0.38987	0.31616	0.39379	0.22401

Table 94. Multiplicative Desirability Inference Observation 16, (x_1, x_2) = (-1,0.5) Empirical Coverage and Average Width

Empirical Coverage								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.9984	0.9504	0.944	0.9044	0.9497	0.9056	0.9492	0.8006
w_1l_2	0.9965	0.9533	0.9473	0.9116	0.9538	0.9073	0.9495	0.8041
w_1l_3	0.9987	0.9635	0.9583	0.9285	0.9637	0.9254	0.9613	0.8253
w_1l_4	0.9968	0.9658	0.9581	0.9353	0.9667	0.9287	0.9617	0.8236
w_2l_1	0.9976	0.9543	0.9451	0.916	0.9544	0.9077	0.9461	0.8069
w_2l_2	0.9943	0.9522	0.9459	0.9103	0.9522	0.9087	0.9474	0.8047
w_2l_3	0.9978	0.9627	0.9518	0.9244	0.9626	0.9178	0.9532	0.8167
w_2l_4	0.9945	0.9614	0.9532	0.9194	0.9607	0.9171	0.9545	0.8145
w_3l_1	0.9995	0.9504	0.9449	0.9055	0.9504	0.9064	0.9467	0.8001
w_3l_2	0.9969	0.9555	0.9468	0.9126	0.9555	0.9077	0.9471	0.8065
w_3l_3	0.9997	0.9614	0.9581	0.9253	0.9614	0.9227	0.9576	0.8169
w_3l_4	0.9972	0.9665	0.9579	0.934	0.9653	0.9266	0.9597	0.8217
Average Width								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.25311	0.14959	0.15739	0.12568	0.14954	0.1321	0.15774	0.09717
w_1l_2	0.25531	0.16576	0.17091	0.13908	0.1657	0.14314	0.17151	0.10517
w_1l_3	0.25354	0.16538	0.17048	0.13255	0.16535	0.1373	0.17039	0.09895
w_1l_4	0.25579	0.18352	0.18719	0.14695	0.18341	0.14995	0.18732	0.10747
w_2l_1	0.22844	0.15498	0.15652	0.12888	0.15504	0.13033	0.15719	0.09529
w_2l_2	0.23012	0.16607	0.16934	0.13804	0.16598	0.14089	0.16994	0.103
w_2l_3	0.22869	0.16186	0.16143	0.13151	0.16186	0.13218	0.16178	0.09606
w_2l_4	0.2304	0.17343	0.17496	0.14079	0.17341	0.14312	0.17553	0.10384
w_3l_1	0.1965	0.10939	0.11721	0.09143	0.10933	0.09803	0.11734	0.07196
w_3l_2	0.18128	0.11558	0.11971	0.09665	0.11555	0.0999	0.12006	0.07315
w_3l_3	0.19682	0.11893	0.12391	0.09547	0.11892	0.10046	0.12377	0.07294
w_3l_4	0.18167	0.12836	0.13196	0.10203	0.12861	0.10509	0.13216	0.07504

Table 95. Multiplicative Desirability Inference Observation 17, (x_1, x_2) = (-0.5,0.5) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9991	0.9525	0.9466	0.9093	0.9531	0.9061	0.9494	0.7952
$w_1 l_2$	0.9971	0.9549	0.9468	0.9156	0.9556	0.9076	0.9478	0.7956
$w_1 l_3$	0.9991	0.9665	0.9576	0.9347	0.9663	0.9283	0.9626	0.831
$w_1 l_4$	0.9973	0.9668	0.9581	0.9375	0.9676	0.9294	0.9623	0.8271
$w_2 l_1$	0.9951	0.9463	0.9498	0.9035	0.9449	0.907	0.9466	0.7988
$w_2 l_2$	0.9935	0.9485	0.9507	0.9071	0.9488	0.9083	0.9478	0.7995
$w_2 l_3$	0.996	0.9633	0.9647	0.9311	0.963	0.933	0.9648	0.8329
$w_2 l_4$	0.9945	0.9652	0.9647	0.9343	0.966	0.9323	0.9643	0.8341
$w_3 l_1$	0.9947	0.9514	0.9459	0.9126	0.9524	0.9093	0.9506	0.8029
$w_3 l_2$	0.992	0.9509	0.9439	0.9094	0.95	0.908	0.9502	0.8022
$w_3 l_3$	0.9948	0.9618	0.953	0.9235	0.962	0.9187	0.9598	0.8095
$w_3 l_4$	0.9921	0.9603	0.9524	0.9208	0.9603	0.9142	0.958	0.8085
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.25623	0.15578	0.16238	0.13089	0.15581	0.13623	0.16278	0.10031
$w_1 l_2$	0.24704	0.16255	0.16745	0.13642	0.16254	0.14024	0.16804	0.10312
$w_1 l_3$	0.25717	0.17591	0.18	0.1404	0.17591	0.14379	0.17946	0.10319
$w_1 l_4$	0.24805	0.18364	0.18749	0.1465	0.18366	0.14909	0.1871	0.10649
$w_2 l_1$	0.33139	0.22359	0.23196	0.19012	0.2236	0.197	0.23229	0.14646
$w_2 l_2$	0.32762	0.23066	0.23535	0.19605	0.23052	0.19975	0.23573	0.14844
$w_2 l_3$	0.33286	0.25886	0.2663	0.2078	0.2586	0.21361	0.26561	0.15359
$w_2 l_4$	0.32915	0.26709	0.27176	0.21424	0.2668	0.21739	0.27101	0.15583
$w_3 l_1$	0.22879	0.16307	0.16417	0.13543	0.163	0.13685	0.16493	0.10008
$w_3 l_2$	0.2248	0.16801	0.17066	0.13953	0.16801	0.14215	0.17128	0.10395
$w_3 l_3$	0.2293	0.17114	0.16997	0.13883	0.1711	0.13917	0.17047	0.10109
$w_3 l_4$	0.22534	0.17633	0.17703	0.14307	0.17563	0.14452	0.17706	0.10476

Table 96. Multiplicative Desirability Inference Observation 18, (x_1, x_2) = (0,0.5) Empirical Coverage and Average Width

Empirical Coverage								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.9989	0.956	0.9538	0.914	0.9567	0.9092	0.952	0.8022
w_1l_2	0.9971	0.9574	0.9524	0.915	0.9579	0.912	0.9524	0.8011
w_1l_3	0.9991	0.9693	0.9618	0.9433	0.9692	0.9377	0.9646	0.85
w_1l_4	0.9973	0.9695	0.9616	0.9462	0.9694	0.9392	0.9636	0.8506
w_2l_1	0.9996	0.955	0.9519	0.9153	0.954	0.9063	0.9482	0.8041
w_2l_2	0.9975	0.9566	0.9523	0.917	0.9573	0.9103	0.9521	0.8044
w_2l_3	0.9998	0.9642	0.9613	0.9413	0.9639	0.9349	0.9606	0.8521
w_2l_4	0.9977	0.9656	0.962	0.9406	0.966	0.9345	0.9618	0.849
w_3l_1	0.9975	0.9529	0.9579	0.9089	0.9541	0.9095	0.9542	0.8023
w_3l_2	0.9957	0.9567	0.9576	0.9124	0.9571	0.9094	0.9541	0.8035
w_3l_3	0.9981	0.9649	0.9682	0.941	0.966	0.942	0.9654	0.866
w_3l_4	0.9967	0.9675	0.9672	0.9442	0.9663	0.9394	0.9626	0.8632
Average Width								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.24939	0.15897	0.16324	0.13333	0.15896	0.13705	0.1637	0.10143
w_1l_2	0.23723	0.16361	0.16772	0.1372	0.16364	0.14058	0.16834	0.10386
w_1l_3	0.25397	0.20254	0.20333	0.15967	0.20262	0.16032	0.20261	0.11342
w_1l_4	0.24204	0.20945	0.21145	0.16477	0.20909	0.16613	0.21082	0.11686
w_2l_1	0.20473	0.1201	0.12309	0.10037	0.12012	0.10303	0.12324	0.07613
w_2l_2	0.1724	0.11704	0.1199	0.09768	0.11696	0.10002	0.12023	0.07363
w_2l_3	0.20846	0.15269	0.15015	0.11978	0.15258	0.11857	0.14998	0.0842
w_2l_4	0.17649	0.15344	0.1545	0.11928	0.15334	0.12006	0.15415	0.08372
w_3l_1	0.30836	0.21417	0.22137	0.1827	0.21404	0.18873	0.22144	0.14205
w_3l_2	0.30149	0.21888	0.22348	0.18676	0.21889	0.19028	0.22349	0.14308
w_3l_3	0.31555	0.29374	0.29909	0.2335	0.29377	0.23722	0.29782	0.16814
w_3l_4	0.30891	0.30029	0.30437	0.23877	0.30081	0.24145	0.30417	0.17016

Table 97. Multiplicative Desirability Inference Observation 19, (x_1, x_2) = (0.5,0.5) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.999	0.9556	0.9487	0.925	0.9555	0.9162	0.9472	0.8168
$w_1 l_2$	0.9961	0.9534	0.9468	0.9212	0.9518	0.9148	0.9449	0.8152
$w_1 l_3$	0.9991	0.9537	0.9327	0.9283	0.952	0.9117	0.9315	0.8381
$w_1 l_4$	0.9964	0.9477	0.9306	0.9254	0.9489	0.9116	0.9347	0.8345
$w_2 l_1$	0.9888	0.9579	0.9502	0.9198	0.9578	0.9132	0.9531	0.8096
$w_2 l_2$	0.982	0.9554	0.9499	0.9161	0.9544	0.9135	0.9532	0.8105
$w_2 l_3$	0.9898	0.9599	0.9403	0.9337	0.961	0.9168	0.9446	0.8345
$w_2 l_4$	0.9837	0.9582	0.941	0.9303	0.9568	0.9186	0.9445	0.8333
$w_3 l_1$	0.9998	0.9609	0.948	0.9362	0.9612	0.9174	0.9487	0.8192
$w_3 l_2$	0.9973	0.9585	0.9479	0.9248	0.9582	0.9184	0.9495	0.8241
$w_3 l_3$	0.9998	0.9567	0.9279	0.9338	0.9576	0.9118	0.9303	0.8432
$w_3 l_4$	0.9976	0.9475	0.9306	0.9227	0.9403	0.9047	0.9277	0.8321
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.28842	0.19844	0.196	0.16302	0.19848	0.16188	0.19662	0.11868
$w_1 l_2$	0.27918	0.21043	0.2118	0.17278	0.21041	0.17475	0.21236	0.12781
$w_1 l_3$	0.3449	0.33778	0.32722	0.27539	0.33755	0.26787	0.32753	0.19387
$w_1 l_4$	0.34125	0.36101	0.35602	0.29437	0.36093	0.29129	0.35641	0.21026
$w_2 l_1$	0.28053	0.22762	0.22344	0.18763	0.22779	0.18581	0.22447	0.13505
$w_2 l_2$	0.27865	0.23743	0.2363	0.19561	0.23735	0.19645	0.2374	0.14274
$w_2 l_3$	0.31092	0.27888	0.26628	0.22751	0.27885	0.21967	0.26754	0.15944
$w_2 l_4$	0.3108	0.29092	0.28224	0.23736	0.29094	0.23267	0.28332	0.16886
$w_3 l_1$	0.23991	0.15055	0.14409	0.12311	0.15055	0.11796	0.14421	0.08606
$w_3 l_2$	0.20686	0.15229	0.15284	0.12381	0.1523	0.12469	0.15322	0.09044
$w_3 l_3$	0.28894	0.26221	0.24449	0.20988	0.26232	0.1963	0.24536	0.14001
$w_3 l_4$	0.26491	0.28499	0.27961	0.22683	0.28412	0.22323	0.27985	0.1576

Table 98. Multiplicative Desirability Inference Observation 20, (x_1, x_2) = (1,0.5) Empirical Coverage and Average Width

Empirical Coverage								
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	1	0.9893	0.9138	0.9805	0.9896	0.8948	0.9197	0.8015
w$_1 l_2$	0.9985	0.9734	0.9447	0.9536	0.9734	0.9246	0.9505	0.8288
w$_1 l_3$	0.9404	0.9622	0.9569	0.9224	0.9607	0.9076	0.9529	0.7935
w$_1 l_4$	0.9057	0.9521	0.9542	0.9099	0.9517	0.9047	0.9488	0.7946
w$_2 l_1$	1	0.9829	0.9474	0.9698	0.9843	0.9341	0.9533	0.8637
w$_2 l_2$	1	0.9729	0.9596	0.9561	0.9727	0.9397	0.9613	0.8646
w$_2 l_3$	0.8984	0.9544	0.9539	0.9101	0.9533	0.9069	0.9513	0.7969
w$_2 l_4$	0.8968	0.9519	0.9523	0.9071	0.9531	0.9057	0.9504	0.799
w$_3 l_1$	0.9964	0.9794	0.9415	0.9543	0.9791	0.9083	0.9458	0.7983
w$_3 l_2$	0.9867	0.9657	0.951	0.9284	0.9648	0.9126	0.9516	0.8017
w$_3 l_3$	0.9858	0.972	0.9552	0.9385	0.9714	0.9145	0.9563	0.8067
w$_3 l_4$	0.9629	0.9653	0.9555	0.9279	0.9643	0.9147	0.9535	0.8021
Average Width								
w$_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w$_1 l_1$	0.33651	0.21578	0.17477	0.17353	0.21579	0.13489	0.17515	0.09454
w$_1 l_2$	0.37391	0.27615	0.26141	0.22088	0.27655	0.20653	0.26139	0.14822
w$_1 l_3$	0.52847	0.40651	0.37463	0.36839	0.40666	0.34036	0.37435	0.2761
w$_1 l_4$	0.71248	0.60792	0.5955	0.55184	0.60785	0.54075	0.59463	0.43859
w$_2 l_1$	0.41362	0.28883	0.25866	0.23587	0.28849	0.20634	0.25815	0.15144
w$_2 l_2$	0.44419	0.351	0.34242	0.28674	0.35102	0.27718	0.341	0.20621
w$_2 l_3$	0.6769	0.55682	0.53355	0.5168	0.55666	0.4993	0.53516	0.41635
w$_2 l_4$	0.81515	0.71481	0.70461	0.66436	0.71517	0.65819	0.70689	0.54797
w$_3 l_1$	0.34266	0.2457	0.21574	0.19875	0.24598	0.17323	0.21638	0.12399
w$_3 l_2$	0.37053	0.30353	0.29179	0.24492	0.30378	0.23607	0.29229	0.17011
w$_3 l_3$	0.50488	0.37447	0.34293	0.32995	0.37444	0.30142	0.34335	0.23528
w$_3 l_4$	0.5988	0.47973	0.46306	0.42263	0.47815	0.40545	0.46201	0.3158

Table 99. Multiplicative Desirability Inference Observation 21, (x_1, x_2) = (-1,1) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9998	0.9859	0.9485	0.9677	0.986	0.9218	0.9524	0.8191
$w_1 l_2$	0.9993	0.9727	0.9535	0.9491	0.9727	0.9224	0.9548	0.8174
$w_1 l_3$	0.9669	0.9535	0.9506	0.9158	0.9539	0.9081	0.9492	0.7969
$w_1 l_4$	0.9656	0.9535	0.9499	0.9113	0.9525	0.9082	0.9483	0.7967
$w_2 l_1$	1	0.9954	0.9392	0.9882	0.9957	0.9121	0.9418	0.8098
$w_2 l_2$	0.9995	0.9745	0.9532	0.9496	0.9756	0.9195	0.9529	0.8144
$w_2 l_3$	0.9691	0.9588	0.9497	0.9223	0.9585	0.9075	0.9483	0.7981
$w_2 l_4$	0.9661	0.9512	0.9503	0.9099	0.9502	0.9042	0.9479	0.795
$w_3 l_1$	0.9999	0.9806	0.9625	0.9558	0.9796	0.93	0.9622	0.8242
$w_3 l_2$	0.999	0.9736	0.9636	0.9437	0.9742	0.9265	0.9613	0.8212
$w_3 l_3$	0.9635	0.9488	0.9495	0.9064	0.9481	0.9045	0.9472	0.7974
$w_3 l_4$	0.9627	0.949	0.9494	0.9051	0.9487	0.9066	0.9476	0.8013
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.27829	0.17579	0.14962	0.14608	0.17593	0.12174	0.14977	0.08783
$w_1 l_2$	0.28688	0.20303	0.19245	0.16845	0.20298	0.15784	0.19254	0.11458
$w_1 l_3$	0.59071	0.44688	0.42913	0.39879	0.44687	0.3837	0.42895	0.30104
$w_1 l_4$	0.67371	0.53192	0.52294	0.47506	0.53181	0.46736	0.52273	0.36631
$w_2 l_1$	0.22209	0.13192	0.0967	0.10897	0.13187	0.07732	0.09713	0.05494
$w_2 l_2$	0.21048	0.14412	0.134	0.1187	0.14408	0.10851	0.13382	0.07832
$w_2 l_3$	0.4958	0.36952	0.34226	0.3244	0.36954	0.30036	0.34186	0.23214
$w_2 l_4$	0.61194	0.49875	0.49049	0.43849	0.49881	0.43026	0.48944	0.33282
$w_3 l_1$	0.34219	0.23431	0.21638	0.19812	0.23451	0.18037	0.21605	0.13438
$w_3 l_2$	0.35124	0.25792	0.25086	0.21797	0.25811	0.2098	0.25009	0.15658
$w_3 l_3$	0.74622	0.58045	0.56876	0.52428	0.58047	0.51574	0.56952	0.40839
$w_3 l_4$	0.81067	0.64541	0.6384	0.58329	0.6444	0.57822	0.63843	0.45687

Table 100. Multiplicative Desirability Inference Observation 22, (x_1, x_2) = (-0.5,1) Empirical Coverage and Average Width

Empirical Coverage								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.9998	0.9823	0.9487	0.9625	0.9822	0.9169	0.9529	0.8075
w_1l_2	0.9986	0.9748	0.952	0.9468	0.9743	0.9123	0.9528	0.8098
w_1l_3	0.9892	0.9553	0.9465	0.9176	0.955	0.909	0.9499	0.806
w_1l_4	0.9872	0.9544	0.9481	0.9144	0.9528	0.9088	0.9498	0.8061
w_2l_1	0.994	0.9631	0.9463	0.9293	0.9628	0.9094	0.9498	0.8003
w_2l_2	0.9908	0.959	0.9468	0.9219	0.958	0.9089	0.9475	0.802
w_2l_3	0.9938	0.9648	0.9516	0.9247	0.9646	0.9055	0.9478	0.7988
w_2l_4	0.993	0.9627	0.95	0.9232	0.9625	0.9054	0.9474	0.7984
w_3l_1	1	0.993	0.9451	0.9834	0.9929	0.9136	0.9487	0.8075
w_3l_2	0.9991	0.973	0.9507	0.9453	0.9724	0.9111	0.9492	0.8043
w_3l_3	0.9917	0.957	0.9467	0.9203	0.9585	0.9055	0.9472	0.7962
w_3l_4	0.988	0.9513	0.9488	0.9107	0.9524	0.9055	0.9486	0.798
Average Width								
w_il_i	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
w_1l_1	0.28594	0.17979	0.15697	0.14941	0.17971	0.1284	0.15713	0.09293
w_1l_2	0.28361	0.19386	0.18355	0.16116	0.19385	0.15062	0.18364	0.1094
w_1l_3	0.63406	0.43291	0.41796	0.37756	0.43294	0.36565	0.41814	0.27808
w_1l_4	0.67468	0.47437	0.46554	0.41379	0.47451	0.40699	0.46566	0.30916
w_2l_1	0.26463	0.2005	0.18872	0.16583	0.20066	0.1561	0.18936	0.11337
w_2l_2	0.2657	0.21133	0.20439	0.1747	0.2112	0.1691	0.20522	0.12288
w_2l_3	0.48171	0.31595	0.30084	0.27298	0.31604	0.25887	0.30117	0.19452
w_2l_4	0.49846	0.3337	0.3222	0.28843	0.33374	0.27724	0.32267	0.20825
w_3l_1	0.23283	0.13525	0.10136	0.11191	0.13524	0.08169	0.10171	0.05852
w_3l_2	0.21072	0.14133	0.1318	0.11669	0.14145	0.10733	0.13176	0.07761
w_3l_3	0.57375	0.40278	0.37921	0.34844	0.40276	0.32885	0.37934	0.24943
w_3l_4	0.63069	0.47189	0.46319	0.40859	0.4722	0.40133	0.46328	0.30376

Table 101. Multiplicative Desirability Inference Observation 23, (x_1, x_2) = (0,1) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9998	0.9792	0.9465	0.9583	0.9789	0.9104	0.9485	0.807
$w_1 l_2$	0.998	0.9718	0.9461	0.9436	0.9711	0.9114	0.9513	0.8051
$w_1 l_3$	0.9899	0.9584	0.9505	0.9223	0.9583	0.9161	0.9531	0.8049
$w_1 l_4$	0.9885	0.9577	0.9497	0.9202	0.9571	0.9146	0.9535	0.8036
$w_2 l_1$	0.9987	0.9708	0.9538	0.9373	0.9707	0.9114	0.9527	0.8009
$w_2 l_2$	0.9972	0.966	0.9526	0.9294	0.9655	0.9131	0.9526	0.7983
$w_2 l_3$	0.9817	0.9498	0.9488	0.9091	0.9486	0.9085	0.9474	0.797
$w_2 l_4$	0.9813	0.9483	0.9487	0.9085	0.9483	0.9084	0.9479	0.7971
$w_3 l_1$	0.9932	0.9654	0.9461	0.9307	0.9655	0.9097	0.9491	0.8013
$w_3 l_2$	0.9902	0.9611	0.9466	0.9236	0.9612	0.9084	0.9493	0.7997
$w_3 l_3$	0.9961	0.9637	0.9496	0.9287	0.9639	0.9098	0.95	0.7987
$w_3 l_4$	0.9954	0.9617	0.9498	0.9272	0.962	0.9061	0.951	0.7931
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.34615	0.21901	0.19486	0.1793	0.21925	0.15673	0.19514	0.11172
$w_1 l_2$	0.34275	0.23295	0.22119	0.19051	0.23274	0.17846	0.22116	0.12758
$w_1 l_3$	0.67667	0.40683	0.3925	0.34828	0.40668	0.33685	0.39142	0.24943
$w_1 l_4$	0.70717	0.43699	0.42835	0.37423	0.43722	0.3673	0.42726	0.27144
$w_2 l_1$	0.43075	0.28347	0.2656	0.23584	0.28361	0.21906	0.26576	0.15926
$w_2 l_2$	0.43078	0.295	0.28547	0.24545	0.29494	0.23567	0.28535	0.17152
$w_2 l_3$	0.80678	0.44445	0.43317	0.3706	0.44423	0.36462	0.43415	0.25752
$w_2 l_4$	0.82959	0.46392	0.45586	0.38751	0.4642	0.38367	0.45699	0.27048
$w_3 l_1$	0.30918	0.2342	0.22073	0.19203	0.2341	0.18127	0.22181	0.13041
$w_3 l_2$	0.30935	0.24378	0.23534	0.19993	0.24393	0.19338	0.23638	0.13924
$w_3 l_3$	0.51969	0.31446	0.29936	0.26942	0.31443	0.25575	0.29996	0.18942
$w_3 l_4$	0.53255	0.32821	0.31643	0.2812	0.32777	0.26963	0.31645	0.1996

Table 102. Multiplicative Desirability Inference Observation 24, (x_1, x_2) = (0.5,1) Empirical Coverage and Average Width

Empirical Coverage								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.9995	0.977	0.9426	0.9524	0.9762	0.9067	0.9445	0.801
$w_1 l_2$	0.9983	0.9683	0.9445	0.9372	0.9683	0.9087	0.947	0.8011
$w_1 l_3$	0.99	0.9608	0.9521	0.9238	0.9595	0.9166	0.9536	0.8059
$w_1 l_4$	0.9885	0.9588	0.9522	0.9199	0.9589	0.9141	0.955	0.8063
$w_2 l_1$	0.9999	0.9919	0.9346	0.979	0.9918	0.9071	0.9401	0.7998
$w_2 l_2$	0.9985	0.9732	0.9449	0.9443	0.9737	0.9103	0.9486	0.8063
$w_2 l_3$	0.9929	0.9616	0.9514	0.9241	0.962	0.9127	0.9529	0.8045
$w_2 l_4$	0.9887	0.9588	0.9522	0.9192	0.9576	0.9139	0.9529	0.8042
$w_3 l_1$	0.999	0.9706	0.9526	0.9404	0.9713	0.9121	0.9544	0.8026
$w_3 l_2$	0.9974	0.9668	0.9515	0.9313	0.9677	0.9138	0.955	0.8051
$w_3 l_3$	0.9824	0.9578	0.9539	0.9165	0.958	0.914	0.9541	0.8056
$w_3 l_4$	0.9816	0.9562	0.9541	0.9156	0.9579	0.9113	0.9535	0.7978
Average Width								
$w_i l_i$	**BW**	**UG**	**MG**	**MVNS**	**MVtS**	**MVNSSig**	**MVtSSig**	**BSR**
$w_1 l_1$	0.51058	0.36722	0.33768	0.28465	0.36721	0.25654	0.33912	0.1719
$w_1 l_2$	0.52789	0.40346	0.38884	0.31204	0.40316	0.29724	0.38983	0.20071
$w_1 l_3$	0.70485	0.38894	0.37054	0.33294	0.38919	0.31849	0.36887	0.2342
$w_1 l_4$	0.75306	0.43227	0.42082	0.37	0.43228	0.36081	0.41856	0.2645
$w_2 l_1$	0.45934	0.33074	0.28504	0.24007	0.33071	0.19838	0.28659	0.12277
$w_2 l_2$	0.46825	0.37569	0.36	0.2701	0.37558	0.25606	0.36145	0.16196
$w_2 l_3$	0.712	0.45251	0.42214	0.38958	0.45251	0.36716	0.42313	0.27607
$w_2 l_4$	0.79702	0.54381	0.5295	0.46856	0.54377	0.45829	0.52992	0.3438
$w_3 l_1$	0.61577	0.45068	0.42907	0.3615	0.45051	0.34134	0.43038	0.23942
$w_3 l_2$	0.63073	0.47866	0.46583	0.3837	0.4784	0.37197	0.46684	0.26144
$w_3 l_3$	0.81396	0.40225	0.38742	0.32994	0.40175	0.32097	0.38637	0.21832
$w_3 l_4$	0.84737	0.42913	0.41822	0.35184	0.42923	0.34689	0.41737	0.23611

Table 103. Multiplicative Desirability Inference Observation 25, (x_1, x_2) = (1,1) Empirical Coverage and Average Width

Appendix C. First Order Max/Min Models: Additive vs Multiplicative Coverage, Width, and Symmetry Plots

C.1 Empirical Coverage and Average Width

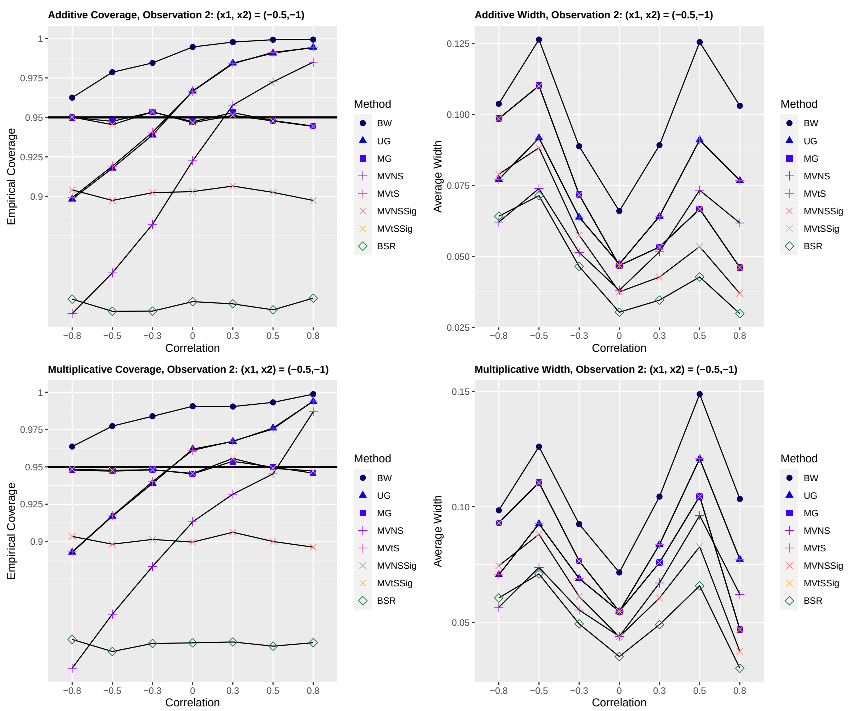


Figure 35. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 2

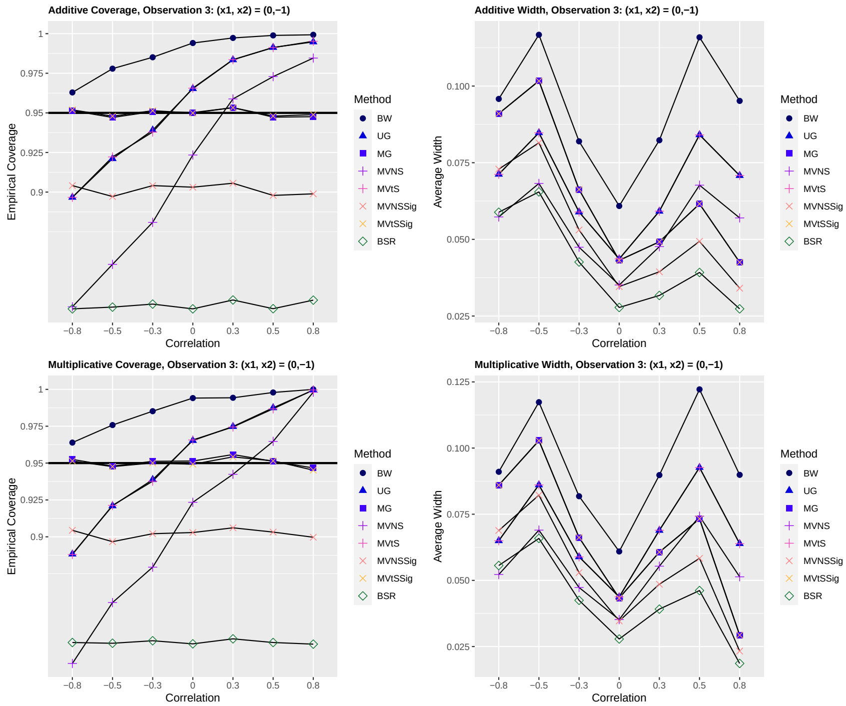


Figure 36. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 3

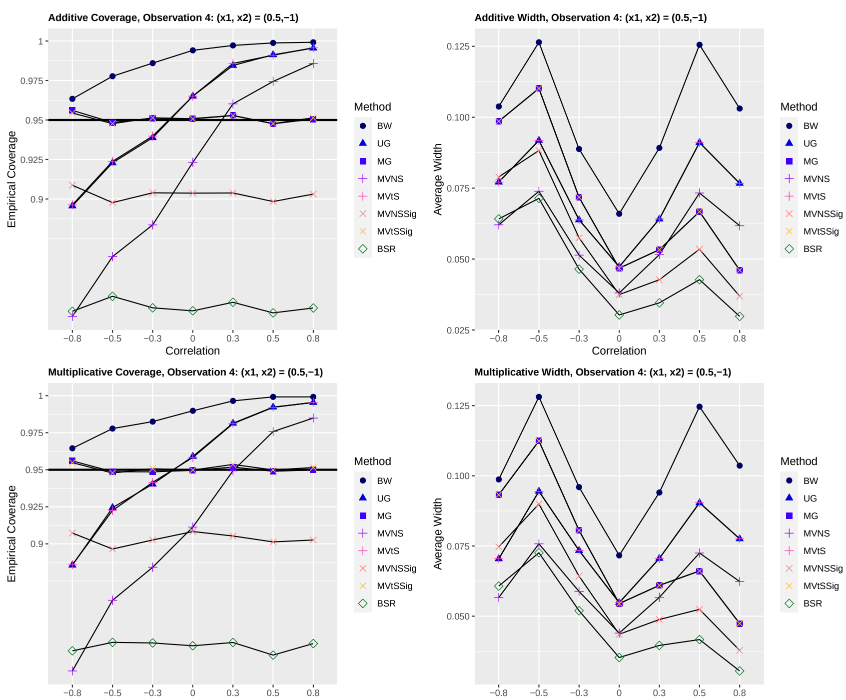


Figure 37. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 4

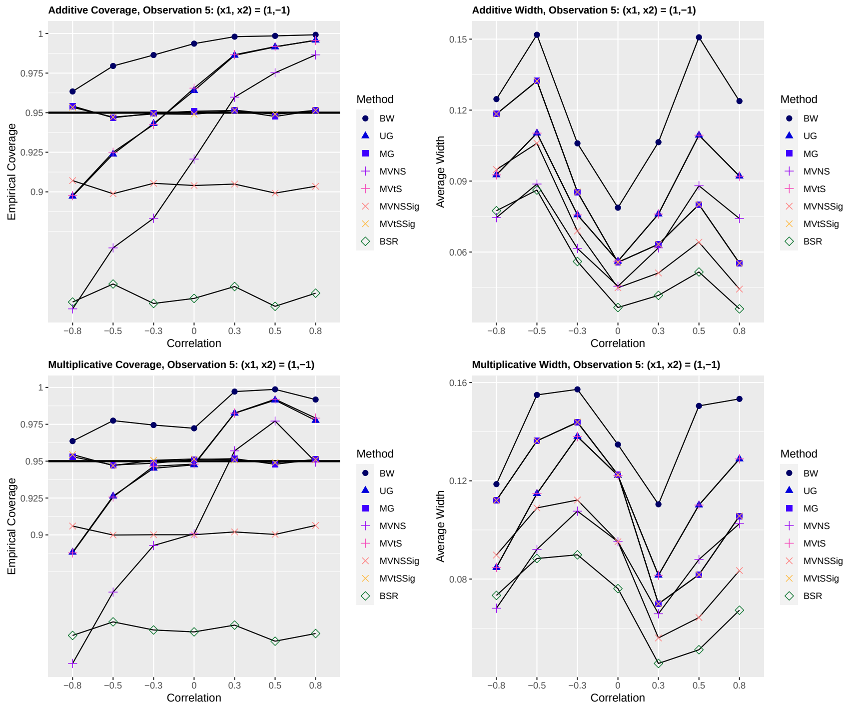


Figure 38. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 5

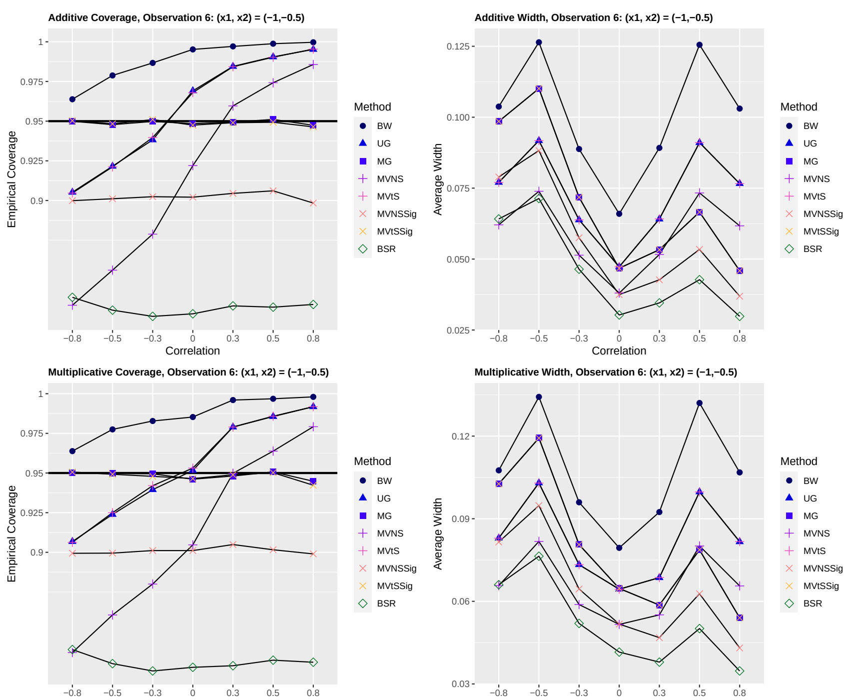


Figure 39. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 6

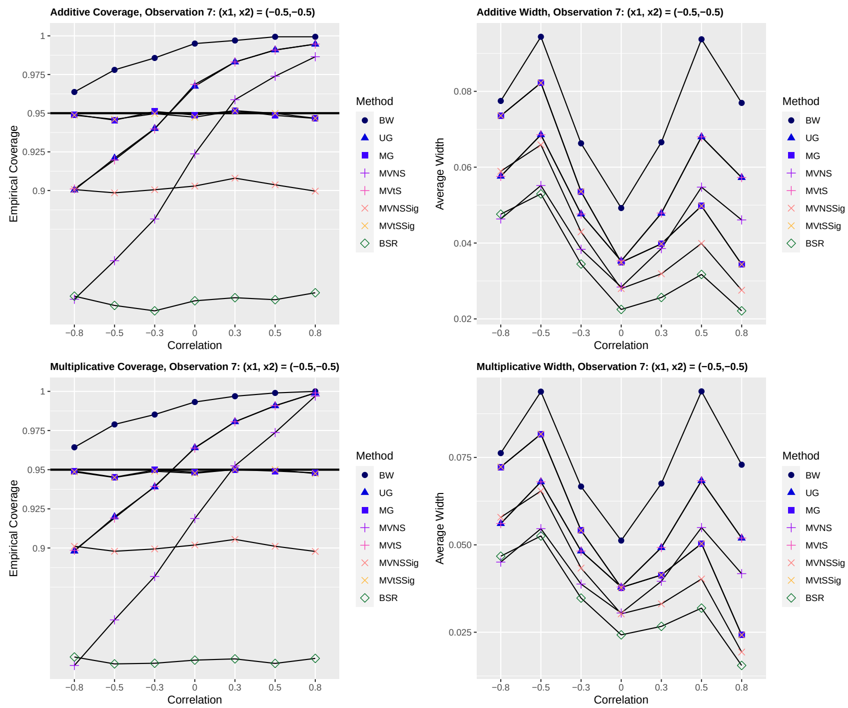


Figure 40. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 7

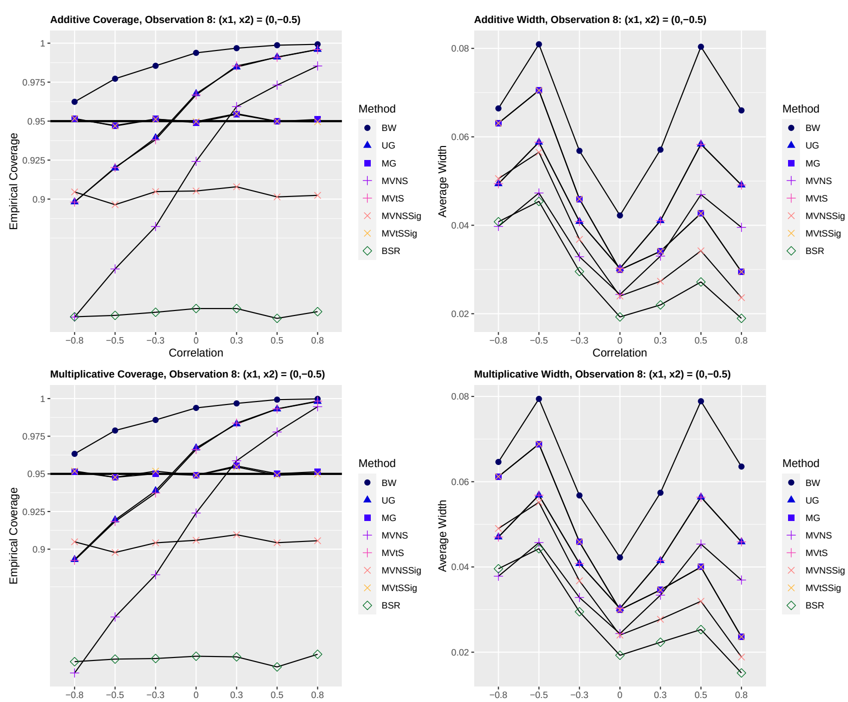


Figure 41. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 8

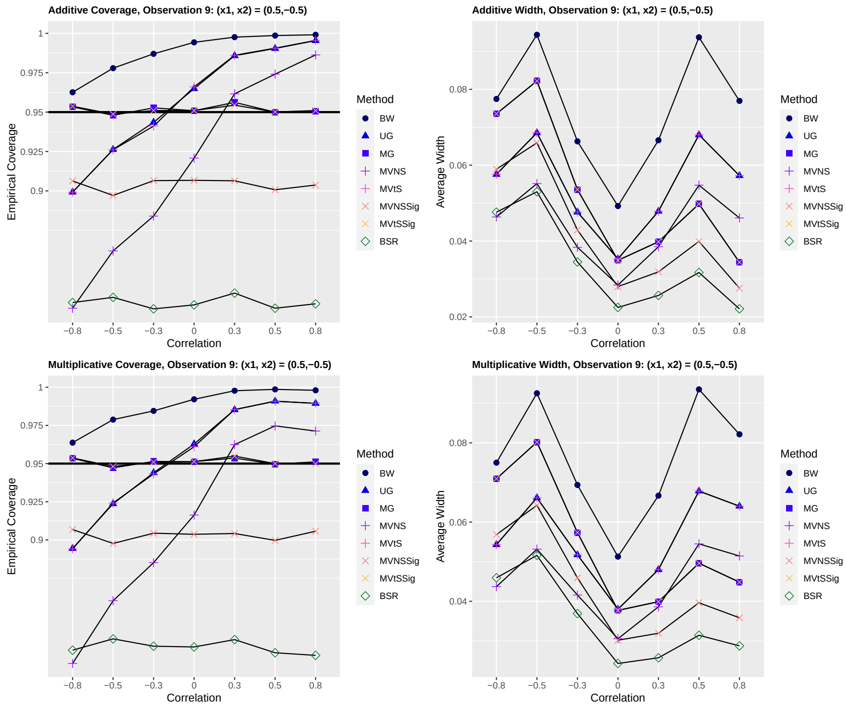


Figure 42. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 9

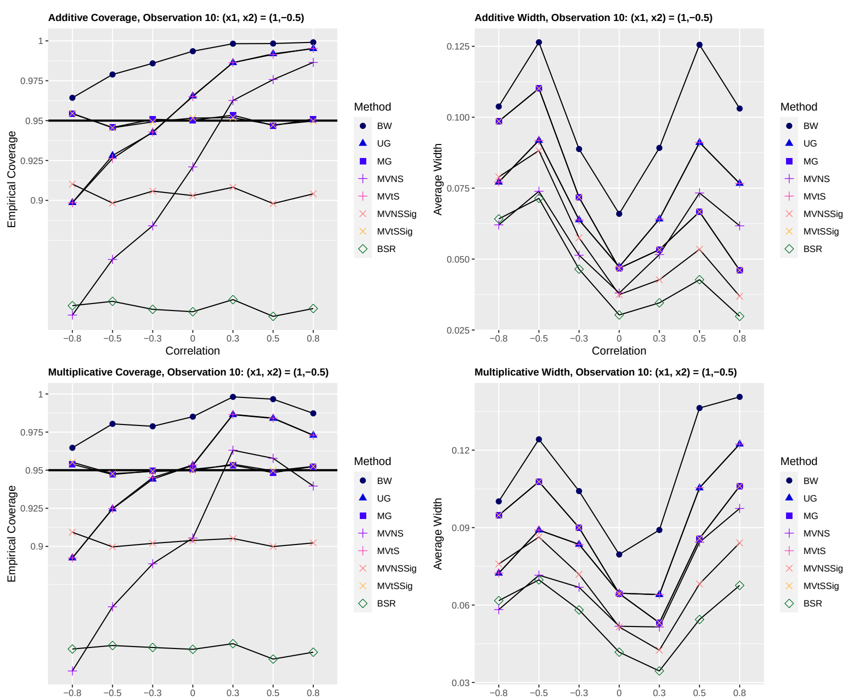


Figure 43. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 10

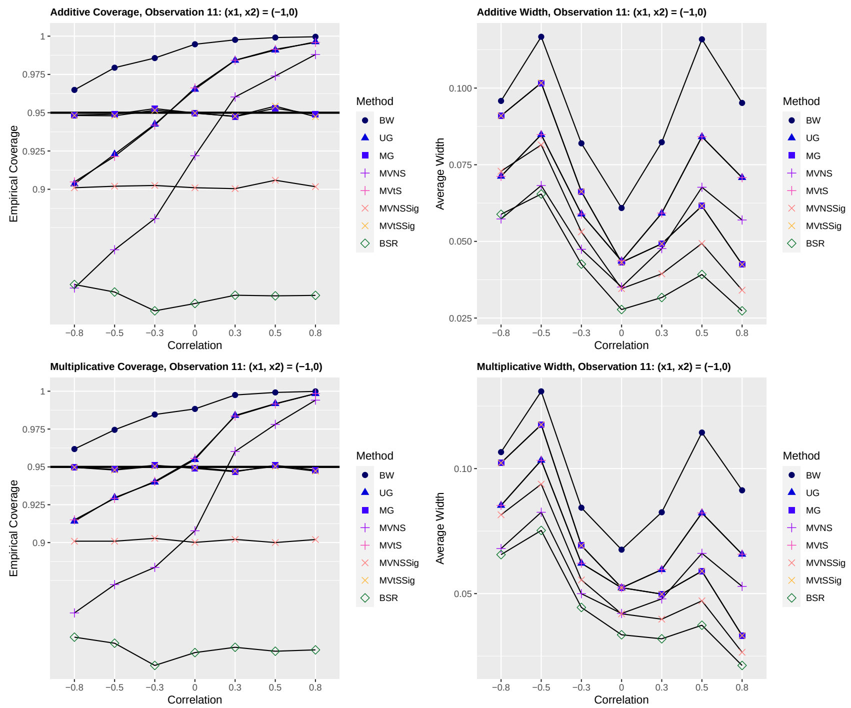


Figure 44. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 11

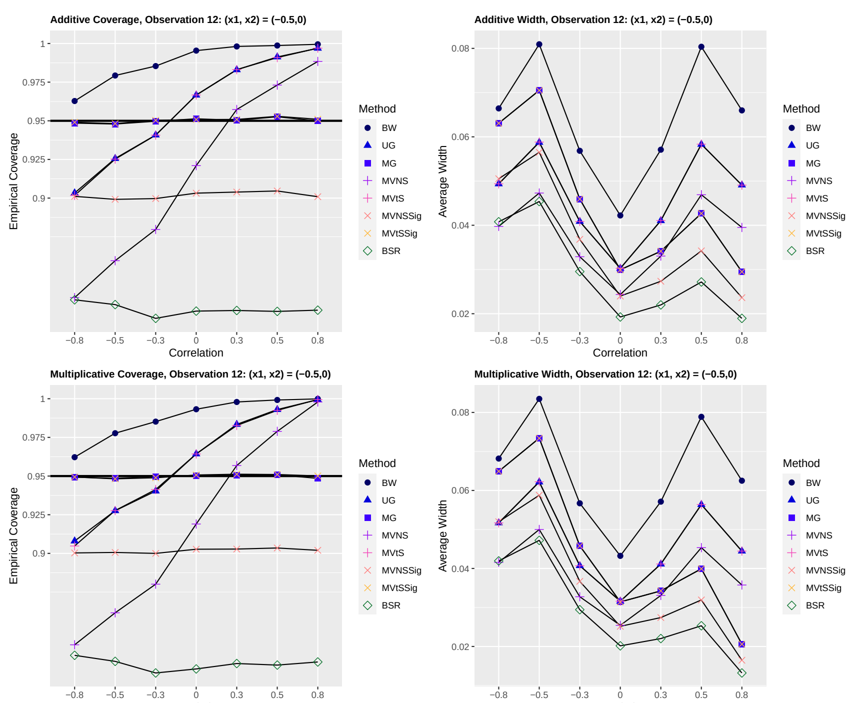


Figure 45. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 12

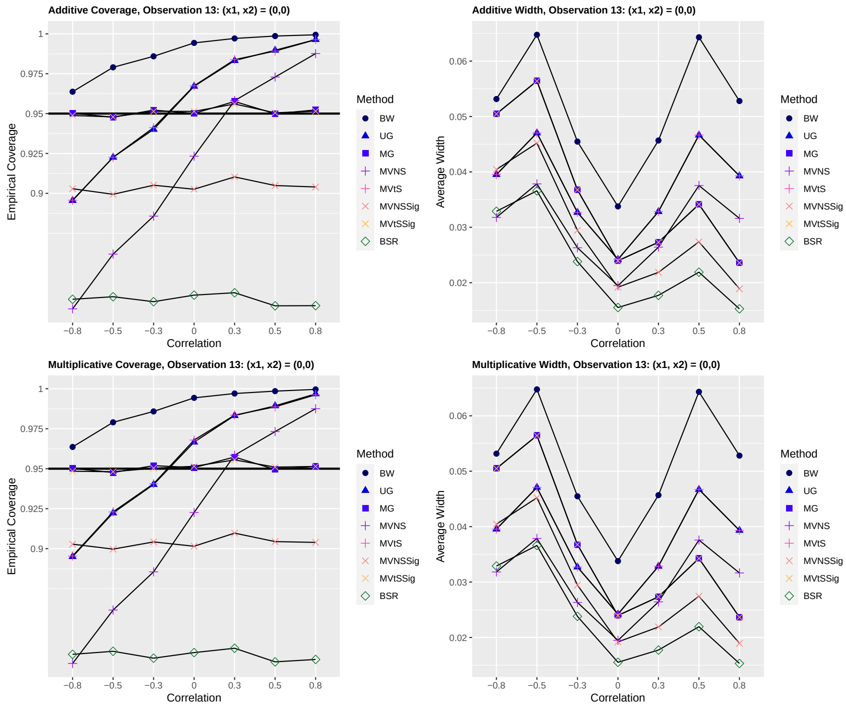


Figure 46. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 13

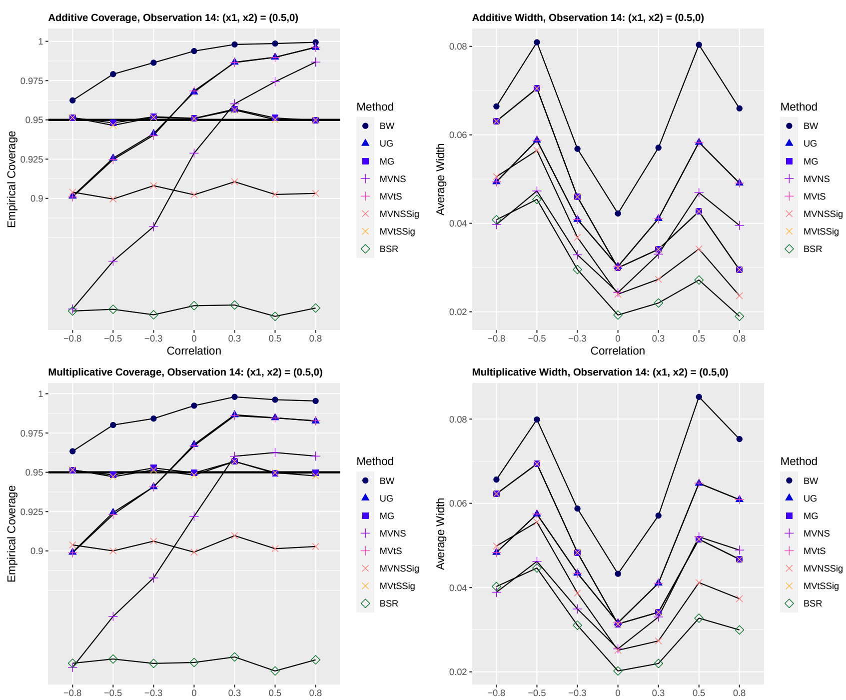


Figure 47. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 14

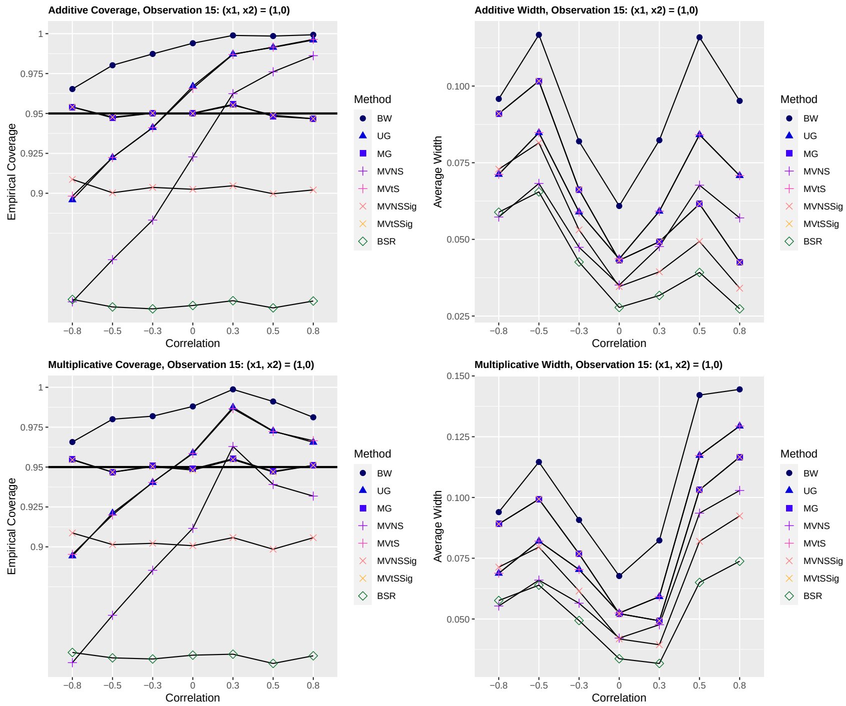


Figure 48. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 15

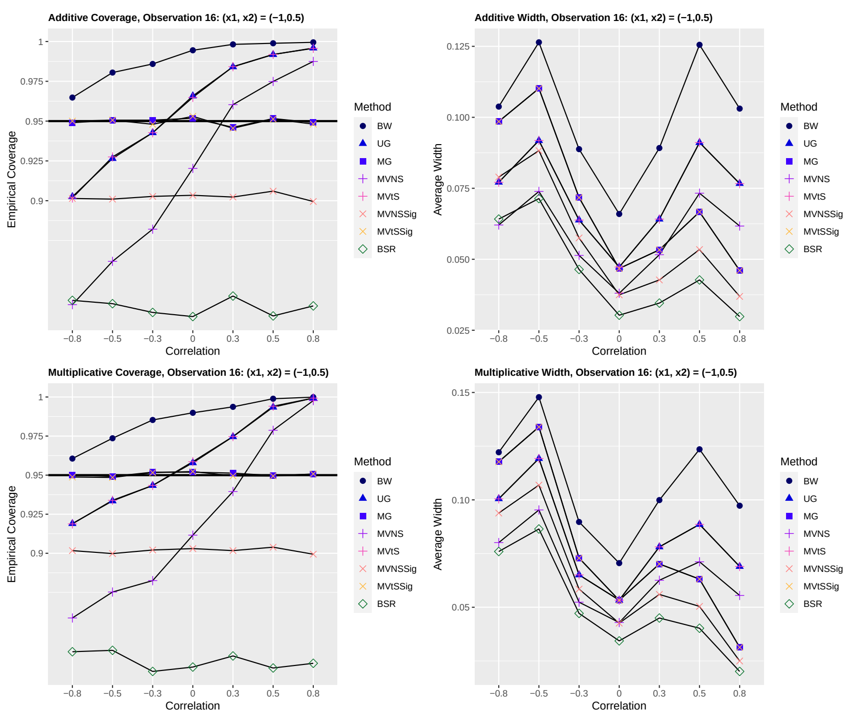


Figure 49. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 16

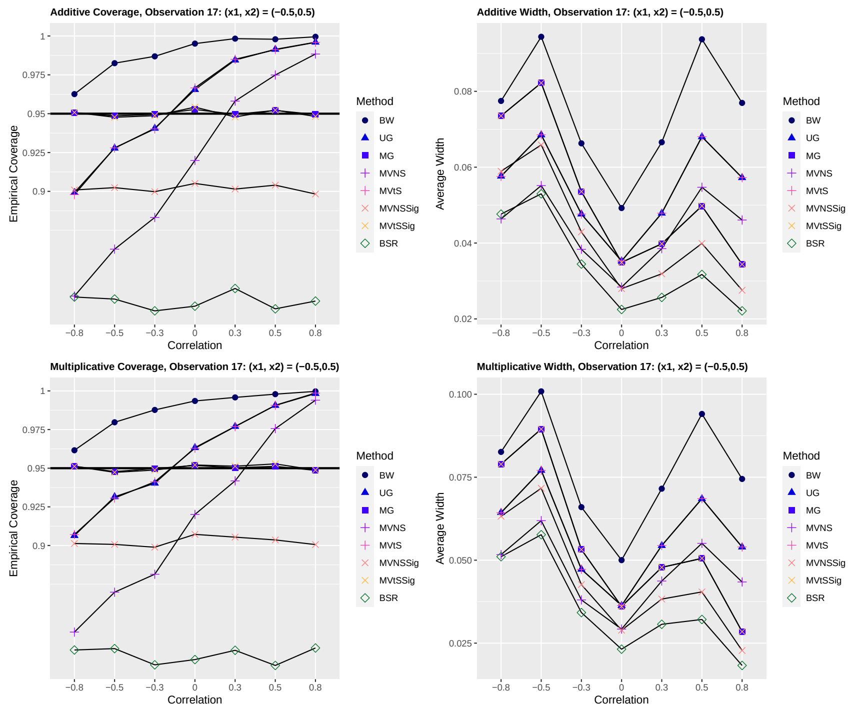


Figure 50. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 17

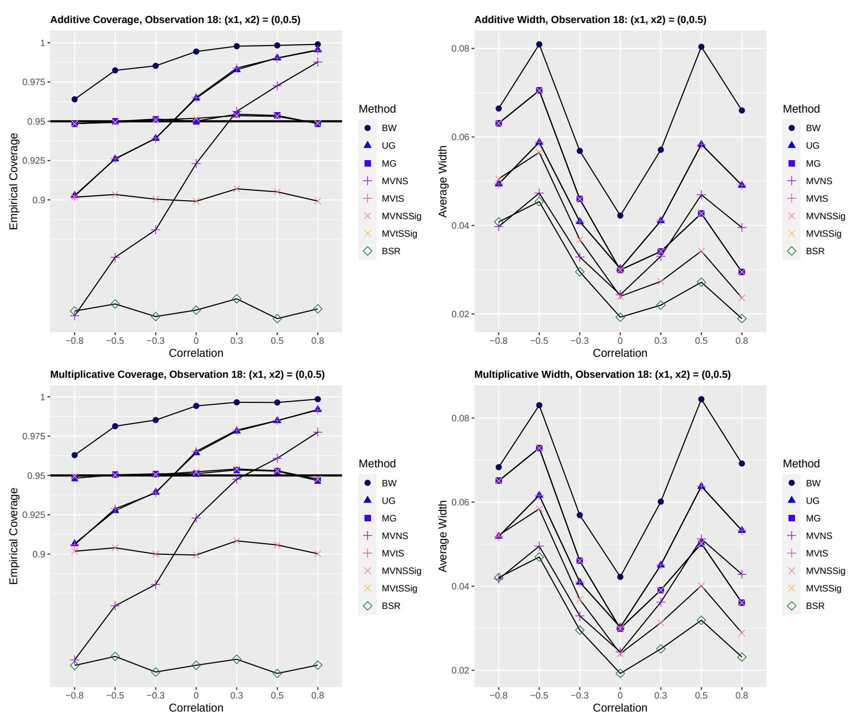


Figure 51. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 18

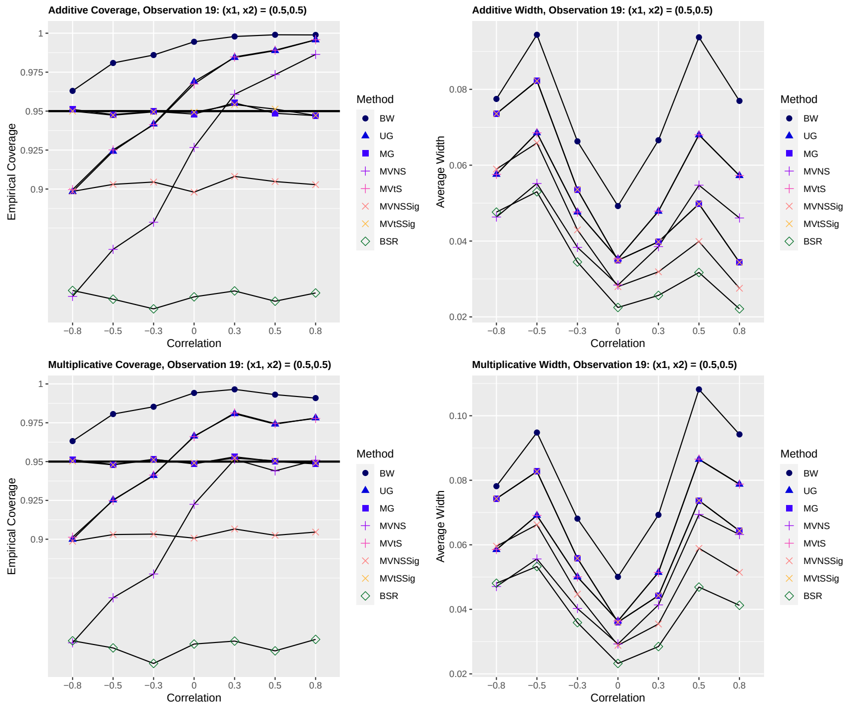


Figure 52. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 19

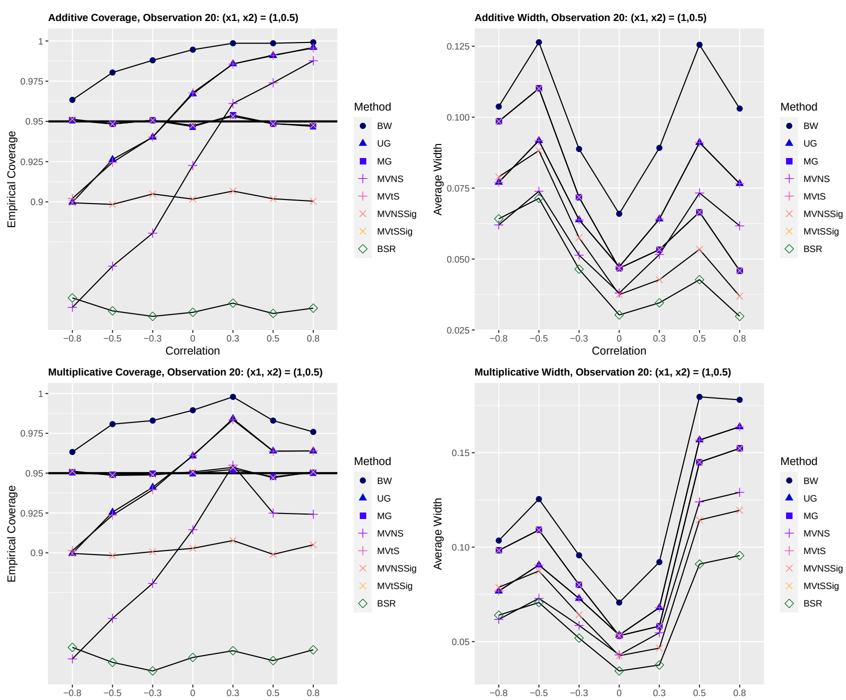


Figure 53. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 20

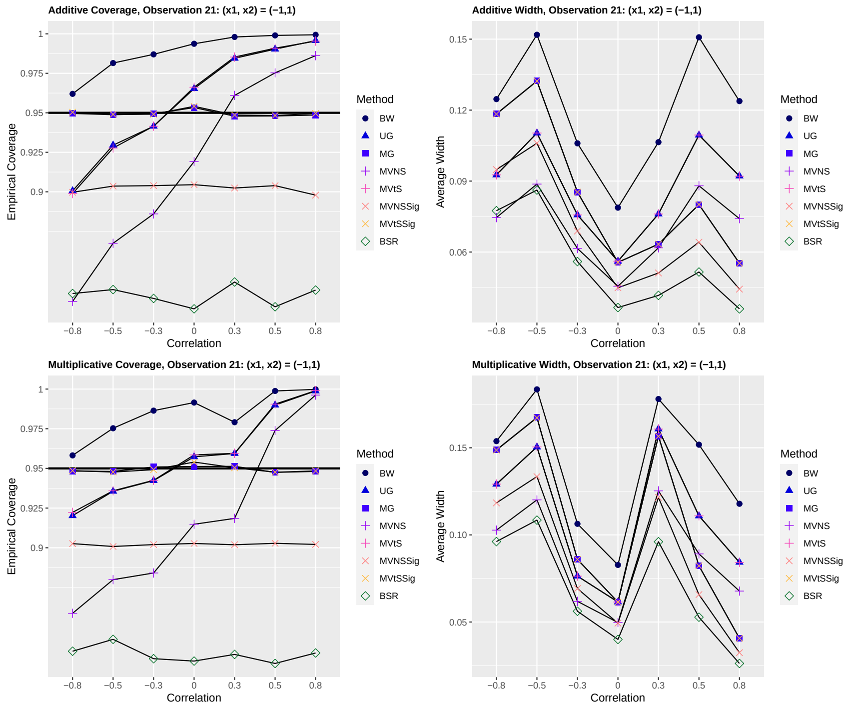


Figure 54. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 21

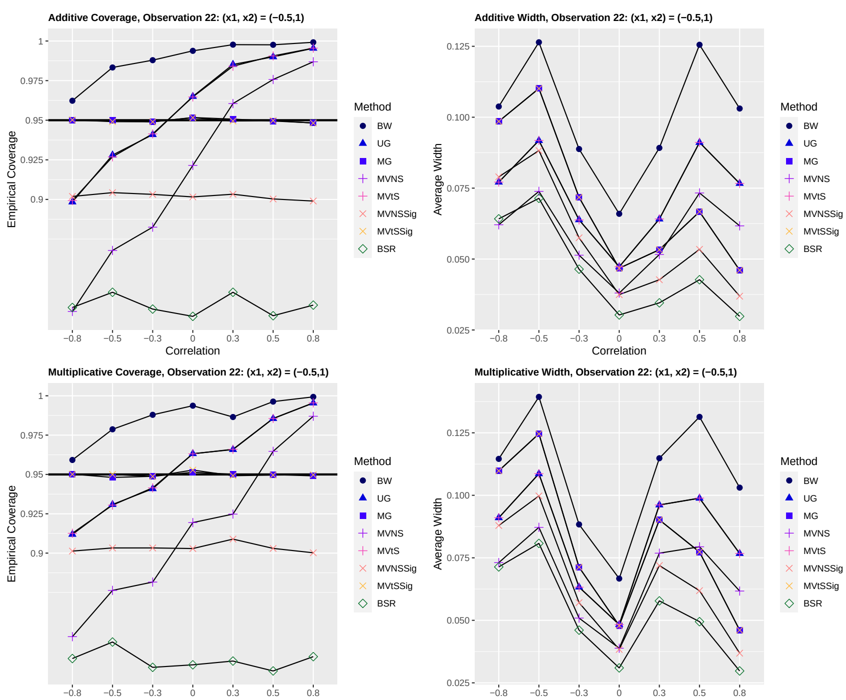


Figure 55. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 22

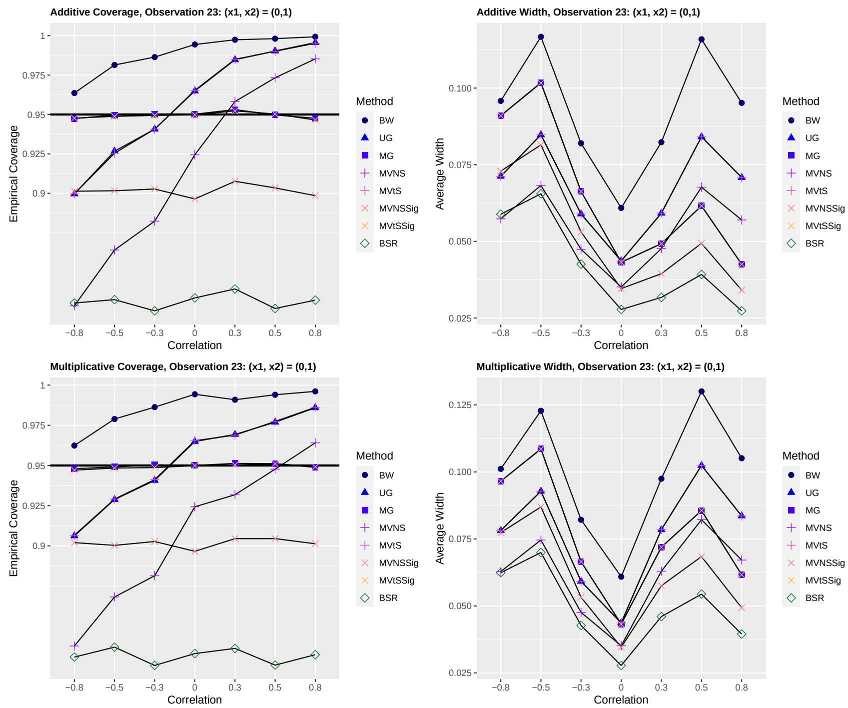


Figure 56. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 23

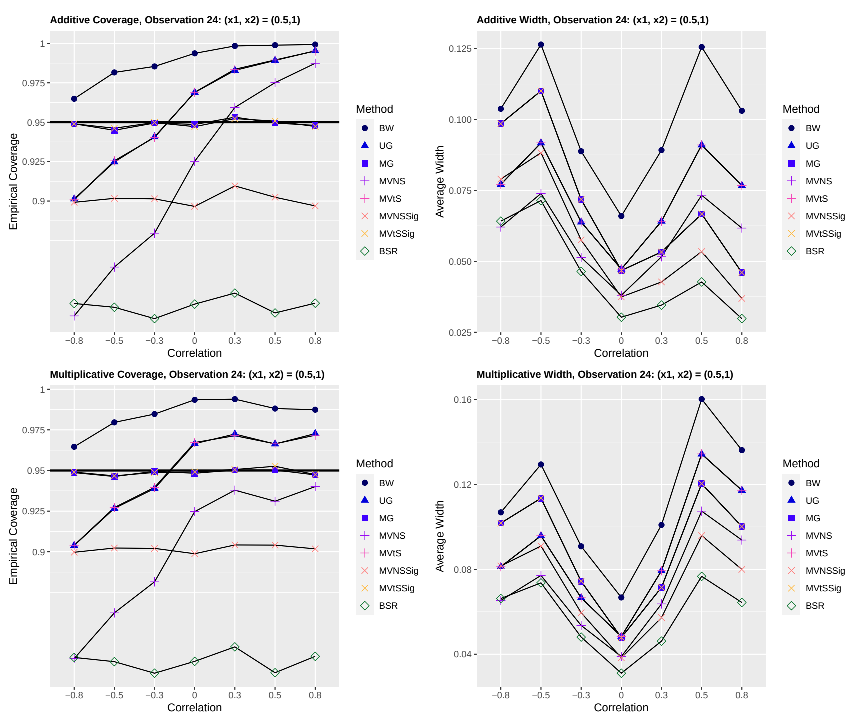


Figure 57. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 24

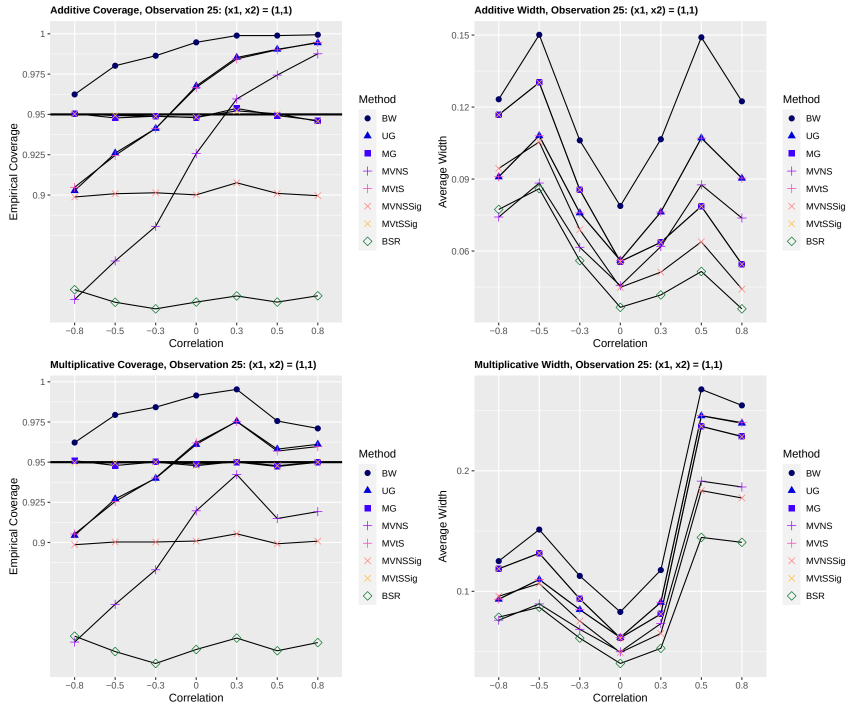


Figure 58. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 25

C.2 Average Symmetry

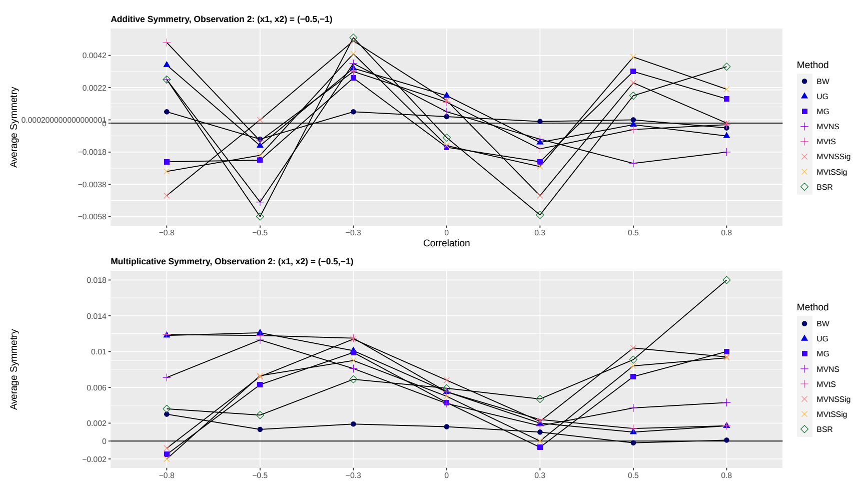


Figure 59. Additive and Multiplicative Plots for Average Symmetry at Observation 2

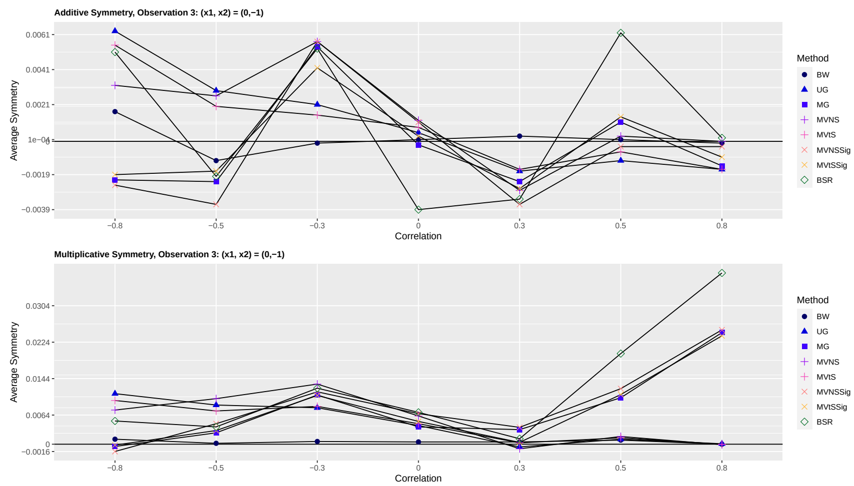


Figure 60. Additive and Multiplicative Plots for Average Symmetry at Observation 3

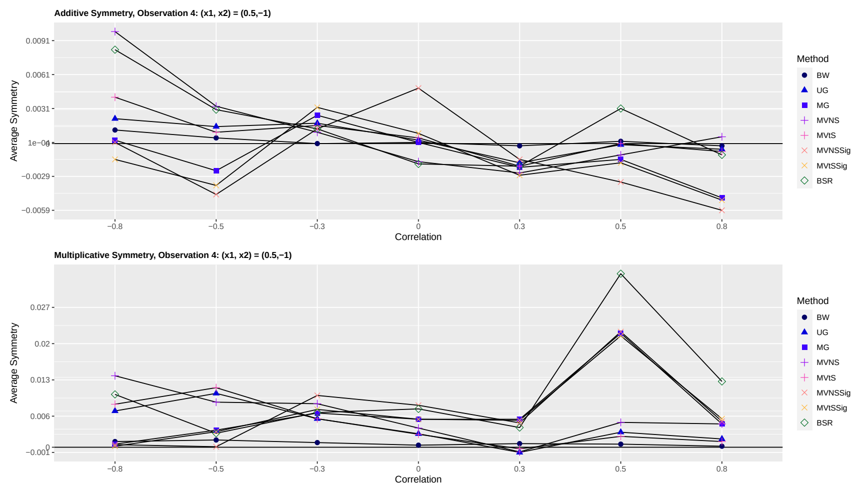


Figure 61. Additive and Multiplicative Plots for Average Symmetry at Observation 4

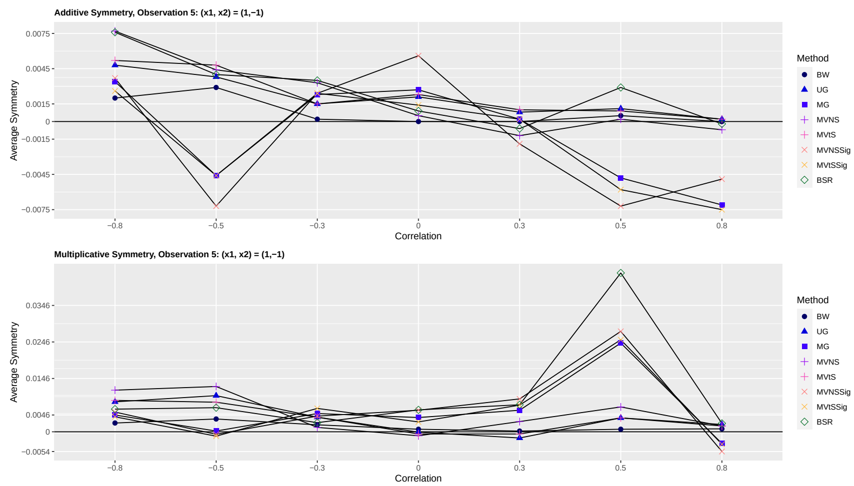


Figure 62. Additive and Multiplicative Plots for Average Symmetry at Observation 5

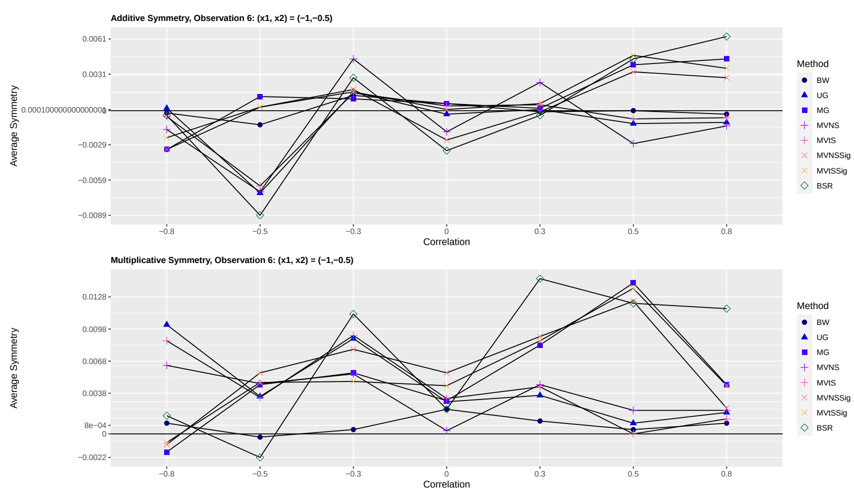


Figure 63. Additive and Multiplicative Plots for Average Symmetry at Observation 6

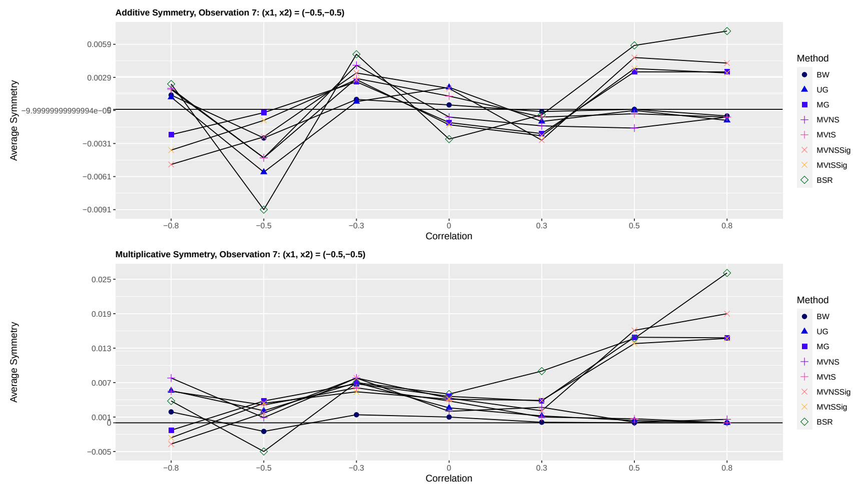


Figure 64. Additive and Multiplicative Plots for Average Symmetry at Observation 7

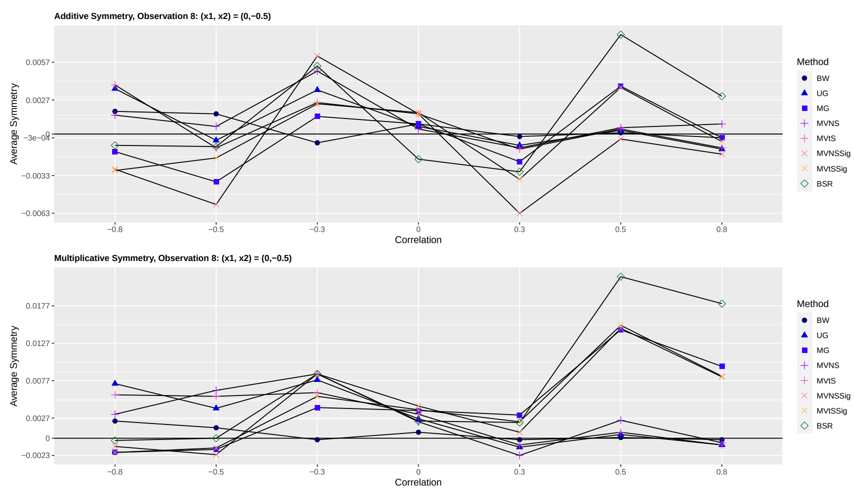


Figure 65. Additive and Multiplicative Plots for Average Symmetry at Observation 8

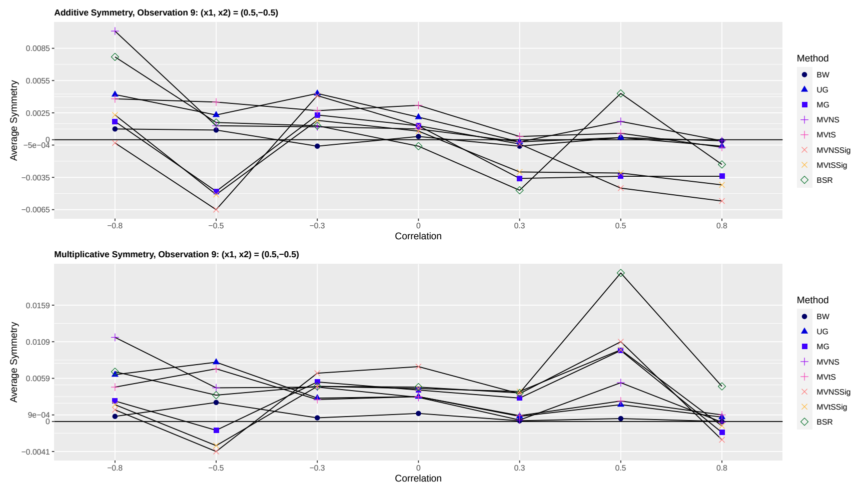


Figure 66. Additive and Multiplicative Plots for Average Symmetry at Observation 9

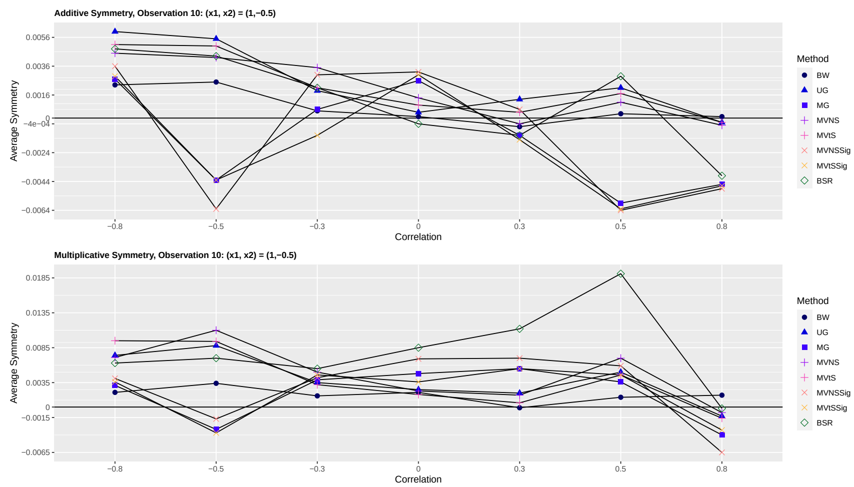


Figure 67. Additive and Multiplicative Plots for Average Symmetry at Observation 10

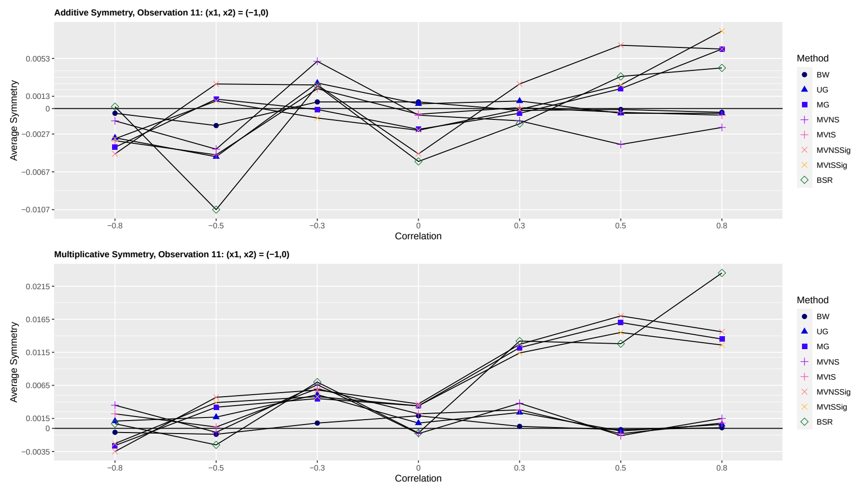


Figure 68. Additive and Multiplicative Plots for Average Symmetry at Observation 11

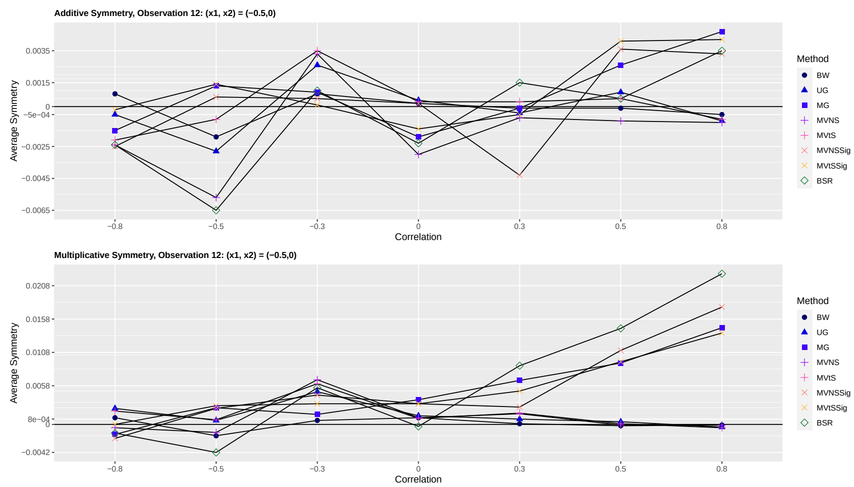


Figure 69. Additive and Multiplicative Plots for Average Symmetry at Observation 12

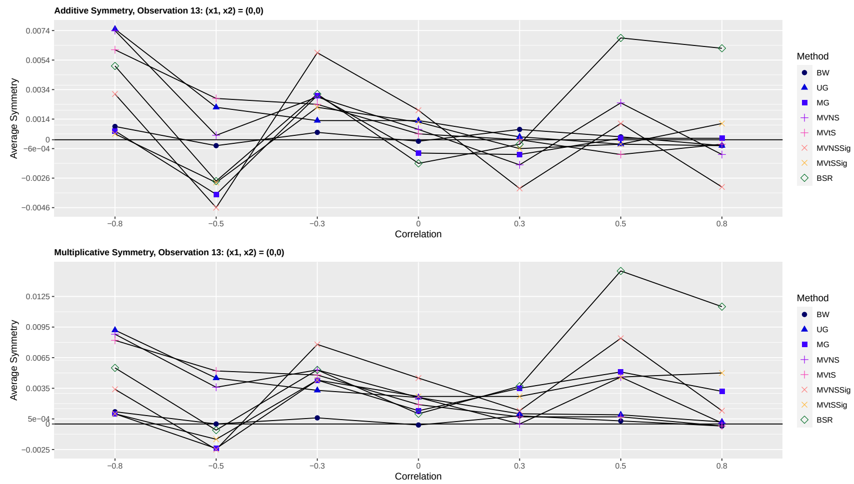


Figure 70. Additive and Multiplicative Plots for Average Symmetry at Observation 13

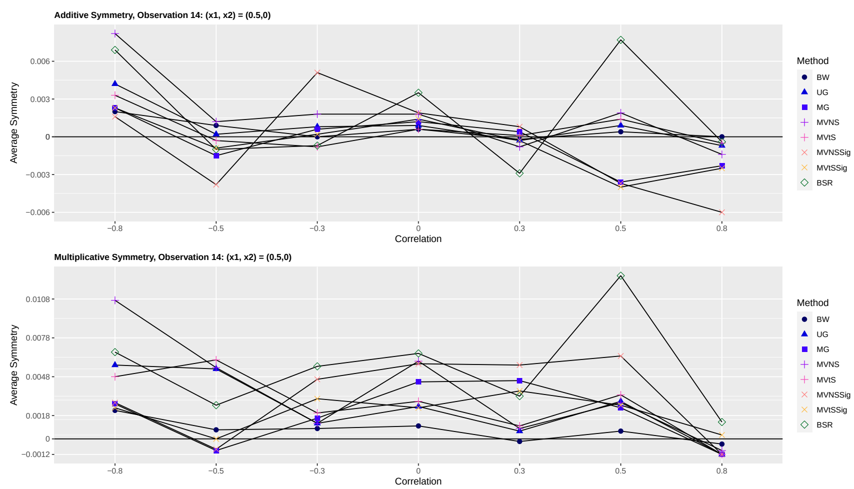


Figure 71. Additive and Multiplicative Plots for Average Symmetry at Observation 14

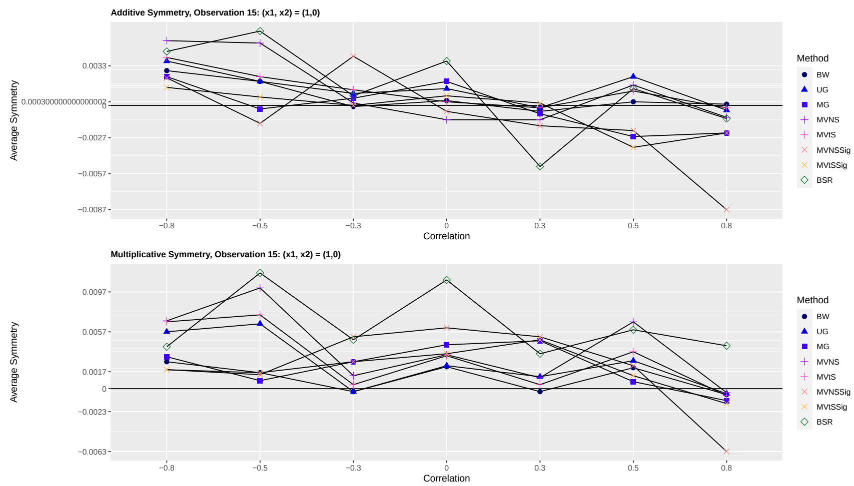


Figure 72. Additive and Multiplicative Plots for Average Symmetry at Observation 15

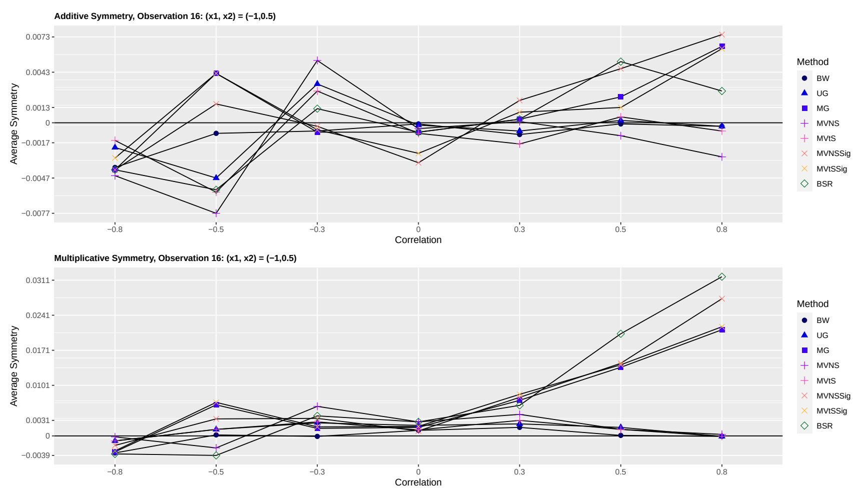


Figure 73. Additive and Multiplicative Plots for Average Symmetry at Observation 16

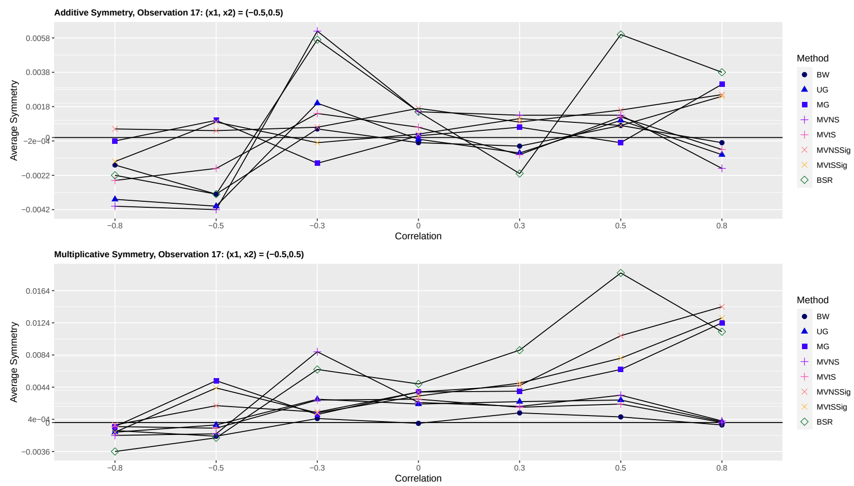


Figure 74. Additive and Multiplicative Plots for Average Symmetry at Observation 17

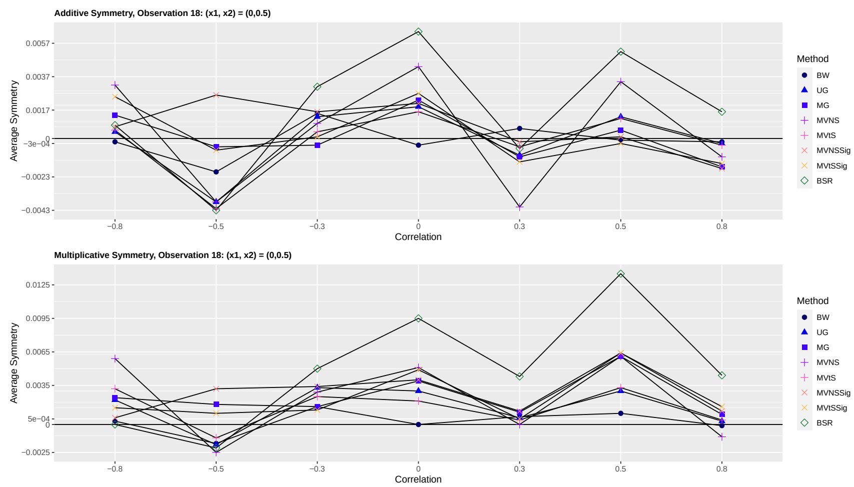


Figure 75. Additive and Multiplicative Plots for Average Symmetry at Observation 18

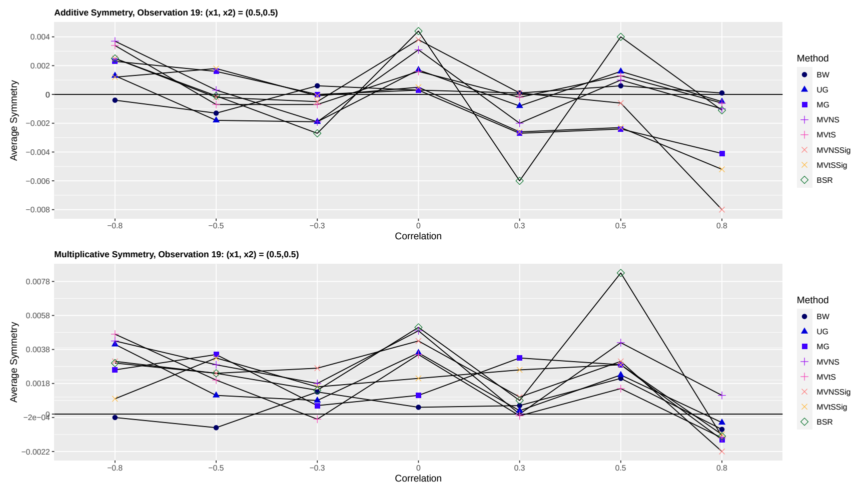


Figure 76. Additive and Multiplicative Plots for Average Symmetry at Observation 19

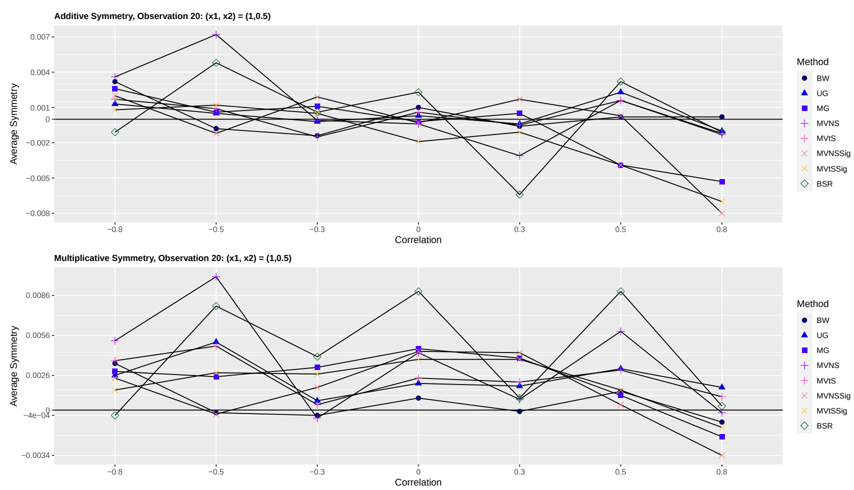


Figure 77. Additive and Multiplicative Plots for Average Symmetry at Observation 20

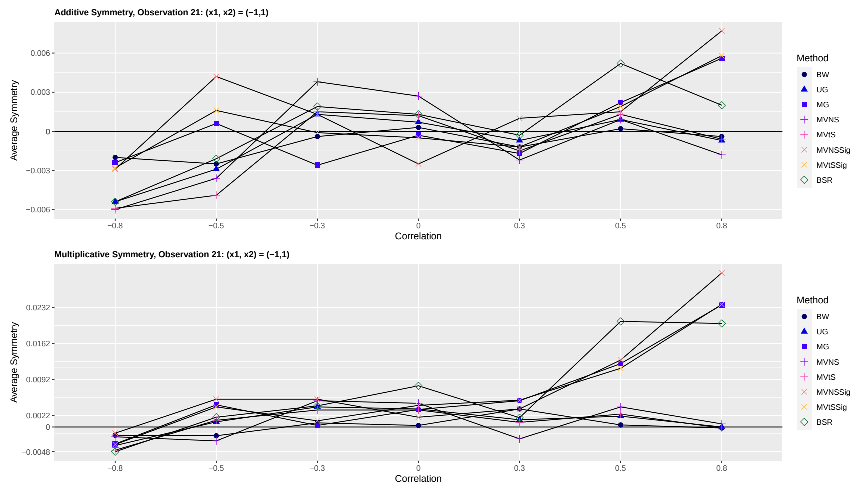


Figure 78. Additive and Multiplicative Plots for Average Symmetry at Observation 21

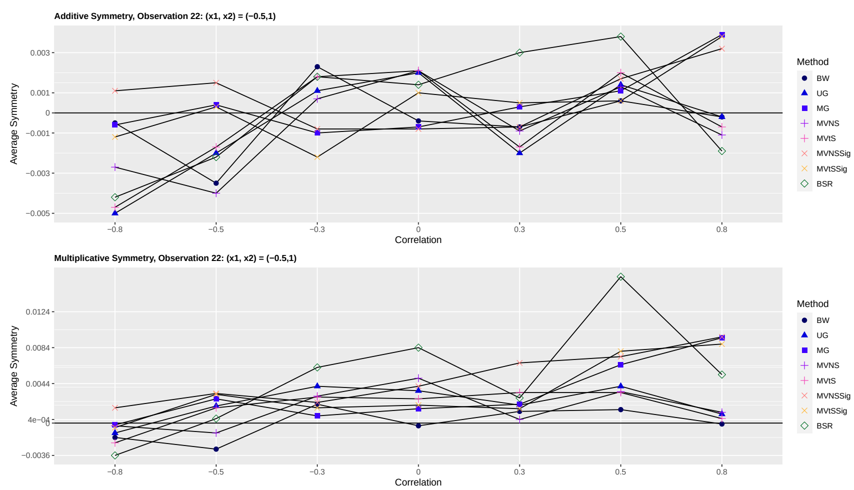


Figure 79. Additive and Multiplicative Plots for Average Symmetry at Observation 22

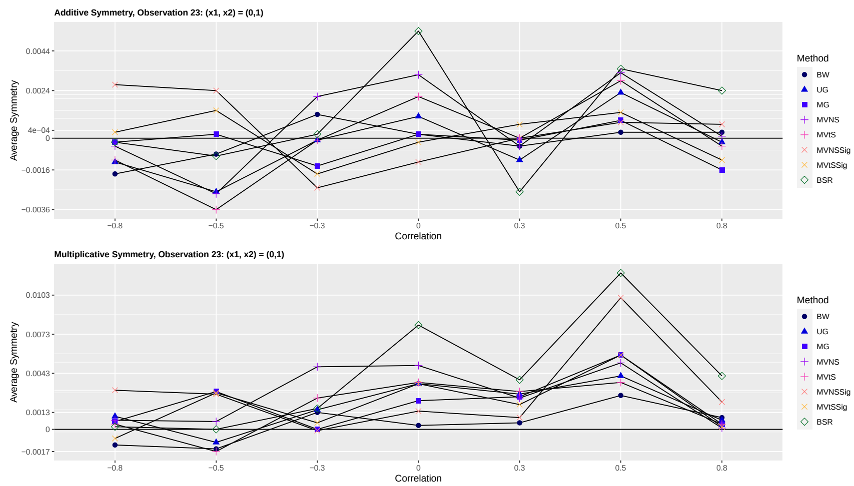


Figure 80. Additive and Multiplicative Plots for Average Symmetry at Observation 23

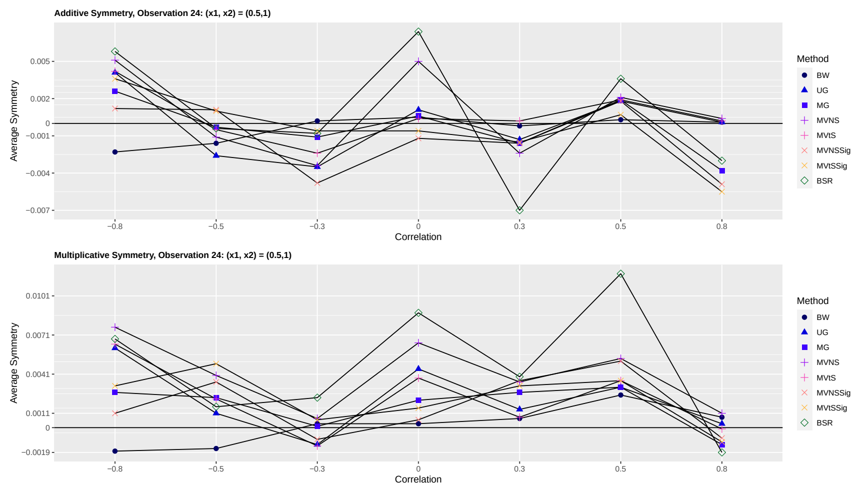


Figure 81. Additive and Multiplicative Plots for Average Symmetry at Observation 24

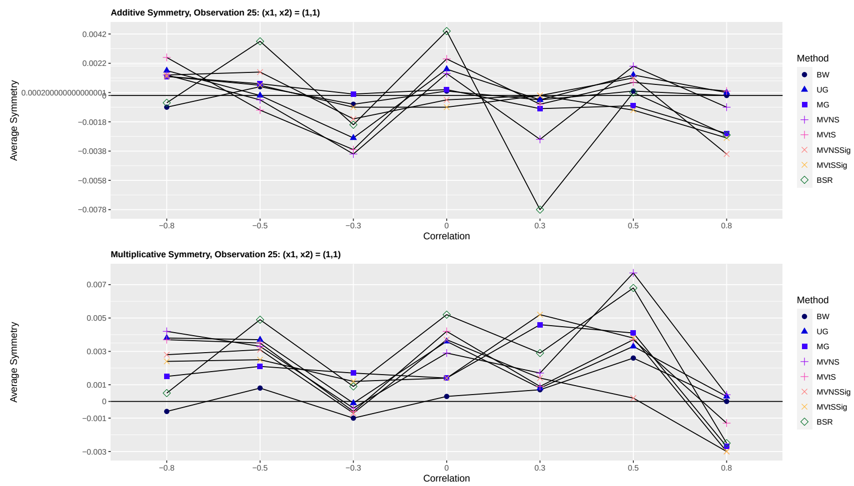


Figure 82. Additive and Multiplicative Plots for Average Symmetry at Observation 25

Appendix D. Second Order Max/Tgt/Min Models: Additive vs Multiplicative Coverage, Width and Symmetry Plots

D.1 Empirical Coverage and Average Width

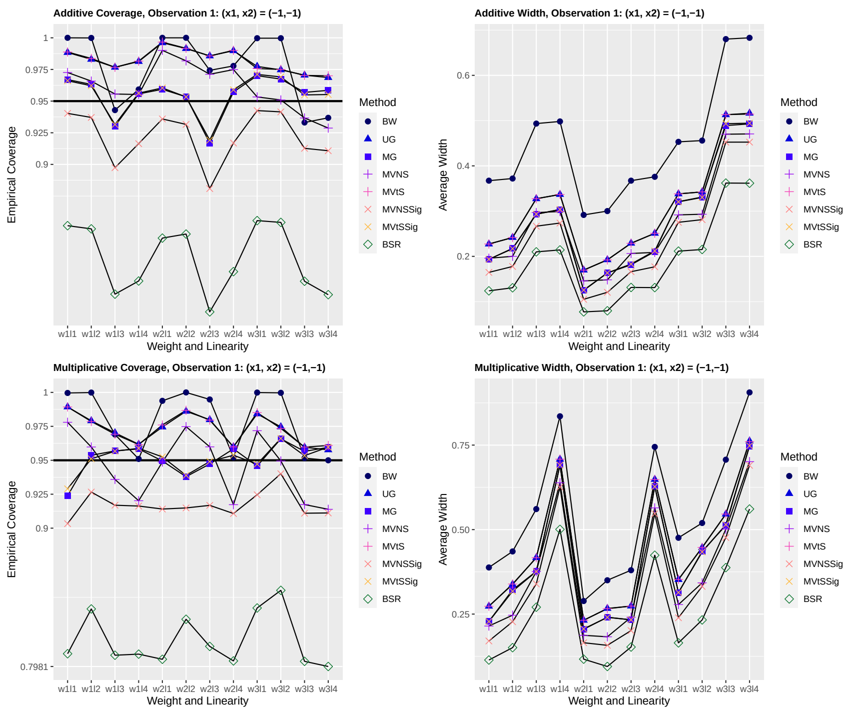


Figure 83. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 1

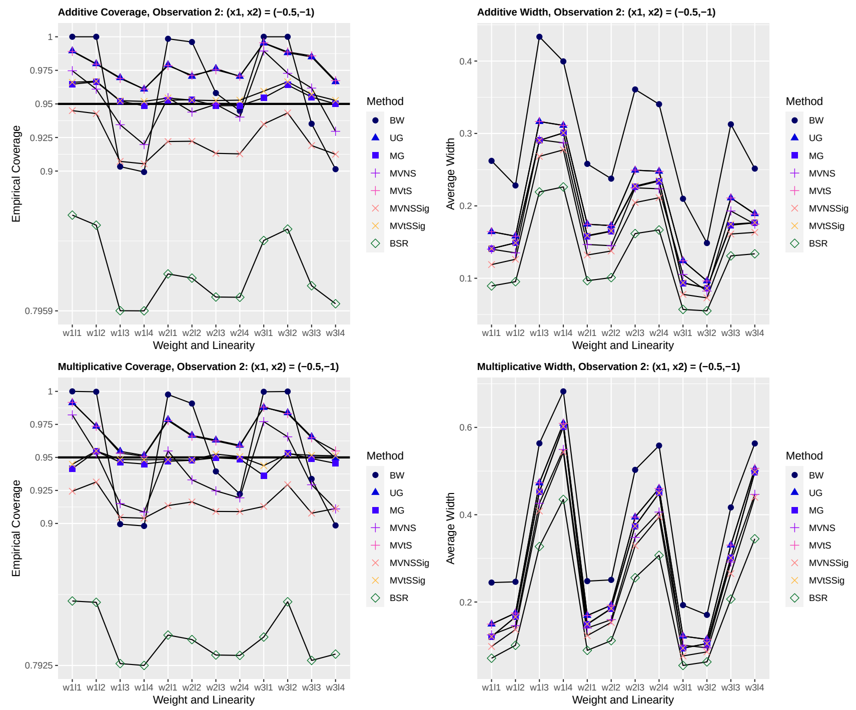


Figure 84. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 2

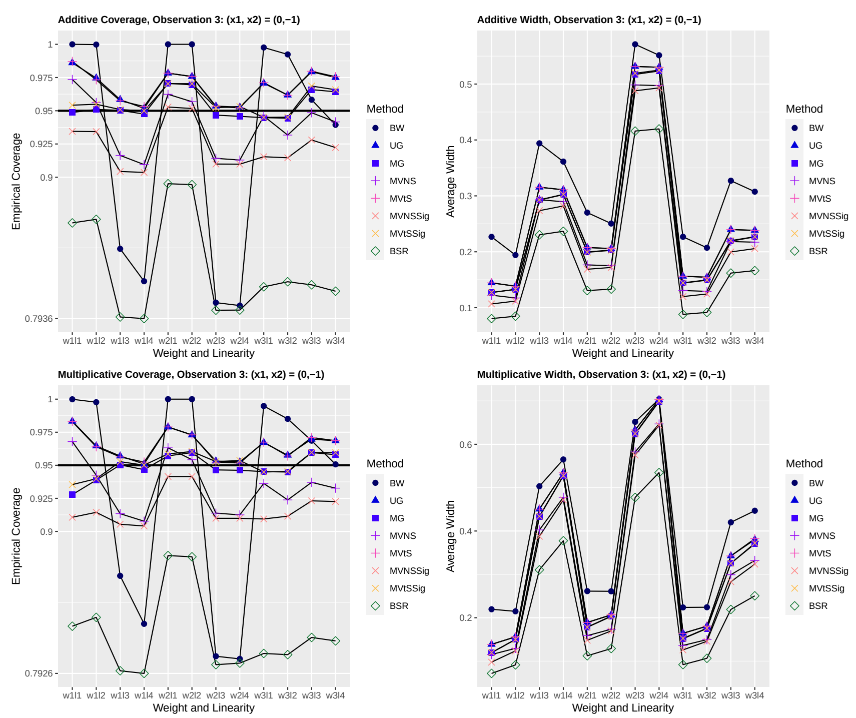


Figure 85. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 3

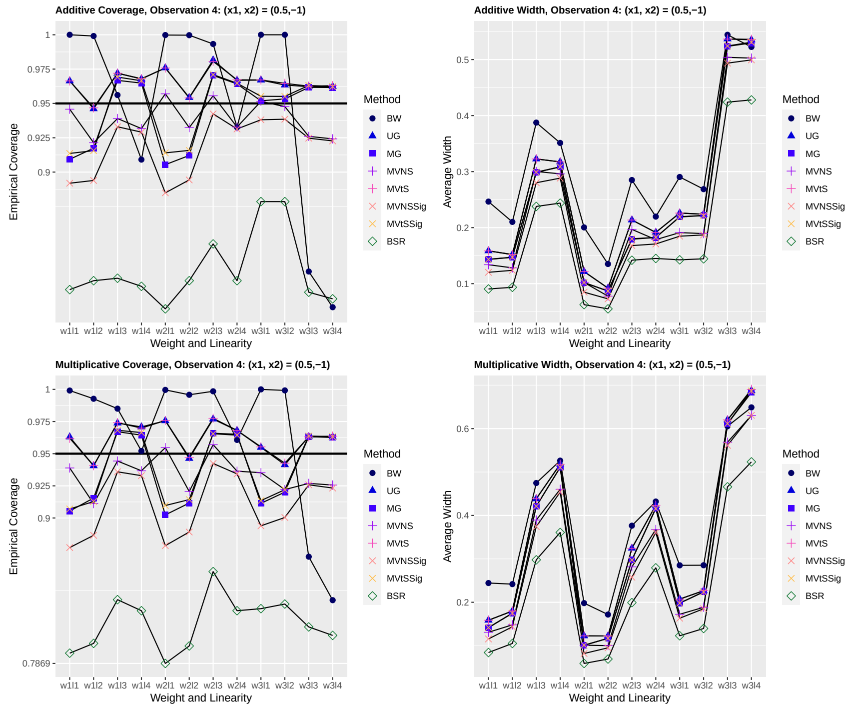


Figure 86. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 4

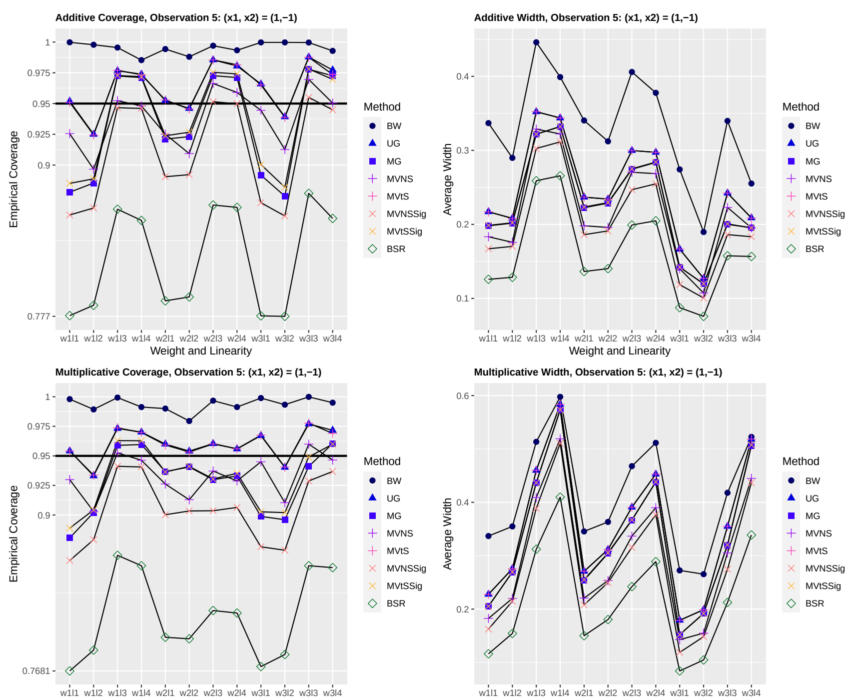


Figure 87. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 5

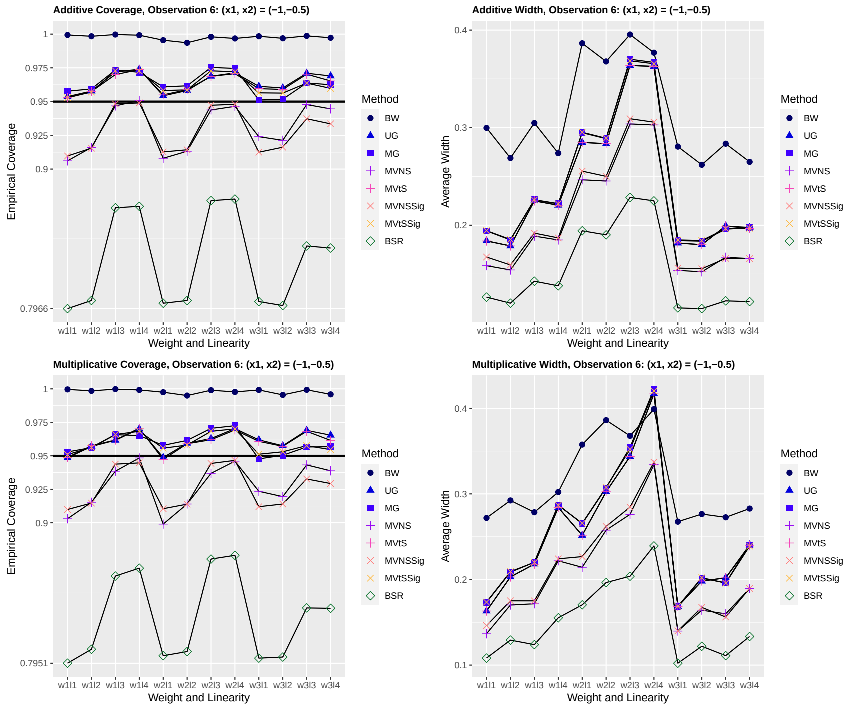


Figure 88. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 6

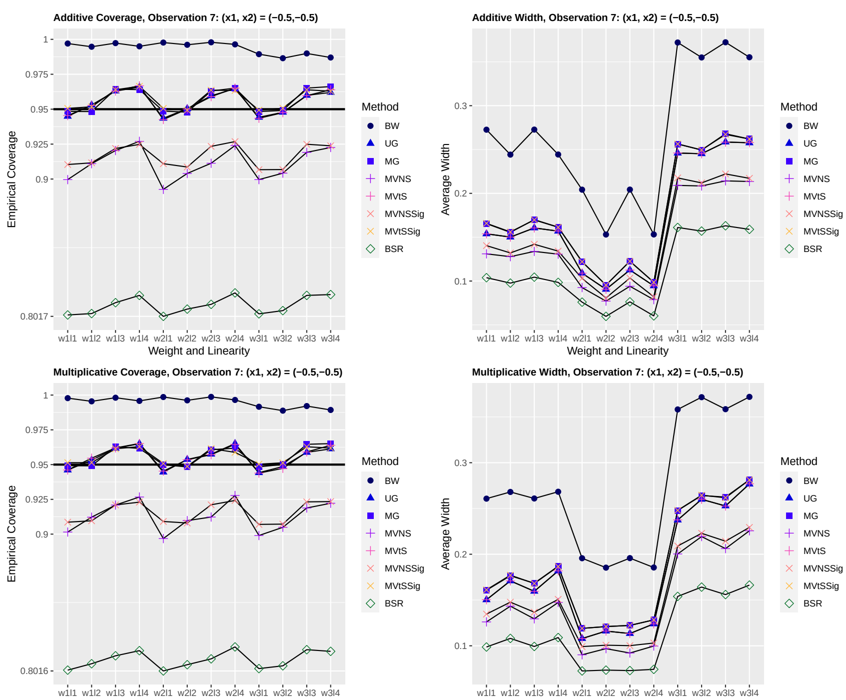


Figure 89. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 7

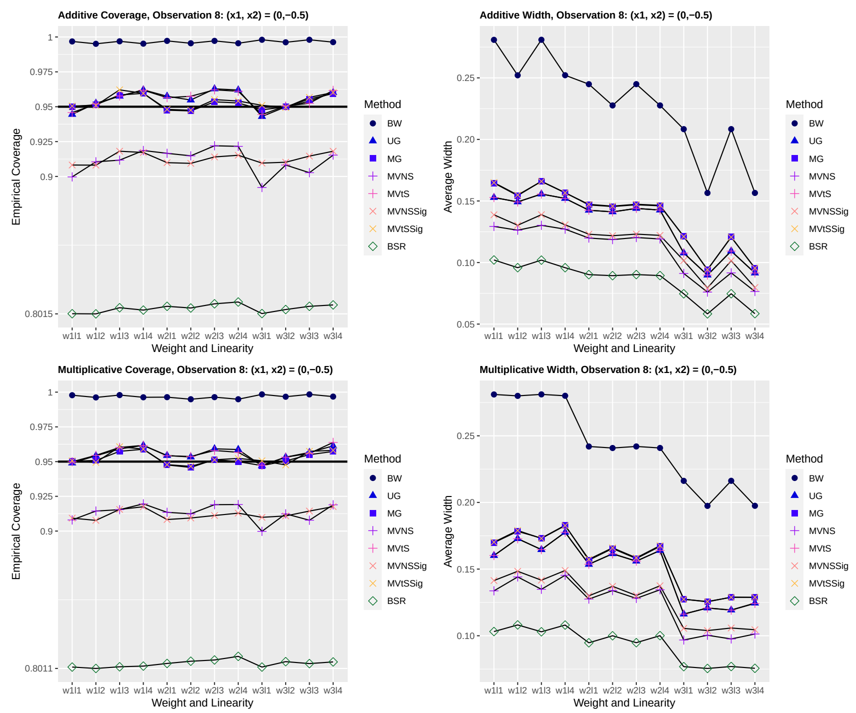


Figure 90. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 8

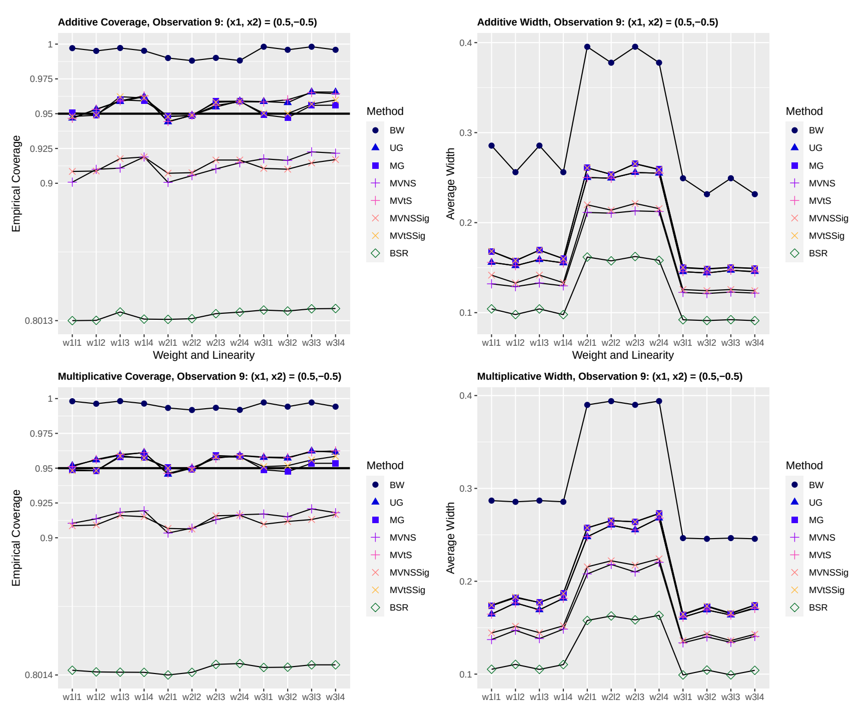


Figure 91. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 9

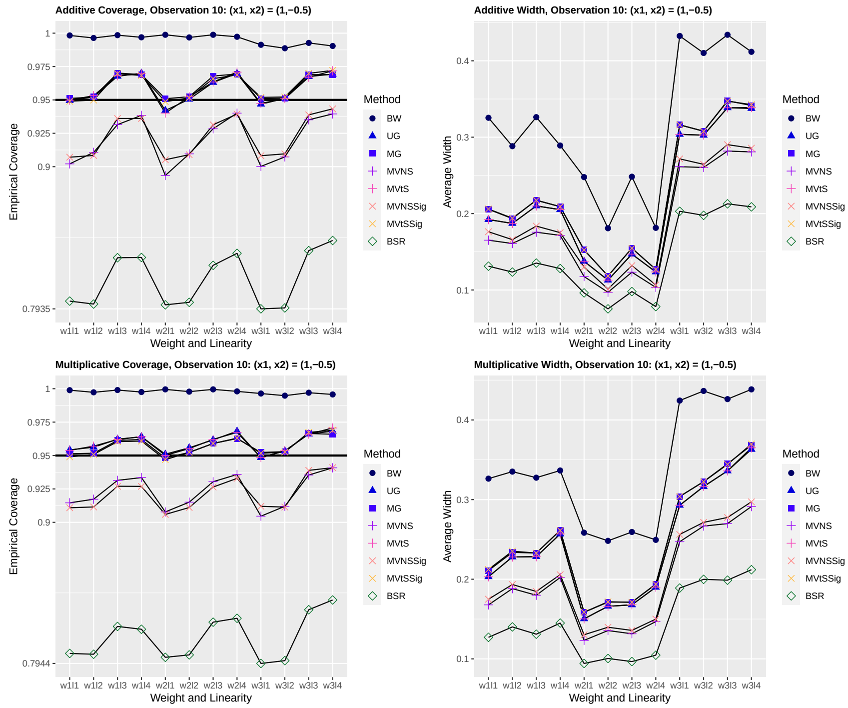


Figure 92. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 10

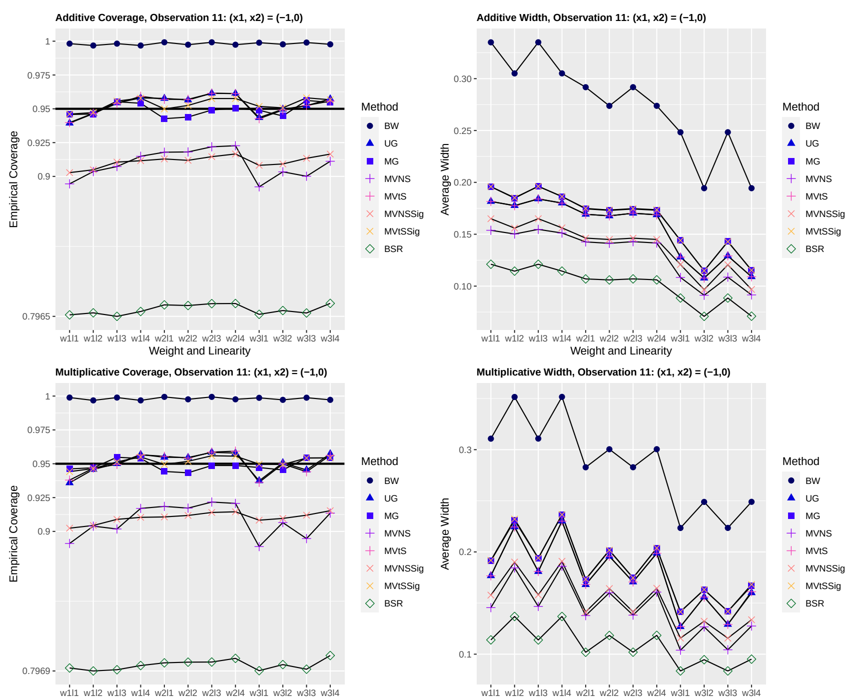


Figure 93. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 11

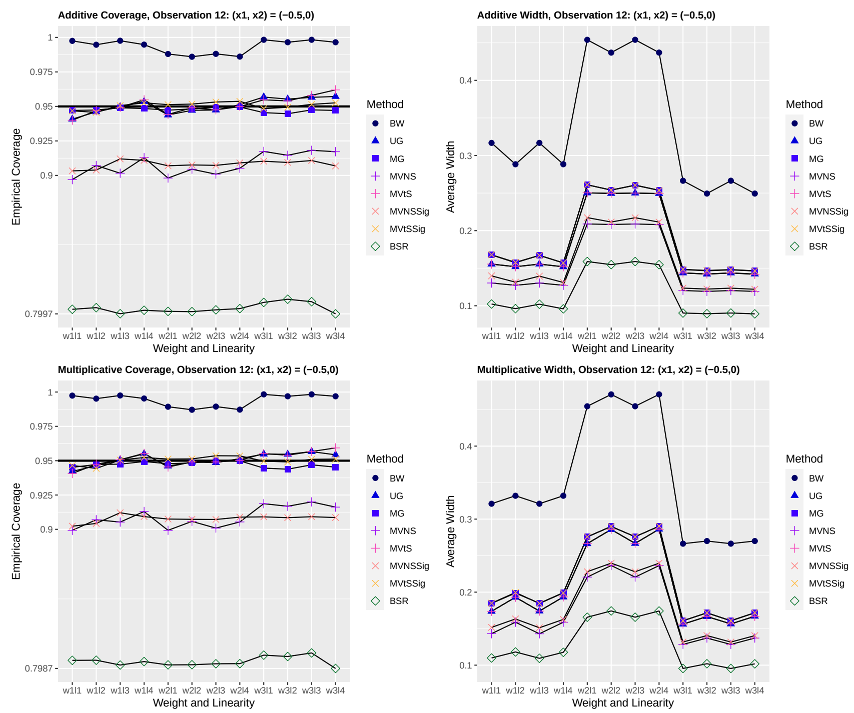


Figure 94. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 12

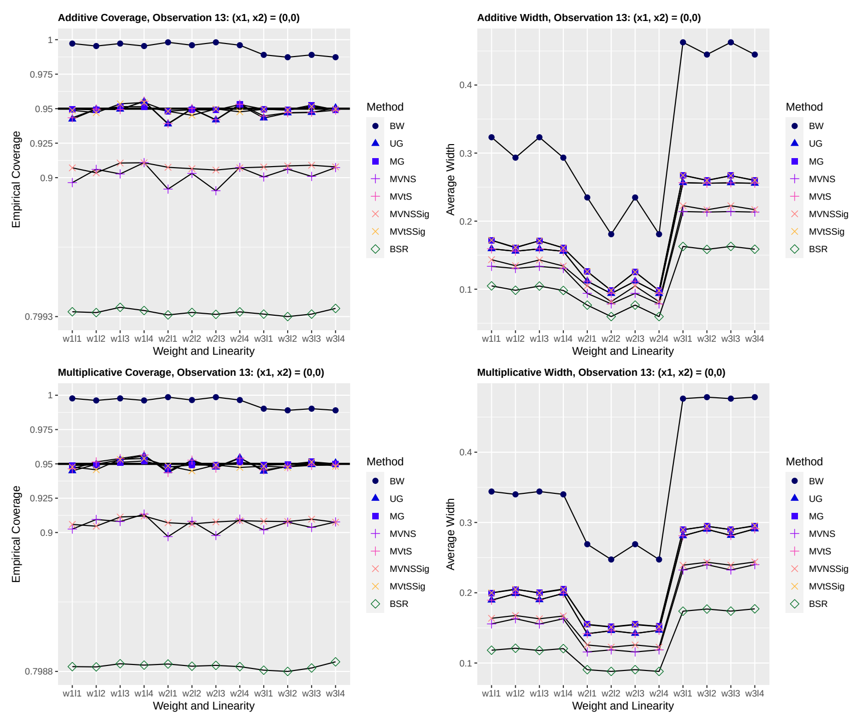


Figure 95. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 13

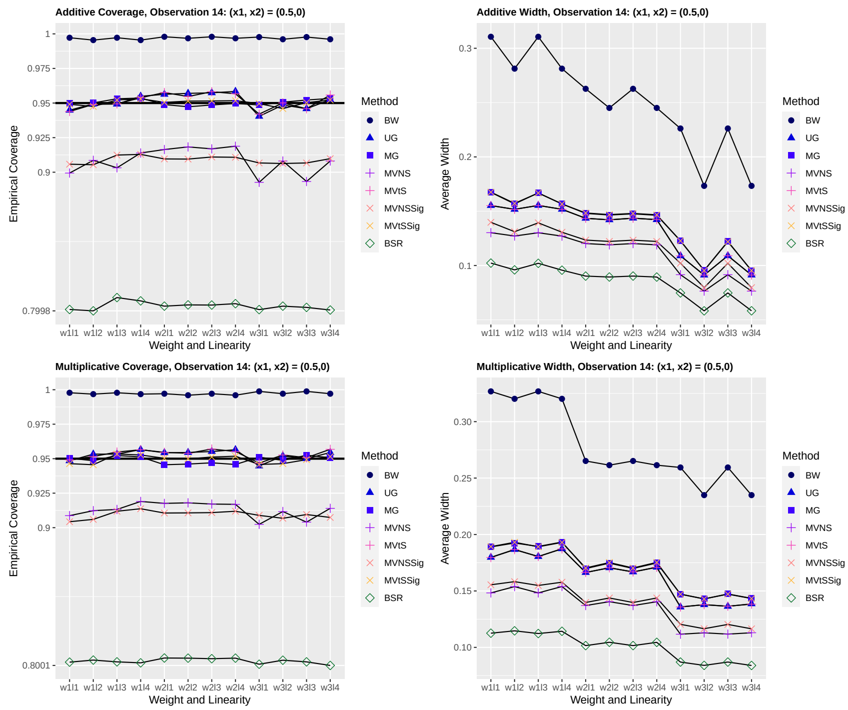


Figure 96. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 14

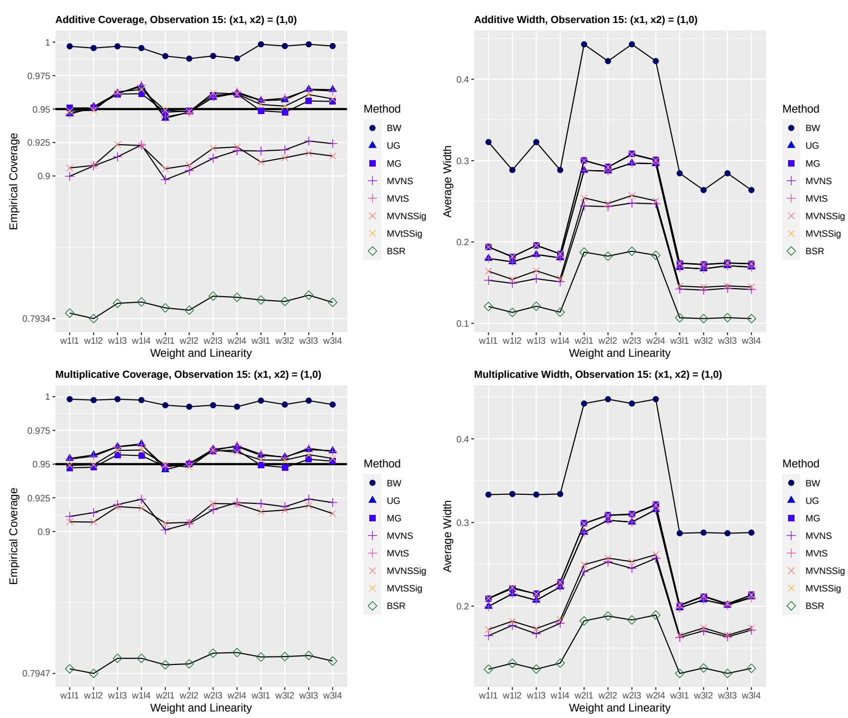


Figure 97. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 15

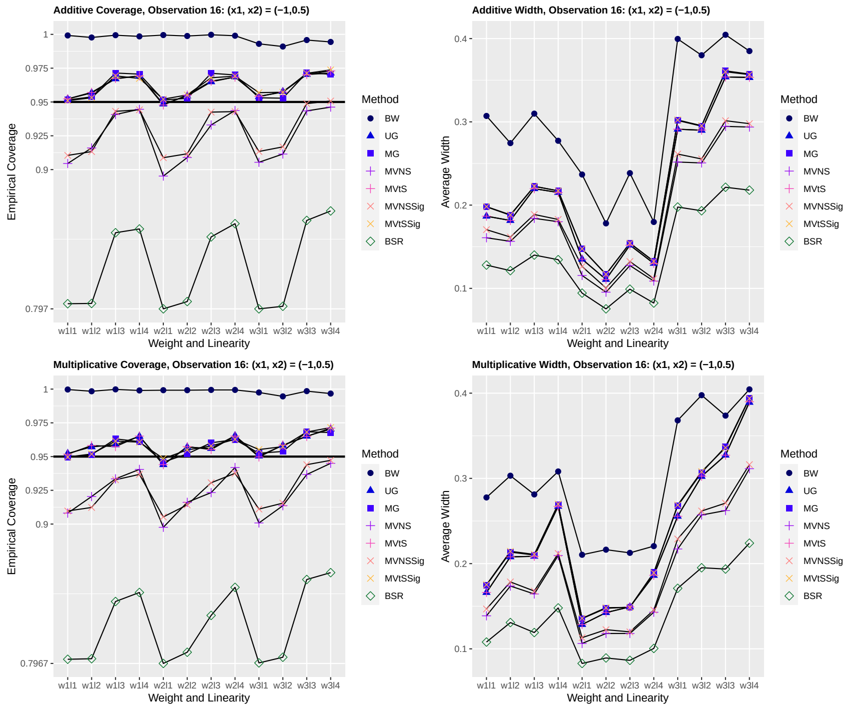


Figure 98. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 16

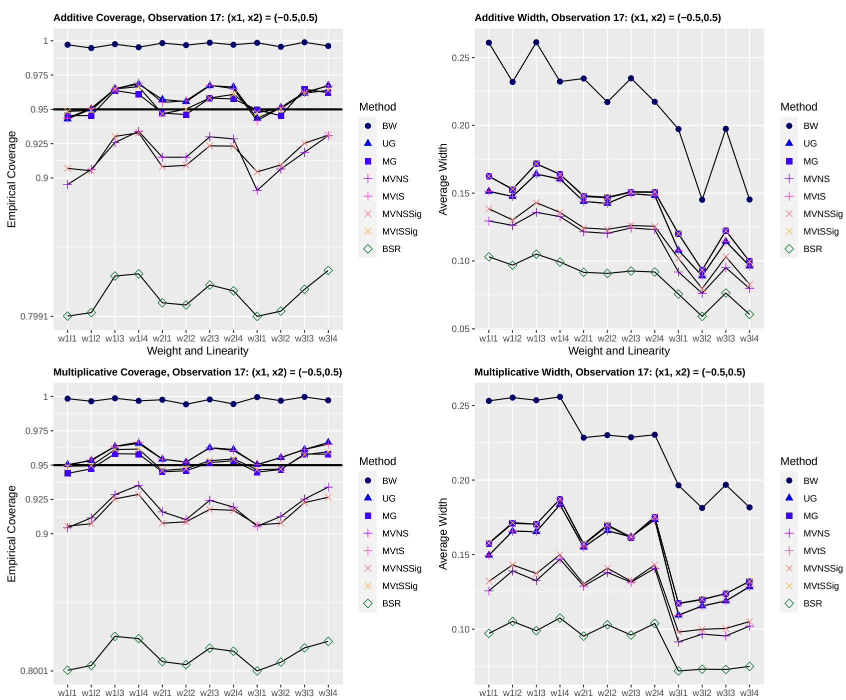


Figure 99. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 17

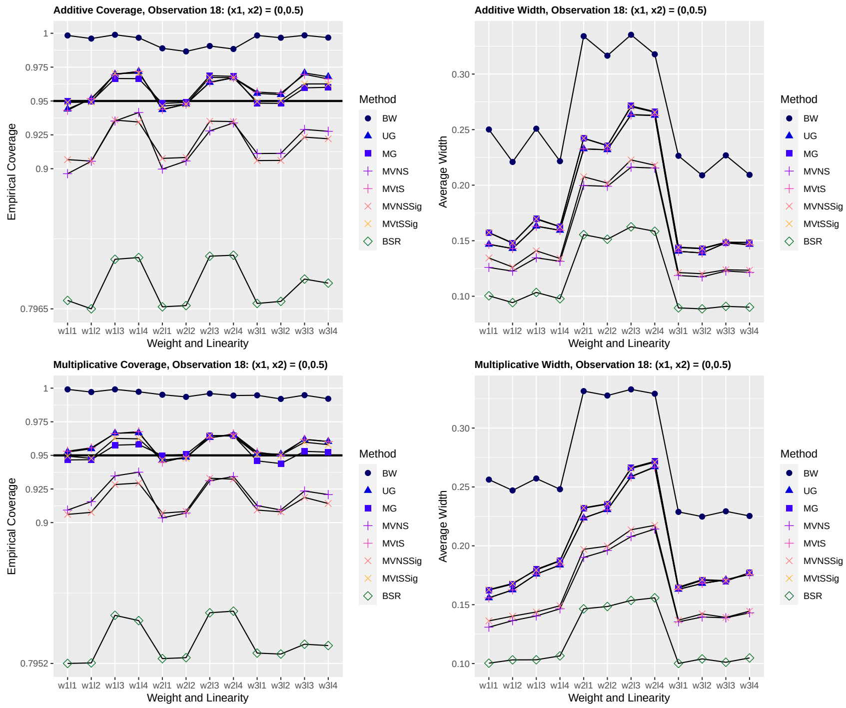


Figure 100. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 18

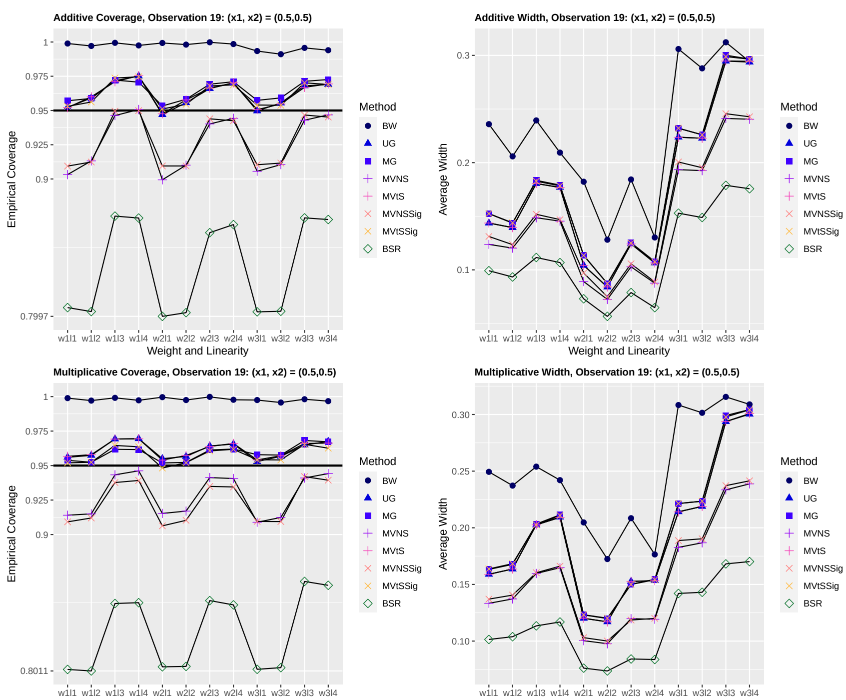


Figure 101. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 19

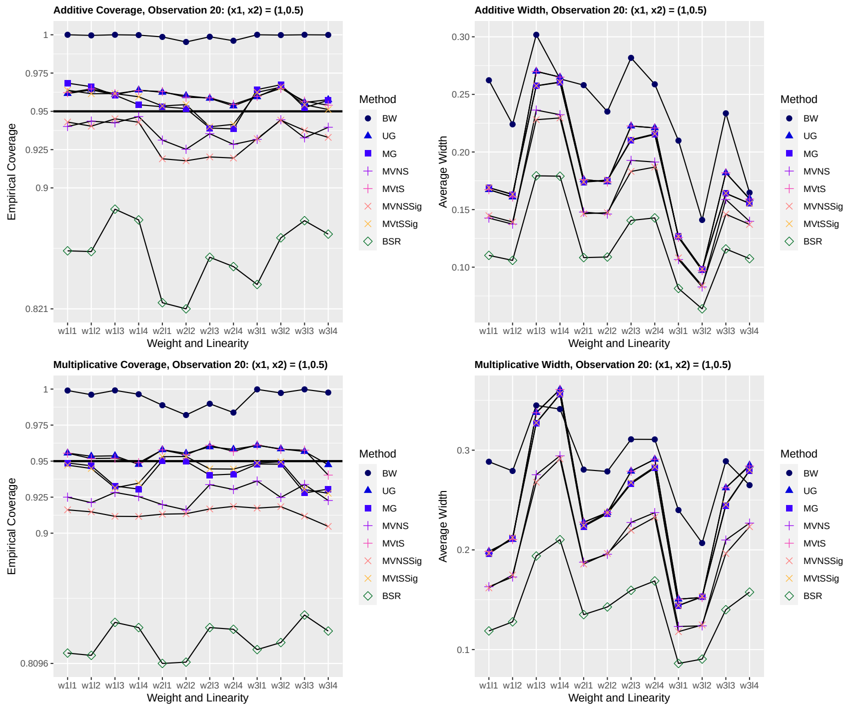


Figure 102. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 20

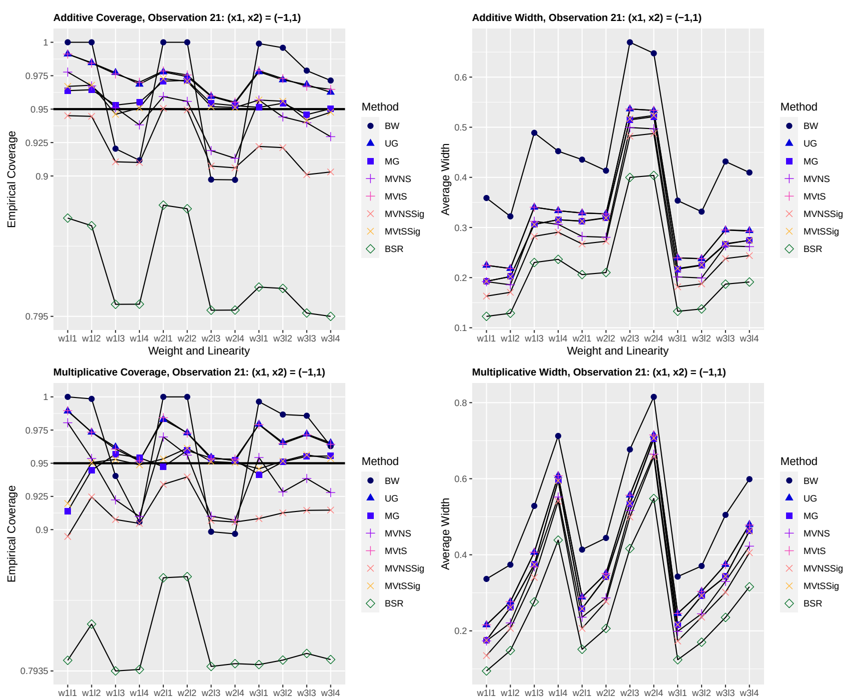


Figure 103. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 21

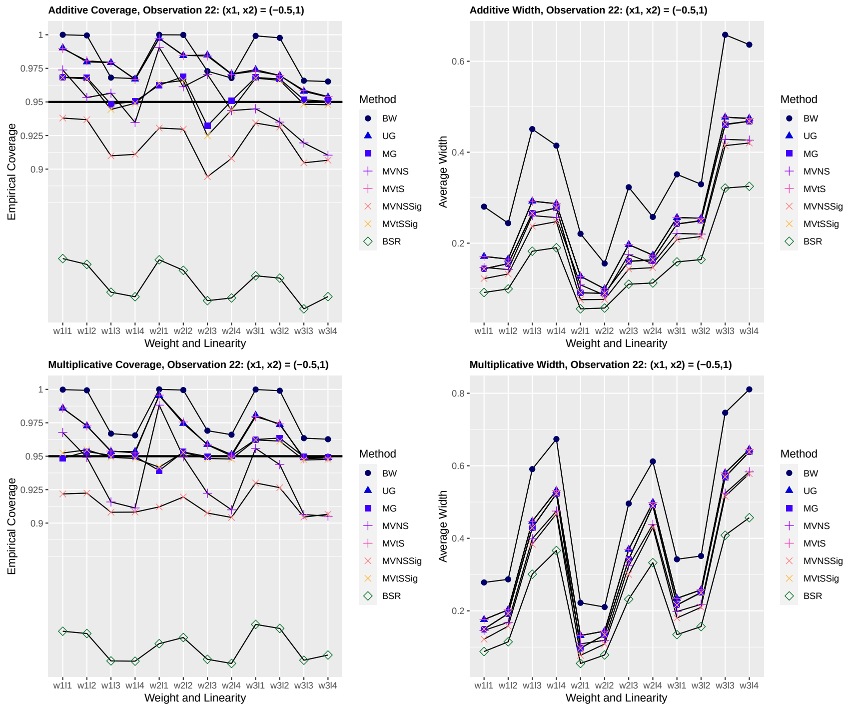


Figure 104. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 22

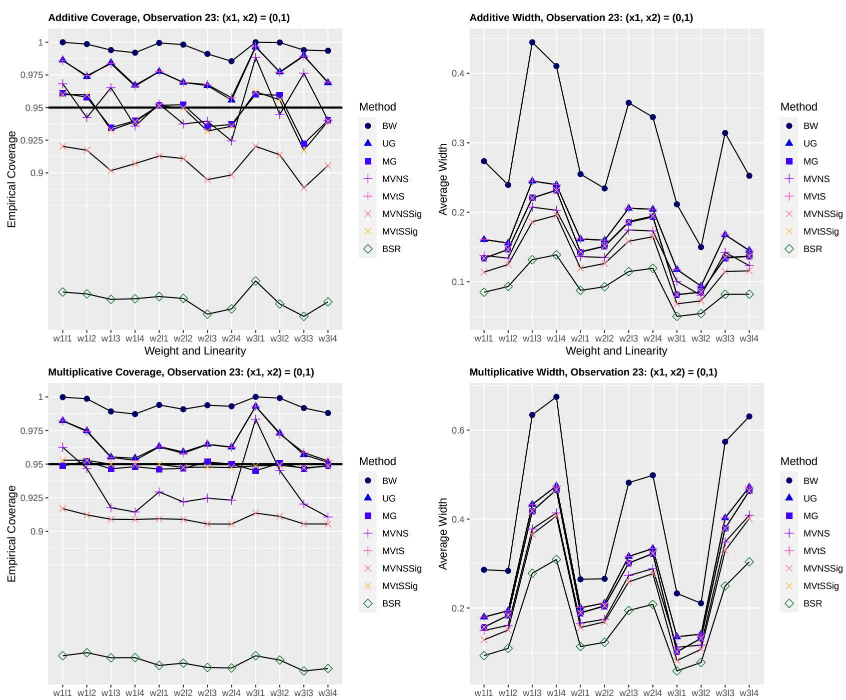


Figure 105. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 23

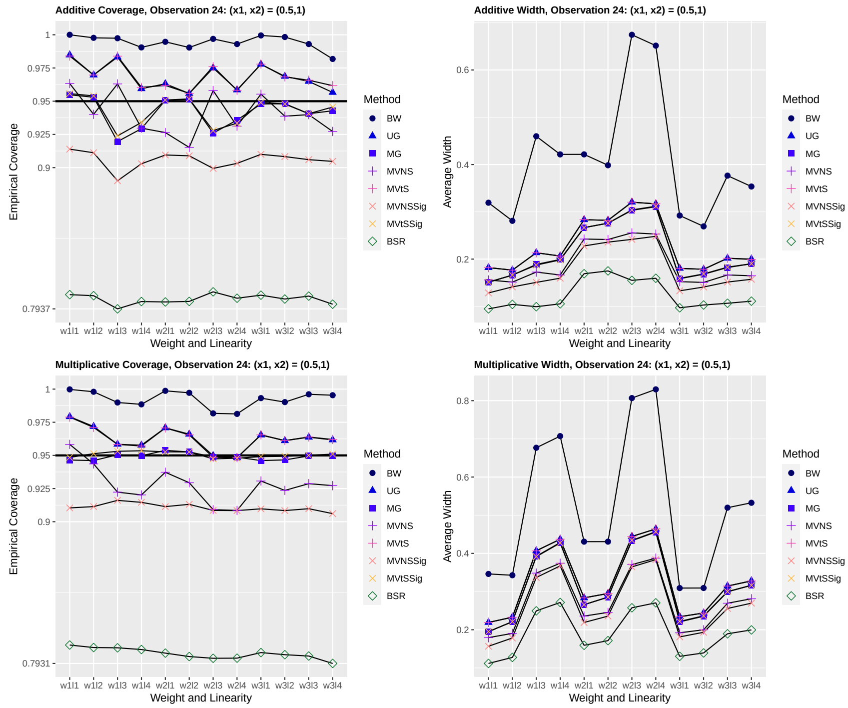


Figure 106. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 24

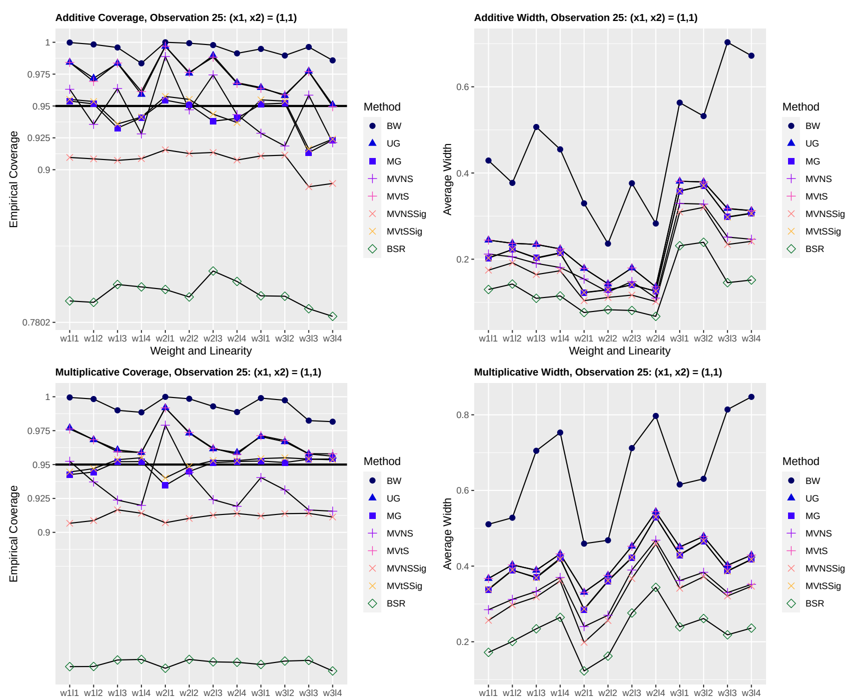


Figure 107. Additive and Multiplicative Plots for Empirical Coverage and Average Width at Observation 25

D.2 Average Symmetry

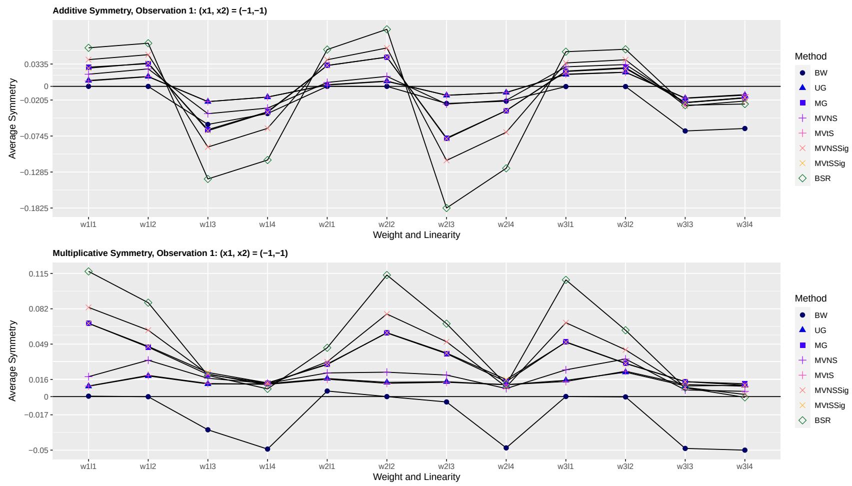


Figure 108. Additive and Multiplicative Plots for Average Symmetry at Observation 1

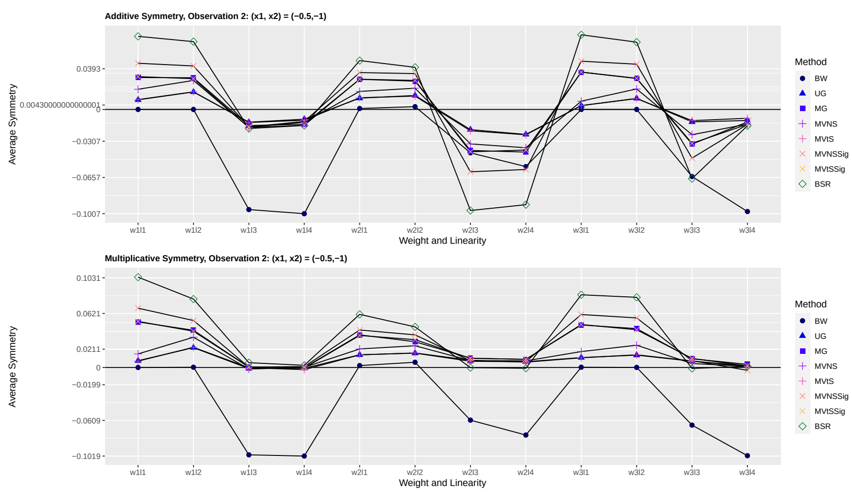


Figure 109. Additive and Multiplicative Plots for Average Symmetry at Observation 2

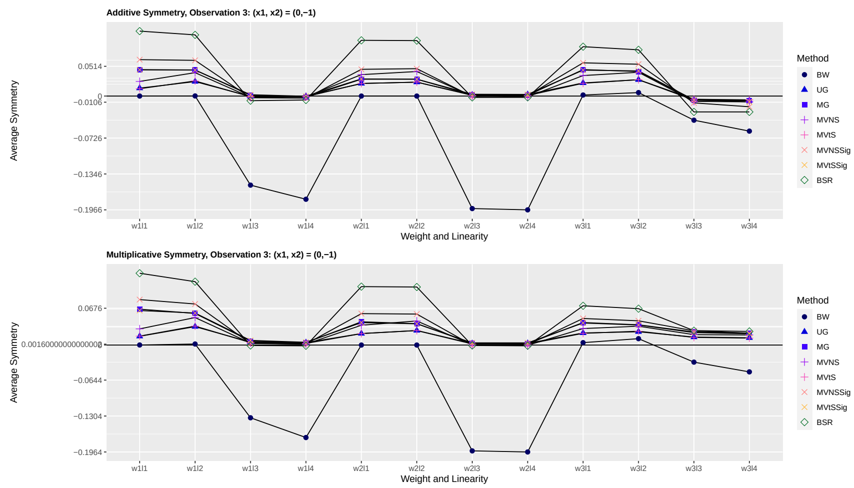


Figure 110. Additive and Multiplicative Plots for Average Symmetry at Observation 3

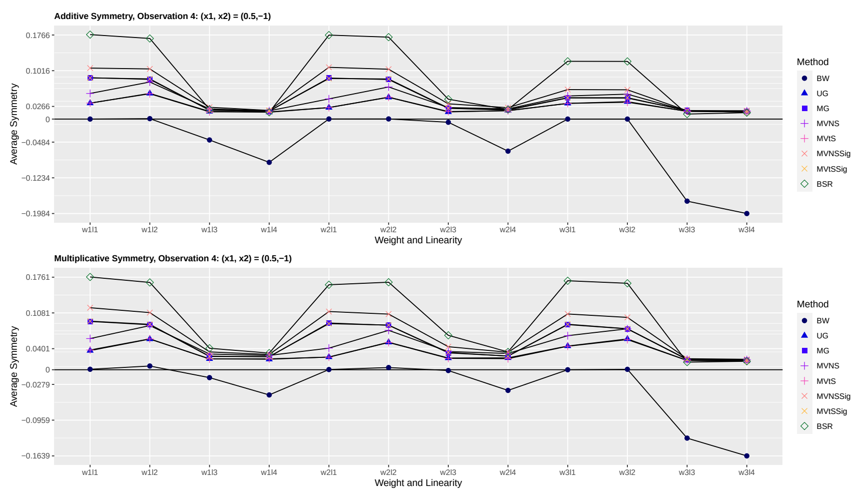


Figure 111. Additive and Multiplicative Plots for Average Symmetry at Observation 4

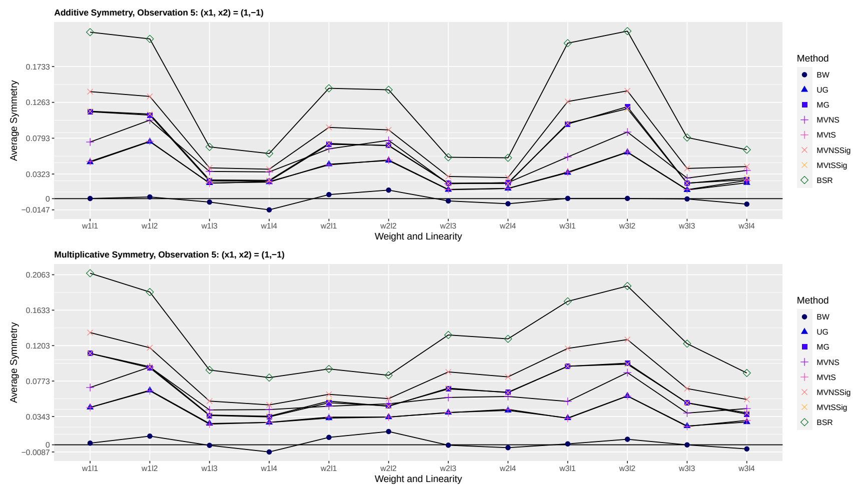


Figure 112. Additive and Multiplicative Plots for Average Symmetry at Observation 5

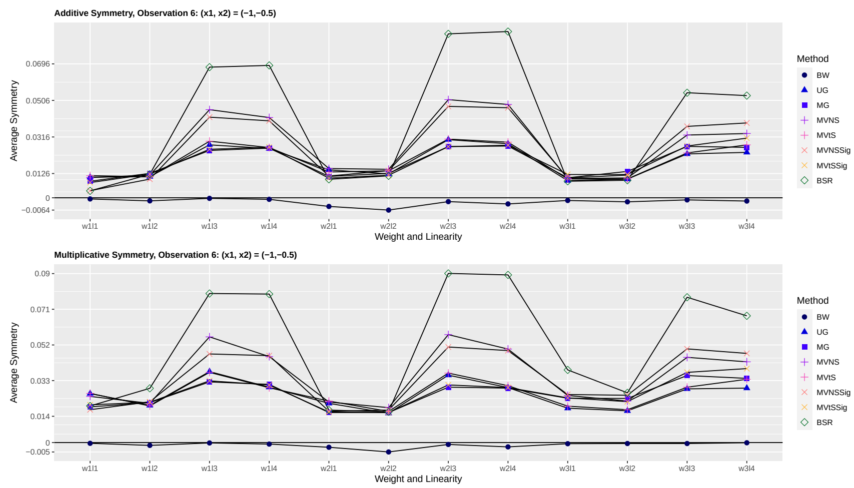


Figure 113. Additive and Multiplicative Plots for Average Symmetry at Observation 6

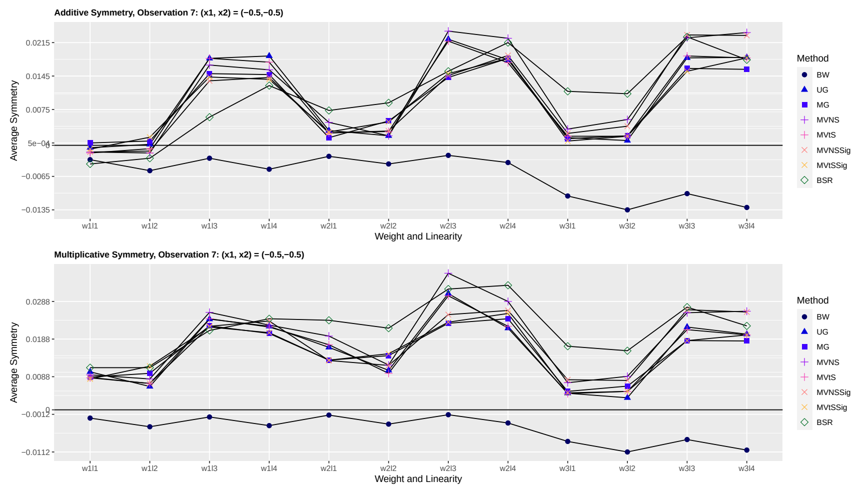


Figure 114. Additive and Multiplicative Plots for Average Symmetry at Observation 7

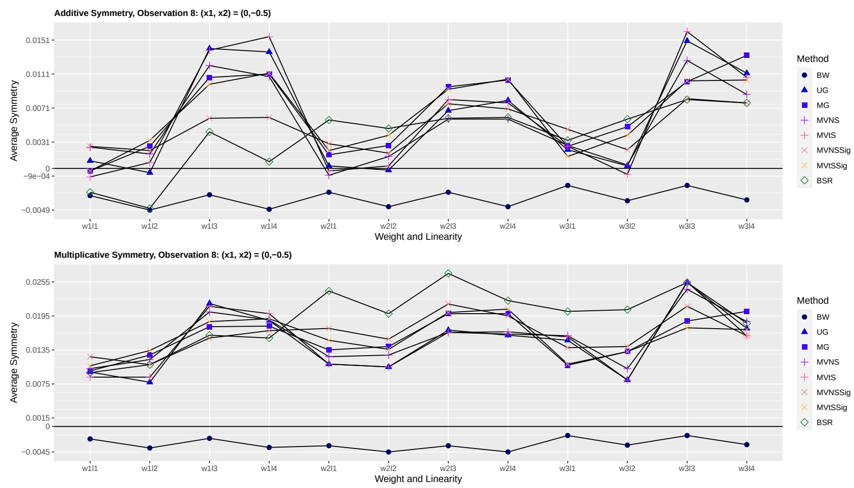


Figure 115. Additive and Multiplicative Plots for Average Symmetry at Observation 8

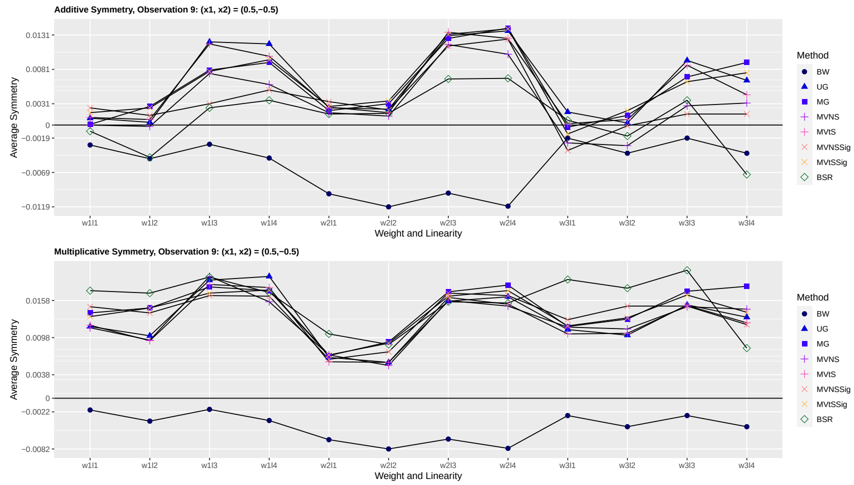


Figure 116. Additive and Multiplicative Plots for Average Symmetry at Observation 9

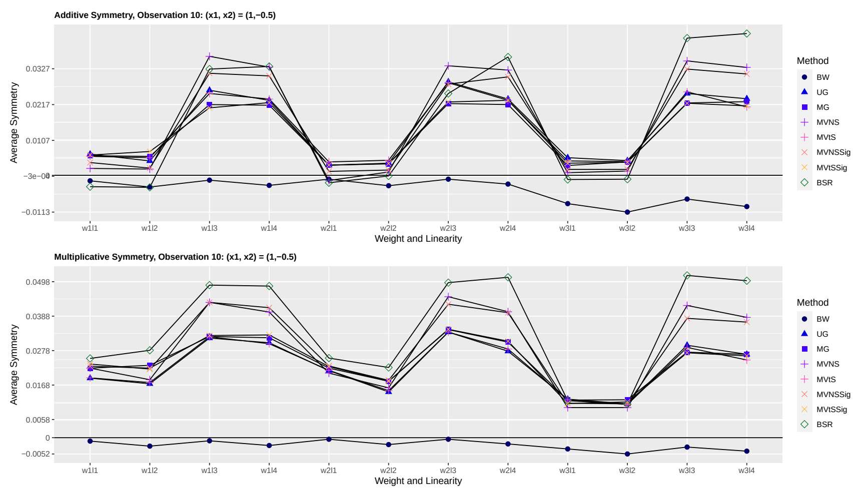


Figure 117. Additive and Multiplicative Plots for Average Symmetry at Observation 10

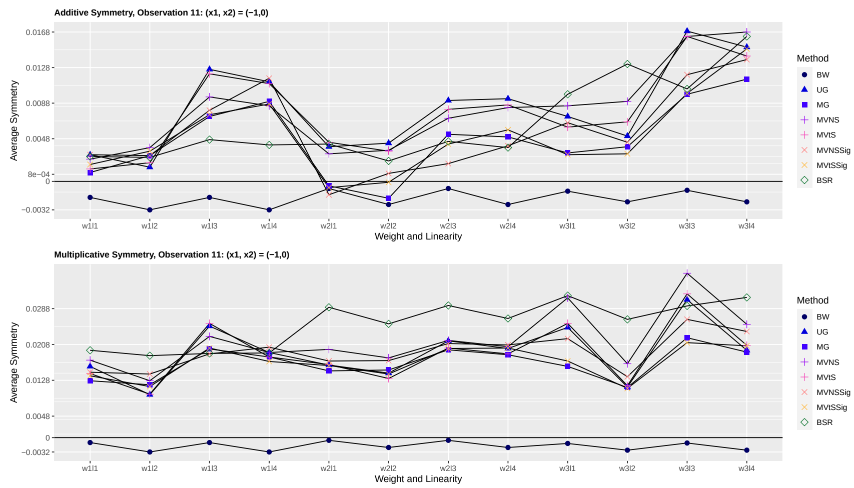


Figure 118. Additive and Multiplicative Plots for Average Symmetry at Observation 11

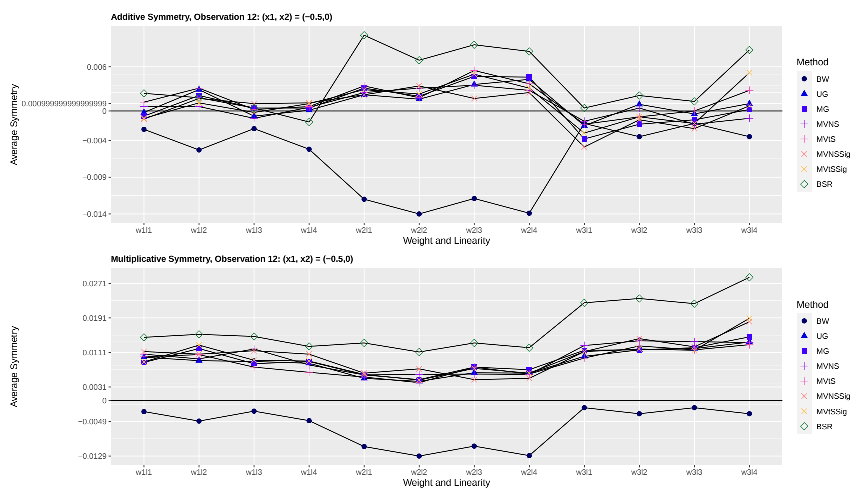


Figure 119. Additive and Multiplicative Plots for Average Symmetry at Observation 12

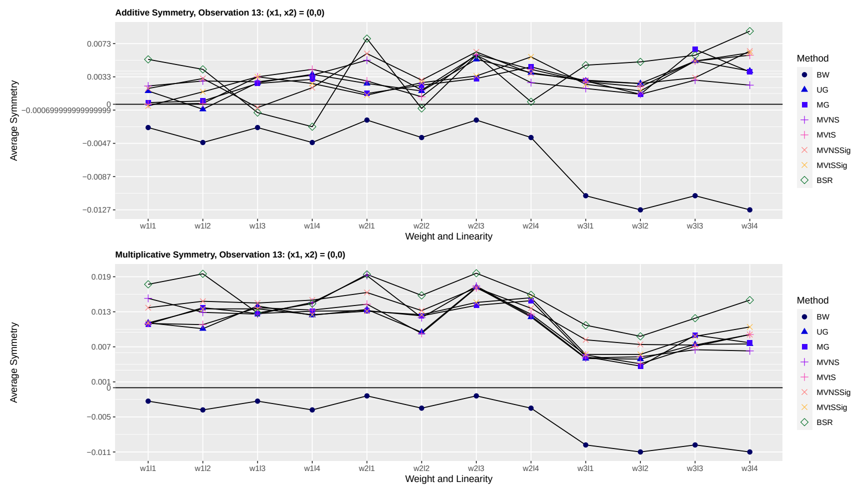


Figure 120. Additive and Multiplicative Plots for Average Symmetry at Observation 13

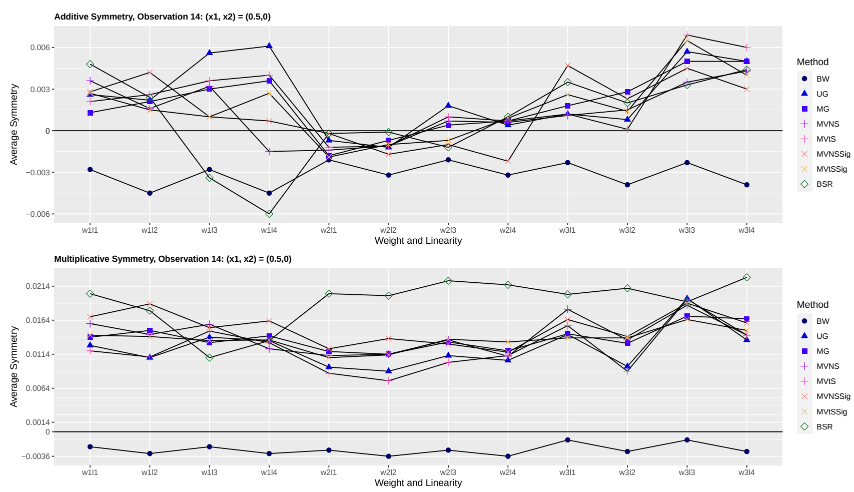


Figure 121. Additive and Multiplicative Plots for Average Symmetry at Observation 14

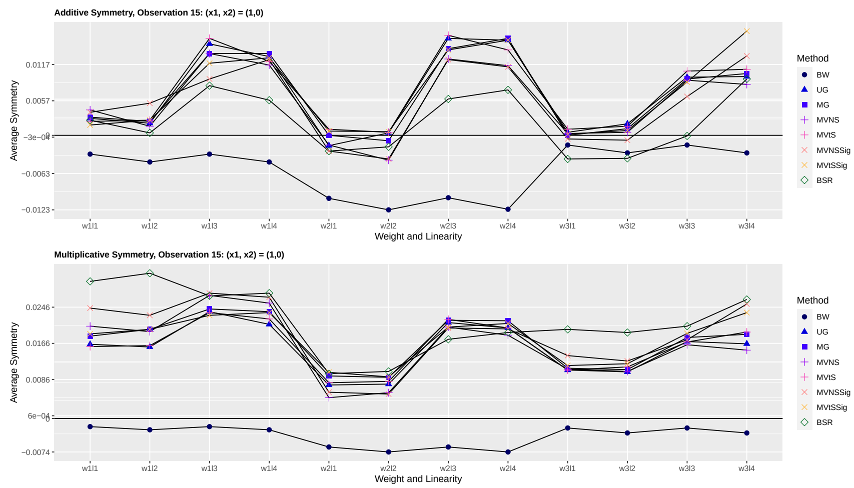


Figure 122. Additive and Multiplicative Plots for Average Symmetry at Observation 15

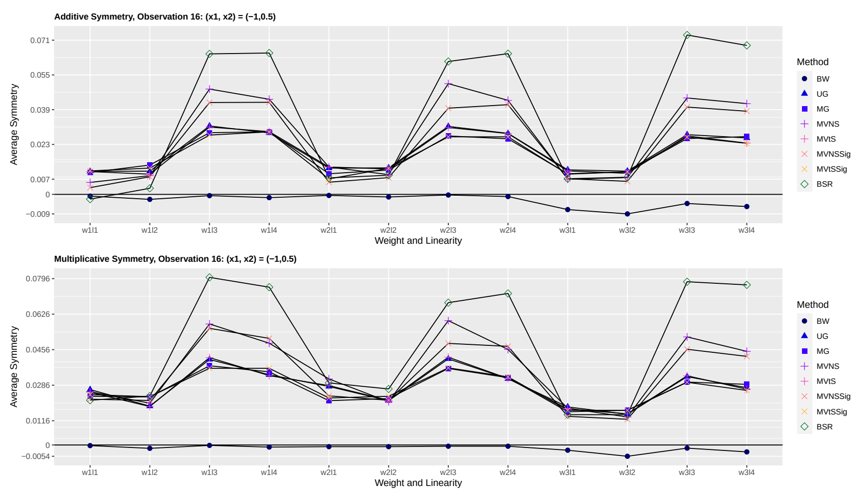


Figure 123. Additive and Multiplicative Plots for Average Symmetry at Observation 16

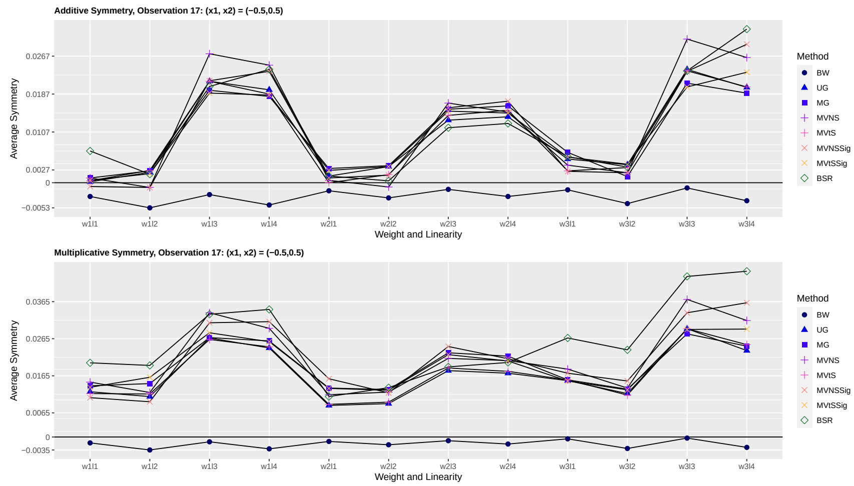


Figure 124. Additive and Multiplicative Plots for Average Symmetry at Observation 17

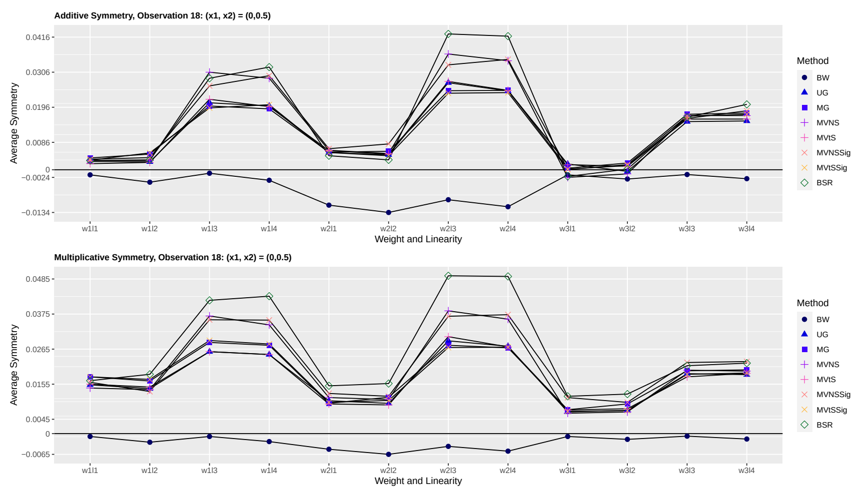


Figure 125. Additive and Multiplicative Plots for Average Symmetry at Observation 18

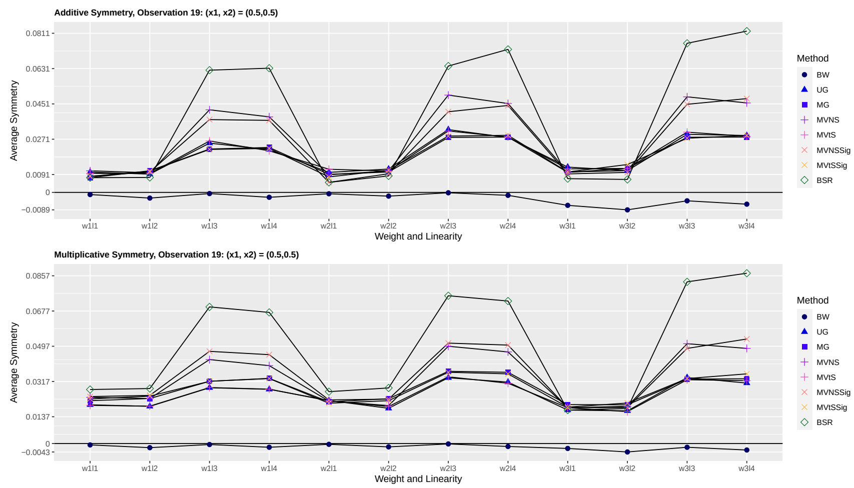


Figure 126. Additive and Multiplicative Plots for Average Symmetry at Observation 19

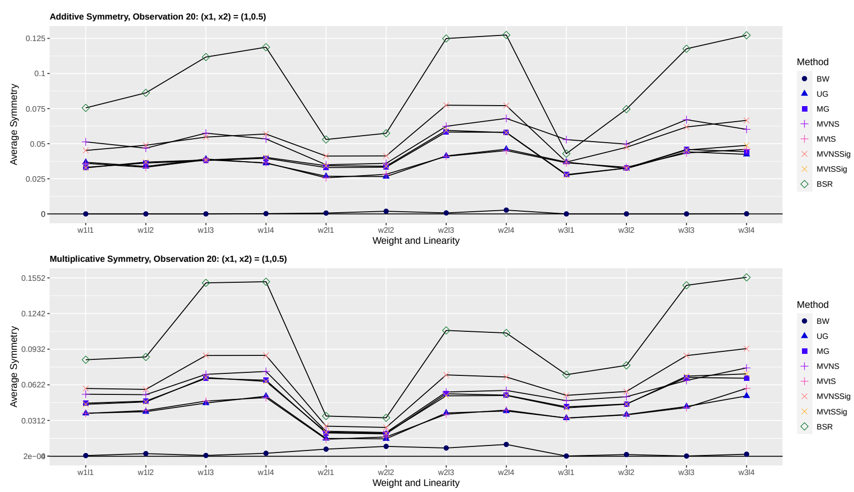


Figure 127. Additive and Multiplicative Plots for Average Symmetry at Observation 20

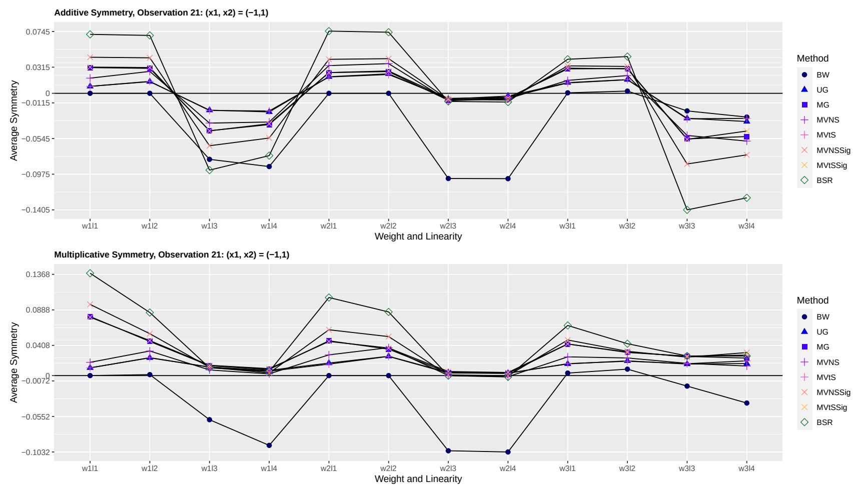


Figure 128. Additive and Multiplicative Plots for Average Symmetry at Observation 21

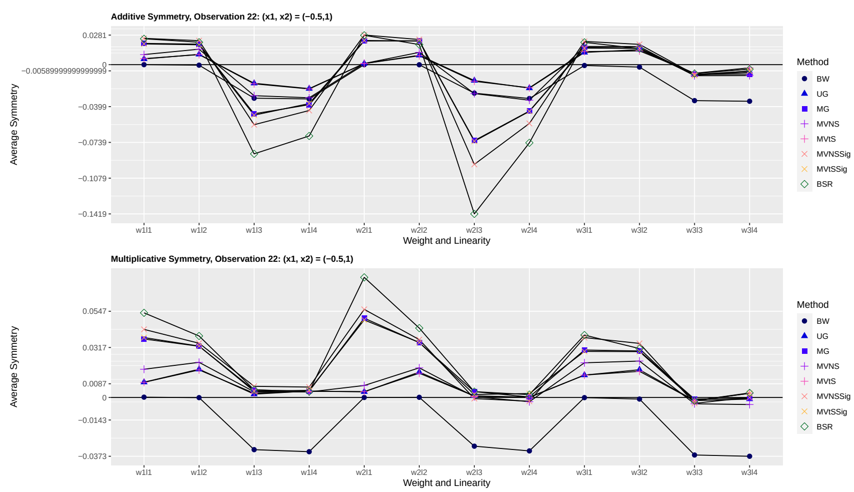


Figure 129. Additive and Multiplicative Plots for Average Symmetry at Observation 22

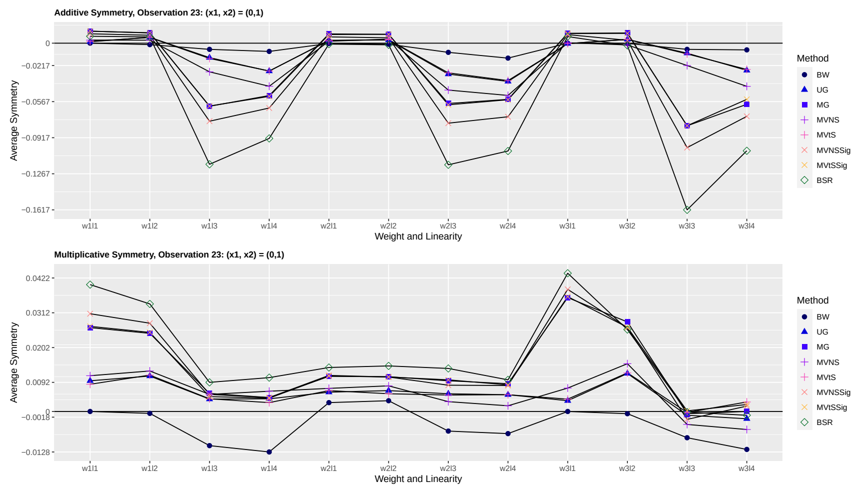


Figure 130. Additive and Multiplicative Plots for Average Symmetry at Observation 23

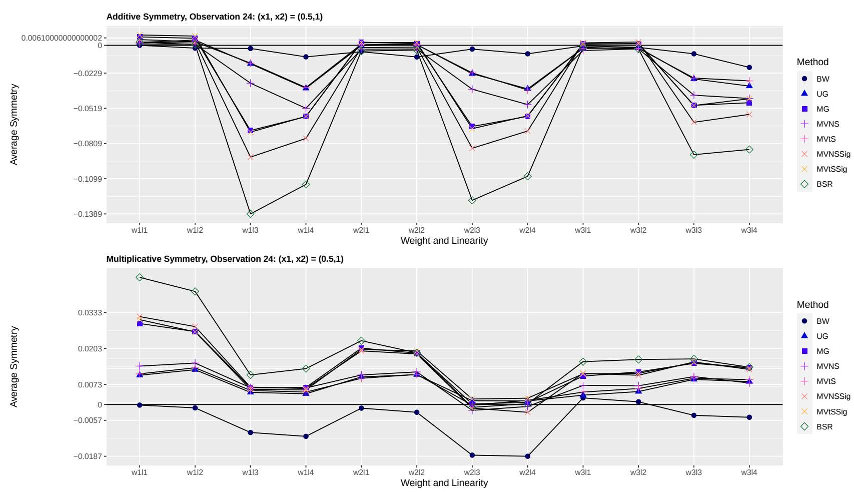


Figure 131. Additive and Multiplicative Plots for Average Symmetry at Observation 24

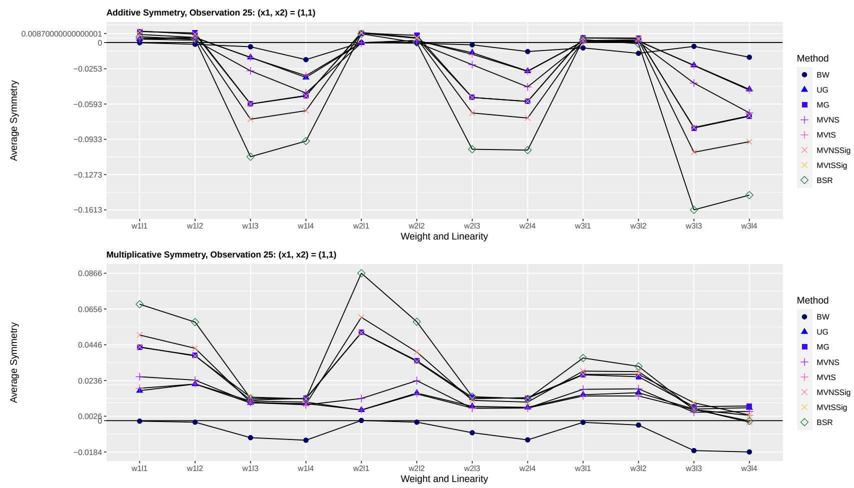


Figure 132. Additive and Multiplicative Plots for Average Symmetry at Observation 25

Appendix E. R Code

```r
library(rsm)
library(mvtnorm)
library(matlib)
library(doParallel)
library(parallel)
library(data.table)
set.seed(1985)

# Import Data (RSM is from Myers_2016)
data.RSM <- read.csv("dataset/RSM_Data.csv", header = TRUE)
alpha <- 0.05

# Form Y and X Data Frames from Data
Y <- data.frame(Y1 = data.RSM$Y1, Y2 = data.RSM$Y2, Y3 = data.RSM$Y3)
X <- data.frame(int = c(replicate(nrow(data.RSM),1)),
                X1 = data.RSM$x1,
                X2 = data.RSM$x2,
                X1X2 = data.RSM$x1*data.RSM$x2,
                X1sq = data.RSM$x1^2,
                X2sq = data.RSM$x2^2)

X.mat <- as.matrix(X)
XpX <- t(X.mat)%*%X.mat
XpXi <- solve(XpX)

grid <- expand.grid(
  seq(-1.5, 1.5, 0.5),
  seq(-1.5, 1.5, 0.5))
grid <- grid[which(grid[,1]^2 + grid[,2]^2 <= 2.0000001),]
rownames(grid) <- c(1:nrow(grid))
colnames(grid) <- c("x1", "x2")
grid.mat <- as.matrix(grid)
grid$Solution <- c(1:nrow(grid))
grid_all <- data.frame(int = rep(1, nrow(grid.mat)), grid.mat, x1x2 = grid[,1]*grid[,2], x1sq = grid[,1]^2, x2sq = grid[,2]^2)
H <- as.matrix(grid_all)%*%solve(XpX)%*%t(as.matrix(grid_all))
hat <- diag(H)

#Develop linear models based on data; Know from past studies that Yield&Visc are SO, MoleWeight is FO but leaving as SO for S_e.
Y1.lm <- rsm(Y1 ~ SO(x1, x2), data = data.RSM)
Y2.lm <- rsm(Y2 ~ SO(x1, x2), data = data.RSM)
Y3.lm <- rsm(Y3 ~ SO(x1, x2), data = data.RSM)
all.lm <- rsm(cbind(Y1,Y2,Y3) ~ SO(x1, x2), data = data.RSM)
MSE1 <- anova(Y1.lm)['Residuals', 'Mean Sq']
MSE2 <- anova(Y2.lm)['Residuals', 'Mean Sq']
MSE3 <- anova(Y3.lm)['Residuals', 'Mean Sq']
df1 <- anova(Y1.lm)['Residuals', 'Df']
df2 <- anova(Y2.lm)['Residuals', 'Df']
df3 <- anova(Y3.lm)['Residuals', 'Df']

Yhat <- data.frame(Yhat1 = Y1.lm$fitted.values,
```

```
                    Yhat2 = Y2.lm$fitted.values,
                    Yhat3 = Y3.lm$fitted.values)
S_e <- as.matrix(t(Y-Yhat))%*%as.matrix(Y-Yhat)/(df1)

# Form 'TRUE' equations for regression; Based on cosine of angle between planes which is equivalent to correlation
Beta1.true <- data.frame(b1 = c(80,1,0.5,0.25,-1.4,-1),
                         b2 = c(70, -0.2, -1, -1.25, -0.7, -7),
                         b3 = c(3400, 200,177,-80,-40,58))
Beta.true <- list(Beta1.true)

# ## Check correlation/angle of surfaces
# a12 <- angle(Beta1.true$b1[-1], Beta1.true$b2[-1])
# a13 <- angle(Beta1.true$b1[-1], Beta1.true$b3[-1])
# a23 <- angle(Beta1.true$b2[-1], Beta1.true$b3[-1])
# a <- data.frame(a12, a13, a23)
# cos(a/180*pi)
# cors <- round(cos(a/180*pi),2)
# ##
Sigma.true <- round(S_e,3)
cov2cor(Sigma.true)

# Generate new data based on 'TRUE' equations
Y.true <- lapply(1:length(Beta.true), function(x) data.frame(X.mat %*% as.matrix(Beta.true[[x]])))
Y.true <- lapply(Y.true,setNames, nm = colnames(Y))
lapply(1:length(Y.true), function(x) cor(Y.true[[x]]))

YX.true <- Y.true
YX.true <- lapply(YX.true, function(x) transform(x, x1 = X$X1))
YX.true <- lapply(YX.true, function(x) transform(x, x2 = X$X2))

Y.all <- lapply(1:length(Beta.true), function(x) data.frame(as.matrix(grid_all) %*% as.matrix(Beta.true[[x]])))
Y.all <- lapply(Y.all,setNames, nm = colnames(Y))
Y.all <- lapply(Y.all, transform, x1 = grid_all$x1)
Y.all <- lapply(Y.all, transform, x2 = grid_all$x2)

# Fit Y.data2 TRUE "regression" models for plotting and such
Y1true.lm <- lapply(1:length(YX.true), function(x) rsm(Y1 ~ SO(x1, x2), data = YX.true[[x]]))
Y2true.lm <- lapply(1:length(YX.true), function(x) rsm(Y2 ~ SO(x1, x2), data = YX.true[[x]]))
Y3true.lm <- lapply(1:length(YX.true), function(x) rsm(Y3 ~ SO(x1, x2), data = YX.true[[x]]))

par(mfrow = c(2,3))
index <- 1
contour(Y1true.lm[[index]], ~ x1 + x2, image = TRUE, main=paste("Y1 Contour"))
contour(Y2true.lm[[index]], ~ x1 + x2, image = TRUE, main=paste("Y2 Contour"))
contour(Y3true.lm[[index]], ~ x1 + x2, image = TRUE, main=paste("Y3 Contour"))
persp(Y1true.lm[[index]], x2 ~ x1, zlab = "y", main=paste("Y1 Perspective"))
persp(Y2true.lm[[index]], x2 ~ x1, zlab = "y", main=paste("Y2 Perspective"))
persp(Y3true.lm[[index]], x2 ~ x1, zlab = "y", main=paste("Y3 Perspective"))


w# Generate Data -- g sampled regression models
g <- 10000
gblocked <- g/100 # This and blocks matter more when g is large (10000) to avoid memory issues, gblocked*blocks
```

```
should equal g
blocks <- g/gblocked

# numCores <- 1 # Use for Windows OS, Do not recommend using windows instead of linux due to large computations
numCores <- detectCores() - 1 # Can be used for Linux OS
Ysim.data <- mclapply(1:g, function(y)
  lapply(1:length(Y.true), function(x)
    data.frame(Y.true[[x]] + rmvnorm(nrow(Y.true[[x]]),
                                     mean = rep(0,ncol(Y.true[[x]])),
                                     sigma = Sigma.true),
               x1 = X$X1,
               x2 = X$X2)),
  mc.cores = numCores)
Y1sim.lm <- mclapply(Ysim.data, function(y) lapply(y, function(x) lm(Y1 ~ SO(x1, x2), data = x)),mc.cores = numCores)
Y2sim.lm <- mclapply(Ysim.data, function(y) lapply(y, function(x) lm(Y2 ~ SO(x1, x2), data = x)),mc.cores = numCores)
Y3sim.lm <- mclapply(Ysim.data, function(y) lapply(y, function(x) lm(Y3 ~ SO(x1, x2), data = x)),mc.cores = numCores)
MSEsim1 <- mclapply(Y1sim.lm, function(y) lapply(y, function(x) anova(x)['Residuals', 'Mean␣Sq']),mc.cores = numCores)
MSEsim2 <- mclapply(Y2sim.lm, function(y) lapply(y, function(x) anova(x)['Residuals', 'Mean␣Sq']),mc.cores = numCores)
MSEsim3 <- mclapply(Y3sim.lm, function(y) lapply(y, function(x) anova(x)['Residuals', 'Mean␣Sq']),mc.cores = numCores)
nusim1 <- mclapply(Y1sim.lm, function(y) lapply(y, function(x) anova(x)['Residuals', 'Df']),mc.cores = numCores)
nusim2 <- mclapply(Y2sim.lm, function(y) lapply(y, function(x) anova(x)['Residuals', 'Df']),mc.cores = numCores)
nusim3 <- mclapply(Y3sim.lm, function(y) lapply(y, function(x) anova(x)['Residuals', 'Df']),mc.cores = numCores)
Ysim.pred <- mclapply(1:g, function(y) lapply(1:length(Beta.true), function(x) data.frame(Y1 = predict(Y1sim.lm[[y]][[x]], newdata = grid_all),
                                                                                        Y2 = predict(Y2sim.lm[[y]][[x]], newdata = grid_all),
                                                                                        Y3 = predict(Y3sim.lm[[y]][[x]], newdata = grid_all),
                                                                                        x1 = grid_all$x1,
                                                                                        x2 = grid_all$x2)), mc.cores = numCores)



MVYsim.lm <- mclapply(Ysim.data, function(y) lapply(y, function(x) lm(cbind(Y1, Y2, Y3) ~ SO(x1, x2), data = x)), mc.cores = numCores)
MVYsimnu <- mclapply(MVYsim.lm, function(y) lapply(y, function(x) x$df.residual), mc.cores = numCores)

MVMSEsim <- mclapply(MVYsim.lm, function(y) lapply(y, function(x) 1/(x$df.residual)*t(x$residuals)%*%x$residuals), mc.cores = numCores)

### Define Derringer and Suich Desirability Index
DS_Max <- function(tgt,L,r,x){
  try(if(tgt < L) stop("Target␣must␣be␣larger␣than␣lower␣bound,␣L."))
  values <- (x - L)/(tgt - L)
  values[values < 0] <- 0
```

```
  values[values > 1] <- 1
  values <- values^r
  return(values)
}
DS_Min <- function(tgt,U,r,x){
  try(if(tgt > U) stop("Target must be smaller than upper bound, U."))
  values <- (U - x)/(U - tgt)
  values[values < 0] <- 0
  values[values > 1] <- 1
  values <- values^r
  return(values)
}
DS_Target <- function(tgt,U,L,r1,r2,x){
  try(if(tgt > U || tgt < L) stop("Target must be between the lower and upper bounds, L and U."))
  X <- data.frame(as.matrix(x),seq(1:nrow(as.matrix(x))))
  B <- ncol(X)-1
  for(i in 1:B){
    values_L <- X[X[,i] <= tgt,c(i,B+1)]
    values_L[,1] <- (values_L[,1] - L)/(tgt - L)
    values_L[values_L[,1] < 0,1] <- 0
    values_L[,1] <- values_L[,1]^r1
    values_U <- X[X[,i] >= tgt,c(i,B+1)]
    values_U[,1] <- (U - values_U[,1])/(U - tgt)
    values_U[values_U[,1] < 0,1] <- 0
    values_U[,1] <- values_U[,1]^r2
    values <- rbind(values_L, values_U)
    values <- values[order(values[,2]),]
    X[,i] <- values[,1]
  }
  return(X[,-(B+1)])
}

DF_Formulas <- function(w, X, type){
  try(if(type != 'add' && type != 'mult') stop("Must specify 'add' or 'mult' for type."))
  try(if(type == 'mult' && sum(X < 0)) stop("Data values must be 0 or larger to use with .ultiplicative form"))
  ncol <- ncol(X)
  nrow <- nrow(X)
  weight <- as.data.frame(matrix(rep(w, nrow), nrow = nrow, ncol = ncol))
  if(type == 'add') {
    DF <- rowSums(X * weight)
  } else {
    DF <- apply(X ^ weight, 1, prod)
  }
  return(DF)
}

tquant <- qt(1-alpha/2, df = nrow(X) - ncol(X))
halfwidth1 <- lapply(1:length(Beta.true), function(x) tquant*sqrt(Sigma.true[1,1]*(hat + 1)))
halfwidth2 <- lapply(1:length(Beta.true), function(x) tquant*sqrt(Sigma.true[2,2]*(hat + 1)))
halfwidth3 <- lapply(1:length(Beta.true), function(x) tquant*sqrt(Sigma.true[3,3]*(hat + 1)))
hw1 <- mclapply(1:g, function(y) lapply(1:length(Beta.true), function(x) tquant*sqrt(MSEsim1[[y]][[x]]*(hat))),
     mc.cores = numCores)
hw2 <- mclapply(1:g, function(y) lapply(1:length(Beta.true), function(x) tquant*sqrt(MSEsim2[[y]][[x]]*(hat))),
     mc.cores = numCores)
```

```
hw3 <- mclapply(1:g, function(y) lapply(1:length(Beta.true), function(x) tquant*sqrt(MSEsim3[[y]][[x]]*(hat))),
      mc.cores = numCores)
Y1_tgt <- lapply(1:length(Beta.true), function(x) max(Y.all[[x]]$Y1 + halfwidth1[[x]]))
Y1_poor <- lapply(1:length(Beta.true), function(x) min(Y.all[[x]]$Y1 - halfwidth1[[x]]))
Y2_tgt <- 65
Y2_poor_L <- lapply(1:length(Beta.true), function(x) min(Y.all[[x]]$Y2 - halfwidth2[[x]]))
Y2_poor_U <- lapply(1:length(Beta.true), function(x) max(Y.all[[x]]$Y2 + halfwidth2[[x]]))
Y3_tgt <- lapply(1:length(Beta.true), function(x) min(Y.all[[x]]$Y3 - halfwidth3[[x]]))
Y3_poor <- lapply(1:length(Beta.true), function(x) max(Y.all[[x]]$Y3 + halfwidth3[[x]]))


## Create hyper grid for weight combinations of desirability function
# DFRange <- expand.grid(
#   seq(0, 1, 1/5),
#   seq(0, 1, 1/5),
#   seq(0, 1, 1/5)
# )
# DFRange <- DFRange[abs(rowSums(DFRange) - 1) <= 0.0001,] # Removes all weight combinations that do not sum to 1
# rownames(DFRange) <- c(1:nrow(DFRange)) # Reestablish row names
# DFRange <- DFRange[c(-1:-7,-9,-11:-12,-15:-16,-18:-21),]
# DFRange <- DFRange[c(-1:-7,-11:-12,-15:-16,-18:-21),]
# rownames(DFRange) <- c(1:nrow(DFRange)) # Reestablish row names
# #DFRange[nrow(DFRange) + 1,] <- c(1/3, 1/3, 1/3)

DFRange <- matrix(data = c(1/3,1/3,1/3,
                           0.6,0.2,0.2,
                           0.2,0.6,0.2), ncol = 3, byrow = TRUE)


rRange <- expand.grid(
  c(1, 0.1),
  c(1, 10),
  c(1),
  c(1)
)
B <- 2000

for(wi in 1:nrow(DFRange)){
  for(ri in 1:nrow(rRange)) {
    # dir.create(paste("results/results_SO_MaxTgtMin/uni_add/w",wi,"r",ri,sep=""))
    # dir.create(paste("results/results_SO_MaxTgtMin/BSR_add/w",wi,"r",ri,sep=""))
    # dir.create(paste("results/results_SO_MaxTgtMin/multi_add/w",wi,"r",ri,sep=""))
    dir.create(paste("results/results_SO_MaxTgtMin/multivect_add/w",wi,"r",ri,sep=""))
    # dir.create(paste("results/results_SO_MaxTgtMin/MVNSurface_add/w",wi,"r",ri,sep=""))
    # dir.create(paste("results/results_SO_MaxTgtMin/MVtSurface_add/w",wi,"r",ri,sep=""))
    # dir.create(paste("results/results_SO_MaxTgtMin/MVNSurfaceSIG_add/w",wi,"r",ri,sep=""))
    # dir.create(paste("results/results_SO_MaxTgtMin/MVtSurfaceSIG_add/w",wi,"r",ri,sep=""))
    # dir.create(paste("results/results_SO_MaxTgtMin/uni_mult/w",wi,"r",ri,sep=""))
    # dir.create(paste("results/results_SO_MaxTgtMin/BSR_mult/w",wi,"r",ri,sep=""))
    # dir.create(paste("results/results_SO_MaxTgtMin/multi_mult/w",wi,"r",ri,sep=""))
    dir.create(paste("results/results_SO_MaxTgtMin/multivect_mult/w",wi,"r",ri,sep=""))
    # dir.create(paste("results/results_SO_MaxTgtMin/MVNSurface_mult/w",wi,"r",ri,sep=""))
    # dir.create(paste("results/results_SO_MaxTgtMin/MVtSurface_mult/w",wi,"r",ri,sep=""))
    # dir.create(paste("results/results_SO_MaxTgtMin/MVNSurfaceSIG_mult/w",wi,"r",ri,sep=""))
```

```r
# dir.create(paste("results/results_SO_MaxTgtMin/MVtSurfaceSIG_mult/w",wi,"r",ri,sep=""))
w <- as.numeric(DFRange[wi,])
r <- as.numeric(rRange[ri,])
print(paste("Weight ", wi, "/", nrow(DFRange), " and Flexibility ", ri, "/", nrow(rRange), " started.", sep = ""))
timestamp()
##################### Desirability of True Surface #####################
d1_d <- lapply(1:length(Beta.true), function(x) DS_Max(Y1_tgt[[x]], Y1_poor[[x]], r[1], Y.all[[x]]$Y1))
d2_d <- lapply(1:length(Beta.true), function(x) DS_Target(Y2_tgt, Y2_poor_U[[x]], Y2_poor_L[[x]], r[2], r[3], Y.all[[x]]$Y2))
d3_d <- lapply(1:length(Beta.true), function(x) DS_Min(Y3_tgt[[x]], Y3_poor[[x]], r[4], Y.all[[x]]$Y3))

Dadd_d <- lapply(1:length(Beta.true), function(x) DF_Formulas(w, data.frame(d1_d[[x]], d2_d[[x]], d3_d[[x]]), type = 'add'))
Dmult_d <- lapply(1:length(Beta.true), function(x) DF_Formulas(w, data.frame(d1_d[[x]], d2_d[[x]], d3_d[[x]]), type = 'mult'))

Results_true <- lapply(1:length(Beta.true), function(x) data.frame(Y.all[[x]],
                                                                   d1 = d1_d[[x]],
                                                                   d2 = d2_d[[x]],
                                                                   d3 = d3_d[[x]],
                                                                   add = Dadd_d[[x]],
                                                                   mult = Dmult_d[[x]]))

df <- data.frame(Results_true[[1]])
plot_ly(df,
        x=df$Y1,
        y=df$Y2,
        z=df$Y3,
        type="scatter3d",
        size = 10,
        markers=list(
          mode = "markers",
          size = 5
        ),
        color = df$add,
        text = ~paste("(", df$x1,",",df$x2,")", sep = ""),
        alpha = 1) %>%
  layout(title = paste("Heat Map of Response Values"))
circles <- data.frame(
  x0 = 0,
  y0 = 0,
  r = 2^(2/4)
)
ggplot(data = df) +
  stat_contour(aes(x = x1, y = x2, z = Y1), color = rgb(255,0,0, maxColorValue = 255), size = 1.0) +
  stat_contour(aes(x = x1, y = x2, z = Y2), color = rgb(0,255,0, maxColorValue = 255), size = 1.0) +
  stat_contour(aes(x = x1, y = x2, z = Y3), color = rgb(0,0,255, maxColorValue = 255), size = 1.0) +
  geom_circle(aes(x0=x0, y0=y0, r=r), data=circles, size = 1) +
  theme_classic() +
  scale_x_continuous(name="X1", limits=c(-1.5, 1.5), breaks = seq(-2,2,0.5)) +
  scale_y_continuous(name="X2", limits=c(-1.5, 1.5), breaks = seq(-2,2,0.5)) +
  theme(axis.line=element_blank(),legend.position = "right")

```

```r
###################### Desirability of Deterministic Regression ######################
# Yhat_d <- lapply(1:length(Beta.true), function(x) data.frame(Ysim.pred[[1]][[x]]))
#
# d1_d <- lapply(1:length(Beta.true), function(x) DS_Max(Y1_tgt[[x]], Y1_poor[[x]], r[1], Yhat_d[[x]]$Y1))
# d2_d <- lapply(1:length(Beta.true), function(x) DS_Target(Y2_tgt[[x]], Y2_poor_U[[x]], Y2_poor_L[[x]], r
    [2], r[3], Yhat_d[[x]]$Y2))
# d3_d <- lapply(1:length(Beta.true), function(x) DS_Min(Y3_tgt[[x]], Y3_poor[[x]], r[4], Yhat_d[[x]]$Y3))
#
# Dadd_d <- lapply(1:length(Beta.true), function(x) DF_Formulas(w, data.frame(d1_d[[x]], d2_d[[x]], d3_d[[x
    ]]), type = 'add'))
# Dmult_d <- lapply(1:length(Beta.true), function(x) DF_Formulas(w, data.frame(d1_d[[x]], d2_d[[x]], d3_d[[x
    ]]), type = 'mult'))
#
# Results_d <- lapply(1:length(Beta.true), function(x) data.frame(Yhat_d[[x]],
#                                                                  d1 = d1_d[[x]],
#                                                                  d2 = d2_d[[x]],
#                                                                  d3 = d3_d[[x]],
#                                                                  add = Dadd_d[[x]],
#                                                                  mult = Dmult_d[[x]]))

###################### Worst/Best Method CIs Univariate ######################
set.seed(1994)
tquant <- qt(1-alpha/2, df = nrow(X) - ncol(X))

d1simUp <- mclapply(1:g, function(y) lapply(1:length(Beta.true), function(x) DS_Max(Y1_tgt[[x]], Y1_poor[[x
    ]], r[1], Ysim.pred[[y]][[x]]$Y1 + hw1[[y]][[x]])), mc.cores = numCores)
d1simLow <- mclapply(1:g, function(y) lapply(1:length(Beta.true), function(x) DS_Max(Y1_tgt[[x]], Y1_poor[[x
    ]], r[1], Ysim.pred[[y]][[x]]$Y1 - hw1[[y]][[x]])), mc.cores = numCores)
d2simWorst <- mclapply(1:g, function(y) lapply(1:length(Beta.true), function(x) DS_Target(Y2_tgt, Y2_poor_U[[
    x]], Y2_poor_L[[x]], r[2], r[3], Ysim.pred[[y]][[x]]$Y2 +
                                                   as.integer(Ysim.pred[[y]][[x]]$Y2 < Y2_tgt)*(-hw2[[y]][[x]]) +
                                                   as.integer(Ysim.pred[[y]][[x]]$Y2 > Y2_tgt)*(hw2[[y]][[x]]) )),
                                                   mc.cores = numCores)
d2simBest <- 1 # Y2 is at tgt = 65 resulting in d_i = 1
d3simUp <- mclapply(1:g, function(y) lapply(1:length(Beta.true), function(x) DS_Min(Y3_tgt[[x]], Y3_poor[[x
    ]], r[4], Ysim.pred[[y]][[x]]$Y3 + hw3[[y]][[x]])), mc.cores = numCores)
d3simLow <- mclapply(1:g, function(y) lapply(1:length(Beta.true), function(x) DS_Min(Y3_tgt[[x]], Y3_poor[[x
    ]], r[4], Ysim.pred[[y]][[x]]$Y3 - hw3[[y]][[x]])), mc.cores = numCores)

DAdd.Best <- mclapply(1:g, function(y) lapply(1:length(Beta.true),
                      function(x) data.frame(DF_Formulas(w, data.frame(d1simUp[[y]][[x]],
                                                                        d2simBest,
                                                                        d3simLow[[y]][[x]]),
```

```
type = 'add'))), mc.cores = numCores)
DAdd.Worst <- mclapply(1:g, function(y) lapply(1:length(Beta.true),
                                              function(x) data.frame(DF_Formulas(w, data.frame(d1simLow[[y]][[x]],
                                                                                              d2simWorst[[y]][[x]],
                                                                                              d3simUp[[y]][[x]]), type = 'add'))), mc.cores = numCores)

DMult.Best <- mclapply(1:g, function(y) lapply(1:length(Beta.true),
                                              function(x) data.frame(DF_Formulas(w, data.frame(d1simUp[[y]][[x]],
                                                                                              d2simBest,
                                                                                              d3simLow[[y]][[x]]), type = 'mult'))), mc.cores = numCores)
DMult.Worst <- mclapply(1:g, function(y) lapply(1:length(Beta.true),
                                               function(x) data.frame(DF_Formulas(w, data.frame(d1simLow[[y]][[x]],
                                                                                               d2simWorst[[y]][[x]],
                                                                                               d3simUp[[y]][[x]]), type = 'mult'))), mc.cores = numCores)

DAdd.CIs <- mclapply(1:g, function(x) lapply(1:length(Beta.true), function(y) {
  data.frame(DAddLow = DAdd.Worst[[x]][[y]], DAddUp = DAdd.Best[[x]][[y]], True = Results_true[[y]]$add)}))
DMult.CIs <- mclapply(1:g, function(x) lapply(1:length(Beta.true), function(y) {
  data.frame(DMultLow = DMult.Worst[[x]][[y]], DMultUp = DMult.Best[[x]][[y]], True = Results_true[[y]]$mult)
    }))
rm(d1simUp, d1simLow, d2simUp, d2simLow, DAdd.Best, DAdd.Worst, DMult.Best, DMult.Worst)
```

```
  gc()
  CoverageAdd <- mclapply(DAdd.CIs, function(x)
    lapply(x, function(y)
      sapply(1:nrow(grid_all), function(z){
        data.frame(Cov = (y[z,1] <= y[z,3])&&(y[z,3] <= y[z,2]),
                   len = (y[z,2] - y[z,1]),
                   above = (y[z,3] > y[z,2]),
                   below = (y[z,3] < y[z,1]))})), mc.cores = numCores)
  CoverageMult <- mclapply(DMult.CIs, function(x)
    lapply(x, function(y)
      sapply(1:nrow(grid_all), function(z){
        data.frame(Cov = (y[z,1] <= y[z,3])&&(y[z,3] <= y[z,2]),
                   len = (y[z,2] - y[z,1]),
                   above = (y[z,3] > y[z,2]),
                   below = (y[z,3] < y[z,1]))})), mc.cores = numCores)

  coveradd <- matrix(0, nrow = nrow(grid_all), ncol = length(Beta.true))
  covermult <- matrix(0, nrow = nrow(grid_all), ncol = length(Beta.true))
  lengthadd <- matrix(0, nrow = nrow(grid_all), ncol = length(Beta.true))
  lengthmult <- matrix(0, nrow = nrow(grid_all), ncol = length(Beta.true))
  aboveadd <- matrix(0, nrow = nrow(grid_all), ncol = length(Beta.true))
  belowadd <- matrix(0, nrow = nrow(grid_all), ncol = length(Beta.true))
  abovemult <- matrix(0, nrow = nrow(grid_all), ncol = length(Beta.true))
  belowmult <- matrix(0, nrow = nrow(grid_all), ncol = length(Beta.true))
  for(i in 1:g) {
    for(j in 1:length(Beta.true)){
      coveradd[,j] <- coveradd[,j] + unlist(as.numeric(CoverageAdd[[i]][[j]][1,]))
      covermult[,j] <- covermult[,j] + unlist(as.numeric(CoverageMult[[i]][[j]][1,]))
      lengthadd[,j] <- lengthadd[,j] + unlist(as.numeric(CoverageAdd[[i]][[j]][2,]))
      lengthmult[,j] <- lengthmult[,j] + unlist(as.numeric(CoverageMult[[i]][[j]][2,]))
      aboveadd[,j] <- aboveadd[,j] + unlist(as.numeric(CoverageAdd[[i]][[j]][3,]))
      abovemult[,j] <- abovemult[,j] + unlist(as.numeric(CoverageMult[[i]][[j]][3,]))
      belowadd[,j] <- belowadd[,j] + unlist(as.numeric(CoverageAdd[[i]][[j]][4,]))
      belowmult[,j] <- belowmult[,j] + unlist(as.numeric(CoverageMult[[i]][[j]][4,]))
    }
  }

  # rm(DAdd.CIs, DMult.CIs, CoverageAdd, CoverageMult)

  write.csv(coveradd/g, file = paste("results/results_SO_MaxTgtMin/Coverage_BestWorst_Additive_SO_w",wi,"_r",ri
      ,".csv", sep = "")) #### USE BACK OF THIS
  write.csv(lengthadd/g, file = paste("results/results_SO_MaxTgtMin/Length_BestWorst_Additive_SO_w",wi,"_r",ri,
      ".csv", sep = ""))
  write.csv(aboveadd, file = paste("results/results_SO_MaxTgtMin/AboveCI_BestWorst_Additive_SO_w",wi,"_r",ri,".
      csv", sep = ""))
  write.csv(belowadd, file = paste("results/results_SO_MaxTgtMin/BelowCI_BestWorst_Additive_SO_w",wi,"_r",ri,".
      csv", sep = ""))
  write.csv(covermult/g, file = paste("results/results_SO_MaxTgtMin/Coverage_BestWorst_Multiplicative_SO_w",wi,
      "_r",ri,".csv", sep = ""))
  write.csv(lengthmult/g, file = paste("results/results_SO_MaxTgtMin/Length_BestWorst_Multiplicative_SO_w",wi,"
      _r",ri,".csv", sep = ""))
  write.csv(abovemult, file = paste("results/results_SO_MaxTgtMin/AboveCI_BestWorst_Multiplicative_SO_w",wi,"_r
      ",ri,".csv", sep = ""))
  write.csv(belowmult, file = paste("results/results_SO_MaxTgtMin/BelowCI_BestWorst_Multiplicative_SO_w",wi,"_r
```

```
",ri,".csv", sep = ""))


# ##################### GENERALIZED METHOD Univariate #####################
# set.seed(1994)
#
#
# system.time(
#   for(i in 1:blocks) {
#     print(paste("Block ", i, " has started. ", (i-1)/(blocks)*100, "% Finished.", sep = ""))
#     mclapply(1:gblocked, function(y) lapply(1:length(Beta.true), function(x){
#       halfwidth1 <- matrix(rt(nrow(grid_all)*B,nusim1[[y + gblocked*(i-1)]][[x]]), nrow = nrow(grid_all), ncol = B)*sqrt(MSEsim1[[y + gblocked*(i-1)]][[x]]*(hat))
#       halfwidth2 <- matrix(rt(nrow(grid_all)*B,nusim2[[y + gblocked*(i-1)]][[x]]), nrow = nrow(grid_all), ncol = B)*sqrt(MSEsim2[[y + gblocked*(i-1)]][[x]]*(hat))
#       halfwidth3 <- matrix(rt(nrow(grid_all)*B,nusim3[[y + gblocked*(i-1)]][[x]]), nrow = nrow(grid_all), ncol = B)*sqrt(MSEsim3[[y + gblocked*(i-1)]][[x]]*(hat))
#       d1sim <- DS_Max(Y1_tgt[[x]], Y1_poor[[x]], r[1], Ysim.pred[[y + gblocked*(i-1)]][[x]]$Y1 - halfwidth1)
#       d2sim <- DS_Target(Y2_tgt, Y2_poor_U[[x]], Y2_poor_L[[x]], r[2], r[3], Ysim.pred[[y + gblocked*(i-1)]][[x]]$Y2 - halfwidth2)
#       d3sim <- DS_Min(Y3_tgt[[x]], Y3_poor[[x]], r[4], Ysim.pred[[y + gblocked*(i-1)]][[x]]$Y3 - halfwidth3)
#       fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim[,z]), type = 'add'))),
#              file = paste("results/results_SO_MaxTgtMin/uni_add/w",wi,"r",ri,"/beta_", x, "_MC_", y + gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE)
#       fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim[,z]), type = 'mult'))),
#              file = paste("results/results_SO_MaxTgtMin/uni_mult/w",wi,"r",ri,"/beta_", x, "_MC_", y + gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE)
#     }), mc.cores = numCores)
#     gc()
#   }
# )
# gc()
#
#
#
# ##################### GENERALIZED METHOD Multivariate #####################
# set.seed(1994)
#
#
# Y1tmp <- matrix(0, nrow = nrow(grid_all), ncol = B)
# Y2tmp <- matrix(0, nrow = nrow(grid_all), ncol = B)
# Y3tmp <- matrix(0, nrow = nrow(grid_all), ncol = B)
# system.time(
#   for(i in 1:blocks) {
#     print(paste("Block ", i, " has started. ", (i-1)/(blocks)*100, "% Finished.", sep = ""))
#     mclapply(1:gblocked, function(y) lapply(1:length(Beta.true), function(x){
#       tmp <- sapply(1:nrow(grid_all), function(j) data.frame(rmvt(B,
#                                                                    delta = as.numeric(Ysim.pred[[y + gblocked*(i-1)]][[x]][j,1:ncol(Y.true[[1]])]),
#                                                                    sigma = MVMSEsim[[y + gblocked*(i-1)]][[x
```

```
]]*hat[[j]],
#                                                           df = MVYsimnu[[y + gblocked*(i-1)]][[x]])
))
#       for(j in 1:nrow(grid_all)) {
#         Y1tmp[j,] <- tmp[,j]$X1
#         Y2tmp[j,] <- tmp[,j]$X2
#         Y3tmp[j,] <- tmp[,j]$X3
#       }
#       d1sim <- DS_Max(Y1_tgt[[x]], Y1_poor[[x]], r[1], Y1tmp)
#       d2sim <- DS_Target(Y2_tgt, Y2_poor_U[[x]], Y2_poor_L[[x]], r[2], r[3], Y2tmp)
#       d3sim <- DS_Min(Y3_tgt[[x]], Y3_poor[[x]], r[4], Y3tmp)
#       fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim[,z]),
 type = 'add'))),
#              file = paste("results/results_SO_MaxTgtMin/multi_add/w",wi,"r",ri,"/beta_", x, "_MC_", y +
gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE )
#       fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim[,z]),
 type = 'mult'))),
#              file = paste("results/results_SO_MaxTgtMin/multi_mult/w",wi,"r",ri,"/beta_", x, "_MC_", y +
gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE)
#     }), mc.cores = numCores)
#     gc()
#   }
# )
# gc()
#
#
# ###################### GENERALIZED METHOD Multivariate VECTORIZED ######################
# set.seed(1994)
#
# Y1tmp <- matrix(0, nrow = nrow(grid_all), ncol = B)
# Y2tmp <- matrix(0, nrow = nrow(grid_all), ncol = B)
# Y3tmp <- matrix(0, nrow = nrow(grid_all), ncol = B)
# system.time(
#   for(i in 1:blocks) {
#     print(paste("Block ", i, " has started. ", (i-1)/(blocks)*100, "% Finished.", sep = ""))
#     mclapply(1:gblocked, function(y) lapply(1:length(Beta.true), function(x){
#       tmp <- data.frame(rmvt(B,
#                               delta = as.vector(unlist(Ysim.pred[[y + gblocked*(i-1)]][[x]][,1:ncol(Y.true
[[1]])])),
#                               sigma = kronecker(MVMSEsim[[y + gblocked*(i-1)]][[x]],H),
#                               df = MVYsimnu[[y + gblocked*(i-1)]][[x]]))
#       Y1tmp <- t(tmp[,1:nrow(grid_all)])
#       Y2tmp <- t(tmp[,(nrow(grid_all)+1):(nrow(grid_all)*2)])
#       Y3tmp <- t(tmp[,-(1:(nrow(grid_all)*2))])
#             d1sim <- DS_Max(Y1_tgt[[x]], Y1_poor[[x]], r[1], Y1tmp)
#             d2sim <- DS_Target(Y2_tgt, Y2_poor_U[[x]], Y2_poor_L[[x]], r[2], r[3], Y2tmp)
#             d3sim <- DS_Min(Y3_tgt[[x]], Y3_poor[[x]], r[4], Y3tmp)
#             fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim
[,z]), type = 'add'))),
#                    file = paste("results/results_SO_MaxTgtMin/multivect_add/w",wi,"r",ri,"/beta_", x, "_MC_
", y + gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE )
#             fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim
[,z]), type = 'mult'))),
#                    file = paste("results/results_SO_MaxTgtMin/multivect_mult/w",wi,"r",ri,"/beta_", x, "_MC
```

```
_", y + gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE)
#           }), mc.cores = numCores)
#           gc()
#         }
#       )
#       gc()
# ##################### BOOTSTRAP METHOD MVNSurfaces #####################
# set.seed(1994)
#
#
# system.time(
#   for(i in 1:blocks) {
#     print(paste("Block ", i, " has started. ", (i-1)/(blocks)*100, "% Finished.", sep = ""))
#     for(j in 1:length(Beta.true)) {
#       mclapply(1:gblocked, function(y) {
#         Beta1 <- rmvnorm(B, mean = Y1sim.lm[[y + gblocked*(i-1)]][[j]]$coefficients, sigma = MSEsim1[[y + gblocked*(i-1)]][[j]]*XpXi)
#         Beta2 <- rmvnorm(B, mean = Y2sim.lm[[y + gblocked*(i-1)]][[j]]$coefficients, sigma = MSEsim2[[y + gblocked*(i-1)]][[j]]*XpXi)
#         Beta3 <- rmvnorm(B, mean = Y3sim.lm[[y + gblocked*(i-1)]][[j]]$coefficients, sigma = MSEsim3[[y + gblocked*(i-1)]][[j]]*XpXi)
#         Beta1_Pred <- as.matrix(grid_all)%*%t(Beta1)
#         Beta2_Pred <- as.matrix(grid_all)%*%t(Beta2)
#         Beta3_Pred <- as.matrix(grid_all)%*%t(Beta3)
#         d1sim <- DS_Max(Y1_tgt[[j]], Y1_poor[[j]], r[1], Beta1_Pred)
#         d2sim <- DS_Target(Y2_tgt, Y2_poor_U[[j]], Y2_poor_L[[j]], r[2], r[3], Beta2_Pred)
#         d3sim <- DS_Min(Y3_tgt[[j]], Y3_poor[[j]], r[4], Beta3_Pred)
#         fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim[,z]), type = 'add'))),
#                file = paste("results/results_SO_MaxTgtMin/MVNSurface_add/w",wi,"r",ri,"/beta_", j, "_MC_", y + gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE )
#         fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim[,z]), type = 'mult'))),
#                file = paste("results/results_SO_MaxTgtMin/MVNSurface_mult/w",wi,"r",ri,"/beta_", j, "_MC_", y + gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE)
#       }, mc.cores = numCores)
#     }
#     gc()
#   }
# )
# gc()
#
#
#
# ##################### BOOTSTRAP METHOD MVtSurfaces #####################
# set.seed(1994)
#
#
# system.time(
#   for(i in 1:blocks) {
#     print(paste("Block ", i, " has started. ", (i-1)/(blocks)*100, "% Finished.", sep = ""))
#     for(j in 1:length(Beta.true)) {
#       mclapply(1:gblocked, function(y) {
#         Beta1 <- rmvt(n = B, delta = Y1sim.lm[[y + gblocked*(i-1)]][[j]]$coefficients, sigma = MSEsim1[[y +
```

```
gblocked*(i-1)]][[j]]*XpXi, df=nusim1[[y + gblocked*(i-1)]][[j]])
#         Beta2 <- rmvt(n = B, delta = Y2sim.lm[[y + gblocked*(i-1)]][[j]]$coefficients, sigma = MSEsim2[[y + gblocked*(i-1)]][[j]]*XpXi, df=nusim2[[y + gblocked*(i-1)]][[j]])
#         Beta3 <- rmvt(n = B, delta = Y3sim.lm[[y + gblocked*(i-1)]][[j]]$coefficients, sigma = MSEsim3[[y + gblocked*(i-1)]][[j]]*XpXi, df=nusim3[[y + gblocked*(i-1)]][[j]])
#         Beta1_Pred <- as.matrix(grid_all)%*%t(Beta1)
#         Beta2_Pred <- as.matrix(grid_all)%*%t(Beta2)
#         Beta3_Pred <- as.matrix(grid_all)%*%t(Beta3)
#         d1sim <- DS_Max(Y1_tgt[[j]], Y1_poor[[j]], r[1], Beta1_Pred)
#         d2sim <- DS_Target(Y2_tgt, Y2_poor_U[[j]], Y2_poor_L[[j]], r[2], r[3], Beta2_Pred)
#         d3sim <- DS_Min(Y3_tgt[[j]], Y3_poor[[j]], r[4], Beta3_Pred)
#         fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim[,z]), type = 'add'))),
#                file = paste("results/results_SO_MaxTgtMin/MVtSurface_add/w",wi,"r",ri,"/beta_", j, "_MC_", y + gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE )
#         fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim[,z]), type = 'mult'))),
#                file = paste("results/results_SO_MaxTgtMin/MVtSurface_mult/w",wi,"r",ri,"/beta_", j, "_MC_", y + gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE)
#       }, mc.cores = numCores)
#     }
#     gc()
#   }
# )
# gc()
#
#
#
# ##################### BOOTSTRAP METHOD MVNSurfaces Covariance Structure #####################
# set.seed(1994)
#
#
# system.time(
#   for(i in 1:blocks) {
#     print(paste("Block ", i, " has started. ", (i-1)/(blocks)*100, "% Finished.", sep = ""))
#     for(j in 1:length(Beta.true)){
#       mclapply(1:gblocked, function(y) {
#         Beta <- rmvnorm(B, mean = c(MVYsim.lm[[y + gblocked*(i-1)]][[j]]$coefficients), sigma = kronecker(MVMSEsim[[y + gblocked*(i-1)]][[j]],XpXi))
#         Beta1_Pred <- as.matrix(grid_all)%*%t(Beta[,1:(ncol(Beta)/3)])
#         Beta2_Pred <- as.matrix(grid_all)%*%t(Beta[,((ncol(Beta)/3)+1):(ncol(Beta)/3*2)])
#         Beta3_Pred <- as.matrix(grid_all)%*%t(Beta[,-(1:(ncol(Beta)/3*2))])
#         d1sim <- DS_Max(Y1_tgt[[j]], Y1_poor[[j]], r[1], Beta1_Pred)
#         d2sim <- DS_Target(Y2_tgt, Y2_poor_U[[j]], Y2_poor_L[[j]], r[2], r[3], Beta2_Pred)
#         d3sim <- DS_Min(Y3_tgt[[j]], Y3_poor[[j]], r[4], Beta3_Pred)
#         fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim[,z]), type = 'add'))),
#                file = paste("results/results_SO_MaxTgtMin/MVNSurfaceSIG_add/w",wi,"r",ri,"/beta_", j, "_MC_", y + gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE )
#         fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim[,z]), type = 'mult'))),
#                file = paste("results/results_SO_MaxTgtMin/MVNSurfaceSIG_mult/w",wi,"r",ri,"/beta_", j, "_MC_", y + gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE)}, mc.cores = numCores)
#       gc()
```

```
#     }
#   }
# )
# gc()
#
#
#
# ##################### BOOTSTRAP METHOD MVtSurfaces Covariance Structure #####################
# set.seed(1994)
#
#
# system.time(
#   for(i in 1:blocks) {
#     print(paste("Block ", i, " has started. ", (i-1)/(blocks)*100, "% Finished.", sep = ""))
#     for(j in 1:length(Beta.true)) {
#       mclapply(1:gblocked, function(y) {
#         Beta <- rmvt(n = B, delta = c(MVYsim.lm[[y + gblocked*(i-1)]][[j]]$coefficients), kronecker(MVMSEsim[[y + gblocked*(i-1)]][[j]],XpXi), df=nusim1[[y + gblocked*(i-1)]][[j]])
#         Beta1_Pred <- as.matrix(grid_all)%*%t(Beta[,1:(ncol(Beta)/3)])
#         Beta2_Pred <- as.matrix(grid_all)%*%t(Beta[,((ncol(Beta)/3)+1):(ncol(Beta)/3*2)])
#         Beta3_Pred <- as.matrix(grid_all)%*%t(Beta[,-(1:(ncol(Beta)/3*2))])
#         d1sim <- DS_Max(Y1_tgt[[j]], Y1_poor[[j]], r[1], Beta1_Pred)
#         d2sim <- DS_Target(Y2_tgt, Y2_poor_U[[j]], Y2_poor_L[[j]], r[2], r[3], Beta2_Pred)
#         d3sim <- DS_Min(Y3_tgt[[j]], Y3_poor[[j]], r[4], Beta3_Pred)
#         fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim[,z]), type = 'add'))),
#                file = paste("results/results_SO_MaxTgtMin/MVtSurfaceSIG_add/w",wi,"r",ri,"/beta_", j, "_MC_", y + gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE )
#         fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim[,z]), type = 'mult'))),
#                file = paste("results/results_SO_MaxTgtMin/MVtSurfaceSIG_mult/w",wi,"r",ri,"/beta_", j, "_MC_", y + gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE)
#       }, mc.cores = numCores)
#       gc()
#     }
#   }
# )
# gc()
#
# ##################### BOOTSTRAP METHOD Residuals #####################
# set.seed(1994)
#
#
# mat <- mclapply(1:(gblocked*blocks), function(y) {
#   return(matrix(sample(nrow(X), nrow(X)*B, replace = TRUE), nrow = nrow(X), ncol = B))
# }, mc.cores = numCores)
# system.time(
#   for(i in 1:blocks) {
#     print(paste("Block ", i, " has started. ", (i-1)/(blocks)*100, "% Finished.", sep = ""))
#     for(j in 1:length(Beta.true)) {
#       mclapply(1:gblocked, function(y) {
#         Yb1 <- matrix(Y1sim.lm[[y + gblocked*(i-1)]][[j]]$fitted.values + Y1sim.lm[[y + gblocked*(i-1)]][[j]]$residuals[mat[[y + gblocked*(i-1)]]], nrow = nrow(X), ncol = B)
#         Yb2 <- matrix(Y2sim.lm[[y + gblocked*(i-1)]][[j]]$fitted.values + Y2sim.lm[[y + gblocked*(i-1)]][[j
```

```
]]$residuals[mat[[y + gblocked*(i-1)]]], nrow = nrow(X), ncol = B)
    #         Yb3 <- matrix(Y3sim.lm[[y + gblocked*(i-1)]][[j]]$fitted.values + Y3sim.lm[[y + gblocked*(i-1)]][[j]]$residuals[mat[[y + gblocked*(i-1)]]], nrow = nrow(X), ncol = B)
    #         d1sim <- DS_Max(Y1_tgt[[j]], Y1_poor[[j]], r[1], as.matrix(grid_all)%*%XpXi%*%t(X.mat)%*%Yb1)
    #         d2sim <- DS_Target(Y2_tgt, Y2_poor_U[[j]], Y2_poor_L[[j]], r[2], r[3], as.matrix(grid_all)%*%XpXi%*%t(X.mat)%*%Yb2)
    #         d3sim <- DS_Min(Y3_tgt[[j]], Y3_poor[[j]], r[4], as.matrix(grid_all)%*%XpXi%*%t(X.mat)%*%Yb3)
    #         fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim[,z]), type = 'add'))),
    #                file = paste("results/results_SO_MaxTgtMin/BSR_add/w",wi,"r",ri,"/beta_", j, "_MC_", y + gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE )
    #         fwrite(data.table(sapply(1:B, function(z) DF_Formulas(w, data.frame(d1sim[,z], d2sim[,z], d3sim[,z]), type = 'mult'))),
    #                file = paste("results/results_SO_MaxTgtMin/BSR_mult/w",wi,"r",ri,"/beta_", j, "_MC_", y + gblocked*(i-1), "_SO_w",wi,"_r",ri,".csv", sep = ""), col.names = FALSE)
    #       }, mc.cores = numCores)
    #       gc()
    #     }
    #   }
    # )
    # gc()

  }
  gc()
}

for(wi in 2:nrow(DFRange)){
  for(ri in 1:nrow(rRange)) {
    ####################### CI FORMATTING ######################
    w <- as.numeric(DFRange[wi,])
    r <- as.numeric(rRange[ri,])
    print(paste("Weight␣", wi, "/", nrow(DFRange), "␣and␣Flexibility␣", ri, "/", nrow(rRange), "␣started.", sep = ""))
    timestamp()
    ###################### Desirability of True Surface #####################
    d1_d <- lapply(1:length(Beta.true), function(x) DS_Max(Y1_tgt[[x]], Y1_poor[[x]], r[1], Y.all[[x]]$Y1))
    d2_d <- lapply(1:length(Beta.true), function(x) DS_Target(Y2_tgt, Y2_poor_U[[x]], Y2_poor_L[[x]], r[2], r[3], Y.all[[x]]$Y2))
    d3_d <- lapply(1:length(Beta.true), function(x) DS_Min(Y3_tgt[[x]], Y3_poor[[x]], r[4], Y.all[[x]]$Y3))

    Dadd_d <- lapply(1:length(Beta.true), function(x) DF_Formulas(w, data.frame(d1_d[[x]], d2_d[[x]], d3_d[[x]]), type = 'add'))
    Dmult_d <- lapply(1:length(Beta.true), function(x) DF_Formulas(w, data.frame(d1_d[[x]], d2_d[[x]], d3_d[[x]]), type = 'mult'))

    Results_true <- lapply(1:length(Beta.true), function(x) data.frame(Y.all[[x]],
                                                                      d1 = d1_d[[x]],
                                                                      d2 = d2_d[[x]],
                                                                      d3 = d3_d[[x]],
                                                                      add = Dadd_d[[x]],
                                                                      mult = Dmult_d[[x]]))
    # ##################### GENERALIZED METHOD Univariate #####################
    # ## Confidence Intervals for Additive and Multiplicative BetaTrue
    # DDist.add <- list()
```

```
# DDist.mult <- list()
# covadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# covmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# aboveadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# abovemult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
#
# registerDoParallel(numCores)
# system.time(
#   for(i in 1:length(Beta.true)) {
#     print(paste("i is", i))
#     DDist.add[[i]] <- vector("list", blocks*gblocked)
#     DDist.mult[[i]] <- vector("list", blocks*gblocked)
#     files.add <- list.files(path = paste("results/results_SO_MaxTgtMin/uni_add/w",wi,"r",ri,"/",sep = ""), pattern = paste("beta_",i, sep = ""))
#     files.mult <- list.files(path = paste("results/results_SO_MaxTgtMin/uni_mult/w",wi,"r",ri,"/",sep = ""), pattern = paste("beta_",i, sep = ""))
#     DDist.add <- foreach(j=1:(blocks*gblocked),
#                          .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]]))) %dopar% {
#                            tmp <- fread(paste("results/results_SO_MaxTgtMin/uni_add/w",wi,"r",ri,"/", files.add[j], sep = ""), header = FALSE)
#                            CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                            CI$YTrue <- Results_true[[i]]$add
#                            CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                            CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                            CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                            CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                            CI <- split(CI, seq(nrow(CI)))
#                            return(CI)
#                          }
#     print(paste("Iteration", i, "additive finished."))
#     DDist.mult <- foreach(j=1:(blocks*gblocked),
#                           .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]]))) %dopar% {
#                             tmp <- fread(paste("results/results_SO_MaxTgtMin/uni_mult/w",wi,"r",ri,"/", files.mult[j], sep = ""), header = FALSE)
#                             CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                             CI$YTrue <- Results_true[[i]]$mult
#                             CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                             CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                             CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                             CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                             CI <- split(CI, seq(nrow(CI)))
#                             return(CI)
#                           }
#     print(paste("Iteration", i, "multiplicative finished."))
#     for (j in 1:nrow(grid_all)) {
#       covadd[j,i] <- sum(DDist.add[[j]]$Coverage)/(blocks*gblocked)
```

```
#       covmult[j,i] <- sum(DDist.mult[[j]]$Coverage)/(blocks*gblocked)
#       lenadd[j,i] <- mean(DDist.add[[j]]$Length)
#       lenmult[j,i] <- mean(DDist.mult[[j]]$Length)
#       aboveadd[j,i] <- sum(DDist.add[[j]]$Above)
#       belowadd[j,i] <- sum(DDist.add[[j]]$Below)
#       abovemult[j,i] <- sum(DDist.mult[[j]]$Above)
#       belowmult[j,i] <- sum(DDist.mult[[j]]$Below)
#     }
#     gc()
#   }
# )
# stopImplicitCluster()
# gc()
#
# write.csv(covadd, file = paste("results/results_SO_MaxTgtMin/Coverage_Univariate_Additive_SO_w",wi,"_r",ri
    ,".csv", sep = ""))
# write.csv(lenadd, file = paste("results/results_SO_MaxTgtMin/Length_Univariate_Additive_SO_w",wi,"_r",ri,".
    csv", sep = ""))
# write.csv(covmult, file = paste("results/results_SO_MaxTgtMin/Coverage_Univariate_Multiplicative_SO_w",wi,"
    _r",ri,".csv", sep = ""))
# write.csv(lenmult, file = paste("results/results_SO_MaxTgtMin/Length_Univariate_Multiplicative_SO_w",wi,"_r
    ",ri,".csv", sep = ""))
# write.csv(aboveadd, file = paste("results/results_SO_MaxTgtMin/AboveCI_Univariate_Additive_SO_w",wi,"_r",ri
    ,".csv", sep = ""))
# write.csv(belowadd, file = paste("results/results_SO_MaxTgtMin/BelowCI_Univariate_Additive_SO_w",wi,"_r",ri
    ,".csv", sep = ""))
# write.csv(abovemult, file = paste("results/results_SO_MaxTgtMin/AboveCI_Univariate_Multiplicative_SO_w",wi
    ,"_r",ri,".csv", sep = ""))
# write.csv(belowmult, file = paste("results/results_SO_MaxTgtMin/BelowCI_Univariate_Multiplicative_SO_w",wi
    ,"_r",ri,".csv", sep = ""))
#
#
#
#
# ###################### GENERALIZED METHOD Multivariate ######################
# DDist.add <- list()
# DDist.mult <- list()
# covadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# covmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# aboveadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# abovemult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
#
# registerDoParallel(numCores)
# system.time(
#   for(i in 1:length(Beta.true)) {
#     print(paste("Iteration", i, "has started."))
#     DDist.add[[i]] <- vector("list", blocks*gblocked)
#     DDist.mult[[i]] <- vector("list", blocks*gblocked)
#     files.add <- list.files(path = paste("results/results_SO_MaxTgtMin/multi_add/w",wi,"r",ri,"/",sep = "")
    , pattern = paste("beta_",i, sep = ""))
```

```
#     files.mult <- list.files(path = paste("results/results_SO_MaxTgtMin/multi_mult/w",wi,"r",ri,"/",sep = ""), pattern = paste("beta_",i, sep = ""))
#     DDist.add <- foreach(j=1:(blocks*gblocked),
#                          .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]]))) %dopar% {
#                            tmp <- fread(paste("results/results_SO_MaxTgtMin/multi_add/w",wi,"r",ri,"/", files.add[j], sep = ""), header = FALSE)
#                            CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                            CI$YTrue <- Results_true[[i]]$add
#                            CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                            CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                            CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                            CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                            CI <- split(CI, seq(nrow(CI)))
#                            return(CI)
#                          }
#     print(paste("Iteration", i, "additive finished."))
#     DDist.mult <- foreach(j=1:(blocks*gblocked),
#                           .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]]))) %dopar% {
#                             tmp <- fread(paste("results/results_SO_MaxTgtMin/multi_mult/w",wi,"r",ri,"/", files.mult[j], sep = ""), header = FALSE)
#                             CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                             CI$YTrue <- Results_true[[i]]$mult
#                             CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                             CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                             CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                             CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                             CI <- split(CI, seq(nrow(CI)))
#                             return(CI)
#                           }
#     print(paste("Iteration", i, "multiplicative finished."))
#     for (j in 1:nrow(grid_all)) {
#       covadd[j,i] <- sum(DDist.add[[j]]$Coverage)/(blocks*gblocked)
#       covmult[j,i] <- sum(DDist.mult[[j]]$Coverage)/(blocks*gblocked)
#       lenadd[j,i] <- mean(DDist.add[[j]]$Length)
#       lenmult[j,i] <- mean(DDist.mult[[j]]$Length)
#       aboveadd[j,i] <- sum(DDist.add[[j]]$Above)
#       belowadd[j,i] <- sum(DDist.add[[j]]$Below)
#       abovemult[j,i] <- sum(DDist.mult[[j]]$Above)
#       belowmult[j,i] <- sum(DDist.mult[[j]]$Below)
#     }
#     gc()
#   }
# )
# stopImplicitCluster()
# gc()
#
# write.csv(covadd, file = paste("results/results_SO_MaxTgtMin/Coverage_Multivariate_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(lenadd, file = paste("results/results_SO_MaxTgtMin/Length_Multivariate_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
```

```
# write.csv(covmult, file = paste("results/results_SO_MaxTgtMin/Coverage_Multivariate_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(lenmult, file = paste("results/results_SO_MaxTgtMin/Length_Multivariate_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(aboveadd, file = paste("results/results_SO_MaxTgtMin/AboveCI_Multivariate_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(belowadd, file = paste("results/results_SO_MaxTgtMin/BelowCI_Multivariate_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(abovemult, file = paste("results/results_SO_MaxTgtMin/AboveCI_Multivariate_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(belowmult, file = paste("results/results_SO_MaxTgtMin/BelowCI_Multivariate_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
#

# ##################### GENERALIZED METHOD Multivariate VECTORIZED #####################
# DDist.add <- list()
# DDist.mult <- list()
# covadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# covmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# aboveadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# abovemult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
#
# registerDoParallel(numCores)
# system.time(
#   for(i in 1:length(Beta.true)) {
#     print(paste("Iteration", i, "has started."))
#     DDist.add[[i]] <- vector("list", blocks*gblocked)
#     DDist.mult[[i]] <- vector("list", blocks*gblocked)
#     files.add <- list.files(path = paste("results/results_SO_MaxTgtMin/multivect_add/w",wi,"r",ri,"/",sep = ""), pattern = paste("beta_",i, sep = ""))
#     files.mult <- list.files(path = paste("results/results_SO_MaxTgtMin/multivect_mult/w",wi,"r",ri,"/",sep = ""), pattern = paste("beta_",i, sep = ""))
#     DDist.add <- foreach(j=1:(blocks*gblocked),
#                          .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]]))) %dopar% {
#                            tmp <- fread(paste("results/results_SO_MaxTgtMin/multivect_add/w",wi,"r",ri,"/", files.add[j], sep = ""), header = FALSE)
#                            CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                            CI$YTrue <- Results_true[[i]]$add
#                            CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                            CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                            CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                            CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                            CI <- split(CI, seq(nrow(CI)))
#                            return(CI)
#                          }
#     print(paste("Iteration", i, "additive finished."))
#     DDist.mult <- foreach(j=1:(blocks*gblocked),
#                           .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]],
```

```
    new[[x]]))) %dopar% {
#                              tmp <- fread(paste("results/results_SO_MaxTgtMin/multivect_mult/w",wi,"r",ri,"/
    ", files.mult[j], sep = ""), header = FALSE)
#                              CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-
    alpha/2)))))
#                              CI$YTrue <- Results_true[[i]]$mult
#                              CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                              CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                              CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                              CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                              CI <- split(CI, seq(nrow(CI)))
#                              return(CI)
#                            }
#     print(paste("Iteration", i, "multiplicative finished."))
#     for (j in 1:nrow(grid_all)) {
#       covadd[j,i] <- sum(DDist.add[[j]]$Coverage)/(blocks*gblocked)
#       covmult[j,i] <- sum(DDist.mult[[j]]$Coverage)/(blocks*gblocked)
#       lenadd[j,i] <- mean(DDist.add[[j]]$Length)
#       lenmult[j,i] <- mean(DDist.mult[[j]]$Length)
#       aboveadd[j,i] <- sum(DDist.add[[j]]$Above)
#       belowadd[j,i] <- sum(DDist.add[[j]]$Below)
#       abovemult[j,i] <- sum(DDist.mult[[j]]$Above)
#       belowmult[j,i] <- sum(DDist.mult[[j]]$Below)
#     }
#     gc()
#   }
# )
# stopImplicitCluster()
# gc()
#
# write.csv(covadd, file = paste("results/results_SO_MaxTgtMin/Coverage_MultivariateVect_Additive_SO_w",wi,"_
    r",ri,".csv", sep = ""))
# write.csv(lenadd, file = paste("results/results_SO_MaxTgtMin/Length_MultivariateVect_Additive_SO_w",wi,"_r
    ",ri,".csv", sep = ""))
# write.csv(covmult, file = paste("results/results_SO_MaxTgtMin/Coverage_MultivariateVect_Multiplicative_SO_w
    ",wi,"_r",ri,".csv", sep = ""))
# write.csv(lenmult, file = paste("results/results_SO_MaxTgtMin/Length_MultivariateVect_Multiplicative_SO_w",
    wi,"_r",ri,".csv", sep = ""))
# write.csv(aboveadd, file = paste("results/results_SO_MaxTgtMin/AboveCI_MultivariateVect_Additive_SO_w",wi,"
    _r",ri,".csv", sep = ""))
# write.csv(belowadd, file = paste("results/results_SO_MaxTgtMin/BelowCI_MultivariateVect_Additive_SO_w",wi,"
    _r",ri,".csv", sep = ""))
# write.csv(abovemult, file = paste("results/results_SO_MaxTgtMin/AboveCI_MultivariateVect_Multiplicative_SO_
    w",wi,"_r",ri,".csv", sep = ""))
# write.csv(belowmult, file = paste("results/results_SO_MaxTgtMin/BelowCI_MultivariateVect_Multiplicative_SO_
    w",wi,"_r",ri,".csv", sep = ""))

# ##################### BOOTSTRAP METHOD MVNSurfaces #####################
# DDist.add <- list()
# DDist.mult <- list()
# covadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# covmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
```

```r
# aboveadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# abovemult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
#
# cl <- makeCluster(numCores)
# registerDoParallel(cl)
# system.time(
#   for(i in 1:length(Beta.true)) {
#     print(paste("Iteration", i, "has started."))
#     DDist.add[[i]] <- vector("list", blocks*gblocked)
#     DDist.mult[[i]] <- vector("list", blocks*gblocked)
#     files.add <- list.files(path = paste("results/results_SO_MaxTgtMin/MVNSurface_add/w",wi,"r",ri,"/",sep = ""), pattern = paste("beta_",i, sep = ""))
#     files.mult <- list.files(path = paste("results/results_SO_MaxTgtMin/MVNSurface_mult/w",wi,"r",ri,"/", sep = ""), pattern = paste("beta_",i, sep = ""))
#     DDist.add <- foreach(j=1:(blocks*gblocked),
#                          .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]])), .export = c('fread', 'data.table')) %dopar% {
#                            tmp <- fread(paste("results/results_SO_MaxTgtMin/MVNSurface_add/w",wi,"r",ri,"/", files.add[j], sep = ""), header = FALSE)
#                            CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                            CI$YTrue <- Results_true[[i]]$add
#                            CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                            CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                            CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                            CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                            CI <- split(CI, seq(nrow(CI)))
#                            return(CI)
#                          }
#     print(paste("Iteration", i, "additive finished."))
#     DDist.mult <- foreach(j=1:(blocks*gblocked),
#                           .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]])), .export = c('fread', 'data.table')) %dopar% {
#                             tmp <- fread(paste("results/results_SO_MaxTgtMin/MVNSurface_mult/w",wi,"r",ri,"/", files.mult[j], sep = ""), header = FALSE)
#                             CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                             CI$YTrue <- Results_true[[i]]$mult
#                             CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                             CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                             CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                             CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                             CI <- split(CI, seq(nrow(CI)))
#                             return(CI)
#                           }
#     print(paste("Iteration", i, "multiplicative finished."))
#     for (j in 1:nrow(grid_all)) {
#       covadd[j,i] <- sum(DDist.add[[j]]$Coverage)/(blocks*gblocked)
#       covmult[j,i] <- sum(DDist.mult[[j]]$Coverage)/(blocks*gblocked)
#       lenadd[j,i] <- mean(DDist.add[[j]]$Length)
#       lenmult[j,i] <- mean(DDist.mult[[j]]$Length)
#       aboveadd[j,i] <- sum(DDist.add[[j]]$Above)
```

```
#       belowadd[j,i] <- sum(DDist.add[[j]]$Below)
#       abovemult[j,i] <- sum(DDist.mult[[j]]$Above)
#       belowmult[j,i] <- sum(DDist.mult[[j]]$Below)
#     }
#     gc()
#   }
# )
# stopImplicitCluster()
# gc()
#
# write.csv(covadd, file = paste("results/results_SO_MaxTgtMin/Coverage_MVNSurface_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(lenadd, file = paste("results/results_SO_MaxTgtMin/Length_MVNSurface_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(covmult, file = paste("results/results_SO_MaxTgtMin/Coverage_MVNSurface_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(lenmult, file = paste("results/results_SO_MaxTgtMin/Length_MVNSurface_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(aboveadd, file = paste("results/results_SO_MaxTgtMin/AboveCI_MVNSurface_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(belowadd, file = paste("results/results_SO_MaxTgtMin/BelowCI_MVNSurface_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(abovemult, file = paste("results/results_SO_MaxTgtMin/AboveCI_MVNSurface_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(belowmult, file = paste("results/results_SO_MaxTgtMin/BelowCI_MVNSurface_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
#
# ##################### BOOTSTRAP METHOD MVtSurfaces #####################
# ## Confidence Intervals for Additive and Multiplicative BetaTrue
# DDist.add <- list()
# DDist.mult <- list()
# covadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# covmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# aboveadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# abovemult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
#
# cl <- makeCluster(numCores)
# registerDoParallel(cl)
# system.time(
#   for(i in 1:length(Beta.true)) {
#     print(paste("Iteration", i, "has started."))
#     DDist.add[[i]] <- vector("list", blocks*gblocked)
#     DDist.mult[[i]] <- vector("list", blocks*gblocked)
#     files.add <- list.files(path = paste("results/results_SO_MaxTgtMin/MVtSurface_add/w",wi,"r",ri,"/",sep = ""), pattern = paste("beta_",i, sep = ""))
#     files.mult <- list.files(path = paste("results/results_SO_MaxTgtMin/MVtSurface_mult/w",wi,"r",ri,"/",sep = ""), pattern = paste("beta_",i, sep = ""))
#     DDist.add <- foreach(j=1:(blocks*gblocked),
#                          .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]])), .export = c('fread','data.table')) %dopar% {
```

```r
#                             tmp <- fread(paste("results/results_SO_MaxTgtMin/MVtSurface_add/w",wi,"r",ri,"/", files.add[j], sep = ""), header = FALSE)
#                             CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                             CI$YTrue <- Results_true[[i]]$add
#                             CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                             CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                             CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                             CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                             CI <- split(CI, seq(nrow(CI)))
#                             return(CI)
#                           }
#     print(paste("Iteration", i, "additive finished."))
#     DDist.mult <- foreach(j=1:(blocks*gblocked),
#                          .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]])), .export = c('fread','data.table')) %dopar% {
#                              tmp <- fread(paste("results/results_SO_MaxTgtMin/MVtSurface_mult/w",wi,"r",ri,"/", files.mult[j], sep = ""), header = FALSE)
#                              CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                              CI$YTrue <- Results_true[[i]]$mult
#                              CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                              CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                              CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                              CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                              CI <- split(CI, seq(nrow(CI)))
#                              return(CI)
#                            }
#     print(paste("Iteration", i, "multiplicative finished."))
#     for (j in 1:nrow(grid_all)) {
#       covadd[j,i] <- sum(DDist.add[[j]]$Coverage)/(blocks*gblocked)
#       covmult[j,i] <- sum(DDist.mult[[j]]$Coverage)/(blocks*gblocked)
#       lenadd[j,i] <- mean(DDist.add[[j]]$Length)
#       lenmult[j,i] <- mean(DDist.mult[[j]]$Length)
#       aboveadd[j,i] <- sum(DDist.add[[j]]$Above)
#       belowadd[j,i] <- sum(DDist.add[[j]]$Below)
#       abovemult[j,i] <- sum(DDist.mult[[j]]$Above)
#       belowmult[j,i] <- sum(DDist.mult[[j]]$Below)
#     }
#     gc()
#   }
# )
# stopImplicitCluster()
# gc()
#
#
# write.csv(covadd, file = paste("results/results_SO_MaxTgtMin/Coverage_MVtSurface_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(lenadd, file = paste("results/results_SO_MaxTgtMin/Length_MVtSurface_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(covmult, file = paste("results/results_SO_MaxTgtMin/Coverage_MVtSurface_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(lenmult, file = paste("results/results_SO_MaxTgtMin/Length_MVtSurface_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
```

```
# write.csv(aboveadd, file = paste("results/results_SO_MaxTgtMin/AboveCI_MVtSurface_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(belowadd, file = paste("results/results_SO_MaxTgtMin/BelowCI_MVtSurface_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(abovemult, file = paste("results/results_SO_MaxTgtMin/AboveCI_MVtSurface_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(belowmult, file = paste("results/results_SO_MaxTgtMin/BelowCI_MVtSurface_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
#
# ##################### BOOTSTRAP METHOD MVNSurfaces Covariance Structure #####################
#
# DDist.add <- list()
# DDist.mult <- list()
# covadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# covmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# aboveadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# abovemult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
#
# cl <- makeCluster(numCores)
# registerDoParallel(cl)
# system.time(
#   for(i in 1:length(Beta.true)) {
#     print(paste("Iteration", i, "has started."))
#     DDist.add[[i]] <- vector("list", blocks*gblocked)
#     DDist.mult[[i]] <- vector("list", blocks*gblocked)
#     files.add <- list.files(path = paste("results/results_SO_MaxTgtMin/MVNSurfaceSIG_add/w",wi,"r",ri,"/", sep = ""), pattern = paste("beta_",i, sep = ""))
#     files.mult <- list.files(path = paste("results/results_SO_MaxTgtMin/MVNSurfaceSIG_mult/w",wi,"r",ri,"/",sep = ""), pattern = paste("beta_",i, sep = ""))
#     DDist.add <- foreach(j=1:(blocks*gblocked),
#                          .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]])), .export = c('fread','data.table')) %dopar% {
#                            tmp <- fread(paste("results/results_SO_MaxTgtMin/MVNSurfaceSIG_add/w",wi,"r",ri,"/", files.add[j], sep = ""), header = FALSE)
#                            CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                            CI$YTrue <- Results_true[[i]]$add
#                            CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                            CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                            CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                            CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                            CI <- split(CI, seq(nrow(CI)))
#                            return(CI)
#                          }
#     print(paste("Iteration", i, "additive finished."))
#     DDist.mult <- foreach(j=1:(blocks*gblocked),
#                           .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]])), .export = c('fread','data.table')) %dopar% {
#                             tmp <- fread(paste("results/results_SO_MaxTgtMin/MVNSurfaceSIG_mult/w",wi,"r",ri,"/", files.mult[j], sep = ""), header = FALSE)
```

```
#                               CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                               CI$YTrue <- Results_true[[i]]$mult
#                               CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                               CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                               CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                               CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                               CI <- split(CI, seq(nrow(CI)))
#                               return(CI)
#                             }
#     print(paste("Iteration", i, "multiplicative finished."))
#     for (j in 1:nrow(grid_all)) {
#       covadd[j,i] <- sum(DDist.add[[j]]$Coverage)/(blocks*gblocked)
#       covmult[j,i] <- sum(DDist.mult[[j]]$Coverage)/(blocks*gblocked)
#       lenadd[j,i] <- mean(DDist.add[[j]]$Length)
#       lenmult[j,i] <- mean(DDist.mult[[j]]$Length)
#       aboveadd[j,i] <- sum(DDist.add[[j]]$Above)
#       belowadd[j,i] <- sum(DDist.add[[j]]$Below)
#       abovemult[j,i] <- sum(DDist.mult[[j]]$Above)
#       belowmult[j,i] <- sum(DDist.mult[[j]]$Below)
#     }
#     gc()
#   }
# )
# stopImplicitCluster()
# gc()
#
# write.csv(covadd, file = paste("results/results_SO_MaxTgtMin/Coverage_MVNSurfaceSIG_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(lenadd, file = paste("results/results_SO_MaxTgtMin/Length_MVNSurfaceSIG_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(covmult, file = paste("results/results_SO_MaxTgtMin/Coverage_MVNSurfaceSIG_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(lenmult, file = paste("results/results_SO_MaxTgtMin/Length_MVNSurfaceSIG_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(aboveadd, file = paste("results/results_SO_MaxTgtMin/AboveCI_MVNSurfaceSIG_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(belowadd, file = paste("results/results_SO_MaxTgtMin/BelowCI_MVNSurfaceSIG_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(abovemult, file = paste("results/results_SO_MaxTgtMin/AboveCI_MVNSurfaceSIG_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(belowmult, file = paste("results/results_SO_MaxTgtMin/BelowCI_MVNSurfaceSIG_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
#
# ##################### BOOTSTRAP METHOD MVtSurfaces Covariance Structure #####################
#
# DDist.add <- list()
# DDist.mult <- list()
# covadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# covmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# aboveadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
```

```
# abovemult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
#
# cl <- makeCluster(numCores)
# registerDoParallel(cl)
# system.time(
#   for(i in 1:length(Beta.true)) {
#     print(paste("Iteration", i, "has started."))
#     DDist.add[[i]] <- vector("list", blocks*gblocked)
#     DDist.mult[[i]] <- vector("list", blocks*gblocked)
#     files.add <- list.files(path = paste("results/results_SO_MaxTgtMin/MVtSurfaceSIG_add/w",wi,"r",ri,"/", sep = ""), pattern = paste("beta_",i, sep = ""))
#     files.mult <- list.files(path = paste("results/results_SO_MaxTgtMin/MVtSurfaceSIG_mult/w",wi,"r",ri,"/",sep = ""), pattern = paste("beta_",i, sep = ""))
#     DDist.add <- foreach(j=1:(blocks*gblocked),
#                          .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]])), .export = c('fread','data.table')) %dopar% {
#                            tmp <- fread(paste("results/results_SO_MaxTgtMin/MVtSurfaceSIG_add/w",wi,"r",ri,"/", files.add[j], sep = ""), header = FALSE)
#                            CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                            CI$YTrue <- Results_true[[i]]$add
#                            CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                            CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                            CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                            CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                            CI <- split(CI, seq(nrow(CI)))
#                            return(CI)
#                          }
#     print(paste("Iteration", i, "additive finished."))
#     DDist.mult <- foreach(j=1:(blocks*gblocked),
#                           .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]])), .export = c('fread','data.table')) %dopar% {
#                             tmp <- fread(paste("results/results_SO_MaxTgtMin/MVtSurfaceSIG_mult/w",wi,"r", ri,"/", files.mult[j], sep = ""), header = FALSE)
#                             CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                             CI$YTrue <- Results_true[[i]]$mult
#                             CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                             CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                             CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                             CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                             CI <- split(CI, seq(nrow(CI)))
#                             return(CI)
#                           }
#     print(paste("Iteration", i, "multiplicative finished."))
#     for (j in 1:nrow(grid_all)) {
#       covadd[j,i] <- sum(DDist.add[[j]]$Coverage)/(blocks*gblocked)
#       covmult[j,i] <- sum(DDist.mult[[j]]$Coverage)/(blocks*gblocked)
#       lenadd[j,i] <- mean(DDist.add[[j]]$Length)
#       lenmult[j,i] <- mean(DDist.mult[[j]]$Length)
#       aboveadd[j,i] <- sum(DDist.add[[j]]$Above)
#       belowadd[j,i] <- sum(DDist.add[[j]]$Below)
#       abovemult[j,i] <- sum(DDist.mult[[j]]$Above)
```

```
#       belowmult[j,i] <- sum(DDist.mult[[j]]$Below)
#     }
#     gc()
#   }
# )
# stopImplicitCluster()
# gc()
#
#
# write.csv(covadd, file = paste("results/results_SO_MaxTgtMin/Coverage_MVtSurfaceSIG_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(lenadd, file = paste("results/results_SO_MaxTgtMin/Length_MVtSurfaceSIG_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(covmult, file = paste("results/results_SO_MaxTgtMin/Coverage_MVtSurfaceSIG_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(lenmult, file = paste("results/results_SO_MaxTgtMin/Length_MVtSurfaceSIG_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(aboveadd, file = paste("results/results_SO_MaxTgtMin/AboveCI_MVtSurfaceSIG_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(belowadd, file = paste("results/results_SO_MaxTgtMin/BelowCI_MVtSurfaceSIG_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(abovemult, file = paste("results/results_SO_MaxTgtMin/AboveCI_MVtSurfaceSIG_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(belowmult, file = paste("results/results_SO_MaxTgtMin/BelowCI_MVtSurfaceSIG_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
#
# ###################### BOOTSTRAP METHOD Residuals ######################
# DDist.add <- list()
# DDist.mult <- list()
# covadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# covmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# lenmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# aboveadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowadd <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# abovemult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
# belowmult <- matrix(NA, nrow = nrow(grid_all), ncol = length(Beta.true))
#
#
# cl <- makeCluster(numCores)
# registerDoParallel(cl)
# system.time(
#   for(i in 1:length(Beta.true)) {
#     print(paste("Iteration", i, "has started."))
#     DDist.add[[i]] <- vector("list", blocks*gblocked)
#     DDist.mult[[i]] <- vector("list", blocks*gblocked)
#     files.add <- list.files(path = paste("results/results_SO_MaxTgtMin/BSR_add/w",wi,"r",ri,"/",sep = ""), pattern = paste("beta_",i, sep = ""))
#     files.mult <- list.files(path = paste("results/results_SO_MaxTgtMin/BSR_mult/w",wi,"r",ri,"/",sep = ""), pattern = paste("beta_",i, sep = ""))
#     DDist.add <- foreach(j=1:(blocks*gblocked),
#                          .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]])), .export = c('fread','data.table')) %dopar% {
#                            tmp <- fread(paste("results/results_SO_MaxTgtMin/BSR_add/w",wi,"r",ri,"/", files
```

```
.add[j], sep = ""), header = FALSE)
#                            CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                            CI$YTrue <- Results_true[[i]]$add
#                            CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                            CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                            CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                            CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                            CI <- split(CI, seq(nrow(CI)))
#                            return(CI)
#                          }
#     print(paste("Iteration", i, "additive finished."))
#     DDist.mult <- foreach(j=1:(blocks*gblocked),
#                           .combine = function(old,new)lapply(1:nrow(grid_all), function(x) rbind(old[[x]], new[[x]])), .export = c('fread','data.table')) %dopar% {
#                             tmp <- fread(paste("results/results_SO_MaxTgtMin/BSR_mult/w",wi,"r",ri,"/", files.mult[j], sep = ""), header = FALSE)
#                             CI <- data.table(t(apply(tmp, 1, function(x) quantile(x, probs = c(alpha/2, 1-alpha/2)))))
#                             CI$YTrue <- Results_true[[i]]$mult
#                             CI$Coverage <- apply(CI, 1, function(x) (x[1] <= x[3]) && (x[3] <= x[2]))
#                             CI$Length <- apply(CI, 1, function(x) x[2] - x[1])
#                             CI$Above <- apply(CI, 1, function(x) x[3] > x[2])
#                             CI$Below <- apply(CI, 1, function(x) x[3] < x[1])
#                             CI <- split(CI, seq(nrow(CI)))
#                             return(CI)
#                           }
#     print(paste("Iteration", i, "multiplicative finished."))
#     for (j in 1:nrow(grid_all)) {
#       covadd[j,i] <- sum(DDist.add[[j]]$Coverage)/(blocks*gblocked)
#       covmult[j,i] <- sum(DDist.mult[[j]]$Coverage)/(blocks*gblocked)
#       lenadd[j,i] <- mean(DDist.add[[j]]$Length)
#       lenmult[j,i] <- mean(DDist.mult[[j]]$Length)
#       aboveadd[j,i] <- sum(DDist.add[[j]]$Above)
#       belowadd[j,i] <- sum(DDist.add[[j]]$Below)
#       abovemult[j,i] <- sum(DDist.mult[[j]]$Above)
#       belowmult[j,i] <- sum(DDist.mult[[j]]$Below)
#   }
#   gc()
# }
# )
# stopImplicitCluster()
# # gc()
#
#
# write.csv(covadd, file = paste("results/results_SO_MaxTgtMin/Coverage_BSR_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(lenadd, file = paste("results/results_SO_MaxTgtMin/Length_BSR_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(covmult, file = paste("results/results_SO_MaxTgtMin/Coverage_BSR_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(lenmult, file = paste("results/results_SO_MaxTgtMin/Length_BSR_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
# write.csv(aboveadd, file = paste("results/results_SO_MaxTgtMin/AboveCI_BSR_Additive_SO_w",wi,"_r",ri,".csv
```

```
", sep = ""))
    # write.csv(belowadd, file = paste("results/results_SO_MaxTgtMin/BelowCI_BSR_Additive_SO_w",wi,"_r",ri,".csv", sep = ""))
    # write.csv(abovemult, file = paste("results/results_SO_MaxTgtMin/AboveCI_BSR_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
    # write.csv(belowmult, file = paste("results/results_SO_MaxTgtMin/BelowCI_BSR_Multiplicative_SO_w",wi,"_r",ri,".csv", sep = ""))
  }
}
```

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14. ABSTRACT

A shortfall of the Derringer and Suich (1980) desirability function is lack of inferential methods to quantify uncertainty. Most articles for addressing uncertainty usually involve robust methods, providing a point estimate that is less affected by variation. Few articles address confidence intervals or bands but not specifically for the Derringer and Suich method. This research provides two valuable contributions to the field of response surface methodology. The first contribution is evaluating the effect of correlation and plane angles on Derringer and Suich optimal solutions. The second contribution proposes and compares 8 inferential methods–both univariate and multivariate–for creating confidence intervals on each desirability function solution for first order and second order models. The effect of the Derringer and Suich method parameters, objective plane angles, and differing correlation between response surfaces are examined through simulation. The 8 proposed methods include a simple best/worst case method, 2 generalized methods, 4 simulated surface methods, and a nonparametric bootstrap method. One of the generalized methods, 2 of the simulated surface methods, and the nonparametric method account for covariance between the response surfaces. Bivariate examples showcase these methods in the first order and second order models. A multivariate real-world case with 3 objectives is also examined. While all 7 novel methods and the best/worst method seem to perform decently on the second order models. The methods which utilize an underlying multivariate-t distribution, Multivariate Generalized (MG) and Multivariate t Simulated Surface (MVtSSig), are recommended methods from this research as they perform well with small samples for both first order and second order models with coverage only becoming unreliable at non-optimal solutions. MG and MVtSSig inference should be used in conjunction with robust methods such as Pareto Front Optimization to help ascertain which solutions are more likely to be optimal before constructing confidence interval.

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