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A NOTE ON THE SOLUTION OF THE CHARACTERISTIC EQUATION OVER THE SYMMETRIZED MAX-PLUS ALGEBRA

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Abstract. *The symmetrized max-plus algebra is an extension of max-plus algebra. One of the problems in the symmetrized max-plus algebra is determining the eigenvalues of a matrix. If the determinant can be defined, the characteristic equation can be formulated as a max-plus algebraic multivariate polynomial equation system. A mathematical tool for solving the problem using operations as in conventional algebra, known as the extended linear complementary problem (ELCP), to determine the solution to the characteristic equation. In this paper, we describe the use of the ELCP in determining the solution to the characteristic equations of matrices over the symmetrized max-plus algebra.*

Keywords: *characteristic equation, determinant, ELCP, symmetrized max-plus algebra.*

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1. INTRODUCTION

It is known that for $\mathbb{R}$, the set of all real numbers, given $\mathbb{R}_\epsilon = \mathbb{R} \cup \{\epsilon\}$ with $\epsilon := -\infty$ and $e \coloneqq 0$ and for every $a, b \in \mathbb{R}_\epsilon$, defined operation $\oplus$ and $\otimes$ as $a \oplus b := max(a,b)$ and $a \otimes b := a + b$, next one, $(\mathbb{R}_\epsilon, \oplus, \otimes)$ is called max-plus algebra [1], [2], [3], [4]. Max-plus algebra is an example of a structure semiring that has no inverse of $\oplus$. In other words, if $a \in \mathbb{R}_\epsilon$ then nothing $b \in \mathbb{R}_\epsilon$ so $a \oplus b = b \oplus a = \epsilon$, unless $a = \epsilon$. In this case, an extension of max-plus algebra has been developed, which is called the symmetrized max-plus algebra and is denoted $(\mathbb{S}, \oplus, \otimes)$ with $\mathbb{S} = ({\mathbb{R}_\epsilon}^2)/\mathcal{B}$ where $\mathcal{B}$ is an equivalence relation [3],[5],[6].

Because the matrix over the symmetrized max-plus algebra can be defined as a determinant [7],[8]. The equation characteristics can be formulated as a system of equations and polynomial inequality multivariate algebra max. To determine solving systems of equations and polynomial inequalities multivariate max algebra can use operations like in conventional algebra through known mathematical tools with the extended complementary linear problem or ELCP [9],[10],[11].

This paper aims to determine the condition of a characteristic equation that has a solution viewed from the matrix over the symmetrized max-plus algebra. In this paper, we mainly concern with a characteristic equation matrix over the symmetrized max-plus algebra. We show that the solution of a characteristic equation on $\mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus} \cup \mathbb{S}^{\cdot}$ can be determined with ELCP. Section 2 will review some basic facts for the symmetrized max-plus algebra. The solution of the characteristic equation of the matrix over $\mathbb{S}$ that determine with ELCP is given in Section 3.

2. RESEARCH METHODS

2.1 The Symmetrized Max-Plus Algebra

In this section, we review some basic facts for max-plus algebra and the symmetrized max-plus algebra. The characteristic of max-plus algebra can be seen in [1],[5],[12]. Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}_\epsilon = \mathbb{R} \cup \{\epsilon\}$ with $\epsilon := -\infty$ as the null element and $e := 0$ as the unit element. For all $a, b \in \mathbb{R}_\epsilon$, we have $a \oplus b = max\ (a,b)$ and $a \otimes b = a + b$. The structure $(\mathbb{R}_\epsilon, \oplus, \otimes)$ is called the max-plus algebra.

Definition 1. [1],[5] *Let* $u = (x,y), v = (w,z) \in {\mathbb{R}_\epsilon}^2$. *We have*

1. *Operators* $\ominus$ *and* $(.)^\bullet$ *are defined as* $\ominus u = (y,x)$ *and* $u^\bullet = u \ominus u$.
2. *An element* u *is called balances with* v, *denoted by* $u \nabla v$, *if* $x \oplus z = y \oplus w$.
3. *A relation* $\mathcal{B}$ *is defined as follows :* $(x,y)\ \mathcal{B}\ (w,z)$ *if* $\begin{cases} (x,y)\ \nabla\ (w,z)\ if\ x \neq y\ and\ w \neq z \\ \quad (x,y) = (w,z), otherwise \end{cases}$.

Because $\mathcal{B}$ is an equivalence relation, we have the set of factor $\mathbb{S} = (\mathbb{R}_\varepsilon^2)/\mathcal{B}$ and the system $(\mathbb{S}, \oplus, \otimes)$ is called the symmetrized max-plus algebra, with the operations of addition and multiplication on $\mathbb{S}$ are defined as follows:

$$\overline{(a,b)} \oplus \overline{(c,d)} = \overline{(a \oplus c, b \oplus d)} \quad \text{and}$$

$$\overline{(a,b)} \otimes \overline{(c,d)} = \overline{(a \otimes c \oplus b \otimes d, a \otimes d \oplus b \otimes c)} \quad \text{for } \overline{(a,b)}, \overline{(c,d)} \in \mathbb{S}.$$

The system $(\mathbb{S}, \oplus, \otimes)$ is a semiring, because $(\mathbb{S}, \oplus)$ is associative, $(\mathbb{S}, \otimes)$ is associative, and $(\mathbb{S}, \oplus, \otimes)$ satisfies both the left and right distributive [7]. Because $\mathbb{S}^{\oplus}$ isomorphic with $\mathbb{R}_\varepsilon$, so it will be shown that for $a \in \mathbb{R}_\varepsilon$, can be expressed by $\overline{(a,\varepsilon)} \in \mathbb{S}^{\oplus}$. Furthermore, it is easily seen that for $a \in \mathbb{R}_\varepsilon$ we have :

1. $a = \overline{(a,\varepsilon)}$ with $\overline{(a,\varepsilon)} \in \mathbb{S}^{\oplus}$.
2. $\ominus a = \ominus \overline{(a,\varepsilon)} = \overline{\ominus (a,\varepsilon)} = \overline{(\varepsilon,a)}$ with $\overline{(\varepsilon,a)} \in \mathbb{S}^{\ominus}$.

   $a^\bullet = a \ominus a = \overline{(a,\varepsilon)} \ominus \overline{(a,\varepsilon)} = \overline{(a,a)} \in \mathbb{S}^\bullet$.

Lemma 2. [1] *For* $a, b \in \mathbb{R}_\varepsilon, a \ominus b = \overline{(a,b)}$.

Let $\mathbb{S}$ be the symmetrized max-plus algebra, a positive integer n, and $M_n(\mathbb{S})$ be the set of all $n \times n$ matrices over $\mathbb{S}$. Operations $\oplus$ and $\otimes$ for matrix over the symmetrized max-plus algebra are defined as follows : $C = A \oplus B \implies c_{ij} = a_{ij} \oplus b_{ij}$ and $C = A \otimes B \implies c_{ij} = \bigoplus_l a_{il} \otimes b_{lj}$.

The $n \times n$ zero matrix over $\mathbb{S}$ is ε_n with $(\varepsilon_n)_{ij} = \varepsilon$ and a $n \times n$ identity matrix over $\mathbb{S}$ is E_n with

$$(E_n)_{ij} = \begin{cases} e \text{ , } if \ i = j \\ \varepsilon \text{ , } if \ i \neq j \end{cases}$$

Definition 3. *We say that the matrix $A \in M_n(\mathbb{S})$ is invertible over $\mathbb{S}$ if $A \otimes B \nabla E_n$ and $B \otimes A \nabla E_n$ for any $B \in M_n(\mathbb{S})$.*

2.2 The Extended Complementary Linear Problem

Some optimization problems that are mathematically modeled can appear as a multiplication between variables or in the form of a square of a variable. Problems with these complementary conditions cannot be solved using the linear complementary problem (LCP). Therefore, the concept of the extended complementary linear problem (ELCP) emerged. The form of the ELCP problem is as follows.

Let a matrix $A \in M_{p\times n}(\mathbb{R}), B \in M_{q\times n}(\mathbb{R}), \bar{c} \in M_{p\times 1}(\mathbb{R})$, and $\bar{d} \in M_{q\times 1}(\mathbb{R})$. Also given the set $\phi_j \subset \{1,2,\dots,p\}$ where $j = 1,2,\dots,m$. Determine the vector $\bar{x} \in M_{n\times 1}(\mathbb{R})$ such that it satisfies:

$$\sum_j^m \prod_{i\in\phi_j} (A\bar{x} - \bar{c})_i = 0 \tag{1}$$

with

$$A\,\bar{x} \geq \bar{c} \text{ and } B\,\bar{x} = \bar{d} \tag{2}$$

Equation (1) is called the complementary condition. The vector set $\bar{x}$ that satisfies Equation (2) will be a solution to that problem. The ELCP form with constrains (2) can be solved by changing the ELCP form (1) and (2) into homogeneous ELCP structures.

The general form of homogeneous ELCP is as follows.
Given matrix $P \in M_{p\times n}(\mathbb{R})$ and $Q \in M_{q\times n}(\mathbb{R})$, and the set $\phi_j \in \{1,2,\dots,p\}$ with $j = 1,2,\dots,m$. Determine the nontrivial vector $\bar{u} \in M_{n\times 1}(\mathbb{R})$ such that it satisfies:

$$\sum_{j=1}^m \prod_{i\in\phi_j} (P\bar{u})_i = 0 \tag{3}$$

with

$$P\,\bar{u} \geq 0 \text{ and } Q\,\bar{u} = 0 \tag{4}$$

The ELCP forms in Equations (1) and (2) can be brought to homogeneous ELCP forms Equations (3) and (4) by taking $\bar{u} = \begin{bmatrix} \bar{x} \\ \alpha \end{bmatrix}$, $P = \begin{bmatrix} A & -\bar{c} \\ 0_{1\times n} & 1 \end{bmatrix}$, and $Q = \begin{bmatrix} B & -\bar{d} \end{bmatrix}$ with $\alpha \geq 0$. All $\bar{u}$ vectors that satisfy Equation (4) then form the solution set of the above problem.

2.3 Research Methods

The method used in preparing this research is first to conduct a literature study on symmetrized max-plus algebra, the matrix over symmetrized max-plus algebra, and the general form of the ELCP problems. This study is necessary because the matrix over symmetrized max-plus algebra can be defined, so from the formulation of the eigenvalues of the matrix over symmetrized max-plus algebra, characteristic equations are formed. And the characteristic equation can be formulated as a system of equations and polynomial inequalities multivariate algebra max-plus.

Determine the solution to a system of equations and inequalities, polynomial multivariate max algebra can use operations as in conventional algebra through a mathematical tool known as the ELCP.

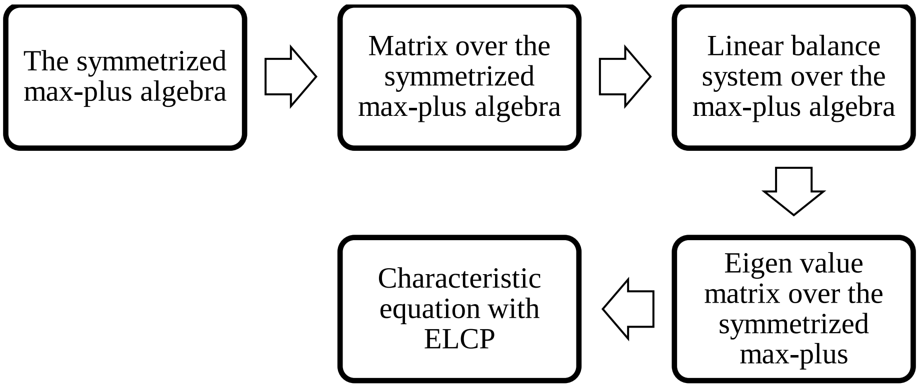

Figure 1. Workflow for the Conceptual Study of Characterization Equation of the Symmetrized Max-Plus Algebra

3. RESULTS AND DISCUSSION

For A, b over the symmetrized max-plus algebra, $A \otimes x \nabla b$ is called the linear balanced system, and $A \otimes x \nabla \epsilon$ is the homogeneous linear balanced system. Note that the operator "∇" and the homogeneous linear balances have the following characterizations.

Theorem 4. [12],[13] *Given $\mathbb{S}$ is a symmetrized max-plus algebra. The following properties apply:*

1. *For all $a, b, c \in C, a \ominus c \nabla b$ if and only if $a \nabla b \oplus c$.*
2. *For all $a, b \in \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus}$, $a \nabla b \Rightarrow a = b$.*
3. *Let $A \in M_n(\mathbb{S})$. The homogeneous linear balance systems $A \otimes x \nabla \varepsilon_{n\times 1}$ has a nontrivial solution in $\mathbb{S}^{\oplus}$ or $\mathbb{S}^{\ominus}$ if and only if $det(A) \nabla \varepsilon_{n\times 1}$.*

The following is the relationship between Systems of Equations and Inequality Multivariate Polynomials in Max-Plus with the ELCP. The general form of a multivariate polynomial system of equations and inequalities in max-plus algebra is as follows:

Given $p_1 + p_2$, with $p_1, p_2 \in \mathbb{N}$, integer $m_1, m_2, \dots, m_{p_1+p_2} \in \mathbb{N} \cup \{0\}$ and $a_{ki}, b_k, c_{kij} \in \mathbb{R}$, for $k = 1,2,\dots, p_1 + p_2, i = 1,2,\dots, m_k$, and $j = 1,2,\dots,n$ and will look for the vector $x \in \mathbb{R}^n$, so that it satisfies:

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} = b_k, \text{ for } k = 1,2,\dots,p_1 \tag{5}$$

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} \leq b_k, \text{ for } k = p_1+1, p_1+2,\dots,p_1+p_2 \tag{6}$$

Based on De Schutter and De Moor's statement, there is the following theorem about necessary and sufficient conditions for the extended complementary linear problem ([14],[15]). A multivariate polynomial system of equations and inequalities in max-plus algebra if and only if it is an ELCP.

From Equations (5) and (6) can be formed matrix $A = \begin{bmatrix} -C_1 \\ -C_2 \\ \vdots \\ -C_{p_1+p_2} \end{bmatrix}$ and $c = \begin{bmatrix} -d_1 \\ -d_2 \\ \vdots \\ -d_{p_1+p_2} \end{bmatrix}$.

If m_k in Equation (5) is equal to 1, then we get: $a_{k1} + \sum_{j=1}^{n} c_{k1j}x_j = b_k$, for $k = 1, 2, \ldots, p_1$. If we add $-a_{k1}$ in two sides and define $d_{1k} = b_k - a_{k1}$, we get $\sum_{j=1}^{n} c_{k1j}x_j = d_{1k}$. So that we get the matrix

$$B = \begin{bmatrix} c_{111} & c_{112} & \cdots & c_{11n} \\ c_{211} & c_{212} & \cdots & c_{21n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p_1 11} & c_{p_1 12} & \cdots & c_{p_1 1n} \end{bmatrix}, \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \bar{d} = \begin{bmatrix} d_{11} \\ d_{21} \\ \vdots \\ d_{p_1 1} \end{bmatrix}.$$

In other words, the equation is obtained in the form of a matrix $B\bar{x} = \bar{d}$.

Defined as much as p_1 set $\phi_1, \phi_2, \cdots, \phi_{p_1}$ such that $\phi_j = \{s_j + 1, s_j + 2, \cdots, s_j + m_j\}$ with $s_1 = 0$ for $j = 1,2, \ldots, p_1$ and $s_{j+1} = s_j + m_j$ for $j = 1,2, \ldots, p_1 - 1$. Thus obtained:

$$\begin{aligned} \phi_1 &= \{s_1 + 1, s_1 + 2, \cdots, s_1 + m_1\} = \{1, 2, \ldots, m_1\} \\ \phi_2 &= \{s_2 + 1, s_2 + 2, \ldots, s_2 + m_2\} = \{s_1 + m_1 + 1, s_1 + m_1 + 2, \ldots, s_1 + m_1 + m_2\} \\ &= \{m_1 + 1, m_1 + 2, \ldots, m_1 + m_2\} \\ &\vdots \\ \phi_{p_1} &= \{s_{p_1} + 1, s_{p_1} + 2, \ldots, s_{p_1} + m_{p_1}\} \\ &= \{m_1 + m_2 + \cdots + m_{\{p_1-1\}} + 1, m_1 + m_2 + \ldots + m_{\{p_1-1\}} + 2, \ldots, m_1 + m_2 + \ldots + m_{\{p_1-1\}} + m_{\{p_1\}}\} \end{aligned}$$

Problems in the form of Equations (5) and (6) are equivalent to the following ELCP.

Given $A, \bar{c}$, and $\phi_1, \phi_2, \ldots, \phi_{p_1}$, define the vector $\bar{x} \in \mathbb{R}^n$ such that $\sum_{j}^{p_1} \prod_{i \in \phi_j} (A\bar{x} - \bar{c})_i = 0$, with $A\bar{x} \geq \bar{c}$. Definition 5 is given a representation of the equation characteristics of a matrix over the symmetrized max-plus algebra.

Definition 5. [3] *Let a matrix* $A \in M_n(\mathbb{S})$. *For* $det\ (\lambda \otimes E_n \ominus A) \nabla \epsilon$, *be in effect*

$$\lambda^{\otimes n} \oplus \bigoplus_{k=1}^{n} a^k \otimes \lambda^{\otimes \{n-k\}} \nabla \epsilon \tag{7}$$

with $a_k = (\ominus\ 0)^{\otimes k} \otimes \bigoplus_{\varphi \in C_n^k} \det A_{\varphi\varphi}$ or

$$a^k = (\ominus\ 0)^{\otimes k} \bigoplus_{i_1, \cdots, i_k \in C_n^k} \bigoplus_{\varphi \in P_k} sgn\ (\sigma) \otimes \bigotimes_{r=1}^{k} a_{i_r i_{\sigma(r)}}, \text{ for } k = 1,2, \ldots, n.$$

The relationship of the eigenvalues of the matrix over the symmetrized max-plus algebra with ELCP is given in the following example.

Example 6

Let a matrix $A = \begin{bmatrix} \ominus 2 & 1 & \epsilon \\ 1 & \ominus 0 & 1 \\ \epsilon & 0 & 2 \end{bmatrix}$. From matrix A, using Definition 5 with the formula:

$$a_k = (\ominus\ 0)^{\otimes k} \otimes \bigoplus_{\varphi \in C_n{}^k} \det A_{\varphi\varphi}$$

As a result, the obtained characteristic equation is

$$\lambda^{\otimes 3} \oplus a_1 \otimes \lambda^{\otimes 2} \oplus a_2 \otimes \lambda^{\otimes 1} \oplus a_3 \nabla \epsilon$$

or

$$\lambda^{\otimes 3} \oplus 2^{\bullet} \otimes \lambda^{\otimes 2} \oplus \ominus 4 \otimes \lambda \oplus 4^{\bullet} \nabla \epsilon$$

Next,

$$\lambda^{\otimes 3} \oplus 2 \otimes \lambda^{\otimes 2} \oplus 4 \nabla 2 \otimes \lambda^{\otimes 2} \oplus 4 \otimes \lambda \oplus 4$$

So

$$\lambda^{\otimes 3} \oplus 2 \otimes \lambda^{\otimes 2} \oplus 4 = 2 \otimes \lambda^{\otimes 2} \oplus 4 \otimes \lambda \oplus 4$$

We have an equation

$$\lambda^{\otimes 3} \oplus 2 \otimes \lambda^{\otimes 2} \oplus 4 = p \tag{8}$$

$$2 \otimes \lambda^{\otimes 2} \oplus 4 \otimes \lambda \oplus 4 = p \tag{9}$$

Then by multiply $p^{\otimes^{-1}}$ in Equations (8) and (9) we obtain

$$\lambda^{\otimes^3} \otimes p^{\otimes^{-1}} \oplus 2 \otimes \lambda^{\otimes^2} \otimes p^{\otimes^{-1}} \oplus 4 \otimes p^{\otimes^{-1}} = 0 \quad (10)$$

$$2 \otimes \lambda^{\otimes^2} \otimes p^{\otimes^{-1}} \oplus 4 \otimes \lambda \otimes p^{\otimes^{-1}} \oplus 4 \otimes p^{\otimes^{-1}} = 0 \quad (11)$$

Equations (10) and (11) can be viewed as **System Multivariate Polynomial Equation in Max-Plus**, consequently obtained:

$$-C_1 = \begin{bmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \end{bmatrix} ; -C_2 = \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}; -\alpha_1 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}; -\alpha_2 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

Next, we form the matrix $A = \begin{bmatrix} -C_1 \\ -C_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ -2 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -d_1 \\ -d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \\ 4 \\ 4 \end{bmatrix}$.

Vector $\bar{x} \in \mathbb{R}^2$ with $\bar{x} = \begin{bmatrix} \lambda \\ p \end{bmatrix}$.

We define ϕ_1 and ϕ_2, so that:

$$\phi_1 = \{s_1 + 1, s_1 + 2, s_1 + 3\} = \{1,2,3\}$$

$$\phi_2 = \{s_2 + 1, s_2 + 2, s_2 + 3\} = \{s_1 + m_1 + 1, s_1 + m_1 + 2, s_1 + m_1 + 3\} = \{4,5,6\}.$$

We form ELCP homogenous with: $P = \begin{bmatrix} A & -C \\ 0_{1\times n} & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \\ -2 & 1 & -2 \\ 0 & 1 & -4 \\ -2 & 1 & -2 \\ -1 & 1 & -4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$; $\bar{u} = \begin{bmatrix} \bar{x} \\ \alpha \end{bmatrix} = \begin{bmatrix} \lambda \\ p \\ \alpha \end{bmatrix}$

$(Pu)_1(Pu)_2(Pu)_3 + (Pu)_4(Pu)_5(Pu)_6 = 0$ with constraints $(Pu) \geq 0$.

With the algorithm in ELCP, we have the set of all eigenvalues $\bar{u} = \kappa \begin{bmatrix} -8 \\ 0 \\ 0 \end{bmatrix}$. In ELCP, $\bar{u} = \begin{bmatrix} \bar{x} \\ \alpha \end{bmatrix} = \begin{bmatrix} \lambda \\ p \\ \alpha \end{bmatrix}$.

Hence, $\lambda = \kappa\,(-8)$. Let $\lambda_1 = 0, \lambda_2 = -2, \lambda_3 = -4$.

The associated eigenvectors for A are:

1). Let $\lambda = 0$. We have the set of all eigenvectors associated with $\lambda = 0$, that is

$$\left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \nabla \begin{bmatrix} t \\ 1 \otimes t \\ 1^{\cdot} \otimes t \end{bmatrix} \middle| t \in \mathbb{S} \right\}$$

2). Let $\lambda = -2$. We have the set of all eigenvectors associated with $\lambda = -2$, that is

$$\left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \nabla \begin{bmatrix} s \\ 1 \otimes s \\ 0^{\cdot} \otimes s \end{bmatrix} \middle| s \in \mathbb{S} \right\}$$

3). Let $\lambda = -4$. We have the set of all eigenvectors associated with $\lambda = -4$, that is

$$\left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \nabla \begin{bmatrix} r \\ 1 \otimes r \\ 0^{\cdot} \otimes r \end{bmatrix} \middle| r \in \mathbb{S} \right\}$$

We construct matrix

$$P = [v_1 \quad v_2 \quad \cdots \quad v_r] = \begin{bmatrix} \ominus 1 & \ominus 1 & \ominus 1 \\ \ominus 2 & \ominus 2 & \ominus 2 \\ 2^{\bullet} & 1^{\bullet} & 1^{\bullet} \end{bmatrix}$$

with $D = \begin{bmatrix} 0 & \epsilon & \epsilon \\ \epsilon & -4 & \epsilon \\ \epsilon & \epsilon & -2 \end{bmatrix}$

We have $A \otimes P = \begin{bmatrix} \ominus 2 & 1 & \epsilon \\ 1 & \ominus 0 & 1 \\ \epsilon & 0 & 2 \end{bmatrix} \otimes \begin{bmatrix} \ominus 1 & \ominus 1 & \ominus 1 \\ \ominus 2 & \ominus 2 & \ominus 2 \\ 2^{\bullet} & 1^{\bullet} & 1^{\bullet} \end{bmatrix} = \begin{bmatrix} 3^{\bullet} & 3^{\bullet} & 3^{\bullet} \\ 3^{\bullet} & 2^{\bullet} & 2^{\bullet} \\ 4^{\bullet} & 3^{\bullet} & 3^{\bullet} \end{bmatrix}$

Hence $P \otimes D = \begin{bmatrix} \ominus 1 & \ominus 1 & \ominus 1 \\ \ominus 2 & \ominus 2 & \ominus 2 \\ 2^{\bullet} & 1^{\bullet} & 1^{\bullet} \end{bmatrix} \otimes \begin{bmatrix} 0 & \epsilon & \epsilon \\ \epsilon & -4 & \epsilon \\ \epsilon & \epsilon & -2 \end{bmatrix} = \begin{bmatrix} \ominus 1 & \ominus (-3) & \ominus (-1) \\ \ominus 2 & \ominus (-2) & \ominus 0 \\ 2^{\bullet} & (-3)^{\bullet} & (-1)^{\bullet} \end{bmatrix}$

Because $A \otimes P \nabla P \otimes D$, so $P^{-1} \otimes A \otimes P \nabla D$.

4. CONCLUSIONS

A mathematical tool for solving the problem using operations as in conventional algebra, known as the ELCP, is used to determine the solution to the characteristic equation. We can form the characteristic equations of matrices over the symmetrized max-plus algebra. The ELCP can be used to system multivariate polynomial equations in max-plus. Because the characteristic equations of matrices over the symmetrized max-plus algebra make to system multivariate polynomial equation in max-plus, we used the ELCP to determine the solution of the characteristic equation of matrices over the symmetrized max-plus algebra.

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