Filling Jars to Measure Time

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Abstract

If water is flowing at the same constant rate through each of $H \geqslant 3$ hoses, so that any one hose will fill any one of $J \geqslant 2$ available jars in exactly one hour, then what are the fillable fractions of a jar, and what are the measurable fractions of an hour? Learning to systematically answer such questions will not only equip readers to fluently use fractions, but also introduce or reintroduce them gently to the Queen of Mathematics – Number Theory.

Keywords: Bisectional fraction; mathematical induction; limits; approximations; factoring; bisectional continued fraction representation

First, we describe an unconventional method of measuring time by filling jars in the simplest, non-trivial case of two jars and three hoses. Oftentimes, before we can begin to measure time, we may have to fill a jar to a desired fraction. Thus, in this paper, filling jars and measuring time go hand in hand. To draw the reader in, we pose a few motivating questions, which we encourage them to answer individually or collectively. Thereafter, as we answer them, we prove the main results:

(i) Any bisectional (to be defined) fraction of a jar is exactly fillable; and

(ii) Any one-third of a bisectional fraction of an hour is exactly measurable.

More specifically, we write down an algorithm that details how to accomplish the above tasks. Furthermore, we prove that, using two jars and three hoses, no other fraction of a jar is fillable, and no other fraction of an hour is measurable! Next, we generalize the above results to two jars and more than three hoses, and thereafter to more than two jars and more than three hoses. Occasionally, we invite the reader to fill in some details, and either improve our solution or prove that it is the best in some sense. We conclude the paper with some discussions.

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1 The Jar Filling Puzzle $(J = 2, H = 3)$

To a camp, where visitors can pay to use electricity, we brought an old microwave to cook our meal. We prepared the ingredients; we put them in a bowl; and we put the bowl inside the microwave. We were about to set the timer to 35 minutes when we found out that the timer on the microwave was not working! We had no watch or timekeeper. Luckily, the campsite had three hoses from which water was flowing at the same constant speed, two differently shaped opaque jars, and a sign that read: “Any one hose will fill any one jar in exactly one hour.” See Figure 1.

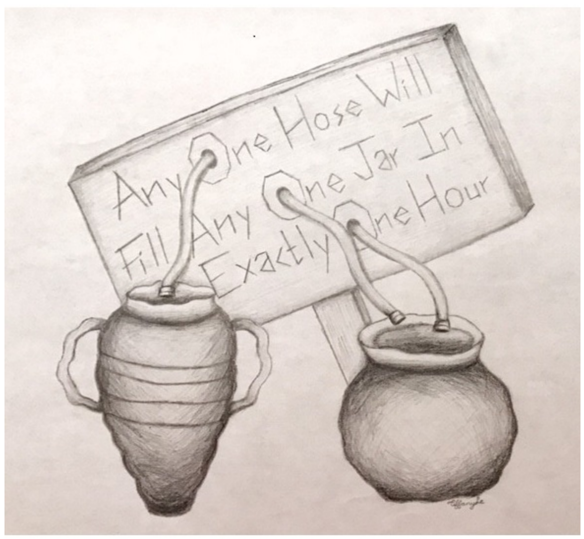

Figure 1: Which time durations are exactly measurable using two jars and three hoses? Which fractions of a jar are exactly fillable?

We want to solve our campsite conundrum and some other extension problems:

Q1 Using these two jars and three hoses, can you determine exactly 35 contiguous minutes so that we can press the START button on the microwave at the beginning and the STOP button at the end of this time interval?

Q2 Which multiples of 5 minutes (between 5 and 55 minutes) can you measure exactly?

Q3 Which multiples of 1 minute (between 1 and 59 minutes) can you measure exactly?

Assumption: In answering Questions Q1–Q3, you may assume that you can empty any completely filled (or partially filled) jar, if needed, in less than a minute.

We recommend that readers answer Questions Q1–Q3 either on their own or with others, before they read our answers. Let us offer a hint by answering a simpler question.

Q0 Which time durations can we measure if there are two jars and only two hoses?

If we assign one hose to each jar, we will fill the two jars simultaneously and measure 1 hour. If we put both hoses to the same jar, we can measure 1/2 hour. We can measure no other time duration with two jars and two hoses. In fact, given two hoses, we need not use the second jar at all! One jar suffices to measure $0, 1/2, 1$ hour. Moreover, 0 and 1 are the only fillable fractions.

2 Answers to Questions Q1–Q3

In this section, we answer the three questions posed in Section 1. Along the way, we define some useful shorthand notation; and prove the main result that characterizes the set of fillable fractions of a jar and the set of measurable fractions of an hour.

2.1 Answer to Question Q1

With both jars empty, simultaneously insert two hoses in Jar 1 and one hose in Jar 2. When Jar 1 is full, 30 minutes have elapsed. At this moment, simultaneously move one hose from Jar 1 to Jar 2 (so that now two hoses are filling Jar 2) and press the START button. While you wait for Jar 2 to fill (it will take 15 minutes more), empty Jar 1. When Jar 2 is full, put all three hoses into Jar 1, which will fill again in another 20 minutes. When Jar 1 is full, press the STOP button. Between START and STOP, the elapsed time duration is 35 minutes.

2.2 Answer to Question Q2

We answer Question Q2 in the form of a proposition. First, let us develop a shorthand notation.

Suppose that we intend to place h_1 hose(s) into Jar 1 and h_2 hose(s) into Jar 2, where integers $h_1, h_2 \geqslant 0$ and $1 \leqslant h_1 + h_2 \leqslant 3$. Let us denote this plan by the notation (h_1, h_2). Thereafter, when one of the jars is full, we may want to empty it. Recall that it takes less than a minute to empty a full (or partially filled) jar. We classify emptying a jar into two distinct types:

e_j Engaged emptying (denoted by e_j) when we empty Jar j while the other jar is being filled by one or more hoses; and

E_j Paused emptying or a pause (denoted by E_j) when we empty Jar j while the other jar is *not* being filled at all (that is, all three hoses are pouring water outside the jars).

Note that we can apply engaged emptying e_j whenever we want. However, since the measured time duration must be contiguous, a paused emptying (or a pause) E_j is not permissible between START (denoted by opening brace {) and STOP (denoted by closing brace }). We do not explicitly rule out pauses anytime earlier.

For example, we write the solution to Question Q1, described above, as

$$(2,1)\{(e_1,2),(3,0)\}$$

Proposition 2.1. *All multiples of 5 minutes are measurable.*

Proof. All multiples of 5 minutes, save 5 and 25, we can measure without using any pause:

$$10=(2,1)\{(0,3)\}\quad 15=(2,1)\{(0,2)\}\quad 20=\{(3,0)\}$$

$$30=\{(2,0)\}\quad 35=(2,1)\{(e_1,2),(3,0)\}\quad 40=\{(3,0),(0,3)\}=\{(2,1),(0,3)\}$$

$$45=\{(2,1),(0,2)\}\quad 50=\{(2,0),(0,3)\}\quad 55=(2,1)\{(e_1,2),(3,e_2),(0,3)\}$$

Using pauses, we measure 5 and 25 minutes as:

$$5=(2,1)E_1(1,2)E_2(1,2)\{(3,0)\}\quad 25=(2,1)E_1(1,2)E_2(1,2)\{(3,e_2),(0,3)\}$$

This completes the proof of Proposition 2.1. □

Remark 1. *If we can measure y minutes, then we can also measure $y+20r+30s$ minutes for any integers $r,s\geqslant 0$. Here is how: As we are filling the last jar, which will take at least one minute more to conclude measuring y minutes, we simultaneously empty the other jar. Then we immediately fill the empty jar using all three hoses (and repeat the process r times), or with two hoses (and repeat the process s times). Of course, we may rearrange these $(r+s)$ operations of emptying and filling of alternating jars in any order.*

Remark 2. *If a pause is not permissible at all, not even before START, then we were unable to measure 5 and 25 minutes using two jars and three hoses. Can you measure them without using a pause? Alternatively, can you prove that they are impossible to measure without using a pause with two jars and three hoses? Nonetheless, without using any pause, we can measure 5 and 25 minutes if we either increase J, or increase H, as shown below:*

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a) If $J = 3, H = 3$*, then* $5 = (2, 1, 0), (e_1, 2, 1), (2, e_2, 1)\{(0, 0, 3)\}$ *and* $25 = (2, 1, 0), (e_1, 2, 1), (2, e_2, 1)\{(0, 0, 3), (0, 3, 0)\}$.

b) If $J = 2, H = 4$*, then* $25 = (3, 1)\{(e_1, 4), (4, 0)\}$.

c) If $J = 2, H = 5$*, then* $5 = (3, 2)\{(0, 4)\}$.

2.3 Fillable fractions and measurable fractions

Let us return to $H = 3$ hoses and $J = 2$ jars. Empowered by the shorthand notation, we jot down many other time durations (in minutes including fractional parts of a minute) that are measurable using two jars and three hoses. For example,

$$7.50 = (2, 1)E_1(1, 2)E_2(1, 2)\{(2, 0)\}$$

$$17.50 = (2, 1)E_1(1, 2)E_2(1, 2)E_2(2, 1)\{(0, 3)\}$$

$$26.25 = (2, 1)E_1(1, 2)E_2(1, 2)E_2(2, 1)\{(0, 2)\}$$

$$16.875 = (2, 1)E_1(1, 2)E_2(1, 2)E_2(2, 1)E_1(1, 2)\{(2, 0)\}$$

$$13.125 = (2, 1)E_1(1, 2)E_2(1, 2)E_2(2, 1)E_1(1, 2)E_2(1, 1)E_1\{(0, 2)\}$$

In view of Remark 1, using the above examples, we can also measure 27.5, 37.5, 47.5, 57.5; 46.25, 56.25; 36.875, 46.875, 56.875; 33.125, 43.125, 53.125 minutes.

Proposition 2.1 proved that all multiples of 5 minutes are measurable. Can we exactly measure all other multiples of one minute? Proposition 2.2 below, which depends on Lemma 1 below, characterizes all measurable times (as fractions of an hour) using $J = 2$ jars and $H = 3$ hoses.

We need one more notation: Let $z(\text{mod } 1)$ denote the fractional part of a number $z > 0$; that is, the remainder when the highest multiple of 1 is subtracted from z. We call $z(\text{mod } 1)$, which takes a value in the interval $[0, 1)$, a proper fraction. In this paper, we focus on only proper fractions of an hour, since we can add any whole number of hours by alternately filling the jars with one hose.

section 2.1. *If x is a fillable fraction, then*

(i) fractions $(1 - x), (1 - x)/2, 2(1 - x)(\text{mod } 1), x/2, 2x(\text{mod } 1), (x + 1/2)(mod 1)$ *are fillable; and*

(ii) time durations $(1 - x), (1 - x)/2, (1 - x)/3, x, x/2, x/3, x + 1/2, x + 1/3, x + 2/3$ *are measurable, provided each expression takes a value in* $[0, 1]$.

Proof. Having filled Jar A to a fraction x and having emptied Jar B, let us list which new fractions are also fillable. Note that we can do one of three things that will lead to filling Jar B (or Jar A) to a fraction other than in the set $\{0, x, 1\}$:

(a) Put one hose in each jar. When Jar A is full, Jar B is filled to $(1-x)$.

(b) Put two hoses in Jar A, and one in Jar B. When Jar A is full, Jar B is filled to $(1-x)/2$.

(c) Put one hose in Jar A, two hoses in Jar B. If $x < 1/2$, when Jar B is full, Jar A is filled to $(x + 1/2)$. Pause (that is, remove all hoses from both jars), and empty Jar B. Again, put one hose in Jar A, and two hoses in Jar B. When Jar A is full, Jar B is filled to $2(1/2 - x) = 1 - 2x = 2(1-x)(\text{mod } 1)$. Likewise, if $x > 1/2$, when Jar A is full, Jar B is filled to $2(1-x)$. Pause and empty Jar A. Again, put one hose in Jar A, and two hoses in Jar B. When Jar B is full, Jar A is filled to $(x + 1/2)(\text{mod } 1)$.

Hence, fractions $(1-x), (1-x)/2, 2(1-x)(\text{mod } 1), (x+1/2)(\text{mod } 1)$ are fillable fractions. In particular, we have proved that if x is fillable, then $(1-x)$ is fillable. Therefore, we can replace x by $(1-x)$ in the above results to show that fractions $x/2, 2x(\text{mod } 1)$ are also fillable. This completes the proof of Part (i).

Next, let us list which fractions of an hour are measurable. Having filled one jar to a fraction x and having emptied the other jar, press the START button and continue to fill the remainder of the partially filled jar with one, two or three hoses. When this jar is full, we press the STOP button. Thus, between START and STOP we can measure any one of three different contiguous time durations: $(1-x), (1-x)/2, (1-x)/3$. Recall that if x is fillable, then $(1-x)$ is fillable. Therefore, we can replace x by $(1-x)$ in the above results to show that the three time durations $x, x/2, x/3$ are also measurable. Finally, recall from Remark 1 that having measured a time duration x (and having emptied the other jar), we can add to it 1/2, 1/3 and 2/3 of an hour, provided the sum is less than one hour. This proves Part (ii). □

Remark 3. *In view of Lemma 1(i), starting from fillable fraction 0 or 1, we note that 1/2 is a new fillable fraction. In fact, 1/2 is the only fillable fraction that requires no emptying of a jar. Likewise, in view of Lemma 1(ii), starting from the fillable fraction 0 or 1, the new measurable fractions of an hour are 1/2, 1/3 and 2/3. Thus, 1/3 (as well as 2/3) of an hour is measurable, but as we shall see in Proposition 2.2(i) below, 1/3 (as well as 2/3) is not a fillable fraction!*

Lemma 1(i) motivates the following definition:

Definition 1. *A fraction of the form $k/2^m$ (for integers $k, m \geqslant 0$) is called a* bisectional fraction.

Proposition 2.2. *Suppose that there are 3 hoses and 2 jars. Then*

(i) the set of all fillable (proper) fractions is $\{k/2^m : m \geqslant 0, 0 \leqslant k \leqslant 2^m\}$; and

(ii) the set of all measurable time durations is $\{k/(3 \cdot 2^m) : m \geqslant 0, 0 \leqslant k \leqslant 3 \cdot 2^m\}$.

Proof. From Lemma 1(i) recall that if x is fillable, then so are $x/2, x + 1/2$ and $(1 - x)$. Clearly, 1 is a fillable fraction, and so is $1/2$. Starting from fillable fraction $1/2$, we see that $1/4$ is fillable, and so is $1/4 + 1/2 = 3/4$ (or the 1-complement of $1/4$). Thus, all multiples of one-quarter, namely $k/4$ ($0 \leqslant k \leqslant 4$), are fillable. Next, their halves, and their halves plus $1/2$ (or the 1-complements of these halves), namely $k/8$ ($0 \leqslant k \leqslant 8$), are fillable. Next, their halves, and their halves plus $1/2$ (or the 1-complements of these halves), $k/16$ ($0 \leqslant k \leqslant 16$), are all fillable fractions. Continuing in this manner, we show that for any $m \geqslant 0$ and $0 \leqslant k \leqslant 2^m$, the fraction $k/2^m$ is fillable. Thus, every bisectional fraction is fillable.

Next, we must show that any other fraction, with a denominator having a prime factor bigger than 2, is not fillable. Recall, from the proof of Lemma 1(i), that after filling a jar to a fillable fraction x, we have paused to empty the other jar, and refilled the jars using three possible actions which led to newly generated fillable fractions $(1-x), (1-x)/2, 2(1-x) (\text{mod } 1), x/2, 2x (\text{mod } 1)$, and $(x+\frac{1}{2}) (\text{mod } 1)$. All such fractions have denominators that are some powers of 2. Repeating this process as many times as we want, we see that fillable fractions are only those whose denominators are powers of 2. This completes the proof of Part (i).

Since $k/2^m$ is a fillable fraction for all $m \geqslant 0, 0 \leqslant k \leqslant 2^m$, by Lemma 1(ii), $k/(3 \cdot 2^m) \leqslant 1/3$ is measurable for all $m \geqslant 0, 0 \leqslant k \leqslant 2^m$. Furthermore, by Lemma 1(ii), whenever $x \leqslant 1/3$ is measurable, so is $x + 1/3 \in [0, 1]$ and $x + 2/3 \in [0, 1]$. Hence, $k/(3 \cdot 2^m)$ is a measurable fraction of an hour for all $m \geqslant 0$, $0 \leqslant k \leqslant 3 \cdot 2^m$.

Next, we must show that any other fraction, with a denominator divisible by 3^2 or any prime bigger than 3, is not measurable. Again, since after filling a jar to a bisectional fillable fraction x (which are the only fillable fractions), we have paused and used three possible actions which led to newly measurable hours

$$(1 - x), (1 - x)/2, (1 - x)/3, x, x/2, x/3, x + 1/2, x + 1/3, x + 2/3$$

. Such fractions (after reduction) have denominators that are either some power of 2, or three times some power of 2. Repeating this process as many times as we want, we see that measurable fractions are only those whose numerators are arbitrary and denominators are three times some power of 2. This completes the proof of Part (ii). $\square$

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Answer to Question Q3

In view of Proposition 2.2 (ii), no multiple of one minute is measurable unless it is also a multiple of 5 minutes. That is, if $1 \leqslant k \leqslant 59$ is not a multiple of 5, then $k/60$ has a denominator divisible by 5, which exceeds 3; hence, $k/60$ is not measurable. This answers Question Q3.

3 How to Fill and How to Measure

Here we describe methods to fill a jar to a desired fraction (or to measure a desired fraction of an hour) exactly, whenever possible; or approximately, when it is impossible to do so exactly.

3.1 Filling exactly and measuring exactly

Proposition 2.2 tells us which fractions in $[0,1]$ are fillable, and which fractions of an hour are measurable. These results are existential. However, how do we actually fill a jar to a fillable fraction, and how do we actually measure a measurable fraction of an hour? Below we describe a constructive method to accomplish these tasks. Let us begin with a couple of examples.

Example 1. *How to measure 7/24 of an hour—not by trial and error—but systematically?*

Since $7/24 = (1-1/8)/3$*, first we must fill a jar to 1/8. To do so, we write*

$$\frac{1}{8} = \frac{1}{2}\left(1 - \frac{3}{4}\right); \quad \frac{3}{4} = \frac{1}{2} + \frac{1}{4}, \text{ or } \frac{3}{4} = 1 - \frac{1}{4}; \quad \frac{1}{4} = \frac{1}{2}\left(1 - \frac{1}{2}\right)$$

Then,

$$(2,1)E_1(1,2)E_2(1,2)E_2(2,1)E_1$$

will fill Jar 2 to 1/8, or

$$(2,1)E_1(1,2)E_2(1,1)E_1(1,2)E_2$$

will fill Jar 1 to 1/8.
Next, we press START and continue to fill that same jar already filled to 1/8 using all 3 hoses. When the jar is full we press STOP. Hence, we measure 7/24 of an hour as either

$$(2,1)E_1(1,2)E_2(1,2)E_2(2,1)E_1\{(0,3)\}$$

or

$$(2,1)E_1(1,2)E_2(1,1)E_1(1,2)E_2\{(3,0)\}$$

Henceforth, in order to avoid a multiplicity of solutions, or in order to write a computer code that will always give the same solution, we shall only use two operations: 1-complement (C) and half of 1-complement (H); and we shall not use the third operation: half plus a fraction.

Example 2. *With only operations C and H permitted, how to measure 101/192 of an hour?*

Since $101/192 = 1/3 + (1 - 27/64)/3$*, it suffices to fill a jar to a fraction 27/64. How do we do it? We can write*

$$\frac{27}{64} = \frac{1 - \frac{5}{32}}{2};\ \frac{5}{32} = \frac{1 - \frac{11}{16}}{2};\ \frac{11}{16} = 1 - \frac{5}{16};\ \frac{5}{16} = \frac{1 - \frac{3}{8}}{2};\ \frac{3}{8} = \frac{1 - \frac{1}{4}}{2};\ \frac{1}{4} = \frac{1 - \frac{1}{2}}{2}$$

That is, $27/64 = H(5/32) = HH(11/16) = \ldots = HHCHHHH(0)$*. Hence, we measure* $101/192$ *of an hour as*

$$(2,1)E_1(1,2)E_2(2,1)E_1(1,2)E_2(1,1)E_1(1,2)E_2(2,1)E_1\{(0,3),(3,0)\}$$

Note that an operation C cannot be followed by another operation C, because CC is an identity operation; and as such, it can be dropped altogether! Therefore, an operation C must be followed by an operation H (or nothing at all). Furthermore, $\mathrm{H}(y) = (1-y)/2$ and $\mathrm{CH}(y) = 1 - \mathrm{H}(y) = (1+y)/2$. Thus, both H and CH doubles the denominator in y; and the two expressions differ only by a sign—minus or plus—in the numerator. So, we prefer to replace H by q, and CH by $p = 1 - q$. We call q the *negative bisection function* and p the *positive bisection function*.

We now generalize the method used in Example 2 to formulate a constructive algorithm to measure $k/[3 \cdot 2^m]$ of an hour. Note that adding $1/3$ (or $2/3$) of an hour contiguously to any measurable time duration is a straight-forward task: Simply augment $(0,3)$ (or $(e_1,3)(3,0)$) just before the closing brace }. Hence, without loss of generality, we can replace k by $k \pmod{2^m}$; or equivalently, we can assume that $1 \leqslant k \leqslant 2^m - 1$. In addition, we can assume that k is odd (for otherwise we can simplify the fraction). Next, if k is divisible by 3 (say, $k = 3\tilde{k}$), then $k/[3 \cdot 2^m]$ simplifies to $\tilde{k}/2^m = 1 - (2^m - \tilde{k})/2^m$; but if k is not divisible by 3, then $k/[3 \cdot 2^m] = (1/3)[1 - (2^m - k)/2^m]$. Therefore, first we fill a jar to a fraction $x = l/2^m$, where $l = 2^m - k$ is an odd number; and then we measure a fraction $(1 - x)$ or $(1 - x)/3$ of an hour. These tasks we do in Algorithm 1 Parts A and B, respectively.

Algorithm 1A (How to fill a jar to a fraction $x = l/2^m \neq 1/2 = q(0)$, where l is odd?)

Step 1. If $x < 1/2$, write $x = q(y) = (1-y)/2$; and if $x > 1/2$, write $x = (1+y)/2 = p(y) = 1 - q(y)$. That is, if $x < 1/2$, then $y = 1 - 2x$; and if $x > 1/2$, then

$y = 2x - 1$. In either case, when simplified, the denominator of y is half as big as that of x.

Step 2. If $y \neq 1/2$, then replace x by y; and repeat Step 1. If $y = 1/2$, then go to Step 3. [The latter condition is guaranteed to hold after exactly $(m-1)$ repetitions of Step 1, so long as $x = l/2^m$ is a bisectional fraction.]

Step 3. Adjoining the results obtained in each repetition of Step 1, write

$$x = v_1 v_2 \cdots v_{m-1}(1/2) = v_1 v_2 \cdots v_{m-1} q(0)$$

where each transform v_i is either p or q.

Step 4. Then, reading $v_1 v_2 \cdots v_{m-1} q(0)$ backwards, translate $q(0)$ as $(2,1)E_1$; each q as either $(1,2)E_2$ if Jar 1 is empty, or $(2,1)E_1$ if Jar 2 is empty; and each p as either $(1,2)E_2(1,1)E_1$ if Jar 1 is empty, or $(2,1)E_1(1,1)E_2$ if Jar 2 is empty.

Rename the jars, if necessary, so that Jar 1 is filled to $x = l/2^m$ and Jar 2 is empty.

Algorithm 1B (Thereafter, how to measure $(1-x)$ or $(1-x)/3$ of an hour?)

Step 5. To measure $(1-x)$ or $(1-x)/3$ of an hour, after filling Jar 1 to x, press START, continue to fill Jar 1 with one or three hoses, and when Jar 1 is full, press STOP. That is, after filling Jar 1 to x, augment $\{(1,0)\}$ or $\{(3,0)\}$, respectively.

Remark 4. *In Step 3 of Algorithm 1A, we express the bisectional fraction $l/2^m$ (where l is odd) as a composition of $(m-1)$ positive and negative bisection functions p and q; and the composition is to be evaluated at $1/2$. We call the resulting composition the* bisectional continued fraction *representation of the bisectional fraction $l/2^m$. In addition, we call Steps 1–3 the* bisectional algorithm. *See [Sar19] for details. In fact, for any $m > 1$, there is a bijection (or a one-to-one correspondence) between bisectional fractions $\{l/2^m : 1 \leqslant l \leqslant 2^m - 1, l \text{ odd}\}$ and compositions of exactly $(m-1)$ bisection functions p and q, there being 2^{m-1} elements in each set.*

Example 3. *To illustrate Algorithm 1, let us measure 25/96 of an hour (15 minutes and 37.5 seconds). Since $25/96 = (1 - 7/32)/3$, we first fill a jar to a fraction $7/2^5$. Using Algorithm 1A,*

$$7/32 = q(9/16) = qp(1/8) = qpq(3/4) = qpqp(1/2) = qpqpq(0)$$

Thus, using five operations we can fill one jar to a fraction $7/2^5$, and then we measure 25/96 of an hour as $(2,1)E_1$; $(1,2)E_2(1,1)E_1$; $(1,2)E_2$; $(2,1)E_1(1,1)E_2$; $(2,1)E_1\{(0,3)\}$, (where we have inserted semi-colons to help the human reader follow the translation). We will see some more examples of Algorithm 1 in the next subsection.

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3.2 Filling approximately and measuring approximately

Proposition 2.2 says that using three hoses, bisectional fractions (fractions of the form $l/2^m$), and no other fractions, are fillable; and Algorithm 1A helps us write down the operations needed to do so. However, we can just as well apply Algorithm 1A to a non-bisectional fraction. The difference is that in Step 2 we will never reach $y = 1/2$; that is, the process will not terminate! Nonetheless, we can continue the process to any desired number of steps, and thereby obtain an approximate bisectional continued fraction representation for any fraction f. Therefore, any non-bisectional fraction f, though not fillable, can be approximated by a bisectional fraction within a specified error of Δ. Simply choose an m large enough so that $2^{-m} < \Delta$; expand f as a bisectional continued fraction up to m steps; and evaluate it at 0. See the next example.

Example 4. *Although $2/5$ is not a fillable fraction with three hoses, we can approximate $2/5$ within an error of 0.001 by a bisectional fraction. Since $2^{-10} < .001 < 2^{-9}$, we expand $2/5$ in bisectional continued fraction representation to $m = 10$ places (using Steps 1–3 of Algorithm 1A); and then evaluate that expansion at 0 to get*

$$\frac{2}{5} = qq\ pq\ pq\ pq\ pq\left(\frac{3}{5}\right) \approx qq\ pq\ pq\ pq\ pq(0) = \frac{409}{1024}$$

with an error of $-.0005859$. Using Step 4 of Algorithm 1A, we obtain the sequence of operations

$$(2,1)E_1(1,2)E_2(1,1)E_1\ (1,2)E_2(2,1)E_1(1,1)E_2\ (2,1)E_1(1,2)E_2(1,1)E_1$$
$$(1,2)E_2(2,1)E_1(1,1)E_2\ (2,1)E_1(1,2)E_2$$

which will fill Jar 1 to a fraction $409/1024$.

Likewise, using three hoses, we can approximately measure any desired fraction of an hour, even when it is not a bisectional fraction or one-third of a bisectional fraction. See the next example.

Example 5. *To measure $2/9$ of an hour correct to one second (or $1/3600 = .000278$ of an hour), we note that $2/9 = (1 - 1/3)/3$. Hence, first we must fill a jar to a fraction $1/3$ within an error of $1/1200$. Since $1/3 = qqq\ldots(0)$, and the sequence $q(0) = 1/2, q^2(0) = 1/4, q^3(0) = 3/8, \ldots$, yields terms that are progressively closer to $1/3$, we may approximate $1/3$ by $q^9(0) = 171/512$, which has an error of $1/1536$. Therefore, we approximate 2/9 of an hour by the measurable fractional hour $(1 - 171/512)/3 = 341/1536$, which has an error of $(341/1536 - 2/9) = -1/4608 = -0.000217$. By Step 4 of Algorithm 1A, we fill Jar 2 to a fraction $171/512$, using the sequence of operations*

$$(2,1)E_1(1,2)E_2(2,1)E_1\ \ (1,2)E_2(2,1)E_1(1,2)E_2\ \ (2,1)E_1(1,2)E_2(2,1)E_1$$

Next, following Algorithm 1B, we press START; we continue to fill Jar 2 with 3 hoses; and when Jar 2 is full, we press STOP. That is, we augment $\{(0,3)\}$ *to measure* $341/1536$ *of an hour.*

4 The Jar Filling Puzzle $(J = 2, H \geqslant 3)$

In this section, we allow three or more hoses, but we still restrict to only $J = 2$ jars. For example, if $H = 5$ hoses are available, then using only one jar we can measure 60, 30, 20, 15, 12 minutes when filling the jar with 1–5 hoses, respectively. Moreover, we can measure every multiple of 1 minute (between 1 and 59 minutes), as shown below. This was an impossible task with $H = 3$ hoses (see Remark 3). Following the same reasoning, we can prove this task is impossible with $H = 4$ hoses. However, as shown below, the situation changes when $H = 5$ hoses are available.

Example 6. *Given* $J = 2$ *jars and* $H = 5$ *hoses, we list below how to measure all multiples of 1 minute between 1 and 11 minutes:*

$$1 = (3,2)E_1(4,1)E_1\{(0,5)\} \quad 2 = (3,1)E_1(2,1)E_1\{(0,5)\} \quad 3 = (4,1)E_1(1,1)E_2\{(5,0)\}$$

$$4 = (3,2)E_1\{(0,5)\} \quad 5 = (3,2)E_1\{(0,4)\} \quad 6 = (2,1)E_1\{(0,5)\}$$

$$7 = (3,1)E_1(4,1)E_1(1,1)E_2\{(5,0)\} = (3,2)E_1(1,2)E_2(1,4)E_2\{(5,0)\} \quad 8 = (3,1)E_1\{(0,5)\}$$

$$9 = (4,1)E_1\{(0,5)\} \quad 10 = (2,1)E_1\{(0,3)\} \quad 11 = (3,2)E_1(1,4)E_2\{(5,0)\}$$

Having measured y *contiguous minutes* $(1 \leqslant y \leqslant 11)$*, we can also measure* $y + 12r$ *minutes for* $1 \leqslant r \leqslant 4$*, by using the simple strategy mentioned in Remark 1. As we are filling the last jar, which will take at least one minute more to conclude measuring* y *minutes, we simultaneously empty the other jar; and then we fill the empty jar using all five hoses (and we repeat the process* r *times). For instance,* $29 = 5 + 12 * 2 = (3,2)E_1\{(0,4),(5,e_2),(0,5)\}$.

Remark 5. *Recall from Proposition 2.1 that we can measure 5 minutes using only 3 hoses as* $5 = (2,1)E_1(1,2)E_2(1,2)\{(3,0)\}$*. Note that this method takes four steps. But with 5 hoses, we can measure 5 minutes more efficiently as* $5 = (3,2)\{(0,4)\}$*; that is, only two steps suffice!*

4.1 Fillable fractions and measurable fractions of an hour

More generally, when we have $J = 2$ jars and $H \geqslant 3$ hoses, we write down the set of all fillable fractions and the set of all measurable fractions of an hour in 1. Theorem 4.1 is an easy extension of Proposition 2.2. Hence, it does not require a separate proof.

Theorem 4.1. *Suppose that there are $J = 2$ jars and $H \geqslant 3$ hoses. Then*

(i) the set $\mathcal{F}$ of all fillable fractions is $\left\{ \dfrac{k}{t} : t = \prod_{p<H} p^{m_p}, m_p \geqslant 0; 0 \leqslant k \leqslant t \right\}$*; and*

(ii) the set $\mathcal{M}$ of all measurable fractions of an hour is

$$\left\{ \frac{k}{Ht} : t = \prod_{p<H} p^{m_p}, m_p \geqslant 0; 0 \leqslant k \leqslant Ht \right\}$$

where the product runs over all prime numbers p less than H.

Remark 6. *If we specialize Theorem 4.1 to $H = 2$, the only fillable fractions are 0 and 1, and the only measurable time durations are $0, 1/2, 1$. This is in agreement with the answer to Question Q0. If we specialize Theorem 4.1 to $H = 3$, we recover Proposition 2.2.*

Let us illustrate the use of Theorem 4.1.

Example 7. *Suppose that we want to measure 37 minutes using $J = 2$ jars and $H = 5$ hoses. Since $37/60 = 3/5 + (1 - 11/12)/5$ hour, first we fill Jar 1 to a fraction 11/12=2/3+1/4 using $(2,3)E_2(1,4)E_2$. Thereafter, we measure 37/60 of an hour using $\{(5,0),(e_1,5),(5,e_2),(0,5)\}$. Furthermore, in view of Theorem 4.1, we know that 37% of an hour is impossible to measure with $J = 2$ jars and $H = 5$ hoses, because the denominator 100 is divisible not only by 5, but also by 5^2. However, with $H = 6$ hoses, we can measure 37% of an hour: First, we write $37\% = 1/5 + (1 - 3/20)/5$; then we fill Jar 1 to fraction 3/20 using $(4,1)E_1(1,5)E_2$; finally, we measure the desired time duration using $\{(5,0),(0,5)\}$.*

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4.2 How many hoses do we need?

We can read Theorem 4.1 in reverse to determine the number of hoses needed to fill one of two jars to a desired fraction (or to measure a desired fraction of an hour). The recipe is given in Corollary 4.1 below. But first, since Theorem 4.1 requires us to factor the denominator t of a target fraction (to fill or to measure), let us recall the most basic result of Number Theory—the Fundamental Theorem of Arithmetic or the *unique prime factorization theorem* (see [Gau86]), which says that any integer t can be uniquely factored as the product of prime powers; that is,

$$t = q_1^{n_1} q_2^{n_2} \cdots q_a^{n_a}$$

where the bases $q_1 < q_2 \cdots < q_a$ are distinct prime numbers and the indexes $n_1, n_2, \ldots, n_a \geqslant 1$ are natural numbers. For example, $18 = 2 \cdot 3^2$; $60 = 2^2 \cdot 3 \cdot 5$; $96 = 2^5 \cdot 3$; $168 = 2^3 \cdot 3 \cdot 7$.

Corollary 4.1. *Let there be $J = 2$ jars. Let Q denote the largest prime factor of t. Then*

(i) to fill a jar to a fraction k/t, we need at least $H = Q + 1$ hoses; and

(ii) to measure k/t of an hour, we need at least $H = Q + 1$ hoses, if Q^2 divides t; otherwise, $H = Q$ hoses will suffice.

We will see several illustrations of Corollary 4.1 in the next subsection.

4.3 How to fill and how to measure?

We leave it to the reader to imitate Algorithm 1A and 1B to write a more general algorithm that will describe how to fill a jar to a desired fraction and how to measure a specific fraction of an hour requiring $H > 3$ hoses. We only give several examples. In each example, we use two jars, and we use the fewest number of hoses (as determined by Corollary 4.1). Furthermore, given the required number of hoses, we strive to use the fewest number of steps necessary to accomplish the task. We leave open the question of proving that these are indeed the fewest number of steps.

Example 8. *(Fill a jar and measure time using the fewest number of hoses as determined by Corollary 4.1):*

(8.1) To measure 2019 seconds, or $2019/(2^4 \cdot 3^2 \cdot 5^2)$ hour, we need $H = 6$ hoses; and we use

$$(2,1)E_1(4,1)E_1(1,5)E_2(1,2)E_2(5,1)E_1\{(0,4),(3,0)\}$$

(8.2) *To measure 2020 seconds, or* $2020/(2^4 \cdot 3^2 \cdot 5^2) = 101/(2^2 \cdot 3^2 \cdot 5)$ *hour, we need* $H = 5$ *hoses; and we use*

$$(3,2)E_1(1,2)E_2(3,1)E_1\{(0,2),(5,0)\}$$

(8.3) *To fill a jar to a fraction 101/180=1/9+1/4+1/5, we need* $H = 6$ *hoses; and we use*

$$(3,2)E_1(1,3)E_2(1,4)E_2(1,5)$$

5 More Than Two Jars

If more than two jars are available, can we fill a jar to a newer fraction, which we could not do with only two jars? Likewise, can we measure some other fraction of an hour, which we could not measure with only two jars?

5.1 The jar filling puzzle ($J > 2, H = 3$)

In case of $J = 2$ jars and $H = 3$ hoses, from Theorem 4.1, we know that all bisectional fractions (and only bisectional fractions) are fillable. Now we claim that there is nothing to be gained if a third jar were available; that is, we cannot fill any jar to a new fraction using a third jar above and beyond what we can do with two jars.

Suppose that three jars are given. Whenever one jar is full, suppose that the other two jars are filled to fractions $x_1 \geqslant x_2$, respectively. At this point, we empty the full jar, so that the three jars are filled to $(x_1, x_2, 0)$ with $x_1 \geqslant x_2 > 0$. Suppose that we assign (h_1, h_2, h_3) hoses to the three jars, where $h_j \geqslant 0$ and $h_1 + h_2 + h_3 \leqslant 3$. If either $h_1 = 0$ or $h_2 = 0$, then we are essentially using only two jars. This cannot lead to any new fraction that was not fillable with two jars. Therefore, we assume $h_1 \geqslant 1, h_2 \geqslant 1$. For every triplet (h_1, h_2, h_3), we document in Table 1 how much the other jars will be filled as soon as one of the jars becomes full.

Note that in every case, each fraction filled is a bisectional fraction, as long as $x_1 \geqslant x_2$ are both bisectional fractions. Therefore, no jar can be filled to a fraction other than a bisectional fraction. However, from Proposition 2.2, we know that all bisectional fractions are fillable with only $J = 2$ jars. Therefore, the third jar does not help us fill any jar to a fraction other than bisectional.

Similarly, one can argue that the third jar will not help you measure any new fraction of an hour other than one-third of a bisectional fraction. We leave the details to the reader.

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(h_1, h_2, h_3)	Jar filled first	Condition	Fractions filled
$(2, 1, 0)$	1	always	$\left(1, x_2 + \frac{1-x_1}{2}, 0\right)$
$(1, 2, 0)$	2	if $x_1 < \frac{1+x_2}{2}$	$\left(x_1 + \frac{1-x_2}{2}, 1, 0\right)$
$(1, 2, 0)$	1	if $x_1 \geqslant \frac{1+x_2}{2}$	$(1, x_2 + 2(1 - x_1), 0)$
$(1, 1, 0)$	1	always	$(1, x_2 + 1 - x_1, 0)$
$(1, 1, 1)$	1	always	$(1, x_2 + 1 - x_1, 1 - x_1)$

Table 1: Fillable fractions when up to three hoses are used to fill three jars.

5.2 The jar filling puzzle $(J > 2, H > 3)$

Now suppose that we have $J > 2$ jars and $H > 3$ hoses. In fact, we can assume that $J \leqslant H - 1$, because we must put at least one hose into each jar; but we must not put exactly one hose in every jar—at least one jar must be filled with two or more hoses. Therefore, Jars $H, H+1, \ldots, J$ are irrelevant. In particular, when $H = 3$, Jars $3, 4, \ldots, J$ are irrelevant; and only two jars suffice. A surprising truth is that when $H > 3$ hoses are available, Jars $3, 4, \ldots, (H - 1)$ are still irrelevant! Just two jars suffice—the availability of more jars does not help us fill any jar to a fraction other than those fillable with only two jars, as described in Theorem 4.1. We prove it in Theorem 5.1.

Theorem 5.1. *Suppose that there are $J \geqslant 3$ jars and $H > J$ hoses. Then the set of all fillable fractions and the set of all measurable time durations are the same as the corresponding sets $\mathcal{F}$ and $\mathcal{M}$ given in Theorem 4.1, where we had only $J = 2$ jars.*

Proof. When any one jar is full, let us pause and empty that full jar. Let us relabel the jars so that they are filled to fractions $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{J-1} \geqslant 0$. Continue to fill the jars after assigning $(h_1, h_2, \ldots, h_J)$ hoses to the J jars. As reasoned earlier, without loss of generality, let us assume that $h_j \geqslant 1$ (for all $j = 1, 2, \ldots, J$) and $h_1 + h_2 + \cdots + h_J \leqslant H$. This, in particular, implies that $1 \leqslant h_j \leqslant H - J + 1 < H$ (for all $j = 1, 2, \ldots, J$).

Suppose that the first jar to get filled is Jar j^*. When that happens, for any j ($1 \leqslant j \leqslant J$), Jar j is filled to fraction $x_j + (1 - x_{j^*})h_j/h_{j^*}$, which belongs to $\mathcal{F}$ as long as x_j and x_{j^*} both belong to $\mathcal{F}$. Hence, every time a jar is full, the other jars are filled to fractions that belong to $\mathcal{F}$. Therefore, even with $J \geqslant 3$ jars available, the set of fillable fractions is $\mathcal{F}$, which consists of those fractions that are fillable using only two jars and H hoses.

Since the set of all fillable fractions is $\mathcal{F}$, the set of all measurable fractions of an hour is $\mathcal{M}$. This completes the proof of the theorem. □

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Conclusion

We have taken the reader through an intellectual excursion, which started with a simple-minded campsite conundrum and culminated in Theorem 4.1, describing all fillable fractions and all measurable fractions of an hour using $J = 2$ jars and $H \geqslant 3$ hoses, when "Any one hose will fill any one jar in exactly one hour." We also constructed an algorithm that details how to fill a jar to a desired fraction and how to measure a specific fraction of an hour, when $J = 2$ and $H = 3$; and we invited readers to extend the algorithm to $J = 2$ jars and $H > 3$ hoses.

We confess that initially Theorem 5.1, which states that any time duration measurable with $J \geqslant 3$ jars and $H > J$ hoses is also measurable with only two jars and H hoses, came to us as counterintuitive! However, on careful reflection, we realized that since we can empty and reuse the two jars as many times as we need, effectively we have an infinite supply of jars of equal volume, even though in reality we have only two jars!

The jar-filling puzzle analyzed and generalized here bear some resemblance to other puzzles we have studied in previous papers [Sar19] and [Sar20]. In fact, we formulated our jar-filling puzzle by merging some features and modifying other features of those two previous puzzles. Let us summarize those other puzzles and explain how they differ from the jar-filling puzzle.

In the "glass filling puzzle" in [Sar19], given g identical and transparent glasses, we can fill any one or more glasses to a proper fraction k/t, where all prime factors of t are at most g; and it is impossible to fill a glass to a proper fraction k/s, if s has a prime factor bigger than g. On the other hand, what happens if the glasses are not transparent? Then, of course, we cannot call them glasses anymore; we call them jars. While we drop identicalness and transparency, we do two things: (1) we preserve an old feature—the jars are of equal volume; and (2) we introduce a new feature—we allow three or more hoses, through each of which water is flowing at the same constant rate. Even with these changes, we can reproduce the results of the glass-filling puzzle.

Similarly, in the "rope burning puzzle" in [Sar20], we have $n \geqslant 1$ ropes, which burn in exactly one hour each, but at unknown, uneven, unequal rates. By burning n ropes, either from one or both ends, but never from an intermediate point, we can exactly measure any bisectional fraction of an hour (a fraction of the form $k/2^n$, where $1 \leqslant k \leqslant 2^n$ is an integer). We can also approximately measure any proper fraction k/t of an hour (or, for that matter, any irrational fraction of an hour) within a desired level of accuracy using sufficiently many ropes. While the ropes burn unevenly, in our jar-filling puzzle each hose fills a jar at the same rate, causing the filling rate of each jar to be proportional to the number of hoses filling them. In both puzzles, the objective is to measure some contiguous stretch of time. While we can burn the ropes only once, we can empty and reuse the jars as often as we want. Thus, while the given number of ropes is a hard upper bound, the number of jars is not. On the other hand, while we can simultaneously burn as many ropes as we want, each from one or both ends, the number

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of hoses we can use at any particular time has a hard upper bound. Thus, a restricted parameter in one puzzle is an unrestricted parameter in the other puzzle and vice versa. The set of measurable fractions of an hour in our jar-filling puzzle is much larger than the corresponding set in the rope-burning puzzle.

We encourage readers to toy with other mathematical puzzles and always follow-up the solutions with thoughtful reflections. We hope they too will discover hidden truths and be pleasantly surprised. A good source of many easy-to-understand problems is [Per79]. Happy hunting!

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