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Modern Approaches and Theoretical Extensions to the Multivariate Kolmogorov Smirnov Test

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MODERN APPROACHES AND THEORETICAL EXTENSIONS TO THE MULTIVARIATE KOLMOGOROV SMIRNOV TEST

DISSERTATION

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Abstract

Big data has become a common and normal practice for any organization. Most statistical tests and methods are fully developed for univariate data, and as such, a common approach to account for multiple variables is to perform projections or reductions in order to invoke a hypothesis test in univariate space. But when inference is required for multivariate data, these reductions to a univariate space risk information and interpretability loss. One of the most commonly used statistical tests to determine differences among data features is the Kolmogorov Smirnov (KS) test based upon the cumulative probability distributions of features, however, there are limitations in its current implementations. The purpose of this research is to develop a modern approach and theoretical extensions to the multivariate KS test that seeks to improve statistical power and incorporate correlation inherent in multivariate data. Specifically, this dissertation 1) derives a modified test that extends the KS test for 2 dimensions and into m-dimensions, 2) derives the small sample critical values for the 2 and 3 dimensional KS test that are not reliant on sample size simulations or correlation between variables, 3) extends large sample estimations and current KS implementations to larger sample sizes, and 4) provides sample size and power calculations in order to enable experimental design with respect to testing for differences in distributions. Through extensive simulation, we demonstrate for 2 and 3 dimensional data that our new modified multidimensional KS test generally has more power to detect differences than other methods for smaller sample sizes and comparable power for larger sample sizes (n=100 and larger) and maintains desirable statistical properties. Furthermore, we demonstrate that our revised critical values and methods improve and extend current implementations of the KS test to sample sizes upwards of n = 5000. Finally, we demonstrate how to compute critical values for implementations of our method to any size dimensional data and provide power and sample size criterion for designing studies using 2 and 3 dimensional distributions. These results enable statistical testing of multidimensional features, irrespective of correlation, thus improving our ability to understand large data sets for rapid and efficient decision making and analysis.

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Gonzalo Hernando

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1. Introduction

1.1. General Issue

Big data has become a common and normal practice for any organization. There are many methods for mining information or creating statistical and machine learning models to answer any question imaginable. When trying to understand features and their underlying random variables we have many statistical measures of information that can capture and summarize different characteristics to compare them against another dataset or against a theoretical distribution. From distance measures to probabilistic measures, the issue is that most of these measures are univariate and do not account for the relationship between the features. Statistical tests and multivariate hypothesis testing are falling behind the available quantity of features in many datasets.

1.2. Motivation

In statistics, a test of hypothesis revolves around testing a population parameter against either a hypothetical value or a parameter from a different population. Most statistical tests and methods are fully developed for univariate data, and as such, a common approach to account for multiple variables is to perform projections or reductions in order to invoke a hypothesis test in univariate space. But when inference is required for multivariate data, these reductions to a univariate space risk information loss and interpretability.

Examples in common analytical and predictive methodologies where this could become a problem are in decision tree algorithms (tree splitting criteria), topological data analysis or clustering algorithms; in such methods, the need to compare groups of data with respect to several variables is very important. Simple univariate tests to see if, for instance, the means of continuous random variables are statistically different from a hypothetical value, or from each other, is not sufficient to capture all of the information the groups of data might contain. Further, mathematical projections of continuous random variables to a univariate space may provide a means for inference, but disparate variable patterns may be masked when conducting variable reduction leading to increases in the type II (or possibly type I) error.

Most of the methods used to find similarity of continuous random variables within datasets, between datasets or against a theoretical distribution fall into two categories, i.e., mathematical distances and probabilistic measures [1]. A common factor among these measures is that they are usually univariate and therefore do not consider the correlation between variables within each group. Many studies and algorithms have addressed the multivariate implications by performing dimensionality reduction processes on the dataset such as principal component analysis (PCA) or linear discriminant analysis (LDA). Once the data is projected to a univariate or independent bivariate space, the same univariate algorithms can be used to compute the mathematical distances and probabilistic measures [2]. However, information may be lost and interpretability of the new space might be impossible.

There is a need to expand these univariate measurements to the multivariate space while correctly deriving their hypothesis tests, corresponding critical values and theory, instead of reducing the dimensionality of the data and performing a univariate test. There have been some multivariate implementations that extend these measures, but they may be lacking in theory, estimations or attempt to account for correlation with ad hoc adjustments and corrections. Ideally, as we expand univariate measures into multiple dimensions, the measurements should be more representative of the multivariate data, with more accurate hypothesis testing and without losing interpretability or data information.

1.3. Air Force Impact

One of the five strategic capabilities that the Air Force Science and Technology Strategy 2030 emphasizes is the "Rapid, Effective Decision-Making" ([3]). Some of the technological opportunities for this strategy include machine learning and machine-based reasoning, data fusion and visualization. There are many techniques and methods that fall inside these technological opportunities including topological data analysis and statistical hypothesis testing. The Air Force Research Laboratory Airman Systems Directorate (AFRL/RH) has applied these techniques to various large dataset including Air Force suicide data as well as COVID-19 data. The goal of these analyses was to visualize, interpret and provide insight into the feature space of the data in order to better predict the phenomenon of interest and allow leadership to make fast and informed decisions. One of the tools used to provide insight into these feature spaces was the one-dimensional Kolmogorov Smirnov test which was used univariately across all continuous features to score and compare the topological decomposition of the data. However, predictors of the phenomenon of interest are likely correlated and this univariate application was lacking at times in discriminatory power. The work in this dissertation extends the nonparametric Kolmogorov Smirnov test into multiple dimensions and provides higher discriminatory power than existing methods, allowing for multiple continuous features to be tested simultaneously and to account for correlation between them.

1.4. The Kolmogorov Smirnov Test

The 1-dimensional Kolmogorov Smirnov (1D KS) test is a nonparametric and distribution free test based upon the maximum distance between two Cumulative Distribution Functions (CDFs). This test is known to be one of the most important of the general goodness of fit tests after the chi-square goodness of fit test [4]. Modern applications of this test seek to extend and utilize this test for multivariate distributions. In 1983, Peacock extended the 1D KS test into two dimensions [5], but was unable to create a table of derived critical values due to the computational infeasibility of the combinatorics, therefore, Peacock estimated the asymptotic critical values empirically by fitting a curve to simulated data and deriving a correction for small samples. He found that these critical values were similar to the 1D KS test with the addition of an offset. In 1987, Gosset expanded the KS to three dimensions (using the same methodology of Peacock) as well as derived new asymptotic equations to estimate the critical values for both the 2D and 3D KS tests, thereby updating and seemingly improving on the 2D KS asymptotic equation developed by Peacock [6]. Furthermore, both Peacock and Gosset extended the one sample (1S) 2D KS test to two samples by looking at the maximum distance between two Empirical Cumulative Distribution Functions (ECDFs) and using the same 1D KS test standardization. Nevertheless, the computational burden of the multi-dimensional KS test as developed by both Peacock and Gosset is significant due to a reliance on evaluating maximum distances at every possible location where the ECDFs change, or jump, in probability. This is why in 1987 Fasano and Franceschini proposed a method of evaluating the max distances only at the observed data points (a subset of the data grid considered by Peacock's test) in order to address the computational time issue. Fasano and Franceschini claim that there is minor difference in power between Peacock's method and their method when data is uncorrelated and that their method has higher power when data is highly correlated [7], but offered no proof. As a result, unlike the evaluation method used by Peacock, the evaluation method of Fasano and Franceschini must account for correlation when deriving the null distribution in order to compute the correct critical values based on correlation (see table of critical values in the appendix of [7]). Other multivariate KS methods have been developed by transforming and projecting the data to the univariate space before computing the maximum distance [8], however, these methods will not be considered in this work due to the potential loss of information and interpretability.

There are several limitations in the current literature for the method derived by Peacock and extended by Gosset: 1) when computing the maximum distance at a particular location, not all directions (approaching the point from the "left" or "right" for both x and y) were considered and therefore, not all distances as a result of the jumps in the empirical CDFs are considered, 2) derived calculations for small sample critical values were not computed and instead a transformation from small sample to large was generated using simulated data, and 3) the asymptotic equations proposed are not mathematically derived, but fitted to (what is now considered) small samples of simulated data. With respect to the latter limitation, Peacock's largest sample size was 50 with 5,000 repetitions, while Gosset's, although not explicitly stated, appears to be on the same order of magnitude as Peacock. On the other hand, Fasano and Franceschini generated tables of critical values (with various degrees of correlation) and developed a generalized formula for computing the critical values using simulated data with sample sizes as large as 5,000 with 500 repetitions. However, the critical values generated by Fasano and Franceschini only apply to their method given that, as a subset of the locations evaluated in Peacock's work, it generates a different distribution. The work in this dissertation seeks to overcome these limitations and generate a methodology to apply the KS test not only to small and large samples, but to 2, 3, and *m*-dimensional data.

1.5. Research Objectives

The purpose of this research is to develop a modern approach and theoretical extensions to the multivariate (KS) test that seeks to capture the true maximum distance between an ECDF and a theoretical CDF. This research will focus on 1) developing a new approach to the multivariate KS test by deriving a modified test that extends the original 1S 2D KS test approach proposed by Peacock for both 2 dimensions and into m-dimensions, 2) mathematically deriving the small sample critical values for the 1S 2D KS test that are not reliant on sample size simulations or correlation between variables, and 3) comparing the performance of this new method against similar approaches for both ECDF evaluations (locations and directions). Achieving these objectives will lead to better understanding of best practices and sample size recommendation for the KS test. Furthermore, reaching these objectives will lead to an approach for this statistical test that results in higher power for the multivariate KS test and an extended table of critical values for the original 2D KS method.

2. Background

2.1. Background and Method

Before developing the new theoretical extensions and modern approach to conducting the KS test in multiple dimensions, we begin by briefly discussing the history of the 1S/2S 1D KS test as well as necessary assumptions and requirements. Furthermore, we will be reviewing the current extensions into multiple dimensions for the KS test, mainly Peacock's approach (the original extension) and Fasano and Franceschini's modifications of Peacock.

2.2. One Dimensional (1D) KS Test

The 1D KS test was first introduced to compare sample data against a theoretical null distribution (known as the one sample KS test). This nonparametric test uses a statistic based upon the maximum distance between estimates of a Cumulative Distribution Function (CDF) from the empirical CDF (ECDF) generated by the data and a theoretical CDF with the goal of testing whether the sample of data follows (is the same as) the theoretical distribution. The null hypothesis (H_0) for the test states that there is no difference in distribution between the sample and theoretical distributions (see Equation (2.1)). While the alternative (H_a) states that the sample and theoretical distributions are different (see Equation (2.2)).

$$H_0: F_n(x) = F(x) \tag{2.1}$$

$$H_a: F_n(x) \neq F(x) \tag{2.2}$$

The one sample (1S) 1D KS test was first developed by Kolmogorov in 1933 [4], [9] and its statistic is given in Equation (2.3):

$$D_n^{(1)} = \sup_x |F_n(x) - F(x)| \tag{2.3}$$

where $F_n(x)$ is the ECDF of a univariate random variable X, *n* references the sample size for the single sample of observed data, and $F(x)$ is the theoretical (null) CDF. The superscript (1) in the statistic denotes the dimension of the test data.

Historically, the CDF and ECDF of a distribution, $F(x)$ and $F_n(x)$ respectively, as used in Equation (2.3), is defined as $P(X \leq x)$ and therefore, has the following properties: 1) right continuity, 2) non-decreasing function, and 3) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$. In addition to these properties, $F(x)$ is smooth and continuous everywhere and therefore $\lim_{x \to a} F(x)$ exists for all $a \in \mathbb{R}$, meaning that approaching a point *a* from the "right" or the "left" achieves the same value. On the other hand, $F_n(x)$, since it is a discretized estimate of the CDF for the random variable, will contain jumps as data is observed and therefore, will have different values as you approach a point *a* from the "left" (denoted here as x^-) and the "right" (usually denoted as x^+, or here simply as *x*). Due to this discontinuity, it is necessary to evaluate not just Equation (2.3), but also Equation (2.4) in order to capture both distances in the ECDF:

$$D_n^{(1)} = \sup_x |F_n(x^-) - F(x)|, \tag{2.4}$$

where $F_n(x^-)$ represents approaching the ECDF from the left.

Soon after the one sample 1D KS test was developed, in 1939 Smirnov extended this test to two samples establishing the commonly known name of the Kolmogorov-Smirnov (KS) test which (perhaps confusingly) refers to both the 1S and the 2S versions of this test for differences in CDFs based on maximum distances [10]. Equation (2.5) provides the test statistic for the 2S version of the KS test, which has the modified null hypothesis $H_0: F_{n_1}(x) = G_{n_2}(y)$ and with the alternative hypothesis of $H_a: F_{n_1}(x) \neq G_{n_2}(y)$.

$$D_n^{(1)} = \sup_{x,y} \left|F_{n_1}(x) - G_{n_2}(y)\right| \; \forall \; x = y \tag{2.5}$$

where $F_{n_1}(x)$ is the ECDF estimated from the first sample of data for one random variable, X, with sample size n_1, and $G_{n_2}(x)$ is the ECDF estimated from the second sample of data for the second random variable, Y with sample size n_2. Notice that for the two-sample case there is no need to evaluate when the ECDFs approach from the left, given that we assume that observations between the samples are unique (being continuous random variables) and therefore, the distance estimated by $\sup_{x,y}\left|F_{n_1}(x) - G_{n_2}(y)\right|$ should be captured by the previous jump in $G_{n_2}(y)$ or $F_{n_1}(x)$.

Furthermore, Smirnov developed a table of critical values needed for the 2D 1S test [10], [11]. Other tables of critical values for various levels of α, sample sizes and inverse tables have been generated by various authors [12]–[14]. To generate the exact critical values for this test, one needs to develop the distribution of the test statistic in Equation (2.5) (the two sample KS distribution) based on the specific sample sizes, n_1 and n_2. In other words, one needs to look at all possible orderings for the specified sample sizes and calculate the KS critical value. For example, with samples of size 2 (observed from random variable X) and 3 (observed from random variable Y) one would need to find the max distances between the CDFs for all 10 orderings of these two sample sizes in order to build the two sample KS distribution and calculate, then, the probabilities associated with each possible maximum distance. This becomes computationally quite difficult as sample size increases which is why Kolmogorov in his original paper [9], as well as others, also developed the limiting distribution for the test statistic (see Equation (2.6)):

$$P\left(Z_n^{(1)} \leq z\right) = 1 - 2\sum_{k=1}^{\infty}(-1)^{k-1}\, e^{-2k^2z^2} \qquad (2.6)$$

where $Z_n^{(1)} = \sqrt{n}D_n^{(1)}$, n is the number of observations (for two samples $n = \frac{n_1 n_2}{n_1+n_2}$) and (1) superscript stands for the dimension of the test data. Kolmogorov [9], [15] and Smirnov [10], [16] separately and with different approaches derived Equation (2.6), but it was Feller [17] that unified and solidified the derivation of the limiting distribution for the KS statistic. Doob [18] provided a heuristic approach to the proof while more recently it has been presented and explained by Vrbik [19]). On the other hand, it was Birnbaum [12] who tabulated the critical values of the KS statistic.

The KS limiting distribution provided in Equation (2.7) is commonly used for large sample sizes, which approximates the infinite sum from Equation (2.6).

$$P\left(Z_n^{(1)} \leq z\right) = 2e^{-2z^2} \tag{2.7}$$

One of the comments made by Vrbik [16] and Marsaglia [19], [20] is the accuracy of Equation (2.7) for small samples, given that the limiting distribution is reached after the sample size is well over a thousand observations. Several corrections to Equation (2.7) have been made to improve the probability estimates for smaller samples, the latest corrections being [21]–[23] (see Equation (2.8)):

$$P\left(Z_n^{(1)} \leq z\right) = 1 - 2\sum_{k=1}^{\infty}(-1)^{k-1}\, e^{-2k^2\left(z+\frac{1}{6\sqrt{n}}+\frac{z-1}{4n}\right)^2} \tag{2.8}$$

Peacock [5] took a different approach for correcting the small sample accuracy issues by creating a correction ($\delta_n^{(1)}$) based on a ratio of critical values between standardized and large sample critical values. Peacock treated the large sample critical values as the asymptotic ones, which he referred to as $Z_\infty^{(1)}$, the standardized distance which we know to be $Z_n^{(1)} = \sqrt{n}D_n^{(1)}$. When fitting a set of critical values based on sample size, α and using Equation (2.9). Peacock determined the best fit line to be $0.2n^{-0.6}$. This function was based only on sample size and by solving for $z_\infty^{(1)}$, one can correct any standardized distance (see Equation (2.10)) in order to use the large sample critical values regardless of sample size. This correction was not derived but instead approximated using Birnbaum tabulated results [5].

$$\delta_n^{(1)} = 1 - \frac{z_n^{(1)}}{z_\infty^{(1)}} \tag{2.9}$$

$$z_\infty^{(1)} = \frac{z_n^{(1)}}{1 - 0.2n^{-0.6}} \tag{2.10}$$

2.3. 2D and 3D KS Test – Peacock's Implementation

Before introducing Peacock's extensions to the 1S 2D KS test, we will introduce a common nomanclature that will be used across all different methods and dimensions of the test throughout this document. Similar to the 1D case, $D_{n,method}^{(2)}$ refers to the KS test statistic for sample size *n*, "method" will be the two letter initial of the various methods presented in the next two sections, and the superscript (2) represents the number of dimensions for the random variable(s) of interest. Furthermore, $D_{n,method}^{(2)I}$ specifies the cumulation ordering (in this case, I). For the 2D KS test, there are a total of four cumulation orderings represented by roman numerals which coincide with the quadrants in the cartesian plane: I – $X \geq x, Y \geq y$, II – $X \leq x, Y \geq y$, III – $X \leq x, Y \leq y$ and IV – $X \geq x, Y \leq y$. Moving forward we will refer to these as orientations (a detailed explanation of these orientations is provided in section 3.1). Finally, $D_{n,method}^{(2)I++}$ apart from specifying the dimension and orientation, also specifies the approaching direction for the evaluation point, for *x* and *y* respectively, i.e., whether the evaluation point is approached from the positive or negative horizontal (*x*) and positive or negative vertical (*y*) direction.

In 1983, Peacock extended the 1S 1D KS test to 2D by looking at the maximum distance of the 2D CDFs for bivariate random variables [5]. Although the definition was not stated, Peacock, as well as other authors, appear to rely on the standard extended defintition for the CDF of a 2D random variable given by $P(X \leq x, Y \leq y)$ which is orientation III. However, many authors recognized that there is no fixed way to order 2D data, and therefore considered the cumulation in both the x and y directions by looking at the following cumulations in each orientation $(X > x, Y > y)$, $(X < x, Y > y)$, $(X < x, Y < y)$, and $(X > x, Y < y)$. The original cumulations that Peacock proposed had no equalities when describing the quadrants to cummulate, but when Gosset summarized the work he added equalities consistent with the standard extended defintition for the CDF of a 2D random variable given by $P(X \leq x, Y \leq y)$ (see Equation (2.11)). This is most relevant for small samples given that as sample size increases the difference in the estimates of the probabilities when using sample data between using equalities and inequalities will be $\frac{1}{n}$. In this work we will continue with the equalities when referencing these earlier works and refer to this method with the equalities as the *Partial Orientation method*.

$$(X > x, Y > y),\ (X \leq x, Y > y),\ (X \leq x, Y \leq y), \text{ and } (X > x, Y \leq y) \quad (2.11)$$

It is important to recognize that the Partial Orientation method uses a combination of equalities and inequalities for each orientation creating the following method to evaluate the maximum difference of the CDFs at all four orientations and evaluated at all possible locations, creating the following test statistic in Equation (2.12):

$$D_{n,pg}^{(2)} = \max\left(D_{n,pg}^{(2)I}, D_{n,pg}^{(2)II}, D_{n,pg}^{(2)III}, D_{n,pg}^{(2)IV}\right) \quad (2.12)$$

where $D_{n,pg}^{(2)}$ represents the test statistic, '*pg*' represents the method Partial Orientation Grid, and:

$$D_{n,pg}^{(2)I} = \left(D_{n,g}^{(2)I--}\right), D_{n,pg}^{(2)II} = \left(D_{n,g}^{(2)II-+}\right), D_{n,pg}^{(2)III} = \left(D_{n,g}^{(2)III++}\right), \qquad (2.13)$$

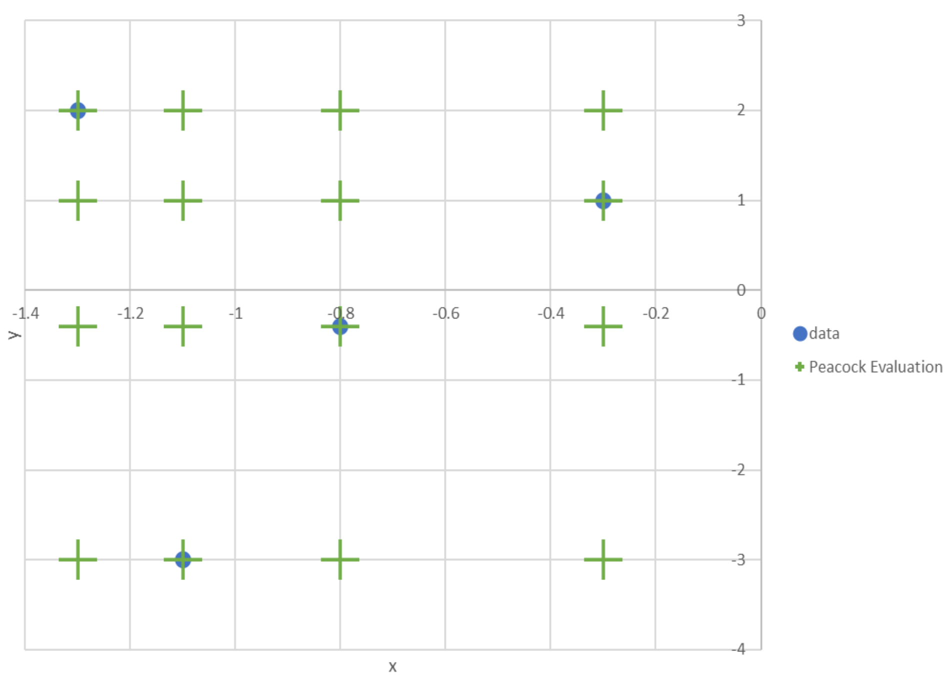
$$D_{n,pg}^{(2)IV} = \left(D_{n,g}^{(2)IV+-}\right)$$

the "p" for Partial Orientation method in the subscript represents the direction that is used for each orientation and for example:

$$D_{n,g}^{(2)III++} = \sup_{all\ x,y} \left| F_n^{(2)III}(x^+, y^+) - F^{(2)III}(x, y) \right| \qquad (2.14)$$

where $F_n^{(2)III}(x^+, y^+)$ is the ECDF in orientation III with direction (x^+, y^+) and $F^{(2)III}(x, y)$ is the theoretical CDF in the specific orientation. The "g" for grid in the subscript represents the evaluation location for the supremum, in this case all x, y. To find the true maximum difference we would have to evaluate $D_{n,pg}^{(2)}$ everywhere in the cartesian plane, but it is sufficient to only look at the locations where the ECDF changes (the grid created by the (x, y) pairs: $(x_i, y_j)\ \forall\ i, j = 1, \dots, n$ where n is the sample size). Figure 2.1 shows an example of the

Figure 2.1: 1S 2D KS Test Evaluation Location



locations where Peacock evaluates the ECDF. The four blue dots represent the sample data, while the green crosses represent all the evaluation locations. Therefore, the number of computations needed to compute $D_{n,pg}^{(2)}$ totals $4n^2$ (4 for each direction and n^2 for the grid generated by the data of size n). As mentioned earlier, we will refer to this approach as Partial Orientation Grid method, where partial orientation refers to the types of cumulation in each orientation with specific directions (see Equations (2.12) and (2.13)) while "grid" represents the evaluation locations (the $4n^2$ locations created by the (x, y) pairs: $\left(X_i, Y_j\right) \forall\, i, j = 1, \ldots, n$).

In order to perform the hypothesis test for the test statistic $D_{n,pg}^{(2)}$ (see Equation (2.12)), Peacock developed a table of critical values for sample sizes ranging from 3 to 50 observations (see table in [5]). Furthermore, an asymptotic equation for the test was fitted to the simulation data (see Equation (2.15)):

$$P\left(Z_{\infty}^{(2)} > z\right) = 2e^{-2(z-0.5)^2}, \tag{2.15}$$

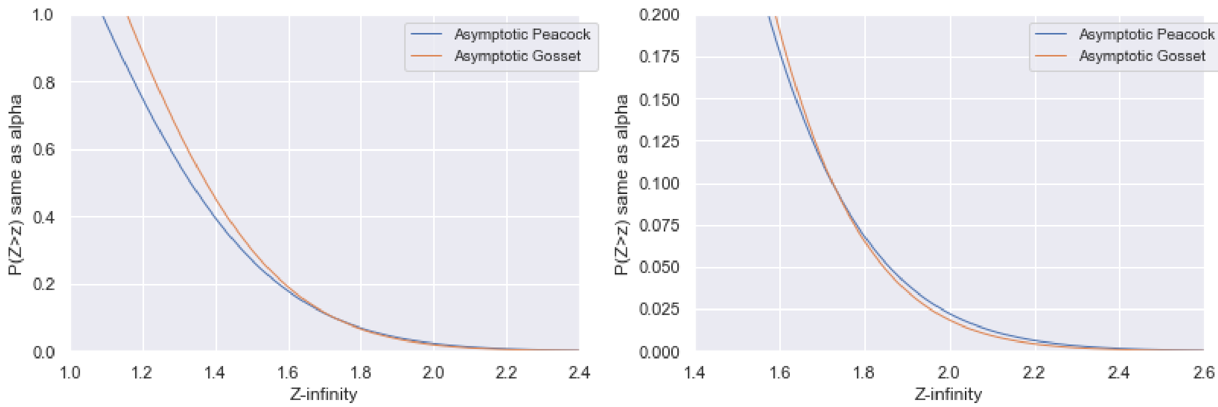
where $Z_{\infty}^{(2)}$, similar to the 1D case, represents what Peacock calls the asymptotic standarized distance of $D_{n,pg}^{(2)}$, which requires a correction

$$\delta_{pg}^{(2)} = 1 - \frac{z_{n,pg}^{(2)}}{z_{\infty}^{(2)}} = 0.53n^{-0.9}, \tag{2.16}$$

where $Z_{n,pg}^{(2)} = \sqrt{n} D_{n,pg}^{(2)}$ (for the two samples 2D test, the only difference is that $n = \frac{n_1 n_2}{n_1 + n_2}$).

The main limitation with Peacock's approach is that he relied on simulated data to estimate his asymptotic equations, with the largest simulation sample of data being 50 with only 5,000 repetitions. Furthermore, to ensure adequate fit of the asymptotic equation, Peacock only fit the asmpytotic equation to the range of α values from 0 – 0.2 which could limit the total accuracy of the general fit especially for large α values, albeit focusing the fit of the asymptotic

Figure 2.2: Peacock and Gosset Asymptotic Equations



equation in the range of commonly used α values. Due to all the constraints of the fit of the asymptotic equation by Peacock in 1983; Gosset, using the same 1S 2D KS test approach as Peacock, fitted his own asymptotic Equation (2.17) to his simulation to generate a different asymptotic equation for the 1S 2D KS test [6]:

$$P\left(Z_{\infty,pg}^{(2)} > z\right) = 2e^{-2.5(z-0.63)^2} \qquad (2.17)$$

Although not explicitly stated, it seems that Gosset might have had more simulations and/or larger sample sizes when fitting his asymptotic equation.

These two fitted asymptotic equations (see Equations (2.15) and (2.17)) are remarkably similar, but there are slight differences that can be seen in Figure 2.2, mainly the shape of the curve and the differences for higher α values. Specifically, we can see that Gosset has higher $z_\infty^{(1)}$ values than Peacock for α values greater than 0.2 while for α values between 0.2 and 0.05 Gosset is only slighly higher and below 0.05 Peacock has slighly higher values. These differences point to the bigger issue of fitting the estimated distribution of a test statistic using simulated data.

In addition to correcting the asymptotic equation of Peacock, Gosset extended the 2D KS test to 3 dimensions by following the same methods as Peacock, but considering all eight orientations, which represent the 8 quadrants of the cartesian coordinates in 3 dimensions defined by: I – $X \geq x, Y \geq y, Z \leq z$, II – $X \leq x, Y \geq y, Z \leq z$, III – $X \leq x, Y \leq y, Z \leq z$, IV – $X \geq x, Y \leq y$, V – $X \geq x, Y \geq y, Z \geq z$, VI – $X \leq x, Y \geq y, Z \geq z$, VII – $X \leq x, Y \leq y, Z \geq z$, and VIII – $X \geq x, Y \leq y, Z \geq z$. Gosset followed his same logic to determine the equalities and inequalities in each orientation (see Equation (2.18)).

$$(X \leq x, Y \leq y, Z \leq z), (X \leq x, Y \leq y, Z > z), (X \leq x, Y > y, Z \leq z),$$
$$(X > x, Y \leq y, Z \leq z), (X \leq x, Y > y, Z > z), (X > x, Y \leq y, Z > z),$$
$$(X > x, Y > y, Z \leq z), (X > x, Y > y, Z > z) \tag{2.18}$$

Furthermore, similar to the 2D test, we now must consider approaching *x*, *y*, and *z* from the "left: or the "right" directions (for example (x^{+}, y^{-}, z^{+}) approaching *x* and *z* from the "right" direction and *y* from the "left" direction). The proposed 3D test follows the same approach as Peacock where only one combination of approaching direction per orientation/quadrant is used (with equalities as seen in Equation (2.18)), that is, not all directions in all orientations are considered. The statistic is defined in Equation (2.19):

$$D_{n,pg}^{(3)} = max(\, D_{n,pg}^{(3)I}\,, D_{n,pg}^{(3)II}\,, D_{n,pg}^{(3)III}\,, D_{n,pg}^{(3)IV}\,, D_{n,pg}^{(3)V}\,, D_{n,pg}^{(3)VI}\,, D_{n,pg}^{(3)VII}\,, D_{n,pg}^{(3)VIII}\,) \tag{2.19}$$

where $D_{n,pg}^{(3)}$ represents the test statistic, and:

$$D_{n,pg}^{(3)I} = \left(D_{n,g}^{(3)I--+}\right), D_{n,g}^{(3)II} = \left(D_{n,g}^{(3)II+-+}\right), D_{n,pg}^{(3)III} = \left(D_{n,g}^{(3)III+++}\right),$$
$$D_{n,pg}^{(3)IV} = \left(D_{n,g}^{(3)IV-++}\right), D_{n,pg}^{(3)V} = \left(D_{n,g}^{(3)V---}\right),\ D_{n,pg}^{(3)VI} = \left(D_{n,g}^{(3)VI+--}\right),$$
$$D_{n,pg}^{(3)VII} = \left(D_{n,g}^{(3)VII++-}\right), and\ D_{n,g}^{(3)VIII} = \left(D_{n,g}^{(3)VIII-+-}\right) \tag{2.20}$$

where the roman numeral represents the orientation in 3D as seen in Equation (2.18) and for example:

$$D_{n,g}^{(3)III+++} = \sup_{all\ x,y,z} \left| F_n^{(3)III}(x^+, y^+, z^+) - F^{(3)III}(x, y, z) \right|, \quad (2.21)$$

where $F_n^{(3)III}(x^+, y^+, z^+)$ is the ECDF in orientation I with direction (x^+, y^+, z^+) and $F^{(3)III}(x, y, z)$ is the theoretical CDF in the specific orientation. For both the 2D and 3D cases, Gosset used the grid approach for evaluating the CDFs, that is, he calculated the maximum distance at all locations where the CDFs change, namely, the 3D grid generated by $\left(X_i, Y_j, Z_k\right)\ \forall\, i, j, k = 1, \ldots, n$, where n is the sample size, causing the number of computations needed to run the test in 3D to be $8n^3$. We can quickly see how this grid approach becomes computationally intense as the number of dimensions (and the sample size) increases. The same procedure for standarizing the 2D case was applied here: $Z_{n,pg}^{(3)} = \sqrt{n} D_{n,pg}^{(3)}$ (for the two sample case $n = \frac{n_1 n_2}{n_1 + n_2}$).

Gosset also developed an associated asymptotic equation (see Equation (2.22)) for the distribution of the test statistic for the 1S 3D KS test, a correction for small samples (see Equation (2.23)) and a table of critical values (see [6]). These solutions were developed using simulated data with largest sample size of data being n = 100 replicated 5,000 times.

$$P\left(Z_{\infty,pg}^{(3)} > z\right) \cong 2e^{-3(z-1.05)^2} \quad (2.22)$$

$$\delta_{pg}^{(3)} = 1 - \frac{Z_{n,pg}^{(3)}}{Z_{\infty,pg}^{(3)}} = 0.75n^{-0.9} \quad (2.23)$$

2.4. 2D and 3D KS Test Fasano and Franceschini Implementation

Several years after Peacock developed the extension to the 1S and 2S 2D KS test, Fasano and Franceschini [7] determined that it was sufficient to estimate the maximum distance using only values computed at locations where data was observed: $\left(X_i, Y_i\right)\ \forall\, i = 1, \ldots, n$ where n is the sample size, rather than at all grid locations. Referencing Figure 2.1, we see all the locations the grid method evaluates to find the maximum distance which includes all 16 locations marked with data and +'s. Fasano and Franceschini, instead of evaluating at all grid locations, focused only on locations where the data is located (4 data points in Figure 2.1). Thus, Fasano and Franceschini used the same definitions for the maximum distances from which to estimate the CDFs, calculated from the locations and directions as defined in Peacock's work, but restricted the evaluation locations. The original work by Fasano and Franceschini used the same orientations and directions as Peacock, namely, $(X > x, Y > y)$, $(X < x, Y > y)$, $(X < x, Y < y)$, and $(X > x, Y < y)$ but again, when summarized by Gosset, the equalities were introduced into these expressions (see Equation (2.11)). The work presented here will maintain the equalities and refer to this method as Partial Orientation Sample with the test statistic defined as Equation (2.24):

$$D_{n,ps}^{(2)} = \max\left(D_{n,ps}^{(2)I}, D_{n,ps}^{(2)II}, D_{n,ps}^{(2)III}, D_{n,ps}^{(2)IV}\right) \tag{2.24}$$

where $D_{n,ps}^{(2)}$ represents the test statistic, 'ps' represents the Partial Orientation Sample method, and:

$$D_{n,ps}^{(2)I} = \left(D_{n,s}^{(2)I--}\right), D_{n,ps}^{(2)II} = \left(D_{n,s}^{(2)II-+}\right), D_{n,ps}^{(2)III} = \left(D_{n,s}^{(2)III++}\right), \quad D_{n,ps}^{(2)IV} = \left(D_{n,s}^{(2)IV+-}\right) \tag{2.25}$$

the "*p*" for Partial Orientation method in the subscript represents the direction that is used for each orientation and for example:

$$D_{n,s}^{(2)III++} = \sup_{(x_i, y_i)\ \forall\ i=1,\dots,n} \left| F_n^{(2)III}(x^+, y^+) - F^{(2)III}(x, y) \right| \tag{2.26}$$

where $F_n^{(2)III}(x^+, y^+)$ is the ECDF in orientation III with direction (x^+, y^+) and $F^{(2)III}(x, y)$ is the theoretical CDF in the specific orientation. The "*s*" for sample in the subscript represents the evaluation location for the supremum, in this case only locations where observations are present: $(X_i, Y_i)\ \forall\ i = 1, \dots, n$.

The one change of restricting the evaluation locations, that according to Fasano and Franceschini has similar performance and power as the original method, causes a substantial decrease in the number of evaluations required to find the maximum distance from *4n*^{2} to *4n*. For simplicity, this method of evaluation will be referred to as Partial Orientation Sample (and annotated in the statistic as "*ps*") where, partial orientation refers to the type of cumulation used in each orientation while sample represents the restricted evaluation locations proposed by Fasano and Franceschini.

As an aside, Fasano and Franceschini claimed that to perform the 2S 2D KS test, the only two things that needed to change were to use $n = \frac{n_1 n_2}{n_1 + n_2}$ and take the average of the maximum distance when calculating the test statistic using the data observed in sample 1 $\left(D_{n,ps}^{(2)}\right)_1$ and sample 2 $\left(D_{n,ps}^{(2)}\right)_2$: $Z_{n,ps}^{(2)} = \sqrt{n}\frac{\left(D_{n,ps}^{(2)}\right)_1 + \left(D_{n,ps}^{(2)}\right)_2}{2}$.

In addition to the 2D implementation of the method developed by Fasano and Franceschini, they extended their method to 3D which follows the same implementation and definitions as the 2D case but for 3 variables, with the only difference being the evaluation locations which now consist of $\left(X_i, Y_i, Z_i\right)\ \forall\ i = 1, \dots, n$, not the grid of all n^3 locations, but only the *n* locations where data is observed. The test statistic $D_{n,ps}^{(3)}$ is the same as Equations (2.19) and (2.20) with the difference that instead of "grid" the statistic uses "sample" meaning that, for instance, Equation (2.21) changes to Equation (2.27) (limiting the supremum).

$$D_{n,s}^{(3)III+++} = \sup_{(x_i, y_i, z_i)\ \forall\ i=1,\dots,n} \left| F_n^{(3)III}(x^+, y^+, z^+) - F^{(3)III}(x, y, z) \right|, \qquad (2.27)$$

The main limitation with both the 2D and 3D Partial Orientation Sample methods is that, due to the restricted evalution locations, correlation needs to be considered when generating the critical values or using the tables provided in [7], this is because when Fasano and Franceschini tried to show sufficiently distribution free property using various well known distributions for their method, they noticed that the test statistic null distribution for highly correlated data did not have the same distribution as the null distribution for non-correlated distributions. Furthermore, similar to Peacock's and Gosset's derivation of critical values, Fasano and Franceschini used their simulated data (in their case up to sample size 5,000 with 500 repetitions) to generate their critical value tables and formulas. Using the 1S 2D KS tables generated by Fasano and Franceschini, Press and Teukolsky created an asymptotic equation (see Equation (2.28) and generated the 2D KS test algorithm that is most often used in various open source codes [24]. The asymptotic equation that Press and Teukolsky developed uses the 1D KS test equation, where $Z_{n,ps}^{(2)}$ is defined as $\sqrt{n}D_{n.ps}^{(2)}$, scaled by the correlation and sample size (see Equation (2.25)).

$$P\left(Z_{n,ps}^{(2)} > z\right) = Q_{KS}(\frac{\sqrt{n}D_{n,ps}^{(2)}}{1+\sqrt{(1+r^2)}\left(.25-\frac{.75}{\sqrt{n}}\right)}) \qquad (2.28)$$

where $Q_{KS} = 2e^{z^2}$ and $r = \frac{\sum_i (x_i - \bar{x})(y_i - y)}{\sqrt{\sum_i (x_i - \bar{x})^2}\sqrt{\sum_i (y_i - \bar{y})^2}}$ (the estimated coefficient of correlation).

2.5. Current Software Implementations

There are several 1S and 2S 2D KS test implementations in various software, however, there is no consistency among the implementations and furthermore, when computing critical values or p-values, different equations and methods have been coded into the software functions providing different results. Table 2.1 shows a summary of current software implementations as well as the main differences between these implementations. Most notably, there is an inconsistency when cummulating the data in each orientation which will affect the maximum distance. For instance, when computing the maximum distance at a point, some implementations use all equalities to count the number of observations in a quadrant, some only use inequalities and others split the point across all orientation equally (1/4 each). Furthermore, there are different implementations to computing the critical values used to conduct the hypothesis test, some use the asymptotic equations presented in the literature, while also allowing a bootstrap implementation. The bootstrap implementation allows the user to input the number of iterations with which to sample the data (with replacement) to generate a *p*-value. Only the software implementations by Gabinou and Syrte follow the original method of Fasano and Franceschini as provided in their original paper, while the Matlab and R implementations have some differences that affect the test statistic.

Table 2.1: Summary of 2D KS Test Software Implementation

	Python Gabinou [25]	Python Syrte [26]	Matlab [27], [28]	R [29], [30]
Partial Orientation Method				
Grid 1S	-	-	-	-
Grid 2S	-	-	available	available
Sample 1S	available	-	-	-
Sample 2S	available	available	available	available
Method used to calculate critical values	Equation (2.28) (Press)	Equation (2.28) (Press) or bootstrap	Equation (2.15) (Peacock) and (2.28) (Press)	Equation (2.15) (Peacock) and (2.28) (Press) or bootstrap
comments	1. Does not use equalities when calculating CDFs. 2. For 2S tests, $D_n = \frac{D_1 + D_2}{2}$ (max D from sample 1 and sample 2)	1. For 2S test $D_n = \frac{D_1 + D_2}{2}$ (max D from sample 1 and sample 2)	1. Does not use equalities when calculating CDFs. 2. Subtracts 1 from sample 2 when computing the quadrants	1. 0.25 is added to every orientation cumulation where data is observed (instead of counting it only for quadrant III)

- Indicates method for conducting the KS test is not available in the software

3. 1S Multivariate KS Test Orientation Method

In this chapter we will introduce a more complete method for calculating the maximum difference between CDFs. Prior to discussing the method, it is necessary to discuss our definition of a multi-dimensional CDF given that it is not as simple as in the case of a single random variable in which probability is cumulated in one dimension (along the x axis).

3.1. Multivariate Cumulative Distribution Functions

In comparison to building the 1-dimensional CDF, higher dimensions require us to look at not just the cumulation of probability in one dimension, but in the other dimensions as well. Therefore, when building the CDF in two or more dimensions, an orientation is required to cumulate probabilities. It is conceivable that probability cumulates differently depending on the orientation. For a 2-dimensional, bivariate random variable, there is a total of four orientations defined for all $x, y \in \mathbb{R}$:

1) $(x \to -\infty, y \to -\infty)$ as orientation I with CDF defined as $F^{I}_{XY}(x, y) = P(X \geq x, Y \geq y)$,

2) $(x \to -\infty, y \to \infty)$ as orientation II with CDF defined as $F^{II}_{XY}(x, y) = P(X \geq x, Y \leq y)$,

3) $(x \to \infty, y \to \infty)$ as orientation III with CDF defined as $F^{III}_{XY}(x, y) = P(X \leq x, Y \leq y)$, and

4) $(x \to \infty, y \to -\infty)$ as orientation IV with CDF defined as $F^{IV}_{XY}(x, y) = P(X \leq x, Y \geq y)$.

The most common and well accepted definition of the 2-dimensional CDF is orientation III which defines a joint CDF for two continuous random variables X and Y as seen in Equation (3.1):

$$F_{XY}(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(u, v) du\, dv \qquad (3.1)$$

where $f_{XY}(x, y)$ is the continuous joint probability density function. Table 3.1 shows the properties that satisfy the definition of the 2-dimensional CDF by orientation which extend from the 1-dimensional case. The nomanclature used here, attemps to follow the notation of limits and continuity commonly seen in statistics textbooks where, for example, for a fixed point (x_0, y_0) in $\mathbb{R}^2$ approaching from the "right" in the x direction is represented by superscript "+" resulting in: x_0^+.

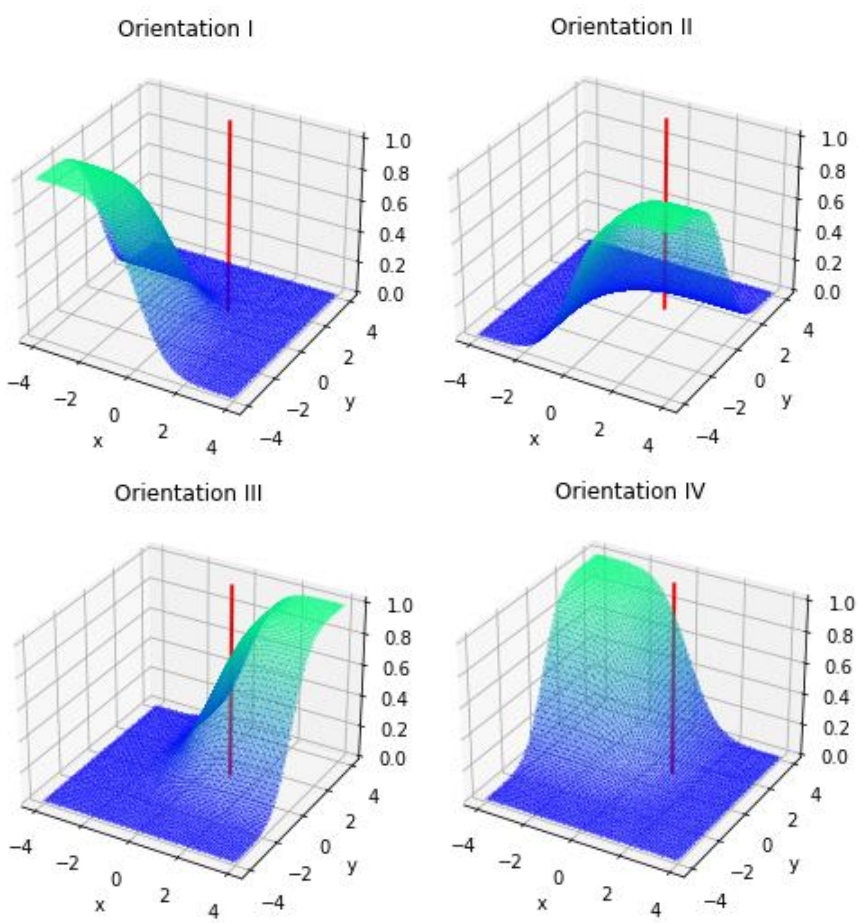
Table 3.1: Properties that satisfy the definition of the 2-dimensional CDF

	Orientation I		Orientation II		Orientation III		Orientation IV	
j	$F^{I}_{XY}(x, y) = P(X \geq x,\ Y \geq y)$		$F^{II}_{XY}(x, y) = P(X \geq x,\ Y \leq y)$		$F^{III}_{XY}(x, y) = P(X \leq x,\ Y \leq y)$		$F^{IV}_{XY}(x, y) = P(X \leq x,\ Y \geq y)$	
	$\lim_{x \to \cdot}$	$\lim_{y \to \cdot}$	$\lim_{x \to \cdot}$	$\lim_{y \to \cdot}$	$\lim_{x \to \cdot}$	$\lim_{y \to \cdot}$	$\lim_{x \to \cdot}$	$\lim_{y \to \cdot}$
1.1 – Boundaries $\lim_{x \to \cdot} \lim_{y \to \cdot} F^{j}_{XY}(x, y) = 1$	$-\infty$	$-\infty$	$-\infty$	∞	∞	∞	∞	$-\infty$
1.2 – Boundaries $\lim_{x \to \cdot} F^{j}_{XY}(x, y) = 0$	∞		∞		$-\infty$		$-\infty$	
1.3 – Boundaries $\lim_{y \to \cdot} F^{j}_{XY}(x, y) = 0$		∞		$-\infty$		$-\infty$		∞
2 - Nondecreasing	Within a given orientation (I, II, III or IV), $F^{*}_{XY}(x, y)$ is a nondecreasing function of x and y, for $-\infty < x, y < \infty$.							
3 - Range	The range of $F_{XY}(x, y)$ is between 0 and 1, since $F_{XY}(x, y)$ is a probability.							
	$\lim_{x \to x_0^{\cdot}}$	$\lim_{y \to y_0^{\cdot}}$	$\lim_{x \to x_0^{\cdot}}$	$\lim_{y \to y_0^{\cdot}}$	$\lim_{x \to x_0^{\cdot}}$	$\lim_{y \to y_0^{\cdot}}$	$\lim_{x \to x_0^{\cdot}}$	$\lim_{y \to y_0^{\cdot}}$
4 - right continuous at $x = x_0$ and $y = y_0$ within orientation $\lim_{x \to x_0^{\cdot}} \lim_{y \to y_0^{\cdot}} F^{j}_{XY}(x, y) = F^{j}_{XY}(x_0, y_0)$	x_0^-	y_0^-	x_0^-	y_0^+	x_0^+	y_0^+	x_0^+	y_0^-

* is either I, II, III, or IV

One of the consequences of defining the CDF for each orientation is that at a fixed point (x_0, y_0) in $\mathbb{R}^2$, the sum of all four CDFs at each orientation will always add up to 1 (under continuity), meaning at point (x_0, y_0): $F_{XY}^{I}(x_0, y_0) + F_{XY}^{II}(x_0, y_0) + F_{XY}^{III}(x_0, y_0) + F_{XY}^{IV}(x_0, y_0) = 1$, but only in some situations will all orientations equal each other, e.g., for symmetric distributions in both x and y such as $BVN(\mathbf{0}, \boldsymbol{I})$ at (0, 0). In fact, the cumulated probability of the CDFs from a bivariate standard Normal distribution ($BVN(\mathbf{0}, \boldsymbol{I})$) are different depending on the orientation (Figure 3.1). For example, looking at the point $(x, y) = (1,1)$ in Figure 3.1, there are four different CDF values depending on the orientation (see Table 3.2). Similarly, consider a set of 10 observations drawn from a $BVN(\mathbf{0}, \boldsymbol{I})$. Even though we have now discretized a continuous distribution (we have a finite sample) we can still use the definition of a CDF to estimate the

Figure 3.1: $BVN(\mathbf{0}, \boldsymbol{I})$ CDF orientations with red vertical line at (1,1)

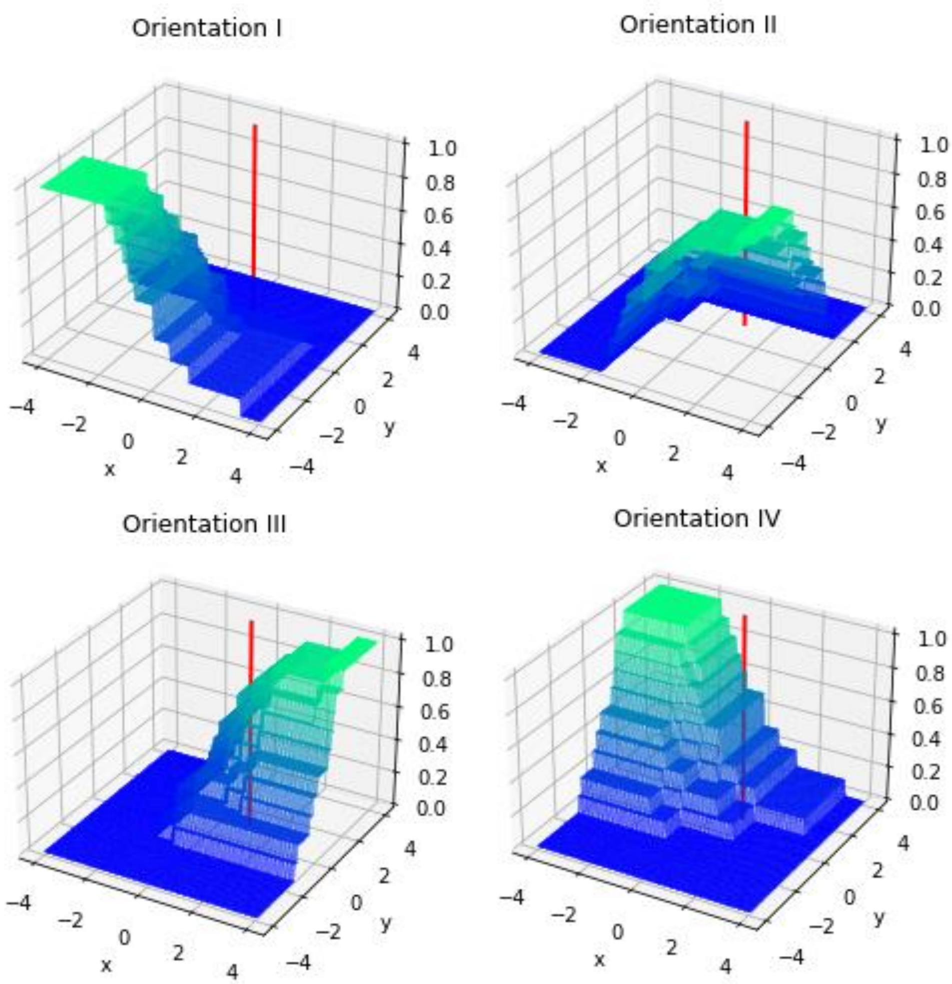


CDF, through the ECDF, by observing the proportion of data points observed according to the orientation definition. However, because the data sample is finite, the ECDF is a step function containing discontinuities, but maintaining the properties of a CDF (see Figure 3.2). As a consequence of the discontinuity along with the property of right continuity within an orientation the sum of all for orientations at a point (x_0, y_0): $F_{XY}^{I}(x_0, y_0) + F_{XY}^{II}(x_0, y_0) + F_{XY}^{III}(x_0, y_0) + F_{XY}^{IV}(x_0, y_0)$ will not necessarily be equal to 1 for the ECDF. When we compare the CDF and the ECDF across orientations we can see how the cumulated probability compared across the orientations varies greatly between 0.0256 and 0.7088 for the CDF and 0.1 to 0.6 for the ECDF (see Table 3.2) although within orientation these values differ by no more than about 0.1. Clearly, the larger the number of observations drawn, the closer the ECDF values will be to the CDF values within an orientation, but major differences in cumulated probability will remain when comparing across orientations.

Table 3.2: CDF and ECDF values by orientation for $(x, y) = (1,1)$ using a BVN(**0**,**I**) (CDF) and a single sample of 10 random draws from a BVN(**0**,**I**) (ECDF)

Orientation	CDF	ECDF
I	0.0256	0.1
II	0.1337	0.2
III	0.7088	0.6
IV	0.1333	0.2

Figure 3.2: ECDF for 10 observations drawn from a $BVN(\mathbf{0}, \mathbf{I})$ with red



3.2. 1-sample 2-dimensional KS Test Orientation Method

The 1-sample 2-dimensional Kolmogorov Smirnov (1S 2D KS) test statistic, like the 1D test statistic, is calculated as the maximum distance between the ECDF estimated from a sample of data and the appropriate null hypothesis based continuous CDF. However, as demonstrated in Figure 3.1 and Figure 3.2, these vary by orientation. Therefore, for the 1S 2D KS test using the Orientation method, all four orientations are evaluated in order to compute the maximum difference within each orientation and then the maximum of these four differences. Furthermore, like the 1S 1D case, when evaluating the ECDF at a given location where there is a data point, (x, y), it is necessary to evaluate the location from every direction: (x^+, y^+), (x^+, y^-), (x^-, y^+) and (x^-, y^-). These evaluations are demonstrated for a simple example in Figure 3.3 for a ECDF calculated within a single orientation (III) and for a sample of three data points. In this example, at point $(3, 3)$ we see 3 possible values of the ECDF for one orientation: $F_n^{III}(3^+, 3^+) = 1$, $F_n^{III}(3^-, 3^-) = \frac{1}{3}$ and $F_n^{III}(3^+, 3^-) = F_n^{III}(3^-, 3^+) = \frac{2}{3}$. Therefore, following the same notation as we did for Partial Orientation Grid method, we define the Orientation Grid method and the corresponding test statistic for the 1S 2D KS test as seen in Equation (3.2):

$$D_{n,og}^{(2)} = \max\left(D_{n,og}^{(2)I}, D_{n,og}^{(2)II}, D_{n,og}^{(2)III}, D_{n,og}^{(2)IV}\right) \tag{3.2}$$

where $D_{n,og}^{(2)}$ is the test statistic, "*og*" represents the Orientation Grid method, and for example:

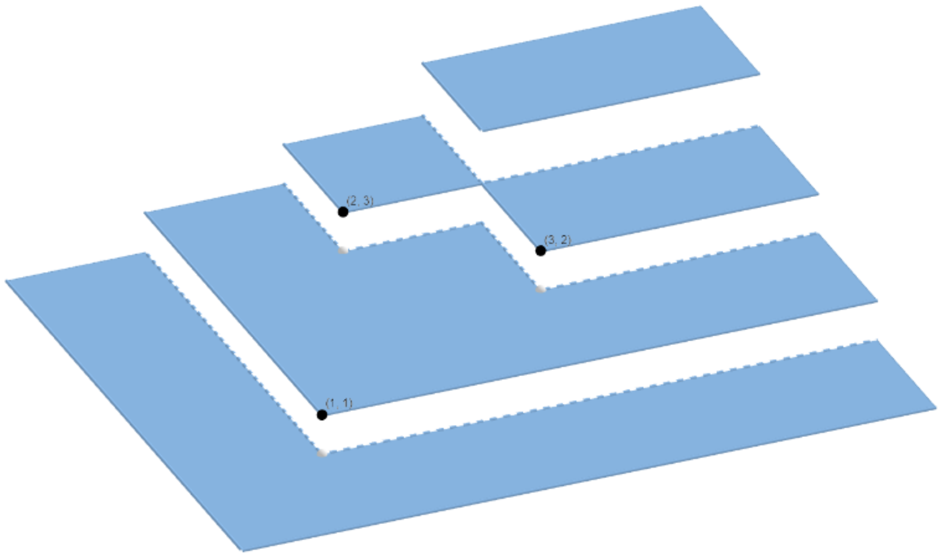
$$D_{n,og}^{(2)\,I} = \max\left(D_{n,g}^{(2)I++}, D_{n,g}^{(2)I+-}, D_{n,g}^{(2)I-+}, D_{n,g}^{(2)I--}\right) \tag{3.3}$$

where the "*o*" for Orientation method in the subscript represents the direction that is used for each orientation (all four directions), and for example:

$$D_{n,g}^{(2)I++} = \sup_{all\ x,y}\left|F_n^{(2)I}(x^+, y^+) - F^{(2)I}(x, y)\right| \tag{3.4}$$

where $F_n^{(2)I}(x^+, y^+)$ is the ECDF in orientation I with direction (x^+, y^+) and $F^{(2)I}(x, y)$ is the theoretical CDF in orientation I. Similar to the proposed method by Peacock, and denoted by the "g" subscript, we can narrow our evaluation locations to the grid generated by the 2D dataset: $(X_i, Y_j)\ \forall\ i, j = 1, \dots, n$ where n is the sample size. Therefore, the number of computations needed to compute the test statistic $D_{n,og}^{(2)}$ totals $16n^2$. As mentioned earlier, we will refer to this method as the 1S 2D KS test Orientation Grid method, where orientation refers to computing the maximum distance from all four orientations and all four directions, while grid as previously defined, consists of the evaluation locations (the grid generated by the 2D dataset).

Figure 3.3: Orientation III CDF for 3 points (1,1), (2,3), and (3,2)



When comparing the test statistic of the Orientation method against the Partial Orientation method (Peacock's method with equalities as proposed by Gosset) we can see how one is a subset of the other. If we combine Equation (3.2) and (3.3) from the Orientation method we get Equation (3.5). On the other hand, if we combine Equation (2.12) and (2.13) we get Equation (3.6) which is the maximum of four of the sixteen total distances that the Orientation method calculates.

$$D_{n,og}^{(2)} = \max\left(D_{n,g}^{(2)I++}, D_{n,g}^{(2)I+-}, D_{n,g}^{(2)I-+}, D_{n,g}^{(2)I--}, \ldots, D_{n,g}^{(2)IV++}, D_{n,g}^{(2)IV+-}, D_{n,g}^{(2)IV-+}, D_{n,g}^{(2)IV--}\right) \quad (3.5)$$

$$D_{n,pg}^{(2)} = \max\left(D_{n,g}^{(2)I--}, D_{n,g}^{(2)II-+}, D_{n,g}^{(2)III++}, D_{n,g}^{(2)IV+-}\right) \quad (3.6)$$

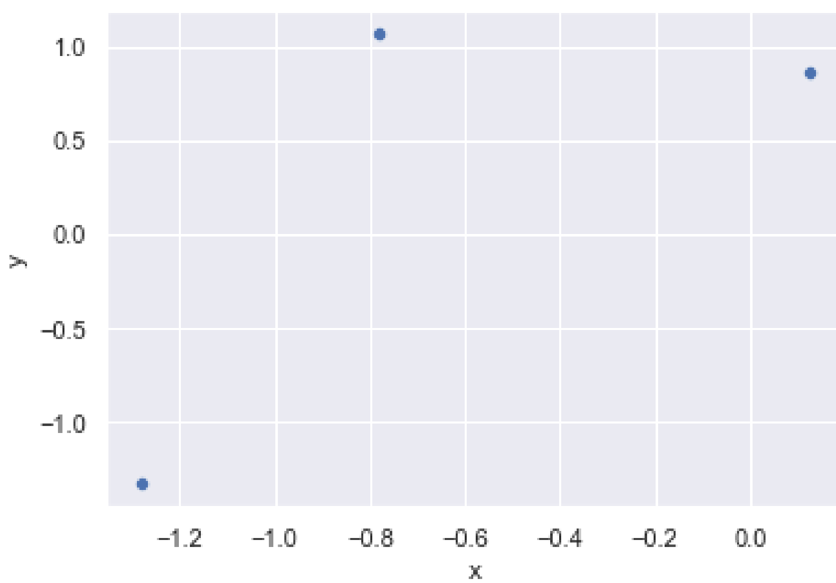
Table 3.3: Summary of 1S 2D KS Test Methods

		2D KS Test Methods	
		Orientation method	Partial Orientation method
Evaluation Locations	Grid	-All 4 orientations: $D_{n,og}^{(2)I}, D_{n,og}^{(2)II}, D_{n,og}^{(2)III}, D_{n,og}^{(2)IV}$ -All 4 directions per orientation: $(x^+, y^+), (x^+, y^-), (x^-, y^+), and\ (x^-, y^-)$ -Evaluates at the grid generated by the data -Number of evaluations: *16n*^{2}	-All 4 orientations: $D_{n,pg}^{(2)I}, D_{n,pg}^{(2)II}, D_{n,pg}^{(2)III}, D_{n,pg}^{(2)IV}$ - 1 direction per orientation: $I - (x^-, y^-), II - (x^-, y^+), III - (x^+, y^+), and\ IV - (x^+, y^-)$ -Evaluates at the grid generated by the data - Number of evaluations: *4n*^{2}
	Sample	-All 4 orientations: $D_{n,os}^{(2)I}, D_{n,os}^{(2)II}, D_{n,os}^{(2)III}, D_{n,os}^{(2)IV}$ -All 4 directions per orientation: $(x^+, y^+), (x^+, y^-), (x^-, y^+), and\ (x^-, y^-)$ -Evaluates at data (observed) - Number of evaluations: *16n*	-All 4 orientations: $D_{n,ps}^{(2)I}, D_{n,ps}^{(2)II}, D_{n,ps}^{(2)III}, D_{n,ps}^{(2)IV}$ - 1 direction per orientation: $I - (x^-, y^-), II - (x^-, y^+), III - (x^+, y^+), and\ IV - (x^+, y^-)$ -Evaluates at data (observed) - Number of evaluations: *4n*

Similar to how Fasano and Franceschini limited the number of evaluation locations but maintained the same method to finding the maximum as Peacock (therefore creating what we are refering to Partial Orientation Sample), we can also limit the evaluation locations of the Orientation Grid method creating Orientation Sample ("*os*"). The Orientation Sample method considers all four orientations and all four directions, but only evaluates the maximum distance in places were data is observed, decreasing the number of evaluations to *16n*. Table 3.3 summarizes all four methods and what process and evaluation locations each one uses. As we can see, the complete method is Orientation Grid, while all others are a subset of this complete method.

The following simple example (see Figure 3.4 for data, and Table 3.4 for computations) shows all the computations needed to calculated $D_{n,og}^{(2)}$ for a sample size $n = 3$ drawn from a $BVN(\mathbf{0}, \boldsymbol{I})$ compared against a continuous $BVN(\mathbf{0}, \boldsymbol{I})$. Given that Orientation Sample, Partial Orientation Grid/Sample methods are a subset of the Orientation Grid method by displaying all calculations from the Orientation Grid we can see what information each method captures. Looking at the whole table and finding the maximum difference at each orientation and direction equates to the Orientation Grid method, while Partial Orientation Grid would only look at the maximum difference in the blue shaded cells. Both sample methods (Orientation and Partial Orientation) would only look at the first three rows (the points where data was observed, shaded grey). Using Orientation Grid the maximum distance would be 0.725, while Partial Orientation Grid would have a maximum distance of 0.528. On the other hand, Orientation Sample equals 0.560 while Partial Orientation Sample equals 0.48. This example accentuates the difference in methods and the amount of information that each captures. As sample size increases, we conjecture that the Grid methods and the Sample methods will converge given that the size of the

Figure 3.4: 1S 2D KS Test Data Example



step in the ECDF will get smaller and smaller and the grid method data locations would be well approximated by the sample method data locations, this claim is supported by Figure 3.5, which shows the null distribution (drawn from a $BVN(\mathbf{0}, \boldsymbol{I})$) of all four methods for sample size 50 and 1,000. These figures were generated using seaborn in Python along with the kernel density function which scales the y axis to ensure that the area under the curve is equal to 1. The details on simulation procedures are explained later in the chapter.

Figure 3.5: 1S 2D KS Test Statistic Probability Distributions by Methods with 10,000

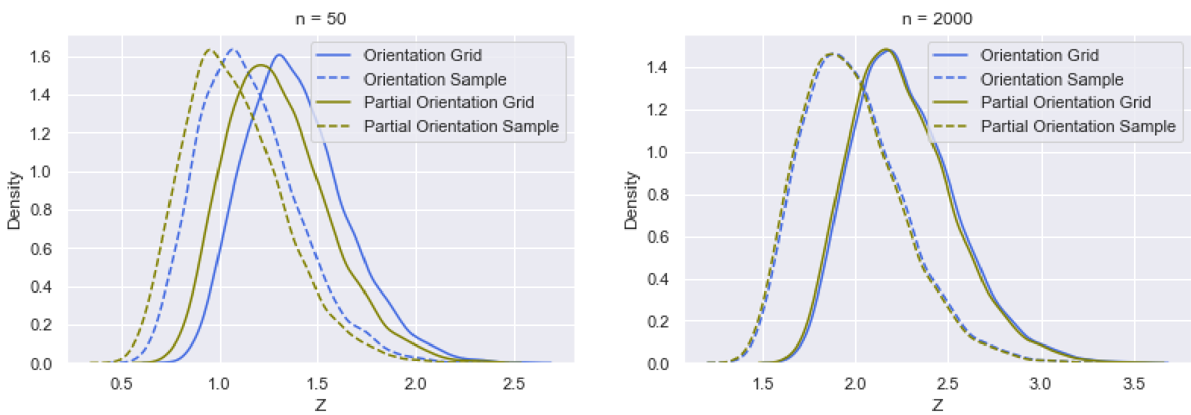


Table 3.4: 1S 2D KS Test Computations for Example

Locations	$F^{III}(x,y)$	$F_n^{III}(x,y)$		Difference		$F^{IV}(x,y)$	$F_n^{IV}(x,y)$		Difference		$F^{II}(x,y)$	$F_n^{II}(x,y)$		Difference		$F^{I}(x,y)$	$F_n^{I}(x,y)$		Difference	
		$-\ +$	$+\ +$				$-\ +$	$+\ +$				$-\ +$	$+\ +$				$-\ +$	$+\ +$		
		$-\ -$	$+\ -$				$-\ -$	$+\ -$				$-\ -$	$+\ -$				$-\ -$	$+\ -$		
[-0.78 1.07]	0.19	0.33	0.67	0.15	0.48	0.03	0.00	0.33	0.03	0.30	0.67	0.33	0.67	0.34	0.00	0.11	0.00	0.33	0.11	0.22
		0.33	0.33	0.15	0.15		0.00	0.00	0.03	0.03		0.33	0.33	0.34	0.34		0.00	0.00	0.11	0.11
[-1.28 - 1.33]	0.01	0.00	0.33	0.01	0.32	0.09	0.00	0.33	0.09	0.24	0.08	0.00	0.33	0.08	0.25	0.82	0.67	1.00	0.15	0.18
		0.00	0.00	0.01	0.01		0.00	0.00	0.09	0.09		0.00	0.00	0.08	0.08		0.67	0.67	0.15	0.15
[0.13 0.86]	0.44	0.33	0.67	0.11	0.22	0.11	0.33	0.67	0.23	0.56	0.36	0.00	0.33	0.36	0.03	0.09	0.00	0.33	0.09	0.25
		0.33	0.33	0.11	0.11		0.33	0.33	0.23	0.23		0.00	0.00	0.36	0.36		0.00	0.00	0.09	0.09
[-1.28 0.86]	0.08	0.00	0.33	0.08	0.25	0.02	0.00	0.00	0.02	0.02	0.73	0.33	0.67	0.39	0.06	0.18	0.67	0.67	0.49	0.49
		0.00	0.33	0.08	0.25		0.00	0.00	0.02	0.02		0.00	0.33	0.73	0.39		0.33	0.33	0.16	0.16
[-1.28 1.07]	0.09	0.00	0.33	0.09	0.25	0.01	0.00	0.00	0.01	0.01	0.77	0.67	1.00	0.11	0.23	0.13	0.33	0.33	0.21	0.21
		0.00	0.33	0.09	0.25		0.00	0.00	0.01	0.01		0.33	0.67	0.44	0.11		0.00	0.00	0.13	0.13
[-0.78 - 1.33]	0.02	0.33	0.33	0.31	0.31	0.20	0.33	0.67	0.14	0.47	0.07	0.00	0.00	0.07	0.07	0.71	0.33	0.67	0.38	0.04
		0.00	0.00	0.02	0.02		0.00	0.33	0.20	0.14		0.00	0.00	0.07	0.07		0.33	0.67	0.38	0.04
[-0.78 0.86]	0.18	0.33	0.33	0.16	0.16	0.04	0.00	0.33	0.04	0.29	0.63	0.33	0.33	0.30	0.30	0.15	0.33	0.67	0.18	0.52
		0.33	0.33	0.16	0.16		0.00	0.33	0.04	0.29		0.00	0.00	0.63	0.63		0.00	0.33	0.15	0.18
[0.13 - 1.33]	0.05	0.33	0.33	0.28	0.28	0.50	0.67	1.00	0.17	0.50	0.04	0.00	0.00	0.04	0.04	0.41	0.00	0.33	0.41	0.08
		0.00	0.00	0.05	0.05		0.33	0.67	0.17	0.17		0.00	0.00	0.04	0.04		0.00	0.33	0.41	0.08
[0.13 1.07]	0.47	0.67	1.00	0.20	0.53	0.08	0.33	0.33	0.26	0.26	0.39	0.00	0.33	0.39	0.05	0.06	0.00	0.00	0.06	0.06
		0.33	0.67	0.14	0.20		0.00	0.00	0.08	0.08		0.00	0.33	0.39	0.05		0.00	0.00	0.06	0.06

*grey cells – sample methods, blue cells – Partial Orientation methods, red cell – maximum difference

3.3. Simulation Procedures

The procedure for gathering the simulated data is as follows and applies to all simulations unless otherwise specified. First, draw n samples from a specified theoretical distribution with appropriate parameters (most simulations were run using the Bivariate Normal distribution). Second, run the KS test method comparing the ECDF against the appropriate continuous CDF distribution. Repeat this process 10,000 times, with the following random seeds: seed 0 for the first 1,000, seed 1 for the second 1,000 up to seed 9 for the last 1,000 repetitions. All simulations and analyses were conducted on Python 3.7.9 using numpy 1.20.3 to generate the random data. Apart from the acceptable machine error, some rounding error was introduced when computing the CDF. In order to not compute double integrals for each data point and for each repetition in the 2D analyses, a file was created with all the $BVN(\mathbf{0}, \boldsymbol{I})$ CDF values for each orientation where x and y ranged from -6 to +6 (rounded to two decimal places). This additional rounding error was introduced when looking for the CDF value in that the data point being evaluated was rounded to two decimal places in order to retrieve the saved file containing the CDF values. A simplified version of the Python code is available in the appendix with all four methods (without the computational time improvements such as retrieving the CDF values from the saved file). These simulations were used to validate properties and assumptions of the 2D KS methods, compare the methods and to perform power analysis and sample size recommendations.

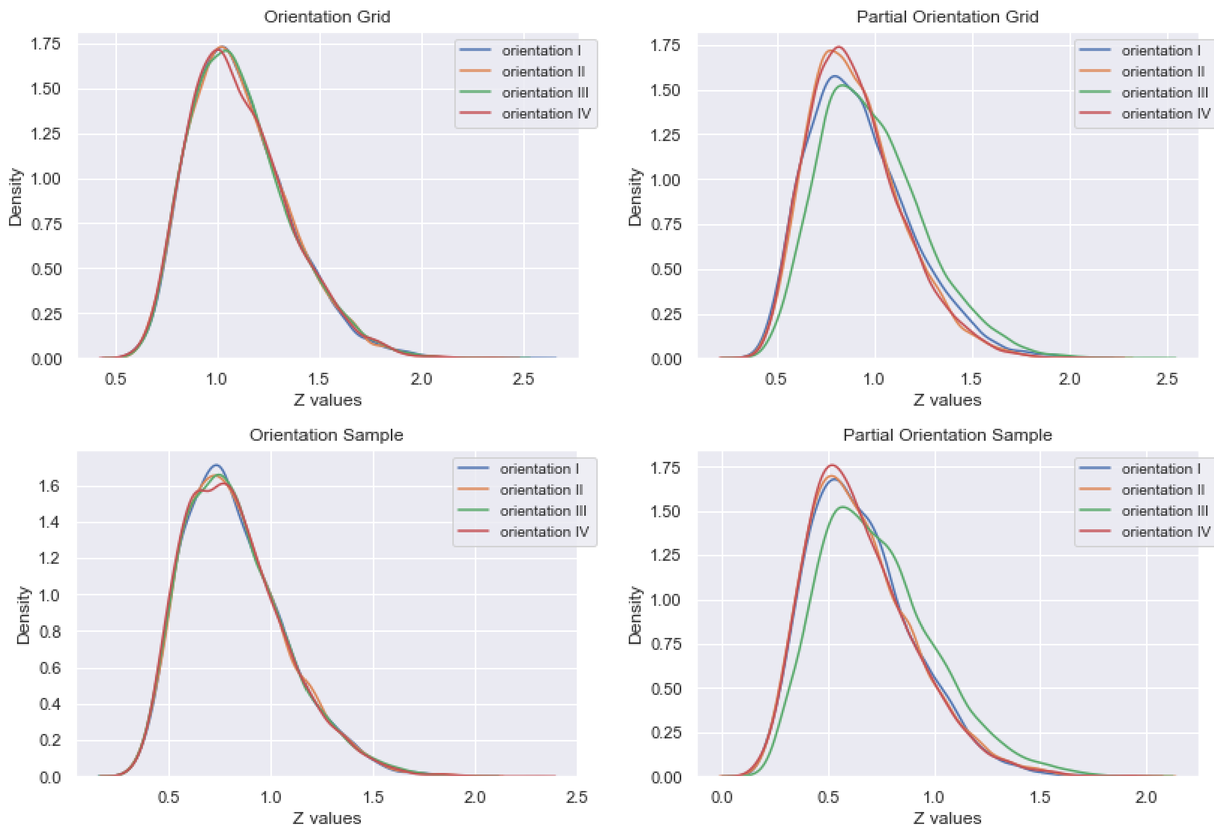
3.4. Properties of Orientation Method

In this section we will discuss several properties and assumptions of the Orientation method such as exchangability and independence of each orientation, as well as the distribution free property of the multi-dimensional test. These properties and assumptions are crucial for the mathematical derivation of the critical values.

3.4.1. Orientation Decomposition of the 1S 2D KS Test

In order to show that the Orientation Grid method has four iid orientations we can look at each orientation that make up $D_{n,og}^{(2)}$. Each orientation has the same four operations (all four directions per orientation) and therefore they are all identical, which is something that both Partial Orientation methods do not have given that a different direction is used for each orientation. On the other hand, by considering the maximum difference in each orientation, it is clear that the maximum occuring in one orientation will not affect the probability of finding the max in any other orientation, therefore we can argue that each orientation is independent (note that directions within an orientation are not independent). Furthermore, these claims were justified after running a simulation using $BVN(\mathbf{0},\boldsymbol{I})$ for the null distribution in which the maximum distance for random samples of size n=10 in each orientation was estimated separately. The distributions of the maximum distances for each orientation and each method is plotted in Figure 3.6. Only test statistic distributions for the Orientation methods (grid and sample) maintain the same distribution across all four orientations (I, II, III, and IV) while both Partial Orientation methods (grid and sample) do not. Even though both Orientation methods have this property, the Orientation Sample method fails to be sufficiently distribution free as shown in the next section, which will leave us with only one method that satisfies all requirements. An interesting consequence of the Partial Orientation methods, is that by defining orientation III as $P(x \leq X, y \leq Y)$ means that the probability of the maximum being reached in this orientation is larger than the other orientations (see Figure 3.6 the green curve "orientation III" in both Partial Orientation methods).

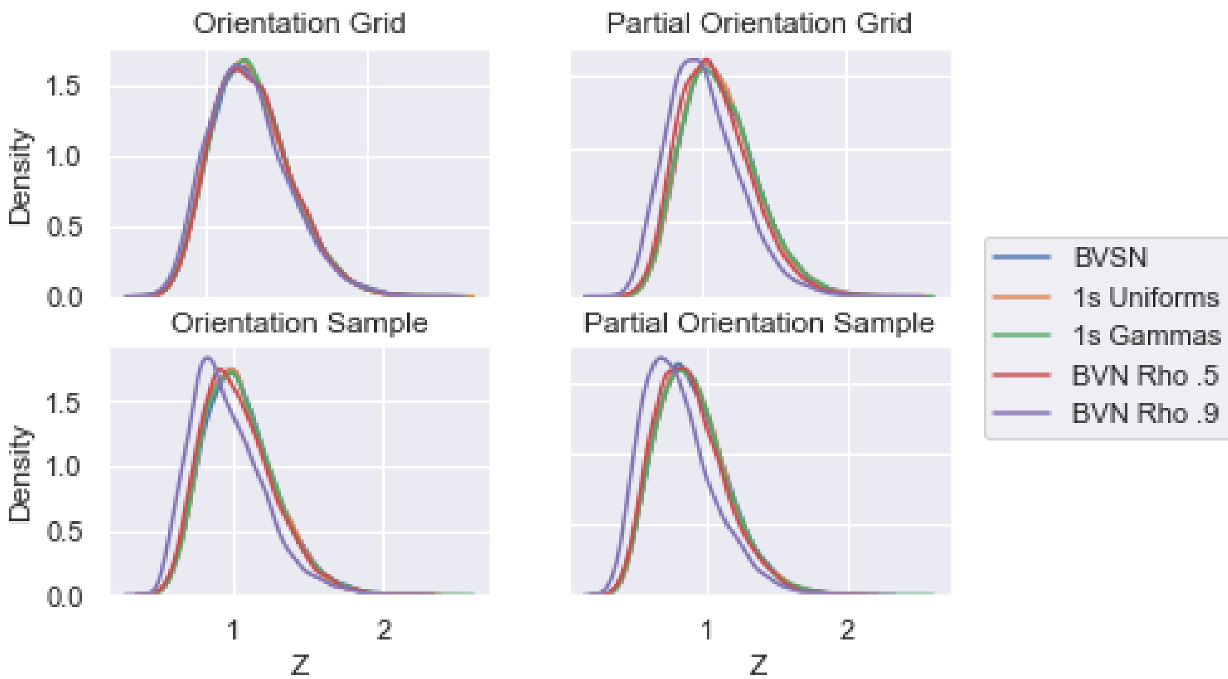
Figure 3.6: 1S 2D KS Distribution Decomposition of Orientations for n = 10



3.4.2. Distribution Free

Both Peacock [5] and Fasano and Franceschini [7] address the issue of the 1S 2D KS test being distribution free. In the 1-dimensional case, this property is inherited given that the ordering is not affected by one-to-one transformations [6], but this is not the case for the 2-dimensional case where ordering can happen in the x or the y direction. In his paper, Peacock shows that the 1S 2D case is sufficiently distribution free (except for high correlated data) by performing the KS test several times with each time having the ECDF drawn from a different distribution, while Fasano and Franceschini show that the statistic is sufficiently distribution free if the correlation coefficient is considered (tables of critical values based on correlation can be found in their paper). In a similar fashion, given that the 1S 2D KS test is not inherintely distribution free, we test the idea of sufficiently distribution free properties by drawing from a speficied distribution and compare against the same theoretical distribution.

Figure 3.7: Null distribution for different probability distributions for n = 10 (10,000 repetitions)



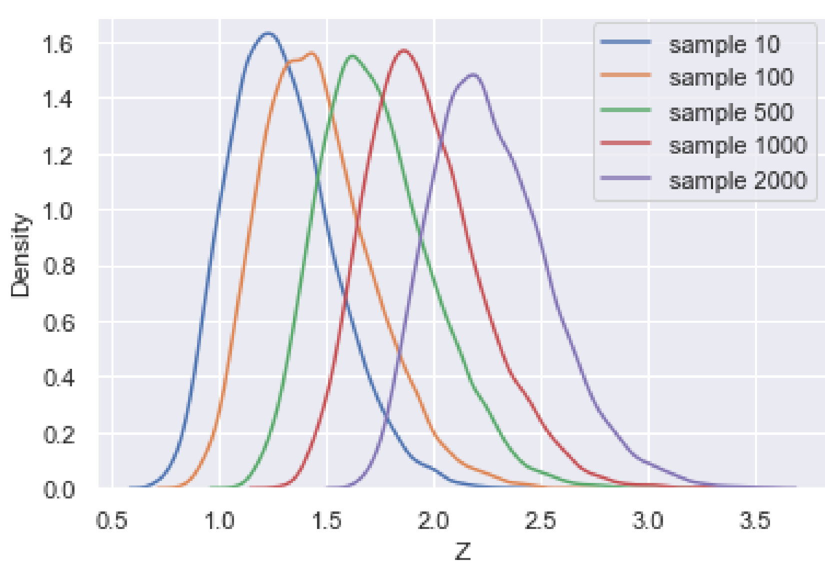
BVSN = $BVN(\mathbf{0}, \boldsymbol{I})$, Uniforms = two independent $U(0,1)$, Gammas = two independent $gamma(2,2)$, , Rho .5 = $BVN(\mathbf{0}, \begin{bmatrix} 1 & .5 \\ .5 & 1 \end{bmatrix})$, Rho .9 = $BVN(\mathbf{0}, \begin{bmatrix} 1 & .9 \\ .9 & 1 \end{bmatrix})$.

Figure 3.7 shows the 1S 2D KS test for sample of size 10 with 10,000 repeptitions, when sample distributions include $BVN(\mathbf{0}, \boldsymbol{I})$, two independent $U(0,1)$, two independent $gamma(2,2)$, a $BVN(\mathbf{0}, \begin{bmatrix} 1 & .5 \\ .5 & 1 \end{bmatrix})$ and a $BVN(\mathbf{0}, \begin{bmatrix} 1 & .9 \\ .9 & 1 \end{bmatrix})$. Based on our simulation, we can claim that all methods appear to be sufficiently distribution free when the correlation of the data is not extremely high. On the other hand, only the Orientation Grid method is sufficiently distribution free regardless of the correlation, even for highly correlated data. This allows us to use the Orientation Grid method to detect differences in distribution without computing the sample correlation and require different critical values based on the correlation. Furthermore, even if the data is highly correlated, we can rest assured that no additional error or variability will be introduced to the critical values.

3.4.3. Orientation Method Null Distribution

Now that we have shown that only the Orientation Grid method appears to have both properties of independent and identically distributed (iid) and distribution free, this section will explore the null distribution of the Orientation Grid method. By sampling from a $BVN(\mathbf{0}, \boldsymbol{I})$ for sample sizes ranging from 3-2,000 and comparing against a continuous $BVN(\mathbf{0}, \boldsymbol{I})$ to compute the maximum difference in the CDFs, repeated 10,000 times, we can generate a smooth distribution for the distance $D_{n,og}^{(2)}$ while this is not the mathematical derivation, it provides us

Figure 3.8: 1S 2D KS test Orientation Grid Z distance



with an understanding of the behavior of the null distribution as sample size increases. After standarizing the distance to $Z_{n,og}^{(2)} = \sqrt{n}D_{n,og}^{(2)}$ we see in Figure 3.8 that the distribution of this standarized distance shifts as sample size increases, but maintains a similar shape. Even for sample size 2,000 there is no clear proof that we have reached the asymptotic distribution of the 1S 2D KS test. Therefore, fitting the asymptotic equation to our simulation will limit the generalization of the equation and prevent us from using it for samples larger than what was used as the asymptotic limit (similar to what Peacock and Gosset did). In the next section, we will show a method for mathematically deriving the critical values for small sample and comment on the difficulty of deriving the asymptotic equation.

3.5. Orientation Grid Analysis

Now that we have established the Orientation Grid 2D KS test, its properties and how to perform its simulations, we can begin to estimate the critical values and evaluated the power of this test. In this section, we will show a derivation for the critical values for the 2D KS test, explore the simulated critical values and determine a correction for large sample (similar to Peacock's approach) and finally compare the power of the Orientation Grid method using derived, simulated critical values and large sample simulated critical values with correction.

3.5.1. Orientation Grid Derived Critical Values

This section follows closely the 1D KS test derivation presented in [4], [17], [19] with the understanding that even though we are still looking for a one-dimensional maximum distance, our observations now lie on a surface instead of a line and we have n^2 evaluation locations. We will assume independence (no correlation) between X and Y given we are deriving the null distribution. Furthermore, we have shown previously that orientations in this method are iid. Thus, without loss of generality, we can focus on a single orientation; we will use orientation III. The goal is to find the maximum in Equation (3.7), we focus on Equation (3.10) and once we have the distribution of one of the orientations, we can find the marginal pdf of Y_n, seen in Equation (3.8).

$$P\left(D_{n,og}^{(2)} > d\right) = \max\left(P\left(D_{n,og}^{(2)I}\right), P\left(D_{n,og}^{(2)II}\right), P\left(D_{n,og}^{(2)III}\right), P\left(D_{n,og}^{(2)IV}\right)\right) \quad (3.7)$$

$$Y_n = X_{(n)} = \max(X_1, \ldots, X_n) = n f_x(y_1)[F_x(y_1)]^{n-1} \quad (3.8)$$

where $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f_X(x)$.

We start by defining our random variable as seen in Equation (3.9):

$$T_{ii'} = n^2(F_e(d_{ii'}) - d_{ii'}) \quad (3.9)$$

where $F_e(d_{ii'})$ is the ECDF, $d_{ii'} = \left(\frac{i}{n}\right)\left(\frac{i'}{n}\right)$, $i,\ i' = 0, 1, 2, \ldots, n$, and given independence $n^2(F_e(d_{ii'}))$ represents the number of observations where $x \le X_i$ and $y \le Y_{i'}$.

Consider the sample space of all possible $T_{11}, T_{12}, \ldots, T_{1n-1}, \ldots, T_{nn}$. Let $A_{ii'}$ and $B_{ii'}$ be the events where $T_{ii'}$ reaches a fixed integer J or -J respectively ($T_{ii'}$ does not reach either J or -J with probability 0).

$$P\left(D_{n,og}^{(2)III} > d\right) = \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1} P(A_{ii'}) + \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1} P(B_{ii'}) \quad (3.10)$$

Using the formula of total probability $P(A) = P(B)P(A|B) + P(B^c)P(A|B^c)$, we can write the $P(T_{kk'} = J)$ for any $k,\ k'$ between 1 and $n-1$

$$P(T_{kk'} = J) = \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1} P(A_{ii'})P(T_{kk'} = J|A_{ii'}) + \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1} P(B_{ii'})P(T_{kk'} = J|B_{ii'}) \quad (3.11)$$

And for -J:

$$P(T_{kk'} = -J) = \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1} P(A_{ii'})\, P(T_{kk'} = -J | A_{ii'}) \, + \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1} P(B_{ii'})\, P(T_{kk'} = -J | B_{ii'}) \quad (3.12)$$

With a similar logic as the 1D case where the probabilities correspond to a binomial experiment (either $T_{ii'}$ reached J or it did not), the probability of success now corresponds to $\frac{kk'}{n^2}$ with n^2 trials. We can represent the following probabilities as binomial distributions: $P(T_{kk'} = J)$

$$P(T_{kk'} = J) = \mathbb{B}_{kk'+J}^{n^2}\left(\frac{kk'}{n^2}\right) \quad (3.13)$$

$$P(T_{kk'} = J | A_{ii'}) = \mathbb{B}_{kk'-ii'}^{n^2-ii'-J}\left(\frac{kk' - ii'}{n^2 - ii'}\right) \quad (3.14)$$

$$P(T_{kk'} = J | B_{ii'}) = \mathbb{B}_{kk'-ii'+2J}^{n^2-ii'+J}\left(\frac{kk' - ii'}{n^2 - ii'}\right) \quad (3.15)$$

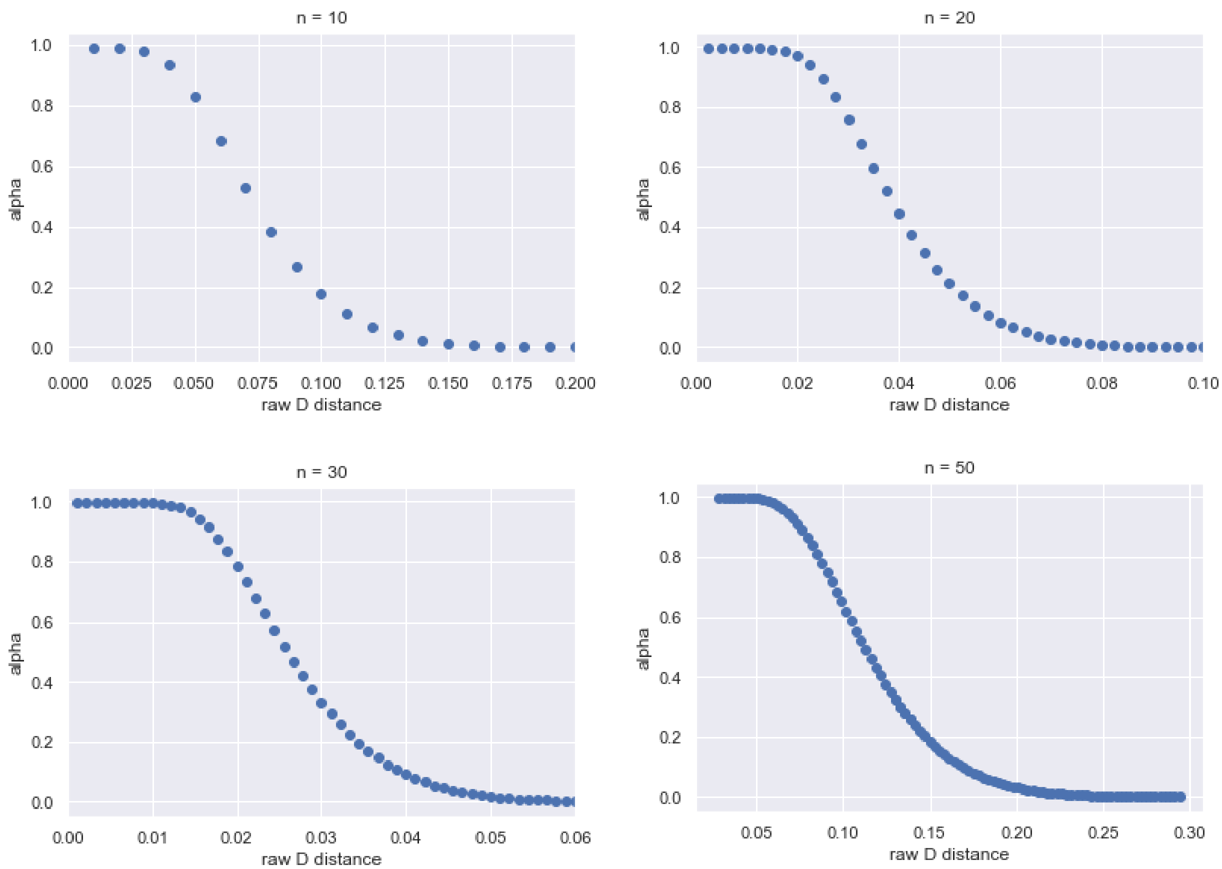
Using (3.13), (3.14), and (3.15) and substituting into (3.11), similar process for (3.12):

$$\mathbb{B}_{kk'+J}^{n^2}\left(\frac{kk'}{n^2}\right) = \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1} P(A_{ii'})\, \mathbb{B}_{kk'-ii'}^{n^2-ii'-J}\left(\frac{kk' - ii'}{n^2 - ii'}\right) + \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1} P(B_{ii'})\, \mathbb{B}_{kk'-ii'+2J}^{n^2-ii'+J}\left(\frac{kk' - ii'}{n^2 - ii'}\right) \quad (3.16)$$

$$\mathbb{B}_{kk'-J}^{n^2}\left(\frac{kk'}{n^2}\right) = \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1} P(A_{ii'})\, \mathbb{B}_{kk'-ii'-2J}^{n^2-ii'-J}\left(\frac{kk' - ii'}{n^2 - ii'}\right) + \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1} P(B_{ii'})\, \mathbb{B}_{kk'-ii'}^{n^2-ii'+J}\left(\frac{kk' - ii'}{n^2 - ii'}\right) \quad (3.17)$$

Using (3.16) and (3.17) we can create a $2(n-1)^2$ system of linear equations with $2(n-1)^2$ unknowns. Solving the system of equation for $P(A_{ii'})$ and $P(B_{ii'})$ provides us with the values necessary to solve Equation (3.10). The issue in the 2D case that is not present in the 1D case is the fact that regardless of sample size, the matrix is singular with an infinite number of solutions. In practice, given that we are only interested in the sum of the solution, the free variables of the solution of system of equations cancel each other and still provide a numerical solution. For example, if the solution was $\begin{bmatrix} .23 \\ .4 - x_3 \\ x_3 \end{bmatrix}$ where x_3 is the free variable, then the sum will still provide a numerical answer. Therefore, it is sufficient to solve the system of equations using a least-square algorithm to solve for $A\boldsymbol{x} = B$. Figure 3.9 shows the solution of the system of equations for various sample sizes after we convert from the J values to D_n using $D_{n,og}^{(2)} = \sqrt{n}\frac{J}{n^2}$. The number of blue points represents the converted J integer values starting with $J = 1$ and increases as sample size increases. To standardize $D_{n,og}^{(2)}$ we can use the transformation of $Z_{n,og}^{(2)} = \sqrt{n}D_{n,og}^{(2)}$.

Figure 3.9: Binomial derivation of raw D distance for 2D KS test

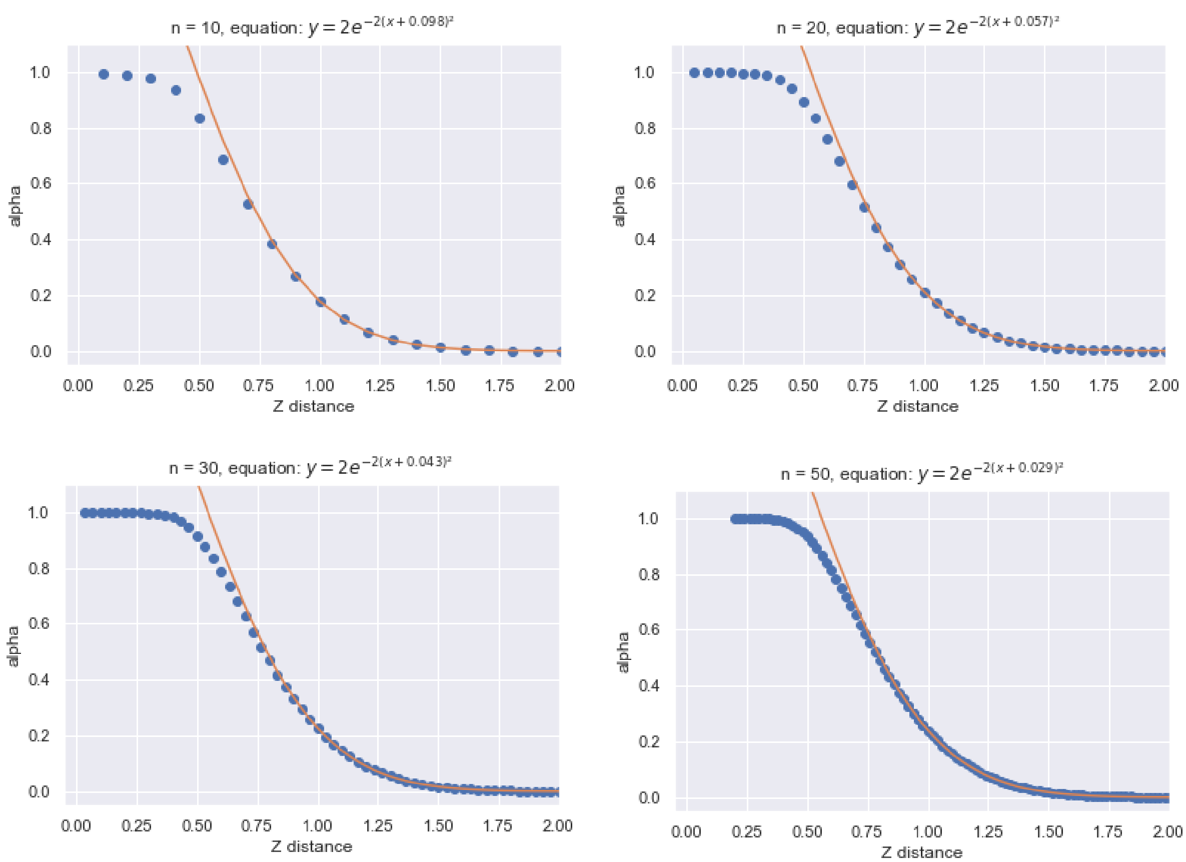


Once the distribution of one orientation is solved and standardized, we can fit an exponential/Gaussian equation such as Equation (3.18) or a variation of the logistic regression equation such as Equation (3.19) in order to provide a functional form that can be used to estimate the probabilities. We will be focusing on the exponential fit because even though the logistic fit is a better fit for the whole curve, including the boundaries (see Figure 3.11) it fails to fit properly when we calculate the marginal pdf using the logistic fit, mainly, the derivative of the logistic equation fails to maintain the desired shape in order to fit with the simulated null distribution. Furthermore, the exponential fit has additional merit given that the form is remarkably similar to the 1D case, and was mathematically derived.

$$F(x) = 2\, e^{-2(x-b)^2} \tag{3.18}$$

$$F(x) = c + \frac{1}{(1 + e^{-a(x-b)})^t} \tag{3.19}$$

Figure 3.10: Standardized 1S 2D KS distance of Binomial derivation and exponential fitted



Even though the exponential fit is not the best fit for the whole CDF of one orientation, when we focus on functional α values of interest, e.g., $\alpha < 0.2$, we get a root mean square error ranging from 0.0001 – 0.0003 for the various sample sizes (see Figure 3.10). Because we are using $P(Z_n > z)$ as the $F_x(y_1)$ for the marginal pdf which is $1 - P(Z_n \leq z)$ the derivative of the $F_x(y_1)$ (the pdf $f_x(y_1)$) will be negative. Using this fit we, find the marginal pdf of the four orientations $P(Z_n > z)$ by using Equation (3.8) where $F_x(y_1) = 2\, e^{-2(x-b)^2}$:

$$P\left(Z_{n,og}^{(2)} > z\right) = 4(-1)\big(-4(x-b)\big)e^{-2(x-b)^2}\left[1 - e^{-2(x-b)^2}\right]^3 \qquad (3.20)$$

$$= 16(x-b)e^{-2(x-b)^2}\left[1 - e^{-2(x-b)^2}\right]^3$$

Figure 3.11: Standardized KS distance of Binomial derivation and Logistic fitted equation

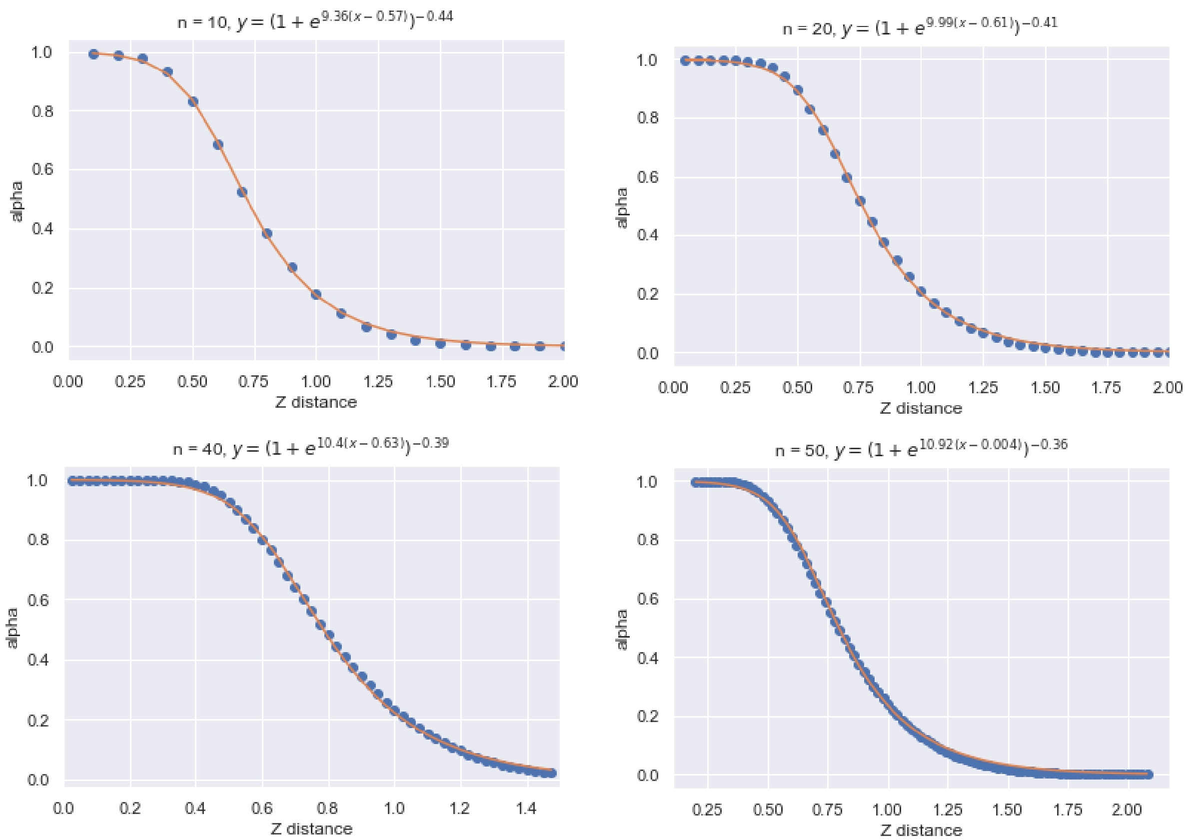


Table 3.5: 1S 2D KS Test Critical Values from derived solution using exponential fit

n/ α	0.01	0.05	0.1	0.2
10	1.95	1.77	1.67	1.57
20	1.99	1.81	1.72	1.61
30	2.01	1.83	1.73	1.62
40	2.02	1.84	1.74	1.63
50	2.02	1.84	1.74	1.64
100	2.03	1.85	1.76	1.65

We can see in Table 3.5 various derived critical values using the method outlined above with the exponential fit (for a more comprehensive list of sample sizes see Table 7.1 in the appendix).

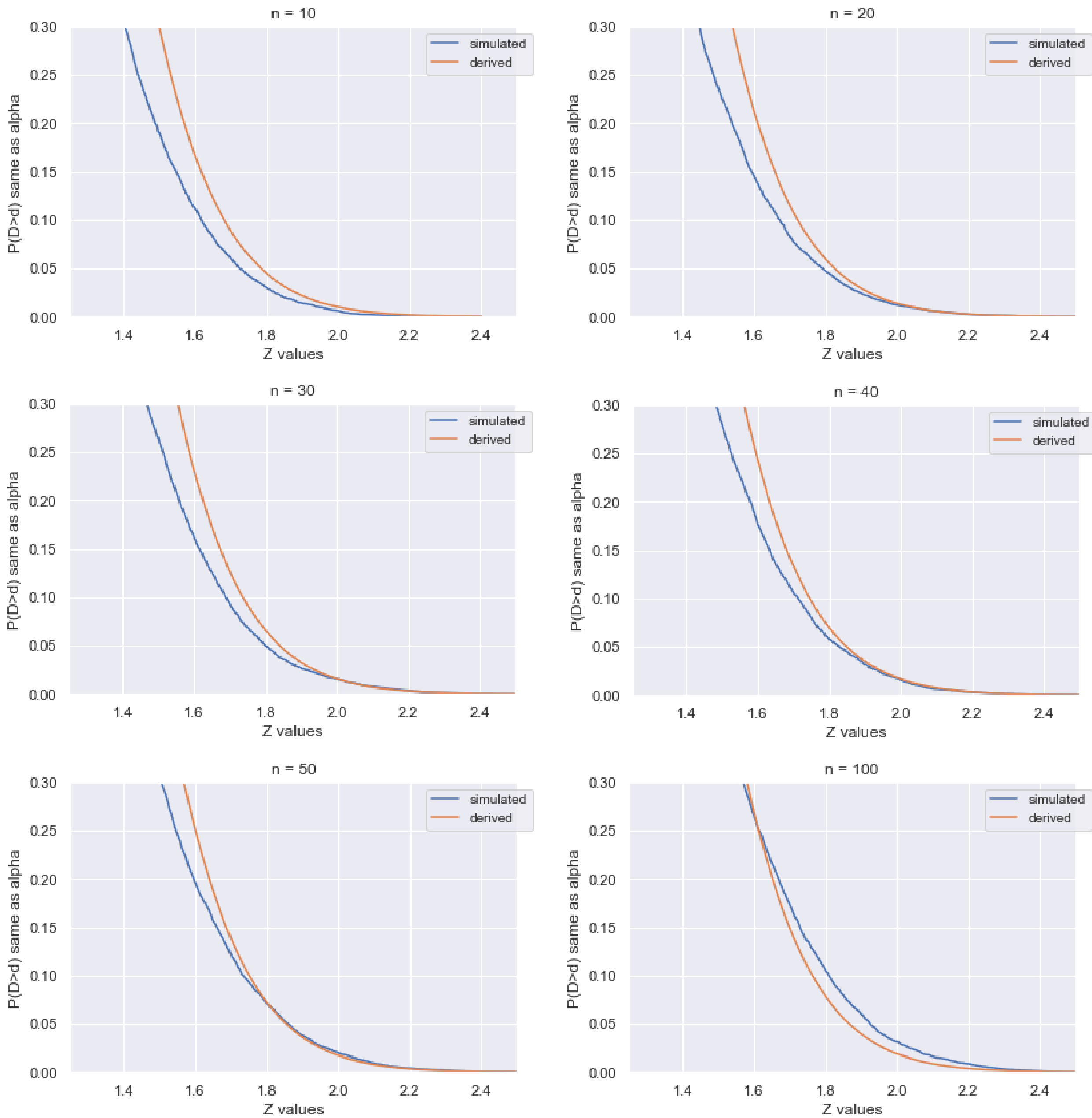
Table 3.6: 1S 2D KS Test Critical Values for Orientation Grid from Simulation

n/ α	0.01	0.05	0.1	0.2
10	1.95	1.73	1.62	1.49
20	2.03	1.79	1.67	1.54
30	2.06	1.80	1.69	1.56
40	2.05	1.83	1.71	1.58
50	2.09	1.86	1.73	1.60
100	2.17	1.92	1.81	1.67
1000	2.70	2.45	2.32	2.17
2000	3.04	2.77	2.64	2.50
5000	3.70	3.44	3.30	3.14

3.5.2. Orientation Grid Simulated Critical Values

Another approach for finding the critical values of the Orientation Grid method, is to use our null simulated distribution and finding the appropriate cutoff x-value that provides the desired α. Table 3.6 shows a few of the critical values based on our simulated null distribution (for a more comprehensive list of sample sizes please see Table 7.2 in the appendix). When we compare these values to the derived critical values, we can see that the average difference between all the values in both tables is less than 0.044 with a maximum difference of 0.14 for sample size 100 and α of 0.01. The distribution of $P\left(Z_{n,og}^{(2)} > z\right)$ for both derived and simulated curves can be seen in Figure 3.12. For sample sizes 20, 30, 40, and 50 the derived and simulated distributions are very close to each other for α values of less than 0.1.

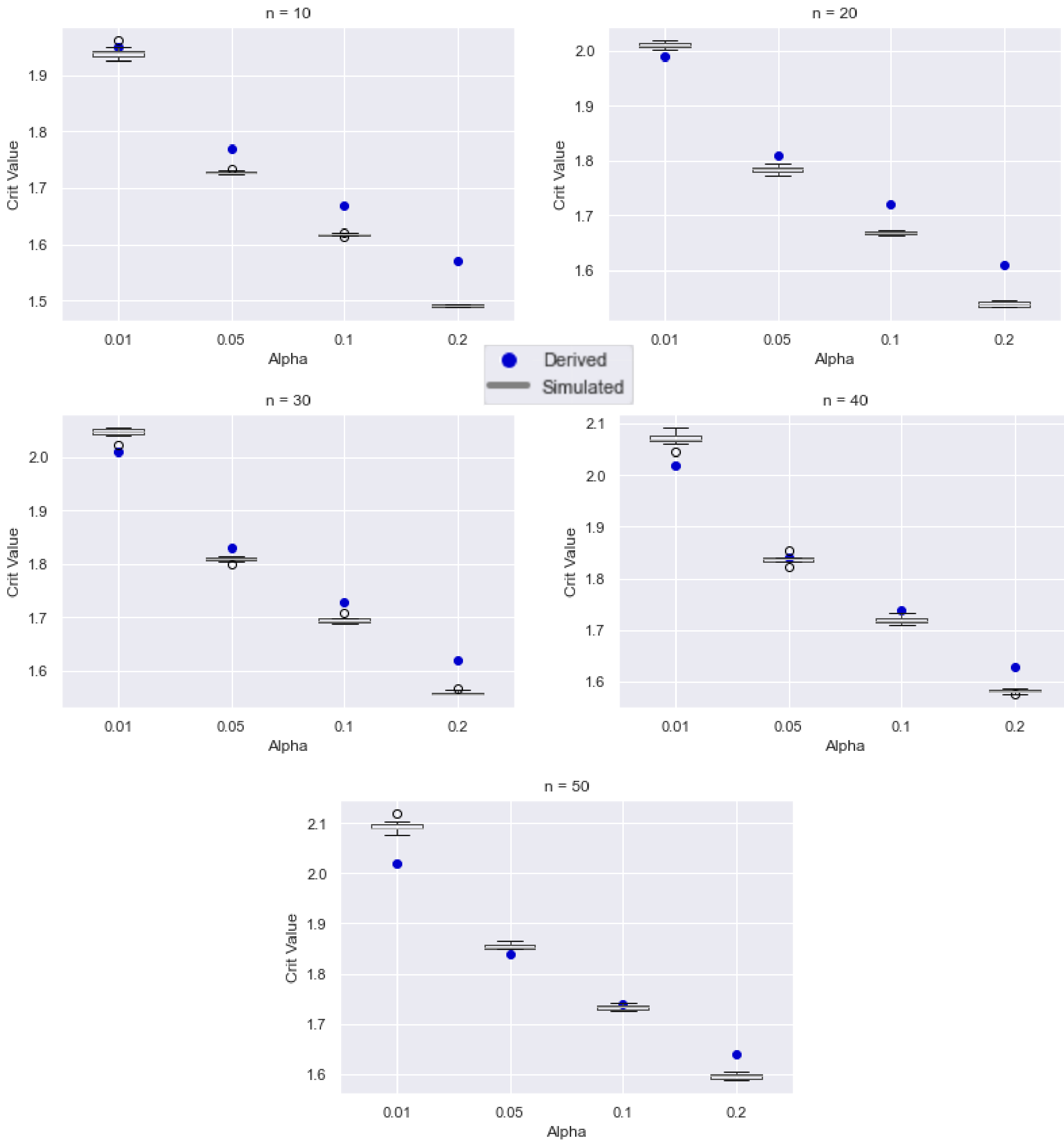
Figure 3.12: 1S 2D KS Test Orientation Grid Method Derived (using exponential fit) vs Simulated



For α values greater than 0.1 the difference is much larger and as α increases so will the difference given that the derived values used a fitted equation to only α values less than 0.2. On the other hand, for sample size 100 there are larger differences between the derived and the simulated critical values, the simulated values should have fairly small variability, but the error that is introduced by fitting an equation to the J values from solving the system of equations might be compounded as we increase sample size.

Similarly, for sample of size 10 there seems to be a larger error between the two curves with a maximum difference of 0.08. It is possible that this error is due to the exponential fit for the derived critical values (smaller number of possible J values) but also from the simulation due to the variability in the maximum distance for small samples. To address the question of variability for small sample sizes, we looked at 10 Orientation Grid method simulated null distributions with 10,000 repetitions each, found the critical values for α values of interest and compared against the derived critical value. As we can see in Figure 3.13, the boxplots represent the 10 samples, while the blue dot represents the derived critical value. The variability of the simulated null distribution is minimal, but when compared to the derived critical values we see an inconsistent small error. It is important to note that when rounding to two decimal places, the error that is introduced by the estimation, equation fitting, machine precision is present, but minimal. Further, when examining the scale of the plots in Figure 3.13, it can be seen that the error between the estimates (median simulated value and derived critical value) is generally no larger than about 0.05. Nevertheless, mathematically deriving the critical values for one orientation and finding the marginal pdf of all orientations (maximum of the four random variables) by using the fitted equation of one orientation (using the α values of interest) provides us with an accurate method for calculating critical values for small sample sizes. Errors and improvement in the derived values rely heavily on the fitted functions to the derived values.

Figure 3.13: 1S 2D KS Test Orientation Grid Method Derived and Simulated (10,000 repetitions 10 times) Critical Values for Various Sample Sizes



3.5.3. Large Sample and Correction

Solving the system of equations for the derived solution of critical values for sample sizes larger than 100 becomes computationally infeasible, furthermore simulating all sample sizes between 100 to 5,000 could take on the order of years (especially if we want to have 10,000 repetitions per sample size). However, as shown in Figure 3.8, the critical values do not converge for values of less than 5,000 and therefore, there is a need to develop a correction so that the large sample critical value can be used regardless of sample size. The largest sample in this research is 5,000 with 10,000 repetitions, but we infer that even with that sample size, we believe we have not reached the asymptotic distribution for the 1S 2D KS test (see Figure 3.14 for a graph of the simulated critical values for the null distribution based on sample size). But given the computation burden of running this test for large samples (for sample size 5000, the test takes several hours to complete one iteration of the test for a single sample and the computational time continues to grow exponentially), if there is a need for the distribution of

Figure 3.14: 1S 2D KS Orientation Grid Method Standardized Null Distribution Critical Values

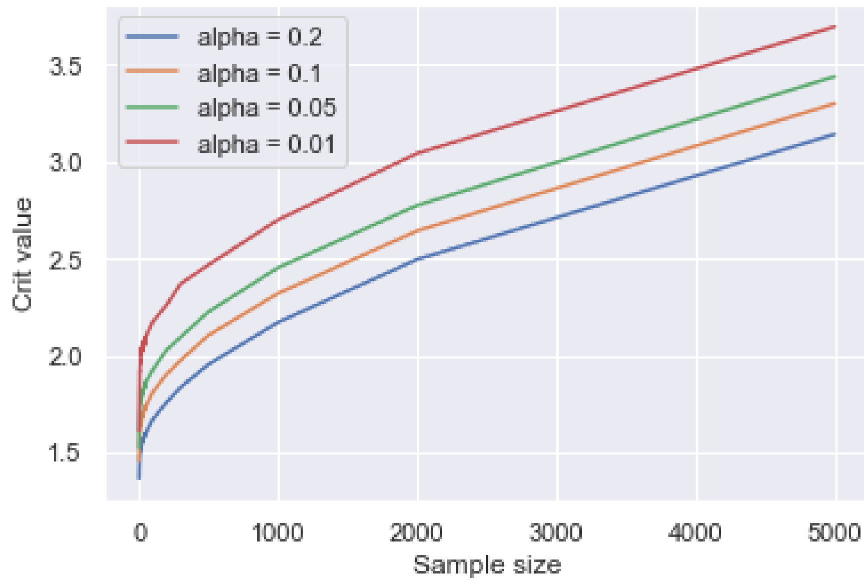
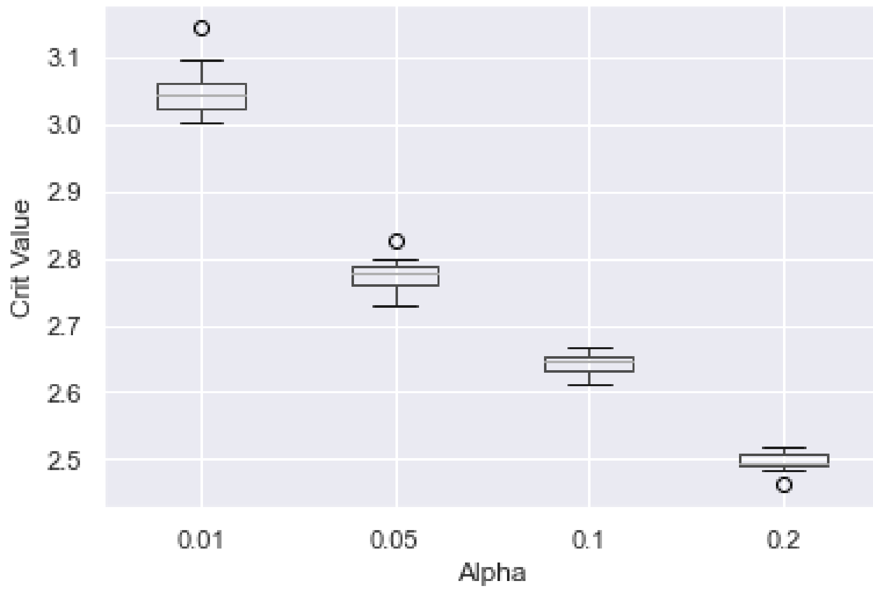


Figure 3.15: Variability of critical values for n=2,000 with 1,000 sub-samples computed each of 10 times



larger samples to be computed, we are confident that less repetitions are sufficient to still achieve high fidelity of the critical values. To prove this claim, we separated the sample from $Z_{2000}^{(2)}$ into 10 equal sub-samples (1,000 each). The critical values achieved by these sub-samples are close to each other with the largest difference being 0.1. Figure 3.15 shows the boxplot of the 10 sub-samples for specified α values. Regardless, using the sample sizes available to us, we can still find a correction and use the large sample critical values to perform the test of hypothesis up to a sample size of 5000.

Using a similar delta method as Peacock [5], we can find the correction (see Equation (3.21)) needed to shift any statistic to large sample and utilize the large sample critical values:

$$\delta_{og}^{(2)} = 1 - \frac{z_{n,og}^{(2)}}{z_{\infty,og}^{(2)}} \qquad (3.21)$$

where $z_{\infty,og}^{(2)}$ represents the hypothetical asymptotic critical values (in our case, sample of size 5,000). Therefore, any sample can be corrected to the large sample approximation by implementing Equation (3.22):

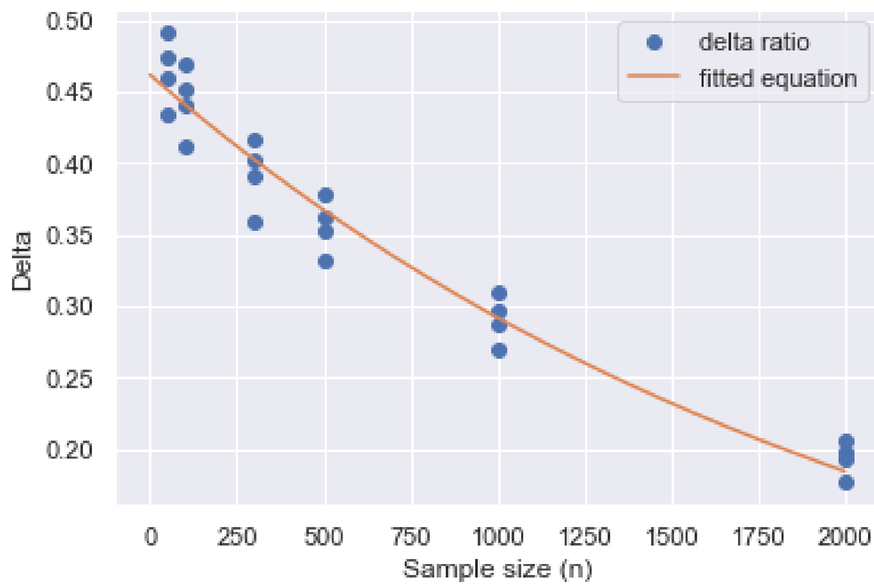
$$Z_{\infty,og}^{(2)} = \frac{Z_{n,og}^{(2)}}{1 + \delta_{og}^{(2)}} \quad (3.22)$$

Before we can use Equation (3.22), we sought to find the relationship of $\delta_{og}^{(2)}$ to sample size by fitting an exponential equation to the critical values of 0.2, 0.1, 0.05, 0.01 for sample sizes 50, 100, 300, 500, 1000, 2000 and used the critical values from sample size 5,000 as $z_{\infty,og}^{(2)}$. Figure 3.16 shows the fitted equation to the critical values based on the four α values (sample size 5,000 is not included because delta would be zero). Using the fitted Equation (3.22) and (3.23) we can correct any sample size.

$$\delta_{og}^{(2)} = 0.43e^{-0.0005(n-165.71)} \quad (3.23)$$

Although these equations extend the 2D KS test from a sample size of 2000 to 5000, the limitation with this method (same limitation Peacock and Gosset had) is that this method will only be accurate for sample sizes smaller than 5,000 (2,000 using Peacock and Gosset

Figure 3.16: 1S 2D KS Orientation Grid Delta ratio fitted equation for $\boldsymbol{\alpha}$ values: 0.2, 0.1, 0.05, 0.01 using null distribution from BVN(0,I)

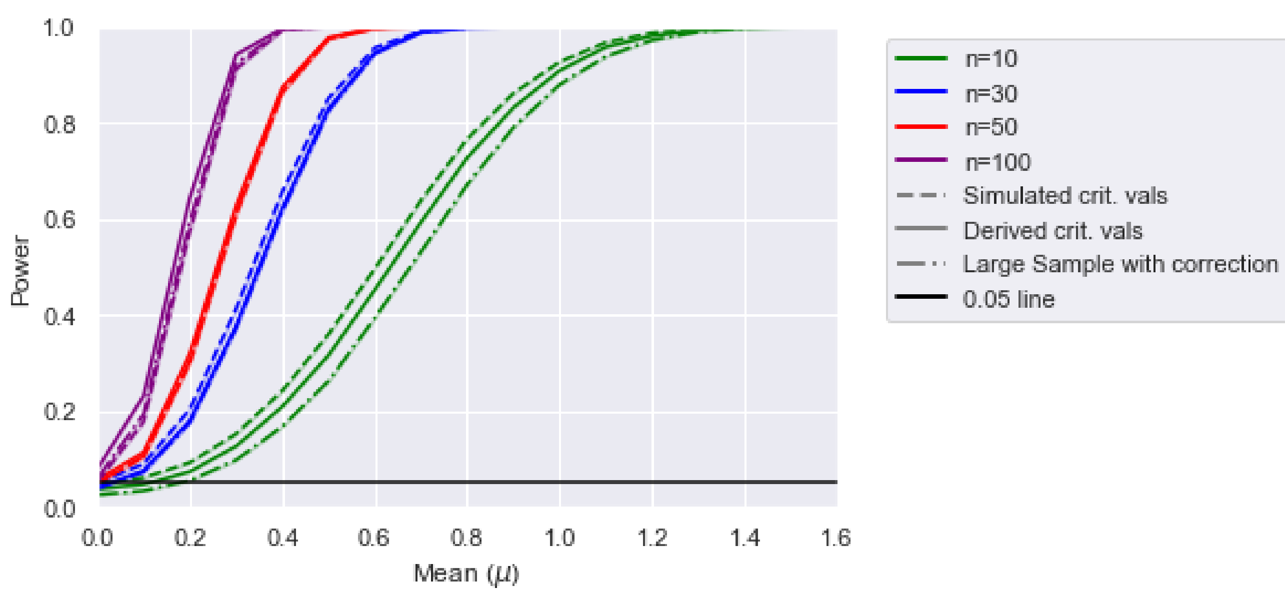


equations). For larger sample sizes, these equations will require updates so as to avoid additional error with unknown magnitude.

3.5.4. Orientation Grid Power Comparison between Critical Values

We have now established three separate ways to run the test of hypothesis for the 1S 2D KS test Orientation Grid method: 1) using derived critical values, 2) using simulated critical values or 3) using large sample critical values with a correction. In this section we will explore how well each of these three critical value methods are able to detect differences in the mean, variance, and correlation shifts for a sample distribution by estimating statistical power from various shifts in either the means, variances, or correlation. Assuming $\alpha = 0.05$, the Orientation Grid 1S 2D KS test was conducted using all three critical value methods for several sample sizes (n = 10, 30, 50 and 100). The null distribution was assumed to be BVN($\mathbf{0}$,$\mathbf{I}$). To determine the power to detect differences in means, 10,000 draws for each sample size were used to conduct

Figure 3.17: 1S 2D KS test Orientation Grid mean power α 0.05 for the three critical value methods



the KS test assuming the following vector of means $\begin{bmatrix}\mu\\ \mu\end{bmatrix}$ where $\mu = 0, \ldots 2$ by 0.1 increments. Similarly, to determine the power to detect differences in variances, draws from a BVN with mean vector of $\begin{bmatrix}0\\ 0\end{bmatrix}$and variance/covariance matrices of $\begin{bmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{bmatrix}$ where $\sigma = 1, \ldots, 3$ by 0.1 increments. Finally, to determine the power in detecting differences in correlation, means and variances for the BVN were fixed to 0 and 1.0 respectively for each variable, however, correlations were varied as $\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}$ where $\rho = -1, \ldots, 1$ by 0.1 increments. First, we will examine power of the Orientation Grid 1S 2D Orientation Grid KS test for shifts in means, then variances, and finally correlation.

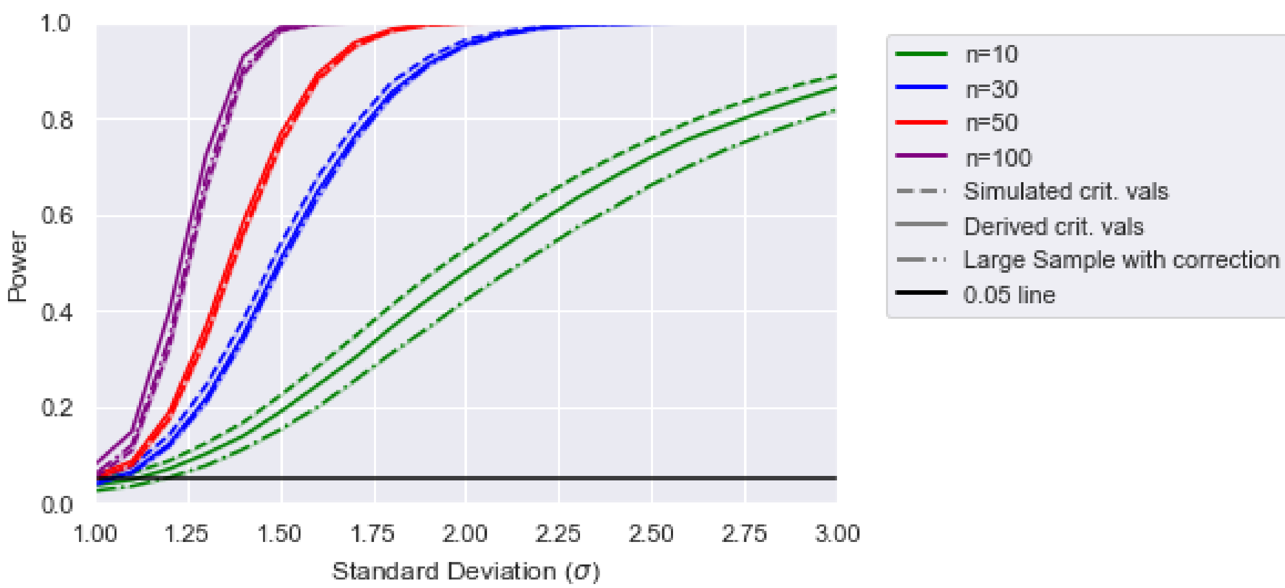
Table 3.7: 1S 2D KS test Orientation Grid achieving power $\alpha = 0.05$ for the three critical value methods

Sample Size	Simulated	Derived	Large Sample
10	0.05	0.04	0.02
30	0.05	0.04	0.04
50	0.05	0.06	0.05
100	0.05	0.08	0.06

As we can see in Figure 3.17, all three ways to compute the critical values provide similar power, but only the simulated critical values truly achieve α for the null distribution (this makes sense given that the simulated critical values came from the simulated null distribution see Table 3.7). For smaller sample sizes, the simulated critical values provide higher power than either derived or the large sample with correction critical values. Further, the large sample with correction provide the lowest power at smaller sample sizes (n = 10), however, as sample size increases, the difference between the three critical value methods gets smaller (for a complete table of power values please see the appendix).

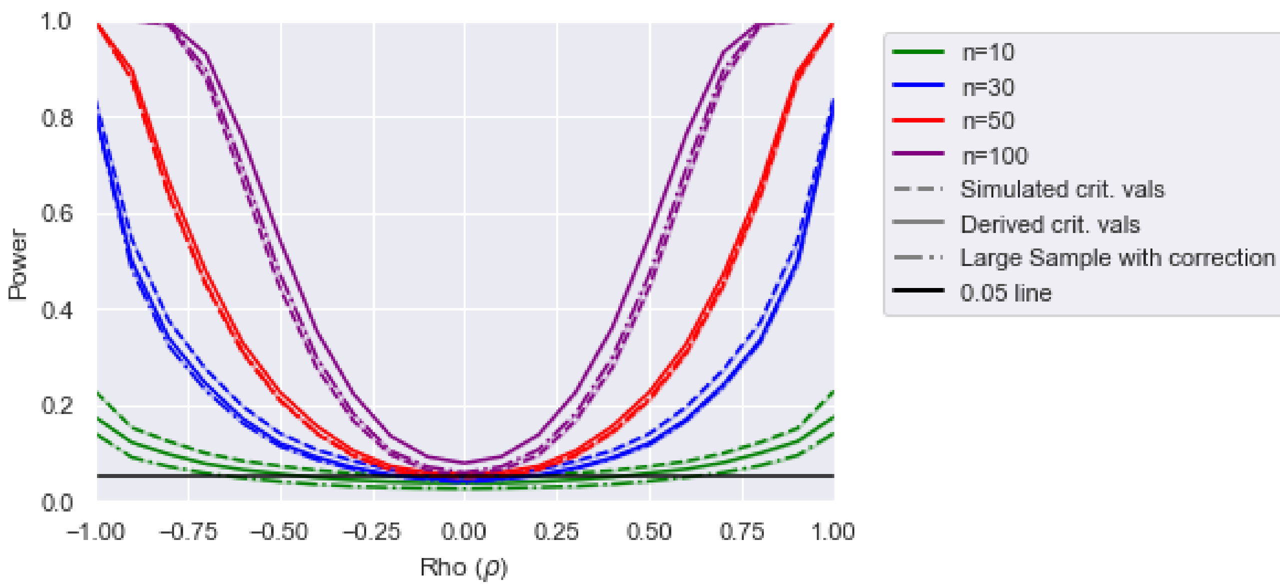
Similar trends were observed for detecting standard deviation shifts and correlation shifts (see Figure 3.18 and Figure 3.19 for power curves). Only the simulated critical values achieve α, while all three methods get closer to one another as sample size increases. Differences in the detectable differences assuming a power of 0.80 or higher were much greater between the three critical value methods for n=10; there was about a difference of 0.60 between what the simulated critical value could detect and what the large sample with correction could detect for variances when n = 10. Fewer differences were seen for sample sizes of n = 30 or larger in variances. Detecting only a difference in correlation was difficult for n = 10, which never achieved adequate power (Figure 3.5.4.3).

Figure 3.18: 1S 2D KS test Orientation Grid standard deviation power $\boldsymbol{\alpha}$ = 0.05 for the three critical value methods



Furthermore, the large sample size with correction method provided lower power than either other critical value method except when n = 100, however, even this sample size showed higher power with the derived values. Given that we are usually concerned with power of 0.8 or higher we can see that especially for small samples (size 10 and 30) the simulated critical values have higher or remarkably similar power to the other two methods, therefore, when possible, the simulated critical values should be used for the test of hypothesis.

Figure 3.19: 1S 2D KS test Orientation Grid correlation power α 0.05 for the three critical value methods



3.6. Orientation Sample Analysis

In contrast to Orientation Grid method, Orientation Sample is unable to use the derived critical values, and to fully utilize the large sample with a correction, correlation would need to be considered given that the null distribution is not robust against changes in correlation (refer back to Figure **3.7** where the 1S 2D KS test when sampling from a distribution with high correlation provides a different null distribution). Therefore, this section will focus on the Orientation Sample method and its simulated critical values, as well as large sample with correction (knowing that this large sample method has issues for very high correlation, but not terribly biased for moderate and low correlation). Furthermore, we will analyze the power of both critical values and provide recommendations. Finally, a detailed power comparison between Orientation Sample and Orientation Grid will be presented to determine overall performance and provide recommendations.

Table 3.8:1S 2D KS Test Critical Values for Orientation Sample from Simulation

n/ α	0.01	0.05	0.1	0.2
10	1.73	1.49	1.38	1.24
20	1.78	1.54	1.42	1.29
30	1.80	1.55	1.45	1.31
40	1.82	1.59	1.46	1.33
50	1.84	1.61	1.49	1.35
100	1.91	1.67	1.56	1.42
1000	2.43	2.18	2.04	1.90
2000	2.77	2.49	2.36	2.22
5000	3.40	3.13	3.00	2.85

3.6.1. Orientation Sample Simulated Critical Values

The Orientation Sample critical values were computed in a similar fashion to Orientation Grid critical values. The same algorithm and random seed procedure was used for drawing the sample data as specified in Section 3.3, the only difference is that the evaluation location was limited to only places where data was observed per the “sample” procedure. Table 3.8 shows a few of the simulated critical values computed for the Orientation Sample method. As expected, when compared with the Orientation Grid critical (see Table 3.6 and Table 3.8) values they are all shifted left (smaller) which is a consequence of limiting the evaluation locations: the maximum might not be captured and could occur in one of the jumps where data is not observed.

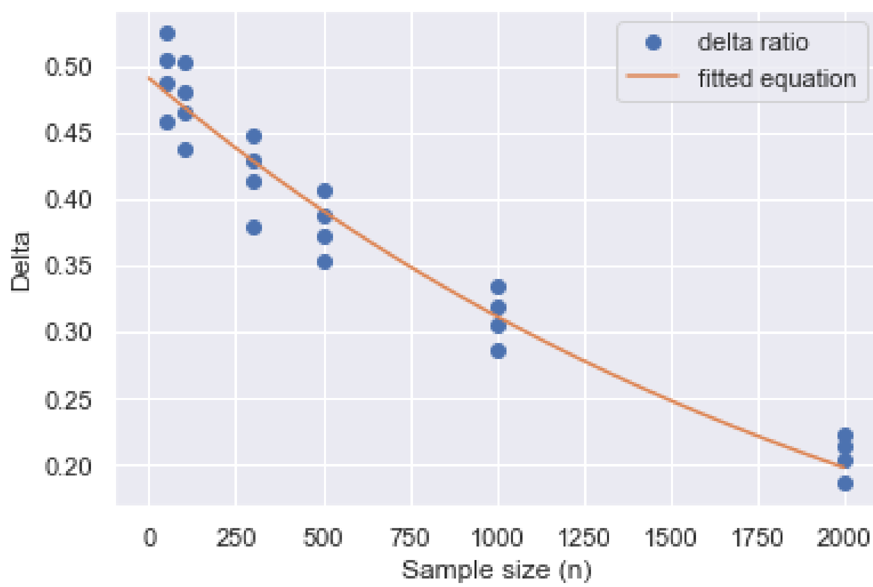
More importantly, though, is the effect these different values have on power and in maintaining α.

3.6.2. Orientation Sample Large Sample Critical Values

Using the same procedure as seen in Orientation Grid large sample, we can find the best exponential fit for a set of Orientation Sample critical values and then apply Equation (3.22) to correct any sample size less than 5,000 and use the critical values from the large sample simulation. In addition to the sample size restriction, because this is the sample method and correlation was not considered, this correlation and critical values only apply to hypothesis testing that compares a sample against a null distribution with no correlation. Figure 3.20 and Equation (3.24) show the exponential fit/equation needed for the correction.

$$\delta_{os}^{(2)} = 0.47e^{-0.0005(n-114.81)} \qquad (3.24)$$

Figure 3.20: 1S 2D KS Orientation Sample Delta ratio fitted equation for $\boldsymbol{\alpha}$ values: 0.2, 0.1, 0.05, 0.01 using null distribution from BVN(**0**,**I**)



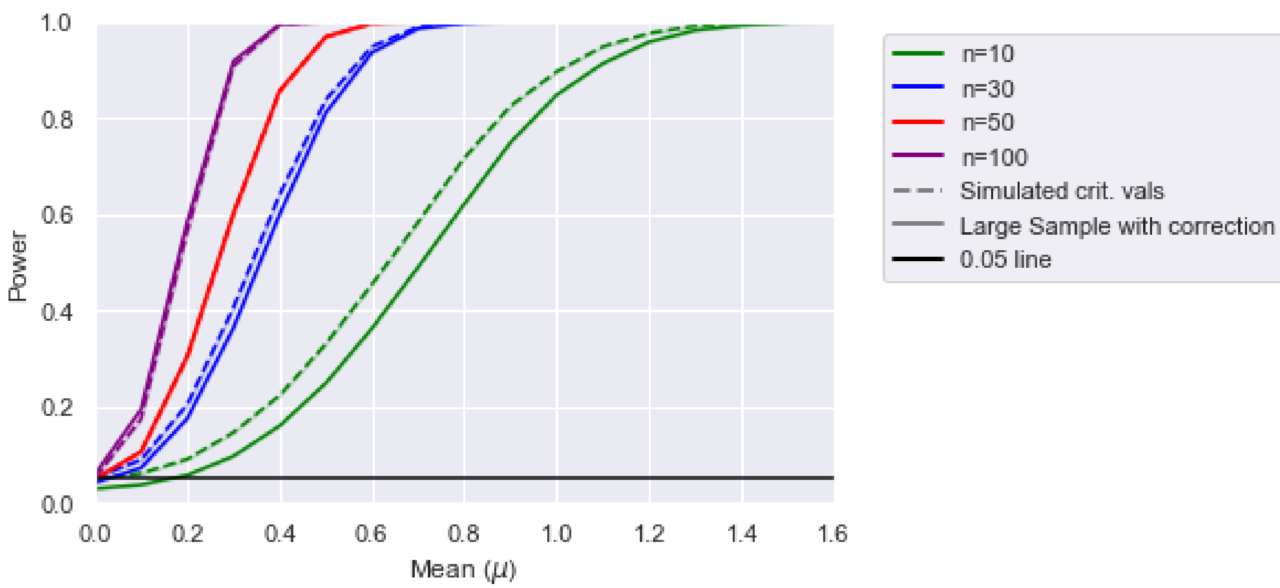
3.6.3. Orientation Sample Power Comparison between Critical Values

Comparisons between the two critical value methods for the Orientation Sample method resulted in similar patterns in power as for the Orientation Grid method when detecting differences in the mean, variance and correlation. As we can see in Figure 3.21, Figure 3.22, and Figure 3.23 the simulated critical values provide more power for small samples than the large sample critical values. Furthermore, the simulated values achieve α when there is no correction to larger sample, but error is no greater than .02 (see Table 3.9).

Table 3.9: 1S 2D KS test Orientation Sample achieving power $\alpha = 0.05$ for the two critical value methods

Sample Size	Simulated	Large Sample
10	0.05	0.03
30	0.05	0.04
50	0.05	0.05
100	0.05	0.06

Figure 3.21: 1S 2D KS test Orientation Sample mean power $\boldsymbol{\alpha} = 0.05$ for the two critical value methods



It is clear that for sample sizes of 100 or greater, the large sample critical value is more than adequate to detect power and the methods are similar at a sample size of 50.

Figure 3.22: 1S 2D KS test Orientation Sample standard deviation power $\boldsymbol{\alpha} = 0.05$ for the two critical value methods

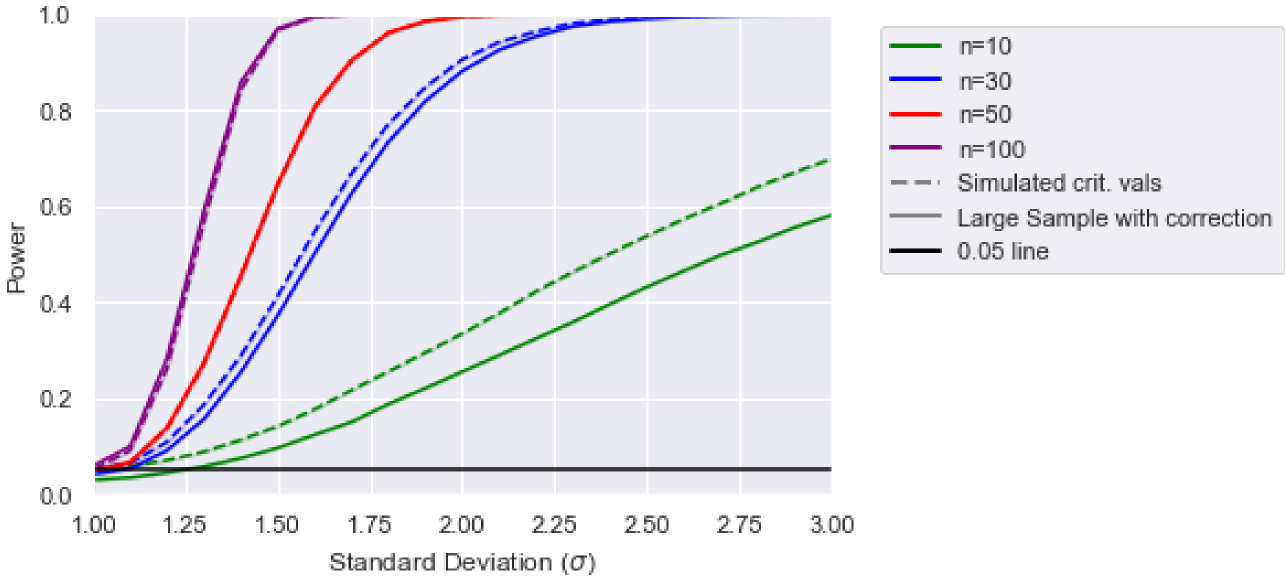
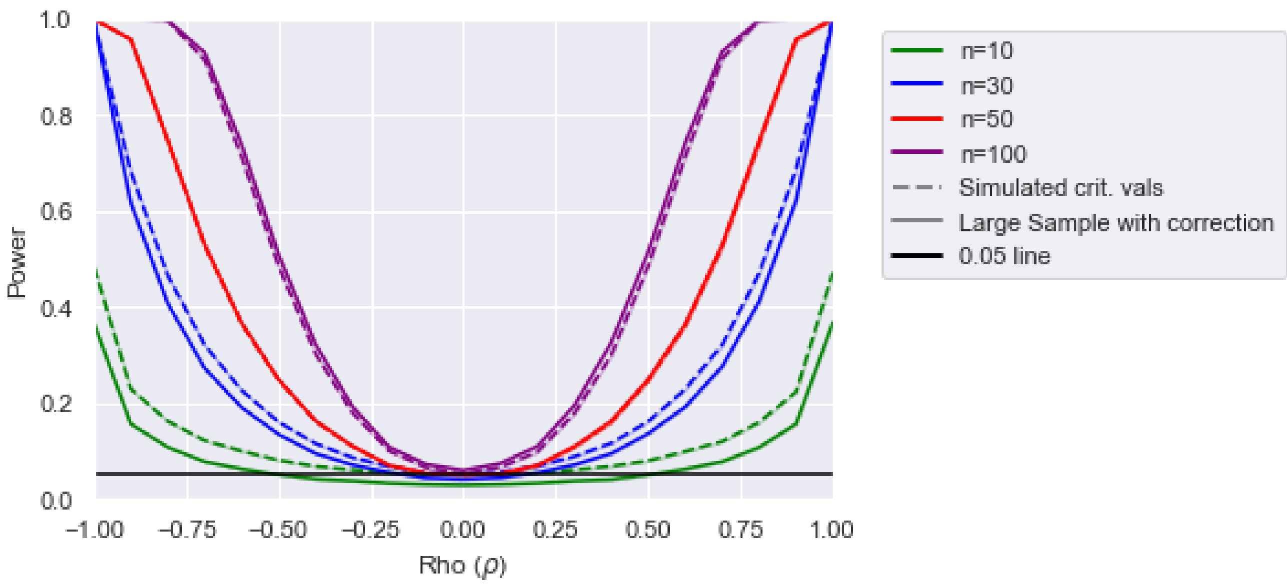


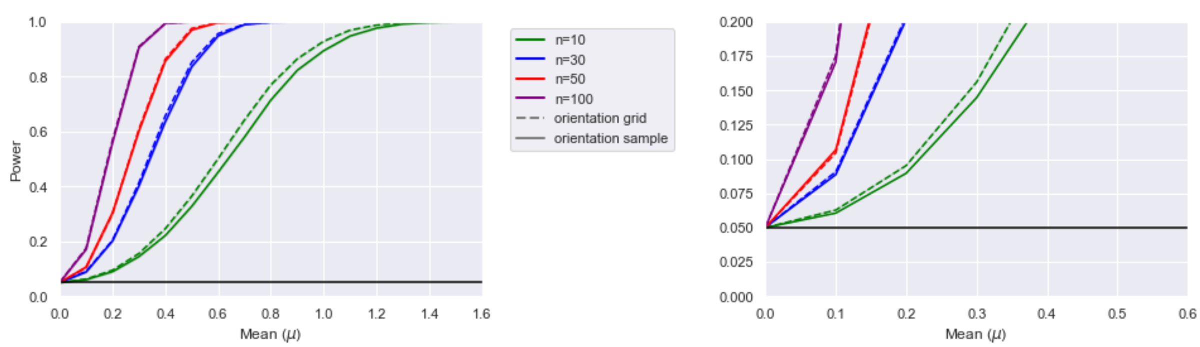
Figure 3.23: 1S 2D KS test Orientation Sample correlation power $\boldsymbol{\alpha} = 0.05$ for the two critical value methods



3.6.4. Power Comparison between Orientation Grid and Sample

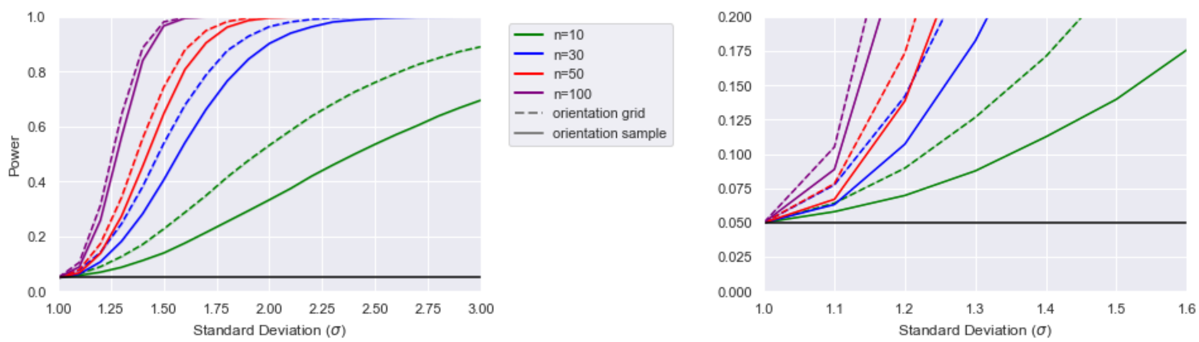
Now that we have established the best critical values to use for both the Orientation Grid and Sample method, comparisons in power were made between the Orientation Grid and Orientation Sample methods. For both methods, the simulated values had better performance for small samples and maintained α (by design) for the null distribution. Comparing the power of each method to detect a mean difference, Orientation Grid has higher power for small samples (n = 10, 30) and performs almost identically for larger samples (see Figure 3.24). For sample of size 10, we can detect mean difference of 0.85 with power of 0.82 for the Orientation Grid method, while Orientation Sample does not achieve 0.80 power to detect a mean difference of 0.85 (power = 0.77). On the other hand, the power difference for sample of size 100 is less than 0.01 for all differences in the means, making the methods comparable with respect to power, although Orientation Grid tends to be the higher power.

Figure 3.24: 1S 2D KS test Orientation Grid and Sample mean power $\boldsymbol{\alpha} = 0.05$



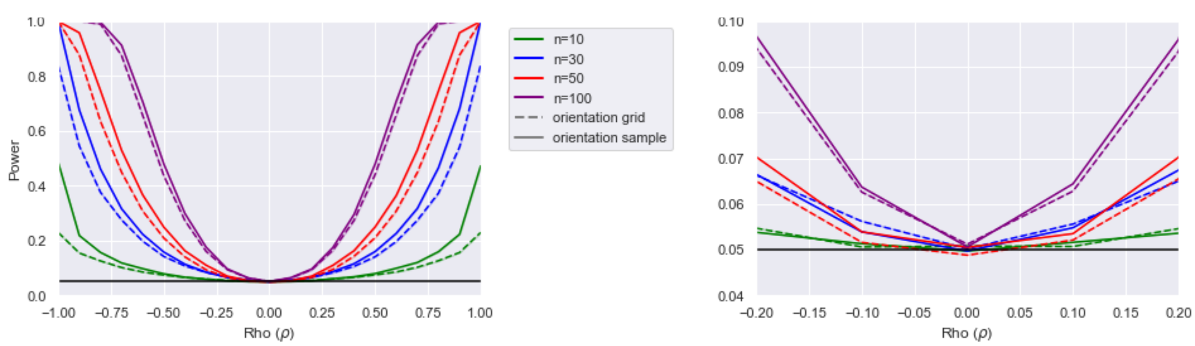
When trying to detect differences in standard deviation, however, the Orientation Grid method maintains higher power for all sample sizes (Figure 3.25). Especially for small samples (n = 10), the Orientation Grid method eventually achieves a power of 0.8, and much earlier than the Orientation Sample method which does not achieve 0.80 power even when standard deviation is three times higher than the null standard deviation (see Figure 3.25). For sample of size 10, we can detect a standard deviation difference of 2.6 to achieve a power of 0.8 for the grid method, while the sample method only has power of 0.57. Large differences still persist in that Orientation Grid method can detect smaller shifts in standard deviation at higher power consistently more so than Orientation Grid method up to n = 100.

Figure 3.25: 1S 2D KS test Orientation Grid and Sample standard deviation power $\boldsymbol{\alpha} = 0.05$



In contrast, when trying to detect differences in correlation, the Orientation Sample has higher power across all sample sizes although differences are minimal at larger sample sizes (see Figure 3.26). Regardless, both methods have a challenging time detecting correlation changes of less than 0.5 and will require larger sample sizes to achieve an acceptable power level.

Figure 3.26: 1S 2D KS test Orientation Grid and Sample correlation power $\boldsymbol{\alpha} = 0.05$



In conclusion, Orientation Grid method has higher power to detect differences in means and standard deviations especially for small sample, while Orientation Sample has slightly higher power for detecting correlation changes, but both methods will require quite large samples to detect slight changes in correlation. In general, for samples less than 100, Orientation Grid method is a more powerful method overall, even though it requires $16n^2$ instead of $16n$, the computational burden is justified due to its better performance.

3.7. Comparing All Four Methods

This section will focus on comparing the power of the orientation and Partial Orientation Grid methods as well as the orientation and Partial Orientation Sample methods. Furthermore, we will analyze computational time to provide recommendations on benefits and drawbacks of these methods.

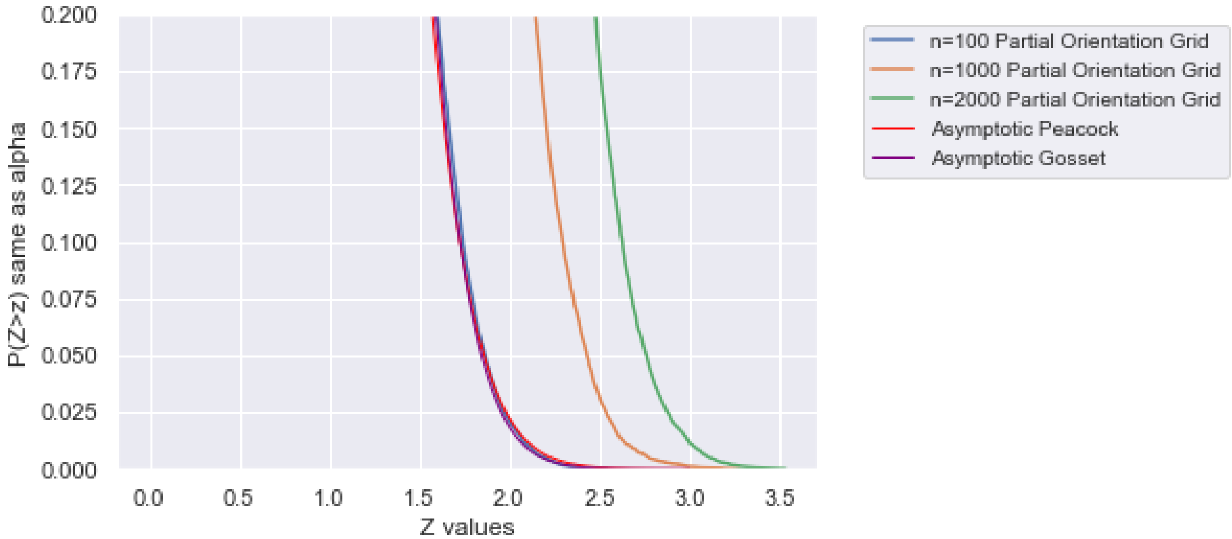
3.7.1. Grid Methods

There are two grid methods that we will be comparing: Orientation and Partial Orientation. Recall that the main difference between these two methods is that the Orientation Grid method evaluates the ECDP using all the directions to approach a data point in 2D (4 total directions) while Partial Orientation grid uses a subset of those directions to approach a data point.

3.7.1.1. Partial Orientation Grid

Before we can compare the two grid methods, we must first determine which set of critical values for Partial Orientation Grid is the best and appropriate set to use. Given that this was the original extension to multiple dimensions for the KS test, we now have five different sets of critical values: 1) the original Peacock critical values, 2) the original Peacock large sample critical values with correction, 3) the Gosset large sample critical values with correction, 4) our own simulated critical values, and finally 5) our large sample critical values with correction.

Figure 3.27: 1S 2D KS Test Partial Orientation Grid for various samples (10,000 repetitions)



As previously stated, both large sample critical values for Peacock and Gosset cannot be used for sample sizes greater than 100 due to their equation and correction being fitted to samples less than or equal to 100. In Figure 3.27 we can see how for sample size 100, our own simulated data matches very closely to the asymptotic equations proposed by Peacock and Gosset, but as we increase sample size, the simulation values are increasing. Therefore, we could make the argument that what Peacock and Gosset call the asymptotic equation, even though lacks in number of repetitions, are fairly accurate for only sample size 100 (their correspoding corrections from small sample lack data points as well as repetitions). Furthemore, we believe it is no longer accurate to consider them asymptotic equations given that even for sample size of 5,000 the null distribution has not converged to the theoretical and unknown asymptotic distribution, however, we do recognize that prior to our simulation including a sample size of 5000, these were the best estimates available.

Table 3.10: 1S 2D KS Test Critical Values for Partial Orientation Grid from Simulation

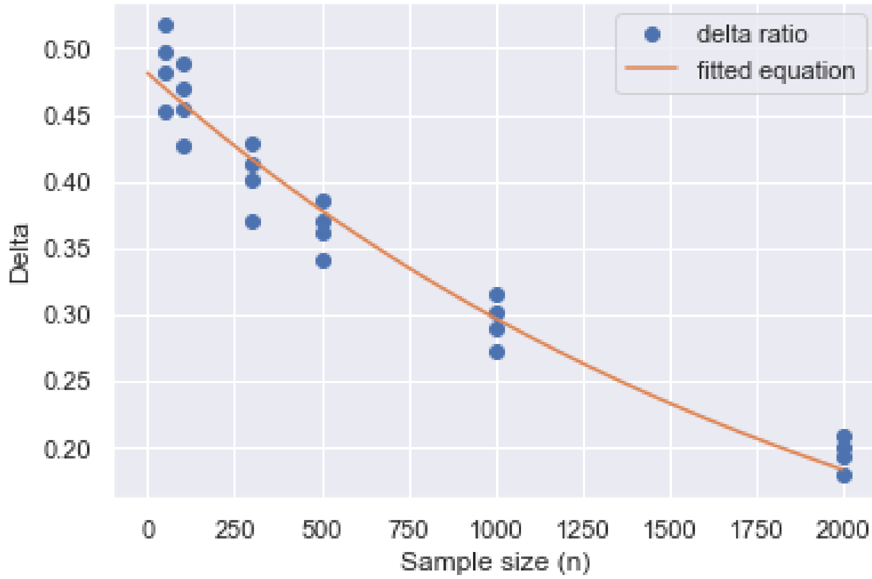
n/ α	0.01	0.05	0.1	0.2
10	1.82	1.58	1.47	1.33
20	1.93	1.67	1.55	1.41
30	1.96	1.69	1.58	1.45
40	1.98	1.74	1.61	1.48
50	2.02	1.77	1.65	1.51
100	2.11	1.87	1.74	1.6
1000	2.67	2.43	2.3	2.14
2000	3.02	2.76	2.63	2.48
5000	3.68	3.42	3.29	3.13

3.7.1.2. Partial Orientation Grid Simulated and Large Sample Critical Values

Based on the lack of sample sizes and repetitions used by Peacock and Gosset, we have extended their work by using sample sizes ranging from 10 to 5,000 with 10,000 replications to accurately depic the 1S 2D KS test null distribution. Using the same simulation method and random seeds as stated previously, we computed the critical values found in Table 3.10 while a more complete list can be found in Table 7.4 in the appendix. Using these values and the same procedure as shown previously to find the correction needed for large sample we found the exponential fit for the correction to large sample as graphed in Figure 3.28 and provided in Equation (3.25). This fit is based on the selected α values (0.01, 0.05, 0.1, and 0.2) and sample sizes (50, 100, 300, 500, and 1000). In the next section we will compare the simulated critical values against the large sample critical values with correction.

$$\delta_{os,ratio}^{(2)} = 0.458e^{-0.0005(n-101.37)} \quad (3.25)$$

Figure 3.28: 1S 2D KS Partial Orientation Grid Delta ratio fitted equation for $\boldsymbol{\alpha}$ values: 0.2, 0.1, 0.05, 0.01 using null distribution from



3.7.1.3. Partial Orientation Grid Power Comparison between Critical Values

For this comparison we will be using our own simulated critical values as well as the large sample critical values with correction. The other critical value methods will not be considered based on the significant shortcomings stated in the previous section. When trying to detect differences in the mean, Figure 3.29 shows that the simulated critical values have higher power for small samples, comparable power for larger samples (n = 100) and also achieve α for the null distribution (see Table 3.11). Similar to trying to detect differences in the standard deviation, the simulated critical values have a much larger power for small sample when compared to the large sample critical values with the proper correction, is the only method to achieve 0.80 power in less than a three times difference in standard deviations and has comparable power for a sample size of n = 100 (see Figure 3.30).

Figure 3.29: 1S 2D KS test Partial Orientation Grid mean power $\boldsymbol{\alpha} = 0.05$ for the two critical value methods

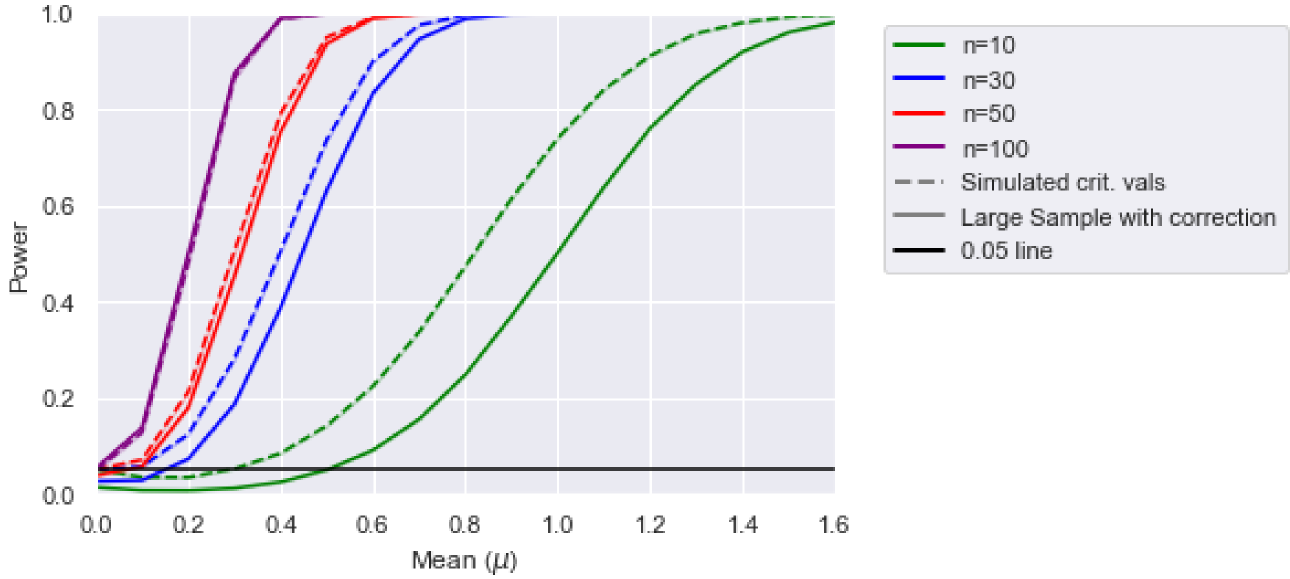
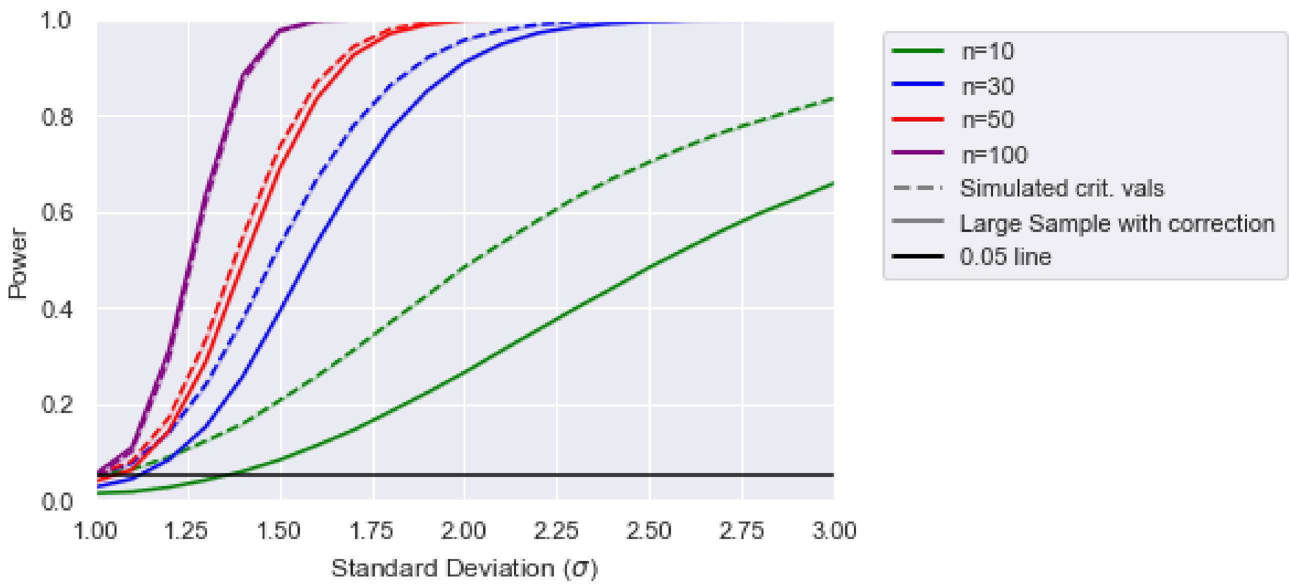


Figure 3.30: 1S 2D KS test Partial Orientation Grid standard deviation power $\boldsymbol{\alpha} = 0.05$ for the two critical value methods



Finally, for detecting correlation, simulated values have higher power in general up through sample sizes of 50, although smaller sample sizes (n = 10, 30) were unable to achieve a power of at least 0.8. Further, larger samples have remarkably similar power regardless of which critical values we use (see Figure 3.31). Therefore, based on this analysis we can conclude that, when possible, the simulated critical values should be used for hypothesis testing, but for samples larger than 100 both sets of critical values have remarkably similar power to detect differences in mean, standard deviation and correlation.

Figure 3.31: 1S 2D KS test Partial Orientation Grid correlation power $\alpha = 0.05$ for the two critical value methods

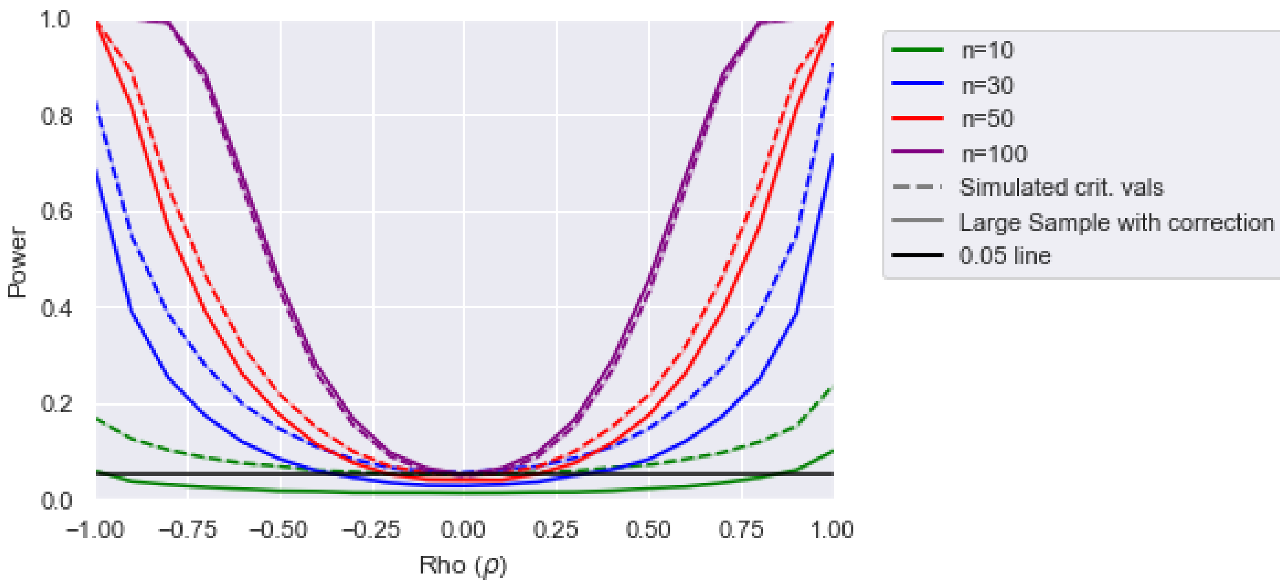


Table 3.11: 1S 2D KS test Partial Orientation Grid achieving power $\alpha = 0.05$ for the two critical value methods

Sample Size	Simulated	Large Sample
10	0.05	0.01
30	0.05	0.03
50	0.05	0.04
100	0.05	0.05

3.7.1.4. Power Comparison between Orientation and Partial Orientation Grid

The final comparison for the grid methods is to compare the Orientation Grid method using its simulated critical values and the Partial Orientation Grid method using its simulated critical values. The results are similar to what has been seen previously where, Orientation Grid method has higher or similar power as the Partial Orientation Grid. When trying to detect differences in the mean, we see that for small samples Orientation Grid has much higher power than Partial Orientation Grid and slightly higher power for large sample sizes (see Figure 3.33).

Figure 3.33: 1S 2D KS Test Grid Methods Mean Power $\boldsymbol{\alpha} = 0.05$

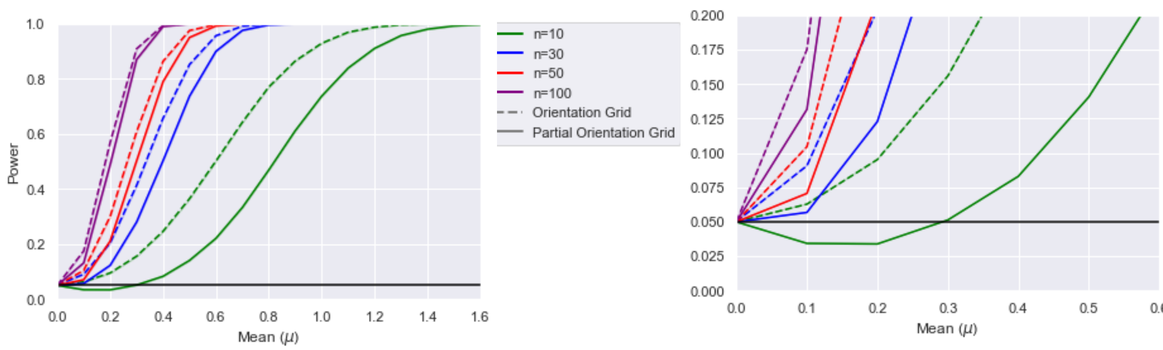
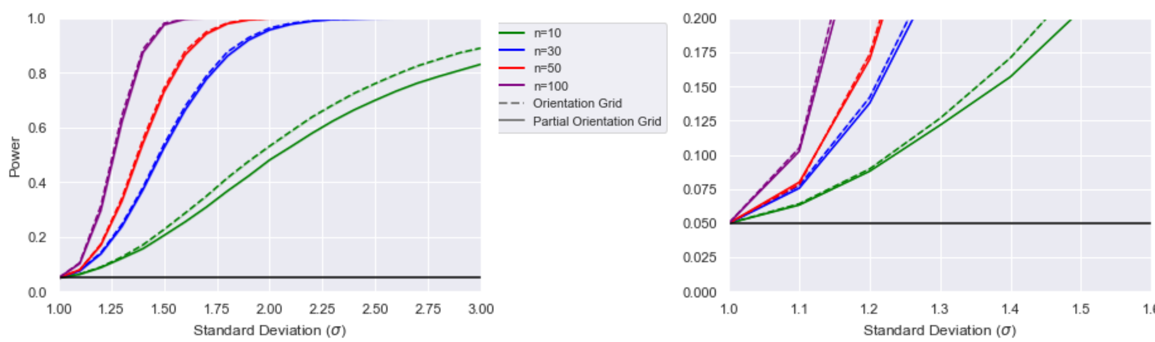
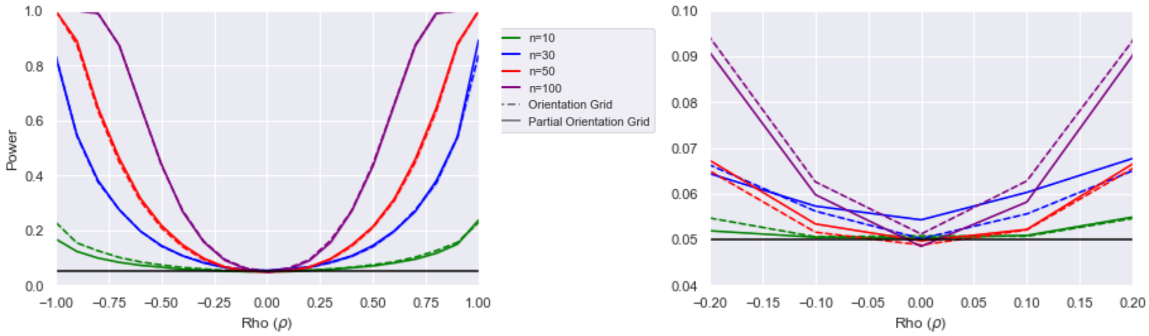


Figure 3.32: 1S 2D KS Test Grid Methods Standard Deviation Power $\boldsymbol{\alpha} = 0.05$



For example, for $n = 10$ Orientation Grid method can detect differences in the mean with at least 80 percent power of 0.85 mean difference, while Partial Orientation Grid for the same power can only detect a 1.1 mean difference. Comparable results, albeit not as drastic, can be seen when

Figure 3.34: 1S 2D KS Test Grid Methods Correlation Power $\alpha = 0.05$



comparing power for differences in standard deviation when sample size is small (see Figure 3.36). For example, for n = 10, Orientation Grid method can detect a difference in the standard deviations of 2.6 with at least 80 percent power, while Partial Orientation Grid for the same power can only detect a difference of 2.9. For the larger sample sizes, power is comparable between Orientation and Partial Orientation Grid. Similarly, when examining the power for detecting differences in correlation, power is almost identical between Orientation and Partial Orientation methods (see Figure 3.34). Furthermore, given that for small sample we are unable to achieve a power of at least 80 percent, we can conclude that to detect small correlation changes an exceptionally large sample will be needed regardless of the method.

In conclusion, when we compare the grid methods, we find that the Orientation Grid using the simulated critical values has an overall higher power especially for small samples compared against the Partial Orientation Grid when we use our own simulated critical values. If the sample size is large and computational time is of significant importance, then using Partial Orientation Grid would provide comparable results (but if computational time is that important at that point, we might recommend one of the sample methods for large samples).

3.7.2. Orientation and Partial Orientation Sample Methods

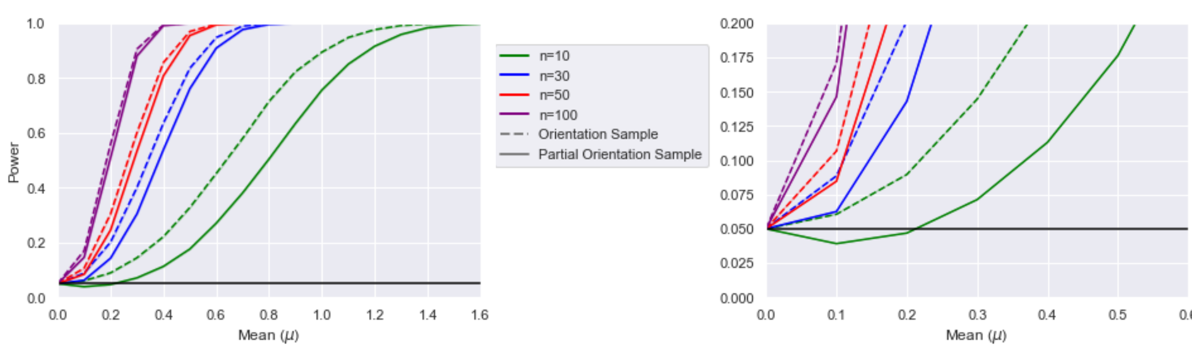
Now that we have established best practices for the grid methods, we can now look at the sample methods and compare the power associated with Orientation Sample and Partial Orientation Sample.

3.7.2.1. Compare Orientation Sample and Partial Orientation Sample

When comparing the sample methods, we can see that Orientation Sample has higher power when detecting mean and correlation differences, but less power when detecting differences in standard deviation. For example, when n = 10, Orientation Sample method can detect differences in the mean with at least 80 percent power of 0.85 difference, while Partial Orientation Sample for the same power can only detect 1.05 difference (see

Figure 3.35). In contrast, for detecting differences in the standard deviation, Partial Orientation Sample has higher power than Orientation Sample (see Figure 3.36). We believe this is a consequence of the different maximum distances each method can capture when the sample has high variance, combined with the inherited variability of the variance for small samples. Furthermore, for small sample sizes variability is higher and less detectable when we account for all four directions per orientation. Regardless, when sample size is large, the differences are

Figure 3.35: 1S 2D KS Test Sample Methods Mean Power $\boldsymbol{\alpha} = 0.05$



minimal. Finally, when looking at differences in the correlation, it is interesting to note that Orientation Sample has higher power for negative correlation whereas power is comparable between Orientation Sample and Partial Orientation Sample for positive correlation. Again, though, as sample size increased, differences in power became smaller, further, larger samples are still likely necessary to detect more minor differences in correlation (see Figure 3.37).

Figure 3.36: 1S 2D KS Test Sample Methods Standard Deviation Power $\boldsymbol{\alpha} = 0.05$

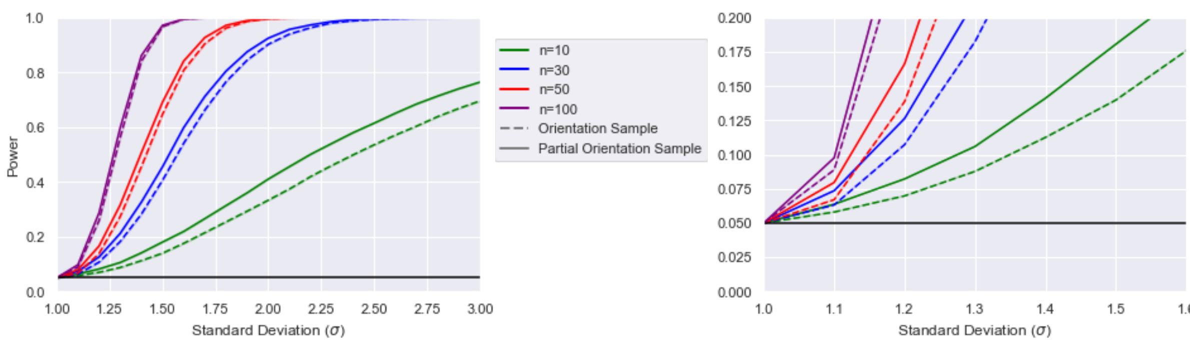
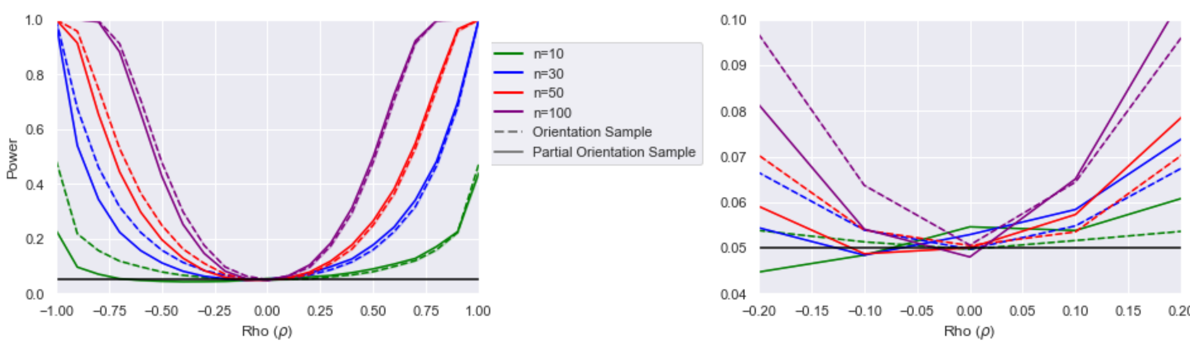


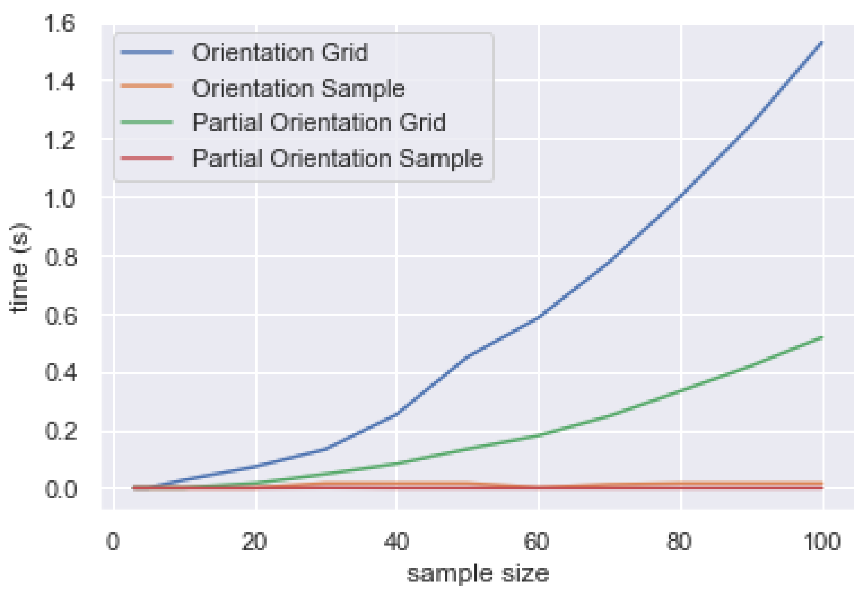
Figure 3.37: 1S 2D KS Test Sample Methods Correlation Power $\boldsymbol{\alpha} = 0.05$



3.8. Power Summary

In addition to comparing power, it is important to consider computation time of each method, especially when the sample size is in the thousands. This was the original concern of Fasano and Franceschini when looking at the original Peacock 2D KS implementation. As we can see in Figure 3.38, Orientation Grid method grows significantly faster than all other methods. This performance is relative to the device being used, in this case an intel i7 10th gen @ 1.30 GHz with 16.0 GB RAM Windows 11 machine was used, as well as the performance improvements of the code (saving the CDF values of the $BVN(\mathbf{0},\boldsymbol{I})$ instead of computing the double integrals). Regardless Figure 3.38 shows how quickly both grid methods become infeasible compared to the sample methods. Of course, this is only when running 10,000 repetitions or more, but if the critical values are available then running one instance of Orientation Grid even for large sample would only take a few minutes.

Figure 3.38: Computation time (seconds) for 1S 2D KS test based on sample size



In conclusion, after doing pair comparisons between these methods, if computational time is not a significant issue and sample size is less than 100 then the method with the most power is Orientation Grid with reasonable computational time. Furthermore, using the simulated critical values provides accurate power while still achieving α for the null distribution. Complete power analysis tables of values for all four methods using the simulated critical values can be found in the appendix, for α values of:

0.2 – Table 8.19, Table 8.23, and Table 8.27,

0.1 – Table 8.20, Table 8.24, and Table 8.28,

0.05 – Table 8.21, Table 8.25, and Table 8.29,

0.01 – Table 8.22, Table 8.26, and Table 8.30

for mean, standard deviation and correlation respectively.

On the other hand, if Partial Orientation Grid method needs to be used, we have corrected and extended the table of critical values as well as provide a correction to large sample that is accurate up to sample size 5,000 (original paper only reached sample size 100). Finally, if computational time is of the outmost importance, then for small sample the Orientation Sample method would be the proper method to use (especially when trying to detect differences in the mean) while Partial Orientation Sample would be reasonable when trying to detect differences in the variance.

4. 1S multi-dimensional KS Test Orientation Grid Method

In this section we will discuss the natural extension from 2D to 3D and then to m-dimensions. Similar to the 1S 2D KS test, the definition of the multi-dimensional CDF is the same, except the number of orientations increases exponentially. For example, in 3D there are 8 possible orientations (the 8 quadrants generated by the x, y and z axis); for m-dimensions we have 2^m orientations.

4.1. 3D Orientation Grid Method

In higher dimensions building the CDF can become difficult to name the orientations, but nevertheless, the same logic applies with a total number of orientations equal to 2^m where m is the dimensions. Furthermore, the properties from the table of properties for the 2D CDF (see Table 3.1) would hold for higher dimensions when extending the Orientation method.

Table 4.1: 3D CDF Definition

Orientation	Limits
$F^{I}_{XYZ}(x,y,z) = P(X \geq x, Y \geq y, Z \leq z)$	$(x \to -\infty, y \to -\infty, z \to +\infty)$
$F^{II}_{XYZ}(x,y,z) = P(X \leq x, Y \geq y, Z \leq z)$	$(x \to +\infty, y \to -\infty, z \to +\infty)$
$F^{III}_{XYZ}(x,y,z) = P(X \leq x, Y \leq y, Z \leq z)$	$(x \to +\infty, y \to +\infty, z \to +\infty)$
$F^{IV}_{XYZ}(x,y,z) = P(X \geq x, Y \leq y, Z \leq z)$	$(x \to -\infty, y \to +\infty, z \to +\infty)$
$F^{V}_{XYZ}(x,y,z) = P(X \geq x, Y \geq y, Z \geq z)$	$(x \to -\infty, y \to -\infty, z \to -\infty)$
$F^{VI}_{XYZ}(x,y,z) = P(X \leq x, Y \geq y, Z \geq z)$	$(x \to +\infty, y \to -\infty, z \to -\infty)$
$F^{VII}_{XYZ}(x,y,z) = P(X \leq x, Y \leq y, Z \geq z)$	$(x \to +\infty, y \to +\infty, z \to -\infty)$
$F^{VIII}_{XYZ}(x,y,z) = P(X \geq x, Y \leq y, Z \geq z)$	$(x \to -\infty, y \to +\infty, z \to -\infty)$

For the 1-sample 3-dimensional Kolmogorov Smirnov (1S 3D KS) test, similar to the 1D and 2D tests, the goal is to find the maximum 1D distance between an ECDF and a continuous CDF in 3D. Given that the CDF is now defined in three dimensions, we must consider all eight orientations as defined in Table 4.1. Apart from evaluating all 8 orientations, we need to evaluate all 8 directions as well:

$(x^+, y^+, z^+),\ (x^+, y^+, z^-),\ (x^+, y^-, z^+),\ (x^-, y^+, z^+),\ (x^+, y^-, z^-),\ (x^-, y^+, z^-),\ (x^-, y^-, z^+),\ (x^-, y^-, z^-)$.

Therefore, the 1S 3D KS test is defined in (4.1):

$$D_{n.og}^{(3)} = \max\left(D_{n.og}^{(3)I}, D_{n.og}^{(3)II}, D_{n.og}^{(3)III}, D_{n.og}^{(3)IV}, D_{n.og}^{(3)V}, D_{n.og}^{(3)VI}, D_{n.og}^{(3)VII}, D_{n.og}^{(3)VIII}\right) \qquad (4.1)$$

where for example:

$$D_{n.og}^{(3)I} = \max\left(D_{n.og}^{(3)I+++}, D_{n.og}^{(3)I++-}, D_{n.og}^{(3)I+-+}, \ldots, D_{n.og}^{(3)I---}\right) \qquad (4.2)$$

and for example:

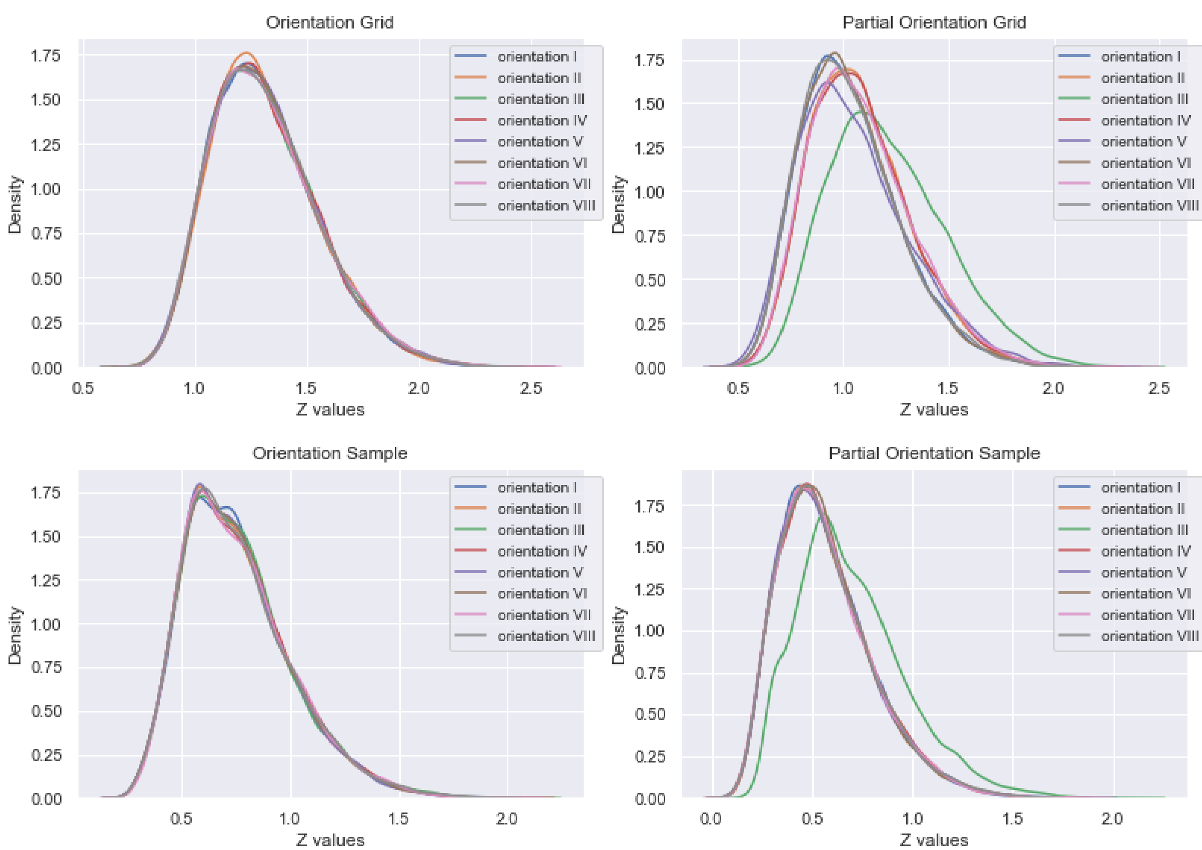
$$D_{n.o}^{(3)I+++} = \sup_{all\ x,y,z}\left|F_n^{(3)I}(x^+, y^+, z^+) - F^{(3)I}(x^+, y^+, z^+)\right| \qquad (4.3)$$

where $F_n^{(3)I}(x^+, y^+, z^+)$ is the 3D ECDF in orientation I with direction (x^+, y^+, z^+) and $F^{(3)I}(x^+, y^+, z^+)$ is the theoretical 3D CDF in orientation I. For the grid method, the evaluation locations become all places where the 3D ECDF changes value: $(X_i, Y_j, Z_k)\ \forall\ i, j, k = 1, \ldots, n$ where n is the sample size. Therefore, the number of compuations needed to compute $D_{n,og}^{(3)}$ totals $64n^3$. In comparison, the 3D Orientation Sample method would only be evaluated for: $(X_i, Y_i, Z_i)\ \forall\ i = 1, \ldots, n$ providing $64n$ computations.

4.1.1. Orientation Decomposition of the 1S 3D KS Test

Similar to the 2D Orientation Grid method, in order to show that the 3D Orientation Grid method has eight iid orientations we can look at each orientation that make up $D_{n,og}^{(3)}$. A similar argument about the *iid* nature can be made where each of the eight orientations has the same four operations (all four directions per orientation) and the maximum occuring in one orientation will not affect the probability of finding the maximum in any of the other seven orientations. Furthermore, these claims were justified after running a simulation using $BVN(\mathbf{0}, \boldsymbol{I})$ for the null distribution in which the maximum distance for random samples of size *n* = 10 in each orientation was estimated separately. The distributions of the maximum distances for each orientation and each method is plotted in Figure 4.1. Only test statistic distributions for the

Figure 4.1: 1S 3D KS Distribution Decomposition of Orientations for *n* = 10



Orientation methods (grid and sample) maintain the same distribution across all eight orientations (I, II, III, IV, V, VI, VII, VIII). On the other hand, the orientation sample has a shape that does not align with the Kolmogorov distribution which has been consistent in the 1D and 2D case.

4.1.2. 1S 3D Orientation Grid Critical Values

Similar to the derivation for the 1S 2D case we define our random variable as

$$T_{ii'i''} = n^3(F_e(d_{ii'i''}) - d_{ii'i''}) \tag{4.4}$$

where $d_{ii'i''} = \left(\frac{i}{n}\right)\left(\frac{i'}{n}\right)\left(\frac{i''}{n}\right)$, $i,\ i',\ i'' = 0, 1, 2, \ldots, n$, and given independence, $n^3(F_e(d_{ii'i''}))$ represents the number of observations where $x \le X_i,\ y \le Y_{i'}\ and\ z \le Z_{i''}$.

Considering the sample space of all possible $T_{111}, T_{112}, \ldots, T_{11\,n-1}, \ldots, T_{1\,n-1\,n-1}, \ldots, T_{nnn}$. Let $A_{ii'i''}$ and $B_{ii'i''}$ be the event where $T_{ii'i''}$ reaches a fixed integer J or -J respectively (if $T_{ii'i''}$ does not reach either J or -J then its probability is 0). Using the formula of total probability $P(A) = P(B)P(A|B) + P(B^c)P(A|B^c)$, we can write the $P\left(T_{kk'k''} = J\right)$ for any $k,\ k',\ k''$ between 1 and $n-1$

$$P(T_{kk'k''} = J) = \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1}\sum_{i''=1}^{n-1} P(A_{ii'i''})P(T_{kk'k''} = J|A_{ii'i''}) + \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1}\sum_{i''=1}^{n-1} P(B_{ii'i''})P(T_{kk'k''} = J|B_{ii'i''}) \tag{4.5}$$

And for -J:

$$P(T_{kk'k''} = -J) = \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1}\sum_{i''=1}^{n-1} P(A_{ii'i''})P(T_{kk'k''} = -J|A_{ii'i''}) + \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1}\sum_{i''=1}^{n-1} P(B_{ii'i''})P(T_{kk'k''} = -J|B_{ii'i''}) \tag{4.6}$$

With a similar logic as the 1D and 2D case where the probabilities correspond to a binomial experiment (either $T_{ii'i''}$ reached J or it did not) but where the probability of success now corresponds to $\frac{kk'k''}{n^3}$ with n^3 trials, we can represent the following probabilities as binomial distributions:

$$P(T_{kk'k''} = J) = \mathbb{B}^{n^3}_{kk'k''+J}\left(\frac{kk'k''}{n^3}\right) \tag{4.7}$$

$$P(T_{kk'k''} = J|A_{ii'i''}) = \mathbb{B}^{n^3-ii'i''-J}_{kk'k''-ii'i''}\left(\frac{kk'k''-ii'i''}{n^3-ii'i''}\right) \tag{4.8}$$

$$P(T_{kk'k''} = J|B_{ii'i''}) = \mathbb{B}^{n^3-ii'i''+J}_{kk'k''-ii'i''+2J}\left(\frac{kk'k''-ii'i''}{n^3-ii'i''}\right) \tag{4.9}$$

Using Equations (4.7),(4.8) and (4.9) and substituting into Equations (4.5) and (4.6) we get:

$$\mathbb{B}^{n^3}_{kk'k''+J}\left(\frac{kk'k''}{n^3}\right) = \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1}\sum_{i''=1}^{n-1} P\left(A_{ii'i''}\right) \mathbb{B}^{n^3-ii'i''-J}_{kk'k''-ii'i''}\left(\frac{kk'k''-ii'i''}{n^3-ii'i''}\right) + \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1}\sum_{i''=1}^{n-1} P\left(B_{ii'i''}\right) \mathbb{B}^{n^3-ii'i''+J}_{kk'k''-ii'i''+2J}\left(\frac{kk'k''-ii'i''}{n^3-ii'i''}\right) \tag{4.10}$$

$$\mathbb{B}^{n^3}_{kk'k''+J}\left(\frac{kk'k''}{n^3}\right) = \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1}\sum_{i''=1}^{n-1} P\left(A_{ii'i''}\right) \mathbb{B}^{n^3-ii'i''-J}_{kk'k''-ii'i''-2J}\left(\frac{kk'k''-ii'i''}{n^3-ii'i''}\right) + \sum_{i=1}^{n-1}\sum_{i'=1}^{n-1}\sum_{i''=1}^{n-1} P\left(B_{ii'i''}\right) \mathbb{B}^{n^3-ii'i''+J}_{kk'k''-ii'i''}\left(\frac{kk'k''-ii'i''}{n^3-ii'i''}\right) \tag{4.11}$$

Using Equation (4.10) and (4.11) we can create a $2(n-1)^3$ system of linear equations with $2(n-1)^3$ unknowns. Solving the system of equations $\boldsymbol{A}x = \boldsymbol{b}$ using the least squares method and summing *x*, we can find the probability associated with J. To transform the J values to the raw distance: $D^{(3)}_{n,og} = \sqrt{n^2}\,\frac{J}{n^3}$, and to standardized the raw distance: $Z^{(3)}_{n,og} = \sqrt{n}D^{(3)}_{n,og}$. These solutions are for one orientation; therefore, we must use the Equation (3.8) to find the marginal pdf which equals $P(D_{n,og}^{(3)} > d)$.

$$Y_n = X_{(n)} = \max(X_1, \dots, X_n) = nf_x(y_1)[F_x(y_1)]^{n-1} \quad (3.5.2)$$

where $X_1, X_2, \dots, X_n \overset{iid}{\sim} f_X(x)$.

Fitting an exponential equation with the form seen in Equation (4.12) (using two parameters for a more flexible fit), we get the exponential equations and fit seen in Figure 4.2. In order to convert the J values to D we use $D_{n,og}^{(3)} = \sqrt{n^2 \frac{J}{n^3}}$ and to standardize $D_{n,og}^{(3)}$ we can use the standard transformation of $Z_{n,og}^{(3)} = \sqrt{n} D_{n,og}^{(3)}$.

Using that form of the exponential fit on the standarized and converted J values, we can find the marginal pdf using Equation (4.13).

$$y = 2e^{-b(x-c)^2} \quad (4.12)$$

$$Y_n = n((-1)(-b(x-c)))2e^{-b(x-c)^2}\left[1 - 2e^{-b(x-c)^2}\right]^{n-1} \quad (4.13)$$

Similar to the 2D case, given that we are working with $P(Z_n > z)$ as the $F_x(y_1)$ there is an extra negative sign for the derivative of $F_x(y_1)$ which is incorporated into Equation (4.13).

Figure 4.2: Standardized 1S 3D KS distance of Binomial derivation and exponential fitted

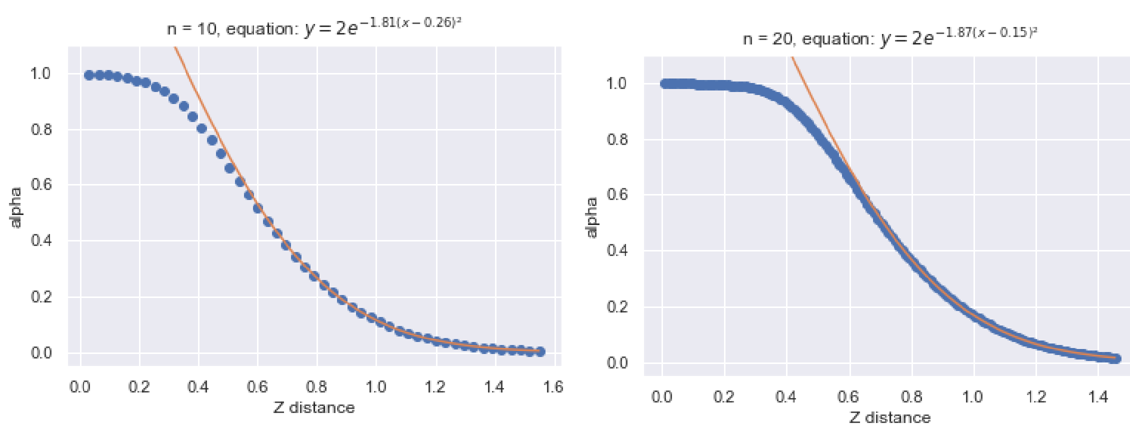
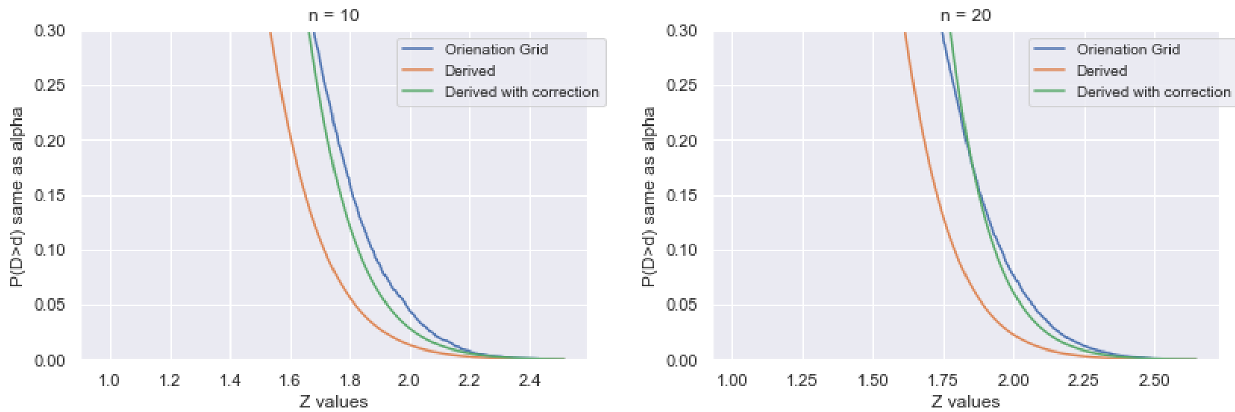


Figure 4.3: 1S 3D KS Test Derived (using exponential fit) vs Simulated



The critical values that result from this method when compared to the simulated data for the 1S 3D KS test null distribution seem to have bias associated with it. Looking at Figure 4.3 we can see that the derived (orange line) and the simulated (blue line) have a similar shape, but are off each other by a small margin. Multiplying the derived values by $\sqrt[3]{n}$ provides the green line which is close and follows the simulated values. Table 4.2 shows the critical values for various α values for the standardized derived values as well as the standardized derived values with correction.

Table 4.2: 1S 3D KS Test Critical Values for Various α Values from Derived and Derived w/ correction

	n/ α	0.01	0.05	0.1	0.2
Derived	10	2.03	1.82	1.71	1.60
	20	2.10	1.89	1.79	1.68
w/ correction	10	2.12	1.92	1.83	1.73
	20	2.22	2.03	1.93	1.84

On the other hand, Table 4.3 shows the simulated critical values for all four methods when drawing *n* samples from a BVN(**0**,**I**) and find the maximum difference in the CDFs against a theoretical BVN(0,I) repeated 10,000 times.

Table 4.3: 1S 3D KS Test Critical Values for Various α Values from Simulation

	$\alpha = 0.2$				$\alpha = 0.1$			
	Orientation		Partial Orientation		Orientation		Partial Orientation	
n	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
10	1.76	1.30	1.57	1.12	1.88	1.42	1.70	1.24
20	1.83	1.34	1.68	1.21	1.96	1.46	1.81	1.33
30	1.86	1.36	1.73	1.25	1.99	1.48	1.86	1.39
40	1.90	1.39	1.78	1.30	2.02	1.52	1.91	1.43
50	1.91	1.40	1.81	1.32	2.05	1.53	1.94	1.45
	$\alpha = 0.05$				$\alpha = 0.01$			
	Orientation		Partial Orientation		Orientation		Partial Orientation	
n	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
10	1.99	1.53	1.81	1.35	2.17	1.73	2.02	1.57
20	2.07	1.57	1.92	1.44	2.27	1.79	2.14	1.67
30	2.10	1.59	1.97	1.49	2.32	1.84	2.20	1.73
40	2.13	1.63	2.02	1.55	2.34	1.84	2.23	1.76
50	2.16	1.64	2.04	1.56	2.37	1.87	2.26	1.78

4.2. 1S 3D KS Test Power Analysis

Using the simulated critical values for all four methods we can see in Figure 4.4 and Table 4.4 the power each of the four methods when detecting differences in the mean. Overall, Orientation Grid method has more power with significant higher power for small samples. For example, for *n* = 10, Orientation Grid method can detect a difference in the mean of 0.8 with at least 80 percent power, while Orientation Sample method for the same mean difference can only achieve 70 percent power. On the other hand, when trying to detect that same mean difference

(0.8) both Partial Orientation methods have less than 60 percent power. For n = 30 we start to see closer power between the methods, but it is still clear that Orientation Grid outperforms all the other three methods.

Figure 4.4: 1S 3D KS Test Sample Methods Mean Power $\boldsymbol{\alpha} = 0.05$

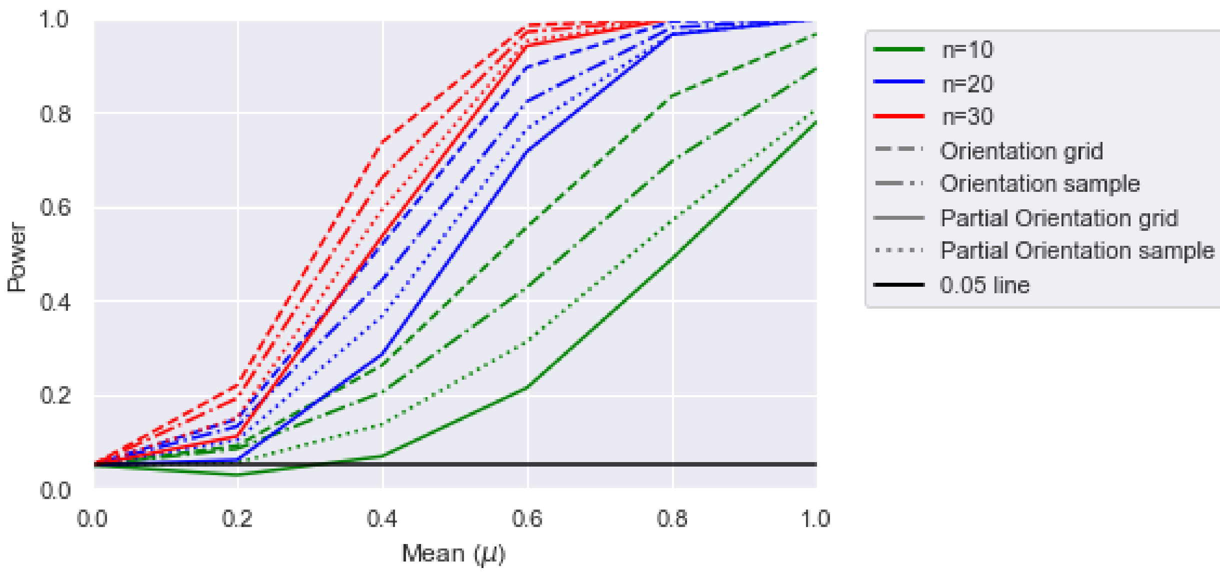


Table 4.4: 1S 3D KS test four methods power curves for mean shift for $\alpha = 0.05$ using simulated critical values

	Sample size 10				Sample size 20				Sample size 30			
	Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation	
mean	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
0	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
0.2	0.09	0.08	0.03	0.06	0.15	0.13	0.06	0.10	0.22	0.19	0.11	0.15
0.4	0.26	0.20	0.07	0.14	0.52	0.44	0.29	0.37	0.74	0.66	0.54	0.59
0.6	0.56	0.43	0.21	0.31	0.90	0.82	0.72	0.77	0.99	0.97	0.94	0.95
0.8	0.83	0.70	0.49	0.57	0.99	0.98	0.97	0.97	1.00	1.00	1.00	1.00
1	0.97	0.89	0.78	0.81	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

4.3. Multi-Dimensional Method

The generalization to the 1S m-dimensional (mD) KS test follows from the 2D and 3D cases, where we examine 2^m orientations and 2^m directions per orientations where *m* is the number of dimensions; with the 1S mD KS test defined in Equation (4.14):

$$D_{n.og}^{(m)} = \max\left(D_{n.og}^{(m)I}, D_{n.og}^{(m)II}, \dots, D_{n.og}^{(m)2^m}\right) \qquad (4.14)$$

where for example:

$$D_{n.og}^{(m)I} = \max\left(D_{n.og}^{(m)I++\dots+}, D_{n.og}^{(m)I++\dots+-}, D_{n.og}^{(m)I++\dots-+}, \dots, D_{n.og}^{(m)I--\dots-}\right) \qquad (4.15)$$

and for example:

$$D_{n.og}^{(m)I++\dots+} = \sup_{all\ x_1,x_2,\dots,x_{2^p}} \left|F_n^{(m)I}(x_1^+, x_2^+, \dots, x_{2^m}^+) - F^{(m)I}(x_1^+, x_2^+, \dots, x_{2^m}^+)\right| \qquad (4.16)$$

where $F_n^{(m)I}(x_1^+, x_2^+, \dots, x_{2^m}^+)$ is the mD ECDF in orientation I with direction $(x_1^+, x_2^+, \dots, x_{2^m}^+)$ and $F^{(m)I}(x_1^+, x_2^+, \dots, x_{2^m}^+)$ is the theoretical mD CDF in orientation I. For the grid method our evaluation locations become $\left(X_{1,i_1}, X_{2,i_2}, \dots, X_{2^p,i_{2^m}}\right)\ \forall\ i_1, i_2, \dots, i_{2^m} = 1, \dots, n$ where *n* is the sample size. Therefore, the number of compuations needed to compute $D_{n,og}^{(m)}$ totals $2^m 2^m n^m$. In comparison the mD Orientation Sample method would only be evaluated: $\left(X_{1,i}, X_{2,i}, \dots, X_{2^p,i}\right)\ \forall\ i = 1, \dots, n$ providing us with $2^m 2^m n$ computations. This extension to multivariate space can be applied to any of the other 3 methods discussed: Orientation Sample, Partial Orientation Grid and Partial Orientation Sample by modifying either the evaluation location or the subset of directions from which to approach a point.

4.4. mD Orientation Grid Critical Values

Using the same derivation as the 2D and 3D cases. Let $k_{total} = \prod_{m=1}^{p} k_m$ and $i_{total} = \prod_{m=1}^{p} i_m$ then:

$$\mathbb{B}_{k_{total}+J}^{n^p}\left(\frac{k_{total}}{n^p}\right) = \sum_{i_1=1}^{n-1} \dots \sum_{i_p=1}^{n-1} P\left(A_{i_{total}}\right) \mathbb{B}_{k_{total}-i_{total}}^{n^3-i_{total}-J}\left(\frac{k_{total}-i_{total}}{n^p-i_{total}}\right) + \sum_{i_1=1}^{n-1} \dots \sum_{i_p=1}^{n-1} P\left(B_{i_{total}}\right) \mathbb{B}_{k_{total}-i_{total}+2J}^{n^p-i_{total}+J}\left(\frac{k_{total}-i_{total}}{n^p-i_{total}}\right) \quad (4.17)$$

$$\mathbb{B}_{k_{total}+J}^{n^p}\left(\frac{k_{total}}{n^p}\right) = \sum_{i_1=1}^{n-1} \dots \sum_{i_p=1}^{n-1} P\left(A_{i_{total}}\right) \mathbb{B}_{k_{total}-i_{total}-2J}^{n^3-i_{total}-J}\left(\frac{k_{total}-i_{total}}{n^p-i_{total}}\right) + \sum_{i_1=1}^{n-1} \dots \sum_{i_p=1}^{n-1} P\left(B_{i_{total}}\right) \mathbb{B}_{k_{total}-i_{total}}^{n^p-i_{total}+J}\left(\frac{k_{total}-i_{total}}{n^p-i_{total}}\right) \quad (4.18)$$

Equations (4.17) and (4.18) are the only equations needed to build the $2(n-1)^m$ system of linear equations with $2(n-1)^m$ unknowns. Solving the system of equations $\boldsymbol{A}x = \boldsymbol{b}$ using the least squares method and summing *x*, we can find the probability associated with J. Following a similar pattern to the 2D and 3D case, in order to convert the J values to $D_n^{(m)} = \left(\sqrt{n}\right)^{m-1} \frac{J}{n^m}$ with the standardization $Z_n^{(m)} = \sqrt{n} D_n^{(m)}$. Just like for the 3D case, a correction of $\sqrt[m]{n}$ might be required.

5. 2-Sample 2-dimensions KS Test Orientation Method

5.1. Chapter Overview

The purpose of this chapter is to show how the 2S 2D KS Orientation Grid method is equivalent to the 2S 2D KS Partial Orientation Grid method. This is a consequence of having two samples that have jumps/discontinuities and evaluating the grid generated for all the observations (both sample 1 and sample 2).

Up to this point, we have discussed the 1 sample case in detail (see Figure 3.3 to see the sample CDF for orientation III) where we have observations from one sample and we compare against a theoretical, continuous distribution. On the other hand, for the two-sample case there is no theoretical distribution. Instead, we compare against another ECDF from a second sample. Figure 5.1 shows two ECDFs for orientation III where sample one (blue) has three points: (1, 1), (2, 3) and (3, 2) and sample two (red) has two points: (1.5, 1.5) and (4, 3.5). Similar to the one-sample case, this figure shows the need to not only search for the maximum where data is observed, but also where the ECDFs change.

Figure 3.3: Orientation III CDF for 3 points (1,1), (2,3), and (3,2)

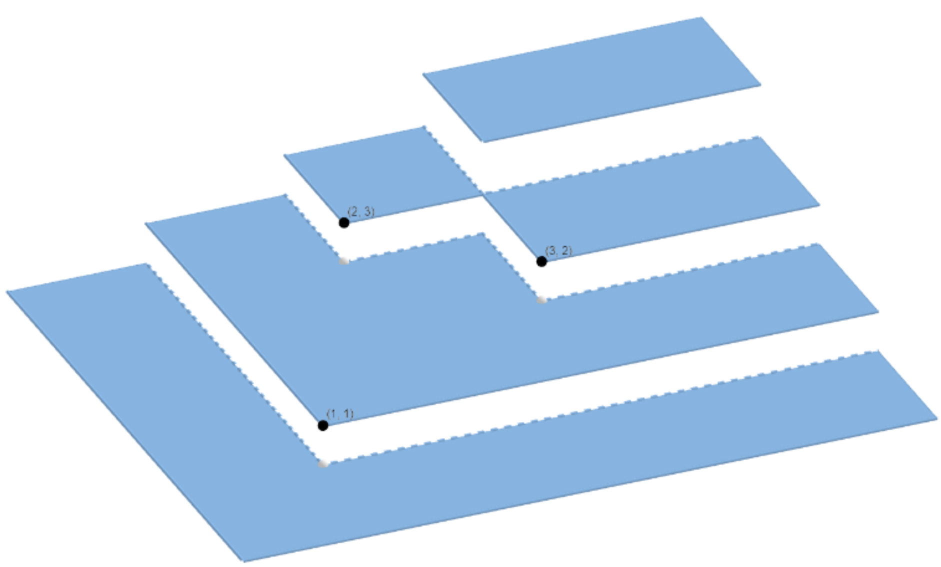
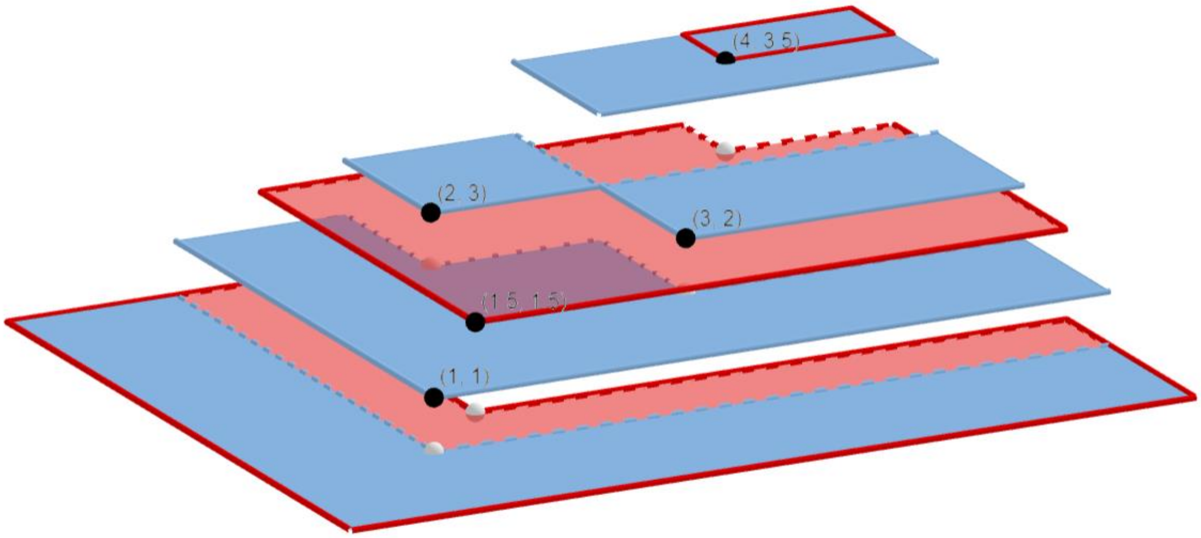


Figure 5.1: Orientation III CDF for two ECDFs



5.2. Theorem I

Theorem I : The 2S 2D KS distance from the Orientation Grid method ($D_{n,og}^{(2)}$) is equivalent to the 2S 2D KS distance from the Partial Orientation Grid method ($D_{n,os}^{(2)}$) for any two random samples.

Proof :

Let $(\mathrm{X_N}, Y_N) = \{(x_{N1}, y_{N1}), \ldots, (x_{Nn}, y_{Nn})\}$ be the samples from the bivariate sample 1 with size *n*, and $(\mathrm{X_M}, Y_M) = \{(x_{M1}, y_{M1}), \ldots, (x_{Mn}, y_{Mn})\}$ be the samples from the bivariate sample 2 with size *m*. Furthermore, let A_i be the unique joint values from X_N and X_M, and similarly B_j be the unique joint values from Y_N and Y_M, where $i, j = 1, \ldots, (n+m)$ then the grid where the 2S 2D KS test will be evaluated consists of (A_i, B_j) $\forall$ $i, j = 1, \ldots, (n+m)$. Using the definition of the 2S 2D KS test Orientation Grid method where the test statistic $D_{n,m}^{(2)}$ is defined in Equation (5.1):

$$D_{n,m}^{(2)} = \max\left(D_{n,m}^{(2)I}, D_{n,m}^{(2)II}, D_{n,m}^{(2)III}, D_{n,m}^{(2)IV}\right) \quad (5.1)$$

where for example:

$$D_{n,m}^{(2)\,I} = \max\left(D_{n,m}^{(2)I++}, D_{n,m}^{(2)I+-}, D_{n,m}^{(2)I-+}, D_{n,m}^{(2)I--}\right) \quad (5.2)$$

and for example:

$$D_{n,m}^{(2)I++} = \sup_{all\ x,y}\left|F_N^{(2)I}(x^+, y^+) - F_M^{(2)I}(x^+, y^+)\right| \quad (5.3)$$

Without loss of generality, we will focus on orientation III and show that $D_{n,m}^{(2)III+-}, D_{n,m}^{(2)III-+}$, $D_{n,m}^{(2)III--}$ are always captured by $D_{n,m}^{(2)III++}$ (which is the definition of $D_{n,m}^{(2)\,I}$ for Partial Orientation Grid method see Equation (2.13) as the 1-sample equivalent) at the same evaluation location or adjacent evaluation locations. The trivial cases occur when the first point is observed when cumulating which causes $D_{n,m}^{(2)III+-} = D_{n,m}^{(2)III-+} = D_{n,m}^{(2)III--} = 0$ as well as the last point cumulated which causes $D_{n,m}^{(2)III++} = 0$.

There are two additional scenarios to consider. First, consider an evaluation location in (A_i, B_j) where an observation is seen, then $F_N^{(2)III}(x_i^+, y_j^+) = \frac{k_1}{n}$ and $F_M^{(2)III}(x_i^+, y_j^+) = \frac{k_2}{m}$ where $\frac{k_1}{n}$ and $\frac{k_2}{m}$ are the proportions at the specified point for each sample. Given that the samples come from continuous distributions, we cannot have another observation equal in either x_i or y_j, therefore there are two options: either (x_{i-1}, y_{j-1}) has an observation or (x_{i-1}, y_{j+1}) has an observation. There is no need to consider (x_{i+1}, y_{j-1}) and (x_{i+1}, y_{j+1}) because we are looking at orientation III. Therefore,

$$D_{n,m}^{(2)III++}(x_{i-1},y_{j-1}) = \left|F_N^{(2)III}(x_{i-1}^+,y_{j-1}^+) - F_M^{(2)III}(x_{i-1}^+,y_{j-1}^+)\right| = \left|\frac{k_1-1}{n} - \frac{k_2-1}{m}\right|$$

If we evaluate the other three directions at (x_i, y_j) they will all equal $D_{n,m}^{(2)III++}(x_{i-1}, y_{j-1})$:

$$D_{n,m}^{(2)III+-}(x_i,y_j) = \left|F_N^{(2)III}(x_i^+,y_j^-) - F_M^{(2)III}(x_i^+,y_j^-)\right| = \left|\frac{k_1-1}{n} - \frac{k_2-1}{m}\right|$$

$$D_{n,m}^{(2)III-+}(x_i,y_j) = \left|F_N^{(2)III}(x_i^-,y_j^+) - F_M^{(2)III}(x_i^-,y_j^+)\right| = \left|\frac{k_1-1}{n} - \frac{k_2-1}{m}\right|$$

$$D_{n,m}^{(2)III--}(x_i,y_j) = \left|F_N^{(2)III}(x_i^-,y_j^-) - F_M^{(2)III}(x_i^-,y_j^-)\right| = \left|\frac{k_1-1}{n} - \frac{k_2-1}{m}\right|$$

Second, consider an evaluation location in (A_i, B_j) where there is no observation. Then there are two cases: 1) (x_{i-1}, y_j) and (x_i, y_{j-1}) are the evaluation locations with observations (sample 1 and sample 2 in that order) or 2) (x_{i+1}, y_j) and (x_i, y_{j-1}) are the evaluation locations with observations (sample 1 and sample 2 in that order). The same logic applies if the observations are from the same sample or the order is inverted.

For case 1)

$$D_{n,m}^{(2)III+-}(x_i,y_j) = \left|F_N^{(2)III}(x_i^+,y_j^-) - F_M^{(2)III}(x_i^+,y_j^-)\right| = \left|\frac{k_1-1}{n} - \frac{k_2}{m}\right| = D_{n,m}^{(2)III++}(x_i,y_{j-1})$$

$$D_{n,m}^{(2)III-+}(x_i,y_j) = \left|F_N^{(2)III}(x_i^-,y_j^+) - F_M^{(2)III}(x_i^-,y_j^+)\right| = \left|\frac{k_1}{n} - \frac{k_2-1}{m}\right| = D_{n,m}^{(2)III++}(x_{i-1},y_j)$$

$$D_{n,m}^{(2)III--}(x_i,y_j) = \left|F_N^{(2)III}(x_i^-,y_j^-) - F_M^{(2)III}(x_i^-,y_j^-)\right| = \left|\frac{k_1-1}{n} - \frac{k_2-1}{m}\right|$$

$$= D_{n,m}^{(2)III++}(x_{i-1},y_{j-1})$$

Same logic can be used to show case 2, where:

$$D_{n,m}^{(2)III+-}(x_i,y_j) = \left|F_N^{(2)III}(x_i^+,y_j^-) - F_M^{(2)III}(x_i^+,y_j^-)\right| = \left|\frac{k_1}{n} - \frac{k_2}{m}\right| = D_{n,m}^{(2)III++}(x_i,y_{j-1})$$

$$D_{n,m}^{(2)III-+}(x_i, y_j) = \left|F_N^{(2)III}(x_i^-, y_j^+) - F_M^{(2)III}(x_i^-, y_j^+)\right| = \left|\frac{k_1 - 1}{n} - \frac{k_2}{m}\right|$$

$$= D_{n,m}^{(2)III++}(x_{i-1}, y_{j-1})$$

$$D_{n,m}^{(2)III--}(x_i, y_j) = \left|F_N^{(2)III}(x_i^-, y_j^-) - F_M^{(2)III}(x_i^-, y_j^-)\right| = \left|\frac{k_1 - 1}{n} - \frac{k_2}{m}\right|$$

$$= D_{n,m}^{(2)III++}(x_{i-1}, y_{j-1})$$

We know that $D_{n,m}^{(2)III++}(x_{i-1}, y_{j-1})$ exists given that (x_i, y_j) is not the first point seen in this orientation.

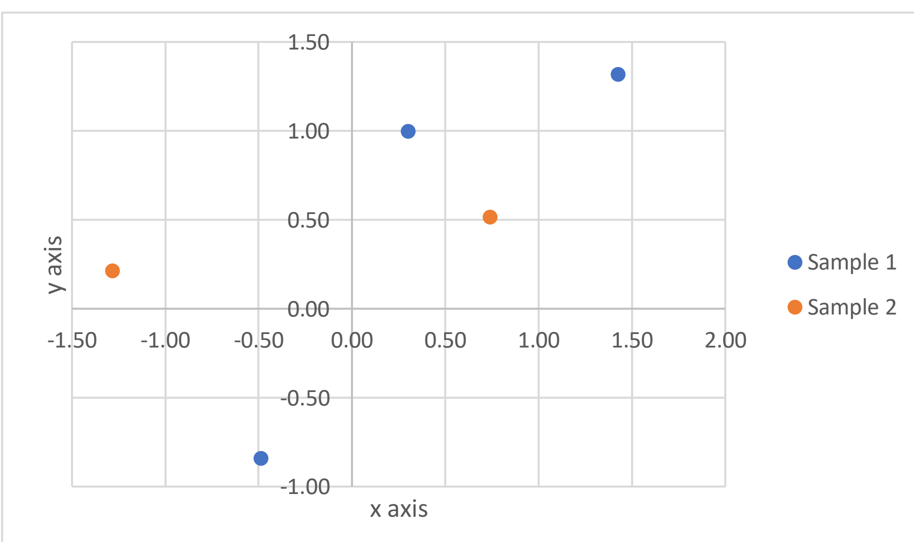
Q.E.D.

In the next section we provide a small sample example where we show how the additional directions in the Orientation method are unnecessary given the repetitions.

5.3. Example with Table

The following example shows (see Figure 5.2 for data sets) how $D_{n,m}^{(2)III++}$ is sufficient to find the maximum distance for orientation III, and how the other directions are simply

Figure 5.2: 2S 2D KS example data sets



repetitions. Similar computations would be done for the other orientations with same results. To highlight the repetitions (see Table 5.1), looking at point (1.74, 1.0) and the shaded cells as well as the shaded cells from (0.3, 1.0), (0.3, 0.51), and (0.74, 0.51) we see how (x_i^+, y_j^+) captures all the distances.

Table 5.1: 2S 2D KS Test Computations Orientation III

		$\left\|F_N^{(2)III}(x_i, y_j) - F_M^{(2)III}(x_i, y_j)\right\|$			
x	y	(x_i^+, y_j^+)	(x_i^+, y_j^-)	(x_i^-, y_j^+)	(x_i^-, y_j^-)
-0.49	-0.84	0.33	0.00	0.00	0.00
1.43	-0.84	0.33	0.00	0.33	0.00
0.30	-0.84	0.33	0.00	0.33	0.00
-1.28	-0.84	0.00	0.00	0.00	0.00
0.74	-0.84	0.33	0.00	0.33	0.00
-0.49	1.32	0.17	0.17	0.50	0.50
1.43	1.32	0.00	0.33	0.33	0.33
0.30	1.32	0.17	0.17	0.17	0.17
-1.28	1.32	0.50	0.50	0.00	0.00
0.74	1.32	0.33	0.33	0.17	0.17
-0.49	1.00	0.17	0.17	0.50	0.50
1.43	1.00	0.33	0.67	0.33	0.67
0.30	1.00	0.17	0.17	0.17	0.17
-1.28	1.00	0.50	0.50	0.00	0.00
0.74	1.00	0.33	0.67	0.17	0.17
-0.49	0.22	0.17	0.33	0.50	0.00
1.43	0.22	0.17	0.33	0.17	0.33
0.30	0.22	0.17	0.33	0.17	0.33
-1.28	0.22	0.50	0.00	0.00	0.00
0.74	0.22	0.17	0.33	0.17	0.33
-0.49	0.51	0.17	0.17	0.50	0.50
1.43	0.51	0.67	0.17	0.67	0.17
0.30	0.51	0.17	0.17	0.17	0.17
-1.28	0.51	0.50	0.50	0.00	0.00
0.74	0.51	0.67	0.17	0.17	0.17

5.4. Summary

The proof and example in this section show how the 2S 2D KS test Orientation Grid method is equal to the 2S 2D KS test Partial Orientation Grid method, and therefore, it is unnecessary to perform $16n^2$ operations and it is sufficient to only perform the $4n^2$ operations from the Partial Orientation Grid method. The implication of this equivalence between methods is that the critical values computed for the Orientation Grid method are only valid for the 1S case and the mathematical derivation cannot be done given that we would not have *iid* for the random variables from each orientation. Further, the critical values extended for the Partial Orientation Grid are valid and extend current implementations of the 2S 2D KS test .

6. Conclusions and Future Work

In this section we will summarize the results of our work, as well as discuss recommendations on when each KS method and implementation is appropriate to use. Furthermore, we will discuss the direction of this research and future derivations and implementation of this work.

6.1. Contributions of Research

We have shown how the 1S 2D KS test Orientation Grid method generally is more powerful than any of the other three methods examined, especially for sample sizes less than 30. Furthermore, this is the only method of the four that maintains the following properties: 1) is sufficiently distribution free even for highly correlated data, 2) the maximum distance for each of the orientations distributions are independent and identically distributed, and 3) captures the true maximum distance between two distributions. Furthermore, the Orientation Grid method is the only method that is robust against high correlation, and does not require additional computations or critical values based on correlation. Last but not least, power tables for the mean, standard deviation and correlation are available for all four 1S 2D KS test methods for various samples. The only potential drawback of the Orientation Grid method is potentially the computational time, which grows exponentially as sample size and dimension increase.

If the original extension proposed by Peacock (Partial Orientation Grid) must be used (with equalities), we have extended the table of critical values up to 5,000 (all with 10,000 repetitions for the simulation). Furthermore, we have presented a new correction to the large sample which will allow the use of the large sample critical values for any sample size less than 5,000.

Further, we demonstrated that the 2S 2D KS Orientation Grid method is equivalent to the 2S 2D KS partial Orientation Grid method and therefore, there is no need to complete all $16n^2$ computations (where $n = n_1 + n_2$), that is, the Orientation Grid method does not capture additional information lost when conducting the 2S KS test.

Finally, we have computed power tables for sample size calculations to be used in designing experiments testing for differences in distributions when considering differences in the means, variances, or correlations of the variables. These are the first extensive tables for all three critical value methods for 1S 2D KS tests.

6.2. Significance of Research

This extension to multiple dimensions for the KS test will allow us to perform test of hypothesis to see if there are differences in distribution for multivariate datasets without having to do dimensionality reduction, projections or some sort of univariate hypothesis test with a weighted average. Further, the addition of the Orientation Grid method provides a more powerful test for 1S 2D and 3D applications with power curves that can be used to design studies and plan sample sizes in addition to being invariant to correlation in the features being examined.

6.3. Recommendations for Future Research

One of the main difficulties of these four methods, especially as dimensions get larger, is the infeasibility of computing the critical values. Therefore, there is a need to explore alternate methods such as advanced numerical methods to solve for the derived critical values and the accuracy of bootstrapping the KS test in high dimensions and comparing against hypothesis test with known critical values. Furthermore, the optimal bootstrapping sample size and sample size recommendations would need to be researched, such extensions include appropriately estimating and accounting for the correlation among features within the dataset. Along the same lines, the need to fix the multivariate convolution for the mathematical derivation of the critical values is a great next step that would allow us to determine the asymptotic equation for the multivariate KS test as well as the correction needed to large sample that would apply to any sample size. Once the multivariate convolution is solved, any dimension of the KS test would have an asymptotic equation. Of course, there would be a need to test the power of the hypothesis test using this asymptotic equation and determining at what sample size do we converge to the equation. Lastly, completing power and sample size tables for smaller sample sizes in the 1S 3D KS test would be useful for advanced experimental design.

7. Appendix A: 1S 2D KS Test Methods Critical Values

Table 7.1: 1S 2D KS Test Critical Values for Orientation Grid from Derived Solution

n\ α	0.01	0.05	0.1	0.2		n\ α	0.01	0.05	0.1	0.2
3	1.82	1.65	1.55	1.44		18	1.99	1.81	1.71	1.6
4	1.87	1.69	1.6	1.49		19	1.99	1.81	1.71	1.6
5	1.9	1.72	1.63	1.52		20	1.99	1.81	1.72	1.61
6	1.91	1.74	1.64	1.53		21	1.99	1.82	1.72	1.61
7	1.92	1.75	1.65	1.54		22	1.99	1.82	1.72	1.61
8	1.94	1.76	1.66	1.55		23	1.99	1.82	1.72	1.61
9	1.95	1.77	1.67	1.56		24	2.00	1.82	1.72	1.61
10	1.95	1.77	1.67	1.57		25	2.00	1.82	1.72	1.61
11	1.96	1.78	1.68	1.57		26	2.00	1.82	1.73	1.62
12	1.96	1.79	1.69	1.58		27	2.00	1.82	1.73	1.62
13	1.97	1.79	1.69	1.58		28	2.00	1.83	1.73	1.62
14	1.97	1.79	1.7	1.59		29	2.00	1.83	1.73	1.62
15	1.98	1.8	1.7	1.59		40	2.02	1.84	1.74	1.63
16	1.98	1.8	1.71	1.6		50	2.02	1.84	1.74	1.64
17	1.98	1.8	1.71	1.6		100	2.03	1.85	1.76	1.65

Table 7.2: 1S 2D KS Test Critical Values for Orientation Grid from Simulation

n\ α	0.01	0.05	0.1	0.2		n\ α	0.01	0.05	0.1	0.2
3	1.61	1.52	1.46	1.37		30	2.06	1.80	1.69	1.56
4	1.74	1.60	1.51	1.39		31	2.03	1.81	1.69	1.57
5	1.81	1.64	1.55	1.43		32	2.05	1.82	1.70	1.57
6	1.84	1.66	1.56	1.44		33	2.07	1.83	1.70	1.57
7	1.89	1.69	1.58	1.46		34	2.06	1.83	1.70	1.57
8	1.92	1.70	1.60	1.47		35	2.06	1.82	1.71	1.57
9	1.92	1.72	1.61	1.49		36	2.07	1.82	1.70	1.57
10	1.95	1.73	1.62	1.49		37	2.07	1.83	1.71	1.57
11	1.96	1.73	1.63	1.49		38	2.06	1.83	1.71	1.58
12	1.97	1.74	1.64	1.51		39	2.07	1.83	1.71	1.58
13	1.98	1.75	1.64	1.51		40	2.05	1.83	1.71	1.58
14	1.96	1.75	1.64	1.51		41	2.08	1.84	1.72	1.58
15	2.00	1.76	1.64	1.52		42	2.09	1.85	1.72	1.58
16	2.01	1.77	1.65	1.52		43	2.08	1.85	1.72	1.59
17	2.00	1.76	1.65	1.53		44	2.08	1.85	1.73	1.59
18	2.03	1.78	1.67	1.53		45	2.07	1.84	1.72	1.59
19	2.02	1.79	1.67	1.54		46	2.09	1.85	1.72	1.59
20	2.03	1.79	1.67	1.54		47	2.09	1.86	1.73	1.59
21	2.02	1.78	1.66	1.54		48	2.09	1.86	1.74	1.59
22	2.04	1.79	1.67	1.54		49	2.06	1.85	1.73	1.60
23	2.03	1.80	1.68	1.55		50	2.09	1.86	1.73	1.60
24	2.02	1.80	1.69	1.55		100	2.17	1.92	1.81	1.67
25	2.05	1.80	1.68	1.55		200	2.26	2.03	1.90	1.76
26	2.03	1.80	1.69	1.56		300	2.37	2.09	1.97	1.83
27	2.05	1.81	1.69	1.56		500	2.47	2.22	2.10	1.95
28	2.05	1.81	1.69	1.56		1000	2.70	2.45	2.32	2.17
29	2.05	1.80	1.69	1.56		2000	3.04	2.77	2.64	2.50
						5000	3.70	3.44	3.30	3.14

Table 7.3: 1S 2D KS Test Critical Values for Orientation Sample from Simulation

n\ α	0.01	0.05	0.1	0.2		n\ α	0.01	0.05	0.1	0.2
3	1.56	1.42	1.32	1.18		30	1.80	1.55	1.45	1.31
4	1.62	1.42	1.32	1.20		31	1.79	1.56	1.45	1.32
5	1.64	1.44	1.33	1.21		32	1.82	1.57	1.46	1.33
6	1.66	1.45	1.35	1.22		33	1.83	1.58	1.46	1.33
7	1.68	1.47	1.36	1.23		34	1.83	1.58	1.45	1.33
8	1.70	1.48	1.37	1.24		35	1.82	1.58	1.47	1.33
9	1.71	1.48	1.37	1.25		36	1.82	1.57	1.46	1.33
10	1.73	1.49	1.38	1.24		37	1.85	1.58	1.46	1.33
11	1.72	1.51	1.38	1.25		38	1.82	1.58	1.47	1.33
12	1.74	1.50	1.39	1.25		39	1.82	1.59	1.48	1.34
13	1.75	1.51	1.39	1.26		40	1.82	1.59	1.46	1.33
14	1.76	1.52	1.40	1.26		41	1.82	1.60	1.48	1.34
15	1.76	1.53	1.41	1.28		42	1.84	1.59	1.48	1.35
16	1.78	1.53	1.41	1.27		43	1.83	1.60	1.48	1.34
17	1.76	1.52	1.40	1.28		44	1.82	1.60	1.50	1.35
18	1.78	1.54	1.42	1.29		45	1.86	1.60	1.47	1.34
19	1.75	1.54	1.42	1.28		46	1.85	1.60	1.48	1.35
20	1.78	1.54	1.42	1.29		47	1.85	1.61	1.49	1.35
21	1.76	1.54	1.42	1.29		48	1.85	1.61	1.49	1.35
22	1.80	1.54	1.42	1.29		49	1.82	1.59	1.48	1.35
23	1.79	1.55	1.43	1.29		50	1.84	1.61	1.49	1.35
24	1.79	1.55	1.44	1.30		100	1.91	1.67	1.56	1.42
25	1.79	1.55	1.43	1.31		200	2.01	1.77	1.65	1.51
26	1.79	1.56	1.45	1.31		300	2.11	1.84	1.71	1.57
27	1.79	1.56	1.45	1.31		500	2.20	1.97	1.84	1.69
28	1.82	1.56	1.45	1.31		1000	2.43	2.18	2.04	1.90
29	1.80	1.56	1.46	1.32		2000	2.77	2.49	2.36	2.22
						5000	3.40	3.13	3.00	2.85

Table 7.4: 1S 2D KS Test Critical Values for Partial Orientation Grid from Simulation

n\ α	0.01	0.05	0.1	0.2		n\ α	0.01	0.05	0.1	0.2
3	1.56	1.40	1.30	1.16		30	1.96	1.69	1.58	1.45
4	1.65	1.44	1.33	1.21		31	1.93	1.71	1.59	1.45
5	1.68	1.50	1.39	1.25		32	1.94	1.72	1.60	1.46
6	1.71	1.51	1.40	1.27		33	1.99	1.73	1.60	1.47
7	1.76	1.53	1.42	1.29		34	1.98	1.73	1.60	1.46
8	1.77	1.56	1.45	1.31		35	1.96	1.73	1.61	1.47
9	1.79	1.57	1.46	1.32		36	1.98	1.72	1.60	1.47
10	1.82	1.58	1.47	1.33		37	1.97	1.74	1.61	1.47
11	1.82	1.60	1.47	1.35		38	1.98	1.73	1.61	1.47
12	1.84	1.60	1.49	1.35		39	1.97	1.73	1.62	1.48
13	1.85	1.62	1.50	1.36		40	1.97	1.74	1.61	1.48
14	1.83	1.62	1.51	1.36		41	1.98	1.76	1.62	1.48
15	1.86	1.63	1.52	1.38		42	1.99	1.75	1.63	1.49
16	1.88	1.64	1.52	1.38		43	1.98	1.76	1.63	1.49
17	1.88	1.65	1.53	1.39		44	2.00	1.75	1.64	1.50
18	1.90	1.66	1.54	1.40		45	1.99	1.75	1.64	1.49
19	1.90	1.66	1.54	1.41		46	2.02	1.76	1.63	1.49
20	1.93	1.67	1.55	1.41		47	2.01	1.77	1.64	1.50
21	1.90	1.66	1.55	1.41		48	1.98	1.76	1.64	1.50
22	1.92	1.68	1.56	1.42		49	1.99	1.76	1.64	1.50
23	1.92	1.68	1.56	1.42		50	2.02	1.77	1.65	1.51
24	1.91	1.68	1.57	1.43		100	2.11	1.87	1.74	1.60
25	1.93	1.69	1.57	1.43		200	2.20	1.98	1.86	1.71
26	1.94	1.69	1.58	1.44		300	2.32	2.05	1.93	1.79
27	1.96	1.70	1.58	1.45		500	2.43	2.19	2.07	1.92
28	1.95	1.70	1.58	1.45		1000	2.67	2.43	2.29	2.14
29	1.95	1.70	1.59	1.45		2000	3.02	2.76	2.63	2.48
						5000	3.68	3.42	3.29	3.13

Table 7.5: 1S 2D KS Test Critical Values for Partial Orientation Sample from Simulation

n\ α	0.01	0.05	0.1	0.2		n\ α	0.01	0.05	0.1	0.2
3	1.44	1.23	1.10	0.99		30	1.72	1.47	1.36	1.23
4	1.45	1.25	1.14	1.00		31	1.71	1.48	1.36	1.23
5	1.52	1.28	1.17	1.04		32	1.71	1.49	1.38	1.24
6	1.52	1.30	1.19	1.05		33	1.75	1.50	1.38	1.24
7	1.55	1.32	1.20	1.07		34	1.76	1.49	1.37	1.24
8	1.57	1.34	1.22	1.08		35	1.73	1.50	1.39	1.25
9	1.58	1.35	1.23	1.10		36	1.73	1.50	1.38	1.24
10	1.58	1.36	1.24	1.11		37	1.76	1.50	1.38	1.24
11	1.61	1.37	1.26	1.11		38	1.74	1.50	1.39	1.26
12	1.62	1.39	1.26	1.13		39	1.74	1.52	1.40	1.25
13	1.63	1.40	1.27	1.13		40	1.74	1.51	1.39	1.25
14	1.62	1.40	1.27	1.14		41	1.75	1.51	1.39	1.26
15	1.63	1.41	1.29	1.15		42	1.78	1.52	1.40	1.26
16	1.69	1.42	1.30	1.16		43	1.76	1.52	1.41	1.27
17	1.66	1.42	1.30	1.17		44	1.74	1.53	1.41	1.27
18	1.67	1.44	1.31	1.18		45	1.78	1.53	1.40	1.27
19	1.66	1.43	1.31	1.18		46	1.79	1.53	1.41	1.28
20	1.67	1.45	1.32	1.18		47	1.78	1.53	1.41	1.28
21	1.68	1.43	1.32	1.19		48	1.77	1.54	1.42	1.28
22	1.69	1.45	1.33	1.19		49	1.76	1.53	1.41	1.28
23	1.70	1.46	1.33	1.20		50	1.78	1.54	1.42	1.28
24	1.69	1.45	1.34	1.20		100	1.85	1.63	1.51	1.36
25	1.70	1.46	1.34	1.21		200	1.97	1.73	1.61	1.46
26	1.70	1.47	1.35	1.22		300	2.07	1.81	1.68	1.54
27	1.71	1.48	1.35	1.22		500	2.16	1.94	1.81	1.66
28	1.72	1.48	1.36	1.23		1000	2.40	2.16	2.02	1.87
29	1.71	1.47	1.37	1.23		2000	2.75	2.47	2.35	2.20
						5000	3.39	3.12	2.99	2.84

8. Appendix B: Power Tables

Table 8.1: 1S 2D KS test Orientation Grid method for 3 critical values power curves for mean shift for $\alpha = 0.01$

Mean	Sample 10			Sample 30			Sample 50			Sample 100		
	Orientation			Orientation			Orientation			Orientation		
	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample
0	0.01	0.01	0.01	0.01	0.01	0.02	0.01	0.02	0.02	0.01	0.03	0.02
0.1	0.01	0.01	0.01	0.02	0.03	0.03	0.03	0.05	0.05	0.06	0.11	0.10
0.2	0.02	0.02	0.02	0.06	0.08	0.08	0.13	0.16	0.17	0.31	0.46	0.43
0.3	0.05	0.05	0.04	0.17	0.21	0.22	0.36	0.43	0.44	0.75	0.86	0.84
0.4	0.09	0.09	0.08	0.37	0.43	0.44	0.68	0.74	0.75	0.97	0.99	0.99
0.5	0.15	0.15	0.13	0.62	0.67	0.68	0.90	0.93	0.93	1.00	1.00	1.00
0.6	0.24	0.24	0.22	0.83	0.86	0.87	0.98	0.99	0.99	1.00	1.00	1.00
0.7	0.37	0.37	0.34	0.95	0.96	0.96	1.00	1.00	1.00	1.00	1.00	1.00
0.8	0.51	0.51	0.47	0.99	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00
0.9	0.65	0.65	0.62	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	0.77	0.77	0.74	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.1	0.86	0.86	0.84	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.2	0.93	0.93	0.91	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.3	0.97	0.97	0.96	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.4	0.99	0.99	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.5	1.00	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.2: 1S 2D KS test Orientation Grid method for 3 critical values power curves for mean shift for $\alpha = 0.05$

Mean	Sample 10			Sample 30			Sample 50			Sample 100		
	Orientation			Orientation			Orientation			Orientation		
	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample
0	0.05	0.04	0.02	0.05	0.04	0.04	0.05	0.06	0.05	0.05	0.08	0.06
0.1	0.06	0.05	0.03	0.09	0.08	0.06	0.10	0.11	0.10	0.18	0.23	0.18
0.2	0.09	0.07	0.05	0.20	0.18	0.16	0.30	0.32	0.29	0.57	0.64	0.58
0.3	0.15	0.13	0.09	0.41	0.38	0.35	0.61	0.63	0.59	0.91	0.94	0.92
0.4	0.24	0.21	0.15	0.66	0.62	0.59	0.86	0.87	0.85	0.99	1.00	0.99
0.5	0.36	0.32	0.24	0.85	0.83	0.81	0.98	0.98	0.97	1.00	1.00	1.00
0.6	0.50	0.45	0.37	0.96	0.95	0.94	1.00	1.00	1.00	1.00	1.00	1.00
0.7	0.64	0.59	0.50	0.99	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00
0.8	0.77	0.73	0.65	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	0.86	0.83	0.77	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	0.93	0.91	0.86	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.1	0.97	0.96	0.93	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.2	0.99	0.98	0.97	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.3	1.00	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.3: 1S 2D KS test Orientation Grid method for 3 critical values power curves for mean shift for $\alpha = 0.1$

Mean	Sample 10			Sample 30			Sample 50			Sample 100		
	Orientation			Orientation			Orientation			Orientation		
	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample
0	0.10	0.07	0.04	0.10	0.08	0.06	0.10	0.10	0.07	0.10	0.13	0.09
0.1	0.12	0.09	0.05	0.15	0.13	0.10	0.19	0.18	0.14	0.27	0.31	0.24
0.2	0.17	0.13	0.08	0.31	0.27	0.22	0.44	0.42	0.36	0.68	0.74	0.66
0.3	0.25	0.20	0.13	0.53	0.49	0.44	0.74	0.73	0.67	0.96	0.97	0.95
0.4	0.36	0.30	0.21	0.76	0.72	0.68	0.93	0.92	0.90	1.00	1.00	1.00
0.5	0.50	0.43	0.32	0.91	0.89	0.86	0.99	0.99	0.98	1.00	1.00	1.00
0.6	0.64	0.57	0.46	0.98	0.97	0.96	1.00	1.00	1.00	1.00	1.00	1.00
0.7	0.77	0.71	0.60	1.00	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00
0.8	0.86	0.82	0.73	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	0.93	0.90	0.83	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	0.97	0.95	0.91	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.1	0.99	0.98	0.96	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.2	1.00	0.99	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.3	1.00	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.4: 1S 2D KS test Orientation Grid method for 3 critical values power curves for mean shift for $\alpha = 0.2$

Mean	Sample 10			Sample 30			Sample 50			Sample 100		
	Orientation			Orientation			Orientation			Orientation		
	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample
0	0.20	0.13	0.07	0.20	0.15	0.10	0.20	0.17	0.12	0.20	0.22	0.14
0.1	0.22	0.15	0.08	0.27	0.21	0.15	0.32	0.27	0.21	0.42	0.44	0.33
0.2	0.29	0.21	0.12	0.45	0.38	0.30	0.59	0.54	0.46	0.81	0.83	0.75
0.3	0.39	0.30	0.19	0.68	0.61	0.53	0.84	0.81	0.76	0.98	0.98	0.97
0.4	0.52	0.42	0.29	0.86	0.82	0.76	0.97	0.96	0.94	1.00	1.00	1.00
0.5	0.65	0.56	0.42	0.96	0.94	0.91	1.00	0.99	0.99	1.00	1.00	1.00
0.6	0.78	0.69	0.56	0.99	0.99	0.98	1.00	1.00	1.00	1.00	1.00	1.00
0.7	0.87	0.81	0.70	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.8	0.93	0.89	0.81	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	0.97	0.95	0.89	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	0.99	0.98	0.95	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.1	1.00	0.99	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.2	1.00	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.5: 1S 2D KS test Orientation Grid method for 3 critical values power curves for STDev shift for $\alpha = 0.01$

STDev	Sample 10			Sample 30			Sample 50			Sample 100		
	Orientation			Orientation			Orientation			Orientation		
	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample
1	0.01	0.01	0.01	0.01	0.01	0.02	0.01	0.02	0.02	0.01	0.03	0.02
1.1	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.03	0.03	0.02	0.06	0.05
1.2	0.02	0.02	0.02	0.04	0.05	0.05	0.06	0.08	0.08	0.11	0.22	0.19
1.3	0.03	0.03	0.03	0.08	0.10	0.11	0.14	0.19	0.19	0.34	0.51	0.47
1.4	0.05	0.05	0.05	0.15	0.18	0.19	0.29	0.36	0.37	0.66	0.81	0.78
1.5	0.08	0.08	0.07	0.25	0.29	0.30	0.48	0.56	0.56	0.89	0.95	0.94
1.6	0.11	0.11	0.09	0.37	0.43	0.44	0.67	0.74	0.75	0.97	0.99	0.99
1.7	0.14	0.14	0.12	0.50	0.56	0.57	0.82	0.87	0.87	1.00	1.00	1.00
1.8	0.18	0.18	0.16	0.62	0.68	0.69	0.91	0.94	0.94	1.00	1.00	1.00
1.9	0.23	0.23	0.20	0.73	0.78	0.79	0.96	0.98	0.98	1.00	1.00	1.00
2	0.27	0.27	0.25	0.82	0.86	0.87	0.99	0.99	0.99	1.00	1.00	1.00
2.1	0.32	0.32	0.29	0.88	0.91	0.92	1.00	1.00	1.00	1.00	1.00	1.00
2.2	0.37	0.37	0.34	0.93	0.95	0.95	1.00	1.00	1.00	1.00	1.00	1.00
2.3	0.41	0.41	0.38	0.96	0.97	0.97	1.00	1.00	1.00	1.00	1.00	1.00
2.4	0.45	0.45	0.42	0.98	0.98	0.98	1.00	1.00	1.00	1.00	1.00	1.00
2.5	0.49	0.49	0.46	0.99	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00
2.6	0.53	0.53	0.50	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.7	0.57	0.57	0.54	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.8	0.60	0.60	0.57	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.9	0.63	0.63	0.60	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3	0.66	0.66	0.64	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.6: 1S 2D KS test Orientation Grid method for 3 critical values power curves for STDev shift for $\alpha = 0.05$

STDev	Sample 10			Sample 30			Sample 50			Sample 100		
	Orientation			Orientation			Orientation			Orientation		
	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample
1	0.05	0.04	0.02	0.05	0.04	0.04	0.05	0.06	0.05	0.05	0.08	0.06
1.1	0.06	0.05	0.03	0.08	0.06	0.06	0.08	0.09	0.07	0.11	0.15	0.11
1.2	0.09	0.07	0.04	0.14	0.12	0.11	0.17	0.19	0.16	0.32	0.40	0.33
1.3	0.12	0.10	0.07	0.25	0.22	0.19	0.34	0.37	0.33	0.65	0.73	0.67
1.4	0.17	0.14	0.10	0.38	0.35	0.32	0.56	0.59	0.53	0.89	0.93	0.90
1.5	0.22	0.19	0.13	0.54	0.51	0.47	0.74	0.77	0.73	0.98	0.99	0.98
1.6	0.28	0.25	0.18	0.68	0.65	0.61	0.88	0.89	0.87	1.00	1.00	1.00
1.7	0.35	0.30	0.23	0.79	0.76	0.73	0.95	0.96	0.94	1.00	1.00	1.00
1.8	0.41	0.37	0.28	0.88	0.86	0.83	0.98	0.99	0.98	1.00	1.00	1.00
1.9	0.47	0.43	0.34	0.93	0.91	0.90	1.00	1.00	0.99	1.00	1.00	1.00
2	0.53	0.48	0.39	0.96	0.95	0.94	1.00	1.00	1.00	1.00	1.00	1.00
2.1	0.58	0.53	0.44	0.98	0.98	0.97	1.00	1.00	1.00	1.00	1.00	1.00
2.2	0.64	0.59	0.49	0.99	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00
2.3	0.68	0.64	0.54	1.00	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00
2.4	0.72	0.68	0.59	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.5	0.76	0.72	0.63	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.6	0.79	0.76	0.67	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.7	0.82	0.79	0.70	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.8	0.85	0.82	0.74	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.9	0.87	0.84	0.77	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3	0.89	0.87	0.79	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.7: 1S 2D KS test Orientation Grid method for 3 critical values power curves for STDev shift for $\alpha = 0.1$

STDev	Sample 10			Sample 30			Sample 50			Sample 100		
	Orientation			Orientation			Orientation			Orientation		
	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample
1	0.10	0.07	0.04	0.10	0.08	0.06	0.10	0.10	0.07	0.10	0.13	0.09
1.1	0.12	0.09	0.05	0.14	0.11	0.09	0.15	0.14	0.11	0.18	0.22	0.16
1.2	0.16	0.12	0.07	0.23	0.19	0.15	0.29	0.28	0.23	0.46	0.51	0.42
1.3	0.21	0.17	0.10	0.37	0.31	0.26	0.49	0.48	0.42	0.77	0.82	0.74
1.4	0.27	0.22	0.14	0.53	0.47	0.41	0.71	0.70	0.64	0.95	0.96	0.94
1.5	0.34	0.28	0.19	0.68	0.62	0.56	0.86	0.85	0.81	0.99	1.00	0.99
1.6	0.42	0.35	0.25	0.80	0.75	0.70	0.95	0.94	0.91	1.00	1.00	1.00
1.7	0.48	0.42	0.30	0.88	0.85	0.81	0.98	0.98	0.97	1.00	1.00	1.00
1.8	0.55	0.49	0.37	0.94	0.92	0.89	1.00	1.00	0.99	1.00	1.00	1.00
1.9	0.61	0.54	0.43	0.97	0.96	0.94	1.00	1.00	1.00	1.00	1.00	1.00
2	0.66	0.60	0.49	0.99	0.98	0.97	1.00	1.00	1.00	1.00	1.00	1.00
2.1	0.71	0.66	0.54	0.99	0.99	0.98	1.00	1.00	1.00	1.00	1.00	1.00
2.2	0.76	0.71	0.59	1.00	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00
2.3	0.80	0.75	0.64	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.4	0.83	0.78	0.68	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.5	0.86	0.82	0.72	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.6	0.88	0.85	0.76	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.7	0.90	0.87	0.79	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.8	0.92	0.89	0.82	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.9	0.93	0.91	0.84	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3	0.94	0.92	0.87	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.8: 1S 2D KS test Orientation Grid method for 3 critical values power curves for STDev shift for $\alpha = 0.2$

STDev	Sample 10			Sample 30			Sample 50			Sample 100		
	Orientation			Orientation			Orientation			Orientation		
	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample
1	0.20	0.13	0.07	0.20	0.15	0.10	0.20	0.17	0.12	0.20	0.22	0.14
1.1	0.23	0.16	0.09	0.25	0.19	0.14	0.27	0.23	0.17	0.32	0.34	0.24
1.2	0.29	0.20	0.12	0.36	0.30	0.22	0.44	0.40	0.32	0.63	0.66	0.54
1.3	0.35	0.26	0.16	0.53	0.45	0.36	0.66	0.62	0.53	0.89	0.90	0.83
1.4	0.43	0.32	0.22	0.68	0.61	0.52	0.84	0.81	0.74	0.98	0.99	0.97
1.5	0.50	0.40	0.27	0.81	0.75	0.67	0.94	0.92	0.88	1.00	1.00	1.00
1.6	0.58	0.47	0.34	0.90	0.85	0.79	0.98	0.97	0.95	1.00	1.00	1.00
1.7	0.64	0.54	0.41	0.95	0.92	0.88	1.00	0.99	0.99	1.00	1.00	1.00
1.8	0.70	0.61	0.48	0.98	0.96	0.94	1.00	1.00	1.00	1.00	1.00	1.00
1.9	0.76	0.66	0.53	0.99	0.98	0.97	1.00	1.00	1.00	1.00	1.00	1.00
2	0.81	0.72	0.59	1.00	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00
2.1	0.84	0.77	0.65	1.00	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00
2.2	0.88	0.80	0.70	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.3	0.90	0.84	0.74	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.4	0.92	0.87	0.78	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.5	0.94	0.89	0.81	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.6	0.95	0.91	0.84	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.7	0.96	0.93	0.86	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.8	0.97	0.94	0.89	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.9	0.98	0.95	0.90	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3	0.98	0.96	0.92	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.9: 1S 2D KS test Orientation Grid method for 3 critical values power curves for correlation shift for $\alpha = 0.01$

Rho	Sample 10			Sample 30			Sample 50			Sample 100		
	Orientation			Orientation			Orientation			Orientation		
	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample
-1.00	0.08	0.08	0.05	0.51	0.55	0.57	0.97	1.00	1.00	1.00	1.00	1.00
-0.90	0.05	0.05	0.04	0.21	0.26	0.27	0.51	0.65	0.65	1.00	1.00	1.00
-0.80	0.03	0.03	0.03	0.12	0.16	0.17	0.30	0.41	0.41	0.83	0.95	0.94
-0.70	0.03	0.03	0.02	0.08	0.10	0.11	0.18	0.25	0.26	0.55	0.74	0.71
-0.60	0.02	0.02	0.02	0.05	0.07	0.07	0.11	0.16	0.16	0.32	0.50	0.46
-0.50	0.02	0.02	0.01	0.03	0.04	0.05	0.06	0.09	0.10	0.17	0.30	0.28
-0.40	0.02	0.02	0.01	0.02	0.03	0.03	0.04	0.06	0.06	0.09	0.17	0.15
-0.30	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.04	0.04	0.04	0.09	0.08
-0.20	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.03	0.02	0.05	0.04
-0.10	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.01	0.03	0.03
0.00	0.01	0.01	0.01	0.01	0.01	0.02	0.01	0.02	0.02	0.01	0.03	0.02
0.10	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.01	0.03	0.03
0.20	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.03	0.02	0.05	0.04
0.30	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.04	0.04	0.04	0.09	0.08
0.40	0.02	0.02	0.01	0.02	0.03	0.03	0.04	0.06	0.06	0.09	0.17	0.15
0.50	0.02	0.02	0.01	0.03	0.04	0.05	0.06	0.10	0.10	0.17	0.30	0.28
0.60	0.02	0.02	0.02	0.05	0.07	0.07	0.11	0.16	0.16	0.32	0.50	0.47
0.70	0.03	0.03	0.02	0.08	0.10	0.11	0.18	0.26	0.26	0.55	0.75	0.71
0.80	0.03	0.03	0.03	0.12	0.16	0.17	0.30	0.41	0.41	0.83	0.95	0.94
0.90	0.05	0.05	0.04	0.21	0.26	0.27	0.51	0.65	0.65	1.00	1.00	1.00
1.00	0.08	0.08	0.06	0.50	0.56	0.56	0.97	1.00	1.00	1.00	1.00	1.00

Table 8.10: 1S 2D KS test Orientation Grid method for 3 critical values power curves for correlation shift for $\alpha = 0.05$

Rho	Sample 10			Sample 30			Sample 50			Sample 100		
	Orientation			Orientation			Orientation			Orientation		
	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample
-1.00	0.23	0.18	0.14	0.84	0.81	0.81	1.00	1.00	1.00	1.00	1.00	1.00
-0.90	0.15	0.12	0.09	0.54	0.50	0.46	0.88	0.90	0.86	1.00	1.00	1.00
-0.80	0.12	0.10	0.07	0.38	0.34	0.31	0.64	0.67	0.62	0.99	1.00	0.99
-0.70	0.10	0.08	0.05	0.27	0.24	0.22	0.45	0.48	0.43	0.88	0.93	0.89
-0.60	0.08	0.07	0.04	0.20	0.17	0.15	0.31	0.33	0.29	0.67	0.76	0.68
-0.50	0.07	0.06	0.04	0.14	0.12	0.11	0.21	0.23	0.20	0.45	0.54	0.46
-0.40	0.06	0.05	0.03	0.11	0.09	0.08	0.14	0.16	0.13	0.28	0.35	0.29
-0.30	0.06	0.04	0.03	0.08	0.07	0.06	0.09	0.10	0.09	0.17	0.22	0.17
-0.20	0.05	0.04	0.03	0.07	0.06	0.05	0.06	0.07	0.06	0.10	0.14	0.10
-0.10	0.05	0.04	0.02	0.06	0.05	0.04	0.05	0.06	0.05	0.06	0.09	0.07
0.00	0.05	0.04	0.02	0.05	0.04	0.03	0.05	0.06	0.04	0.05	0.08	0.06
0.10	0.05	0.04	0.02	0.06	0.05	0.04	0.05	0.06	0.05	0.06	0.09	0.07
0.20	0.05	0.04	0.02	0.06	0.05	0.05	0.07	0.07	0.06	0.10	0.14	0.10
0.30	0.06	0.04	0.03	0.08	0.07	0.06	0.09	0.10	0.09	0.17	0.23	0.17
0.40	0.06	0.05	0.03	0.11	0.09	0.08	0.14	0.16	0.13	0.28	0.36	0.29
0.50	0.07	0.06	0.04	0.14	0.12	0.11	0.21	0.23	0.20	0.45	0.55	0.46
0.60	0.08	0.07	0.04	0.20	0.17	0.15	0.31	0.33	0.29	0.66	0.76	0.68
0.70	0.10	0.08	0.05	0.27	0.24	0.22	0.45	0.47	0.43	0.88	0.93	0.89
0.80	0.12	0.10	0.07	0.37	0.34	0.31	0.63	0.66	0.62	0.99	1.00	0.99
0.90	0.15	0.12	0.09	0.54	0.50	0.46	0.88	0.89	0.87	1.00	1.00	1.00
1.00	0.23	0.18	0.13	0.84	0.82	0.80	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.11: 1S 2D KS test Orientation Grid method for 3 critical values power curves for correlation shift for $\alpha = 0.1$

Rho	Sample 10			Sample 30			Sample 50			Sample 100		
	Orientation			Orientation			Orientation			Orientation		
	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample
-1.00	0.33	0.24	0.18	0.98	0.97	0.85	1.00	1.00	1.00	1.00	1.00	1.00
-0.90	0.25	0.20	0.13	0.73	0.66	0.57	0.98	0.98	0.93	1.00	1.00	1.00
-0.80	0.21	0.16	0.10	0.55	0.48	0.40	0.83	0.82	0.73	1.00	1.00	1.00
-0.70	0.18	0.14	0.08	0.41	0.36	0.29	0.64	0.63	0.54	0.96	0.98	0.94
-0.60	0.16	0.12	0.07	0.32	0.27	0.21	0.48	0.47	0.38	0.81	0.86	0.78
-0.50	0.14	0.10	0.06	0.24	0.20	0.16	0.35	0.34	0.27	0.61	0.68	0.57
-0.40	0.12	0.09	0.05	0.18	0.15	0.12	0.26	0.25	0.19	0.41	0.48	0.38
-0.30	0.11	0.08	0.04	0.15	0.12	0.09	0.19	0.18	0.13	0.27	0.32	0.24
-0.20	0.10	0.08	0.04	0.12	0.10	0.07	0.13	0.13	0.09	0.17	0.21	0.15
-0.10	0.10	0.07	0.04	0.11	0.08	0.06	0.11	0.10	0.08	0.11	0.14	0.10
0.00	0.10	0.07	0.04	0.10	0.08	0.06	0.10	0.10	0.07	0.10	0.13	0.09
0.10	0.10	0.07	0.04	0.11	0.08	0.06	0.11	0.10	0.07	0.11	0.14	0.10
0.20	0.10	0.08	0.04	0.12	0.09	0.07	0.13	0.13	0.09	0.17	0.21	0.15
0.30	0.11	0.08	0.05	0.15	0.12	0.09	0.19	0.18	0.13	0.27	0.32	0.24
0.40	0.13	0.09	0.05	0.19	0.15	0.12	0.26	0.25	0.19	0.41	0.48	0.38
0.50	0.14	0.10	0.06	0.24	0.20	0.15	0.35	0.34	0.27	0.61	0.68	0.58
0.60	0.16	0.12	0.07	0.32	0.27	0.21	0.49	0.47	0.39	0.81	0.87	0.78
0.70	0.18	0.14	0.08	0.41	0.36	0.29	0.64	0.63	0.54	0.96	0.98	0.94
0.80	0.21	0.16	0.10	0.55	0.48	0.40	0.83	0.82	0.73	1.00	1.00	1.00
0.90	0.26	0.20	0.13	0.74	0.66	0.57	0.98	0.98	0.93	1.00	1.00	1.00
1.00	0.34	0.24	0.18	0.98	0.96	0.87	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.12: 1S 2D KS test Orientation Grid method for 3 critical values power curves for correlation shift for $\alpha = 0.2$

Rho	Sample 10			Sample 30			Sample 50			Sample 100		
	Orientation			Orientation			Orientation			Orientation		
	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample	Simulated	Derived	Large Sample
-1.00	0.51	0.38	0.24	1.00	0.99	0.98	1.00	1.00	1.00	1.00	1.00	1.00
-0.90	0.43	0.31	0.19	0.91	0.84	0.72	1.00	1.00	0.99	1.00	1.00	1.00
-0.80	0.37	0.26	0.16	0.76	0.67	0.54	0.95	0.92	0.85	1.00	1.00	1.00
-0.70	0.32	0.23	0.13	0.62	0.53	0.40	0.83	0.78	0.67	0.99	1.00	0.98
-0.60	0.29	0.20	0.11	0.50	0.41	0.31	0.68	0.62	0.52	0.94	0.95	0.88
-0.50	0.26	0.18	0.10	0.40	0.33	0.24	0.54	0.48	0.38	0.80	0.82	0.70
-0.40	0.24	0.16	0.09	0.33	0.26	0.18	0.42	0.36	0.28	0.62	0.64	0.50
-0.30	0.22	0.15	0.08	0.27	0.21	0.14	0.32	0.28	0.21	0.44	0.47	0.34
-0.20	0.21	0.14	0.07	0.23	0.17	0.11	0.26	0.22	0.15	0.31	0.33	0.22
-0.10	0.20	0.13	0.07	0.20	0.15	0.10	0.22	0.18	0.12	0.22	0.24	0.16
0.00	0.20	0.13	0.07	0.20	0.15	0.10	0.20	0.16	0.12	0.20	0.22	0.14
0.10	0.20	0.13	0.07	0.20	0.15	0.10	0.21	0.18	0.12	0.22	0.24	0.16
0.20	0.21	0.14	0.07	0.23	0.17	0.11	0.26	0.21	0.15	0.31	0.33	0.23
0.30	0.22	0.14	0.08	0.27	0.20	0.14	0.32	0.27	0.21	0.44	0.47	0.34
0.40	0.24	0.16	0.09	0.33	0.25	0.18	0.42	0.36	0.28	0.61	0.65	0.50
0.50	0.26	0.18	0.10	0.40	0.32	0.24	0.54	0.48	0.38	0.80	0.83	0.70
0.60	0.29	0.20	0.11	0.50	0.41	0.31	0.68	0.62	0.52	0.94	0.95	0.88
0.70	0.32	0.23	0.13	0.62	0.52	0.41	0.83	0.77	0.67	0.99	1.00	0.98
0.80	0.36	0.26	0.16	0.76	0.66	0.54	0.95	0.92	0.85	1.00	1.00	1.00
0.90	0.42	0.31	0.19	0.91	0.84	0.72	1.00	1.00	0.99	1.00	1.00	1.00
1.00	0.49	0.38	0.24	1.00	0.99	0.97	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.13: 1S 2D KS test Orientation Sample method for 2 critical values power curves for Mean shift for $\alpha = 0.05$

Mean	Sample 10		Sample 30		Sample 50		Sample 100	
	Orientation		Orientation		Orientation		Orientation	
	Simulated	Large Sample	Simulated	Large Sample	Simulated	Large Sample	Simulated	Large Sample
0.0	0.05	0.03	0.05	0.04	0.05	0.05	0.05	0.06
0.1	0.06	0.03	0.09	0.07	0.11	0.11	0.17	0.20
0.2	0.09	0.05	0.20	0.17	0.30	0.30	0.56	0.59
0.3	0.15	0.09	0.41	0.36	0.60	0.60	0.91	0.92
0.4	0.22	0.15	0.64	0.59	0.86	0.85	0.99	1.00
0.5	0.33	0.24	0.84	0.81	0.97	0.97	1.00	1.00
0.6	0.45	0.35	0.95	0.93	1.00	1.00	1.00	1.00
0.7	0.58	0.48	0.99	0.99	1.00	1.00	1.00	1.00
0.8	0.72	0.61	1.00	1.00	1.00	1.00	1.00	1.00
0.9	0.83	0.74	1.00	1.00	1.00	1.00	1.00	1.00
1.0	0.90	0.84	1.00	1.00	1.00	1.00	1.00	1.00
1.1	0.95	0.91	1.00	1.00	1.00	1.00	1.00	1.00
1.2	0.98	0.95	1.00	1.00	1.00	1.00	1.00	1.00
1.3	0.99	0.98	1.00	1.00	1.00	1.00	1.00	1.00
1.4	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00
1.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.14: 1S 2D KS test Orientation Sample method for 2 critical values power curves for Standard Deviation shift for $\alpha = 0.05$

STDev	Sample 10		Sample 30		Sample 50		Sample 100	
	Orientation		Orientation		Orientation		Orientation	
	Simulated	Large Sample	Simulated	Large Sample	Simulated	Large Sample	Simulated	Large Sample
1.0	0.05	0.03	0.05	0.04	0.05	0.05	0.05	0.06
1.1	0.06	0.03	0.07	0.05	0.07	0.07	0.09	0.10
1.2	0.07	0.04	0.11	0.09	0.14	0.14	0.26	0.29
1.3	0.09	0.05	0.19	0.15	0.27	0.27	0.57	0.59
1.4	0.11	0.07	0.29	0.25	0.45	0.45	0.84	0.86
1.5	0.14	0.09	0.41	0.36	0.64	0.64	0.97	0.97
1.6	0.18	0.12	0.55	0.49	0.81	0.81	1.00	1.00
1.7	0.22	0.15	0.67	0.62	0.90	0.90	1.00	1.00
1.8	0.26	0.18	0.77	0.73	0.96	0.96	1.00	1.00
1.9	0.30	0.21	0.85	0.81	0.99	0.99	1.00	1.00
2.0	0.34	0.25	0.91	0.88	1.00	1.00	1.00	1.00
2.1	0.38	0.28	0.94	0.92	1.00	1.00	1.00	1.00
2.2	0.42	0.32	0.96	0.95	1.00	1.00	1.00	1.00
2.3	0.46	0.35	0.98	0.97	1.00	1.00	1.00	1.00
2.4	0.50	0.38	0.99	0.99	1.00	1.00	1.00	1.00
2.5	0.54	0.42	0.99	0.99	1.00	1.00	1.00	1.00
2.6	0.57	0.45	1.00	0.99	1.00	1.00	1.00	1.00
2.7	0.61	0.48	1.00	1.00	1.00	1.00	1.00	1.00
2.8	0.64	0.51	1.00	1.00	1.00	1.00	1.00	1.00
2.9	0.67	0.54	1.00	1.00	1.00	1.00	1.00	1.00
3.0	0.70	0.56	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.15: 1S 2D KS test Orientation Sample method for 2 critical values power curves for Correlation shift for $\alpha = 0.05$

Rho	Sample 10		Sample 30		Sample 50		Sample 100	
	Orientation		Orientation		Orientation		Orientation	
	Simulated	Large Sample	Simulated	Large Sample	Simulated	Large Sample	Simulated	Large Sample
-1.0	0.49	0.37	1.00	1.00	1.00	1.00	1.00	1.00
-0.9	0.23	0.15	0.68	0.61	0.96	0.96	1.00	1.00
-0.8	0.16	0.10	0.47	0.40	0.75	0.75	1.00	1.00
-0.7	0.12	0.07	0.32	0.27	0.53	0.53	0.92	0.93
-0.6	0.10	0.06	0.23	0.19	0.36	0.36	0.71	0.74
-0.5	0.08	0.05	0.16	0.13	0.25	0.25	0.49	0.51
-0.4	0.07	0.04	0.12	0.09	0.16	0.16	0.30	0.33
-0.3	0.06	0.04	0.09	0.07	0.11	0.11	0.18	0.19
-0.2	0.05	0.03	0.07	0.05	0.07	0.07	0.10	0.11
-0.1	0.05	0.03	0.06	0.04	0.05	0.05	0.07	0.07
0.0	0.05	0.03	0.05	0.04	0.05	0.05	0.05	0.06
0.1	0.05	0.03	0.06	0.04	0.05	0.05	0.07	0.07
0.2	0.05	0.03	0.07	0.05	0.07	0.07	0.10	0.11
0.3	0.06	0.04	0.09	0.07	0.11	0.11	0.18	0.20
0.4	0.07	0.04	0.12	0.09	0.16	0.16	0.30	0.33
0.5	0.08	0.05	0.16	0.13	0.25	0.25	0.49	0.52
0.6	0.10	0.06	0.23	0.19	0.36	0.36	0.71	0.74
0.7	0.12	0.07	0.32	0.27	0.53	0.52	0.92	0.93
0.8	0.16	0.10	0.47	0.40	0.74	0.74	1.00	1.00
0.9	0.22	0.15	0.69	0.61	0.96	0.96	1.00	1.00
1.0	0.47	0.37	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.16: 1S 2D KS test Partial Orientation Grid method for 2 critical values power curves for Mean shift for $\alpha = 0.05$

Mean	Sample 10		Sample 30		Sample 50		Sample 100	
	Orientation		Orientation		Orientation		Orientation	
	Simulated	Large Sample	Simulated	Large Sample	Simulated	Large Sample	Simulated	Large Sample
0.0	0.05	0.01	0.05	0.03	0.05	0.04	0.05	0.05
0.1	0.04	0.01	0.06	0.03	0.07	0.06	0.13	0.14
0.2	0.03	0.01	0.12	0.07	0.21	0.18	0.48	0.50
0.3	0.05	0.01	0.28	0.19	0.51	0.45	0.87	0.88
0.4	0.09	0.03	0.50	0.39	0.79	0.75	0.99	0.99
0.5	0.14	0.05	0.74	0.63	0.95	0.94	1.00	1.00
0.6	0.22	0.09	0.90	0.83	0.99	0.99	1.00	1.00
0.7	0.34	0.16	0.98	0.95	1.00	1.00	1.00	1.00
0.8	0.47	0.25	1.00	0.99	1.00	1.00	1.00	1.00
0.9	0.62	0.37	1.00	1.00	1.00	1.00	1.00	1.00
1.0	0.74	0.50	1.00	1.00	1.00	1.00	1.00	1.00
1.1	0.84	0.64	1.00	1.00	1.00	1.00	1.00	1.00
1.2	0.91	0.76	1.00	1.00	1.00	1.00	1.00	1.00
1.3	0.96	0.85	1.00	1.00	1.00	1.00	1.00	1.00
1.4	0.98	0.92	1.00	1.00	1.00	1.00	1.00	1.00
1.5	0.99	0.96	1.00	1.00	1.00	1.00	1.00	1.00
1.6	1.00	0.98	1.00	1.00	1.00	1.00	1.00	1.00
1.7	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00
1.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.17: 1S 2D KS test Partial Orientation Grid method for 2 critical values power curves for Standard Deviation shift for $\alpha = 0.05$

STDev	Sample 10		Sample 30		Sample 50		Sample 100	
	Orientation		Orientation		Orientation		Orientation	
	Simulated	Large Sample	Simulated	Large Sample	Simulated	Large Sample	Simulated	Large Sample
1.0	0.05	0.01	0.05	0.03	0.05	0.04	0.05	0.05
1.1	0.06	0.02	0.08	0.04	0.08	0.06	0.10	0.11
1.2	0.09	0.03	0.14	0.08	0.17	0.14	0.29	0.31
1.3	0.12	0.04	0.24	0.15	0.33	0.29	0.62	0.64
1.4	0.16	0.06	0.38	0.26	0.55	0.49	0.87	0.88
1.5	0.21	0.08	0.53	0.39	0.73	0.69	0.98	0.98
1.6	0.26	0.11	0.67	0.53	0.87	0.83	1.00	1.00
1.7	0.31	0.15	0.78	0.66	0.94	0.93	1.00	1.00
1.8	0.37	0.18	0.86	0.77	0.98	0.97	1.00	1.00
1.9	0.43	0.22	0.92	0.85	0.99	0.99	1.00	1.00
2.0	0.49	0.27	0.96	0.91	1.00	1.00	1.00	1.00
2.1	0.53	0.31	0.98	0.95	1.00	1.00	1.00	1.00
2.2	0.58	0.35	0.99	0.97	1.00	1.00	1.00	1.00
2.3	0.63	0.40	1.00	0.99	1.00	1.00	1.00	1.00
2.4	0.67	0.44	1.00	0.99	1.00	1.00	1.00	1.00
2.5	0.70	0.48	1.00	1.00	1.00	1.00	1.00	1.00
2.6	0.74	0.52	1.00	1.00	1.00	1.00	1.00	1.00
2.7	0.77	0.56	1.00	1.00	1.00	1.00	1.00	1.00
2.8	0.79	0.60	1.00	1.00	1.00	1.00	1.00	1.00
2.9	0.81	0.63	1.00	1.00	1.00	1.00	1.00	1.00
3.0	0.84	0.66	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.18: 1S 2D KS test Partial Orientation Grid method for 2 critical values power curves for Correlation shift for $\alpha = 0.05$

Rho	Sample 10		Sample 30		Sample 50		Sample 100	
	Orientation		Orientation		Orientation		Orientation	
	Simulated	Large Sample	Simulated	Large Sample	Simulated	Large Sample	Simulated	Large Sample
-1.0	0.17	0.06	0.83	0.69	1.00	1.00	1.00	1.00
-0.9	0.13	0.04	0.55	0.39	0.89	0.82	1.00	1.00
-0.8	0.10	0.03	0.38	0.25	0.65	0.57	0.99	0.99
-0.7	0.09	0.02	0.28	0.17	0.47	0.39	0.87	0.89
-0.6	0.07	0.02	0.20	0.12	0.32	0.26	0.65	0.67
-0.5	0.07	0.02	0.15	0.08	0.22	0.18	0.44	0.46
-0.4	0.06	0.02	0.11	0.06	0.15	0.11	0.26	0.28
-0.3	0.06	0.01	0.08	0.04	0.10	0.08	0.16	0.17
-0.2	0.05	0.01	0.07	0.03	0.07	0.05	0.09	0.10
-0.1	0.05	0.01	0.06	0.03	0.05	0.04	0.06	0.06
0.0	0.05	0.01	0.06	0.03	0.05	0.04	0.05	0.05
0.1	0.05	0.01	0.06	0.03	0.05	0.04	0.06	0.06
0.2	0.06	0.01	0.07	0.04	0.07	0.05	0.09	0.10
0.3	0.06	0.01	0.09	0.05	0.10	0.08	0.15	0.17
0.4	0.07	0.02	0.11	0.06	0.15	0.12	0.27	0.29
0.5	0.07	0.02	0.15	0.08	0.22	0.17	0.43	0.46
0.6	0.08	0.03	0.20	0.12	0.32	0.26	0.65	0.67
0.7	0.10	0.03	0.27	0.17	0.46	0.39	0.87	0.88
0.8	0.12	0.04	0.39	0.25	0.65	0.57	0.99	0.99
0.9	0.15	0.06	0.55	0.39	0.89	0.81	1.00	1.00
1.0	0.24	0.10	0.91	0.72	1.00	1.00	1.00	1.00

Table 8.19: 1S 2D KS test four methods power curves for mean shift for $\alpha = 0.01$ using simulated critical values

Mean	Sample 10				Sample 30				Sample 50				Sample 100			
	Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation	
	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
0	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
0.1	0.01	0.01	0.01	0.01	0.02	0.02	0.01	0.01	0.03	0.03	0.02	0.02	0.06	0.06	0.04	0.05
0.2	0.02	0.02	0.01	0.01	0.06	0.06	0.03	0.04	0.13	0.12	0.07	0.09	0.31	0.31	0.25	0.28
0.3	0.05	0.04	0.01	0.02	0.17	0.17	0.09	0.11	0.36	0.35	0.24	0.28	0.75	0.75	0.68	0.71
0.4	0.09	0.07	0.02	0.03	0.38	0.37	0.24	0.27	0.68	0.66	0.56	0.58	0.97	0.97	0.95	0.96
0.5	0.15	0.13	0.04	0.06	0.62	0.61	0.46	0.50	0.90	0.89	0.83	0.84	1.00	1.00	1.00	1.00
0.6	0.25	0.21	0.07	0.10	0.83	0.81	0.70	0.72	0.98	0.98	0.96	0.97	1.00	1.00	1.00	1.00
0.7	0.38	0.31	0.13	0.16	0.95	0.94	0.88	0.89	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.8	0.51	0.43	0.21	0.25	0.99	0.99	0.97	0.97	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	0.66	0.56	0.32	0.36	1.00	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	0.78	0.69	0.45	0.49	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.1	0.87	0.80	0.59	0.62	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.2	0.93	0.88	0.72	0.73	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.3	0.97	0.94	0.82	0.83	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.4	0.99	0.97	0.90	0.90	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.5	1.00	0.99	0.95	0.95	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.6	1.00	1.00	0.98	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.7	1.00	1.00	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.20: 1S 2D KS test four methods power curves for mean shift for α = 0.05 using simulated critical values

Mean	Sample 10				Sample 30				Sample 50				Sample 100			
	Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation	
	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
0	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
0.1	0.06	0.06	0.03	0.04	0.09	0.09	0.06	0.06	0.11	0.11	0.07	0.09	0.18	0.17	0.13	0.15
0.2	0.10	0.09	0.03	0.05	0.20	0.20	0.12	0.14	0.30	0.31	0.21	0.25	0.57	0.56	0.49	0.51
0.3	0.16	0.15	0.05	0.07	0.41	0.40	0.28	0.31	0.61	0.60	0.50	0.53	0.91	0.91	0.87	0.88
0.4	0.25	0.22	0.08	0.11	0.66	0.64	0.50	0.54	0.86	0.86	0.79	0.81	0.99	0.99	0.99	0.99
0.5	0.37	0.33	0.14	0.18	0.85	0.84	0.74	0.76	0.98	0.97	0.95	0.96	1.00	1.00	1.00	1.00
0.6	0.50	0.45	0.22	0.27	0.96	0.95	0.90	0.91	1.00	1.00	0.99	0.99	1.00	1.00	1.00	1.00
0.7	0.64	0.58	0.33	0.38	0.99	0.99	0.98	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.8	0.77	0.72	0.47	0.51	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	0.87	0.82	0.61	0.64	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	0.93	0.90	0.74	0.76	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.1	0.97	0.95	0.84	0.85	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.2	0.99	0.98	0.91	0.92	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.3	1.00	0.99	0.96	0.96	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.4	1.00	1.00	0.98	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.5	1.00	1.00	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.21: 1S 2D KS test four methods power curves for mean shift for $\alpha = 0.1$ using simulated critical values

Mean	Sample 10				Sample 30				Sample 50				Sample 100			
	Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation	
	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
0	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
0.1	0.12	0.11	0.07	0.09	0.16	0.15	0.11	0.12	0.18	0.19	0.13	0.15	0.27	0.27	0.22	0.24
0.2	0.17	0.16	0.08	0.10	0.31	0.30	0.20	0.23	0.43	0.43	0.32	0.37	0.68	0.68	0.62	0.64
0.3	0.25	0.23	0.10	0.13	0.54	0.52	0.39	0.43	0.73	0.73	0.63	0.66	0.96	0.95	0.93	0.94
0.4	0.36	0.33	0.16	0.20	0.76	0.75	0.63	0.66	0.93	0.92	0.87	0.89	1.00	1.00	1.00	1.00
0.5	0.50	0.46	0.23	0.29	0.91	0.90	0.83	0.85	0.99	0.99	0.98	0.98	1.00	1.00	1.00	1.00
0.6	0.64	0.58	0.35	0.40	0.98	0.97	0.95	0.95	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.7	0.77	0.71	0.48	0.52	1.00	1.00	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.8	0.86	0.82	0.62	0.65	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	0.93	0.90	0.75	0.77	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	0.97	0.95	0.84	0.86	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.1	0.99	0.98	0.92	0.92	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.2	1.00	0.99	0.96	0.96	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.3	1.00	1.00	0.98	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.4	1.00	1.00	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.22: 1S 2D KS test four methods power curves for mean shift for $\alpha = 0.2$ using simulated critical values

Mean	Sample 10				Sample 30				Sample 50				Sample 100			
	Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation	
	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
0	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
0.1	0.23	0.22	0.16	0.18	0.27	0.27	0.20	0.22	0.32	0.32	0.25	0.27	0.42	0.42	0.36	0.38
0.2	0.29	0.28	0.16	0.19	0.46	0.45	0.33	0.36	0.59	0.59	0.49	0.52	0.81	0.81	0.76	0.77
0.3	0.39	0.38	0.20	0.24	0.68	0.66	0.54	0.58	0.85	0.84	0.77	0.78	0.98	0.98	0.97	0.97
0.4	0.52	0.50	0.27	0.33	0.87	0.85	0.76	0.79	0.97	0.96	0.94	0.95	1.00	1.00	1.00	1.00
0.5	0.66	0.62	0.38	0.44	0.96	0.96	0.91	0.92	1.00	1.00	0.99	0.99	1.00	1.00	1.00	1.00
0.6	0.78	0.74	0.51	0.56	0.99	0.99	0.98	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.7	0.87	0.84	0.65	0.68	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.8	0.93	0.91	0.77	0.80	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	0.97	0.96	0.86	0.88	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	0.99	0.98	0.93	0.93	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.1	1.00	0.99	0.97	0.97	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.2	1.00	1.00	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.23: 1S 2D KS test four methods power curves for standard deviation shift for $\alpha = 0.01$ using simulated critical values

STDev	Sample 10				Sample 30				Sample 50				Sample 100			
	Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation	
	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
1.1	0.01	0.01	0.02	0.02	0.02	0.01	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.03
1.2	0.02	0.02	0.02	0.02	0.04	0.03	0.04	0.03	0.05	0.04	0.05	0.05	0.11	0.10	0.11	0.11
1.3	0.04	0.02	0.03	0.03	0.08	0.05	0.08	0.07	0.14	0.11	0.13	0.12	0.34	0.29	0.34	0.32
1.4	0.06	0.03	0.05	0.05	0.15	0.10	0.14	0.12	0.29	0.22	0.26	0.24	0.65	0.59	0.65	0.63
1.5	0.08	0.04	0.07	0.06	0.25	0.17	0.23	0.20	0.47	0.38	0.44	0.41	0.88	0.84	0.88	0.87
1.6	0.11	0.06	0.10	0.08	0.38	0.27	0.35	0.31	0.67	0.56	0.63	0.59	0.97	0.96	0.97	0.97
1.7	0.15	0.08	0.13	0.11	0.50	0.38	0.48	0.42	0.81	0.72	0.78	0.75	1.00	0.99	1.00	1.00
1.8	0.19	0.10	0.16	0.14	0.63	0.49	0.60	0.54	0.91	0.85	0.89	0.86	1.00	1.00	1.00	1.00
1.9	0.23	0.12	0.20	0.17	0.74	0.60	0.71	0.65	0.96	0.92	0.95	0.93	1.00	1.00	1.00	1.00
2	0.28	0.15	0.24	0.20	0.83	0.70	0.80	0.74	0.99	0.96	0.98	0.97	1.00	1.00	1.00	1.00
2.1	0.33	0.17	0.28	0.23	0.89	0.78	0.87	0.81	1.00	0.99	0.99	0.99	1.00	1.00	1.00	1.00
2.2	0.37	0.20	0.32	0.26	0.93	0.84	0.92	0.87	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00
2.3	0.41	0.23	0.36	0.29	0.96	0.89	0.95	0.91	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.4	0.46	0.26	0.40	0.32	0.98	0.93	0.97	0.94	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.5	0.50	0.29	0.45	0.35	0.99	0.95	0.98	0.96	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.6	0.54	0.32	0.48	0.39	0.99	0.97	0.99	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.7	0.57	0.34	0.52	0.42	1.00	0.98	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.8	0.61	0.37	0.56	0.44	1.00	0.99	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.9	0.64	0.40	0.59	0.47	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3	0.67	0.42	0.62	0.49	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.24: 1S 2D KS test four methods power curves for standard deviation shift for $\alpha = 0.05$ using simulated critical values

STDev	Sample 10				Sample 30				Sample 50				Sample 100			
	Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation	
	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
1	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
1.1	0.06	0.06	0.06	0.06	0.08	0.06	0.08	0.07	0.08	0.07	0.08	0.08	0.11	0.09	0.10	0.10
1.2	0.09	0.07	0.09	0.08	0.14	0.11	0.14	0.13	0.17	0.14	0.17	0.17	0.31	0.26	0.30	0.29
1.3	0.13	0.09	0.12	0.11	0.25	0.18	0.24	0.21	0.34	0.27	0.33	0.32	0.64	0.56	0.62	0.60
1.4	0.17	0.11	0.16	0.14	0.38	0.28	0.37	0.33	0.56	0.46	0.54	0.51	0.89	0.84	0.88	0.86
1.5	0.23	0.14	0.21	0.18	0.54	0.41	0.53	0.46	0.74	0.65	0.73	0.69	0.98	0.97	0.98	0.97
1.6	0.29	0.18	0.26	0.22	0.68	0.54	0.67	0.60	0.88	0.81	0.87	0.84	1.00	1.00	1.00	1.00
1.7	0.35	0.22	0.31	0.27	0.79	0.66	0.78	0.71	0.95	0.91	0.94	0.93	1.00	1.00	1.00	1.00
1.8	0.42	0.25	0.37	0.31	0.88	0.77	0.86	0.81	0.98	0.96	0.98	0.97	1.00	1.00	1.00	1.00
1.9	0.48	0.29	0.42	0.36	0.93	0.85	0.92	0.88	1.00	0.99	0.99	0.99	1.00	1.00	1.00	1.00
2	0.53	0.33	0.48	0.41	0.96	0.90	0.96	0.93	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.1	0.59	0.37	0.53	0.46	0.98	0.94	0.98	0.96	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.2	0.64	0.42	0.58	0.50	0.99	0.96	0.99	0.97	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.3	0.68	0.46	0.62	0.54	1.00	0.98	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.4	0.73	0.50	0.67	0.58	1.00	0.99	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.5	0.76	0.54	0.70	0.62	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.6	0.80	0.57	0.73	0.65	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.7	0.83	0.60	0.76	0.69	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.8	0.85	0.64	0.79	0.71	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.9	0.87	0.67	0.81	0.74	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3	0.89	0.70	0.83	0.77	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.25: 1S 2D KS test four methods power curves for standard deviation shift for $\alpha = 0.1$ using simulated critical values

STDev	Sample 10				Sample 30				Sample 50				Sample 100			
	Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation	
	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
1	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
1.1	0.13	0.11	0.12	0.12	0.14	0.12	0.14	0.14	0.15	0.13	0.15	0.15	0.18	0.16	0.18	0.18
1.2	0.16	0.13	0.16	0.15	0.23	0.18	0.22	0.21	0.28	0.24	0.27	0.27	0.46	0.39	0.45	0.43
1.3	0.22	0.16	0.20	0.19	0.37	0.28	0.35	0.32	0.49	0.41	0.47	0.45	0.77	0.70	0.76	0.74
1.4	0.28	0.19	0.25	0.23	0.53	0.41	0.51	0.46	0.70	0.61	0.69	0.66	0.95	0.92	0.94	0.94
1.5	0.35	0.23	0.31	0.28	0.68	0.54	0.66	0.60	0.86	0.79	0.84	0.82	0.99	0.99	0.99	0.99
1.6	0.42	0.27	0.38	0.33	0.80	0.68	0.78	0.73	0.94	0.90	0.93	0.92	1.00	1.00	1.00	1.00
1.7	0.49	0.32	0.44	0.38	0.88	0.78	0.87	0.82	0.98	0.96	0.98	0.97	1.00	1.00	1.00	1.00
1.8	0.55	0.36	0.51	0.44	0.94	0.86	0.93	0.89	1.00	0.99	0.99	0.99	1.00	1.00	1.00	1.00
1.9	0.61	0.41	0.57	0.49	0.97	0.91	0.96	0.94	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	0.67	0.46	0.62	0.54	0.99	0.95	0.98	0.97	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.1	0.72	0.51	0.67	0.58	0.99	0.97	0.99	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.2	0.76	0.55	0.71	0.63	1.00	0.99	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.3	0.80	0.59	0.75	0.67	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.4	0.83	0.63	0.79	0.71	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.5	0.86	0.67	0.82	0.74	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.6	0.88	0.70	0.85	0.77	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.7	0.90	0.73	0.87	0.80	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.8	0.92	0.76	0.89	0.82	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.9	0.93	0.78	0.91	0.84	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3	0.94	0.81	0.92	0.86	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.26: 1S 2D KS test four methods power curves for standard deviation shift for $\alpha = 0.2$ using simulated critical values

STDev	Sample 10				Sample 30				Sample 50				Sample 100			
	Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation	
	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
1	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
1.1	0.23	0.21	0.23	0.23	0.25	0.23	0.25	0.25	0.27	0.25	0.26	0.26	0.32	0.29	0.32	0.31
1.2	0.29	0.24	0.27	0.27	0.37	0.31	0.36	0.34	0.45	0.39	0.44	0.42	0.63	0.58	0.63	0.61
1.3	0.35	0.28	0.34	0.32	0.53	0.43	0.52	0.48	0.67	0.58	0.66	0.62	0.89	0.85	0.89	0.87
1.4	0.43	0.32	0.41	0.37	0.69	0.58	0.67	0.63	0.84	0.76	0.83	0.80	0.98	0.97	0.98	0.98
1.5	0.50	0.37	0.48	0.43	0.81	0.71	0.79	0.76	0.94	0.89	0.93	0.91	1.00	1.00	1.00	1.00
1.6	0.58	0.42	0.55	0.49	0.90	0.81	0.88	0.85	0.98	0.96	0.98	0.97	1.00	1.00	1.00	1.00
1.7	0.64	0.47	0.61	0.55	0.95	0.89	0.94	0.91	1.00	0.99	1.00	0.99	1.00	1.00	1.00	1.00
1.8	0.71	0.53	0.67	0.61	0.98	0.94	0.97	0.96	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.9	0.76	0.58	0.73	0.66	0.99	0.97	0.99	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	0.81	0.62	0.77	0.71	1.00	0.99	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.1	0.85	0.66	0.81	0.75	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.2	0.88	0.71	0.84	0.79	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.3	0.90	0.75	0.87	0.82	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.4	0.92	0.78	0.89	0.84	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.5	0.94	0.81	0.92	0.87	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.6	0.95	0.83	0.93	0.89	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.7	0.96	0.86	0.94	0.91	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.8	0.97	0.88	0.95	0.92	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.9	0.98	0.90	0.96	0.93	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3	0.98	0.91	0.97	0.94	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.27: 1S 2D KS test four methods power curves for correlation shift for $\alpha = 0.01$ using simulated critical values

Mean	Sample 10				Sample 30				Sample 50				Sample 100			
	Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation	
	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
-1	0.08	0.22	0.05	0.10	0.52	0.83	0.44	0.75	0.96	1.00	0.96	1.00	1.00	1.00	1.00	1.00
-0.9	0.05	0.07	0.03	0.03	0.21	0.31	0.21	0.22	0.50	0.66	0.52	0.54	1.00	1.00	1.00	1.00
-0.8	0.04	0.05	0.03	0.02	0.13	0.18	0.12	0.11	0.30	0.39	0.31	0.31	0.83	0.90	0.84	0.87
-0.7	0.03	0.03	0.02	0.01	0.08	0.11	0.08	0.07	0.17	0.23	0.18	0.17	0.54	0.63	0.55	0.58
-0.6	0.02	0.02	0.02	0.01	0.05	0.07	0.05	0.04	0.10	0.13	0.10	0.09	0.31	0.37	0.32	0.33
-0.5	0.02	0.02	0.01	0.01	0.03	0.04	0.03	0.03	0.06	0.08	0.06	0.05	0.17	0.20	0.18	0.18
-0.4	0.02	0.01	0.01	0.01	0.02	0.03	0.02	0.02	0.04	0.05	0.04	0.03	0.09	0.10	0.09	0.09
-0.3	0.02	0.01	0.01	0.01	0.02	0.02	0.02	0.01	0.02	0.03	0.02	0.02	0.04	0.05	0.04	0.04
-0.2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.01	0.02	0.03	0.02	0.02
-0.1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
0	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
0.1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02
0.2	0.01	0.01	0.01	0.02	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.02	0.03	0.02	0.03
0.3	0.02	0.01	0.01	0.02	0.02	0.02	0.02	0.02	0.02	0.03	0.02	0.03	0.04	0.05	0.04	0.06
0.4	0.02	0.01	0.01	0.02	0.02	0.03	0.02	0.03	0.04	0.05	0.04	0.05	0.09	0.10	0.09	0.12
0.5	0.02	0.02	0.02	0.03	0.03	0.04	0.03	0.05	0.06	0.08	0.06	0.08	0.17	0.20	0.17	0.22
0.6	0.02	0.02	0.02	0.03	0.05	0.07	0.05	0.07	0.11	0.13	0.11	0.14	0.32	0.37	0.32	0.40
0.7	0.03	0.03	0.03	0.04	0.08	0.11	0.08	0.11	0.18	0.23	0.18	0.24	0.54	0.62	0.55	0.66
0.8	0.04	0.05	0.03	0.06	0.13	0.18	0.13	0.18	0.30	0.39	0.31	0.40	0.82	0.90	0.83	0.92
0.9	0.05	0.07	0.05	0.09	0.21	0.31	0.21	0.32	0.51	0.66	0.52	0.66	1.00	1.00	1.00	1.00
1	0.08	0.22	0.08	0.24	0.50	0.84	0.50	0.87	0.97	1.00	0.98	1.00	1.00	1.00	1.00	1.00

Table 8.28: 1S 2D KS test four methods power curves for correlation shift for $\alpha = 0.05$ using simulated critical values

Mean	Sample 10				Sample 30				Sample 50				Sample 100			
	Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation	
	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
-1	0.23	0.49	0.17	0.23	0.84	1.00	0.83	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
-0.9	0.16	0.22	0.12	0.10	0.55	0.68	0.55	0.54	0.88	0.96	0.89	0.92	1.00	1.00	1.00	1.00
-0.8	0.13	0.16	0.10	0.07	0.38	0.46	0.38	0.34	0.64	0.75	0.65	0.66	0.99	1.00	0.99	0.99
-0.7	0.10	0.12	0.08	0.05	0.28	0.32	0.28	0.22	0.45	0.53	0.46	0.45	0.88	0.91	0.87	0.88
-0.6	0.09	0.10	0.07	0.05	0.20	0.22	0.20	0.16	0.31	0.37	0.32	0.30	0.66	0.71	0.65	0.66
-0.5	0.07	0.08	0.07	0.05	0.14	0.16	0.15	0.11	0.21	0.25	0.22	0.20	0.44	0.48	0.44	0.43
-0.4	0.07	0.07	0.06	0.04	0.11	0.11	0.11	0.08	0.14	0.16	0.15	0.13	0.27	0.30	0.27	0.25
-0.3	0.06	0.06	0.06	0.04	0.08	0.09	0.08	0.06	0.09	0.11	0.10	0.08	0.16	0.18	0.16	0.15
-0.2	0.06	0.05	0.05	0.05	0.07	0.07	0.06	0.05	0.07	0.07	0.07	0.06	0.09	0.10	0.09	0.08
-0.1	0.05	0.05	0.05	0.05	0.06	0.05	0.06	0.05	0.05	0.05	0.05	0.05	0.06	0.06	0.06	0.05
0	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
0.1	0.05	0.05	0.05	0.05	0.06	0.06	0.06	0.06	0.05	0.05	0.05	0.06	0.06	0.06	0.06	0.07
0.2	0.06	0.05	0.06	0.06	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.08	0.09	0.10	0.09	0.10
0.3	0.06	0.06	0.06	0.07	0.08	0.09	0.08	0.10	0.10	0.11	0.10	0.12	0.16	0.17	0.16	0.19
0.4	0.07	0.07	0.06	0.08	0.11	0.12	0.11	0.13	0.14	0.16	0.15	0.18	0.27	0.30	0.27	0.31
0.5	0.08	0.08	0.07	0.09	0.14	0.16	0.15	0.18	0.21	0.25	0.21	0.27	0.44	0.48	0.44	0.50
0.6	0.09	0.10	0.08	0.11	0.20	0.23	0.20	0.24	0.31	0.36	0.31	0.38	0.66	0.71	0.66	0.73
0.7	0.10	0.12	0.10	0.13	0.27	0.32	0.27	0.34	0.45	0.53	0.46	0.55	0.88	0.91	0.87	0.92
0.8	0.13	0.16	0.12	0.17	0.37	0.47	0.38	0.48	0.64	0.75	0.65	0.77	0.99	1.00	0.99	1.00
0.9	0.16	0.22	0.15	0.23	0.54	0.68	0.54	0.70	0.88	0.96	0.88	0.97	1.00	1.00	1.00	1.00
1	0.23	0.47	0.24	0.44	0.84	1.00	0.89	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.29: 1S 2D KS test four methods power curves for correlation shift for $\alpha = 0.1$ using simulated critical values

Mean	Sample 10				Sample 30				Sample 50				Sample 100			
	Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation	
	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
-1	0.33	0.63	0.29	0.37	0.98	1.00	0.97	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
-0.9	0.26	0.33	0.23	0.17	0.74	0.84	0.75	0.74	0.98	1.00	0.98	0.98	1.00	1.00	1.00	1.00
-0.8	0.21	0.25	0.19	0.13	0.56	0.63	0.57	0.52	0.83	0.89	0.82	0.83	1.00	1.00	1.00	1.00
-0.7	0.18	0.20	0.16	0.11	0.42	0.47	0.43	0.37	0.64	0.71	0.64	0.63	0.96	0.98	0.96	0.97
-0.6	0.16	0.17	0.14	0.10	0.32	0.35	0.32	0.27	0.48	0.53	0.48	0.45	0.81	0.85	0.82	0.82
-0.5	0.14	0.14	0.13	0.09	0.25	0.26	0.24	0.20	0.35	0.38	0.35	0.32	0.61	0.64	0.62	0.61
-0.4	0.13	0.13	0.12	0.09	0.19	0.20	0.19	0.15	0.25	0.28	0.25	0.22	0.41	0.44	0.43	0.41
-0.3	0.11	0.11	0.11	0.09	0.15	0.15	0.15	0.12	0.18	0.20	0.18	0.16	0.27	0.28	0.27	0.25
-0.2	0.11	0.10	0.11	0.10	0.12	0.12	0.12	0.11	0.13	0.14	0.13	0.12	0.17	0.17	0.17	0.15
-0.1	0.10	0.10	0.10	0.10	0.11	0.11	0.11	0.09	0.11	0.11	0.10	0.10	0.11	0.12	0.11	0.11
0	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
0.1	0.10	0.10	0.10	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.10	0.12	0.11	0.12	0.12	0.13
0.2	0.10	0.10	0.11	0.12	0.12	0.12	0.12	0.13	0.13	0.14	0.13	0.15	0.17	0.17	0.17	0.19
0.3	0.11	0.11	0.11	0.13	0.15	0.15	0.15	0.17	0.18	0.20	0.18	0.21	0.27	0.28	0.27	0.31
0.4	0.13	0.13	0.12	0.14	0.19	0.20	0.19	0.22	0.25	0.28	0.25	0.30	0.41	0.44	0.42	0.47
0.5	0.14	0.14	0.13	0.17	0.25	0.26	0.24	0.29	0.35	0.38	0.35	0.41	0.61	0.64	0.62	0.68
0.6	0.16	0.17	0.15	0.19	0.32	0.35	0.32	0.38	0.48	0.53	0.47	0.55	0.81	0.85	0.82	0.87
0.7	0.18	0.20	0.17	0.23	0.42	0.47	0.42	0.50	0.64	0.71	0.63	0.73	0.96	0.98	0.97	0.98
0.8	0.21	0.25	0.21	0.28	0.55	0.63	0.55	0.66	0.83	0.89	0.82	0.91	1.00	1.00	1.00	1.00
0.9	0.26	0.33	0.24	0.36	0.74	0.84	0.73	0.86	0.98	1.00	0.98	1.00	1.00	1.00	1.00	1.00
1	0.35	0.63	0.36	0.62	0.98	1.00	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8.30: 1S 2D KS test four methods power curves for correlation shift for $\alpha = 0.2$ using simulated critical values

Mean	Sample 10				Sample 30				Sample 50				Sample 100			
	Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation		Orientation		Partial Orientation	
	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample	Grid	Sample
-1	0.51	0.83	0.48	0.62	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
-0.9	0.43	0.52	0.42	0.31	0.91	0.96	0.92	0.91	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
-0.8	0.37	0.42	0.36	0.25	0.76	0.83	0.76	0.73	0.95	0.98	0.96	0.96	1.00	1.00	1.00	1.00
-0.7	0.32	0.35	0.32	0.21	0.63	0.68	0.63	0.57	0.83	0.88	0.83	0.82	1.00	1.00	1.00	1.00
-0.6	0.29	0.31	0.29	0.20	0.51	0.54	0.51	0.45	0.68	0.73	0.69	0.65	0.94	0.96	0.94	0.94
-0.5	0.26	0.28	0.26	0.19	0.41	0.43	0.41	0.35	0.54	0.58	0.54	0.50	0.80	0.83	0.81	0.80
-0.4	0.24	0.24	0.24	0.19	0.33	0.34	0.33	0.28	0.42	0.44	0.42	0.38	0.62	0.65	0.63	0.60
-0.3	0.22	0.22	0.22	0.19	0.27	0.28	0.27	0.23	0.33	0.33	0.33	0.29	0.44	0.46	0.45	0.43
-0.2	0.21	0.21	0.21	0.19	0.23	0.24	0.23	0.21	0.26	0.27	0.26	0.23	0.31	0.32	0.32	0.29
-0.1	0.20	0.21	0.20	0.19	0.21	0.21	0.21	0.20	0.22	0.22	0.21	0.20	0.22	0.23	0.23	0.22
0	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
0.1	0.20	0.21	0.20	0.21	0.21	0.21	0.21	0.22	0.22	0.22	0.21	0.23	0.22	0.23	0.22	0.24
0.2	0.21	0.21	0.21	0.22	0.23	0.24	0.23	0.26	0.26	0.27	0.26	0.28	0.31	0.32	0.32	0.34
0.3	0.22	0.22	0.21	0.24	0.27	0.28	0.28	0.30	0.33	0.33	0.32	0.36	0.44	0.46	0.45	0.49
0.4	0.24	0.24	0.23	0.27	0.33	0.34	0.33	0.38	0.42	0.44	0.42	0.47	0.61	0.65	0.63	0.68
0.5	0.26	0.28	0.25	0.30	0.41	0.43	0.41	0.47	0.54	0.57	0.54	0.60	0.80	0.83	0.82	0.85
0.6	0.29	0.31	0.28	0.34	0.50	0.54	0.51	0.58	0.69	0.72	0.68	0.75	0.94	0.96	0.94	0.97
0.7	0.32	0.35	0.31	0.39	0.63	0.67	0.62	0.71	0.83	0.88	0.84	0.90	0.99	1.00	1.00	1.00
0.8	0.37	0.42	0.34	0.45	0.77	0.82	0.76	0.86	0.95	0.98	0.96	0.98	1.00	1.00	1.00	1.00
0.9	0.42	0.53	0.39	0.55	0.91	0.96	0.91	0.97	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	0.49	0.83	0.54	0.85	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

9. Appendix C: 1S 2D KS Test Code

Simplified Python code for reference, contains all four methods: Orientation Grid/sample and Partial Orientation Grid/sample.

```
# Necessary imports
import numpy as np
import scipy  as sp


def FuncQuads_1s_2d(x, y, dist_func):
  """
  Computes the double integrals (CDF value) for the specified bivariate distribution function
  using the coordinates (x,y) and the limits for x and y of -6 to 6.

  Args:
    x (): the x coordinate for calculating the CDF value.
    y (): the y coordinate for calculating the CDF value.
    dist_func (): the pdf of a bivariate distribution function to compute the double integral (CDF
value).

  Returns: a tuple object (LL, LH, HL, HH) with the four values associated with each orientation
    of the CDF.  For example, LL as the double integral from negative infinity to x, y.

  """
  xlim = [-6, 6]
  ylim = [-6, 6]
  point = [x, y]
  LL = sp.integrate.dblquad(dist_func, np.amin(xlim), point[0],
        lambda x: np.amin(ylim),
        lambda x: point[1])[0]
  LH = sp.integrate.dblquad(dist_func, np.amin(xlim), point[0],
        lambda x: point[1],
        lambda x: np.amax(ylim))[0]
  HL = sp.integrate.dblquad(dist_func, point[0], np.amax(xlim),
        lambda x: np.amin(ylim),
        lambda x: point[1])[0]
  HH = 1 - LL - LH - HL #Due to continuity we know that the sum of each orientation equals 1
  return (LL, LH, HL, HH)


##################################################################################
######  1 sample 2D KS Orientation Grid/sample

def ks2d1s_orientation_grid(xx, yy, dist_func):
  """
```

```
  Computes the 1 sample 2 dimensional Kolmogorov-Smirnov test Orientation Grid methods
  for the specified theoretical distribution (dist_func)

  Args:
    xx (): The x values of the samples.
    yy (): The y values of the samples.
    dist_func (): the pdf of a bivariate distribution function to compute the double integral (CDF
value).

  Returns: the raw distance d from computing the 1 sample 2 dimensional Kolmogorov-Smirnov
test Orientation Grid method

  """
  iterate_x = np.unique(xx)
  iterate_y = np.unique(yy)
  d = -1
  for x in iterate_x:
    for y in iterate_y:
    d = orientation_computations(d, x, y, xx, yy, dist_func)
  return d


def ks2d1s_orientation_sample(xx, yy, dist_func):
  """
  Computes the 1 sample 2 dimensional Kolmogorov-Smirnov test Orientation Sample methods
  for the specified theoretical distribution (dist_func)

  Args:
    xx (): The x values of the samples.
    yy (): The y values of the samples.
    dist_func (): the pdf of a bivariate distribution function to compute the double integral (CDF
value).

  Returns: the raw distance d from computing the 1 sample 2 dimensional Kolmogorov-Smirnov
test Orientation Sample method

  """
  d = -1
  for i in range(len(xx)):
    x = xx[i]
    y = yy[i]
    d = orientation_computations(d, x, y, xx, yy, dist_func)
  return d


def orientation_computations(d, x, y, xx, yy, dist_func):
  """
  Computes the 1 sample 2 dimensional Kolmogorov-Smirnov test for a specified point (x,y)
  for all 4 orientations and all 4 directions
```

```
  for the specified theoretical distribution (dist_func).

  Args:
    d (): the current maximum distance d.
    x (): the x coordinate for calculating the ECDF value.
    y (): the y coordinate for calculating the ECDF value.
    xx (): The x values of the samples.
    yy (): The y values of the samples.
    dist_func (): the pdf of a bivariate distribution function to compute the double integral (CDF
value).

  Returns: the raw distance d from computing the maximum distance using Orientation method
for the specified point (x,y)

  """
  cLL, cLH, cHL, cHH = FuncQuads_1s_2d(x, y, dist_func)
  n = len(xx)
  # Orientation LL
  ix1 = np.less_equal(xx, x)
  ix2 = np.less_equal(yy, y)
  fpp2 = np.sum(ix1 & ix2) / n
  ix1 = np.less_equal(xx, x)
  ix2 = np.less(yy, y)
  fpm2 = np.sum(ix1 & ix2) / n
  ix1 = np.less(xx, x)
  ix2 = np.less_equal(yy, y)
  fmp2 = np.sum(ix1 & ix2) / n
  ix1 = np.less(xx, x)
  ix2 = np.less(yy, y)
  fmm2 = np.sum(ix1 & ix2) / n
  d = np.maximum(d, np.abs(cLL - fpp2))
  d = np.maximum(d, np.abs(cLL - fpm2))
  d = np.maximum(d, np.abs(cLL - fmp2))
  d = np.maximum(d, np.abs(cLL - fmm2))
  # Orientation LH
  ix1 = np.less_equal(xx, x)
  ix2 = np.greater_equal(yy, y)
  fpp2 = np.sum(ix1 & ix2) / n
  ix1 = np.less_equal(xx, x)
  ix2 = np.greater(yy, y)
  fpm2 = np.sum(ix1 & ix2) / n
  ix1 = np.less(xx, x)
  ix2 = np.greater_equal(yy, y)
  fmp2 = np.sum(ix1 & ix2) / n
  ix1 = np.less(xx, x)
  ix2 = np.greater(yy, y)
  fmm2 = np.sum(ix1 & ix2) / n
  d = np.maximum(d, np.abs(cLH - fpp2))
  d = np.maximum(d, np.abs(cLH - fpm2))
```

```
  d = np.maximum(d, np.abs(cLH - fmp2))
  d = np.maximum(d, np.abs(cLH - fmm2))
  # Orientation HL
  ix1 = np.greater_equal(xx, x)
  ix2 = np.less_equal(yy, y)
  fpp2 = np.sum(ix1 & ix2) / n
  ix1 = np.greater_equal(xx, x)
  ix2 = np.less(yy, y)
  fpm2 = np.sum(ix1 & ix2) / n
  ix1 = np.greater(xx, x)
  ix2 = np.less_equal(yy, y)
  fmp2 = np.sum(ix1 & ix2) / n
  ix1 = np.greater(xx, x)
  ix2 = np.less(yy, y)
  fmm2 = np.sum(ix1 & ix2) / n
  d = np.maximum(d, np.abs(cHL - fpp2))
  d = np.maximum(d, np.abs(cHL - fpm2))
  d = np.maximum(d, np.abs(cHL - fmp2))
  d = np.maximum(d, np.abs(cHL - fmm2))
  # Orientation HH
  ix1 = np.greater_equal(xx, x)
  ix2 = np.greater_equal(yy, y)
  fpp2 = np.sum(ix1 & ix2) / n
  ix1 = np.greater_equal(xx, x)
  ix2 = np.greater(yy, y)
  fpm2 = np.sum(ix1 & ix2) / n
  ix1 = np.greater(xx, x)
  ix2 = np.greater_equal(yy, y)
  fmp2 = np.sum(ix1 & ix2) / n
  ix1 = np.greater(xx, x)
  ix2 = np.greater(yy, y)
  fmm2 = np.sum(ix1 & ix2) / n
  d = np.maximum(d, np.abs(cHH - fpp2))
  d = np.maximum(d, np.abs(cHH - fpm2))
  d = np.maximum(d, np.abs(cHH - fmp2))
  d = np.maximum(d, np.abs(cHH - fmm2))
  return d


################################################################################
######  1 sample 2D KS Partial Orientation Grid/sample
def ks2d1s_partial_grid(xx, yy, dist_func):
  """
  Computes the 1 sample 2 dimensional Kolmogorov-Smirnov test Partial Orientation Grid
methods
  for the specified theoretical distribution (dist_func).

  Args:
    xx (): The x values of the samples.
```

```
    yy (): The y values of the samples.
    dist_func (): the pdf of a bivariate distribution function to compute the double integral (CDF
value).

  Returns: the raw distance d from computing the 1 sample 2 dimensional Kolmogorov-Smirnov
test Partial Orientation Grid method

  """
  iterate_x = np.unique(xx)
  iterate_y = np.unique(yy)
  d = -1
  for x in iterate_x:
    for y in iterate_y:
    d = partial_computations(d, x, y, xx, yy, dist_func)
  return d


def ks2d1s_partial_sample(xx, yy, dist_func):
  """
  Computes the 1 sample 2 dimensional Kolmogorov-Smirnov test Partial Orientation Sample
methods
  for the specified theoretical distribution (dist_func).

  Args:
    xx (): The x values of the samples.
    yy (): The y values of the samples.
    dist_func (): the pdf of a bivariate distribution function to compute the double integral (CDF
value).

  Returns: the raw distance d from computing the 1 sample 2 dimensional Kolmogorov-Smirnov
test Partial Orientation Sample method

  """
  d = -1
  for i in range(len(xx)):
    x = xx[i]
    y = yy[i]
    d = partial_computations(d, x, y, xx, yy, dist_func)
  return d


def partial_computations(d, x, y, xx, yy, dist_func):
  """
  Computes the 1 sample 2 dimensional Kolmogorov-Smirnov test for a specified point (x,y)
  for all 4 orientations and one direction
  for the specified theoretical distribution (dist_func).

  Args:
    d (): the current maximum distance d.
```

```
    x (): the x coordinate for calculating the ECDF value.
    y (): the y coordinate for calculating the ECDF value.
    xx (): The x values of the samples.
    yy (): The y values of the samples.
    dist_func (): the pdf of a bivariate distribution function to compute the double integral (CDF
value).

  Returns: the raw distance d from computing the maximum distance using Partial Orientation
method for the specified point (x,y)

  """
  cLL, cLH, cHL, cHH = FuncQuads_1s_2d(x, y, dist_func)
  n = len(xx)
  # Orientation LL
  ix1 = np.less_equal(xx, x)
  ix2 = np.less_equal(yy, y)
  fpp2 = np.sum(ix1 & ix2) / n
  d = np.maximum(d, np.abs(cLL - fpp2))
  # Orientation LH
  ix1 = np.less_equal(xx, x)
  ix2 = np.greater(yy, y)
  fpm2 = np.sum(ix1 & ix2) / n
  d = np.maximum(d, np.abs(cLH - fpm2))
  # Orientation HL
  ix1 = np.greater(xx, x)
  ix2 = np.less_equal(yy, y)
  fmp2 = np.sum(ix1 & ix2) / n
  d = np.maximum(d, np.abs(cHL - fmp2))
  # Orientation HH
  ix1 = np.greater(xx, x)
  ix2 = np.greater(yy, y)
  fmm2 = np.sum(ix1 & ix2) / n
  d = np.maximum(d, np.abs(cHH - fmm2))
  return d
```

Bibliography

[1] L. Rokach and O. Maimon, "Top-down induction of decision trees classifiers - a survey," *IEEE Transactions on Systems, Man, and Cybernetics, Part C (Applications and Reviews)*, vol. 35, no. 4, pp. 476–487, 2005, doi: 10.1109/TSMCC.2004.843247.

[2] K. Sasaki, D. Bruder, and E. A. Hernandez-Vargas, "Topological data analysis to model the shape of immune responses during co-infections," *Commun Nonlinear Sci Numer Simul*, vol. 85, p. 105228, Jun. 2020, doi: 10.1016/j.cnsns.2020.105228.

[3] "U.S. Air Force Science and Technology Strategy," 2019.

[4] M. Kendall and A. Stuart, *The advanced theory of statistics. Vol.2: Inference and relationship*. 1979. [Online]. Available: https://ui.adsabs.harvard.edu/abs/1979ats..book.....K

[5] J. A. Peacock, "Two-dimensional goodness-of-fit testing in astronomy," *Monthly Notices of the Royal Astronomical Society*, vol. 202, no. 3, pp. 615–627, Mar. 1983, doi: 10.1093/mnras/202.3.615.

[6] E. Gosset, "A three-dimensional extended Kolmogorov-Smirnov test as a useful tool in astronomy," *Astronomy and Astrophysics*, vol. 188, pp. 258–264, Dec. 1987, [Online]. Available: https://ui.adsabs.harvard.edu/abs/1987A&A...188..258G

[7] G. Fasano and A. Franceschini, "A multidimensional version of the Kolmogorov–Smirnov test," *Monthly Notices of the Royal Astronomical Society*, vol. 225, no. 1, pp. 155–170, Mar. 1987, doi: 10.1093/mnras/225.1.155.

[8] A. Justel, D. Peña, and R. Zamar, "A multivariate Kolmogorov-Smirnov test of goodness of fit," *Statistics & Probability Letters*, vol. 35, no. 3, pp. 251–259, Oct. 1997, doi: 10.1016/S0167-7152(97)00020-5.

[9] A. Kolmogorov, "Sulla determinazione empirica di una lgge di distribuzione," *Inst. Ital. Attuari, Giorn.*, vol. 4, pp. 83–91, 1933.

[10] N. V Smirnov, "On the estimation of the discrepancy between empirical curves of distribution for two independent samples," *Bull. Math. Univ. Moscou*, vol. 2, no. 2, pp. 3–14, 1939.

[11] N. Smirnov, "Table for Estimating the Goodness of Fit of Empirical Distributions," *The Annals of Mathematical Statistics*, vol. 19, no. 2, pp. 279–281, Jul. 1948.

[12] Z. W. Birnbaum, "Numerical Tabulation of the Distribution of Kolmogorov's Statistic for Finite Sample Size," *J Am Stat Assoc*, vol. 47, no. 259, pp. 425–441, Sep. 1952, doi: 10.1080/01621459.1952.10501182.

[13] L. H. Miller, "Table of percentage points of Kolmogorov statistics," *J Am Stat Assoc*, vol. 51, no. 273, pp. 111–121, 1956.

[14] F. J. Massey, "The Kolmogorov-Smirnov Test for Goodness of Fit," *J Am Stat Assoc*, vol. 46, no. 253, pp. 68–78, Aug. 1951, doi: 10.2307/2280095.

[15] A. Kolmogorov, "Über die Grenzwertsätze der Wahrscheinlichkeitsrechnung," *Bulletin de l'Académie des Sciences de l'URSS. Classe des sciences mathématiques et na*, vol. 3, pp. 363–372, 1933.

[16] N. V Smirnov, "Ob uklonenijah empiriceskoi krivoi raspredelenija," *Recueil Mathe- matique (Matematiceskii Sbornik)*, vol. 6, no. 48, pp. 13–26, 1939.

[17] W. Feller, “On the Kolmogoro-Smirnov Limit Theorems for Empirical Distributions,” *Annals of Mathematical Statistics*, vol. 19, pp. 735–749, 1948.

[18] J. Doob, “Heuristic Approach to the Kolmogorov-Smirnov Theorems,” *Annals of Mathematical Statistics*, vol. 20, pp. 393–403, 1949.

[19] J. Vrbik, “Deriving CDF of Kolmogorov-Smirnov Test Statistic,” *Applied Mathematics*, vol. 11, pp. 227–246, Jan. 2020, doi: 10.4236/am.2020.113018.

[20] G. Marsaglia, W. W. Tsang, and J. Wang, “Evaluating Kolmogorov’s Distribution,” *Journal of Statistical Software, Articles*, vol. 8, no. 18, pp. 1–4, 2003, doi: 10.18637/jss.v008.i18.

[21] W. Pelz and I. J. Good, “Approximating the Lower Tail-Areas of the Kolmogorov–Smirnov One-Sample Statistic,” *Journal of the Royal Statistical Society: Series B (Methodological)*, vol. 38, no. 2, pp. 152–156, Jan. 1976, doi: https://doi.org/10.1111/j.2517-6161.1976.tb01579.x.

[22] J. Vrbik, “Small-Sample Corrections to Kolmogorov-Smirnov Test Statistic,” *Pioneer Journal of Theoretical and Applied Statistics*, vol. 15, no. 15, p. 23, 2018.

[23] L. C. Chang, “On the Exact Distribution of the Statistics of A. N. Kolmogorov and Their Asymptotic Expansion,” *Acta Mathematica Sinica*, vol. 6, no. 55, p. 81, 1956.

[24] W. H. Press and S. A. Teukolsky, “Kolmogorov-Smirnov Test for Two-Dimensional Data,” *Computers in Physics*, vol. 2, no. 4, pp. 74–77, Jul. 1988, doi: 10.1063/1.4822753.

[25] Gabinou, “2 Dimensional Kolmogorov-Smirnov test for goodness-of-fit.”

[26] Syrte, “multi-dimensional statistical test with Python.”

[27] B. Lau, “2-d Kolmorogov-Smirnov test, n-d energy test, Hotelling T^2 test,” 2022.

[28] D. Muir, “kstest_2s_2d(x1, x2, alpha).” 2022.

[29] Y. Xiao, “Two and Three Dimensional Kolmogorov-Smirnov Two-Sample Tests.” 2016.

[30] E. Puritz, Connor and Ness-Cohn, “Fasano-Franceschini Test.” 2022.

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14. ABSTRACT

Most statistical tests are fully developed for univariate data, but when inference is required for multivariate data, univariate tests risk information loss and interpretability. This research 1) derives and extends the multivariate Komolgorov Smirnov test for 2 and into m-dimensions, 2) derives small sample critical values for the KS test that are not reliant on sample size simulations or correlation between variables, 3) extends large sample estimations and current KS implementations, and 4) provides sample size and power calculations in order to enable experimental design with respect to testing for differences in distributions. Through extensive simulation, we demonstrate that our new multidimensional KS test generally has more power for smaller sample sizes and comparable power for larger sample sizes and maintains desirable statistical properties. Furthermore, we improve and extend current implementations of the KS test to sample sizes upwards of n = 5000. Finally, we demonstrate how to compute critical values to any size dimensional data and provide power and sample size criterion for designing studies using 2 and 3 dimensional distributions. These results enable statistical testing of multidimensional features, irrespective of correlation, thus improving our ability to understand large data sets for rapid and efficient decision making and analysis.

15. SUBJECT TERMS

Komogorov Smirnov Test, Cumulative Distribution Function, multivariate data, multivariate empirical distribution estimation, statistical power

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