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THE HARMONIOUS, ODD HARMONIOUS, AND EVEN HARMONIOUS LABELING

Ahmad Lasim[1], Ikhsanul Halikin[2], Kristiana Wijaya[3*]

[1,2,3]Graph and Algebra Research Group, Mathematics Department, Faculty of Mathematics and Natural Sciences, Universitas Jember Jl. Kalimantan 37 Jember, 68121, Jawa Timur, Indonesia

Corresponding author's e-mail: [3] kristiana.fmipa@unej.ac.id*

Abstract. *Suppose G is a simple and connected graph with q edges. A harmonious labeling on a graph G is an injective function $f: V(G) \to \{0, 1, 2, \dots, q-1\}$ so that there exists a bijective function $f^*: E(G) \to \{0, 1, 2, \dots, q-1\}$ where $f^*(uv) = f(u) + f(v) (mod\ q)$, for each $uv \in E(G)$. An odd harmonious labeling on a graph G is an injective function f from $V(G)$ to non-negative integer set less than $2q$ so that there is a function $f^*(uv) = f(u) + f(v)$ where $f^*(uv) \in \{1, 3, 5, \dots, 2q-1\}$ for every $uv \in E(G)$. An even harmonious labeling on a graph G is an injective function $f: V(G) \to \{0, 1, 2, \dots, 2q\}$ so that there is a bijective function $f^*: E(G) \to \{0, 2, 4, \dots, 2q-2\}$ where $f^*(uv) = f(u) + f(v) (mod\ 2q)$ for each $uv \in E(G)$. In this paper, we discuss how to build new labeling (harmonious, odd harmonious, even harmonious) based on the existing labeling (harmonious, odd harmonious, even harmonious).*

Keywords: *even harmonious labeling, harmonious labeling, odd harmonious labeling.*

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1. INTRODUCTION

Sedlacek first introduced graph labeling in 1963. Graph labeling is a function that maps a set of vertices or edges of a graph to a non-negative integer that fulfills specific properties/rules. This particular nature/rule leads to different types of labeling. Thousands of papers on graph labeling have been produced. Gallian [1] surveyed this. In this paper, the research focus is on harmonious, odd harmonious labeling, and even harmonious labeling, all three of which are vertex labeling.

A harmonious labeling was first introduced by Graham and Sloane [2] in 1980. A graph G with q edges is said to be harmonious if there is a one-to-one (injective) function f that maps the set of vertices $V(G)$ to the set $\{0,1,2,\dots,q\text{-}1\}$ so that each edge gets the label of the sum of the two vertices labels incident with that edge in modulo q, which are all different. When G is a tree, G has $(q+1)$ vertices, so it is clear that the one-to-one function condition is not satisfied. Some researchers give exceptions to trees by changing the existence of a one-to-one function into a surjective function [1]. Therefore, in this paper, the focus of the discussion is only on connected graphs that are not trees. It means that the graph always contains a closed path. Not many graphs can be labeled harmoniously [1]. Graham and Sloane [2] proved that only a cycle graph with an odd length is a harmonious graph; the wheel W_n is a harmonious graph for every n. Other results regarding harmonious labeling can be seen in [1].

Furthermore, harmonious labeling inspires the emergence of odd and even harmonious labeling. It is said to be odd harmonious because the label of each edge must be odd. It is said to be even harmonious because all of its edges must be labeled even. Odd harmonious labeling was introduced by Liang and Bai in 2009. A graph G with q edges is said to be odd harmonious if there is a one-to-one function f that maps every element in $V(G)$ to the set of non-negative integers less than $2q$, so that each edge gets an odd label that is all different from the set $\{1,3,5,\dots,2q\text{-}1\}$, which is obtained from the sum of the two vertices labels incident to that edge. The following is a theorem related to odd harmonious labeling.

Theorem 1. (Liang and Bai [3])

(1) *If G is an odd harmonious graph, then G is a bipartite graph.*

(2) *Let G be a graph with p vertices and q edges. If G is an odd harmonious graph, then* $2\sqrt{q} \leq p \leq 2q\text{-}1$.

Several other studies on odd harmonious labeling that have been carried out can be seen, among others: [4], [5], [6], [7], [8], [9], [10].

An even harmonious labeling of a graph was introduced by Sarasija and Binthiya [11] in 2011. The graph G is an even harmonious if there is a one-to-one function $f{:}V(G)\rightarrow\{0,1,2,\dots,2q\}$ so that each edge gets a different even label in the set of integers modulo $2q$ as the result of the sum of the two labels of the vertices incident to that edge. Even harmonious graphs that have been studied can be seen in [11], [12], [13], [14], [15].

The graph studied in this paper was a simple and connected. The basic theory for the graph can be seen in Chartrand and Zhang [16]. In this paper, we study more deeply about harmonious, odd harmonious, and even harmonious labeling on a graph. The previous studies emphasize more on the class of graphs studied. Then, this study emphasizes the labeling itself, namely how to build a new label based on the current labeling. In particular, if it is given labeling of harmonious, odd harmonious, or even harmonious on a graph, a new label will be built based on that labeling so that the resulting labeling fulfills the labeling requirements for harmonious, odd harmonious, or even harmonious labeling.

2. RESEARCH METHOD

The method in this research was exploratory research which aimed to build new labeling from the current labeling of harmonious, odd, and even harmonious. The research was begun by analyzing the definitions of the three types of labeling and what properties each labeling must fulfill. Next, a new label based on the existing labeling was built. The final step was proving that the new labeling met the rules of the labeling, whether the labeling was harmonious, odd harmonious, or even harmonious.

3. RESULT AND DISCUSSION

In this section, three types of vertex labeling were discussed, namely harmonious labeling, odd harmonious, and even harmonious labeling. From the three labels given, a new function was built based on the existing labeling so that the resulting labeling meets the labeling properties of harmonious, odd harmonious, or even harmonious labeling.

3.1. Harmonious Labeling

A harmonious labeling on a graph G with q edges is a one-to-one function (injective) $f: V(G) \to \{0, 1, 2, \dots, q-1\}$ so that it produces a one-to-one correspondence (bijective function) $f^*: E(G) \to \{0,1,2,\dots,q-1\}$ with $f^*(uv) = \big(f(u) + f(v)\big)\ (mod\ q)$, for each $uv \in E(G)$. A graph is said to be harmonious if it can be labeled with the harmonious labeling rule. The following two theorems explain how to construct a new harmonious label based on the known harmonious labeling.

Theorem 2. *Let G be a graph with q edges and f be a harmonious labeling of G. Suppose that the function* $g_f: V(G) \to \{0, 1, \dots, q-1\}$ *with* $g_f(u) = q - f(u)$ *for each* $u \in V(G)$. *Therefore,* g_f *is a harmonious labeling on the graph G.*

Proof. Suppose f is a harmonious labeling on a graph G with q edges and the function $g_f: V(G) \to \{0, 1, \dots, q-1\}$ with $g_f(u) = q - f(u)$ for each $u \in V(G)$. First, it will be shown that g_f is a one-to-one function. Since f is a one-to-one function, it is clear that any two distinct vertices in G will be labeled differently by the g_f function. Thus g_f is a one-to-one function.

Then, it will be proven that for each edge $uv \in E(G)$ get a different label from $\{0, 1, 2, \dots, q-1\}$ by the rules $g_f^*(uv) = g_f(u) + g_f(v)\ (mod\ q)$.

$$\begin{aligned} g_f^*(uv) &= \big(g_f(u) + g_f(v)\big)\ (mod\ q) \\ &= \big((q - f(u)) + (q - f(v))\big)\ (mod\ q) \\ &= 2q - \big(f(u) + f(v)\big)\ (mod\ q) \\ &= 2q - f^*(uv)\ (mod\ q) \\ &= -f^*(uv)\ (mod\ q) \end{aligned}$$

This means g_f^* is the summation inverse of f^* on the set of integers modulo q. This shows that each edge $uv \in E(G)$ get a different label from $\{0,1,2,\dots,q-1\}$. Because each vertex in G gets a different label with the function rule g_f, each edge in G also gets a different label in $\{0, 1, 2, \dots, q-1\}$ by the rule $g_f^*(uv) = g_f(u) + g_f(v)\ (mod\ q)$. Therefore, it is proven g_f is a harmonious labeling on the graph G. ∎

For example, in Figure 1 (a), it is given a labeling of a wheel with 8 edges, W_4. Furthermore, if the label of vertices in Figure 1 (a) is changed by the rule $8 - f(u)$ for each $u \in V(W_4)$, it will produce the harmonious labeling in Figure 1 (b).

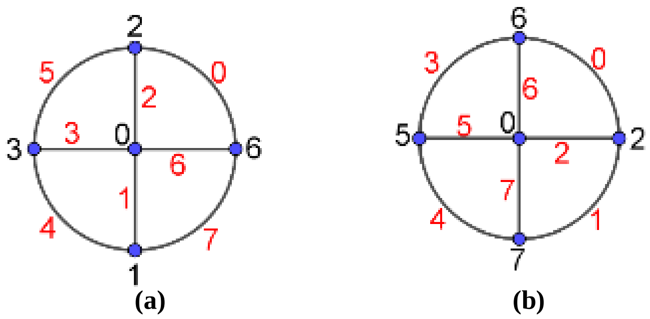

Figure 1. Two harmonious labelings of wheel W_4 to illustrate Theorem 2

Theorem 3. *Suppose the function f is a harmonious labeling on a graph G with q edges. Then the function* $g_f: V(G) \to \{0, 1, \dots, q-1\}$ *with* $g_f(u) = (f(u) + k)\ (mod\ q)$ *for each* $u \in V(G)$ *and* $k \in [1, q-1]$ *is a harmonious labeling on the graph G.*

Proof. First, it will be proved that g_f is a one-to-one function of the set $V(G)$ to the set $\{0, 1, 2, \ldots, q-1\}$. Take any $u, v \in V(G)$ that fulfills $g_f(u) = g_f(v)$. Therefore, $(f(u)+k)(mod\ q) = (f(v)+k)\ (mod\ q)$ for $k \in \{1, 2, \ldots, q-1\}$. Thus, we get $f(u) = f(v)$. Because f is a one-to-one function, then $u = v$. Therefore, g_f is a one-to-one function.

Next, we will prove that with the function g_f^*, each edge in G gets a different label from $0, 1, 2, \ldots, q-1$. Suppose there are two different edges uv and u_0v_0 in $E(G)$ which has the same label, i.e. $g_f^*(uv) = g_f^*(u_0v_0)$. It is obtained $(f^*(uv) + 2k)\ (mod\ q) = (f^*(u_0v_0) + 2k)\ (mod\ q)$. Consequently, $f^*(uv) = f^*(u_0v_0)$. This is contrary to f^* one-to-one function. It is proven that $g_f^* : V(G) \to \mathbb{Z}_q$ is bijective function. Thus, g_f is a harmonious labeling on the graph *G*. ■

For example, consider the harmonious labeling of the 15-edges Petersen graph in Figure 2(a). If the labels of each vertex in Figure 2 (a) are added to 5 in modulo 15, then the harmonious labeling is obtained as in Figure 2 (b).

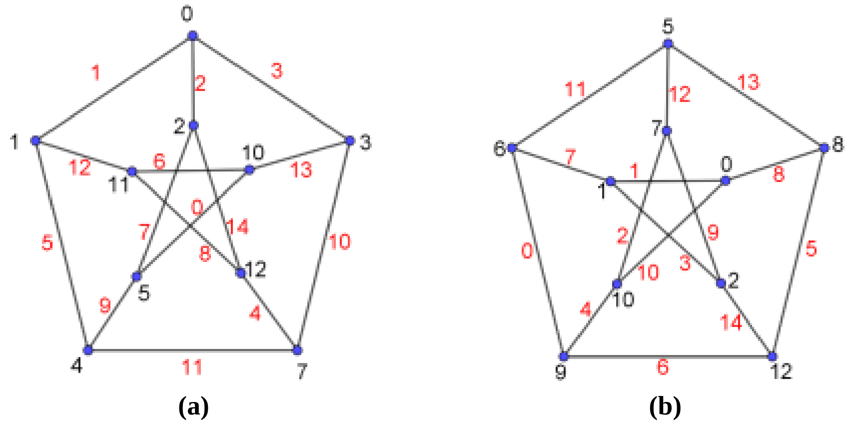

Figure 2. Two harmonious labelings of a Petersen graph to illustrate Theorem 3

3.2. Odd Harmonious Labeling

An odd harmonious labeling on a graph G with q edges is a one-to-one function $f: V(G) \to \{0, 1, 2, \ldots, 2q-1\}$ so that there is a one-to-one correspondence $f^*: E(G) \to \{1, 3, 5, \ldots, 2q-1\}$ with $f^*(uv) = f(u) + f(v)$ for each $uv \in E(G)$ [8]. The following is a theorem related to odd harmonious labeling. According to Theorem 1, an odd harmonious graph is a bipartition graph. Since a bipartition graph does not contain odd cycles, therefore an odd harmonious graph will not contain odd cycles. In addition, based on the definition of odd harmonious labeling that each edge *e* = *uv* gets a label from the number of labels of two vertices incident to that edge, it is clear that the edge labeled 1 is only obtained from the vertex with label 0 and the vertex with label 1. Thus, the vertex with a label 0 must be adjacent to the vertex with label 1. Furthermore, the following theorem discusses odd harmonious labeling, which is built from existing odd harmonious labeling.

Theorem 4. *Let G be a graph with q edges. If f is an odd harmonious labeling on a graph G, then the function* $g_f: V(G) \to \{0, 1, \ldots, 2q-1\}$ *which is defined as*

$$g_f(u) = \begin{cases} f(u) - 1 & \text{if } f(u) \text{ odd}, \\ f(u) + 1 & \text{if } f(u) \text{ even}, \end{cases}$$

for each $u \in V(G)$ *is an odd harmonious labeling on graph G with the same edge label as labeling f.*

Proof. First, it will be shown that the function g_f is one-to-one. Based on the definition of g_f, label $g_f(u)$ is odd if and only if $f(u)$ is even and vice versa, label $g_f(u)$ is even if and only if $f(u)$ is odd. Thus, since f is a one-to-one function, obviously a function g_f is also one-to-one. Next, it will be proved that each edge $uv \in E(G)$ gets odd labels that are all different from $1, 3, \ldots, 2q-1$ by the rule $g_f^*(uv) = g_f(u) + g_f(v)$. Because $g_f^*(uv)$ must be odd, one of $g_f(u)$ and $g_f(v)$ must be odd and others even. Based on the function definition g_f, label $g_f(u)$ is odd if and only if $f(u)$ is even and vice versa, label $g_f(u)$ is even if and only if $f(u)$ is odd. Without loss of generality, if $g_f(u)$ is odd and $g_f(v)$ is even, it is obtained $g_f^*(uv) = g_f(u) + g_f(v) = (f(u) - 1) + (f(v) + 1) = f(u) + f(v) = f^*(uv)$. Because f^* is a bijective function that maps each edge $uv \in E(G)$ to the set $\{1, 3, \dots, 2q-1\}$, so that g_f^* is also a bijective function. Therefore, it is proven that g_f is an odd harmonious labeling on a graph G, with the same edge label as the function f. ■

For example, in Figure 3 (a) , it is given an odd harmonious label to a star S_8. The odd harmonious labeling in Figure 3 (b) is obtained from Figure 3 (a) with the rules written in Theorem 4, namely the odd label is minus one, while the even label is added one.

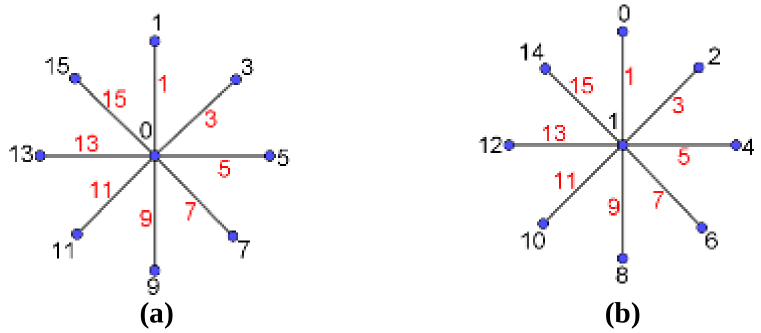

Figure 3. Two odd harmonious labelings with the same edge label on a star to illustrate Theorem 4

3.3. Even Harmonious Labeling

An *even harmonious labeling on a graph G is a one-to-one function* $f: V(G) \to \{0, 1, 2, \dots, 2q\}$ so that there is a one-to-one correspondence $f^*: E(G) \to \{0, 2, 4, \dots, 2q-2\}$ with $f^*(uv) = \big(f(u) + f(v)\big)\ (mod\ 2q)$ for each edge $uv \in E(G)$ [13]. A graph G is said to be even harmonious if the set of vertices and edges in graph G can be labeled with an even harmonious labeling rule. It is said to be even harmonious because all of its edges are labeled even. Because even numbers are obtained from the sum of both even or odd numbers, all vertices must be labeled even or all vertices must be labeled odd in this even harmonious labeling. Thus, when G is a tree, all vertices of G must be labeled even. In the following two theorems, it is proved that if f is even harmonious labeling on a graph G with q edges, then the two labels constructed from f, namely for each $u \in V(G)$, $g_f(u) = (f(u) + k)\ (mod\ 2q)$ for $k \in [1, 2q-1]$ or $g_f(u) = 2q - f(u)$, are even harmonious labeling.

Theorem 5. *Let G be a graph that has q edges. Suppose f is an even harmonious labeling on G and the function* $g_f: V(G) \to \{0, 1, \dots, 2q\}$ *is defined as* $g_f(u) = (f(u) + k)\ (mod\ 2q)$ *for each* $u \in V(G)$ *and* $k \in [1, 2q-1]$. *If* $f : V(G) \to \{1, 2, \dots, 2q-1\}$ *so* g_f *is an even harmonious labeling on graph* G.

Proof. Suppose $f : V(G) \to \{1, 2, \dots, 2q-1\}$ is even harmonious labeling on a graph G. Suppose again that $g_f(u) = (f(u) + k)\ (mod\ 2q)$ for each $u \in V(G)$ and $k \in [1, 2q-1]$. Suppose g_f is not one-to-one function, there are two distinct vertices u and v in G so that $g_f(u) = g_f(v)$, namely $(f(u) + k)\ (mod\ 2q) = (f(v) + k)(mod\ 2q)$ for some $k \in [1, 2q-1]$. Since the sum is in modulo $2q$, this equation is fulfilled when $f(u) = 0$ and $f(v) = 2q$ or vice versa. However, this contradicts to $f : V(G) \to \{1, 2, \dots, 2q-1\}$. Therefore, g_f must be one-to-one function. Next, we consider $g_f^*: E(G) \to \{0, 2, \dots, 2q-2\}$ with $g_f^*(uv) = \big(g_f(u) + g_f(v)\big)\ (mod\ 2q)$ for each edge $uv \in E(G)$. Therefore, $g_f^*(uv) = \big(g_f(u) + g_f(v)\big)\ (mod\ 2q) = (f(u) + f(v) + 2k)\ (mod\ 2q) = (f^*(uv) + 2k)\ (mod\ 2q)$. This shows that the edge label is based on the function rule g_f^* different to $2k$ with edge label based on the function rules f. As a note, the edge label of both functions will be the same when $k = q$. ■

As an illustration, in the Figure 4 is given an even harmonious labeling on cycle C_5 with different vertex label but produces the same edge labels. The labeling of the vertices in Figure 4 (b) is obtained from the labeling rules in Theorem 5, namely the labeling of the vertices in Figure 4 (a) plus 5 in modulo 10, while the labeling of the edges do not change.

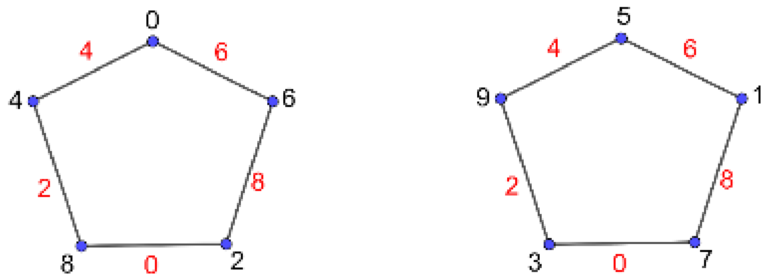

Figure 4. Two Even haramonious labelings with different vertex labels on a cycle C_5

Theorem 6. *Suppose f is an even harmonious labeling on a graph G with q edges. Then, the function $g_f: V(G) \rightarrow \{0, 1, \dots, 2q\}$ with $g_f(u) = 2q - f(u)$ for each $u \in V(G)$ is an even harmonious labeling on the graph G.*

Proof. Since f is a one-to-one function from the set of vertices *V(G)* to the set of labels $\{0, 1, \dots, 2q\}$, it is obvious that function $g_f(u) = 2q - f(u)$ for each $u \in V(G)$, is one-to-one. Next, for each edge $uv \in E(G)$ will get label $g_f^*(uv) = \left(g_f(u) + g_f(v)\right) \ (mod\ 2q) = \left(2q - f^*(uv)\right) (mod\ 2q) = -f^*(uv)$. This means the edge label based on function rules g_f is the inverse of addition in modulo *2q* of the edge label according to the rule of function *f*. Thus, each edge in *G* gets a different label and even. Therefore, it is proven that g_f is an even harmonious labeling on the graph *G*. ∎

As an illustration of Theorem 6, if given an even harmonious labeling in Figure 5 (a), then we can construct an even harmonious labeling by rule 10 – *f*(*u*) for each vertex label, as shown in Figure 5 (b). It can be seen that the edge label on the graph in Figure 5 (b) is the inverse of the integer sum modulo 10 of the edge label on the graph in Figure 5 (a).

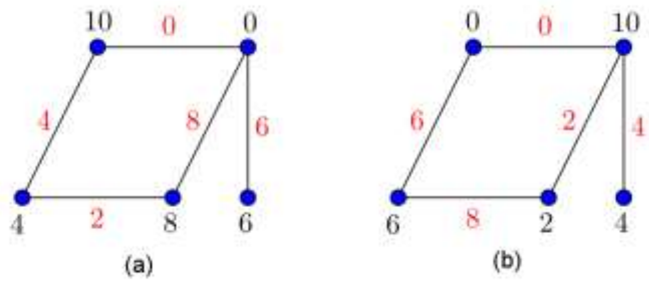

Figure 5. Two even harmonious labelings on a 5-edges graph to illustrate Theorem 6

4. CONCLUSION

In this paper, we have constructed two new labelings (harmonious and even harmonious) from the existing (harmonious and even harmonious) labeling. However, only one new labeling (odd harmonious) is obtained from the existing (odd harmonious) labeling. It is due to the edge label in odd harmonious obtained from the addition of two integers that are not in modulo *2q*, with *q* is the number of edges in the graph, so the vertex labeled zero must be adjacent to the vertex labeled one. However, it is possible to construct another odd harmonious labeling. Therefore, this study is still open to other researchers to construct a new labeling from the existing labeling according to the harmonious labeling.

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