12-1991

Controlling wildlife damage by diffusing beaver population : a bioeconomic application of the distributed parameter control model

Mahadev G. Bhat

To the Graduate Council:

I am submitting herewith a dissertation written by Mahadev G. Bhat entitled "Controlling wildlife damage by diffusing beaver population : a bioeconomic application of the distributed parameter control model." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Agricultural Economics.

Luther Keller, Major Professor

We have read this dissertation and recommend its acceptance:

Ray Huffaker, Bill Park

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Luther Keller, Major Professor

Ray Huffaker

Ray Huffaker, Co-Major Professor

We have read this dissertation
and recommend its acceptance:

Suzanne Lenhart

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Accepted for the Council:

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Associate Vice Chancellor
and Dean of The Graduate School

CONTROLLING WILDLIFE DAMAGE BY DIFFUSING BEAVER POPULATION: A BIOECONOMIC APPLICATION OF THE DISTRIBUTED PARAMETER CONTROL MODEL

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Mahadev G. Bhat
December 1991

DEDICATION

This dissertation is dedicated to my parents

Mr. Ganapati Mahadev Bhat
(ಶ್ರೀ ಗಣಪತಿ ಮಹಾದೇವ ಭಟ್ಟ)

and

Ms. Seeta Bhat
(ಶ್ರೀಮತಿ ಸೀತಾ ಭಟ್ಟ).

ACKNOWLEDGEMENT

There were many individuals who helped me make this dissertation possible. I would like to begin by thanking Dr. Suzanne Lenhart, Department of Mathematics, who was of immense help in completing this study. She taught and helped me in understanding the theory behind control models in partial differential equations. She offered her constructive criticisms and friendly advice throughout this study.

My special thanks are due to Dr. Ray Huffaker, who influenced strongly my interest and pursuit in the problems of natural resource economics in general and beaver damage control in particular. Ray, who left this University on a different assignment in the middle of my study, never felt bothered from my odd hour phone calls for advice. His valuable comments helped to clarify my views and sharpen my thinking regarding the economic scope of this problem. I would like to thank Dr. Luther Keller, whose enthusiasm, support and ready sense of humor helped me throughout this project. He always made me cautioned against the fact that how easily one could lose touch with the economic reality while building high-tech mathematical model. My thanks are due to Dr. Bill Park for his valuable advice and willingness to serve in my doctoral committee.

I would like to thank Mr. Bob Gotie, New York Department of Environmental Conservation, Cortland, New York, for giving me the valuable field data on beaver population. I richly benefited from his field experience about the beaver biology and management problems.

I sincerely acknowledge the financial support given by the Tennessee Agricultural Experiment Station for my graduate program. Part of this study was supported by a grant form National Science Foundation, courtesy Suzanne Lenhart.

There were other two individuals whose association was highly rewarding. Dr. Burton English offered me a special research assistantship and teaching opportunity during the latter part of my program. I extend my heart-felt thanks for his support and encouragement. Mr. Robert Alexander deserves my special thanks for his stimulating dialogues regarding subjects of economics and computers.

I owe him the aesthetic look of this dissertation.

I would also like to thank my parents, Sri Ganapati Bhat and Smt. Seeta Bhat, and my brothers for their love, affection and excellent educational opportunity. Finally, I thank my wife, Usha, for her cheerful support and encouragement throughout my study. The warmest thanks go out to my daughter, Nivedita, for sustaining my long hours away home while completing this study.

ABSTRACT

The beaver population in the Southeastern United States has caused severe damage to valuable timber land through dam-building and flooding of bottom-land forest. Traditionally, beavers have been trapped by small group of people as a source of their livelihood. The low pelt price in the recent years has failed to stimulate adequate trapping pressure, and thus, resulted in increased beaver population and damage losses. The low trapping pressure has left the burden of nuisance control on property owners. Since the beaver population is mobile, extermination of beavers from affected parcels results in migration of beavers from neighboring less controlled parcels to less populated controlled parcels. This backward migration of beavers from uncontrolled habitat to controlled habitat imposes a negative *diffusion externality* on the owners of controlled parcels because they have to incur the future cost of trapping immigrating beavers. Unless all the land owners agree to control the beaver population simultaneously, the *diffusion externality* could result in a low incentive for control of beaver population on the part of individual land owners, causing a wedge between social and private needs for controlling beaver population.

This study attempts to develop a bioeconomic model that incorporates dispersive population dynamics of beavers into the design of a cost-minimizing trapping strategy. While recognizing the need for several management options, depending on the land owners attitude about beavers, this study focusses its attention on the situation where all the land owners in a given habitat share common interest of controlling beaver nuisance, and collectively agree to place the area-wide control decision in the hands of a public agency, on a cost sharing basis. The model is based on the notion that the public manager attempts to minimize the present value combined costs of beaver damage and trapping over a finite period of time subject to spatiotemporal dynamics of beaver population. The time and spatial dynamics of beaver population is summarized by the parabolic diffusive Volterra-Lotka partial differential equation. Thus, the current problem is a typical *distributed parameter control* problem.

The cost-minimizing area-wide trapping model is capable of characterizing the beaver control strategy that leaves enough beavers after taking into account the net migration at each location and time, so as to strike the optimal balance between timber damage and trapping cost. The marginality condition governing this tradeoff requires that the marginal damage savings from the beavers trapped at each location equal the marginal costs of trapping. The marginal savings from trapping activity, in turn, is measured as the imputed nuisance value (shadow price) of the beaver stock in a unit area.

The optimality system for this problem that characterizes the optimal control is solved numerically. The validity of the theoretical model is empirically examined using the bioeconomic data collected for the Wildlife Management Regions of the New York State Department of Environmental Conservation. The empirical simulation generated discrete values for the optimal beaver densities and trapping rates across all the individual operational units over time. The entire distribution of optimal beaver densities does gradually and smoothly decline over the period of time. The unevenness of the initial population distribution smoothes out eventually across the beaver habitat. At each geographical location, towards the end of the planning period optimal trapping rate will become zero, whereas the population density asymptotically approaches zero.

The sensitivity analysis where the cost and damage parameters of the model are alternated between high and low values indicates that an increase in the damage potential of beavers could substantially increase the net present value total cost. On the other hand, an increase in the cost of beaver trapping adds only marginally to the total cost, conserving more number of beavers. The geographical variation in the beaver damage potential has a noticeable reflection on the spatial distribution of trapping rates, with little impact on the optimal densities. The areas with higher beaver damage potentials require more intensive trapping operation.

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CHAPTER I

INTRODUCTION

Beavers, *caster canadensis*, were saved from almost complete elimination in the middle of this century by regulations controlling trapping season, method, and numbers. Under this protection the beaver population has increased alarmingly in the last 25 years across the Southeastern United States (Arner and Dubose; Bullock and Arner). In many parts of this region, beavers have caused severe damage to valuable timber land through dam-building and resultant flooding of bottom-land forest. Other nuisance activities include blocking of roadside culvert flooding highways, building dams on streams where shore-line developments are flooded, and destruction of ornamental trees and shrubs in urban and suburban areas (Hill). A recent survey report by Miller on damage from vertebrates to southern forests states that ... *without question the beaver is the vertebrate animal causing the most damage to southern forests at the present time* (p. 13).

Reported losses to local/state economies due to beaver damage has been alarming. In Tennessee alone, a survey by the Forest Department indicated that more than 81,000 acres of dead and fading forest land required beaver control and drainage of standing water, with an associated loss of over $26 million (Tennessee Forestry Department). Bullock and Arner estimated direct and indirect economic losses to Mississippi economy of approximately of $2.4 billion for the period 1975 to 1985. Substantial timber damage estimates have been reported for other states (Hill).

Documented benefits have been cited for beaver as a conservator of nature and source of recreation. In addition, beavers may have economic value in terms of fur, meat and caster and oil glands. However, the high degree of beaver nuisance has earned them a status of pest, at least in the southern range limits.

Beavers as Common Property and Economics of Overpopulation

Beaver as an important wildlife species has remained a property of the public domain. In most cases, each state has mandated the respective state wildlife protection agency to ensure the overall balance of beaver population. Traditionally, beavers have been trapped by small group of people as a source of their livelihood. A primary responsibility of the management agencies is to restrict the number of trappers in a specified region through quotas or seasons. In areas where beavers are known to contribute direct economic and ecological benefit, management goals are to *increase long-term mean population levels by preventing destruction of habitat by themselves* (Todd, p. 119). In other areas where they are both beneficial and a nuisance, management is faced with conflicting goals. Management strategy is to allow enough number of trappers to keep the damage under control while at the same time to guard the population from overharvest. Conversely, in regions like the southeast where beavers are treated as a pest, wildlife agencies have adopted an extreme harvest strategy. There are no restrictions on trapping in most southern states. In fact, some states like Alabama, Tennessee and Kentucky have passed legislation to pay bounties to trappers (Hill).

The level of beaver trapping, the only effective means of controlling populations, is mostly driven by the market value of beaver pelts. By simple economic intuition, trappers are induced to undertake trapping as long as the expected pelt prices are high enough to yield economic profits. Thus, pelt price determined by the pelt quality is the main economic factor that could regulate the beaver population along with appropriate regulation against overharvest. Historically, prices of beaver pelt coming from southern range limits have remained very low. Hill and Novakowski point out that in Canada the demand for trapping responds instantaneously to pelt price whereas in the United States, trapping pressure is more price-inelastic. This seems to be particularly true in the southeastern states. As a result, beaver trapping has not been financially attractive in this region. The pelt market which has been successful in the northern part of this continent in controlling beaver population has failed in the southeast to stimulate adequate trapping pressure.

Ecological Externality and Management Scenarios

The immediate question that follows is how property owners are responding to low trapping pressure. More than 80 percent of the land affected by beavers in the southeast is under private ownership (personal communication with McMahan). It is obvious that beaver control must be the concern of private land owners and incur the cost of trapping beavers in order to prevent damage to their economically valuable timber land.

There have been many myopic attempts on a limited scale to eradicate beavers from the affected parcels. But these attempts have failed to keep the beaver population under control for several reasons which are ecologically and economically intuitive. Experiences from isolated eradication efforts have demonstrated that beavers from neighboring parcels tend to immigrate continually into less populated controlled parcels (Houston). This migration imposes a negative externality, hereafter called *diffusion externality,* on the owners of controlled parcels because they incur the future costs of eradicating beavers that are currently on uncontrolled (or less controlled) neighboring parcels but that will fill the vacuum created by extermination on controlled parcel. Because of their mobility, the dynamics of beaver populations within a controlled parcel depends upon the total population levels in the region as a whole rather than on the levels within a decision unit. Unless there is some arrangement to control the beaver population on the entire habitat, individuals owning part of the beaver habitat may not be able to control the beaver population and the associated damage effectively. This is a typical feature of a common property resource that could drive a wedge between private and public incentives for controlling the nuisance.

The level of control effort by owner of a beaver-affected parcel, however, depends upon the degree of cooperation by the neighboring land owners. The owners of neighboring parcels may have mixed economic objectives. Based on their economic motives, the beaver-affected land owners may find themselves under three different scenarios. First, a neighboring parcel may be a public land where beavers may not be viewed as a nuisance, and/or beavers are protected to provide source of recreation. Under such circumstances, owners of beaver-affected parcels have to face extreme non-cooperation from their neighbors. They are aware that there would be no control effort on the neighboring parcels, and that they have to exercise adequate control efforts unilaterally, keeping in view the possible immigration of beavers from uncontrolled adjacent parcels, following the control effort on their beaver resident land. Second, the neighboring land owners may view beavers as a valuable capital stock and try to increase their population. The other possibility is that neighboring owners may be acting as *free riders* enjoying the savings in damage from, and trapping costs of, beavers migrating into controlled parcels. The owners of controlled parcels incurring trapping costs have no means to exclude non-acting neighbors from receiving the benefit of beaver control efforts on the resident land. Thus, two or more contiguous land owners in a given beaver habitat may hold mixed or diametrically opposite views about beaver population.

Finally, all land owners in a given watershed may have a similar objective of controlling the beaver nuisance problem. Any individual timber land owner might be affected by beavers as badly as others in the watershed. But, any single owner is less likely to exercise enough control effort because of the *diffusion externality.* However, because of the common problem experienced by all the land owners, there might be a consensus among owners to collectively control the beaver population in the entire habitat.

Management Implications

There is no single management strategy that can provide a solution to the beaver nuisance problem under all the scenarios. The management strategy depends on the situation faced by the land owners. Under the first scenario, a land owner doesn't anticipate any cooperation from the neighbor. The management of beaver population under this circumstance needs to be modelled under the framework of single species harvesting from two ecologically dependent species population. This type of problem has been most popular in the fishery economic literature. Under the second scenario, land owners in a given watershed have diametrically opposite objectives. Such a management problem can be simulated as differential game planning with two or more players.

The third scenario, where all the land owners in a beaver habitat suffer similar damage and thus, are interested in controlling the beaver population if other members mutually respond, warrants totally different management strategy. The obvious strategy open to land owners under this circumstance is to organize among themselves to develop a collective trapping strategy that aims to minimize the beaver damage with all the direct operational costs and externalities internalized. As individual decision making land owners are unable to control the population of the entire beaver habitat, they would better serve their common interests by collective action and placement of the responsibility of region-wide regulation in the hands of a single, public manager, on a cost sharing basis. Such a policy enables the public manager to explicitly consider costs of operation and externalities stemmed from their natural dispersal behavior into a management strategy in addition to damage reduction goal.[1]

Since the focus of this study is controlling timber damage inflicted by beavers on large timber lands controlled by multiple land owners, this study concentrates on a cooperative beaver management strategy. The development of beaver trapping models under all the three scenarios is beyond the scope of this study. The analyses under other two scenarios are being undertaken in separate studies.

Beaver Management and Spatiotemporal Optimization

Cooperative management of beavers is basically a problem of managing a renewable capital resource over a period of time and space. Since growth of this biological capital is dynamic in nature, present harvesting at a given location can affect future availability and biological productivity of the stock throughout the entire beaver habitat. Further, the damage savings from and/or cost of current trapping may accrue in the future at all locations. Therefore, an economically optimal harvesting policy constitutes *simultaneous choices of present and future harvesting* [for all the spatial points], *and the optimal choices at different times* [and space] *are interrelated* (Arrow and Kurz). This type of temporal and spatially distributed control strategy could be evolved using a complicated mathematical analysis

[1]See Feder and Regev for a discussion of similar economic view point in the context of multiple pest species control problem.

called *distributed parameter control.*

The idea of distributed control problem is similar to classical optimal control theory. Here capital stock and control (harvesting) variables are treated as functions of time and space. The time and spatial evolution of a state variable is expressed in terms of a partial differential equation. Some pay-off function is optimized over time and space domain, subject to the above partial differential equation of motion, to obtain area-wide distributed optimal control and state values. Essentially, the need for distributed control arises when capital stock has more than one attribute.

In economic literature, there are only few studies which have attempted to consider multiple attributes of capital simultaneously (Haurie, Sethi and Hartl; Robson; Bensoussan, Nissen and Tapiero). Manzell, in the context of agricultural pest management, introduced the idea of *classical trapping models* that described the pest diffusion process based on a partial differential equation. However, these models were developed to analyze the ecological behavior of a pest population which was subject to trapping, rather than explore economic implications for pest control. The harvesting model of diffusive population developed by mathematicians, Leung and Stojanovic, in the recent years is a theoretical modeling exercise, with no real life application. No work has been located in economic literature that integrates dispersive population dynamics of a small-mammal into an optimization framework capable of characterizing cost-minimizing spatial trapping strategies over time. Such an integration would have not only the practical importance outlined above, but also would make an interesting addition to bioeconomic research on optimal harvest of diffusive species. Clark, and Hamalainen and Kaitala considered diffusion dynamics in the context of optimal harvesting-for-sale of fish populations migrating between only two adjacent patches. Their models are not adequate to handle small mammal populations like beavers which continuously migrate over a broad geographical area. Another feature that distinguishes the current problem from their study is the design of the institutional framework within which the management decisions would be made.

It is essential to realize that a bioeconomically optimal spatial trapping strategy that may result in the complete eradication of beavers is also not acceptable politically. Such a strategy would incur strong opposition from advocates of animal rights and esthetics. Therefore, caution should be exercised in modeling beaver management to see that optimal trapping does not result in species extinction in the region.

Objectives of the Study

Beaver damage to timber lands in the Southeastern United States is a pressing concern for private and public timber land owners and wildlife agencies. Despite severe loss to timber industry, no serious attempt has been made to evolve a suitable management plan for controlling beaver population. The migratory behavior of beavers, which has a direct impact on the cost effectiveness of any control program, seems to have been ignored by the management agencies. The beaver population dynamics depends on the population levels of the entire habitat on which an individual owner suffering beaver damage losses has no control. The resulting *diffusion externality* draws a gap between public and private incentives for controlling the beaver population. This study attempts to develop a unified bioeconomic trapping model that incorporates dispersal behavior of the beavers into an economic framework. The analysis focuses on the bioeconomic implications of an area-wide centralized control policy, emphasizing the common property feature of the beaver nuisance problem. The economic conflict between density dependent beaver damage, trapping cost and diffusion-related externality are explicitly considered in the model.

The key assumption of this study is that all the owners operating in a given beaver habitat share common interest of controlling the beaver problem, and that they collectively agree to place the area-wide control decision in the hands of a public agency and share the cost. Following Clark, we adopt the notion that a public manager acting on behalf of land owners attempts to maximize the community welfare, which is similar to a sole owner's profit-maximizing behavior.

An attempt is made to see that the optimal trapping strategy does not totally eliminate the beaver population from the region. This study, without loss of generality, ignores potential environmental and economic benefit (Hill) that beavers might have in the southeast at low levels of population. The model developed here would need only slight modification to include possible beaver benefits. Finally, the validity of the theoretical model is empirically examined using economic and biological data from the Wildlife Management Regions of the New York State Department of Environmental Conservation.

The organization of the study is as follows. A brief sketch of biology of beavers, nature of their damage, and current beaver management strategy followed in various parts of the United States and Canada is provided in the next chapter. Also included in the following chapter is an overview of modeling spatiotemporal dynamics of diffusing animals and optimal control models for harvesting structured biological species. The third chapter develops the area-wide *distributed control* model evolving cost-minimizing beaver trapping strategy. The optimal trapping model developed in this study was a complicated nonlinear control model which required rigorous numerical simulation before making practical use of the same. The numerical simulation of the *distributed-control* model and its empirical application is presented in the fourth chapter. The last chapter summarizes the study and adds management implications.

CHAPTER II

REVIEW OF LITERATURE

Before developing a bioeconomic model for beaver management, it is essential to a have basic understanding of various components of the modelling process. These components include (1) beaver ecology (e.g., birth, mortality and migratory behavior), (2) nature of beaver damage, (3) techniques and methods of control available, and (4) economics of damage and control. Based on a survey on vast beaver biology literature, these components are introduced in this chapter. This chapter also reviews several ecological diffusion models that provide a mathematical framework for developing beaver trapping strategy. Standard mathematical/economic terminologies that are used in subsequent chapters are also introduced. These models depict how technological and ecological attributes of a biological resources can be incorporated into a decision maker's economic optimization framework.

Dynamics of Beaver Population

The beaver has been the most widely studied wildlife species by field biologists. A recent review article by Hill provides a detailed description of beaver biology. The following sections draw heavily upon his study and references cited therein.

Reproduction and Mortality

Beavers are monogamous animals and reproduce once a year generally during late fall through the winter season. Sexual maturity occurs at an age of 2 to 4 depending on environmental factors and population levels (Semyonoff). A typical litter would be 3 to 4 youngs with a wide range of one to nine (Hill). Beaver mortality is generally caused by *predation by mammalian predators such as coyote and timber wolf.* Predation is more likely during periods of food shortage when beavers tend to range over larger areas. Other minor predators and water-borne disease can also cause beaver mortality.

Payne estimated the annual birth and natural mortality for Newfoundland beavers at 0.536 and 0.188. This gives a net annual population growth rate of 0.3479. He also pointed out that growth of the beaver population is *compensatory* in nature. That is, population growth responds to the current population level and mortality rate (natural and harvesting). Lancia and Bishir showed that observed data on beaver population in Massachusetts from 1952 to 1978 followed a logistic growth function. From the estimated logistic growth function, a maximum average annual growth rate of .335 was obtained. These results support the view that the growth of beaver population is compensatory in nature, and that logistic growth function is an adequate approximation of the temporal growth process of beavers.

Beaver Habitat

Beavers are commonly found in large rivers, impoundments, lakes streams, tributaries and seepage. High quality habitat with abundant vegetation can harbor large number of beaver colonies. A colony unit, as Bradt defines, constitutes of *a group of beavers occupying a pond or stretch of stream in common utilizing a common food supply, and maintaining a common dam or dams*. Hill describes the members of a typical colony to include *the adult pair, two to four kits from the previous spring litter, two or three yearlings, and occasionally one or more that are about 2.5 years old*. The size of a colony depends upon the habitat quality. Several studies have provided estimates of colony size. Denney estimated an average of 5.2 beavers per colony in the United States and 5 in Canada.

Beavers are known to construct dams across flowing water which consequently raises the water level. They prefer to build food reserves for the winter season in the water. Lodges or bank dens are constructed for shelter near the water and food source.

Migratory Behavior

Spatial diffusion of the beaver is an important aspect of their population dynamics. Bergerud and Miller identified the following four types of beaver movements: (1) movement of the entire colony between ponds within its territory, (2) wandering of yearlings, (3) dispersal of two-year-old beaver to establish new colonies, and (4) miscellaneous movement of adults who likely have lost their mates. The last three categories of beaver movement constitute actual inter-territorial immigration or emigration (Hodgdon). The substantial portion of territorial migration is caused by the third factor, dispersal of two-year-old beavers to establish new colonies. There exists an *innate tendency* (among two-years-old) *to leave their home colony* (Leege; Bergerud and Miller). It has also been found that habitat quality may have some influence on the rate of migration. In good quality habitat, less and less beavers are found to emigrate (Gunson). Hill cites several studies which documented the distance travelled by transplanted beavers. In some instances, beavers have moved more than 200 km.

Beaver Damage and Control

Beaver Damage

Beaver damage was reported as early as the 1950s, soon after restocking beavers in the continent. The nature of beaver damage varies widely across regions. Todd, while making an argument for beaver trapping, explains the several ways they can cause damage. During periods of shortage of their preferred food like aspen and willow, beavers can inflict huge damage to the logging industry by cutting timber and flooding timber land with their dams. Failure to control the beaver population will lead to their dispersal into other agricultural land where they can be found in direct conflict with human interest (Parsons and Brown, 1978). There are instances where wide spread dam building activity has caused water stagnation which hindered fish migration and spawning movements (Todd; Knudsen). Beavers are also known to be potential carrier of disease, called Tularemia, which can affect other wildlives and domestic animals including man.

Houston, in a personal communication, mentioned that damage of the habitat by beavers is a dynamic process of interaction between dam building activities and siltation. Beavers generally select tributaries or creeks joining the main streams or rivers, and build dams restructuring stream flow. Over time, the back-up water collects silt, raising the upstream water level. As beaver population increases over time, individuals find it much easier to go to the fringes of the back-up water and construct more dams across streams joining the existing water pond. This process could continue until a vast area becomes flooded.

Obviously, beavers have many direct and indirect harmful effects on human beings and other wildlife. As beaver population increases, the financial loss (direct and indirect) seems to increase more than proportional. While several beaver damage estimates have been made, no systematic attempt has been made to compare the beaver population and the associated damage levels. Damage estimates and population estimates for a wildlife management unit in New York State reported by Purdy *et al.* are partial but provide some basis for damage cost to be used in the current study.

Control Techniques and Costs

Byford discussed various types of beaver control techniques in practice. In some cases, beaver dams and lodges have been destructed manually or by using dynamite. However, this technique has not been very effective since beavers can reestablish easily dams and lodges. Altering the beaver habitat might be effective in some cases. Removing certain tree species which are favorite food for beavers has been found to be an effective way of checking their population. This method, of course, is not universal, but only site specific. Toxicants have been used in some cases to poison the beaver food source. Since beavers are mobile and may not return to the same food source for 2 to 3 weeks, this method may also be ineffective (Byford). The cost of using toxicants would likely be similar to costs of trapping operation (Hill). Also, the pelt and meat obtained from poisoned beavers would be of no use value. Other techniques available, which are considered less effective, include shooting, using alligators as predators, live-trapping and translocating.

Trapping is considered to be the most effective means of beaver control (Gotie; Hill; Byford). This method has been found more useful and reliable when seasons and a quota system of management need to be enforced. The conibear trap, size 330, has been found to be an extremely effective trap.

This trap is suitable for either shallow or deep water. The traps are generally set in the dams, burrow, or lodge entrance, in narrow channels, in runs in front of drain pipes, or beneath slides (Byford). When beavers encounter the trap, they are killed instantly.

Only limited information is available relative to the costs and resource requirement of a beaver trapping operation. Estimate by Hill was based on a two years survey of trappers in Alabama. During the 1972-73 trapping season, an average beaver trapper employed 20 Conibear traps and spent 58 days to capture 50 beavers which meant 23.2 days of conibear-trap time (20 times 58 divided by 50), and 1.16 days of trapper's time per beaver. During the next year, an average trapper employed only 16 Conibear-traps and spent 32.2 days capturing 39.7 beavers, which amounts to 12.98 days of conibear-trap time and 0.81 days of trapper's time per beaver. Though this information is limited and region specific, it is indicative of the underlying cost structure of the industry. The higher level of trapping operation in the first year required more resources in terms of equipment and labor compared to that of the second year. The input requirement reported did not include other resources needed like fuel and vehicle. This leads to the conclusion that the unit cost of trapping is an increasing function of level of trapping operation. This has an implication on the owner's management decision while measuring beaver damage relative to cost of trapping.

Population Management

Although, currently, there is no systematic beaver population management effort in the southeast, a knowledge of beaver management strategy that are in vogue in other parts of the United States and Canada may be of interest. Management objectives differ from region to region. Maintaining a sustained harvest of population is an important objective in most Canadian provinces (Todd) whereas striking a balance between land use conflict and biological carrying capacity has been the main concern in the northern United States (Gotie). In almost every state or provinces with beaver management programs, a quota system has been used for controlling the beaver population. The quota system requires taking certain number of beavers from each active colony. Quotas are established on the basis of predicted population or active colonies, level of nuisance, land owners tolerance to beaver damage (Gotie) and previous year trapping experience. In recent years, more state wildlife agencies, such as New York, have been air-surveying every year to take inventory of the number of active beaver colonies. Based on the beaver survey, the harvest quotas for the coming season would be determined.

Hill and Novakowski have reported on the various systems through which quotas have been enforced. Under the Registered Trap Line System, which has been popular in Canada, an individual biologist, trapper, or group is assigned a specified geographic area. The registered trappers are expected to harvest according to the quota fixed by the jurisdictional agency. Control on the harvest is administered by sealing or tagging pelts recovered in the area. In some part of the United States, state agencies fix zonal quotas, and the total harvest is distributed among the trapper-permittees. Generally, the trapping operation is restricted to a certain season. Control regulations are enforced only when state agencies have reason to believe that the population is being subjected to overtrapping. Even in areas where beavers are managed for damage control, no specific information on damage estimates, trapping cost, nor migration is considered in fixing quotas. Quotas in each management unit are established mostly on the basis of qualitative measures of land owners' beaver damage tolerance level generally assessed by land owners' opinion surveys.

Dynamic Diffusion-Interaction Population Models

The development and understanding of an appropriate population model is a key to the success and reliability of a management model for biological renewable resources. For the sake of mathematical simplicity, most biological resource management models are based on highly simplified ecological assumptions (eg. single species, temporal variation, general production, and deterministic growth) (Clark). The real life management problems call for more realistic, though complex, modelling efforts. The complexity may be added due to species-specific biological variation from the basic modelling framework, for instance, age-structure, spatial heterogeneity, and size-specific harvesting. Spatial distribution and diffusion are the key ecological considerations that warrant explicit consideration in characterizing beaver management strategy. In the spirit of its importance, the following subsection is devoted to a brief survey of the development of diffusion models in population ecology literature.

Diffusion and Random Walk

The classical theory of the diffusion of a biological population is founded on the popular theory known as *random walk* (Skellam). Skellam was the first to draw an analogy between the random motion of molecules and that of organisms (Edelstein-Keshet). According to the *random walk* hypothesis, an organism on a line moves one place to the left or right, with equal chance of occupance. If this random jump process continues, after many jumps, the distribution of the probabilities of occupance of the organism at different points is binomial. For a larger number of movements with smaller steps, the distribution tends to be normal. For a particle suffering random displacement ϵ on one dimensional space x at regular intervals of time ω, Skellam showed that the probability density (ψ) of the particle at different location and time must satisfy the following partial differential equation,

$$(1) \qquad \frac{\partial\psi}{\partial t} = \frac{1}{2}\frac{\epsilon^2}{\omega}\frac{\partial^2\psi}{\partial x^2}$$

The above equation derived from the random walk models is analogous to classical diffusion process. The classical diffusion process is readily applicable to process of population migration. The conservation law as applied to the movement of particles or individual organisms can be represented as (Edelstein-Keshet),

$$(2) \qquad \begin{pmatrix} \textit{rate of change} \\ \textit{of particle} \\ \textit{population} \\ \textit{in } (x, x+h) \\ \textit{per unit time} \end{pmatrix} = \begin{pmatrix} \textit{rate of} \\ \textit{entry} \\ \textit{into } (x, x+h) \\ \textit{per unit time} \end{pmatrix} - \begin{pmatrix} \textit{rate of} \\ \textit{departure} \\ \textit{from } (x, x+h) \\ \textit{per unit time} \end{pmatrix}.$$

where h is small interval on the linear space x. The two terms on the right hand side are inward and outward flux of particles or individuals, respectively. The inward and outward flux can be determined by classical Fick's law. The law states that *the amount of transport of matter in the x direction across a unit normal area in a unit time, i.e., the flux, is proportional to the gradient of the concentration of matter.* The gradient is measured by the difference or variation in the concentration or density of individuals across unit interval. Let D represent constant of proportionality of flux to gradient, and $C(x,t)$ the concentration of particles at point x and time t. Now utilizing Fick's law, each of the flux terms on RHS of equation (2) can be expressed in terms of gradients to obtain

$$(3) \qquad \left(\frac{C(x,t+\delta)-C(x,t)}{\delta}\right)h = D\left(\frac{C(x+h,t)-C(x,t)}{h}\right) - D\left(\frac{C(x,t)-C(x-h,t)}{h}\right)$$

Simplifying (3) further, we obtain

$$(4) \qquad \left(\frac{C(x,t+\delta)-C(x,t)}{\delta}\right) = D\left(\frac{C(x+h,t)-2C(x,t)+C(x-h,t)}{h^2}\right)$$

Taking a limit of this equation as $h \to 0$ and $\delta \to 0$, that is, as respectively space and time intervals get vanishingly small, we arrive at the following parabolic equation of diffusion:

$$(5) \qquad \frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}.$$

Notice that equation (5) is same as equation (1). Skellam's rationale behind using the random walk-based equation (1) as a measure of population dynamics of diffusing biological population seems to be quite justified. He suggested that for a large population reproducing continuously according to Pearl-Verhulst logistic law, an appropriate model would be

$$(6) \qquad \frac{\partial \Psi}{\partial t} = D\frac{\partial^2 \Psi}{\partial x^2} + \Psi(a - b\Psi)$$

where $F(\Psi) = \Psi(a-b\Psi)$ is the rate of growth of local population in the absence of dispersal. The D is called dispersion rate or mean square dispersion per unit time. The total time rate of growth of population at any spatial point, $\partial\Psi(x,t)/\partial t$, is an interaction between local temporal growth and the net population flux through that point. The growth term $F(\Psi)$ *not only increases density locally but also causes a faster spatial distribution in the population than that anticipated by diffusion alone* (Edelstein-Keshet). Skellam cited the example of the spread of muskrat population over central Europe over a period of 25 years, and showed that the equation (6) modelled muskrat diffusion fairly well.

The above parabolic partial differential equation (6) of diffusion was used by Fisher as early as 1937 to model movement of genes. Similar diffusion models have been reported in recent literature for studying population dispersal. A more detailed survey of diffusion models is available in Okubu. Two models describing the spread of small insects are worthy of mention here. Ludwig, Aronson and Weinberger modelled the movement of spruce budworm using the following equation,

$$(7) \qquad \frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} + \alpha P\left(1 - \frac{P}{K}\right) - \beta\frac{P^2}{H^2 + P^2}$$

where P is the population at period t and distance x from some initial point, and β is the rate of mortality due to predation. Other parameters have the usual interpretation. Notice that here movement was considered on a single spatial coordinate. Kareiva (1983) applied the following diffusion equation to observed data on spatial movements of several herbivore insects:

$$(8) \qquad \frac{\partial N(x,y,t)}{\partial t} = D\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2}\right)$$

where $N(x,y,t)$ is the population density at t and spatial coordinates (x,y), and D is the diffusion coefficient. He reported that this diffusion model was a fairly good approximation of the spread of various insects he studied. In all of the above models the diffusion coefficient was assumed to be constant across the entire space domain. Attempts have been made to vary the diffusion coefficient with time, weather, density, age and spatial position. Models of density-dependent and spatially varying diffusion were studied by Kareiva (1982), Gureny and Nisbet, and Shigesada.

The above summary indicates that diffusion equations of appropriate form have been applied with reasonable success to model dynamics of migratory populations. The models considered so far represent dynamics of unexploited resources. Suitable harvesting mortality needs to be added into above models to simulate temporal and spatial growth of exploited population. This will permit incorporation of spatial distribution and dispersal aspects into an economic optimization framework.

Optimal Harvesting of Biological Resources

The management of renewable biological resources caught the attention of economists with the pioneering and much cited work of Scott (Clark and Munro). He attempted to cast harvesting of fishery under the framework of a static capital problem. Until the development of optimal control theory in the early 1960s, economics of renewable resources continued to be studied under static terms. Subsequent to the classic work of Pontryagin *et al.*, economists soon realized the strength of control theory in handling complexity that arises with the capital problem due to the explicit consideration of time. Today the biological resource economics literature is rich and profound, and is much influenced by optimal control theory.[1] However, most harvesting models developed so far have considered only time aspects of capital under various economic, institutional and biological circumstances. Nevertheless, these models have added significantly to our understanding of the fundamental mechanics of modern capital-theoretic management of renewable resources.

Inclusion of structural attributes like age or spatial coordinates makes optimal harvesting models quickly complicated. This is likely the reason that bioeconomics researchers have stubbornly resisted modelling of multiple attributes. The limited number of studies that have investigated harvesting of structured populations are mostly from the disciplines of mathematics and ecology, and obviously, lack economic orientation. However, it is useful in the current study to review analytical techniques that have been developed for these types of control problem.

[1]The interested readers are advised to see Clark for a detailed discussion on application of the optimal control theory to harvesting of biological species.

When the population of a biological species is characterized by structural (space, age, size) and temporal dynamics, the entire population can best be considered as a system. Then the optimal control of such a system must be achieved simultaneously across the entire domain of a given attribute. The *distributed parameter control* theory is considered most appropriate for controlling these systems. Under this control theory, the dynamics of the subject population is characterized by a partial differential equation. The optimality conditions, similar to those of conventional optimal control theory, capable of yielding necessary optimal conditions are derived for specific problems (Brokate for age-structured population model; Bensoussan, Nisson and Tapiero for vintage capital model; Robson for vintage housing model). However, unlike Pontryagin's Maximum Principles, the optimality systems derived in these studies lack generalization, and they are very problem specific. See Lions for some generalizations without detailed existence results for the optimal systems. The complexity and nature of the analysis depends on the type of partial differential equation and the associated boundary conditions for each problem. Unfortunately, there are few studies of optimal harvesting of a structured population. More research, though limited in absolute sense, has been reported on age-structured population control (Brokate, Getz, Clark, Gurtin and Murphy) than on spatially distributed population control problem. Leung and Stojanovic is an example study which investigated return-maximizing harvesting of diffusive biological species. Because of its usefulness for our study, a brief discussion of this model seems appropriate.

Harvesting of Diffusive Population

In the optimal harvesting model for diffusive population, Leung and Stojanovic describe population dynamics of the species by

(9) $$\Delta u + u[(a(x) - f(x)) - bu] = 0 \qquad in\ \Omega$$

with no-flux boundary condition,

(10) $$\frac{\partial u}{\partial v} = 0$$

where u is the population density of the species, Δu refers to the Laplacian, which measures spatial rate of change in the density, i.e., diffusion, the $a(x)$ and $f(x)$ refer to spatially dependent intrinsic growth and harvesting rate, respectively ; and b denotes crowding effect which dampens the average annual growth rate. The Ω is the domain of the system in the spatial coordinates. Equation (9) indicates that at every point on the domain, the net rate of change in population density due to migration (Δu) is counter-balanced by local periodic growth, net of harvest mortality. The boundary condition in (10) constrains the migration (flux) to zero on the boundary. Mathematically, this system is called an elliptic partial differential equation.

The objective of the sole owner of this biological resource is to maximize a pay-off function - the difference between gross revenue and total harvest cost across entire region and over time. That is,

$$(11) \qquad J(f) = \int_{\Omega}[Kuf - Mf^2]dx$$

where K is the market price of the product and M the constant parameter of the quadratic total cost function in effort f. By rigorous mathematical proof, it can be shown that the optimal population stock for the above problem would be the solution of the following coupled system of partial differential equations in u and p (adjoint variable):

$$(12) \qquad \begin{aligned} &\Delta u + au - \left(b + \frac{K-p}{2M}\right)u^2 = 0 \\ & && \textit{in } \Omega \\ &\Delta p + (a-2bu)p + \frac{(K-p)^2 u}{2M} = 0 \\ &\textit{with no-flux boundary condition:} \\ &\frac{\partial u}{\partial \nu} = \frac{\partial p}{\partial \nu} = 0 && \textit{on } \partial\Omega \end{aligned}$$

The variable p has the usual interpretation of the costate variable encountered in an optimal control problem with ordinary differential equation (Haurie, Sethi and Hartl). This is the marginal benefit associated with the state variable u. Given the optimal solutions of u and p from the above system, the optimal harvesting for the entire domain was found to be

$$(13) \qquad f = \frac{u}{2M}(K-p) \qquad\qquad in\ \Omega$$

The solution of control and state variables were based on certain assumption on annual growth rate and control variable. It was also proved that biological stock will not go to extinction when optimal harvesting strategy in (13) is exercised.

CHAPTER III

OPTIMAL STRATEGY FOR BEAVER TRAPPING

In the first chapter, it was emphasized that the nuisance beaver populations have typical features of a (harmful) common property resource. As beavers are mobile, the decision of an individual land owner to control the damage will be affected by the population dynamics of the entire region. Since the individual property owners have no means to control the population of the entire watershed, they lack enough incentive to control the population on their respective parcels. Therefore, timber land owners suffering beaver damage losses may better achieve their common goal of nuisance control by an area-wide management response. Institution of a collective management agency to ensure area-wide trapping in the interest of the society is a policy compromise worthy of consideration. In this chapter, a bioeconomic model that integrates diffusive beaver population dynamics with a loss-minimizing trapping strategy of a public manager is suggested.

Development of the Bioeconomic Model

An Ecological Diffusion Model

Consider a large wildlife management region, perhaps of the order of hundreds of square miles, which is a potential habitat for a beaver population. This region is assumed to be closed to outside migration of beaver population. Let us assume that the management region is a continuous two-dimensional domain Ω with spatial axes x and y, i.e. $\Omega \subset \boldsymbol{R}^2$, with smooth boundary of the domain denoted by $\partial\Omega$ (see Figure 1).[1] Let Q represent a three dimensional real domain whose components are a spatial domain Ω and a time domain on the interval $0 \leq t \leq T$, i.e., $Q \subset \boldsymbol{R}^3$. Time T is some

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[1]In general, the problem we develop holds valid for n-dimensional domain also, i.e. $\Omega \subset \boldsymbol{R}^n$.

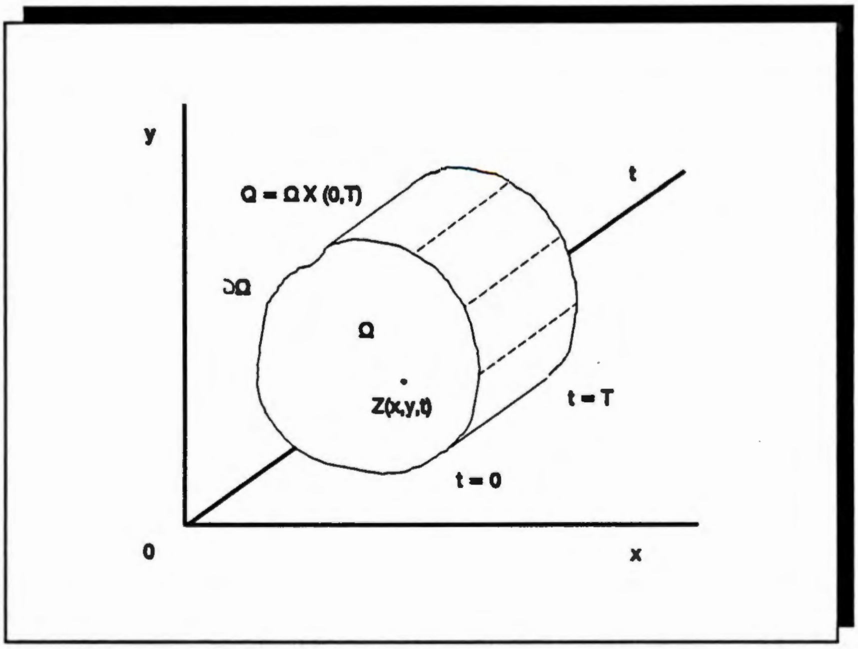

Figure 1. The representative beaver management region.

finite terminal time. Denote $Z(x,y,t)$ as density of beaver population [heads (hd)/square mile (sq mi)] at location (x,y) and time t. Consistent with other standard diffusive population models (e.g. Ludwig, Aronson and Weinberger), it is assumed that beavers encounter a hostile environment at the boundary of the domain $\partial\Omega$ and hence can't survive. As younger members of the beaver colony have the *innate tendency to leave the home colony* (Leege), it is further assumed that the beavers within the management region are mobile. Let $P(x,y,t)$ be the proportion of Z to be trapped at location (x,y) and time t. Then modifying the general framework of Skellam[2] to include human intervention by way of trapping, the dynamics of diffusing beaver population are formalized by the following parabolic partial differential equation:

[2]See section on *Diffusion and Random Walk* in the second chapter for detailed discussion on theoretical development of standard diffusion models.

$$(1) \qquad \frac{\partial Z}{\partial t} = \alpha\left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2}\right) + g(Z) - PZ \qquad in\ Q = \Omega \times (0,T)$$

with initial and side boundary conditions:

$$Z(x,y,0) = Z_0(x,y) \qquad on\ \Omega,\ t = 0$$

$$Z(x,y,t) = 0 \qquad on\ \partial\Omega \times (0,T)$$

where $\partial Z/\partial t$ is net time rate of growth of population density at a given point (x,y) and time t. The constant α is the diffusion coefficient [sq mi/year (yr)] and measures the rate of mean square dispersion. The expression $\alpha\,(\partial^2 Z/\partial x^2 + \partial^2 Z/\partial y^2)$ is the formal representation of net diffusion flux across the spatial coordinate (x,y). For the sake of simplicity and in keeping with many standard diffusion models, assumptions are made that beavers always move from higher density habitat to lower density habitat without backward migration. Though we can't totally reject the possibility that beavers may return to the territory of the home colony, this assumption seems to be relatively realistic. In the short-run beavers might undertake exploratory trips to a potential new colony site and return to the home territory (Bergerud and Miller). But in the long-run, the beaver population tends to move outward and can be expected to expand to a larger area. Furthermore, the rate of dispersion α is assumed constant over the entire region mainly for analytical simplicity. Varying this parameter could be handled with some modification of our analysis.

The function $g(Z)$ is the density-dependent annual biological productivity of beavers, in the absence of dispersion. For analytical simplicity, we ignore other demographic aspects such as age structure, time lags and sex differences. Following Lancia and Bishir, the logistic growth function was considered appropriate to capture the compensatory impacts of population density and available resources on the beaver population growth rate. Hence, we maintain

$$(2) \qquad g(Z) = a(x,y)Z - b(x,y)Z^2$$

where $a(x,y)$ is the maximum possible rate of net recruitment at location (x,y), and $b(x,y)$ the measure of density dependence at (x,y). Notice that these parameters can vary across beaver habitats reflecting the impact of environmental variability on the productivity of the beaver. This function has the following additional features:

(3) $\quad g(Z) > 0 \quad for \quad 0 < Z(x,y) < K(x,y), \qquad g(0) = g(K) = 0 \qquad g''(Z) < 0$

where $K(x,y) = [a(x,y)]/[b(x,y)]$ denotes the carrying capacity of the habitat at location (x,y).

The term PZ is annual trapping density (hd/sq mi/yr) at each location (x,y). Since trapping is the only means of beaver population control, it is assumed that beavers are harvested only through use of conibear traps. The level of trapping at given location can impact current and future productivity of beavers at that given location, as well as at surrounding locations. A physical constraint exists on the maximum trapping attainable, i.e. $0 \leq P(x,y,t) \leq P^{max}$.

The partial differential equation in (1), which is also called the state equation of motion, has an initial boundary condition, $Z(x,y,0) = Z_0(x,y)$. That is, we have definite knowledge of the initial distribution of the beaver population over the entire region. The initial density distribution function $Z_0(x,y)$ is a nonnegative function. The environmental hostility impact on the beaver population along the boundary of the region $\partial\Omega$ at any point in time is represented by the condition, $Z(x,y,t) = 0$ on $\partial\Omega \times (0,T)$. Introducing (2) in (1), we have the following system describing the dynamics of beaver population in the domain Q:

(4) $\quad Z_t = \alpha(Z_{xx}+Z_{yy}) + aZ - bZ^2 - PZ \qquad in\ Q = \Omega \times (0,T) \qquad$ *(state equation)*

with initial and side boundary conditions,

$$Z(x,y,0) = Z_0(x,y) \qquad on\ \Omega,\ t = 0$$

$$Z(x,y,t) = 0 \qquad on\ \partial\Omega \times (0,T)$$

with physical constraint on trapping,

$$0 \leq P \leq P^{max}$$

where, for notational simplicity, $Z_t = \frac{\partial Z}{\partial t}$, $Z_{xx} = \frac{\partial^2 Z}{\partial x^2}$, $Z_{yy} = \frac{\partial^2 Z}{\partial y^2}$, $a = a(x,y)$, and $b = b(x,y)$. Note that from comparison results for parabolic partial differential equations, solutions of (4) are positive (Protter and Weinberger). The functions a and b are bounded (finite) and positive. The term PZ reduces the magnitude of the solution of equation (4). Therefore, the solutions of the above equations are bounded independent of P^{max}. The actual proof of existence of the solution to the state equation is shown in Appendix I.2.

The Economic Optimization Framework

The economic motivation for the problem is based on the society's collective welfare maximization criterion. Suppose that the representative beaver management region is comprised of numerous identical decision units (i.e. timber land owners). Beavers are causing damage in each decision unit in several ways. However, the degree of their damage potential can vary across decision units depending on the type of food habitat and the nature of the affected timber land. In common, beaver damage in all the units increases with their number. Their damage impact can be contained only by reducing their population density. Because of their mobility, the dynamics of beaver population within a decision unit depends upon the total population level in the region as a whole. The total population of the region is exogenous to an individual decision maker, and hence he is aware of his inability to influence to a significant degree the total population. As a result, individual decision makers have little incentive to exercise beaver control on their parcel unless there exists an area-wide combined effort. They know that their individual effort would cause an environmental vacuum and, thus, attracting more beavers to their parcel. The presence of this *diffusion externality* explains why most land owners are currently unwilling to invest significantly in beaver control. Consequently, in the aggregate the decision units are likely to better achieve their common interests by placing the right of region-wide supervision in the hands of a single public decision maker.[3] In the present study, it was assumed that they have agreed for such collective action, and that the individual decision units abide by the supervisory and control regulations stipulated by the public manager for their respective parcels. Following Clark, the public manager is assumed *to view the* [beaver] *stock as a capital asset;* [and thus to manage according to] *the standard cost-benefit criterion of* [minimizing] *present values of net economic* [losses] (pp. 3, 4).

The public manager must deal with two important economic effects relative to this problem: (1) the direct and indirect loss to society through beaver damage to timber land, and (2) the monetary costs incurred in controlling beaver population. The need for decision making emerges because there exists a trade-off between these two economic effects.

Define $D(Z)$ [\$/sq mi/yr] as beaver-inflicted damage loss which is assumed to increase at an increasing rate with increase in Z, i.e., $D'(Z) > 0$ and $D''(Z) > 0$. Beaver density has a compounding effect on the dollar damage. This phenomenon can be represented by the following simple function:

$$(5) \qquad D(Z) = \frac{1}{2}\gamma Z^2$$

where $\gamma = \gamma(x,y)$ is damage parameter (\$ sq mi/yr/hd^2) at location (x,y).

In order to expose the cost structure of the beaver trapping industry, let us first analyze the trapping rate P which is already defined. The P can be viewed as the periodic *rate of production* from the beaver capital resource.[4] Each level of *production* (trapping) *rate* is associated with the specific quantity of a composite beaver control input that may include conibear trap, baits, vehicle time, fuel,

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[3]Similar economic motivation is developed by Feder and Regev in the context of multiple pest species control problem.

[4]The definition of P (as percent or proportionate of Z) is based on real life management practice. Under the quota system of management, quotas are generally defined as fixed number of beaver per colony. Given the colony size, it is easier to express the absolute quotas into per cent quotas and vice versa.

and trapper's labor.[5] Based on the study reported by Hill, we assume that the cost of trapping unit beaver uniformly increases with level of operation, i.e., *production* or trapping rate. As the desired level of operation P is increased, owner would be required to maintain a more than proportionate inventory stock and spend more time to search for beavers and to monitor the trapping operation. The result would be an increase in the unit trapping cost with an increase in P. Bringing additional resources into operation therefore would be possible only at the expense of increased unit cost. The unit cost, and hence total trapping costs, exhibit *adjustment externality*.[6] In other words, the owner would be penalized severely as he tries to adjust resources in order to attain higher levels of trapping operation. Let $C(P)$ (\$/hd) denote unit cost of trapping beavers. The $C(P)$ is assumed to be a linear function of P. Thus, we have

(6) $$C(P) = cP \qquad\qquad C'(P) = c > 0$$

where $c = c(x,y)$ is cost parameter (\$ yr/hd) at location (x,y). Notice that the total cost of trapping ($cPPZ = cP^2Z$) turns out to be quadratic in P, which is commonly seen in many economic problems.

The problem before the public manager is to select a spatiotemporal trapping strategy that minimizes the present value of the sum of beaver-inflicted damage to society and costs of trapping over the entire region (\$), taking into account the beaver population dynamics which explicitly includes intraregional beaver migration. That is, assuming a positive discount rate r reflecting time preference, the goal is to minimize the following total cost functional:

(7) $$J(P) = \int_0^T e^{-rt} \iint_\Omega \left(\frac{1}{2}\gamma Z^2 + cP^2Z\right) dx\,dy\; dt$$

subject to system (4).

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[5]To fix idea in terms of more popular notion prevailing in fishery economics literature (Clark), each level of composite beaver control input can be viewed as certain *effort* level.

[6]Smith discussed other types of recovery cost externalities in the context of production from natural resources. He considered a similar case where total cost of recovery increased at an increasing rate with rate of production.

Derivation of the Optimality System

To find an optimal control P^* for our problem in (7), we need to differentiate the cost functional $J(P)$ with respect to control P. The existence of such an optimal control will be treated in Appendix I.3. Since the state variable Z is contained in the objective functional, and the control P is in the state equation, it is evident that the choice of control P determines Z. We can show that Z depends on P in a differential way. Then we can characterize the optimal control P^* in terms of the unique solution of the optimality system, which consists of the state equation coupled with an adjoint equation.

Suppose that Z^* is the optimal state variable associated with an optimal control P^*. Consider another modified admissible control level $P^\epsilon = P^* + \epsilon h$ with associated state variable Z^ϵ, where $h = h(x,y,t)$ is a variation function and ϵ is a constant parameter. Clearly, as $\epsilon \to 0$, the modified control tends to optimal control P^*. Further, $Z^\epsilon(x,y,0) = Z^*(x,y,0) = Z_0(x,y)$. It can be shown that the solution of state equation Z is differentiable with respect to control P.[7] Mathematically,

(8) $$\lim_{\epsilon \to 0} \frac{Z(P^* + \epsilon h) - Z(P^*)}{\epsilon} = \psi$$

In other words,

(9) $$\lim_{\epsilon \to 0} \frac{Z^\epsilon - Z^*}{\epsilon} = \left.\frac{\partial Z}{\partial P}\right|_{P^*} = \psi$$

where ψ is the variable that measures differential dependence of the state variable on the control. Notice that since $Z^\epsilon(x,y,0) - Z^*(x,y,0) = 0$, $\psi(x,y,0) = 0$. Call ψ hereafter state-differential variable.

[7]See Appendix I.3 for the proof of this result.

By construction both Z^ϵ and Z^* with their respective controls $P^\epsilon = P^* + \epsilon h$ and P^* are solutions of the state equations (4). Hence we have

$$(10) \qquad Z_t^\epsilon = \alpha(Z_{xx}^\epsilon + Z_{yy}^\epsilon) + aZ^\epsilon - b(Z^\epsilon)^2 - (P^* + \epsilon h)Z^\epsilon \qquad \textit{in } Q = \Omega \times (0,T)$$

with initial and side boundary conditions:

$$Z^\epsilon(x,y,0) = Z_0(x,y) \qquad \textit{on } \Omega,\ t = 0$$

$$Z^\epsilon(x,y,t) = 0 \qquad \textit{on } \partial\Omega \times (0,T)$$

and

$$(11) \qquad Z_t^* = \alpha(Z_{xx}^* + Z_{yy}^*) + aZ^* - b(Z^*)^2 - P^*Z^* \qquad \textit{in } Q = \Omega \times (0,T)$$

with initial and side boundary conditions:

$$Z^*(x,y,0) = Z_0(x,y) \qquad \textit{on } \Omega,\ t = 0$$

$$Z^*(x,y,t) = 0 \qquad \textit{on } \partial\Omega \times (0,T)$$

Subtracting (11) from (10) and dividing by parameter ϵ, we obtain

$$(12) \qquad \left(\frac{Z^\epsilon - Z^*}{\epsilon}\right)_t = \alpha\left[\left(\frac{Z^\epsilon - Z^*}{\epsilon}\right)_{xx} + \left(\frac{Z^\epsilon - Z^*}{\epsilon}\right)_{yy}\right] + a\left(\frac{Z^\epsilon - Z^*}{\epsilon}\right) - b\left[\frac{(Z^\epsilon)^2 - (Z^*)^2}{\epsilon}\right] - P^*\left(\frac{Z^\epsilon - Z^*}{\epsilon}\right) - hZ^\epsilon$$

Taking the limit of the above equation as $\epsilon \to 0$ (i.e., as the parameter gets small), and using the differentiability result in (9), we obtain the following partial differential equation which state-differential variable ψ must satisfy:

$$(13) \qquad \psi_t = \alpha(\psi_{xx} + \psi_{yy}) + a\psi - 2bZ^*\psi - P^*\psi - hZ^* \qquad \textit{in } Q = \Omega \times (0,T)$$

with initial and side boundary conditions:

$$\psi(x,y,0) = 0 \qquad \textit{on } \Omega,\ t = 0$$

$$\psi(x,y,t) = 0 \qquad \textit{on } \partial\Omega \times (0,T)$$

Proposition: *For an optimal control P^* and corresponding solution $Z^* = Z(P^*)$, there exists a function $\lambda(x,y,t)$ satisfying the adjoint equation:*

$$(14) \qquad -\lambda_t = \alpha(\lambda_{xx} + \lambda_{yy}) + a\lambda - 2bZ^*\lambda - r\lambda - P^*\lambda + \gamma Z^* + c(P^*)^2 \qquad in\ \Omega \times (0,T)$$

with terminal and side boundary conditions:

$$\lambda(x,y,T) = 0 \qquad on\ \Omega,\ t = T$$

$$\lambda(x,y,t) = 0 \qquad on\ \partial\Omega \times (0,T)$$

and

$$(15) \qquad P^* = \frac{\bar{\lambda}}{2c}$$

where $\lambda(x,y,T) = 0$ is the usual transversality condition.

Proof: Since, by definition, P^* yields the minimum value of the social loss functional $J(P)$, any control other than P^* must be inefficient. That is, $0 \leq J(P^* + \epsilon h) - J(P^*)$. Given this, we can conveniently state that the derivative of the loss functional with respect to the control evaluated at P^* must be greater than or equal to zero. Symbolically,

$$(16) \qquad 0 \leq \lim_{\epsilon \to 0} \frac{J(P^* + \epsilon h) - J(P^*)}{\epsilon}$$

Substituting for J from (7), using the differentiability result obtained in (9), and simplifying the results, we get

$$(17)\qquad 0 \le \lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_0^T\!\!\iint_\Omega e^{-rt}\left\{\left[\frac{1}{2}\gamma(Z^\epsilon)^2 + c(P^* + \epsilon h)^2 Z^\epsilon\right] - \left[\frac{1}{2}\gamma(Z^*)^2 + c(P^*)^2 Z^*\right]\right\}dx\,dy\,dt$$

$$= \lim_{\epsilon\to 0} \int_0^T\!\!\iint_\Omega e^{-rt}\left\{\frac{1}{2}\gamma\left[\left(\frac{Z^\epsilon - Z^*}{\epsilon}\right)(Z^\epsilon + Z^*)\right] + c\left[(P^*)^2\left(\frac{Z^\epsilon - Z^*}{\epsilon}\right) + (2P^*h + \epsilon h^2)Z^\epsilon\right]\right\}dx\,dy\,dt$$

$$= \int_0^T\!\!\iint_\Omega e^{-rt}\{[\gamma Z^* + c(P^*)^2]\psi + 2cP^*hZ^*\}dx\,dy\,dt$$

Rearrange the adjoint equation in (14) as

$$(18)\qquad -\lambda_t - \alpha(\lambda_{xx} + \lambda_{yy}) - a\lambda + 2bZ^*\lambda + r\lambda + P^*\lambda = \gamma Z^* + c(P^*)^2$$

and substitute the result into inequality (17) to obtain

$$(19)\qquad 0 \le \int_0^T\!\!\iint_\Omega e^{-rt}\{[-\lambda_t - \alpha(\lambda_{xx} + \lambda_{yy}) - a\lambda + 2bZ^*\lambda + r\lambda + P^*\lambda]\psi + 2cP^*hZ^*\}dx\,dy\,dt$$

Integrating the right hand side of (19) by parts, above inequality can further be reduced to

$$(20)\qquad 0 \le \int_0^T\!\!\iint_\Omega e^{-rt}\{[\psi_t - \alpha(\psi_{xx} + \psi_{yy}) - a\psi + 2bZ^*\psi + P^*\psi]\lambda + 2cP^*hZ^*\}dx\,dy\,dt$$

Rewriting the partial differential equation in (13) as

$$(21)\qquad \psi_t - \alpha(\psi_{xx} + \psi_{yy}) - a\psi + 2bZ^*\psi + P^*\psi = -hZ^*$$

and substituting the result into inequality (20), we obtain

$$(22)\qquad 0 \le \int_0^T\!\!\iint_\Omega e^{-rt}h(-\lambda Z^* + 2cP^*Z^*)dx\,dy\,dt.$$

Inequality (22) is used to characterize the optimal control. Consider the following three cases:

Case 1: $P^*(x,y,t) = 0$.

Along the optimal path where $P^*(x,y,t) = 0$, we can choose only nonnegative variations h which would ensure $P^\epsilon = P^* + \epsilon h = \epsilon h \geq 0$. If $h \geq 0$, inequality (22) holds if and only if

(23) $\quad - \lambda Z^* + 2cP^*Z^* = Z^*(-\lambda + 2cP^*) \geq 0.$

This implies that $2cP^* = 0 \geq \lambda$. Since λ is a solution of the partial differential equation in (18) which has a positive source term $\gamma Z^* + c(P^*)^2$ on the right hand side, λ must be nonnegative (Protter and Weinberger). For similar reason, $Z^* > 0$ for all (x,y,t) but the boundary $\partial\Omega$. Therefore, the fact that $2cP^* = 0 \geq \lambda$. is a contradiction unless $\lambda = 0$. Thus, we conclude that whenever $P^*(x,y,t) = 0$, $\lambda(x,y,t) = 0$.

Case 2: $0 < P^*(x,y,t) < P^{max}$.

On the above control set, we can choose variation function h with arbitrary sign. When h is arbitrary, only way the inequality (22) holds is by requiring

(24) $\quad - \lambda Z^* + 2cP^*Z^* = Z^*(-\lambda + 2cP^*) = 0.$

Since $Z^* > 0$, this implies that on this set $P^* = \dfrac{\lambda}{2c}$.

Case 3: $P^*(x,y,t) = P^{max}$.

We can choose only nonpositive h on this set, to ensure $P^\epsilon = P^* + \epsilon h = P^{max} + \epsilon h \leq P^{max}$. If $h \leq 0$, the inequality (22) holds if and only if

(25) $\quad -\lambda Z^* + 2cP^*Z^* = Z^*(-\lambda + 2cP^*) \le 0$

This implies that on this control set $P^* = P^{max} \le \dfrac{\lambda}{2c}$.

Putting the three cases together, the optimal control should be

(26) $$P^* = \min\left(P^{max}, \frac{\lambda}{2c}\right)$$

It is easier to obtain a simpler form of the solution for P^* in terms of only the adjoint variable if we go through the following reasoning. Conceptually we may select P^{max} large such that $\lambda/2c$ is always less than P^{max}. Then what we need to show is that λ is always finite and bounded independent of the bounded P^{max}.

<u>Claim</u>: There exists $M > 0$ independent of P^{max} such that $\lambda \le M$ on Q.

Since the bound on Z^* is independent of P^{max}, all the terms not involving P^* in the adjoint equation (14) are finite and do not force the solution λ to be unbounded. Even the terms involving P^* pull the solution down, since they are negative, i.e., $P^*(cP^* - \lambda) \le P^*[c(\lambda/2c) - \lambda] = -P^*\lambda/2 \le 0$. Hence, the solution of the adjoint equation is bounded from above independent of P^{max}.

Now we can choose P^{max} so large that

(27) $$\frac{\lambda}{2c} \le \frac{M}{2c} < P^{max}$$

Making use of the result (27), the optimal control (26) can be conveniently expressed as

(28) $$P^* = \frac{\lambda}{2c}$$

Considering the above relationship between an optimal control and the associated adjoint variable, we now consider the following optimality system (OS):

$$(29) \qquad Z_t^* = \alpha(Z_{xx}^* + Z_{yy}^*) + aZ^* - b(Z^*)^2 - \frac{Z^*\lambda^*}{2c} \qquad in\ Q = \Omega \times (0,T)$$

$$-\lambda_t^* = \alpha(\lambda_{xx}^* + \lambda_{yy}^*) + a\lambda^* - 2bZ^*\lambda^* - r\lambda^* - \frac{\lambda^2}{4c} + \gamma Z^* \qquad in\ Q = \Omega \times (0,T)$$

with initial, terminal, and side boundary conditions:

$$Z(x,y,0) = Z_0(x,y) \qquad on\ \Omega,\ t = 0$$

$$\lambda(x,y,T) = 0 \qquad on\ \Omega,\ t = T$$

$$Z(x,y,t) = \lambda(x,y,t) = 0 \qquad on\ \partial\Omega \times (0,T)$$

Notice that the above OS is obtained by substituting the optimal control P^* from (28) into the state equation (4) and the adjoint equation (14). The proof of existence and uniqueness of the solution to OS is shown in Appendix I.4, which gives the representation of the unique optimal control. The optimal solutions of state and costate variables are the solutions of the above system of two coupled nonlinear partial differential equations. As mentioned before, the optimal beaver population density Z^* will remain positive at all locations (x,y) and time t. This fulfills our goal that the social loss-minimizing optimal beaver trapping strategy should not call for a complete eradication of beaver population from the South.

Economic Interpretations of the Necessary Conditions

In order to gain additional insight into the OS and the necessary conditions that yielded the optimal control, an attempt to provide economic interpretations for various expressions is made here. Following Hartl and Sethi, and Robson the adjoint variable $\lambda(x,y,t)$ is assumed to have usual interpretation of the marginal shadow price of capital, i.e., marginal value of beavers at location (x,y)

and time t. In other words, λ measures the effect of an incremental change in the optimal beaver density $Z^*(x,y,t)$ at a given location and time on the *future* social welfare of *all the locations* in the region.

Given the interpretation of the adjoint variable, the economic meaning of the conditions obtained in (23), (24) and (25) which are used to characterize the optimal control can be sought. Let us examine each of the terms in the above mentioned equations:

$2cP^*Z^*$ — (\$/sq mi) is the derivative of the total cost of trapping $[c(P^*)^2Z^*]$ on a unit area with respect to the rate of trapping P evaluated at optimal P^*. Hence, this is the marginal cost (*MC*) of investment in the beaver trapping activity.

λZ^* — (\$/sq mi) is the derivative of the total potential *nuisance* value (λP^*Z^*) of beavers trapped from a unit area (P^*Z^*) evaluated at its shadow price λ with respect to the rate of trapping P. Intuitively, this is the marginal savings in beaver damage loss to society as a result of the beaver trapping activity, i.e., the marginal rate of return (*MR*) generated by this activity.

Now going back to the necessary condition in (23), we know that the inequality in (23) holds only when $\lambda = 0$, that is, when the marginal potential *nuisance* value of beaver (λZ^*) is zero. We have also seen that in *case 1* corresponding to (23), the optimal control must be zero. This amounts to say that when beavers are not a problem, it is not economically wise to trap them.

Equation (24) represents a situation (*case 2*) where the marginal return from trapping beaver equals the marginal cost of trapping, i.e., $\lambda Z^* = 2cP^*Z^*$. This condition yielded the closed form optimal control $P^* = \lambda/2c$. Finally, the inequality (25) represents a situation (*case 3*) where marginal return is greater than the marginal cost. Under this situation, the public manager has to exercise the maximum control. That is, when the marginal nuisance value of beavers trapped is much higher than the incremental cost of trapping, the public manager can not spare any effort.

Synthesizing the above three situations, we can write the optimal control in terms of following simplified expression:

$$(30) \qquad P^*(x,y,t) = \begin{cases} P^{max} & if \;\; MR > MC \\ \dfrac{\lambda^*}{2C} & if \;\; MR = MC \\ 0 & if \;\; MR = 0 \end{cases}$$

The above expression looks similar to the *synthesized* control obtained for the standard fishery problems discussed by Clark.

Adjoint Equation

The economic meaning of the adjoint equation (14) becomes more lucid if we go through the following routine. Form a current-value Hamiltonian from the integrand of the social loss functional (7) and the state equation (4) as

$$(31) \qquad H^{cv}(Z,P,x,y,t;\lambda) = F(Z,P,x,y,t) + \lambda(x,y,t)[Z_t(Z,P,x,y,t) - \alpha(Z_{xx}+Z_{yy})]$$

where $F(Z,P,x,y,t) = \frac{1}{2}\gamma Z^2 + cP^2Z$. Notice that the above Hamiltonian represents the instantaneous flow of social loss F at a given location plus the future social loss $\lambda[Z_t - \alpha(Z_{xx}+Z_{yy})]$ inflicted by the current recruitees at location (x,y), which is annual increment in the beaver density net of migrants $[Z_t - \alpha(Z_{xx}+Z_{yy})]$. Differentiating the Hamiltonian with respect to the state variable and using equation (4), we obtain

$$(32) \qquad \frac{\partial H^{cv}}{\partial Z} = \frac{\partial F}{\partial Z} + \lambda \frac{\partial}{\partial Z}\left[Z_t - \alpha(Z_{xx}+Z_{yy})\right]$$

$$= \frac{\partial F}{\partial Z} + \lambda \frac{\partial}{\partial Z}\left[aZ - bZ^2 - PZ\right]$$

$$= [\gamma Z^* + c(P^*)^2] + [a\lambda - 2bZ^*\lambda - P^*\lambda]$$

Now, comparing the above result with the adjoint equation in (14), it is readily apparent that

$$(33) \qquad -[\lambda_t + \alpha(\lambda_{xx}+\lambda_{yy})] + r\lambda = \frac{\partial H^{cv}}{\partial Z} = \frac{\partial F}{\partial Z} + \lambda \frac{\partial}{\partial Z}\left[Z_t - \alpha(Z_{xx}+Z_{yy})\right]$$

Now the adjoint equation can be easily interpreted. The $\lambda_t + \alpha(\lambda_{xx}+\lambda_{yy})$ is the change in the marginal valuation of beaver stock due to passage of time, including the impact of net diffusion on the value of beaver stock [i.e., $\alpha(\lambda_{xx}+\lambda_{yy})$]. In the case of conventional capital stock, following Dorfman, this measures the change in the shadow price of the capital stock. But in the case of the beaver population which is a capital asset generating negative return, this has to be carefully interpreted. As discussed before, λ in some sense is the potential marginal reduction in social loss accruing over the entire spatiotemporal domain as a result of trapping a unit beaver at the present time. The dollar value of this potential marginal reduction in loss is likely to depreciate over time. Therefore, the left hand side of the equation (33) represents the discounted shadow price of beaver ($r\lambda$), net of current depreciation in value $\{-[\lambda_t + \alpha(\lambda_{xx}+\lambda_{yy})]\}$. Thus, the left hand side represents the instantaneous *capital dividend rate* (using the terminology of Haurie *et.al*). The terms on the right hand side represent the marginal rate of flow of current and future social loss as a result of a change in the beaver density which can be viewed as the *capital loss rate*. The condition (33) and hence the adjoint equation (14) asserts that the *marginal dividend rate* must equate with the *marginal capital loss rate*, along the optimal path, over the entire spatial and temporal domain.

The meaning of the transversality condition $\lambda(x,y,T) = 0$ should be intuitively obvious. Since we have not assigned any salvage value to beaver stock at T *a priori*, the marginal value $\lambda(T)$ must be zero.

CHAPTER IV

EMPIRICAL SIMULATION OF THE BEAVER TRAPPING MODEL

The bioeconomic beaver trapping model is illustrated in this chapter by numerically deriving an area-wide optimal trapping strategy for a real life problem. Unfortunately, the biological and economic information required by the model was not readily available for the Southeast. However, in order to demonstrate the analytical and practical utility of the model, the same is applied to the beaver population data collected for the Wildlife Management Region 7 of the New York State Department of Environmental Conservation (NYSDEC). This region is selected not because beavers are causing severe problem in the area, but because the most comprehensive biological and economic data required by the model are available for this one region. Since the purpose of this analysis is primarily exploratory, the results obtained from this should be used with caution. The results obtained here may lack reliability to the extent that certain assumptions made in the model may not be exactly valid for the study region. For instance, the assumption of *environmental hostility* at the boundary of this region was imposed. This may be debatable since some parts of the Region 7 are contiguous parts of the neighboring potential beaver habitats which sustain beaver population.

Before getting into the actual exercise, it may be recalled that the optimality system (OS) obtained in the equation (29) of the chapter III is a system of coupled nonlinear parabolic partial differential equations. There is no way to obtain closed-form solutions of this system. A scheme that numerically solves the system is needed. In this Chapter a numerical simulation procedure for solving the optimality system is designed. Then the numerical model is applied to the beaver population data available from the NYSDEC and other sources.

Numerical Solution of the Optimality System

There are many numerical methods developed to derive approximate solutions to partial differential equations (Hall and Porsching; Ames). One of the popular techniques that has been widely used in natural and physical sciences is the *finite difference method*. The basic approach in this method is to approximate a continuous variable/domain by finite number of discrete values/points. The derivatives of any variables are replaced by appropriate difference quotients (eg. first derivative by a first difference quotient).

Finite Differences: A Simple Case[1]

Consider a continuous function $u(x,y)$ on the square domain $\Omega \subset R^2$ which is twice differentiable with respect to x as well as y. As shown in Figure 2, the domain Ω is discretized into uniform mesh of points, $x_0, x_1, x_2 \cdots x_i \cdots x_M$ along x axis and $y_0, y_1, y_2 \cdots y_j \cdots y_N$ along y axis with mesh size Δx and Δy, respectively. Therefore, in general $x_{i+1} = x_i + \Delta x$ and $y_{j+1} = y_j + \Delta y$. The function value $u(x,y)$ near location (x_i,y_j) is approximated by $u(x_i,y_j)$. Using Taylor's series expansion for $u(x_{i+1},y_j)$ about (x_i,y_j) in the variable x, it follows:

$$(1) \qquad u(x_{i+1},y_j) = u(x_i,y_j) + \Delta x \frac{\partial u}{\partial x}(x_i,y_j) + \frac{(\Delta x)^2}{2!} \cdot \frac{\partial^2 u}{\partial x^2}(x_i,y_j) + O[(\Delta x)^3]$$

Divide both sides by Δx to get

$$(2) \qquad \frac{u(x_{i+1},y_j)}{\Delta x} = \frac{u(x_i,y_j)}{\Delta x} + \frac{\partial u}{\partial x}(x_i,y_j) + O(\Delta x)$$

where $O(\Delta x)$ represents all the higher order terms. Simplifying (2) further, we have

$$(3) \qquad \frac{\partial u}{\partial x}(x_i,y_j) = \frac{u(x_{i+1},y_j) - u(x_i,y_j)}{\Delta x} + O(\Delta x)$$

Dropping the higher order terms yields the following approximation:

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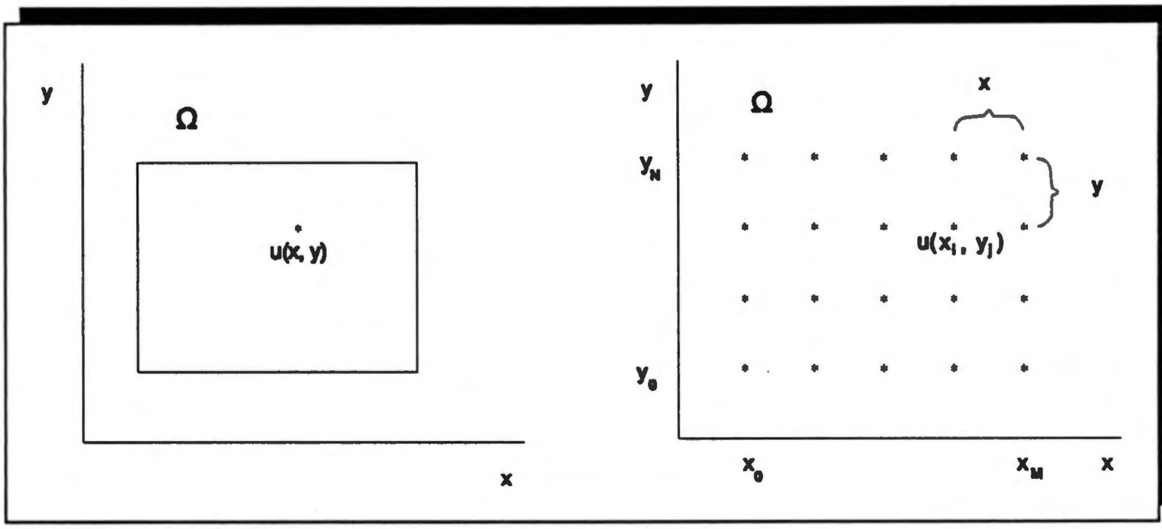
[1]This section is adopted from Ames.



(a) (b)

Figure 2. The discretization of a continuous domain: (a) two dimensional continuous domain, (b) two dimensional discrete domain.

(4) $$\frac{\partial u}{\partial x}(x_i,y_j) \approx \frac{u(x_{i+1},y_j) - u(x_i,y_j)}{\Delta x}$$ *forward difference*

The right hand side in (4) is known as *forward difference* approximation of $\partial u/\partial x$.

Using Taylor's approximation, $\partial u/\partial x$ can also be approximated by *backward* and *centered differences* as under:

(5) $$\frac{\partial u}{\partial x}(x_i,y_j) \approx \frac{u(x_i,y_j) - u(x_{i-1},y_j)}{\Delta x}$$ *backward difference*

(6) $$\frac{\partial u}{\partial x}(x_i,y_j) \approx \frac{u(x_{i+1},y_j) - u(x_{i-1},y_j)}{2\Delta x}$$ *centered difference*

Similarly, the second order partial of u with respect to x can be approximated by

(7) $$\frac{\partial^2 u}{\partial x^2}(x_i,y_j) \approx \frac{u(x_{i+1},y_j) - 2u(x_i,y_j) + u(x_{i-1},y_j)}{(\Delta x)^2}$$ *second difference*

Similar expressions can be obtained for partial derivatives of u with respect to y.

Discretization of the Optimality System

On the above lines, the original problem of the OS can be discretized. For the sake of analytical simplicity, it was assumed that the spatial domain Ω was a rectangle. Subdivide the spatiotemporal domain $Q = \Omega \times (0,T)$ by uniform mesh points denoted by x_i, y_j and t where $i = 0, 1, 2, \cdots M$, $j = 0, 1, 2, \cdots N$ and $t = 0, 1, 2, \cdots T$ with mesh size Δx, Δy and Δt, respectively along x, y and t axes (see Figure 3). The boundary of the domain lies along $i = 0, M$; $j = 0, N$ and $t = 0, T$. Let $Z^t_{i,j}$ and $\lambda^t_{i,j}$ respectively denote the values of the beaver density and the adjoint variable at location (x_i, y_j) and time t. These are the discrete approximations of the values of the continuous functions $Z(x,y,t)$ and $\lambda(x,y,t)$, respectively. Notice that the domain Q is now discretized into $(M+1)(N+1)(T+1)$ number of grid points. Now the state and the adjoint partial differential equations at each mesh point (x_i, y_j, t) can be approximated by replacing the derivatives by suitable finite difference approximations developed in (4), (5), (6) and (7). Thus, the following approximations were chosen.

(8) $$Z_t = \frac{Z^{t+1}_{i,j} - Z^t_{i,j}}{\Delta t} \qquad -\lambda_t = \frac{\lambda^{t-1}_{i,j} - \lambda^t_{i,j}}{\Delta t}$$

$$Z_{xx} = \frac{Z^t_{i+1,j} - 2Z^t_{i,j} + Z^t_{i-1,j}}{(\Delta x)^2} \qquad \lambda_{xx} = \frac{\lambda^t_{i+1,j} - 2\lambda^t_{i,j} + \lambda^t_{i-1,j}}{(\Delta x)^2}$$

$$Z_{yy} = \frac{Z^t_{i,j+1} - 2Z^t_{i,j} + Z^t_{i,j-1}}{(\Delta y)^2} \qquad \lambda_{yy} = \frac{\lambda^t_{i,j+1} - 2\lambda^t_{i,j} + \lambda^t_{i,j-1}}{(\Delta y)^2}$$

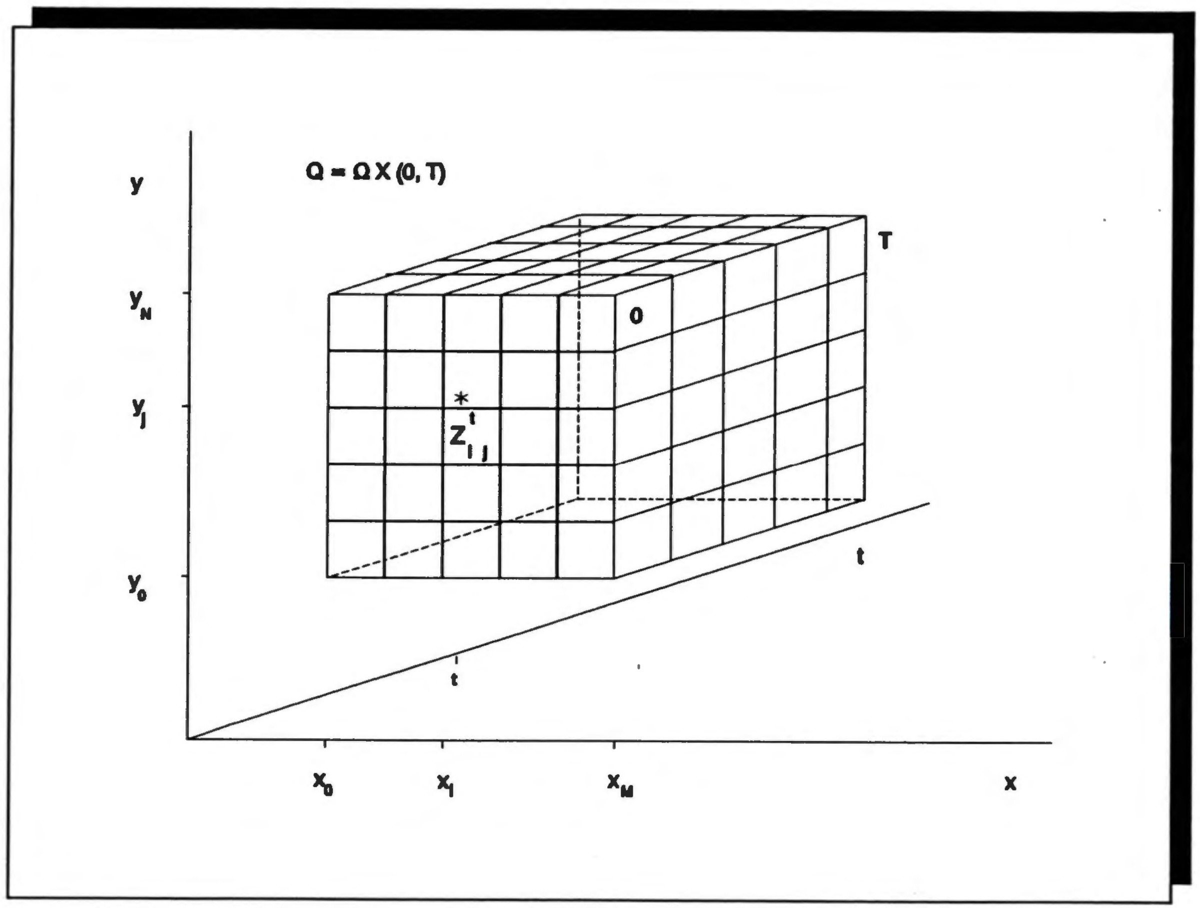


Figure 3. The representation of three dimensional discrete domain.

Notice that backward difference was used for the time derivative of λ. Because, two partial differential equations in OS have opposite orientations. While the state equation moves forward from the initial condition, the adjoint equation moves backward from the terminal condition. In other words, the solution for λ variable evolves through backward steps beginning terminal point. Hence, a backward difference approximation was selected for $-\lambda_t$.

Substituting (8) into the OS, we get

$$(9)\qquad \frac{Z_{i,j}^{t+1} - Z_{i,j}^{t}}{\Delta t} = \alpha\left[\frac{1}{(\Delta x)^2}\left(Z_{i+1,j}^{t} - 2Z_{i,j}^{t} + Z_{i-1,j}^{t}\right) + \frac{1}{(\Delta y)^2}\left(Z_{i,j+1}^{t} - 2Z_{i,j}^{t} + Z_{i,j-1}^{t}\right)\right]$$

$$+ \left(a_{i,j} - b_{i,j}Z_{i,j}^{t} - \frac{\lambda_{i,j}^{t}}{2c_{i,j}}\right)Z_{i,j}^{t} \qquad \textit{for all } i, j, \textit{ and } t$$

with initial and side boundary conditions:

$Z_{i,j}^{0} = Z(x_i, y_j, 0)$ *for all i and j*

$Z_{i,j}^{t} = 0$ *for $i = 0$ and M, $j = 0$ and N; all t*

$$\frac{\lambda_{i,j}^{t-1} - \lambda_{i,j}^{t}}{\Delta t} = \alpha\left[\frac{1}{(\Delta x)^2}\left(\lambda_{i+1,j}^{t} - 2\lambda_{i,j}^{t} + \lambda_{i-1,j}^{t}\right) + \frac{1}{(\Delta y)^2}\left(\lambda_{i,j+1}^{t} - 2\lambda_{i,j}^{t} + \lambda_{i,j-1}^{t}\right)\right]$$

$$+ \left(a_{i,j} - 2b_{i,j}Z_{i,j}^{t} - r\frac{\lambda_{i,j}^{t}}{4c_{i,j}}\right)\lambda_{i,j}^{t} + \gamma_{i,j}Z_{i,j}^{t} \qquad \textit{for all } i, j, \textit{ and } t$$

with terminal and side boundary conditions:

$\lambda_{i,j}^{T} = 0$ *for all i and j*

$\lambda_{i,j}^{t} = 0$ *for $i = 0$ and M, $j = 0$ and N; all t*

Observe that various model parameter values $a(x,y)$, $b(x,y)$, $\gamma(x,y)$ and $c(x,y)$ are also approximated by their respective discrete values.

Simplifying the system of equations in (9) further, the following system of coupled nonlinear difference equations in the state and the adjoint variables can be derived along with the respective boundary conditions mentioned above:

$$
(10) \qquad Z_{i,j}^{t+1} = \Delta t\ \alpha\left[\frac{1}{(\Delta x)^2}\left(Z_{i+1,j}^{t} - 2Z_{i,j}^{t} + Z_{i-1,j}^{t}\right) + \frac{1}{(\Delta y)^2}\left(Z_{i,j+1}^{t} - 2Z_{i,j}^{t} + Z_{i,j-1}^{t}\right)\right]
$$

$$
+ \Delta t\left(a_{i,j} - b_{i,j}Z_{i,j}^{t} - \frac{\lambda_{i,j}^{t}}{2c_{i,j}}\right)Z_{i,j}^{t} + Z_{i,j}^{t} \qquad \textit{for all } i,\ j, \textit{ and } t
$$

$$
\lambda_{i,j}^{t-1} = \Delta t\ \alpha\left[\frac{1}{(\Delta x)^2}\left(\lambda_{i+1,j}^{t} - 2\lambda_{i,j}^{t} + \lambda_{i-1,j}^{t}\right) + \frac{1}{(\Delta y)^2}\left(\lambda_{i,j+1}^{t} - 2\lambda_{i,j}^{t} + \lambda_{i,j-1}^{t}\right)\right]
$$

$$
+ \Delta t\left[\left(a_{i,j} - 2b_{i,j}Z_{i,j}^{t} - r\frac{\lambda_{i,j}^{t}}{4c_{i,j}}\right)\lambda_{i,j}^{t} + \gamma_{i,j}Z_{i,j}^{t}\right] + \lambda_{i,j}^{t} \qquad \textit{for all } i,\ j, \textit{ and } t
$$

The above system constitutes $(M+1)(N+1)(T+1)$ number of difference equations for each of the $Z_{i,j}^{t}$ and $\lambda_{i,j}^{t}$ variables.

The solution of the simultaneous system in (10) is characterized as follows. If the above system of equations for every i, j and t were not coupled, given the values of $Z_{i,j}^{t}$ at all locations for the initial time $t = 0$ known, one could advance the solutions of $Z_{i,j}^{t}$ at $t = 1$, and repeatedly up to $t = T$. Similarly, knowing the terminal values of $\lambda_{i,j}^{T}$, the solutions of the adjoint variable for all the time steps could be derived by advancing backward from $t = T-1$ to $t = 0$. Since the state and adjoint difference equations are coupled, the solutions for $Z_{i,j}^{t+1}$ at any t can not be found unless the corresponding solutions of the adjoint variable $\lambda_{i,j}^{t}$ are known. Similar problem exists in solving for $\lambda_{i,j}^{t-1}$. In order to overcome this problem, the following iteration scheme was followed.[2]

1. Assume some arbitrary values for $\lambda_{i,j}^{t}$ at all the grid points for time $t = 0$ to T ($\lambda_{i,j}^{T} = 0$ by transversality condition).

2. Impose the initial boundary conditions of the state equation, use information on $\lambda_{i,j}^{0}$ from the step (1), and generate the values of $Z_{i,j}^{1}$ for all the locations (x_i, y_j) using difference equations in (10). Of course, the solutions at the side boundary are zero, and hence, need not be computed.

3. Solve successively for $Z_{i,j}^{2} \cdots Z_{i,j}^{T}$ using solutions in the step (2) and information in the step (1).

4. Solve iteratively for $\lambda_{i,j}^{T-1}$ down to $\lambda_{i,j}^{0}$ using the solutions of $Z_{i,j}^{t}$ generated in steps (2) and (3) and terminal condition of the adjoint variable $\lambda_{i,j}^{T} = 0$. Replace the initial arbitrary values of $\lambda_{i,j}^{t}$ in the step (1) by this newly generated array of $\lambda_{i,j}^{t}$.

5. Repeat the steps (2) and (3) to obtain a new array of solutions of $Z_{i,j}^{t}$ using the solutions of $\lambda_{i,j}^{t}$ generated in the step (4),

6. Compare the arrays of the solutions of $Z_{i,j}^{t}$ obtained in the step (3) and (5) and that of $\lambda_{i,j}^{t}$ in the steps (1) and (4). If the difference in the weighted averages of these values at each time t from two successive iterations is greater than certain tolerance limit, repeat the steps (2) to (5).

7. Continue the above iteration until the solutions of $Z_{i,j}^{t}$ and $\lambda_{i,j}^{t}$ converge between iterations.

In order to obtain convergence in the solutions, a *stability condition* for the above problem is needed (Hall and Porsching). Rewrite the state difference equation in (10) to obtain

[2]See Hackbusch for a development of numerical methods for solving parabolic equations with opposite orientations.

(11) $$Z_{i,j}^{t+1} = \left[1 - \frac{2\alpha\Delta t}{(\Delta x)^2} - \frac{2\alpha\Delta t}{(\Delta y)^2}\right] Z_{i,j}^{t} + \left[\frac{\alpha\Delta t}{(\Delta x)^2}\right] Z_{i+1,j}^{t} + \left[\frac{\alpha\Delta t}{(\Delta x)^2}\right] Z_{i-1,j}^{t}$$

$$+ \left[\frac{\alpha\Delta t}{(\Delta y)^2}\right] Z_{i,j+1}^{t} + \left[\frac{\alpha\Delta t}{(\Delta y)^2}\right] Z_{i,j-1}^{t}$$

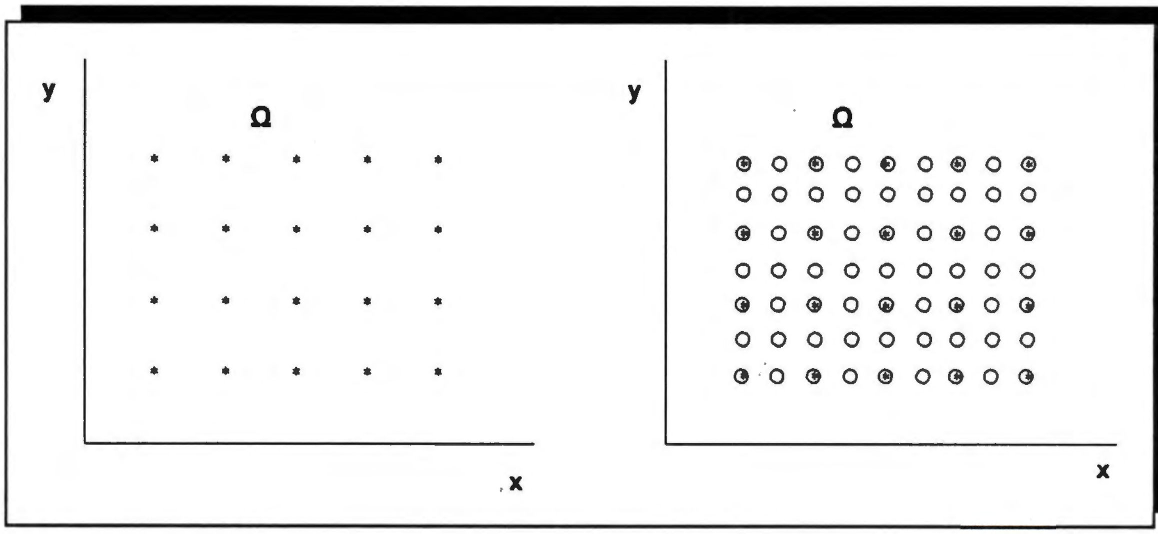
For convergence of the solutions, $Z_{i,j}^{t+1}$ should be a convex combination of various Z values on the right hand side of the above equation. The $Z_{i,j}^{t+1}$ will be the convex combination of all the variables on right hand side only if (1) sum of all the coefficients on the right hand side (bracketed terms) is one, and (2) all the coefficients are greater than or equal to zero. The first condition is fulfilled in the above case. The second condition will be fulfilled only if the values of Δx, Δy and Δt are chosen such that

(12) $$1 - \frac{2\alpha\Delta t}{(\Delta x)^2} - \frac{2\alpha\Delta t}{(\Delta y)^2} \geq 0$$

Simplify (12) further to obtain the following *stability condition* for this problem:

(13) $$\Delta t \leq \frac{(\Delta x)^2(\Delta y)^2}{2\alpha\left[(\Delta x)^2 + (\Delta y)^2\right]}$$

The accuracy of the solutions of the state and adjoint variables from the above numerical method depends on the actual size of Δx, Δy and Δt. The accuracy can be improved by narrowing the mesh size. For instance, assume that the continuous domain Ω is discretized into (5 by 4) grid points initially (see Figure 4a). Then the numerical simulation must be performed on the above grid. Later, suppose that each grid interval is equally subdivided along both x and y axes increasing the number of grid points to (9 by 7) (as shown in Figure 4b). By superimposing the Figure 4a on 4b, every mesh point on the coarse grid coincides with every alternate grid point on the fine grid. Now simulation can be performed for the fine grid. Then the solutions of the fine and coarse grid simulations for the common mesh points should be compared. If there is a wide difference in the solutions of two



(a) (b)

Figure 4. The refinement of numerical simulation by narrowing grid size: (a) coarse grid intervals, (b) fine grid intervals.

simulations across the mesh points, one can further narrow down the mesh intervals and perform the simulation until the solutions converge.

Computer Simulation

A computer algorithm in VAX FORTRAN codes was written to perform the above iterative numerical simulation (see Appendix II). A pictorial representation of this algorithm is given in Figure 5. The program reads the specified inputs concerning model's bioeconomic parameters, initial, terminal and side boundary conditions on beaver density and adjoint variables, grid size, and number of grid points. Then the program initializes user-specified values for the state and adjoint variables over the entire grid. It runs through two loops computing the density of beaver population and the adjoint variable using the discretized optimality system in (10). The values of the state and the adjoint variables generated from the above computation are compared with their respective initial array of values against a user-specified tolerance limit. If the tolerance criterion is not met, the current values will be stored in a buffer, and the program goes through the computation loops again, every time comparing the

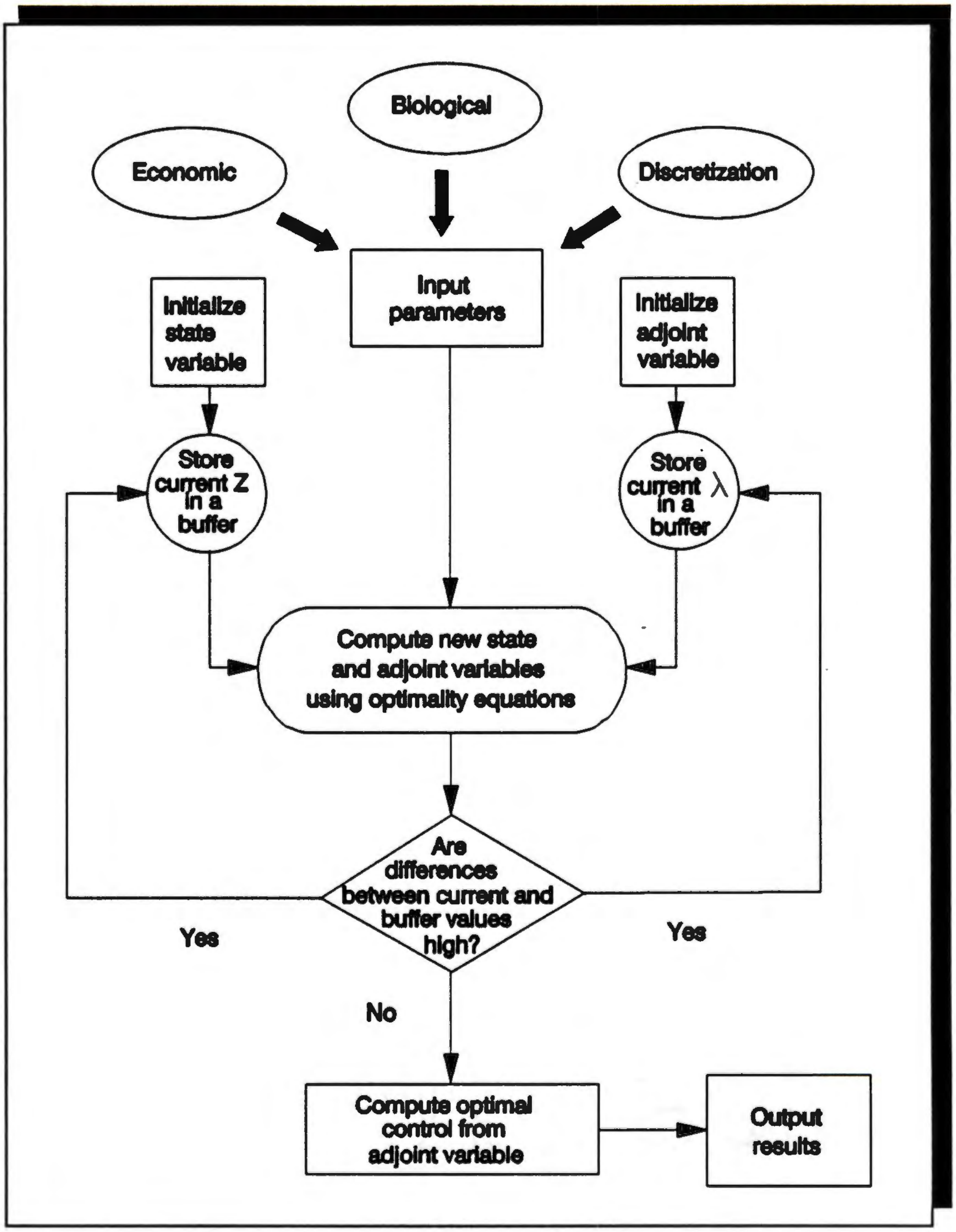


Figure 5. The flow chart showing the computer simulation of the beaver trapping model.

current array solutions with those in the buffer. Once the values of $Z_{i,j}^t$ and $\lambda_{i,j}^t$ converge, the program calculates the optimal control for all the locations (grid points), using the latest solution values of the adjoint variable, and outputs the results.

Empirical Application: Parameterization

Delineation of the Study Area

The administrative Region 7 of the NYSDEC was selected for our study. The study region includes nine counties of central New York with an expanse of 6,296.4 sq mi (see Figure 6). This area cuts across eight ecological zones. More than 50 percent of the study area falls in Central Appalachians ecozone. The landscape is mostly characterized by hills of wide ranging elevations interspersed with farmlands and mixed-species of hardwood forest trees (Purdy *et al.*). The NYSDEC has been conducting aerial surveys to estimate beaver populations in order to form a basis for the design of management regulations. Mr. Gotie of NYSDEC estimated around 6,900 beavers in this region during the year 1988.

Considering the significant role of beavers in increasing wetland area in the state of New York, the DEC attempts to increase the beaver population in this region. However, the Department's goal can't always be fulfilled due to concerns expressed by public and private land owners about the increasing beaver damage. For the purpose of this study, the beneficial value of beavers in the study region is not important and hence ignored.

Analytical Setting

For analytical simplicity and lack of better information, certain assumptions were made regarding the study area. The *study region* is inscribed into a rectangle region Ω (Figure 7). From here onwards, Ω becomes the representative beaver *management region* for the analytical purpose. The

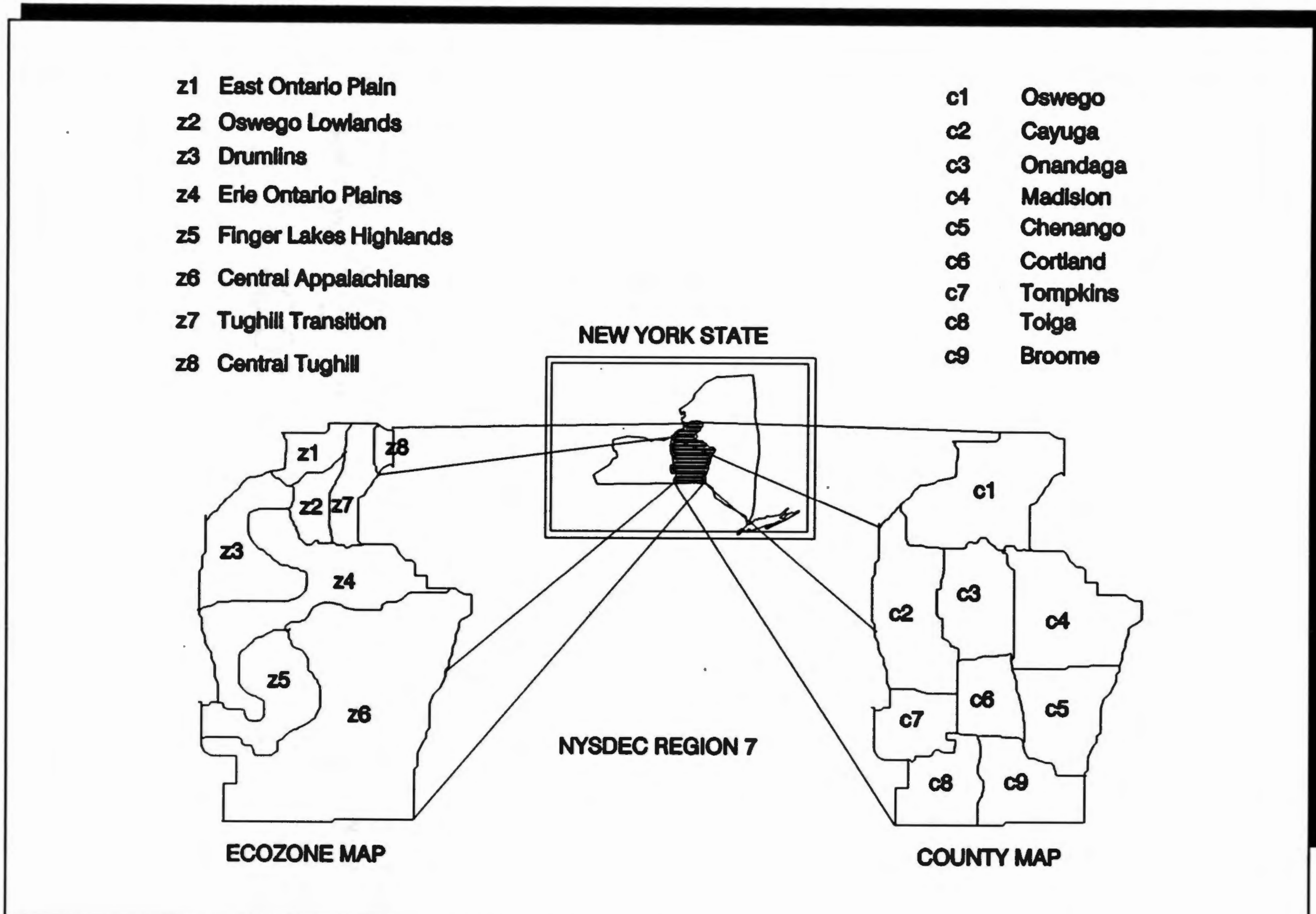


Figure 6. The map of the Region 7 of the New York State Department of Environmental Conservation with ecological zones and counties.

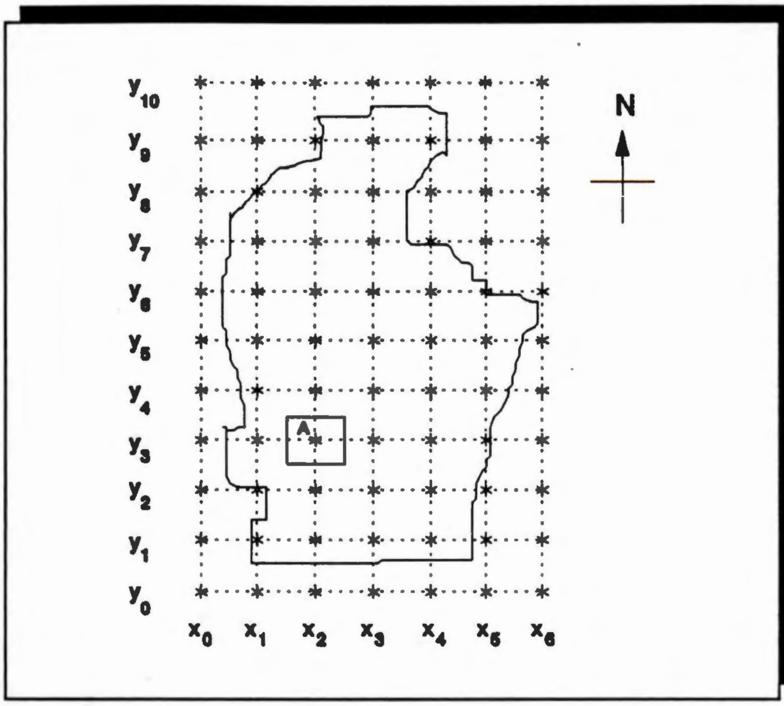


Figure 7. The analytical beaver management region showing the numerical grid points.

perimeter of this rectangle is the boundary of the *management region* along which the density of beaver population was assumed to be zero. The *management region* was arbitrarily subdivided into 60 small cells, 6 cells along x axis (west-to-east direction) and 10 cells along y axis (south-to-north direction). By eyeballing the *study region* was found to occupy two-thirds of the *management region*. As mentioned earlier, the total geographical area of the *study region* is 6,296.4 sq mi. Based on this, the total area of the entire *management region* was found to be 9444.6 sq mi and the individual cells 157.41 sq mi. Given the area of the individual cells, the size of the grid intervals happens to be $\Delta x = \Delta y = 12.55$ miles. Accordingly, the values of M and N are, respectively, 6 and 10. The total number of grid points across the spatial domain including those on the boundary of the region is 77. The points with (*) mark in the figure indicate the locations of the grid points. The density of beaver population on each location (grid point) corresponds to the beaver density (hd/sq mi) on an area of 157.41 sq mi represented by a square cell surrounding the grid point (for instance, the square cell bordering point A in Figure 7).

Baseline Parameter Values

Most of the biological and economic data required by the model are available for the Region 7 of the NYSDEC. Part of the necessary information was obtained from personal contact with state biologist Mr. Gotie, and the other part was obtained from published literature. The information not available for this region was taken from multiple sources as explained below. For the baseline simulation all the biological and economic parameters were assumed constant for the entire region.

<u>Initial population distribution</u> ($Z_{i,j}^{0}$): From the aerial survey data supplied by the NYSDEC, it was found that 1988 was the recent year for which zone-wise population estimates for the entire *study region* were available. Thus the year 1988 was considered as time period zero in this analysis. Using the available data, the beaver density for each of the ecological zones was computed (see Table 1). Using the zonal density estimates, weighted population density for each grid point of the *management region* was determined. The proportionate areas of ecozones falling under a given cell were used as weights. For example, consider the square cell surrounding the point A in Figure 7. The 61 percent of the area of this cell falls in the Central Appalachians ecozone and 39 percent in the Finger Lakes Highlands zone. From Table 1, the beaver densities of these two regions in 1988 were, respectively, in hd/sq mi 0.939 and 0.762. Hence, the weighted density of beavers at grid point A became 0.61 times 0.939 plus 0.39 times 0.762, which was equal to 0.87 hd/sq mi. The densities at the grid points along the boundary of the *management region* were set to zero. The initial distribution of the beaver densities for all the grid points derived in this manner are presented in Table 2.

<u>Population growth parameters (*a* and *b*)</u>: Lancia and Bishir fit observed data on the beaver population in Massachusetts from 1952 to 1978 to a logistic growth function. They estimated a maximum specific

Table 1. **The density of beaver population in various ecological zones of the Region 7 of NYSDEC during the year 1988.**

Ecological Zones	Geographical Area	Number of Beavers*	Density of Beaver Population
	(Square miles)		(Heads/square miles)
East Ontario Plain	124.1	224	1.805
Oswgo Lowlands	280.1	313	1.117
Drumlins	452.7	556	1.228
Erie Ontario Plains	964.4	697	0.723
Finger Lakes Highlands	513.1	391	0.762
Central Appalachians	3,573.8	3,356	0.939
Tughill Transition	328.7	1165	3.544
Central Tughill	59.5	200	3.361

*Estimated by Mr. Gotie, state biologist with NYSDEC, Cortland, New York, based on the aerial survey of active beaver colonies.

growth rate of 0.335. Interestingly, the above estimate is close to another estimate made by Payne for Newfoundland beavers in 1984 (0.347875).[3] Hence, Lancia and Bishir's estimate was taken as reasonable baseline value for natural growth rate parameter *a* in this analysis, and it was assumed that natural growth rate was constant over the entire *management region*.

Based on the number of active colonies and approximate population estimates (provided by Mr. Gotie) for the years 1983 to 1990 average beaver colony size in the *study region* was estimated at 4.755 hd/colony. The number of potential sites in the region was reported to be 5,367. Parsons and Brown (1979) from a study in the part of Northern New York reported that reproduction of beavers ceased when occupancy rate (the ratio of number of active colonies to that of potential colonies) exceeded 40

[3]Payne estimated birth and death rates at 0.536 and 0.355, respectively. The death rate was the sum of natural mortality and harvest mortality; hence, the natural mortality was calculated by netting out the average harvest mortality rate computed from Payne's Table 2 (0.166875). The maximum net proportional birth rate for the above beaver population could be calculated as 0.536 - (0.335 - 0.166875) = 0.347875.

Table 2. **The distribution of initial time beaver population densities used in the optimal beaver trapping simulation.**

Grid Points on y Axis	Grid Points on x Axis						
	0	1	2	3	4	5	6
	(Density of beavers in heads per square miles)						
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.000	0.939	0.939	0.939	0.939	0.939	0.000
2	0.000	0.939	0.939	0.939	0.939	0.939	0.000
3	0.000	0.761	0.870	0.939	0.939	0.939	0.000
4	0.000	0.751	0.793	0.939	0.939	0.939	0.000
5	0.000	0.729	0.854	0.939	0.939	0.939	0.000
6	0.000	1.183	0.930	0.727	0.777	0.777	0.000
7	0.000	1.026	0.886	1.346	2.136	0.723	0.000
8	0.000	1.228	1.308	2.164	3.544	3.544	0.000
9	0.000	0.000	0.000	2.132	3.452	3.361	0.000
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000

percent. Assuming 40 percent occupancy rate to be the reasonable carrying capacity for the *study region*, the number of active colonies at the carrying capacity was worked out as 0.40 times 5,367 which was equal to 2,147 colonies. Given the estimated colony size of 4.755 hd/colony, the number of beaver at the carrying capacity was calculated to be (2147)(4.755) = 10,208. Then the average density at carrying capacity K could be easily worked out as 10,208 divided by total geographical area 6,296.4 sq mi, which yielded K = 1.6212 hd/sq mi. Knowing the values of a and K, the measure of density dependence b was computed as $b = a/K = 0.335/1.621 = 0.2066315$.

<u>Diffusion coefficient (α)</u>: A baseline value for the diffusion coefficient was not readily available. The following procedure was adopted to derive a reasonable estimate of this parameter. Recall the partial differential equation describing beaver population dynamics without trapping as:

$$(14) \qquad Z_t = \alpha(Z_{xx} + Z_{yy}) + aZ - bZ^2$$

Discretize the above equation to obtain the following finite difference equation:

$$(15) \qquad \frac{Z_{i,j}^{t+1} - Z_{i,j}^t}{\Delta t} = \alpha\left[\frac{1}{(\Delta x)^2}\left(Z_{i+1,j}^t - 2Z_{i,j}^t + Z_{i-1,j}^t\right) + \frac{1}{(\Delta y)^2}\left(Z_{i,j+1}^t - 2Z_{i,j}^t + Z_{i,j-1}^t\right)\right]$$

$$+ a_{i,j}Z_{i,j}^t - b_{i,j}(Z_{i,j}^t)^2$$

Solving the above equation for α,

$$(16) \qquad \alpha = \left[\frac{1}{(\Delta x)^2}\left(Z_{i+1,j}^t - 2Z_{i,j}^t + Z_{i-1,j}^t\right) + \frac{1}{(\Delta y)^2}\left(Z_{i,j+1}^t - 2Z_{i,j}^t + Z_{i,j-1}^t\right)\right]^{-1}$$

$$\left[\frac{Z_{i,j}^{t+1} - Z_{i,j}^t}{\Delta t} - a_{i,j}Z_{i,j}^t - b_{i,j}(Z_{i,j}^t)^2\right]$$

The values of a, b, Δx and Δy for the *management region* are already known. In order to obtain an estimate of α, the densities of beaver population at some location (i,j) at two successive time steps $(t+1)$ and t, and densities of beaver population at the four surrounding locations $(i+1,j)$, $(i-1,j)$, $(i,j+1)$ and $(i,j-1)$ at time step t were used. The point A in Figure 7 which lies in the border of Central Appalachians and Finger Lakes Highland ecozones was selected for this purpose. The point A was found suitable because it was on the border of two ecological zones across which there was a wide variation in beaver densities. Such variation in densities should help measure the diffusion phenomenon fairly well. Another reason was that the density estimates for this location and all the required surrounding locations for two continuous years (1984 and 1985) were available. Since the population was measured once a year, the time interval was considered to be $\Delta t = 1$ for the purpose of this calculation. Inserting all these values into (16), the baseline value for the diffusion coefficient was found to be $\alpha = 725.27$ sq mi/yr.

__Discount rate (r)__: The proxy used in this study for the society's time preference rate or the real discount rate is the nominal rate on AAA corporate bonds for June 1987 less the percentage change in price from June 1986-87 (see Federal Reserve Bulletin, Table A24). This rate, $r = 0.056$, was assumed to be a good average of the productivity of low-risk investments in the capital markets.

__Damage function parameter (γ)__: The baseline damage parameter was calculated from the beaver-inflicted damage estimates provided by Purdy *et al.* for this region during the period 1983-1984. They reported 780 damage incidences over a period of two years. Based on this figure, we assumed an average number of incidence per year to be 390. They also provided the average dollar estimate of these damages at \$ 736 per incidence. The average annual total damage in the region was found as (736)(390) = \$ 287,040. This amounted to unit area average damage level of $D(Z) = 287{,}040/6{,}296.4 = \$45.59$/sq mi/yr. Further, the damage estimated above was assumed to be associated with the region's population level of 5,000 beavers (i.e., a beaver density of $Z = 5{,}000/6{,}296.4 = 0.7941$) during

1984 as reported in the above study. Thus, the damage parameter could be calculated as $\gamma = 2D(Z)/(Z)^2 = 2(45.59)/(0.7941)^2 = \$\ 144.59$ sq mi/yr/hd^2.

<u>Cost function parameter (c)</u>: Ermer reported from a trappers' survey for the Region 9 of the NYSDEC, west part of the state, on the beaver trapping time. The reported trapping was 12.79 hours/hd. Assuming a wage rate of \$ 5/hour, the unit trapping labor cost was calculated at \$ 63.95. Notice that this didn't include other overhead costs. For lack of information it was assumed that labor cost was the major component of the trapping cost, and that the unit overhead cost was 20 percent of the labor cost, which resulted into total unit trapping cost $C(P) = 63.95 + 12.79 = \$\ 76.74$/hd. Ermer also considered an average harvest rate of 2 hds/colony for this region. Assuming that the average colony size of 4.755 hds calculated for the *study region* applied to Region 9, the average trapping rate for this region was found out to be $P = 2/4.755 = 0.42$. Assume that this trapping rate was associated with the unit trapping cost $C(P) = \$76.74$. Then the cost parameter was worked out as $c = C(P)/P = \ = 76.74/0.42 = \$\ 182.71$ yr/hd.

<u>Planning period (T) and time step interval (Δt)</u>: The numerical simulation in this analysis assumed ten year period as a reasonable finite planning horizon (T) within which most parameters of the model were likely to remain unchanged. It was presumed that the optimal trapping strategy needs to be revisited after the end of the finite planning period, by reassessing the values of the model parameters and the beaver density distribution existing at that time. It was expected that the continuous beaver trapping on lines with the proposed optimal strategy would change the ecosystem of the beaver habitat, and thus, could affect some of the biological parameters. The economic parameters may also change after a finite period of time. By equation (13) it was required that the time axis should be discretized by choosing each time step Δt based on the spatial intervals Δx and Δy, and the value of α. Thus,

the value of Δt was calculated by the following expression

$$\Delta t = \frac{0.48(\Delta x)^2(\Delta y)^2}{\alpha\left[(\Delta x)^2 + (\Delta y)^2\right]} < \frac{(\Delta x)^2(\Delta y)^2}{2\alpha\left[(\Delta x)^2 + (\Delta y)^2\right]} \tag{17}$$

The computer algorithm calculates the value for Δt automatically given the values of Δx, Δy and α. For the baseline values of $\Delta x = \Delta y = 12.55$ and $\alpha = 725.27$, Δt was found to be 0.05209. Now the time axis t could be rescaled such that each time period (one year) corresponds to 0.05209. Thus, the entire planning horizon of 10 years was equivalent to (0.05209)(10) = 0.5209 on the new time scale.

Baseline simulation was conducted using the above parameters (see Table 3). As suggested before, for more accuracy of the results a second simulation was performed by halving the grid intervals, Δx and Δy, i.e., $\Delta x = \Delta y = 6.27$. As a result, in the second simulation, $\Delta t = 0.013$, $M = 12$, $N = 20$, and $T = 40$. The total number of grid points increased to 11,193. The optimal solutions of the beaver densities and the trapping rates were found to converge on the common grid points in just two iterations. Thus, the values from the second iteration were accepted as the optimal state and control variables.

Table 3. The baseline parameter values.

Symbol	Meaning	Units	Value
a	maximum rate of net recruitment	yr^{-1}	0.335
b	density dependence of beaver stock	(hds/sq mi) yr^{-1}	0.2066315
α	diffusion coefficient	(sq mi) yr^{-1}	725.27
r	real annual discount rate		0.056
γ	damage function parameter	\$ (sq mi)/yr/hd^2	144.59
c	cost function parameter	\$ yr/hd	182.71

Empirical Simulation Results

Baseline Simulation

The optimal densities of beavers and rates of trapping across the *management region* over the entire planning horizon are presented in Appendix III. Figures 8a - c and 9a - c show the three dimensional surface graphs of the spatial distributions of the optimal beaver densities and trapping rates, respectively, at three different points in time. The optimal population levels are shown for the time steps 0, 5 and 10 whereas the trapping rates are presented for the time steps 0, 5 and 9. The trapping operation begins at time 1 and ends at one time step prior to the terminal time. The terminal time trapping rates are zero due to the transversality condition, i.e., zero marginal nuisance value of beavers at the end. For a better understanding of the spatiotemporal changes in the optimal beaver stocks and the control, cross-sectional views of the three dimensional representations (Figure 8 and 9) are presented in Figure 10 and 11. For instance, Figure 10a shows the cross-sectional view of the optimal beaver densities for all the grid points along the south-north direction at location x_1 on west-east direction at three points in time, i.e., $t = 0, 5, 10$ (refer to Figure 7 for identification of the above geographical locations). Similarly, Figures 10b and c represent the optimal densities at locations x_3 and x_5, respectively. Figures 11a - c correspond to the optimal beaver trapping rates at the three respective locations.

The initial distribution of the beaver population (Figure 8a) indicated that the beavers were more concentrated in the northern parts of the *management region* in the year 1988. Their stocks in the southern portion were fairly even distributed. By exercising the optimal trapping strategy over the entire *management region* every year thereafter, the beaver densities seemed to uniformly *smooth out* across the individual management cells. The optimal stock distribution in Figure 8a and c, respectively, for the years 5 and 10 were more even. The above *smoothing out* phenomenon is also demonstrated in Figures 10a - c. Another noteworthy feature was that the optimal densities over a period of time were gradually,

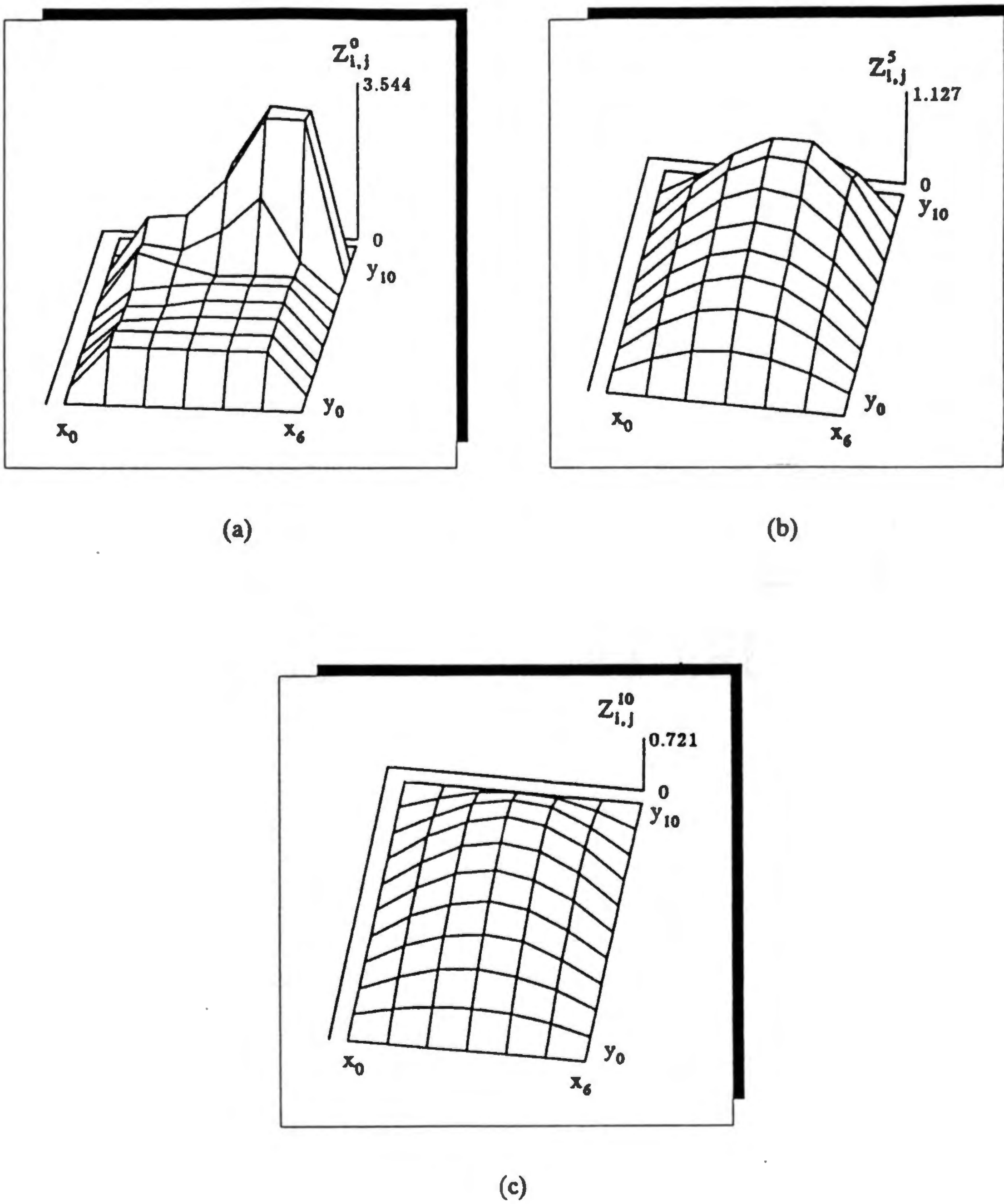
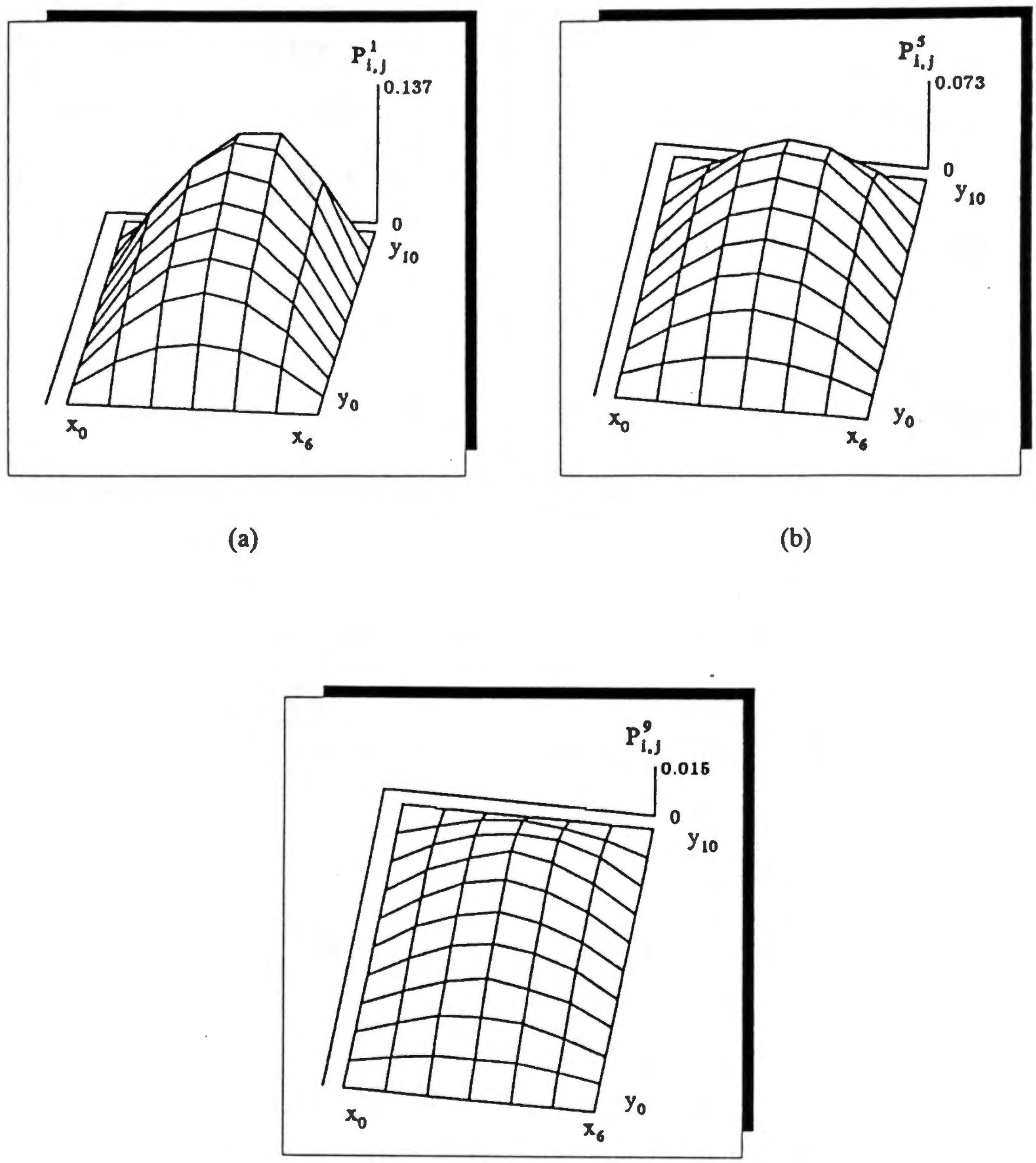
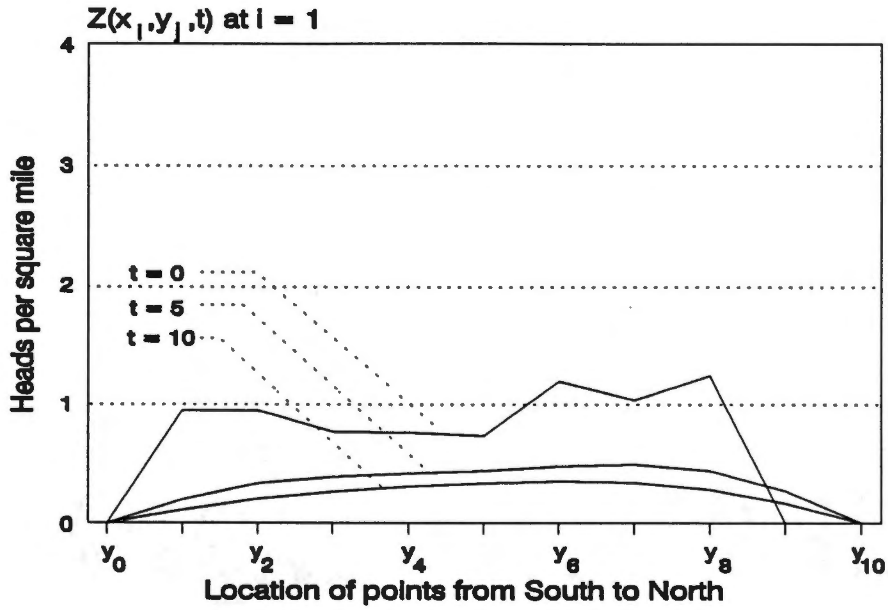


Figure 8. Optimal densities of beaver population in the management region for the baseline simulation: (a) for the initial year (t = 0), (b) for the middle year (t = 5), and (c) for the terminal year (t = 10).

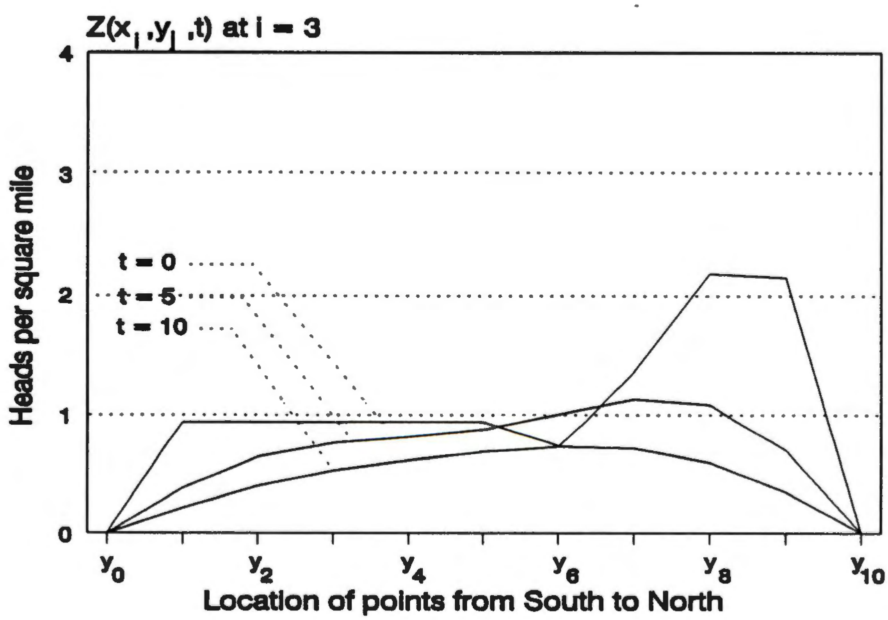


(c)

Figure 9. Optimal trapping rates in the management region for the baseline simulation: (a) for the initial trapping year (t = 1), (b) for the middle trapping year (t = 5), and (c) for the terminal trapping year (t = 9).

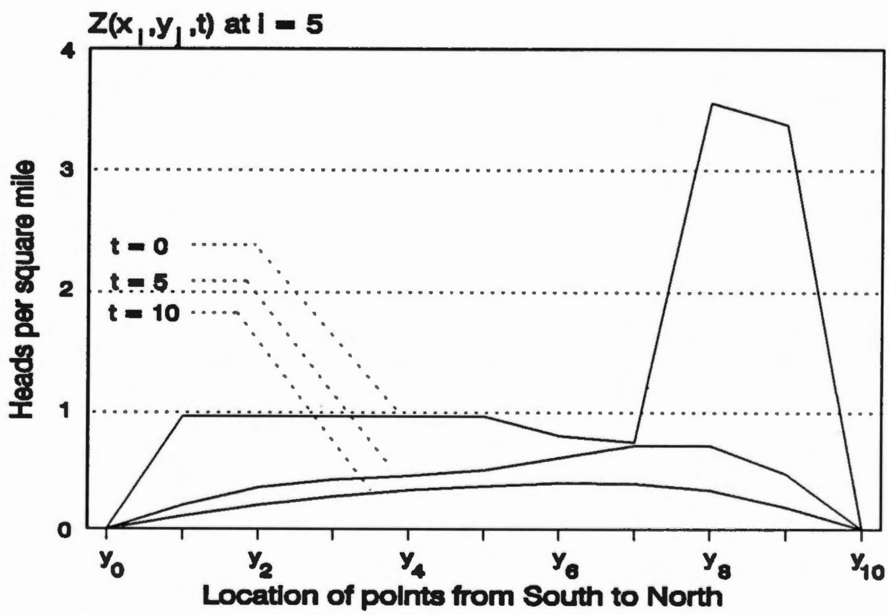


(a)



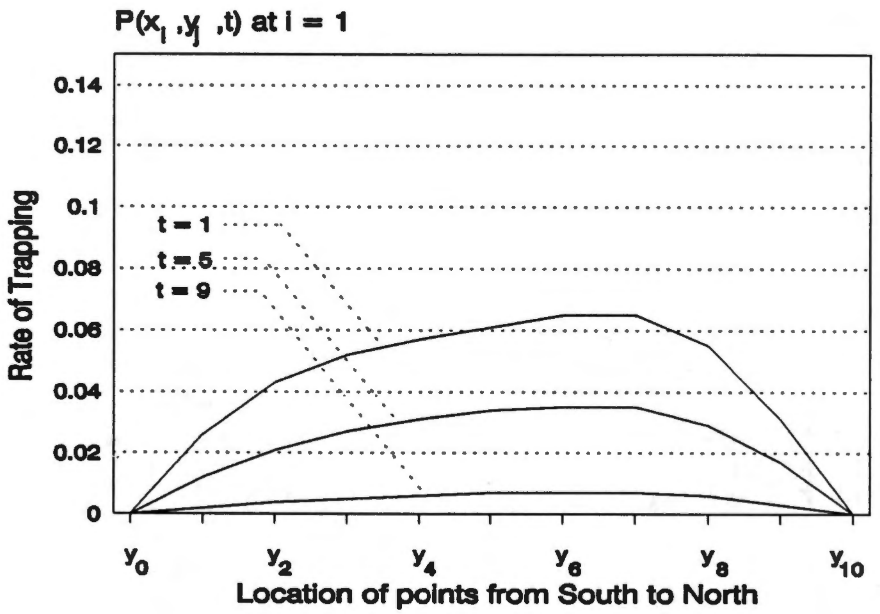
(b)

Figure 10. The optimal beaver densities along the y axis at various cross-sections of x axis: (a) cross-sectional location $i = 1$, (b) cross-sectional location $i = 3$, and (c) cross-sectional location $i = 5$.

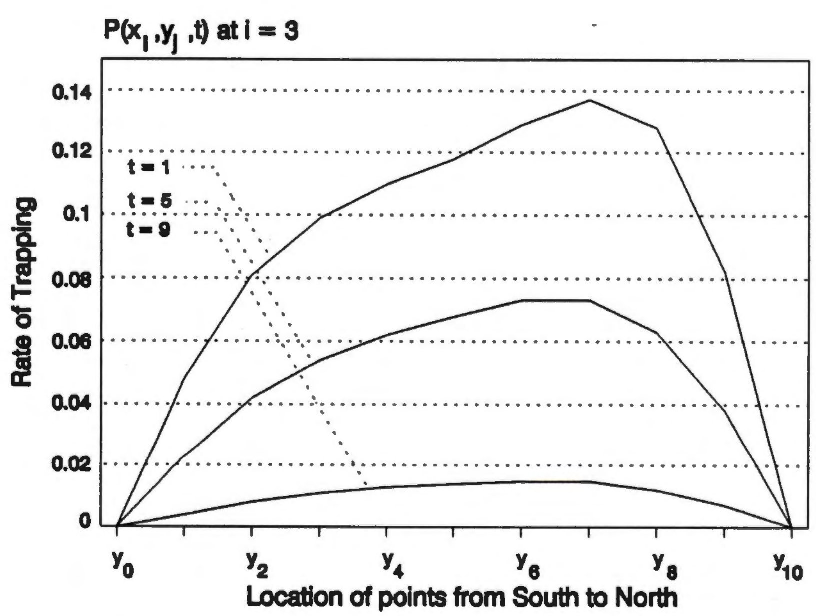


(c)

Figure 10. (continued).

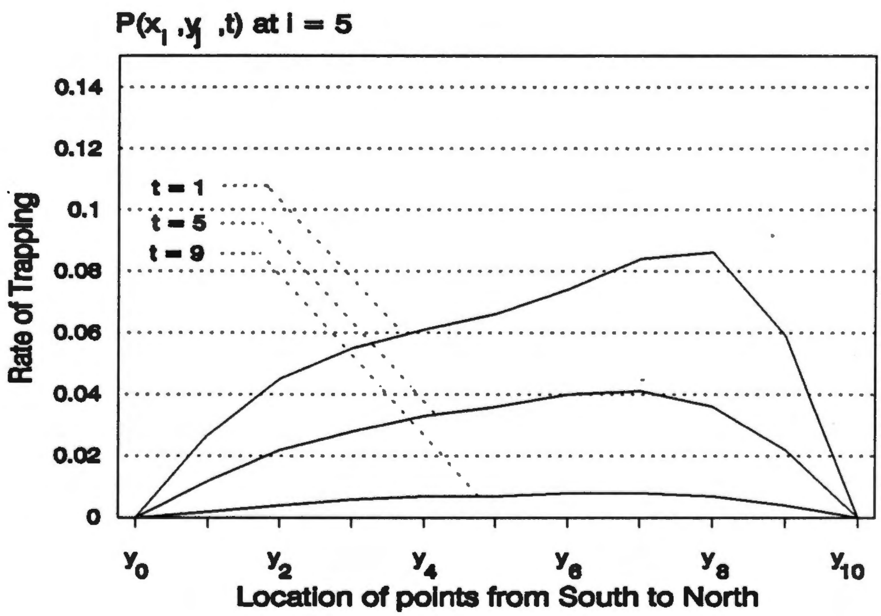


(a)



(b)

Figure 11. The optimal rate of beaver trapping along the y axis at various cross-sections of x axis: (a) cross-sectional location $i = 1$, (b) cross-sectional location $i = 3$, and (c) cross-sectional location $i = 5$.



(c)

Figure 11. (continued).

uniformly declining at all the geographical locations. Towards the terminal time (Figure 8c), beaver densities at all the grid points were asymptotically approaching zero. The densities tapered down to zero towards the boundaries with a *hump* about the center of the region. However, it was important to note that the optimal strategy did not totally eliminate the beaver populations from the region at the end of the planning period.

The optimal trapping rates in the beginning of the planning horizon were more uneven across the management region (Figures 9a and 11a - c). The management strategy required trapping more beavers from the northern parts with higher densities. As the planning period advanced, the area-wide optimal trapping rates became evenly distributed (Figures 9b - c and 11a - c). This was consistent with the even distribution of the densities observed towards the end of the planning period. Also notice that the spatial distribution of the optimal trapping rates gradually approached zero at all the points towards the end of the planning horizon, and in fact, became zero at the terminal point.

Figure 12 presents the temporal behavior of the optimal stock and control at a selected location represented by point A shown in Figure 7. The optimal trapping rate tapered down to zero at the end of the finite planning period where as there were some beavers left in the habitat, the damage potential of which, by the design of the model, was negligible. The similar behavior could be expected for the optimal stocks and controls at all the other grid points.

The discounted total cost of the optimal trapping program over the finite time horizon was \$136,500.80 for the entire *management region*. The total number of beavers trapped following this optimal strategy was 2,314.

Sensitivity Analysis

The optimal trapping strategy was expected to be sensitive to the model's parameters. The model's sensitivity was analyzed for variation in the two key economic parameters, i.e., c and γ. Four simulations were run alternating between high and low values for each. Table 4 summarizes the results of the above simulations. At both the levels of γ, the increase in c increased the total loss marginally

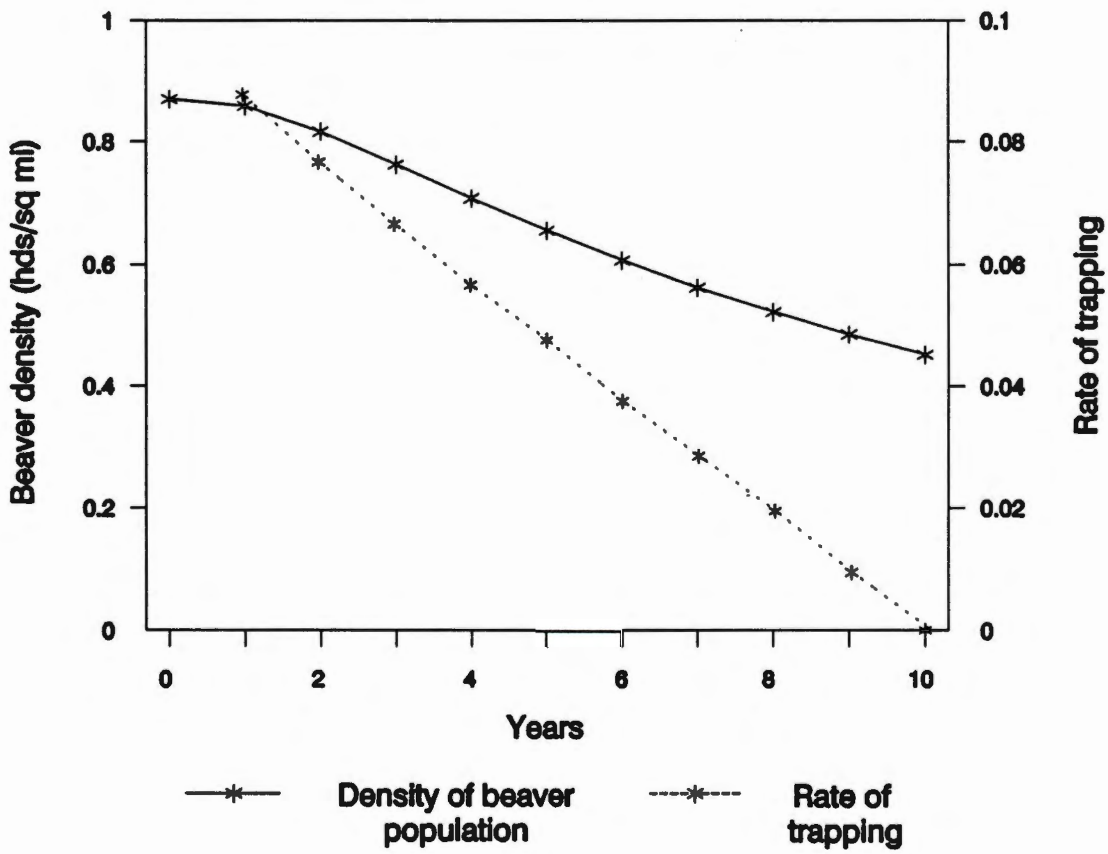


Figure 12. The optimal beaver densities and trapping rates over the entire planning horizon for a selected location point A.

Table 4. **The societal loss and number of beavers trapped under varying levels of cost and damage parameters.**

	Low c	High c
Low γ	c = 100 γ = 100 The value of total = \$93,920.87 loss functional Number of beavers trapped = 2,895	c = 300 γ = 100 The value of total = \$95,498.05 loss functional Number of beavers trapped = 997
High γ	c = 100 γ = 300 The value of total = \$268,922.90 loss functional Number of beavers trapped = 7,939	c = 300 γ = 300 The value of total = \$281,762.40 loss functional Number of beavers trapped = 2,894

(from \$93,920.87 to \$95,495.05 at lower damage potential and \$268,922.90 to \$281,762.40 at high damage potential). On the other hand, there were substantial reductions in the overall beavers trapped in both the cases (2,895 to 997 and 7,939 to 2,894, respectively). The increase in the cost of operation makes beaver trapping less attractive leaving more beavers in the watershed. This would lead to an increase in the beaver damage.

The total loss functional was found to be more sensitive to the damage parameter. At both the levels of c, the increase in γ resulted in substantial jump in the total loss (from \$93,920.87 to \$268,922.90 at low trapping cost and from \$95,498.05 to \$281,762.40 at high trapping cost). The number of beavers trapped also increased more than double in both the cases (2,895 to 7,939 and 997 to 2,894, respectively). This indicates that from society's welfare point of view, the beaver damage potential is more crucial than the cost of trapping operation. However, it is interesting to note that increase in the cost of trapping would conserve beavers with marginal increase in burden to the society. This has a potential policy implication for conserving beaver population at times of environmental adversity in that given any damage potential of beavers, a tax on beaver trapping could be used as a potential instrument in building beaver stocks. This policy instrument could be enforced at nominally low additional burden to the society.

Further, the effect of spatial variation in the damage parameter on the optimal beaver stocks and trapping rates was analyzed. Figure 13 represents the spatial variation in the damage parameter assumed in this simulation. The optimal beaver population and trapping across the *management region* at three different points in time are presented in Figure 14 and 15, respectively. There was not much difference in the spatial behavior of the optimal beaver stocks between the baseline simulation with constant damage parameter (Figure 8) and the current simulation (Figure 14). However, a distinct change in the spatial behavior of the optimal trapping between the baseline simulation and current simulation could be observed (compare Figures 9 and 15). The spatial pattern of the optimal trapping under the current simulation was consistent with the spatial variation in the damage parameter as illustrated in Figure 13. The values of the damage parameter about the middle portion of the management region were less than those of the northern and southern portions of the region. Accordingly, there was a *trough* in the optimal trapping rates in the middle portion with two *humps* at both ends of the region (Figure 15). This indicates that the optimal trapping is directly proportional to the damage parameter. As observed in the baseline simulation, both the distributions of optimal stock and control tapered down over the period of planning horizon. Again at the terminal time, control rates were zero whereas the optimal densities were asymptotically declining.

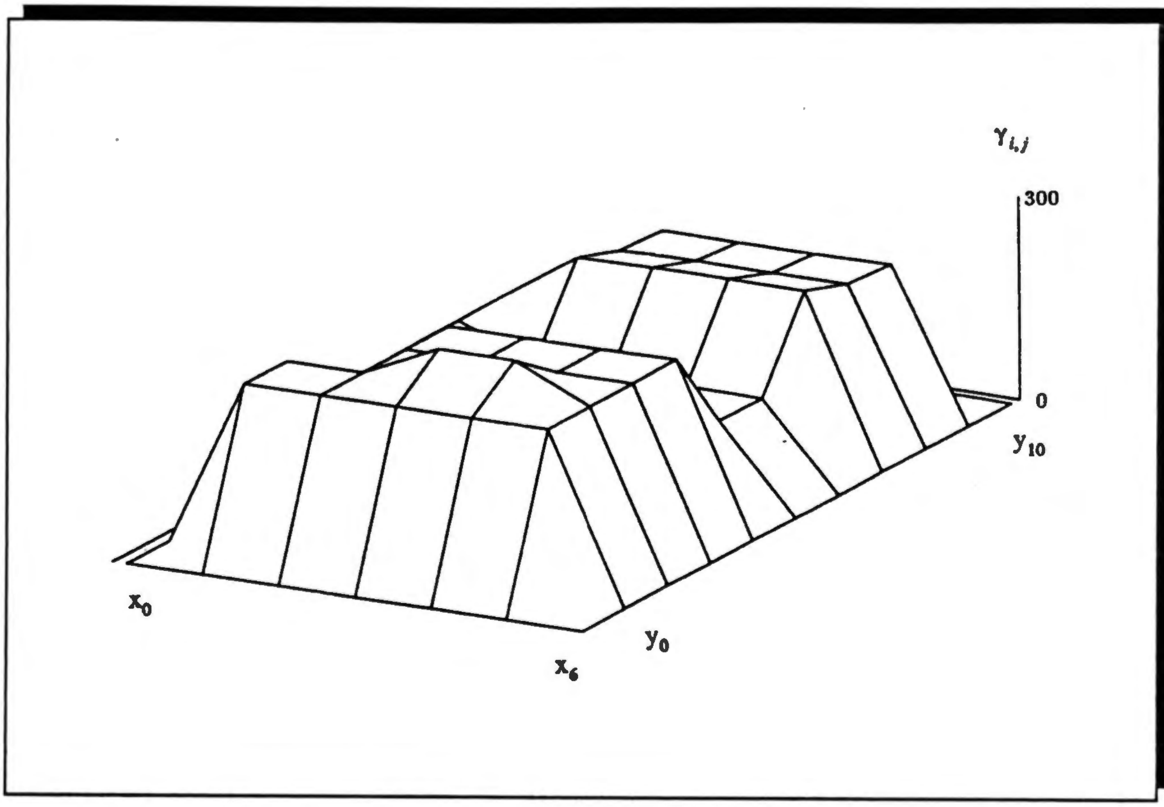


Figure 13. The values of the damage parameter across the management region assumed in the sensitivity analysis.

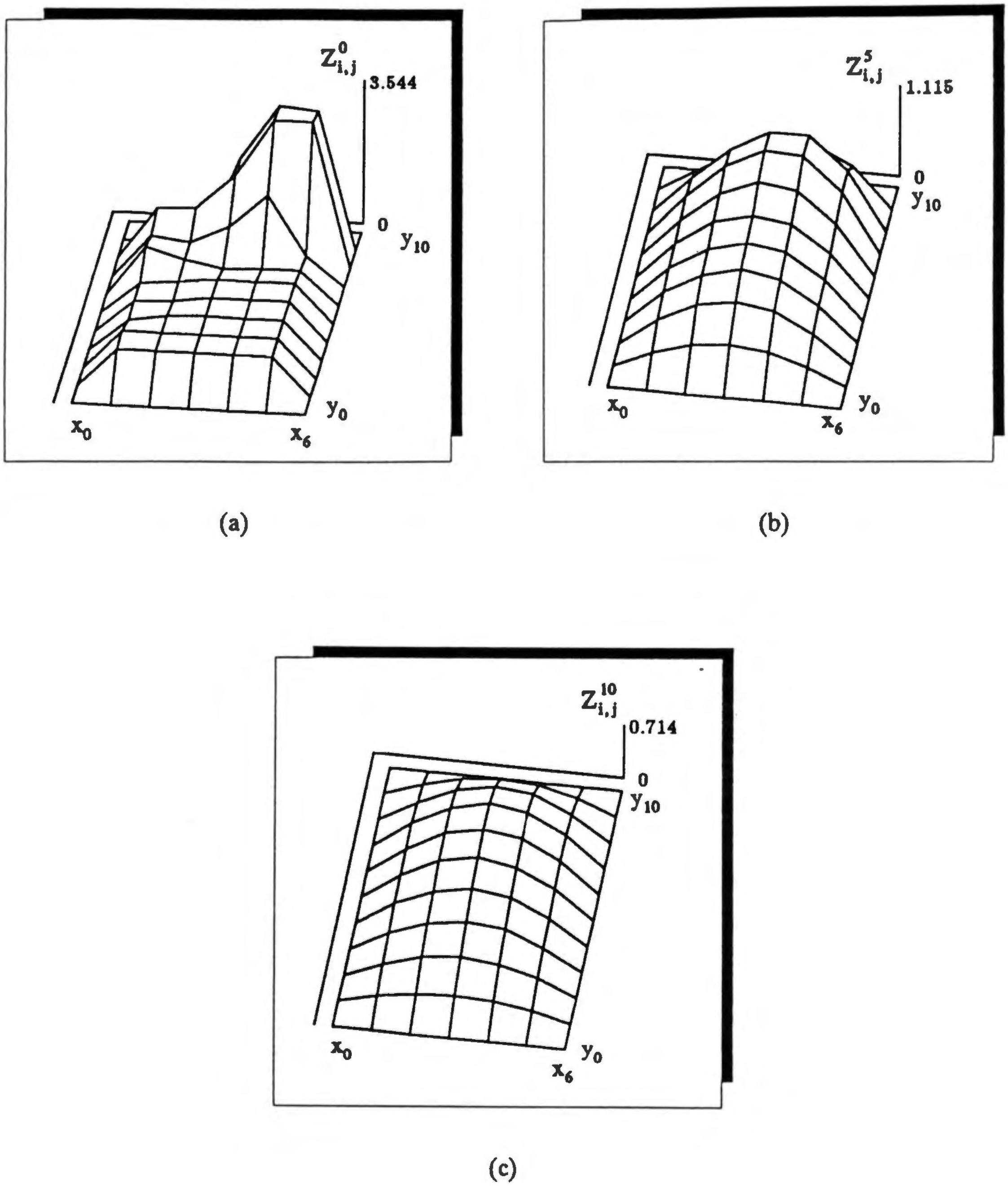
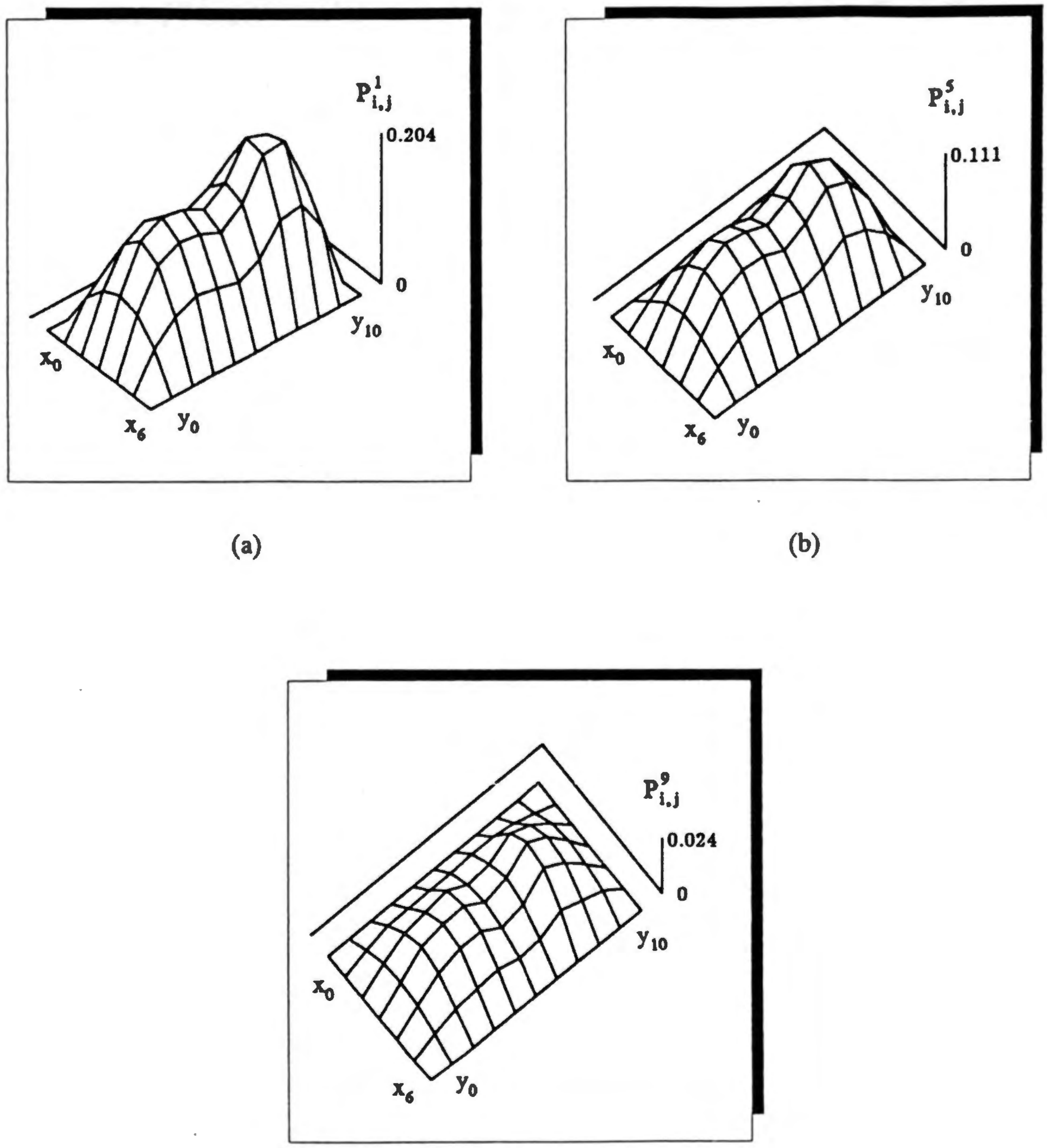


Figure 14. Optimal densities of beaver population in the management region for the simulation with variable damage parameter: (a) for the initial year (t = 0), (b) for the middle year (t = 5), and (c) for the terminal year (t = 10).



(a) (b)

(c)

Figure 15. Optimal trapping rates in the management region for the simulation with variable damage parameter: (a) for the initial trapping year (t = 1), (b) for the middle trapping year (t = 5), and (c) for the terminal trapping year (t = 9).

CHAPTER V

SUMMARY AND MANAGEMENT IMPLICATIONS

Background and Summary

The beaver population in the Southeastern United States has increased alarmingly over the last 25 years and caused severe damage to valuable timber land through dam-building and flooding of bottom-land forest. Miller mentioned in a recent survey report that among all the vertebrate animals, beaver has caused the most damage to southern forests. Though beavers have a potential to be beneficial as a conservator of nature, source of recreation and furbearer, the high degree of beaver nuisance has earned them a status of pest, at least in the southern range limits.

Traditionally, beavers have been trapped by small group of people as a source of their livelihood. The market for beaver pelt, the main driving force inducing the trappers' effort, has been the main factor in governing the population of beavers. The wildlife protection agencies of respective states have the responsibility of protecting beaver populations against overharvest during the periods of high market for the beaver pelts. In the northern parts of the United States and Canada, because of the better quality of beaver pelt, there has been fairly consistent market for beavers, and hence, the management agencies have tried to sustain the long-term productivity of beavers through trapping regulations such as quotas or seasons. Historically, prices for beaver pelt coming from southern range limits have remained low. Hill and Novakowski point out that in Canada the demand for trapping responds instantaneously to pelt price whereas in the United States, trapping pressure is more price-inelastic. This seems to be particularly true in the southern states. The low pelt price in the south has failed to stimulate adequate trapping pressure, and thus, resulted in increased beaver population and nuisance.

The low trapping pressure and the resulting hike in the beaver nuisance activities have forced property owners to start undertaking their own control measures. The experiences from few eradication efforts, however, have demonstrated that beavers from neighboring parcels tend to immigrate continually into less populated controlled parcels (Houston). This backward migration of beavers from uncontrolled habitat to controlled habitat imposes a negative *diffusion externality* on the owners of controlled parcels because they have to incur the future cost of trapping immigrating beavers. The owners incurring trapping costs have no means to exclude the non-acting neighbors from enjoying the benefits of beaver control and damage/cost saving advantage. The problems of *non-exclusion, free riders* and diffusion related *externality* are present in their classical form inherent in most common property resources. The concurrence of these classical problems of a common property nature has severely affected the cost effectiveness of trapping operation. As the beaver population is mobile, the affected owners are aware of their inability to influence the population on their own parcel or the entire region. This has resulted in a low incentive for control of beaver population on the part of individual land owners, causing a wedge between social and private needs for controlling beaver population.

Because of the transboundary nature of the beaver nuisance problem, beaver-affected land owners require to consider how neighboring land owners view the beaver population, while designing their management strategy. While recognizing various management scenarios based on the objectives of neighboring owners, this study concentrates on the situation where all the owners in the given beaver habitat consider the beavers as a nuisance. Under such circumstance, they would better serve their common interests from a collective action by way of placing the responsibility of region-wide regulation in the hands of a single, public manager, on a cost sharing basis. Such a policy enables the public manager to explicitly consider costs of operation and beaver diffusion-related externalities into a suitable management strategy which would maximize the social welfare of all the land owners involved.

A number of studies are available in the literature where the optimal harvesting of biological renewable resources is cast under the capital-theoretic framework. The harvesting of a biological resource has been studied as an intertemporal resource allocation problem. Unlike the classical capital control problem, the current problem needs a special focus on the spatial diffusion aspects of beaver. Because of the mobility of beavers, present harvesting of beaver at a given location can affect future availability and biological productivity of beaver stock through out the entire habitat. The public manager, in charge of controlling timber damage, therefore is required to make simultaneous choices of present and future trapping of beavers at all the locations. Thus, an ideal strategy needs to consider both temporal and spatial dynamics of beaver population together.

In the current study, the dynamic optimal harvesting of structured beaver population was modelled as a *distributed parameter control* problem. The time and spatial evolution of beaver population was simulated by the parabolic diffusive Volterra-Lotka partial differential equation. The diffusion term was added to classical logistic growth equation to capture the beaver diffusion across the spatial locations. The dichotomy between density dependent beaver damage and cost of trapping was expressed in the objective total cost function. The economic goal of the public manager was to minimize the present value combined cost of beaver damage and trapping over a finite period of time subject to the spatiotemporal population dynamics summarized in the above partial differential equation. This optimization model characterized the beaver control strategy that left enough beavers taking into account the net migration at each location and time so as to strike the optimal balance between timber damage and trapping cost. The marginality condition governing this tradeoff required that the marginal damage savings from the beavers trapped at each location equal the marginal costs of trapping. The marginal savings from trapping activity, in turn, was measured as the imputed nuisance value (shadow price) of the beaver stock in a unit area (the product of the adjoint and state variables of the model).

The optimality system of the beaver trapping model developed in this study turned out to be a complicated system of two nonlinear partial differential equations. Therefore, a computer numerical simulation was developed to solve the nonlinear optimality system using the *finite difference* technique. Under this technique, the entire representative management region was subdivided into numerous operational units, and the time evolution of the optimal beaver population and trapping levels were characterized for each unit.

The empirical application of the model was constrained by non-availability of economic and biological information for the beaver population in the Southeast. However, in order to explore model's economic and practical implications, the same was applied to the beaver population in the Wildlife

Management Regions of the New York State Department of Environmental Conservation. The most of the comprehensive data needed for the model were available for this one region. Certain bioeconomic assumptions were made regarding the study area in order to make best use of the available information.

The current study is a rare application of a nonlinear *distributed parameter control* in the area of natural resources. No work has been located that integrates dispersive population dynamics of a small-mammal into an optimization framework capable of characterizing area-wide trapping strategies. Such an integration would make an interesting addition to bioeconomic research on optimal harvesting of diffusive species.

Empirical Results

The empirical simulation of the area-wide distributed beaver trapping model using a set of baseline parameters generated discrete values for the optimal beaver densities and trapping rates across all the individual operational units over time. The entire distribution of optimal beaver densities did gradually and smoothly decline over the period of time. The unevenness of the initial population distribution smoothed out eventually across the beaver habitat. At each geographical location, towards the end of the planning period optimal trapping rate became zero, whereas the population density asymptotically approached zero.

The sensitivity analysis where the cost and damage parameters of the model were alternated between high and low values indicated that an increase in the damage potential of beavers could substantially increase the net present value total cost. On the other hand, an increase in the cost of beaver trapping added only marginally to the total cost, conserving more number of beavers. The geographical variation in the beaver damage potential had a noticeable reflection on the spatial distribution of trapping rates, with little impact on the optimal densities. The areas with higher beaver damage potentials required more intensive trapping operation.

Beaver Management Implications

An ideal beaver management strategy in the southeastern forest needs to recognize beaver stock as a (harmful) common property resource. The *diffusion externality* associated with isolated trapping effort doesn't result in *socially* acceptable levels of beaver damage and control. If timber land owners agree to cooperate, a centralized decision policy that internalizes the *diffusion externality* can be instituted.

There might exist several institutional and political obstacles to the actual implementation of the centralized area-wide control strategy. Details have to be worked out as to how the central authority should be constituted. One possibility is that the land owners might delegate the decision power to the respective wildlife agency of the State. Alternatively, any existing or new timber owners organization might take the responsibility. In either case, the administrative procedure on collecting the service charges from the participating land owners and executing trapping operations needs to be designed.

Coming to the actual functioning of the trapping activity, the land owners have several choices. The individual owners may hire trappers to trap beavers on their respective parcels as per the quotas decided by the central authority. This option may fail if any single owner attempts to breach the agreement. Alternatively, the central authority may itself assume the responsibility of trapping operation on the lands of all the participating members and collect the users' fee. In recent years, professional firms specialized in wildlife damage control have emerged (Braband). The public manager may find it more convenient in hiring such professional firms for the area-wide control, holding the right of supervision.

The proposed centralized trapping policy entails certain initial investment. An extensive inventory of existing beaver population and assessment of biological and economic parameters are essential to institute the control policy. Nevertheless, the economic loss from beaver-inflicted timber damage may far exceed these investments.

Although it was not explicitly dealt in the current beaver trapping model, beaver benefits at low levels of population in the southeast can be considered in the area-wide control policy. Such a modification would be useful in the states where wildlife agencies are managing beaver population with conflicting goals of beaver damage control and overharvest. The results of empirical analysis suggested that in the event of extreme environmental adversity, a suitable tax policy making trapping more expensive helps conserve beavers without much additional burden to the land owners. Since the optimal trapping model is designed for a finite period, reassessment of biological and environmental relationships at the end of every planning period may be essential.

Limitations and Suggested Extensions

The model analyzed in this study, though already complicated, is a simple formulation of the distributed beaver trapping strategy. For theoretical and analytical simplicity, several assumptions were made. This model can be refined in several directions in order to be more realistic. First, the beneficial aspects of beavers in terms of their ability to create wetlands, provide recreation, and supply pelt and meat may be considered. Second, the diffusion coefficient adopted in this study is nonvariant in space. This coefficient may, perhaps will, vary across the ecological zones. Third, the model developed here is purely deterministic. Perhaps, more insight and objectivity can be gained by analyzing the beaver dynamics under the conditions of uncertainty.

Fourth, the distributed centralized strategy is based on the assumption that all the participants agree to cooperate. If any individual participants don't cooperate, the outcome of the centralized policy would be entirely different. This type of noncooperative mode can be analyzed under game-theoretic framework with several ramifications like binding and nonbinding agreements. The game-theoretic analysis of managing transboundary biological resources has become popular in the recent fishery economics literature. To best of our knowledge, little or nothing has been done to design an optimal management strategy for a diffusive small-mammal population the dynamics of which is governed by partial differential equation. Finally, the current model is treated as a finite time problem. An attempt to extend this problem to infinite planning horizon may be worth considering.

From an empirical view point, the task of testing this model for the Southeast still remains undone. This necessitates developing a reliable information base on the damage and cost parameters for forest ranges in this region. Perhaps, this will enable one to evolve location-specific management plans.

Some or all of the above extensions may require more powerful analytical techniques. Until now, the management of beavers has been the concern of land owners, biologists and state agencies. Because of the economic implications and complexity of the problem, it is time that economists and applied mathematicians be involved in designing a suitable control policy.

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APPENDICES

APPENDIX I

Proofs of Existence and Uniqueness of Solutions to the State Equation and the Optimality System

1. Statement of the Problem

The density of the beaver population is governed by the following parabolic partial differential equation with Lotka-Volterra growth term:

$$z_t - \alpha \Delta z = z(a - bz - p) \; in \; Q = \Omega \times (0,T) \tag{1.1}$$

$$z(x,0) = z_0(x) \; on \; \Omega, \; t=0 \tag{1.2}$$

$$z(x,t) = 0 \; on \; \partial\Omega \times (0,T) \tag{1.3}$$

The problem is posed for $\Omega \subset \mathbf{R}^n$. For practical applications, the habitat sits in $\mathbf{R}^2$, i.e. $\Omega \subset \mathbf{R}^2$. The numerical example is set in $\mathbf{R}^2$. The function z_0 gives the initial density distribution of the beaver population. Assuming that beavers do not live on the edge of the region, we have zero boundary conditions on the sides of the region, $\partial\Omega \times (0,T)$. The solution space for the state equation (1.1) - (1.3) is $W_2^{2,1}(Q) \cap \mathring{W}_2^{1,0}(Q)$, using notation from Ladyzenskaya, Solonnikov, and Ural'ceva (p. 5). The control variable is $p(x,t)$, which represents the proportion of the beaver population to be trapped.

Given the control set,

$$U_M = \{ \, p \in L^2(Q) \mid 0 \le p(x,t) \le M \, \},$$

we seek to minimize the cost functional:

$$J(p) = \int_Q e^{-rt} \left(\frac{1}{2} \gamma z^2 + cp^2 z \right) dx\,dt$$

We will characterize the optimal control that minimizes the cost functional:

$$\min_{p \in U_M} J(p).$$

The $\frac{1}{2} \gamma z^2$ term represents the damage due to beavers, and cp^2z term represents the cost of trapping the beavers.

Section 2 gives the existence of solutions to the state equation (1.1)-(1.3) and the existence of an optimal control. In section 3, an optimal control is characterized in terms of the optimality system. Uniqueness and existence results for the optimal system that yield an *analytic* construction of the unique optimal control are presented in section 4.

This method using optimality systems with iteration schemes to construct the optimal control was introduced by Stojanovic. See Leung and Stojanovic for results on optimal control of elliptic Lotka-Volterra type equations with different payoff functional than the one used here. Background concerning diffusive Volterra-Lotka equations and monotone schemes can be found in Fife and Leung.

2. Existence of a Solution to the State Equation and an Optimal Control

To guarantee solutions of the state equation, the following assumptions are made throughout this discussion:

$$z_0 \in H_0^1(\Omega), \quad z_0 \geq 0 \tag{2.1}$$

$$a,\ b,\ c,\ \gamma \in L^\infty(Q),\ a,\ b,\ \gamma \geq 0, \tag{2.2}$$

$$c(x,\ t) \geq c_0 > 0 \tag{2.3}$$

$$\alpha \in \mathbb{R},\quad \alpha > 0 \tag{2.4}$$

$\partial\Omega$ is smooth. See Ladyzenskaya, Solonnikov and Ural'ceva for notation explanations.

The following lemma constructs solutions of the state equation.

<u>Lemma 2.1</u>: Given $p \in U_M$, there exists a solution $z = z(p)$ to (1.1) - (1.3) with z in $W_2^{2,1}(Q) \cap \mathring{W}_2^{1,0}(Q)$.

<u>Proof:</u> Choose $C_1 > 0$ large enough such that $aC_1 - bC_1^2 \leq 0$ and $C_1 \geq |z_0|_{L^\infty(\Omega)}$.
Choose C_2 such that

$$sup(2bC_1) + sup(p-a) < C_2$$

Denote $z^0 = C_1$. Inductively define z^k, $k=1,2,...$ as the solution in $W_2^{2,1}(Q)$ of

$$z_t^k - \alpha\Delta z^k + C_2 z^k = z^{k-1}(a - bz^{k-1} - p) + C_2 z^{k-1} \ in\ Q \tag{2.5}$$

$$z^k = z_0 \quad when\ t=0$$

$$z^k = 0 \quad on\ \partial\Omega\times(0,\ T)$$

The RHS of (2.5) is non-decreasing function of z^k for $0 \leq z^k \leq z^0$. By comparing RHS of (2.5) for z^{k+1} and z^k, we obtain inductively

$$z^{k+1} \leq z^k \leq z^0, \qquad k = 1,2,...$$

Using (2.1) and the uniform boundedness of the RHS of (2.6),

$$|z^k|_{W_2^{2,1}(Q)} \leq C(|z_0|_{H^1}\ ,\ C_1).$$

The weak convergence of the z^k sequence in $W_2^{2,1}$ and $z^k \to z$ strongly in $L^2(Q)$ can be obtained. Using this convergence, z is the desired solution of (1.1) - (1.3).

<u>Remark:</u> Solutions to (1.1) - (1.3) are unique under the given assumptions. Use the construction in the proof of the previous lemma to define the map:

$$p \in U_M \quad \to \quad z=z(p),$$

where $z(p)$ solves (1.1) - (1.3) with p as control. Assume an *a priori* bound on $z(p)$, from the construction,

$$z(p) \le z^0 \le C_1 ,$$

which is independent of the bound M on U_M.

Next a minimizing sequence argument is used to prove the existence of an optimal control.

<u>Lemma 2.2:</u> With the cost functional $J(p)$ defined in (1.4), there exists an optimal control in U_M and corresponding state equation solution, which minimizes the cost functional.

<u>Proof:</u> Let $\{p_k\}$, k=1,2,..., be a minimizing sequence with corresponding solutions $z_k = z(p_k)$. Since the sequences $\{p_k\}$ and $\{z_k\}$, are bounded independent of k, the right hand sides of (1.1) for these sequences are bounded independent of k. Thus we obtain

$$\|z_k\|_{W_2^{2,1}(Q)} \le C \neq C(k).$$

There exists p* and z* such that

$$p_k \to p* \; weakly \;\; in \; L^2(Q)$$

$$z_k \to z* \; weakly \;\; in \; W_2^{2,1}(Q)$$

$$z_k \to z* \; strongly \;\; in \; L^2(Q).$$

In the weak formulation of (1.1) - (1.3),

$$\int_Q ((z_k)_t - \Delta z_k)\, \Phi = \int_Q (az_k - bz_k^2 - p_k z_k)\Phi,$$

we need the strong convergence of z_k for the convergence of the quadratic terms on the RHS. We have $z^* = z(p^*)$. Then using lower semicontinuity of the cost functional,

$$J(p^*) \leq \lim \int e^{-rt}(\frac{1}{2}\gamma z_k^2) + \lim \int e^{-rt} c p_k^2 z_k = \min_p J(p).$$

Thus p^* is a minimizer of $J(p)$.

3. The Derivation of the Optimality System (OS)

Now the optimality system is derived by differentiating the cost functional with respect to the control. Since the state variable z, solution of (1.1) - (1.3), is contained in the functional, we first need to show that z depends on the control in a differentiable way. Then we will characterize the optimal control in terms of the solution of the optimality system, which consists of the state equation coupled with an adjoint equation.

Proposition 3.1: The mapping

$$p \in U_M \to z=z(p) \in \mathring{W}_2^{1,0}(\Omega)$$

is differentiable in the following sense:

$$\frac{z(p+\varepsilon h)-z(p)}{\varepsilon} \to \psi \textit{ weakly in } \mathring{W}_2^{1,0}(Q)$$

as $\varepsilon \to 0$ for any $p \in U_M$, and $h \in L^\infty(Q)$ such that $p+\varepsilon h \in U_M$ for ε small. Also ψ is a solution of the following problem:

$$\psi_t = \alpha\Delta\psi + a\psi - 2bz(p)\,\psi - p\,\psi - hz(p) \textit{ in } Q \tag{3.1}$$

$$\psi = 0 \textit{ on } \Omega\times\{0\} \cup \partial\Omega\times(0,T) \tag{3.2}$$

Proof: For $p \in U_M$, $p+\varepsilon h \in U_M$ *for* ε small, denote $z = z(p)$ and $z^\varepsilon = z(p+\varepsilon h)$. The quotient $\frac{z^\varepsilon - z}{\varepsilon}$ satisfies

$$\left(\frac{z^\varepsilon-z}{\varepsilon}\right)_t - \alpha\Delta\left(\frac{z^\varepsilon-z}{\varepsilon}\right) + \left(\frac{z^\varepsilon-z}{\varepsilon}\right)[-a+b(z^\varepsilon+z)+p] = -hz^\varepsilon \textit{ in } Q \tag{3.2}$$

$$\frac{z^\varepsilon-z}{\varepsilon} = 0 \textit{ on } \Omega\times\{0\} \cup \partial\Omega\times(0,T).$$

Since the $-a + b(z^\varepsilon+z) + p$ is bounded independent of ε, one obtains

$$\left\|\frac{z^{\varepsilon}-z}{\varepsilon}\right\|_{L^2(Q)} + \left\|\nabla\left(\frac{z^{\varepsilon}-z}{\varepsilon}\right)\right\|_{L^2(Q)} \leq C(h)$$

and then following Ladyzenskaya, Solonnikov and Ural'ceva

$$\left\|\left(\frac{z^{\varepsilon}-z}{\varepsilon}\right)_t\right\|_{L^2(Q)} + \sum_{i,j=1}^{n}\left\|\left(\frac{z^{\varepsilon}-z}{\varepsilon}\right)_{x_i x_j}\right\|_{L^2(Q)} \leq C$$

We obtain the desired weak convergence,

$$\frac{z^{\varepsilon}-z}{\varepsilon} \rightharpoonup \psi, \text{ in } \mathring{W}_2^{1,0}(Q).$$

Noticing $z^{\varepsilon} \to z$ strongly in $L^2(Q)$, we have ψ satisfying (3.1).

Proposition 3.2: For an optimal control p^* and corresponding solution $z^*=z(p^*)$, there exists λ *in* $W_2^{2,1}(Q) \cap \mathring{W}_2^{1,0}(Q)$ satisfying the adjoint equation

$$-\lambda_t = \alpha\Delta\lambda + a\lambda - 2bz^*\lambda - r\lambda - p\lambda + \gamma z^* + c(p^*)^2 \text{ in } Q \tag{3.3}$$

$$\lambda(x,T) = 0 \text{ on } \Omega \tag{3.4}$$

$$\lambda(x,t) = 0 \text{ on } \partial\Omega\times(0,T), \tag{3.5}$$

and $p^* = \dfrac{\lambda}{2c}$.

Proof: Suppose p^* is an optimal control and z^* is its corresponding solution of (1.1) - (1.3) whose existence are guaranteed by Lemmas 2.1 and 2.2. Consider control $p^*+\varepsilon h \in U_M$ with associated solution $z^{\varepsilon} = z(p^*+\varepsilon h)$. Since the minimum of the cost functional J is achieved at p^*,

$$0 \leq \frac{J(p^*+\varepsilon h) - J(p^*)}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_Q\int e^{-rt}\left\{\frac{1}{2}\gamma\left[(z^{\varepsilon})^2-(z^*)^2\right] + c\left[(p^*)^2z^{\varepsilon}+2p^*\varepsilon hz^{\varepsilon}+\varepsilon^2h^2z^{\varepsilon}\right] - c(p^*)^2z^*\right\}dxdt$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \iint_Q e^{-rt} \left\{ \frac{1}{2}\gamma\left(\frac{z^\varepsilon - z^*}{\varepsilon}\right)(z^\varepsilon + z^*) + c\left[(p^*)^2\left(\frac{z^\varepsilon - z^*}{\varepsilon}\right) - 2p^* h z^\varepsilon + \varepsilon h^2 z^\varepsilon\right]\right\} dx\,dt$$

$$= \iint_Q e^{-rt} \{ \psi[\gamma z^* + c(p^*)^2] + 2cp^* h z^* \} dx\,dt \tag{3.6}$$

where $\frac{z^\varepsilon - z^*}{\varepsilon} \to \psi$. Using standard linear parabolic results (Ladyzenskaya, Solonnikov and Ural'ceva), there exists a solution $\lambda \epsilon W_2^{2,1}(Q) \cap \mathring{W}_2^{1,0}(Q)$ satisfying the adjoint equation (3.3) - (3.5). Substituting from the adjoint equation into the above inequality (3.5), we obtain

$$0 \le \int_Q\int e^{-rt}\left[\psi\left(-\lambda_t - \alpha\Delta\lambda - a\lambda + 2bz^*\lambda + r\lambda + p^*\lambda\right) + 2cp^* h z^*\right] dx\,dt.$$

Integration by parts together with equations (3.1), (3.2) yields

$$0 \le \int_Q e^{-rt} h z^*(-\lambda + 2cp^*)\, dx\,dt \tag{3.7}$$

Considering the form of the state equation and the adjoint equation, the parabolic maximum principle (Protter and Weinberger) implies

$$0 < z \;\; and \;\; 0 \le \lambda \;\; on \; Q.$$

Consider the following three cases:

(i) On the set $\{(x,t) \in Q \mid p^*(x,t) = 0\}$, we can choose non-negative variations h with support on this set. Inequality (3.7) implies $-\lambda + 2cp^* \ge 0$ *and* $-\lambda \ge 0$ on this set, which is a contradiction unless $\lambda = 0$. Thus $\lambda = 0$ on this set.

(ii) On the set $\{(x,t) \epsilon Q \mid 0 < p^*(x,t) < M\}$, we can choose variations h with arbitrary sign and with support on this set. Thus inequality (3.7) implies

$$-\lambda + 2cp^* = 0, \qquad p^* = \frac{\lambda}{2c}$$

on this set.

(iii) On the set $\{(x,t) \epsilon Q \mid p^*(x,t) = M\}$, we can choose non-positive variations h with support on this set. Using inequality (3.7) gives $-\lambda + 2cp^* \le 0$ on this set. In this case, $M = p^* \le \frac{\lambda}{2c}$.

Putting the three cases together,

$$p^* = \min\left(M, \frac{\lambda}{2c}\right)$$

<u>Claim:</u> There exists $M_1 > 0$ independent of M such that

$$\lambda \le M_1 \quad on \quad Q.$$

Since the bound on z^* is independent of M, the only possible "M" dependence in the λ bound could come from the p^* terms. But the terms involving p^* pull the solution λ down:

$$p^*(cp^* - \lambda) \le p^*\left(c\frac{\lambda}{2c} - \lambda\right) \le 0.$$

Thus, λ is bounded above by the bounded solution of

$$-\Lambda_t = \alpha\Delta\Lambda + a\Lambda + rz^* \quad in \ Q$$

$$\Lambda = 0 \quad on \quad t = T$$

$$\Lambda = 0 \quad on \quad \partial\Omega\times(0,T).$$

This bound of λ finishes the claim.

Now we choose M so large that

$$\frac{\lambda}{2c} \le \frac{M_1}{2c} < M.$$

The relationship between p^* and λ becomes

$$p^* = \frac{\lambda}{2c}.$$

Considering the relationship between an optimal control and the associated adjoint variable from this theorem, we now consider the following optimality system (OS):

$$z_t = \alpha\Delta z + z\left(a - bz - \frac{\lambda}{2c}\right) \text{ in } Q \tag{OS}$$

$$-\lambda_t = \alpha\Delta\lambda + a\lambda - 2bz\lambda - r\lambda - \frac{\lambda^2}{4c} + \gamma z \text{ in } Q$$

$$z(x,0) = z_0(x) \quad on \quad \Omega$$

$$\lambda(x,T) = 0 \quad on \quad \Omega$$

$$z(x,t) = \lambda(x,t) = 0 \quad on \quad \partial\Omega\times(0,T)$$

4. The Existence and Uniqueness Results for the OS

Now we prove the existence and uniqueness results for the OS, which will give us a characterization of the unique optimal control.

<u>Proposition 4.1</u>: When T is sufficiently small, solutions of OS are unique.

<u>Proof</u>: Suppose (z_1, λ_1), (z_2, λ_2) are two solutions. Let

$$e^{\mu t}w_1 = z_1, \quad e^{-\mu t}\Lambda_1 = \lambda_1, \quad e^{\mu t}w_2 = z_2, \quad e^{-\mu t}\Lambda_2 = \lambda_2.$$

where $\mu > 0$ to be chosen below. Let M_2 be an upper bound on z_1, λ_1, z_2 and λ_2. Multiply w_1 PDE by $w_1 - w_2$, Λ_1 PDE by $\Lambda_1 - \Lambda_2$, w_2 PDE by $w_2 - w_1$ and Λ_2 PDE by $\Lambda_2 - \Lambda_1$, integrate on Q, and add all four equations together. The w_1 and Λ_1 equations are written for illustration.

$$-e^{-\mu t}(\Lambda_1)_t - e^{-\mu t}\alpha\Delta(\Lambda_1) + \mu e^{-\mu t}\Lambda_1 = ae^{-\mu t}\Lambda_1 - 2bw_1\Lambda_1 - re^{-\mu t}\Lambda_1 - e^{-2\mu t}\frac{\Lambda_1^2}{4c} + \gamma e^{\mu t}w_1$$

$$-(\Delta_1)_t - \alpha\Delta\Delta_1 + \mu\Delta_1 = (a-r)\Delta_1 - 2bw_1\Delta_1 e^{\mu t} - e^{-\mu t}\frac{\Delta_1^2}{4c} + \gamma e^{2\mu t}w_1$$

$$(w_1)_t - \alpha\Delta w_1 + \mu w_1 = aw_1 - be^{\mu t}w_1^2 - e^{-\mu t}w_1\frac{\Delta_1}{2c}$$

The result after adding the four PDEs is:

$$\int_Q \Big[(w_1-w_2)_t(w_1-w_2)-(\Lambda_1-\Lambda_2)_t(\Lambda_1-\Lambda_2)-\alpha(w_1-w_2)\Delta(w_1-w_2)-\alpha(\Lambda_1-\Lambda_2)\Delta(\Lambda_1-\Lambda_2)$$

$$+\mu(w_1-w_2)^2+\mu(\Lambda_1-\Lambda_2)^2 \Big]$$

$$= \int_Q \Big[a(w_1-w_2)^2+(a-r)(\Lambda_1-\Lambda_2)^2-be^{\mu t}\big(w_1^2-w_2^2\big)(w_1-w_2)$$

$$+(w_2\Lambda_2-w_1\Lambda_1)\frac{e^{-\mu t}}{2c}(w_1-w_2) \tag{4.8}$$

$$+2(w_2\Lambda_2-w_1\Lambda_1)be^{\mu t}(\Lambda_1-\Lambda_2)$$

$$+\gamma e^{2\mu t}(w_1-w_2)(\Lambda_1-\Lambda_2)-\frac{e^{\mu t}}{4c}\big(\Lambda_1^2-\Lambda_2^2\big)(\Lambda_1-\Lambda_2) \Big]$$

Simplifying the time derivative terms,

$$\int_Q \left[(w_1-w_2)_t(w_1-w_2)-(\Lambda_1-\Lambda_2)_t(\Lambda_1-\Lambda_2) \right]$$

$$= \int_Q \left(\frac{(w_1-w_2)^2}{2}\right)_t - \left(\frac{(\Lambda_1-\Lambda_2)^2}{2}\right)_t$$

$$= \frac{1}{2}\int_\Omega \left[(w_1-w_2)^2(x,T)+(\Lambda_1-\Lambda_2)^2(x,0) \right]$$

The Laplacian terms become

$$\int \left(\alpha\,|\nabla(w_1-w_2)|^2 + \alpha\,|\nabla(\Lambda_1-\Lambda_2)|^2\right).$$

The terms on RHS of (4.8),

$$-\ \frac{e^{-\mu t}}{4c}\ \big(\Lambda_1^2-\Lambda_2^2\big)(\Lambda_1-\Lambda_2) \quad \text{and} \quad -be^{\mu t}\big(w_1^2-w_2^2\big)(w_1-w_2)$$

are non-positive. Using the positivity of the time derivative and Laplacian terms,

$$\mu \int_Q ((w_1-w_2)^2+(\Lambda_1-\Lambda_2)^2) \leq \int_Q \Big[a((w_1-w_2)^2+(\Lambda_1-\Lambda_2)^2)$$

$$+\frac{e^{-\mu t}}{2c}|w_2\Lambda_2-w_1\Lambda_1||w_1-w_2|+2be^{\mu t}|w_2\Lambda_2-w_1\Lambda_1||\Lambda_1-\Lambda_2| \tag{4.9}$$

$$+e^{2\mu t}\gamma|w_1-w_2||\Lambda_1-\Lambda_2|\Big]$$

Estimating terms from (4.9), we have

$$\frac{e^{-\mu t}}{2c}|w_2\Lambda_2-w_1\Lambda_1||w_1-w_2| \leq \frac{e^{-\mu t}}{2c}|w_2(\Lambda_2-\Lambda_1)+(w_2-w_1)(\Lambda_1)||w_1-w_2|$$

$$\leq \frac{1}{4c}e^{-\mu t}\ sup\ (|w_2|,|\Lambda_1|)\ ((\Lambda_2-\Lambda_1)^2+3(w_1-w_2)^2)$$

$$\leq \frac{M_2}{4c}((\Lambda_2-\Lambda_1)^2+3(w_1-w_2)^2)$$

$$2be^{\mu t}|w_2\Lambda_2-w_1\Lambda_1||\Lambda_1-\Lambda_2| \leq be^{\mu t}\ sup\ (|w_2|,|\Lambda_1|)\cdot$$

$$\cdot\ (3(\Lambda_2-\Lambda_1)^2+(w_1-w_2)^2)$$

$$\leq\ 3be^{2\mu T}M_2((\Lambda_2-\Lambda_1)^2+(w_1-w_2)^2)$$

$$\gamma e^{2\mu t}|w_1-w_2||\Lambda_1-\Lambda_2| \leq \frac{\gamma}{2}e^{2\mu T}((w_1-w_2)^2+(\Lambda_1-\Lambda_2)^2).$$

Combining these estimates, (4.9) implies

$$\mu \int_Q ((w_1-w_2)^2+(\Lambda_1-\Lambda_2)^2)$$

$$\leq (C_a+e^{2\mu T}M_2C_3)\int_Q ((w_1-w_2)^2+(\Lambda_1-\Lambda_2)^2)$$

where C_3 depends on γ, b, c_0, *and* C_a depends on a. If $\mu > C_a + e^{2\mu T}M_2C_3$ then the solution is unique. Rearranging and taking natural logarithm of both sides,

$$\frac{\ln(\mu-C_a)-\ln(MC_3)}{2\mu} > T,$$

which can be satisfied if μ is large and T is sufficiently small.

Theorem 4.2: If T is sufficiently small, there exists a solution pair z and λ satisfying the optimality system (OS).

Proof: Define

$$f(z,\lambda) = z\left(a-bz-\frac{\lambda}{2c}\right)+Rz$$

and

$$g(\lambda,z,w) = \lambda(a-2bz) + \gamma w + (R-r)\lambda-\frac{\lambda^2}{4c}.$$

For R large enough,

> f is an increasing function of z,
> f is a decreasing function of λ,
> g is an increasing function in λ and in w, and
> g is a decreasing function in z,

for $0 \le z \le C_1$ and $0 \le \lambda \le M_1$ (where C_1, M_1 are the upper bounds on our solutions of the state and adjoint equations.).

Define z^k and λ^k, $k=1,2,...$, as solutions of

$$z_t^k-\alpha\Delta z^k+Rz^k = f(z^{k-2}, \lambda^{k-2}) \text{ in } Q$$

$$z = z_0 \text{ on } t = 0$$

$$z = 0 \text{ on } \partial\Omega\times(0,T)$$

$$-\lambda_t^k - \alpha\Delta\lambda^k + R\lambda^k = g(\lambda^{k-2}, z^k, z^{k-1}) \text{ in } Q$$

$$\lambda = 0 \quad \text{on} \quad t=T$$

$$\lambda = 0 \quad \text{on} \quad \partial\Omega\times(0,T)$$

where $z^0 = 0$, $z^{-1}=C_1$, $\lambda^{-1}=0$, $\lambda^0=M_1$. Comparing the RHS of z^1 PDE and the RHS of z^2 PDE and using the properties of f, we obtain

$$z^2 \le z^1.$$

Notice that all iterates are non-negative. Also

$$z_t^{-1} - \alpha\Delta z^{-1} + Rz^{-1} \ge f(z^{-1}, \lambda^{-1}) = z_t^1 - \alpha\Delta z^1 + Rz^1$$

implies $z^1 \le z^{-1} = C_1$. By choosing M_1 large enough (depending on a, γ, and c), we can similarly obtain

$$\lambda^2 \le \lambda^0.$$

Comparing RHS of λ^1 PDE and RHS of λ^2 PDE and using properties of g, we obtain

$$\lambda^1 \le \lambda^2.$$

Continuing we have

$$z^{2k} \le z^{2k+2}\ ,\ z^{2k+3} \le z^{2k+1}$$

and

$$\lambda^{2k+1} \le \lambda^{2k+3}\ ,\ \lambda^{2k+2} \le \lambda^{2k}$$

With uniform *a priori* estimates as in Lemma 2.1 and Proposition 3.1, we obtain

$$z^{2k} \nearrow z\ \ ,\ \ z^{2k+1} \searrow \bar{z}$$

and

$$\lambda^{2k} \searrow \bar{\lambda}\ ,\ \lambda^{2k+1} \nearrow \lambda$$

and the limiting functions $(z, \bar{\lambda}, \bar{z}, \lambda)$ satisfy

$$z_t - \alpha\Delta z + Rz = f(z, \bar{\lambda}) \quad in \quad Q$$

$$-\bar{\lambda}_t - \alpha\Delta\bar{\lambda} + R\bar{\lambda} = g(\bar{\lambda}, z, \bar{z})$$

$$\bar{z}_t - \alpha\Delta\bar{z} + r\bar{z} = f(\bar{z}, \lambda) \tag{4.10}$$

$$-\lambda_t - \alpha\Delta\lambda + R\lambda = g(\lambda, \bar{z}, z)$$

$$z = \bar{z} = z_0 \quad when \quad t=0$$

$$\lambda = \bar{\lambda} = 0 \quad when \quad t=T$$

$$z = \bar{z} = \lambda = \bar{\lambda} = 0 \quad on \quad \partial\Omega\times(0,T).$$

Notice $(\bar{z}, \lambda, z, \bar{\lambda})$ is also a solution of that system. A uniqueness result, similar to Proposition 4.1, gives that solutions of system (4.10) are unique. Hence, we conclude $\bar{z} = z$ and $\bar{\lambda} = \lambda$, and $(\bar{z}, \bar{\lambda})$ is the desired solution to the optimality system.

Combining Proposition 4.1 and Theorem 4.2, we have our desired characterization of the unique optimal control,

$$p^* = \frac{\lambda}{2c}$$

where (p^*, λ) is the unique solution of OS.

APPENDIX II

The Computer Algorithm for Numerical Simulation

```
*****************************************************************************************************
*     THIS PROGRAM SIMULATES THE GROWTH OF OPTIMAL BEAVER POPULATION AND TRAPPING OVER        *
*     TIME AND SPACE SUBJECT TO THEIR GROWTH AND SPATIAL DIFFUSION FOLLOWING PARTIAL          *
*     DIFFERENTIAL EQUATION IN ORDER TO MINIMIZE DISCOUNTED SOCIAL COST TO SOCIETY.           *
*     POPULATION IN THIS PROGRAM IS DEFINED AS HEADS PER SQUARE MILES.                        *
*     X AND Y ARE SPATIAL CORDINATES (SPACE VARIABLES).                                       *
*****************************************************************************************************

*     VARIABLE DECLARATION

      INTEGER          I, J, K               ! LOOP VARIABLES CORR. TO X, Y, AND TIME VARIABLES
      INTEGER          M, N, T               ! NUMBER OF GRID POINTS ON X, Y, AND TIME VARIABLES[1]
      INTEGER          V                     ! INDEX DENOTING CURRENT AND PREVIOUS ITERATIONS
      INTEGER          II                    ! INDEX COUNTING THE # OF SIMULATIONS
      INTEGER          III                   ! INDEX COUNTING THE TIME PERIODS
      REAL*8           DX, DY, DT            ! CHANGE IN X, Y AND TIME VARIABLES
      REAL*8           Z(40, 40, 200,2)      ! STATE VARIABLE
      REAL*8           LD(40, 40,200,2)      ! COSTATE (ADJOINT) VARIABLE
      REAL*8           P(40, 40,200,2)       ! RATE OF HARVEST (CONTROL VAREABLE)
      REAL             ALPHA, A,B            ! BIOLOGICAL PARAMETERS
      REAL             R, C, GEMA            ! ECONOMIC PARAMETERS
      REAL             ZSUM LSUM             ! COMPARISION VARIABLES FOR Z AND LD
      REAL             SLOS                  ! SOCIAL LOSS FUNCTIONAL VALUE
      REAL             NBT                   ! NUMBER OF BEAVERS TRAPPED

      CHARACTER  INTPAR                      ! INPUT FILE FOR INITIAL PARAMETER
      CHARACTER  INTPOP                      ! INPUT FILE FOR INITIAL POPULATION
      CHARACTER  OPTSOLN                     ! OUTPUT FILE

*     READING INPUT PARAMETERS

      OPEN (7, FILE = 'INTPAR', STATUS = 'OLD')

         READ (7, *) ALPHA, A, B
         READ (7, *) C, GEMA, R
         READ (7, *) M, N, T
         READ (7, *) DX, DY

      CLOSE (7)

      V = 1
      II = 0
      DT = (0.48)*(((DX**2)*(DY**2))/(((DX**2)+(DY**2))*ALPHA))

*     INVOKING  INITIAL POPULATION INPUT FILE. NOTE THAT POPULATIONS ON SIDE
      BOUBDARIES ARE ZERO

      OPEN (8, FILE = 'INTPOP', STATUS = 'OLD')

         K = 1
         DO J = 1, N

            READ (8, *) (Z(I,J,K,V), I = 1, M)

         END DO

      CLOSE (8)

*     INITIALIZING Z AS ZERO ON ALL THE GRID POINTS INCLUDING SIDE BOUNDARIES
*     FROM PERIOD K = 2 TO T SO AS TO PROVIDE A BASIS FOR COMPARISION OF
*     THE RESULTS OF FIRST ROUND ITERATION OF STATE EQUATION

      DO K = 2, T
         DO J = 1, N
```

[1]The actual planning period starts from time zero. But in this program, the initial values of all the loop variables are set to 1. Therefore, the values M, N and T may be set *one* more than the actual number of grid points on each of the axes.

```
         DO I = 1, M

            Z(I,J,K,V) = 0.0

         END DO
      END DO
   END DO

*     INITIALIZING THE VALUES OF COSTATE VARIABLE TO ZERO FOR ALL THE
*     GRID POINTS INCLUDING THE TERMINAL POINTS.

      DO K = 1, T
         DO J = 1, N
            DO I = 1, M

               LD(I,J,K,V) = 0.0

            END DO
         END DO
      END DO

*     TRANSFERING THE ARRAY OF STATE AND COSTATE VARIABLES OF CURRENT ITERATION
*     TO AN INTERMEDIATE BUFFER

30    II = II + 1

      DO K = 1, T
         DO J = 1, N
            DO I = 1, M

               Z(I,J,K,V+1)  = Z(I,J,K,V)
               LD(I,J,K,V+1) = LD(I,J,K,V)

            END DO
         END DO
      END DO

*     SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION IN STATE VARIABLE GENERATING THE OPTIMAL
*     ARRAY OF Z FOR GIVEN INITIAL POPULATION AND CURRENT LEVELS OF LD VALUES.

       DO K = 1, T-1
          DO J = 2, N-1
             DO I = 2, M-1

                Z(I,J,K+1,V) = (ALPHA*DT)*((1.0/(DX**2))*(Z(I+1,J,K,V) -
     1                         2.0*Z(I,J,K,V) + Z(I-1,J,K,V)) + (1.0/
     1                         (DY**2))*(Z(I,J+1,K,V) - 2.0*Z(I,J,K,V)
     1                         + Z(I,J-1,K,V))) + DT*Z(I,J,K,V)*
     1                         (A - B*Z(I,J,K,V) -
     1                         (LD(I,J,K,V)/(2.0*C))) + Z(I,J,K,V)

             END DO
          END DO
       END DO

*     SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION IN LD VARIABLE USING TERMINAL AND SIDE BOUNDARY
*     CONDITIONS FOR LD, AND OPTIMAL Z VALUES OBTAINED FROM THE ABOVE ITERATION.
*     LD VALUES ARE GENERATED BACKWARD FROM T-1 TO 1 PERIOD.

       DO K = 1, T-1
          DO J = 2, N-1
             DO I = 2, M-1

                LD(I,J,T-K,V) = (DT*ALPHA)*((1.0/(DX**2))*
```

```
     1                          (LD(I+1,J,T+1-K,V)
     1                          - 2.0*LD(I,J,T+1-K,V) +
     1                          LD(I-1,J,T+1-K,V))
     1                          + (1.0/(DY**2))*(LD(I,J+1,T+1-K,V) -
     1                          2.0*LD(I,J,T+1-K,V) +
     1                          LD(I,J-1,T+1-K,V)))
     1                          + DT*LD(I,J,T+1-K,V)*(-R + A
     1                          -(2.0*B*Z(I,J,T+1-K,V)) -
     1                          (LD(I,J,T+1-K,V)/(4.0*C)))
     1                          + DT*GEMA*Z(I,J,T+1-K,V)
     1                          + LD(I,J,T+1-K,V)

          END DO
        END DO
      END DO

*     CHECKING FOR CONVERGENCE OF Z AND LD BETWEEN TWO SUCCESSIVE ITERATIONS.  ITERATION STOPS IF
*     DIFFERENCE OF WEIGHTED VALUES OF Z AND LD BETWEEN TWO ITERATIONS IS LESS THAN OR EQUAL TO
*     PRESPECIFIED TOLERANCE LIMIT.

      DO K = 1, T

         ZSUM = 0.0
         LSUM = 0.0

            DO J = 2, N-1
               DO I = 2, M-1

                  ZSUM = ZSUM + ABS(LD(I,J,K,V) - LD(I,J,K,V+1))
                  LSUM = LSUM + ABS(Z(I,J,K,V)-Z(I,J,K,V+1))

               END DO
            END DO

                  IF ( ZSUM*(DX*DY) .GT. .0001) GO TO 30
                  IF ( LSUM*(DX*DY) .GT. 0.001) GO TO 30

      END DO

*     FINDING OPTIMAL RATE OF TRAPPING P

      DO K = 2, T
         DO J = 1, N
            DO I = 1, M

               P(I,J,K,V) = LD(I,J,K,V)/(2.0*C)

            END DO
         END DO
      END DO

*     COMPUTATION OF THE VALUE OF SOCIAL LOSS FUNCTIONAL.

      SLOS = 0.0
      NBT = 0.0
      III = 0

      DO K = 2, T
```

```
         III = III + 1

         DO J = 1, N
            DO I = 1, M

                  SLOS = SLOS + (Z(I,J,K,V)*(0.5*GEMA*Z(I,J,K,V) +
     1                            C*(P(I,J,K,V)**2))*DX*DY*
     1                            DT)*exp(-R*DT*III)

                   NBT = NBT + Z(I,J,K,V)*P(I,J,K,V)*DX*DY

             END DO
         END DO

        END DO

*     SENDING THE RESULTS TO OUTPUT FILE NAMED 'OPTSOLN'.

      OPEN (12, FILE = 'OPTSOLN', STATUS = 'NEW' )

        WRITE (12,*) 'ALPHA =  ', ALPHA, ' A = ', A, 'B = ', B
        WRITE (12,*) 'R = ', R, 'C = ', C, 'GEMA= ', GEMA
        WRITE (12,*) 'M= ', M, 'N = ', N,    ' T = ',T
        WRITE (12,*) 'DX = ', DX, 'DY = ', DY, 'DT = ',DT
        WRITE (12,*) 'NUMBER OF SIMULATIONS = ', II
        WRITE (12,*) 'VALUE OF THE SOCIAL LOSS FUNCTIONAL = ', SLOS
        WRITE (12,*) 'NUMBER OF BEAVERS TRAPPED', NBT

        WRITE (12,*) 'OPTIMAL POPULATION DENSITIES OF BEAVERS, Z(x, y, t)'

        III = 0

        DO K = 1, T

           WRITE (12, *) 'TIME PERIOD =  ', III

             DO J = 1, N

                WRITE (12, 100) J,  (Z (I,J,K,V), I = 1, M)
100             FORMAT (I3,6X,7F8.4)

             END DO

             III = III + 1

        END DO

        WRITE (12,*) 'OPTIMAL TRAPPING RATES OF BEAVERS, P(x, y, t)'

        III = 1

        DO K = 1, T

           WRITE (12,*) 'TIME PERIOD =  ', III

             DO J = 1, N

                WRITE (12, 101) J, (P (I,J,K,V), I = 1, M)
101             FORMAT (I3,6X, 7F7.3)

             END DO

             III = III + 1

        END DO

END
```

APPENDIX III

The Baseline Simulation Values of the Optimal Beaver Population and Trapping Rates

This is the result of baseline simulation with constant parameters and fine grid intervals:

Simulation Parameters:

α = 725.27	a = 0.335	b = 0.2066315
r = 0.056	c = 182.71	γ = 144.59
M = 12	N = 20	T = 40
dx = 6.27	dy = 6.27	dt = 0.01302

Number of Simulations = 7

The value of the Objective Function = $136,500.80

The number of Beavers Trapped = 2,314

Optimal population densities of beavers, $Z(x_i, y_j, t)$

t = 0

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_9	0.0000	0.0000	0.0000	2.1326	3.4528	3.3613	0.0000
y_8	0.0000	1.2282	1.3075	2.1642	3.5443	3.5443	0.0000
y_7	0.0000	1.0260	0.8862	1.3457	2.1335	0.7227	0.0000
y_6	0.0000	1.1827	0.9300	0.7271	0.7768	0.7768	0.0000
y_5	0.0000	0.7290	0.8539	0.9391	0.9391	0.9391	0.0000
y_4	0.0000	0.7510	0.7939	0.9391	0.9391	0.9391	0.0000
y_3	0.0000	0.7613	0.8700	0.9391	0.9391	0.9391	0.0000
y_2	0.0000	0.9391	0.9391	0.9391	0.9391	0.9391	0.0000
y_1	0.0000	0.9391	0.9391	0.9391	0.9391	0.9391	0.0000
y_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

t = 1

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_9	0.0000	0.2584	0.6505	1.4708	2.1511	1.7555	0.0000
y_8	0.0000	0.6592	1.1360	1.9143	2.5738	2.0202	0.0000
y_7	0.0000	0.7775	1.1025	1.4588	1.7292	1.2278	0.0000
y_6	0.0000	0.7263	0.9332	0.9944	1.0307	0.7098	0.0000
y_5	0.0000	0.6295	0.8517	0.8937	0.8973	0.6655	0.0000
y_4	0.0000	0.5815	0.8288	0.9089	0.9192	0.6897	0.0000
y_3	0.0000	0.6113	0.8587	0.9222	0.9236	0.6923	0.0000
y_2	0.0000	0.6526	0.8893	0.9205	0.9088	0.6792	0.0000
y_1	0.0000	0.5025	0.6780	0.6924	0.6793	0.5046	0.0000
y_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

t = 2

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_9	0.0000	0.3059	0.6867	1.1995	1.5261	1.1457	0.0000
y_8	0.0000	0.5609	1.0765	1.6570	1.9900	1.4587	0.0000
y_7	0.0000	0.6470	1.0771	1.4229	1.5652	1.0976	0.0000
y_6	0.0000	0.6076	0.9241	1.0649	1.0643	0.7158	0.0000
y_5	0.0000	0.5494	0.8272	0.9116	0.8765	0.5888	0.0000
y_4	0.0000	0.5222	0.8028	0.8944	0.8619	0.5843	0.0000
y_3	0.0000	0.5326	0.8166	0.9033	0.8652	0.5860	0.0000
y_2	0.0000	0.5219	0.7892	0.8570	0.8107	0.5461	0.0000

y_1	0.0000	0.3602	0.5413	0.5820	0.5460	0.3657	0.0000
y_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

t = 3

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_9	0.0000	0.3061	0.6415	0.9872	1.1355	0.8030	0.0000
y_8	0.0000	0.5135	0.9962	1.4314	1.5851	1.1035	0.0000
y_7	0.0000	0.5759	1.0220	1.3373	1.3952	0.9433	0.0000
y_6	0.0000	0.5437	0.8993	1.0743	1.0472	0.6839	0.0000
y_5	0.0000	0.4983	0.8021	0.9150	0.8570	0.5494	0.0000
y_4	0.0000	0.4774	0.7696	0.8729	0.8117	0.5196	0.0000
y_3	0.0000	0.4738	0.7632	0.8621	0.7981	0.5100	0.0000
y_2	0.0000	0.4351	0.6967	0.7796	0.7156	0.4550	0.0000
y_1	0.0000	0.2806	0.4478	0.4979	0.4539	0.2872	0.0000
y_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

t = 4

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_9	0.0000	0.2890	0.5770	0.8209	0.8794	0.5942	0.0000
y_8	0.0000	0.4746	0.9079	1.2397	1.2945	0.8651	0.0000
y_7	0.0000	0.5291	0.9580	1.2334	1.2363	0.8089	0.0000
y_6	0.0000	0.5029	0.8659	1.0484	1.0013	0.6375	0.0000
y_5	0.0000	0.4634	0.7758	0.9026	0.8323	0.5191	0.0000
y_4	0.0000	0.4419	0.7345	0.8435	0.7680	0.4754	0.0000
y_3	0.0000	0.4269	0.7082	0.8098	0.7335	0.4527	0.0000
y_2	0.0000	0.3736	0.6182	0.7032	0.6334	0.3893	0.0000
y_1	0.0000	0.2302	0.3803	0.4309	0.3863	0.2364	0.0000
y_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

t = 5

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_9	0.0000	0.2658	0.5121	0.6908	0.7031	0.4587	0.0000
y_8	0.0000	0.4372	0.8209	1.0787	1.0788	0.6983	0.0000
y_7	0.0000	0.4914	0.8912	1.1272	1.0956	0.6982	0.0000
y_6	0.0000	0.4722	0.8272	1.0032	0.9420	0.5879	0.0000
y_5	0.0000	0.4368	0.7473	0.8778	0.8010	0.4908	0.0000
y_4	0.0000	0.4125	0.6987	0.8088	0.7274	0.4414	0.0000
y_3	0.0000	0.3883	0.6556	0.7548	0.6749	0.4077	0.0000
y_2	0.0000	0.3274	0.5520	0.6335	0.5640	0.3395	0.0000
y_1	0.0000	0.1952	0.3290	0.3767	0.3343	0.2006	0.0000
y_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

t = 6

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_9	0.0000	0.2413	0.4529	0.5881	0.5764	0.3661	0.0000
y_8	0.0000	0.4007	0.7398	0.9441	0.9141	0.5773	0.0000
y_7	0.0000	0.4571	0.8250	1.0263	0.9739	0.6080	0.0000
y_6	0.0000	0.4455	0.7852	0.9487	0.8781	0.5396	0.0000
y_5	0.0000	0.4141	0.7164	0.8446	0.7644	0.4626	0.0000
y_4	0.0000	0.3871	0.6630	0.7708	0.6881	0.4124	0.0000
y_3	0.0000	0.3557	0.6070	0.7016	0.6223	0.3711	0.0000
y_2	0.0000	0.2910	0.4962	0.5720	0.5056	0.3005	0.0000

y_1	0.0000	0.1693	0.2887	0.3324	0.2930	0.1737	0.0000
y_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

t = 7

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_9	0.0000	0.2177	0.4007	0.5061	0.4822	0.3000	0.0000
y_8	0.0000	0.3659	0.6661	0.8313	0.7854	0.4867	0.0000
y_7	0.0000	0.4246	0.7612	0.9333	0.8693	0.5341	0.0000
y_6	0.0000	0.4204	0.7413	0.8906	0.8146	0.4946	0.0000
y_5	0.0000	0.3931	0.6835	0.8061	0.7246	0.4347	0.0000
y_4	0.0000	0.3641	0.6276	0.7313	0.6495	0.3862	0.0000
y_3	0.0000	0.3276	0.5626	0.6518	0.5752	0.3403	0.0000
y_2	0.0000	0.2615	0.4487	0.5186	0.4561	0.2690	0.0000
y_1	0.0000	0.1492	0.2560	0.2956	0.2595	0.1527	0.0000
y_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

t = 8

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_9	0.0000	0.1959	0.3554	0.4397	0.4101	0.2511	0.0000
y_8	0.0000	0.3335	0.6002	0.7363	0.6826	0.4168	0.0000
y_7	0.0000	0.3936	0.7009	0.8492	0.7794	0.4730	0.0000
y_6	0.0000	0.3960	0.6970	0.8322	0.7538	0.4534	0.0000
y_5	0.0000	0.3728	0.6494	0.7647	0.6835	0.4073	0.0000
y_4	0.0000	0.3428	0.5930	0.6914	0.6118	0.3618	0.0000
y_3	0.0000	0.3029	0.5223	0.6057	0.5328	0.3136	0.0000
y_2	0.0000	0.2369	0.4080	0.4721	0.4140	0.2429	0.0000
y_1	0.0000	0.1330	0.2291	0.2649	0.2319	0.1358	0.0000
y_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

t = 9

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_9	0.0000	0.1763	0.3163	0.3852	0.3535	0.2138	0.0000
y_8	0.0000	0.3036	0.5417	0.6557	0.5992	0.3617	0.0000
y_7	0.0000	0.3642	0.6449	0.7737	0.7018	0.4217	0.0000
y_6	0.0000	0.3720	0.6533	0.7755	0.6967	0.4159	0.0000
y_5	0.0000	0.3528	0.6147	0.7223	0.6424	0.3807	0.0000
y_4	0.0000	0.3228	0.5593	0.6520	0.5752	0.3388	0.0000
y_3	0.0000	0.2810	0.4856	0.5634	0.4945	0.2900	0.0000
y_2	0.0000	0.2159	0.3727	0.4316	0.3777	0.2209	0.0000
y_1	0.0000	0.1196	0.2065	0.2390	0.2088	0.1219	0.0000
y_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

t = 10

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_9	0.0000	0.1588	0.2826	0.3400	0.3082	0.1846	0.0000
y_8	0.0000	0.2765	0.4899	0.5869	0.5303	0.3172	0.0000
y_7	0.0000	0.3365	0.5931	0.7060	0.6345	0.3784	0.0000
y_6	0.0000	0.3486	0.6107	0.7214	0.6438	0.3820	0.0000
y_5	0.0000	0.3330	0.5801	0.6800	0.6022	0.3554	0.0000
y_4	0.0000	0.3037	0.5268	0.6136	0.5400	0.3170	0.0000
y_3	0.0000	0.2613	0.4521	0.5246	0.4596	0.2688	0.0000

y_2	0.0000	0.1978	0.3420	0.3961	0.3461	0.2020	0.0000
y_1	0.0000	0.1084	0.1874	0.2169	0.1892	0.1102	0.0000
y_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Optimal trapping rates of beavers, $P(x_i, y_j, t)$

$t = 1$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_9	0.000	0.031	0.060	0.082	0.087	0.059	0.000
y_8	0.000	0.055	0.099	0.128	0.130	0.086	0.000
y_7	0.000	0.065	0.112	0.137	0.132	0.084	0.000
y_6	0.000	0.065	0.109	0.129	0.119	0.074	0.000
y_5	0.000	0.061	0.103	0.118	0.107	0.066	0.000
y_4	0.000	0.057	0.095	0.110	0.099	0.061	0.000
y_3	0.000	0.052	0.087	0.099	0.089	0.055	0.000
y_2	0.000	0.043	0.071	0.081	0.072	0.045	0.000
y_1	0.000	0.026	0.043	0.048	0.043	0.027	0.000
y_0	0.000	0.000	0.000	0.000	0.000	0.000	0.000

$t = 2$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_9	0.000	0.028	0.052	0.069	0.069	0.045	0.000
y_8	0.000	0.048	0.087	0.110	0.107	0.069	0.000
y_7	0.000	0.056	0.099	0.121	0.114	0.072	0.000
y_6	0.000	0.057	0.098	0.116	0.106	0.065	0.000
y_5	0.000	0.054	0.092	0.107	0.096	0.058	0.000
y_4	0.000	0.050	0.085	0.098	0.088	0.053	0.000
y_3	0.000	0.045	0.076	0.088	0.078	0.047	0.000
y_2	0.000	0.037	0.062	0.070	0.063	0.038	0.000
y_1	0.000	0.021	0.036	0.041	0.036	0.022	0.000
y_0	0.000	0.000	0.000	0.000	0.000	0.000	0.000

$t = 3$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_9	0.000	0.024	0.045	0.057	0.055	0.035	0.000
y_8	0.000	0.041	0.074	0.093	0.089	0.056	0.000
y_7	0.000	0.049	0.086	0.105	0.098	0.061	0.000
y_6	0.000	0.050	0.086	0.102	0.093	0.057	0.000
y_5	0.000	0.047	0.081	0.094	0.084	0.051	0.000
y_4	0.000	0.044	0.075	0.086	0.077	0.046	0.000
y_3	0.000	0.039	0.066	0.076	0.068	0.040	0.000
y_2	0.000	0.031	0.053	0.060	0.053	0.032	0.000
y_1	0.000	0.018	0.030	0.034	0.030	0.018	0.000
y_0	0.000	0.000	0.000	0.000	0.000	0.000	0.000

$t = 4$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_9	0.000	0.021	0.037	0.047	0.044	0.028	0.000
y_8	0.000	0.035	0.063	0.077	0.073	0.045	0.000
y_7	0.000	0.042	0.074	0.089	0.082	0.050	0.000
y_6	0.000	0.043	0.074	0.088	0.079	0.048	0.000

y_5	0.000	0.040	0.070	0.081	0.073	0.044	0.000
y_4	0.000	0.037	0.064	0.074	0.066	0.039	0.000
y_3	0.000	0.033	0.056	0.065	0.057	0.034	0.000
y_2	0.000	0.026	0.044	0.051	0.045	0.026	0.000
y_1	0.000	0.015	0.025	0.029	0.025	0.015	0.000
y_0	0.000	0.000	0.000	0.000	0.000	0.000	0.000

t = 5

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_9	0.000	0.017	0.030	0.038	0.035	0.022	0.000
y_8	0.000	0.029	0.052	0.063	0.058	0.036	0.000
y_7	0.000	0.035	0.061	0.073	0.067	0.041	0.000
y_6	0.000	0.035	0.062	0.073	0.066	0.040	0.000
y_5	0.000	0.034	0.058	0.068	0.061	0.036	0.000
y_4	0.000	0.031	0.053	0.062	0.055	0.033	0.000
y_3	0.000	0.027	0.047	0.054	0.048	0.028	0.000
y_2	0.000	0.021	0.036	0.042	0.037	0.022	0.000
y_1	0.000	0.012	0.020	0.023	0.020	0.012	0.000
y_0	0.000	0.000	0.000	0.000	0.000	0.000	0.000

t = 6

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_9	0.000	0.013	0.024	0.029	0.027	0.016	0.000
y_8	0.000	0.023	0.041	0.049	0.045	0.028	0.000
y_7	0.000	0.028	0.049	0.058	0.053	0.032	0.000
y_6	0.000	0.028	0.050	0.059	0.053	0.032	0.000
y_5	0.000	0.027	0.047	0.055	0.049	0.029	0.000
y_4	0.000	0.025	0.043	0.050	0.044	0.026	0.000
y_3	0.000	0.022	0.037	0.043	0.038	0.022	0.000
y_2	0.000	0.017	0.029	0.033	0.029	0.017	0.000
y_1	0.000	0.009	0.016	0.018	0.016	0.009	0.000
y_0	0.000	0.000	0.000	0.000	0.000	0.000	0.000

t = 7

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_9	0.000	0.010	0.018	0.021	0.020	0.012	0.000
y_8	0.000	0.017	0.030	0.037	0.033	0.020	0.000
y_7	0.000	0.021	0.037	0.044	0.039	0.024	0.000
y_6	0.000	0.021	0.037	0.044	0.040	0.024	0.000
y_5	0.000	0.020	0.035	0.041	0.037	0.022	0.000
y_4	0.000	0.019	0.032	0.038	0.033	0.020	0.000
y_3	0.000	0.016	0.028	0.032	0.028	0.017	0.000
y_2	0.000	0.012	0.021	0.025	0.022	0.013	0.000
y_1	0.000	0.007	0.012	0.014	0.012	0.007	0.000
y_0	0.000	0.000	0.000	0.000	0.000	0.000	0.000

t = 8

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_9	0.000	0.007	0.012	0.014	0.013	0.008	0.000
y_8	0.000	0.011	0.020	0.024	0.022	0.013	0.000
y_7	0.000	0.014	0.024	0.029	0.026	0.016	0.000
y_6	0.000	0.014	0.025	0.030	0.026	0.016	0.000

y_5	0.000	0.014	0.024	0.028	0.025	0.015	0.000
y_4	0.000	0.012	0.022	0.025	0.022	0.013	0.000
y_3	0.000	0.011	0.019	0.022	0.019	0.011	0.000
y_2	0.000	0.008	0.014	0.016	0.014	0.008	0.000
y_1	0.000	0.004	0.008	0.009	0.008	0.005	0.000
y_0	0.000	0.000	0.000	0.000	0.000	0.000	0.000

$t = 9$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_9	0.000	0.003	0.006	0.007	0.006	0.004	0.000
y_8	0.000	0.006	0.010	0.012	0.011	0.007	0.000
y_7	0.000	0.007	0.012	0.015	0.013	0.008	0.000
y_6	0.000	0.007	0.013	0.015	0.013	0.008	0.000
y_5	0.000	0.007	0.012	0.014	0.012	0.007	0.000
y_4	0.000	0.006	0.011	0.013	0.011	0.007	0.000
y_3	0.000	0.005	0.009	0.011	0.009	0.006	0.000
y_2	0.000	0.004	0.007	0.008	0.007	0.004	0.000
y_1	0.000	0.002	0.004	0.004	0.004	0.002	0.000
y_0	0.000	0.000	0.000	0.000	0.000	0.000	0.000

$t = 10$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y_{10}	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_9	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_8	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_7	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_6	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_5	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_4	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_3	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_2	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_1	0.000	0.000	0.000	0.000	0.000	0.000	0.000
y_0	0.000	0.000	0.000	0.000	0.000	0.000	0.000

VITA

Mahadev G. Bhat was born in Puttanamane, Sirsi Taluk, Karnataka State, India, on July 15, 1959. He attended lower primary school in that village and higher primary school in Agasala Bommanahally in Sirsi Taluk. His secondary school education was completed in Shri Sharadamba High School, Bhairumbe, Sirsi, in May 1975. The following June he entered M. M. Arts and Science College, Sirsi, and completed the Pre-University Course in June 1977.

In August 1977, Mr. Bhat was admitted to College of Agriculture, Dharwad (India), for pursuing B.S. degree in Agricultural Marketing and Cooperation. He attended this institution through September 1983 when he earned a master's degree in Agricultural Economics. After this degree, he served for the Dena Bank, a nationalized bank in India, as Agricultural Finance Officer through August 1988.

In August 1988, Mr. Bhat began the doctoral program in Agricultural Economics with concentration in natural resource economics at the University of Tennessee, Knoxville and was awarded the Doctor of Philosophy degree in December 1991. During his doctoral program, he worked on several research projects which included Economics of Biomass Energy Crops, funded by Oak Ridge National Laboratory. He is a member of American Agricultural Economics Association and Phi Kappa Phi. His future career interests include teaching and research in the areas of natural resource economics, agricultural policy and mathematical economics.