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Local well-posedness of a coupled Westervelt–Pennes model of nonlinear ultrasonic heating

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Abstract
High-intensity focused ultrasound (HIFU) waves are known to induce localised heat to a targeted area during medical treatments. In turn, the rise in temperature influences their speed of propagation. This coupling affects the position of the focal region as well as the achieved pressure and temperature values. In this work, we investigate a mathematical model of nonlinear ultrasonic heating based on the Westervelt wave equation coupled to the Pennes bioheat equation that captures this so-called thermal lensing effect. We prove that this quasi-linear model is well-posed locally in time and does not degenerate under a smallness assumption on the pressure data.

Keywords: ultrasonic heating, Westervelt's equation, nonlinear acoustics, Pennes bioheat equation, HIFU
Mathematics Subject Classification numbers: 35L70, 35K05.

(Some figures may appear in colour only in the online journal)

1. Introduction

High-intensity focused ultrasound (HIFU) is an innovative medical tool that relies on focused sound waves to induce localised heating to the targeted tissue [37]. Due to its non-invasive

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Recommended by Dr John Lowengrub.

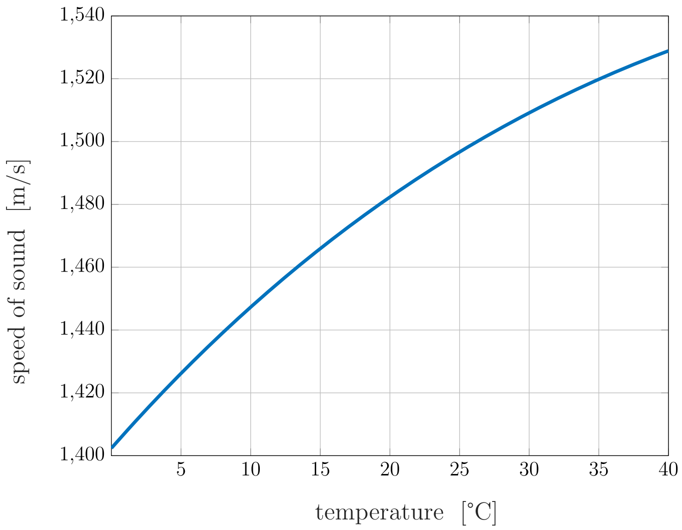

Figure 1. The dependency of the sound speed on the temperature in water.

nature and relatively brief treatment time, it has excellent potential to be used in the therapy of various benign and malignant tumors; see, e.g., [11, 14, 23, 25, 39]. The ability to accurately determine the properties of the pressure and temperature field in the focal region is crucial in these procedures and motivates the research into the validity of the corresponding mathematical models.

It is well-known that the heating of tissue influences the speed of propagation of sound waves and, in turn, the position of the focal region; this effect is commonly referred to as thermal lensing [7, 12, 13]. In this work, we analyse a mathematical model of nonlinear ultrasonic heating that captures this effect. More precisely, we study a coupled problem consisting of the Westervelt wave equation of nonlinear acoustics [38]:

$$p_{tt} - c^2(\Theta)\Delta p - b\Delta p_t = k(\Theta)\big(p^2\big)_{tt} \tag{1.1}$$

and the Pennes bioheat equation [29]:

$$\rho_\mathrm{a} C_\mathrm{a}\Theta_t - \kappa_\mathrm{a}\Delta\Theta + \rho_\mathrm{b} C_\mathrm{b} W(\Theta - \Theta_\mathrm{a}) = \mathcal{Q}(p_t). \tag{1.2}$$

Westervelt's equation (1.1) is given in terms of the acoustic pressure $p = p(x, t)$. The coefficient $c = c(\Theta)$ denotes the speed of sound, which is known to change with the temperature. Experimentally determined values of the speed of sound are usually represented as polynomial functions of the temperature using a least squares fit; see, e.g., [2]. In water, for instance, the speed of sound is taken to be

$$\begin{aligned} c(\Theta) = {} & 1402.39 + 5.0371\Theta - 5.8085\times 10^{-2}\Theta^2 + 3.3420\times 10^{-4}\Theta^3 \\ & - 1.4780\times 10^{-6}\Theta^4 + 3.1464\times 10^{-9}\Theta^5; \end{aligned} \tag{1.3}$$

see [7, section 2.2] and [2] and figure 1.

The term $-b\Delta p_t$ in Westervelt's equation (1.1) accounts for the losses in propagation due to the viscosity and thermal conductivity of the propagation medium. The damping parameter $b > 0$ is called the sound diffusivity [24] as the strong damping $-b\Delta p_t$ is responsible for the parabolic character of the acoustic equation. Assuming harmonic excitation with angular frequency ω, sound diffusivity b is connected to the absorption coefficient α via

$$b = \frac{\alpha c_{\mathrm{a}}^3}{\omega^2},$$

where c_{a} is the ambient speed of sound (in the tissue) [28]. Note that if the attenuation obeys a frequency power law, equation (1.1) generalises to involve a fractional damping term; see, e.g., [28]. This case is thus of interest for future analysis as well, but outside the scope of the current work. The right-hand side coefficient in (1.1) is given by

$$k(\Theta) = \frac{1}{\rho c^2(\Theta)} \beta_{\mathrm{acou}}. \tag{1.4}$$

Here ρ is the medium density and β_{acou} the acoustic coefficient of nonlinearity.

Westervelt's equation (1.1) can be seen as an approximation of the thermoviscous Navier–Stokes–Fourier system of governing equations of sound propagation. In its derivation, it is assumed that the deviations of the involved quantities from their equilibrium values of order three and higher can be neglected. Thus, the nonlinearity in the resulting wave equation is of quadratic type. We refer the reader to, e.g., [8] and [20, chapter 5] for details on this so-called weakly nonlinear approach to acoustic modelling. The right-hand side term in (1.1) can be written out as

$$k(\Theta)(p^2)_{tt} = 2k(\Theta)(pp_{tt} + p_t^2). \tag{1.5}$$

In the course of the analysis one thus needs to handle the nonlinearities pp_{tt} and p_t^2. The first one represents the main challenge as it contributes to the quasilinear character of the equation. This can be seen if we rewrite (1.1) equivalently as

$$(1 - 2k(\Theta)p)p_{tt} - c^2(\Theta)\Delta p - b\Delta p_t = 2k(\Theta)p_t^2.$$

To ensure the validity of this wave model, a well-posedness analysis of Westervelt's equation must guarantee that the leading factor remains positive almost everywhere. This invokes the condition $1 - 2k(\Theta)p > 0$ almost everywhere, which in turn requires $\|p\|_{L^\infty}$ to remain small enough in time. This issue is commonly resolved by using a Sobolev embedding under the assumption of small pressure data, e.g., $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, as in [15]. Note that in practice this condition is less restrictive than it appears since $k(\Theta)$ is proportional to the inverse of speed of sound squared (see (1.4)) and thus relatively small. Values of the involved acoustic coefficients in different thermoviscous fluids can be found in, e.g., [33, chapter 8].

Pennes bioheat equation (1.2) is solved for the temperature $\Theta = \Theta(x, t)$. The function $\mathcal{Q} = \mathcal{Q}(p_t)$ represents the acoustic energy absorbed by the tissue at any given point. The term $\rho_{\mathrm{b}} C_{\mathrm{b}} W(\Theta - \Theta_{\mathrm{a}})$ models the removal of heat by blood circulation. Here, ρ_{b} and C_{b} are the density and specific heat capacity of blood, respectively, and W is the volumetric perfusion rate of the tissue measured in milliliters of blood per milliliter of tissue per second. The values of these material properties in the human tissue can be found, for example, in [7, table 3]. The coefficients ρ_{a} and κ_{a} denote the ambient density and thermal conductivity (i.e., the tissue density and thermal conductivity). C_{a} is the ambient heat capacity and Θ_{a} is the ambient temperature. In the body, the latter is usually taken to be 37°C; see [7].

To the best of our knowledge, this is the first work dealing with a rigorous mathematical analysis of a coupled Westervelt–Pennes model. Westervelt's equation has been extensively studied by now in various settings with constant material parameters; see, e.g., [15, 16, 18, 19, 26] and the references given therein, where results concerning local well-posedness, global well-posedness, and asymptotic behaviour of the solution have been established. The results on the well-posedness of the Westervelt equation with an additional strong nonlinear damping and with $L^\infty(\Omega)$ varying coefficients have been obtained in [4, 27]. We mention that this wave equation can also be rigorously recovered in the limit of a third-order nonlinear acoustic equation for vanishing thermal relaxation time; see the analysis in [3, 17].

A prominent feature of the present quasilinear thermo-acoustic problem is the dependence of propagation speed on the temperature, which we will assume in the analysis to be polynomial and non-degenerate in accordance with (1.3). Our approach in proving the local-in-time well-posedness of the Westervelt–Pennes system relies on an energy method, where an energy analysis of a suitable linearisation is combined with a fixed-point argument under an assumption of smooth and small (with respect to pressure) data. Although the heat equation (1.2) has regularising properties, it does not seem feasible to transfer these to the pressure equation (1.1) and make use of the damping property of heat conduction as in the classical thermo-elastic systems; see, e.g., [21, 22, 31] and the references given therein. This issue arises due to the very weak coupling in the present model, meaning that the coupling is realised through the source term $\mathcal{Q}(p_t)$ only and the linearised model will be decoupled; see section 3 below. In the classical thermoelasticity, the coupling is achieved already in the linear model through terms of the form $\nabla\Theta$ in the elastic equation and $\nabla \cdot p_t$ in the heat equation. Such a coupling allows stabilising the system by using only the damping coming from the heat equation, which is not the case here. In fact, the assumption $b > 0$ will be crucial for obtaining the energy bounds.

As mentioned above, a critical step in any analysis involving Westervelt's equation is handling the higher-order time-derivative of the pressure in the nonlinear term; that is, $k(\Theta)\left(p^2\right)_{tt}$. Due to the temperature-dependent coefficients, we here rely on higher-order energies compared to the analysis of Westervelt equation in homogeneous media in [15] and assume

$$(p, p_t)_{|t=0} = (p_0, p_1) \in H^3(\Omega) \times H^2(\Omega).$$

More precisely, the energy functional for the acoustic pressure used in the analysis will be the sum of the following:

$$\begin{aligned} E_0[p](t) &= \frac{1}{2}\Big\{\|\sqrt{1-2k(\Theta)p(t)}\,p_t(t)\|_{L^2}^2 + \|c(\Theta)\nabla p(t)\|_{L^2}^2\Big\},\\ E_1[p](t) &= \frac{1}{2}\Big\{\|\sqrt{1-2k(\Theta)p(t)}\,p_{tt}(t)\|_{L^2}^2 + \|c(\Theta)\nabla p_t(t)\|_{L^2}^2 + \|c(\Theta)\Delta p(t)\|_{L^2}^2\Big\}, \end{aligned}$$

and

$$E_2[p](t) = \frac{1}{2}\|\sqrt{b}\nabla\Delta p(t)\|_{L^2}^2.$$

Note that in a linear wave equation where $k = 0$ and the speed of sound is constant, E_0 would reduce to the standard energy functional for the wave equation; see, e.g., [9, chapter 7] and [34, chapter 9]. Here due to the quasilinear character of Westervelt' equation we have to involve higher-order (with respect to space and time) energy functionals to handle the nonlinearities in the analysis.

For clarity of exposition, in this work we consider pressure nonlinearities in the form of (1.5) and with Dirichlet boundary data. However, we emphasise that our theoretical framework can be extended in a straightforward manner to nonlinearities in the form of $k(\Theta)f(p, p_t, p_{tt})$ with suitable assumptions on the function f as well as to more general pressure and temperature boundary data, such as Neumann conditions or absorbing boundary conditions for the pressure.

We organise the rest of our exposition as follows. We provide more detailed insight into mathematical bio-acoustic modelling in section 2. Section 3 focuses on the energy analysis of a (partially) linearised uncoupled problem. In section 4, we present the study of the coupled nonlinear model by relying on the result from the previous section and Banach's fixed-point theorem. Our main well-posedness result is contained in theorem 4.1. We conclude the paper with a discussion and an outlook on future work.

2. Theoretical preliminaries

As discussed above, volume coupling of the acoustic pressure p to the temperature field Θ is achieved via appropriate source terms and the use of temperature-dependent acoustic material parameters; [6, 7, 12, 28, 35]. We therefore study the following coupled problem:

$$\begin{cases} p_{tt} - q(\Theta)\Delta p - b\Delta p_t = k(\Theta)\big(p^2\big)_{tt}, & \text{in } \Omega \times (0,T), \\ \rho_\mathrm{a} C_\mathrm{a}\Theta_t - \kappa_\mathrm{a}\Delta\Theta + \rho_\mathrm{b} C_\mathrm{b} W(\Theta - \Theta_\mathrm{a}) = \mathcal{Q}(p_t), & \text{in } \Omega \times (0,T), \end{cases} \tag{2.1a}$$

where we have introduced the function

$$q(\Theta) = c^2(\Theta).$$

We consider (2.1a) together with homogeneous Dirichlet boundary conditions

$$p|_{\partial\Omega} = 0, \qquad \Theta|_{\partial\Omega} = 0, \tag{2.1b}$$

and the initial data

$$(p, p_t)|_{t=0} = (p_0, p_1), \qquad \Theta|_{t=0} = \Theta_0. \tag{2.1c}$$

The constant medium parameters appearing in (2.1) are all assumed to be positive. As discussed, the speed of sound $c = c(\Theta)$ exhibits polynomial dependence on the temperature, so we make the following assumptions on the function q in our analysis. Note that throughout the paper, we use $x \lesssim y$ to denote $x \leqslant Cy$, where $C > 0$ is a generic constant that may depend on Ω, the final time T, and medium parameters.

Assumption 1. Let $q \in C^2(\mathbb{R})$. We assume that there exists $q_0 > 0$, such that

$$q(s) \geqslant q_0 \quad \forall\, s \in \mathbb{R}.$$

Furthermore, there exist $\gamma_1 \geqslant 0$ and $C_1 > 0$, such that

$$|q''(s)| \leqslant C_1(1 + |s|^{\gamma_1}) \quad \forall\, s \in \mathbb{R}.$$

By these assumptions and Taylor's formula, it further follows that

$$|q'(s)| \lesssim 1 + |s|^{\gamma_1 + 1}. \tag{2.2}$$

The function k is assumed to be related to q via (1.4) throughout this work. Therefore, we have

$$|k(\Theta)| \lesssim \frac{1}{q_0}. \tag{2.3}$$

Furthermore, since

$$
\begin{aligned}
|k'(\Theta)| &\lesssim \frac{1}{q_0^2}|q'(\Theta)| \lesssim \frac{1}{q_0^2}(1+|\Theta|^{\gamma_1+1}),\\
|k''(\Theta)| &\lesssim \frac{1}{q_0^2}|q''(\Theta)| + \frac{1}{q_0^3}|q'(\Theta)|^2 \lesssim \frac{1}{q_0^2}(1+|\Theta|^{\gamma_1}) + \frac{1}{q_0^3}(1+|\Theta|^{\gamma_1+1})^2,
\end{aligned}
$$

we conclude that there exists $\gamma_2 > 0$, such that

$$
|k'(\Theta)| \lesssim 1 + |\Theta|^{\gamma_2+1}, \qquad |k''(\Theta)| \lesssim 1 + |\Theta|^{\gamma_2}. \tag{2.4}
$$

Modelling the absorbed acoustic energy. The acoustic energy absorbed by the tissue is represented by the source term $\mathcal{Q} = \mathcal{Q}(p_t)$ in the heat equation. We will make the following general assumptions concerning its properties in our analysis, which allow us to cover important particular cases from the literature.

Assumption 2. The mapping Q is Lipschitz continuous on bounded subsets of the space $L^\infty(0,T;\ L^\infty(\Omega))$ with values in $L^2(0,T;\ L^2(\Omega))$, that is,

$$
\|\mathcal{Q}(u) - \mathcal{Q}(v)\|_{L^2(L^2)} \lesssim (\|u\|_{L^\infty(L^\infty)} + \|v\|_{L^\infty(L^\infty)})\|u-v\|_{L^2(L^2)}, \tag{2.5}
$$

and such that $\mathcal{Q}(0) = 0$. Additionally,

$$
\|\partial_t[\mathcal{Q}(u) - \mathcal{Q}(v)]\|_{L^2(L^2)} \lesssim \|u\|_{L^2(L^\infty)}\|u_t - v_t\|_{L^\infty(L^2)} + \|v_t\|_{L^\infty(L^2)}\|u-v\|_{L^2(L^\infty)}. \tag{2.6}
$$

Note that by plugging in $v = 0$ above, these assumptions further imply that

$$
\begin{aligned}
\|\mathcal{Q}(u)\|_{L^2(L^2)} &\lesssim \|u\|_{L^\infty(L^\infty)}\|u\|_{L^2(L^2)},\\
\|\partial_t[\mathcal{Q}(u)]\|_{L^2(L^2)} &\lesssim \|u\|_{L^2(L^\infty)}\|u_t\|_{L^\infty(L^2)}.
\end{aligned}
$$

In [28, 30], the absorption term is modelled as

$$
\mathcal{Q}(p_t) = \frac{2b}{\rho_a c_a^4} p_t^2,
$$

which clearly satisfies our assumptions if $p_t \in L^\infty(0,T;\ L^\infty(\Omega))$ and $p_{tt} \in L^\infty(0,T;\ L^2(\Omega))$. More commonly, the absorption term appears in the literature averaged over a certain time interval. In, e.g., [7, section 2.2], the absorbed energy is given by

$$
\mathcal{Q}(p_t) = \frac{1}{j\tau}\frac{2b}{\rho_a c_a^4}\int_{t'}^{t'+j\tau} p_t^2\,\mathrm{d}t.
$$

Here j is a positive integer, τ is the period of ultrasound excitation and t' is a sufficient time from the start of the simulation so that a steady-state has been reached. In [12], the absorbed energy is averaged over the whole time interval

$$
\mathcal{Q}(p_t) = \frac{1}{T}\frac{2b}{\rho_a c_a^4}\int_0^T p_t^2\,\mathrm{d}t. \tag{2.7}
$$

Both of these functionals satisfy assumption 2. In case of (2.7), for example, we note that for all $t \in [0, T]$, and by using Minkowski's inequality (see [1, proposition 1.3]),

$$\begin{aligned}\left\|\left\|\frac{1}{T}\int_0^T (u_t^2 - v_t^2)\mathrm{d}t\right\|_{L^2(\Omega)}\right\|_{L^2(0,t)} &\leqslant \left\|\frac{1}{T}\int_0^T \|u_t^2 - v_t^2\|_{L^2(\Omega)}\mathrm{d}t\right\|_{L^2(0,t)} \\ &= \left\|\frac{1}{T}\int_0^T (\|u_t\|_{L^\infty} + \|v_t\|_{L^\infty})\|u_t - v_t\|_{L^2}\mathrm{d}t\right\|_{L^2(0,t)} \\ &\lesssim (\|u_t\|_{L^\infty(L^\infty)} + \|v_t\|_{L^\infty(L^\infty)})\|u_t - v_t\|_{L^2(L^2)}.\end{aligned}$$

In case of a time-averaged absorbed energy, we have $\|\partial_t[\mathcal{Q}(u_t) - \mathcal{Q}(v_t)]\|_{L^2(L^2)} = 0$.

Auxiliary results. We collect here several useful inequalities that are repeatedly used in the analysis below. We assume throughout that $\Omega \subset \mathbb{R}^d$, where $d \in \{1, 2, 3\}$, is a, bounded and sufficiently smooth domain. We will often rely on the Ladyzhenskaya inequality for $u \in H^1(\Omega)$:

$$\|u\|_{L^4} \leqslant C\|u\|_{L^2}^{1-d/4}\|u\|_{H^1}^{d/4}. \tag{2.8}$$

By using (2.8) together with Young's inequality, we further find that for $u \in H_0^1(\Omega)$ and any $\varepsilon > 0$

$$\begin{aligned}\|u\|_{L^4}^2 \lesssim \|u\|_{L^2}^{2(1-d/4)}\|u\|_{H^1}^{d/2} &\lesssim \|u\|_{L^2}^{2(1-d/4)}\|\nabla u\|_{L^2}^{d/2} \\ &\lesssim \frac{1}{\tilde{\varepsilon}^{\frac{4}{4-d}}}\|u\|_{L^2}^2 + \tilde{\varepsilon}^{4/d}\|\nabla u\|_{L^2}^2 = C(\varepsilon)\|u\|_{L^2}^2 + \varepsilon\|\nabla u\|_{L^2}^2\end{aligned} \tag{2.9}$$

with $\varepsilon = C\tilde{\varepsilon}^{4/d}$. This estimate can also be obtained (on bounded domains) by employing Ehrling's lemma; see [32, lemma 8.2].

Further, given $u \in H^{-1}(\Omega)$ and $v \in W^{1,3}(\Omega) \cap L^\infty(\Omega)$, the following bound holds:

$$\|uv\|_{H^{-1}} \lesssim \|u\|_{H^{-1}}(\|\nabla v\|_{L^3} + \|v\|_{L^\infty}). \tag{2.10}$$

To keep the presentation self-contained, we also state here the version of Gronwall's inequality that will be employed in the proofs.

Lemma 2.1. *Let $I = [0, t]$ and let $\alpha : I \to \mathbb{R}$ and $\beta : I \to \mathbb{R}$ be locally integrable functions. Let v be non-negative and integrable. Suppose that $u : I \to \mathbb{R}$ is in $C^1(I)$ and satisfies:*

$$u'(t) + v(t) \leqslant \alpha(t)u(t) + \beta(t), \quad \text{for } t \in I \quad \text{and} \quad u(0) = u_0.$$

Then it holds that

$$u(t) + \int_0^t v(s)\mathrm{d}s \leqslant u_0\,\mathrm{e}^{A(t)} + \int_0^t \beta(s)\mathrm{e}^{A(t)-A(s)}\,\mathrm{d}s,$$

where

$$A(t) = \int_0^t \alpha(s)\mathrm{d}s.$$

Proof. The inequality follows by combining the arguments of [5, appendix B] and [10, lemma 3.1]. □

3. Analysis of a linearised problem

We first analyse a decoupled linearisation of (2.1a), given by

$$\begin{cases} \alpha(x,t)p_{tt} - r(x,t)\Delta p - b\Delta p_t = f_1(x,t), & \text{in } \Omega\times(0,T), \\ \rho_{\mathrm{a}}C_{\mathrm{a}}\Theta_t - \kappa_{\mathrm{a}}\Delta\Theta + \rho_{\mathrm{b}}C_{\mathrm{b}}W(\Theta-\Theta_{\mathrm{a}}) = \mathcal{Q}(p_t) + f_2(x,t), & \text{in } \Omega\times(0,T), \end{cases} \tag{3.1}$$

and supplemented by the boundary (2.1b) and initial (2.1c) conditions. To facilitate the analysis, we make the following regularity and non-degeneracy assumptions on the involved coefficients and source terms.

Assumption 3. Given $T > 0$, the variable coefficients and the source terms satisfy the following assumptions.

(A) Let $\alpha \in L^\infty(0,T;\ L^\infty(\Omega)\cap W^{1,3}(\Omega))$ and $\alpha_t \in L^2(0,T;\ L^3(\Omega))\cap L^\infty(0,T;\ L^2(\Omega))$. Further, we assume that there exist α_0, $\alpha_1 > 0$, such that

$$\alpha_0 \leqslant \alpha(x,t) \leqslant \alpha_1 \quad \text{a.e. in } \Omega\times(0,T).$$

(R) We assume that $r \in L^\infty(0,T;\ L^\infty(\Omega)\cap W^{1,4}(\Omega))$ and $r_t \in L^\infty(0,T;\ L^2(\Omega))$. Further, there exist r_0, $r_1 > 0$, such that

$$r_0 \leqslant r(x,t) \leqslant r_1 \quad \text{a.e. in } \Omega\times(0,T).$$

(F) Let $f_1 \in L^2(0,T;H_0^1(\Omega))$, $\partial_t f_1 \in L^2(0,T;\ H^{-1}(\Omega))$, and $f_2 \in H^1(0,T;\ L^2(\Omega))$.

From the last assumption, by [34, theorem 7.22], we have $f_1 \in C([0,T];\ L^2(\Omega))$ and

$$\max_{0\leqslant t\leqslant T}\|f_1(t)\|_{L^2} \leqslant C_T(\|f_1\|_{L^2(H^1)} + \|\partial_t f_1\|_{L^2(H^{-1})}). \tag{3.2}$$

Energies. To accommodate the energy analysis, we introduce the following lower and higher-order acoustic energies:

$$\begin{aligned} E_0[p](t) &= \frac{1}{2}\Big\{\|\sqrt{\alpha(t)}p_t(t)\|_{L^2}^2 + \|\sqrt{r(t)}\nabla p(t)\|_{L^2}^2\Big\}, \\ E_1[p](t) &= \frac{1}{2}\Big\{\|\sqrt{\alpha(t)}p_{tt}(t)\|_{L^2}^2 + \|\sqrt{r(t)}\nabla p_t(t)\|_{L^2}^2 + \|\sqrt{r(t)}\Delta p(t)\|_{L^2}^2\Big\}, \\ E_2[p](t) &= \frac{1}{2}\|\sqrt{b}\nabla\Delta p(t)\|_{L^2}^2. \end{aligned} \tag{3.3}$$

In the analysis, we will also use the combined acoustic energy

$$\mathcal{E}[p](t) = E_0[p](t) + E_1[p](t) + E_2[p](t), \quad t\in[0,T]$$

with the associated dissipation rate

$$\begin{aligned} \mathcal{D}[p](t) = &\|\sqrt{b}\nabla p_{tt}(t)\|_{L^2}^2 + \|\sqrt{b}\Delta p_t(t)\|_{L^2}^2 \\ &+ \|\sqrt{r}(t)\nabla\Delta p(t)\|_{L^2}^2 + \|\sqrt{b}\nabla p_t(t)\|_{L^2}^2. \end{aligned}$$

The initial acoustic energy is set to

$$\begin{aligned}\mathcal{E}[p](0) = \frac{1}{2}\Big\{ &\|\sqrt{\alpha(0)}p_1\|^2_{L^2} + \|\sqrt{r(0)}\nabla p_0\|^2_{L^2} + \|\sqrt{r(0)}\nabla p_1\|^2_{L^2} \\ &+ \|\sqrt{\alpha(0)}p_{tt}(0)\|^2_{L^2} + \|\sqrt{b}\Delta\nabla p_0\|^2_{L^2} + \|\sqrt{r(0)}\Delta p_0\|^2_{L^2}\Big\}\end{aligned}$$

with

$$p_{tt}(0) = \alpha(0)^{-1}(r(0)\Delta p_0 + b\Delta p_1 + f_1(0)).$$

Further, the heat energy is given by

$$\mathcal{E}[\Theta](t) = \frac{1}{2}\big\{\|\Theta(t)\|^2_{H^2} + \|\Theta_t(t)\|^2_{L^2}\big\}$$

with the associated dissipation

$$\mathcal{D}[\Theta](t) = \|\Theta_t(t)\|^2_{H^1} + \|\Theta_{tt}(t)\|^2_{H^{-1}}.$$

Solution spaces. To formulate the existence result, we also introduce the following solutions spaces for the pressure:

$$\begin{aligned}X_p = \big\{p \in L^\infty(0,T;H^3_\diamondsuit(\Omega))\colon &p_t \in L^\infty(0,T;H^2_\diamondsuit(\Omega)) \cap L^2(0,T;H^3_\diamondsuit(\Omega)),\\ &p_{tt} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)),\\ &p_{ttt} \in L^2(0,T;H^{-1}(\Omega))\big\},\end{aligned}$$

and the temperature:

$$\begin{aligned}X_\Theta = \big\{\Theta \in C([0,T];H^2_\diamondsuit(\Omega))\colon &\Theta_t \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)),\\ &\Theta_{tt} \in L^2(0,T;H^{-1}(\Omega))\big\},\end{aligned}$$

with the short-hand notation

$$\begin{aligned}&H^2_\diamondsuit(\Omega) = H^1_0(\Omega) \cap H^2(\Omega),\\ &H^3_\diamondsuit(\Omega) = \big\{u \in H^3(\Omega) : \mathrm{tr}_{\partial\Omega}\, u = 0,\ \mathrm{tr}_{\partial\Omega}\,\Delta u = 0\big\}.\end{aligned}$$

We claim that the linearised problem is well-posed under the above-made assumptions.

Proposition 3.1. *Let $T > 0$ and let assumption 3 hold. Further, assume that*

$$(p_0, p_1) \in H^3_\diamondsuit(\Omega) \times H^2_\diamondsuit(\Omega), \quad \Theta_0 \in H^2_\diamondsuit(\Omega).$$

Then there exists a unique solution $(p,\Theta) \in X_p \times X_\Theta$ of (3.1). Furthermore, the acoustic pressure satisfies

$$\begin{aligned}&\mathcal{E}[p](t) + \|\Delta p_t(t)\|^2_{L^2} + \int_0^t \mathcal{D}[p](s)\mathrm{d}s + \int_0^t (\|p_{ttt}(s)\|^2_{H^{-1}} + \|\nabla\Delta p_t(s)\|^2_{L^2})\mathrm{d}s\\ &\quad\lesssim \mathcal{E}[p](0)\exp\left(\int_0^t (1+\Lambda(s))\mathrm{d}s\right) + \int_0^t \exp\left(\int_s^t (1+\Lambda(\sigma))\mathrm{d}\sigma\right)\mathbb{F}(s)\mathrm{d}s\end{aligned} \tag{3.4}$$

a.e. in time, with

$$\Lambda(t) = \| r_t(t) \|_{L^2} + \| r_t(t) \|_{L^2}^2 + \| \nabla r(t)\|_{L^4} + \| \nabla r(t)\|_{L^4}^2 + \| \alpha_t(t) \|_{L^2} + \| \alpha_t(t)\|_{L^3}^2 + \| \nabla\alpha(t) \|_{L^3}^2 \tag{3.5}$$

and

$$\mathbb{F}(t) = \|f_1(t)\|_{H^1}^2 + (1 + \|\nabla\alpha(t)\|_{L^3}^2)\|\partial_t f_1(t)\|_{H^{-1}}^2, \tag{3.6}$$

whereas the temperature satisfies

$$\begin{aligned}\mathcal{E}[\Theta](t) + \int_0^t \mathcal{D}[\Theta](s)\mathrm{d}s \leqslant C_T\Big(&\|\Theta_0\|_{H^2_\diamondsuit(\Omega)}^2 + \|f_2\|_{H^1(L^2)}^2 + \|p_t\|_{L^\infty(L^\infty)}^2 \\ &\times \|p_t\|_{L^2(L^2)}^2 + \|p_t\|_{L^2(L^\infty)}^2\|p_{tt}\|_{L^\infty(L^2)}^2 + 1\Big)\end{aligned}$$

for all $t \in [0, T]$.

Proof. Since the system is decoupled, we can analyse the equations in (3.1) sequentially.

Analysis of the pressure equation. The analysis of the pressure equation can be rigorously conducted by employing a Galerkin discretisation in space based on the smooth eigenfunctions of the Dirichlet–Laplacian; see, e.g., [9, chapter 7]. We focus here on presenting the energy analysis.

Energy analysis. Testing the (semi-discrete) pressure equation with p_t, integrating over Ω, and using integration by parts yields the following identity:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\alpha(t)}p_t(t)\|_{L^2}^2 + \|\sqrt{b}\nabla p_t(t)\|_{L^2}^2 = \frac{1}{2}(\alpha_t p_t, p_t)_{L^2} + (r\Delta p, p_t)_{L^2} + (f_1, p_t)_{L^2}$$

a.e. in time. From here, by Hölder's and Young's inequalities, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\alpha(t)}p_t(t)\|_{L^2}^2 + \|\sqrt{b}\nabla p_t(t)\|_{L^2}^2 \lesssim{}& \left\|\frac{\alpha_t(t)}{b}\right\|_{L^2}\|\sqrt{b}p_t(t)\|_{L^4}^2 + \left\|\sqrt{\frac{r(t)}{\alpha(t)}}\right\|_{L^\infty}\Big(\|\sqrt{r(t)}\Delta p(t)\|_{L^2}^2 \\ &+ \|\sqrt{\alpha(t)}p_t(t)\|_{L^2}^2\Big) + \frac{1}{\sqrt{b}}\|f_1(t)\|_{L^2}\|\sqrt{b}p_t(t)\|_{L^2}.\end{aligned}$$

On account of assumption 3, we know that

$$\left\|\sqrt{r(t)/\alpha(t)}\right\|_{L^\infty} \leqslant \sqrt{r_1/\alpha_0} \quad \text{a.e. in time,}$$

and thus for any $\varepsilon > 0$, it holds that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\alpha(t)}p_t(t)\|_{L^2}^2 + \|\sqrt{b}\nabla p_t(t)\|_{L^2}^2 \lesssim{}& \left\|\frac{\alpha_t(t)}{b}\right\|_{L^2}\|\sqrt{b}p_t(t)\|_{L^4}^2 + E_0[p](t) + E_1[p](t) \\ &+ \frac{1}{4\varepsilon}\|f_1(t)\|_{L^2}^2 + \varepsilon\|\sqrt{b}\nabla p_t(t)\|_{L^2}^2,\end{aligned} \tag{3.7}$$

where we have applied Poincare's inequality together with Young's ε-inequality in the estimate of the last term. Note that by fixing $\varepsilon > 0$ small enough, we can absorb the last term in (3.7) by the dissipative term on the left.

By using the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ together with the Poincaré inequality, the first term on the right-hand side of (3.7) can be absorbed by the dissipative term $\|\sqrt{b}\nabla p_t(t)\|^2_{L^2}$ as well if we assume the norm $\|\alpha_t/b\|_{L^\infty(L^2)}$ to be small. However, to avoid this smallness assumption, we use inequality (2.9) instead and split this term into two parts: an energy term and a dissipation term with an arbitrary small factor $\varepsilon > 0$. This idea will be used repeatedly in the proof below. Indeed, by using inequality (2.9), we have

$$\|\sqrt{b}p_t(t)\|^2_{L^4} \lesssim C(\varepsilon)\left\|\frac{b}{\alpha(t)}\right\|_{L^\infty}\|\sqrt{\alpha}p_t(t)\|^2_{L^2} + \varepsilon\|\sqrt{b}\nabla p_t(t)\|^2_{L^2}.$$

Consequently, by recalling assumption 3 and fixing $\varepsilon > 0$ small enough, so that

$$1 - C\varepsilon \sup_{t\in(0,T)} \|\alpha_t(t)/b\|_{L^2} > 0,$$

where C is the hidden constant in (3.7), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_0[p](t) + \|\sqrt{b}\nabla p_t(t)\|^2_{L^2} \lesssim\ & E_0[p](t) + E_1[p](t) + \left\|\frac{\alpha_t(t)}{b}\right\|_{L^2} \\ & \times \|\sqrt{\alpha(t)}p_t(t)\|^2_{L^2} + \|f_1(t)\|^2_{L^2}, \end{aligned} \tag{3.8}$$

where we have also used again the uniform bound on α given in assumption 3.

Estimate (3.8) indicates that further testing is needed to absorb the energy E_1 on the right. Thus, we test the first (semi-discrete) equation in (3.1) with $-\Delta p_t$ and integrate in space, which yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{r(t)}\Delta p(t)\|^2_{L^2} + \|\sqrt{b}\Delta p_t(t)\|^2_{L^2} &= (\alpha(t)p_{tt}, \Delta p_t)_{L^2} + \frac{1}{2}(r_t(t)\Delta p, \Delta p) - (f_1(t), \Delta p_t)_{L^2} \\ &\lesssim \frac{1}{4\varepsilon}\|\sqrt{\alpha(t)}p_{tt}(t)\|^2_{L^2} + \varepsilon\left\|\sqrt{\frac{\alpha(t)}{b}}\right\|_{L^\infty}\|\sqrt{b}\Delta p_t(t)\|^2_{L^2} \\ &\quad + \left\|\frac{r_t(t)}{b}\right\|_{L^2}\|\sqrt{b}\Delta p(t)\|^2_{L^4} + \frac{1}{b}\|f_1(t)\|^2_{L^2} + \varepsilon\|\sqrt{b}\Delta p_t(t)\|^2_{L^2}.\end{aligned}$$

Clearly, by selecting $\varepsilon > 0$ small enough in the above estimate, the second term on the right-hand side will be absorbed by the dissipation on the left. Hence, by choosing $\varepsilon > 0$ as small as needed, keeping in mind that $\Delta p = 0$ on $\partial\Omega$, and using Poincaré's inequality, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{r}\Delta p(t)\|^2_{L^2} + \|\sqrt{b}\Delta p_t(t)\|^2_{L^2} \lesssim E_1[p](t) + \left\|\frac{r_t(t)}{b}\right\|_{L^2} \times \|\sqrt{b}\nabla\Delta p(t)\|^2_{L^2} + \|f_1(t)\|^2_{L^2}. \tag{3.9}$$

To retrieve the energy E_1 on the left, we will next work with the time-differentiated pressure equation. Indeed, on account of the regularity assumptions on the coefficients and source term, we can differentiate the semi-discrete pressure equation with respect to t:

$$\alpha(x,t)p_{ttt} - r(x,t)\Delta p_t - b\Delta p_{tt} = \partial_t f_1(x,t) - \alpha_t(x,t)p_{tt} + r_t(x,t)\Delta p. \tag{3.10}$$

Multiplying (3.10) by p_{tt}, integrating over Ω, and using integration by parts with respect to time in the first term, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|\sqrt{\alpha(t)}p_{tt}(t)\|_{L^2}^2+\|\sqrt{r}(t)\nabla p_t(t)\|_{L^2}^2\Big\}+\|\sqrt{b}\nabla p_{tt}(t)\|_{L^2}^2$$
$$=\frac{1}{2}(\alpha_t p_{tt},p_{tt})_{L^2}-(\nabla r p_t,\nabla p_{tt})_{L^2}+\frac{1}{2}(r_t\nabla p_t,\nabla p_t)_{L^2}$$
$$+\langle\partial_t f_1,p_{tt}\rangle_{H^{-1},H^1}-(\alpha_t p_{tt},p_{tt})_{L^2}+(r_t\Delta p,p_{tt})_{L^2}. \tag{3.11}$$

The first two r terms on the right can be estimated as follows:

$$-(\nabla r p_t,\nabla p_{tt})_{L^2}+\frac{1}{2}(r_t\nabla p_t,\nabla p_t)_{L^2}\leqslant\varepsilon\parallel\sqrt{b}\nabla p_{tt}(t)\|_{L^2}^2+C(\varepsilon)\|\ \frac{1}{\sqrt{r}}\ \|_{L^\infty}^2$$
$$\times\parallel\nabla r\|_{L^4}^2\parallel\sqrt{r}\nabla p_t\|_{L^2}^2+\frac{1}{2}(r_t\nabla p_t,\nabla p_t)_{L^2}$$

for some $\varepsilon>0$, where we have relied on the embedding $H^1(\Omega)\hookrightarrow L^4(\Omega)$. By applying estimate (2.9), we can further bound the last term:

$$\frac{1}{2}(r_t\nabla p_t,\nabla p_t)_{L^2}\lesssim\|r_t\|_{L^2}\|\nabla p_t\|_{L^4}^2$$
$$\lesssim C(\varepsilon)\|r_t\|_{L^2}^2\|r^{-1}\|_{L^\infty}\|\sqrt{r}\nabla p_t\|_{L^2}^2+\varepsilon\|\Delta p_t\|_{L^2}^2,$$

where we have also utilised elliptic regularity (since $\partial\Omega$ is smooth):

$$\|\nabla p_t\|_{H^1}\leqslant\|p_t\|_{H^2}\leqslant C\|\Delta p_t\|_{L^2}.$$

The first and the fifth term on the right-hand side of (3.11) can be estimated as follows:

$$\frac{1}{2}(\alpha_t p_{tt},p_{tt})_{L^2}-(\alpha_t p_{tt},p_{tt})_{L^2}=-\frac{1}{2}(\alpha_t p_{tt},p_{tt})_{L^2}\lesssim\left\|\frac{\alpha_t(t)}{b}\right\|_{L^2}\|\sqrt{b}p_{tt}(t)\|_{L^4}^2. \tag{3.12}$$

We then further estimate the last term above using again inequality (2.9):

$$\|\sqrt{b}p_{tt}(t)\|_{L^4}^2\lesssim C(\varepsilon)\left\|\frac{b}{\alpha(t)}\right\|_{L^\infty}\|\sqrt{\alpha(t)}p_{tt}(t)\|_{L^2}^2+\varepsilon\|\sqrt{b}\nabla p_{tt}(t)\|_{L^2}^2. \tag{3.13}$$

Keeping in mind assumption 3, and plugging (3.13) into (3.12), we have

$$-\frac{1}{2}(\alpha_t p_{tt},p_{tt})_{L^2}\lesssim\left\|\frac{\alpha_t(t)}{b}\right\|_{L^2}\Big(\varepsilon\|\sqrt{b}\nabla p_{tt}(t)\|_{L^2}^2+C(\varepsilon)\|\sqrt{\alpha}p_{tt}(t)\|_{L^2}^2\Big).$$

By using Young's inequality together with the Poincaré's inequality, we find that

$$\langle\partial_t f_1(t),p_{tt}(t)\rangle_{H^{-1},H^1}\lesssim\frac{1}{\sqrt{b}}\|\partial_t f_1(t)\|_{H^{-1}}\|\sqrt{b}p_{tt}(t)\|_{H^1}$$
$$\lesssim 4\varepsilon\frac{1}{b}\|\partial_t f_1(t)\|_{H^{-1}}^2+\varepsilon\|\sqrt{b}\nabla p_{tt}(t)\|_{L^2}^2.$$

Recalling that $\Delta p=0$ on $\partial\Omega$, we can estimate the last term on the right-hand side of (3.11) as follows:

$$(r_t\Delta p, p_{tt})_{L^2} \lesssim \varepsilon\|\sqrt{b}p_{tt}\|_{L^4}^2 + C(\varepsilon)\left\|\frac{r_t}{b}\right\|_{L^2}^2\|\sqrt{b}\Delta p\|_{L^4}^2$$
$$\lesssim \varepsilon\|\sqrt{b}\nabla p_{tt}\|_{L^2}^2 + C(\varepsilon)\left\|\frac{r_t}{b}\right\|_{L^2}^2\|\sqrt{b}\nabla\Delta p\|_{L^2}^2.$$

We see that the first term on the right can be absorbed by the dissipation in (3.11) and the last one is an energy term. By collecting the above estimates with $\varepsilon > 0$ small enough, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\| \sqrt{\alpha}p_{tt}(t)\|_{L^2}^2+ \| \sqrt{r}(t)\nabla p_t(t)\|_{L^2}^2)+ \| \sqrt{b}\nabla p_{tt}(t)\|_{L^2}^2\\
&\quad\lesssim \| \frac{r_t}{b} \|_{L^2}^2 \| \sqrt{b}\nabla\Delta p\|_{L^2}^2 + (\| \nabla r\|_{L^4}^2+ \| r_t\|_{L^2}^2) \| \sqrt{r}\nabla p_t\|_{L^2}^2\\
&\qquad + \| \partial_t f_1(t)\|_{H^{-1}}^2 + \varepsilon \| \Delta p_t\|_{L^2}^2.
\end{aligned} \tag{3.14}$$

Adding (3.14) to (3.9), exploiting assumption 3, using Poincaré's inequality, and possibly reducing ε, so that the ε terms can be absorbed by the left side, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\frac{1}{2}\left[\|\sqrt{r(t)}\Delta p(t)\|_{L^2}^2 + \|\sqrt{\alpha(t)}p_{tt}(t)\|_{L^2}^2 + \|\sqrt{r}(t)\nabla p_t(t)\|_{L^2}^2\right]}_{:=E_1[p](t)}\\
&\qquad + \|\sqrt{b}\nabla p_{tt}(t)\|_{L^2}^2 + \|\sqrt{b}\Delta p_t(t)\|_{L^2}^2\\
&\quad\lesssim \left(1 + \|\nabla r\|_{L^4} + \|r_t\|_{L^2} + \|r_t\|_{L^2}^2\right)E_1[p](t) + \left\|\frac{r_t(t)}{b}\right\|_{L^2}^2\\
&\qquad \times \|\sqrt{b}\nabla\Delta p(t)\|_{L^2}^2 + \|f_1(t)\|_{L^2}^2 + \|\partial_t f_1(t)\|_{H^{-1}}^2.
\end{aligned} \tag{3.15}$$

To be able to absorb the term $\|\sqrt{b}\nabla\Delta p(t)\|_{L^2}^2$ on the right, we should additionally test the pressure equation with $\Delta^2 p$:

$$(\alpha(t)p_{tt} - r(t)\Delta p - b\Delta p_t, \Delta^2 p)_{L^2} = (f_1(t), \Delta^2 p)_{L^2}.$$

Integrating by parts and using the fact that $p_{tt} = \Delta p = \Delta p_t = 0$ on the boundary for smooth Galerkin approximations, as well as that $f_1(t) \in H_0^1(\Omega)$, yields

$$(r\nabla\Delta p + b\nabla\Delta p_t, \nabla\Delta p)_{L^2} = -(\alpha\nabla p_{tt} + p_{tt}\nabla\alpha + \nabla r\Delta p, \nabla\Delta p)_{L^2} + (\nabla f_1, \nabla\Delta p)_{L^2}.$$

Recalling how the energy E_2 is defined in (3.3), from here we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2[p](t) + \|\sqrt{r}(t)\nabla\Delta p(t)\|_{L^2}^2 = -(\alpha\nabla p_{tt} + p_{tt}\nabla\alpha + \nabla r\Delta p, \nabla\Delta p)_{L^2} + (\nabla f_1(x,t), \nabla\Delta p)_{L^2}.$$

By Hölder's inequality, we further have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_2[p](t) + \|\sqrt{r}(t)\nabla\Delta p(t)\|_{L^2}^2 &\lesssim \|\alpha(t)\|_{L^\infty}\|\nabla p_{tt}(t)\|_{L^2}\|\nabla\Delta p(t)\|_{L^2} + \|p_{tt}(t)\|_{L^6}\|\nabla\alpha(t)\|_{L^3}\\
&\quad \times \|\nabla\Delta p(t)\|_{L^2} + \|\nabla r(t)\|_{L^4}\|\Delta p(t)\|_{L^4}\|\nabla\Delta p(t)\|_{L^2} + \frac{1}{4\varepsilon}\|\nabla f_1(t)\|_{L^2}^2\\
&\quad + \varepsilon\|r(t)^{-1}\|_{L^\infty}\|\sqrt{r(t)}\nabla\Delta p(t)\|_{L^2}.
\end{aligned}$$

Using Young's and Poincaré's inequalities, and the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2[p](t) + \|\sqrt{r}(t)\nabla\Delta p(t)\|_{L^2}^2 \lesssim \frac{\varepsilon}{b}\|\alpha(t)\|_{L^\infty}^2\|\sqrt{b}\nabla p_{tt}(t)\|_{L^2}^2 + \frac{1}{4\varepsilon b}\|\sqrt{b}\nabla\Delta p(t)\|_{L^2}^2 + \varepsilon\|\nabla p_{tt}(t)\|_{L^2}^2$$
$$\times\|\nabla\alpha(t)\|_{L^3}^2 + \|\sqrt{b}\nabla\Delta p(t)\|_{L^2}^2 + \|\nabla r(t)\|_{L^4}\|\sqrt{b}\nabla\Delta p(t)\|_{L^2}^2 + \|\nabla f_1(t)\|_{L^2}^2. \tag{3.16}$$

By adding inequalities (3.15) and (3.16), and selecting $\varepsilon > 0$ small enough, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\{E_1[p](t) + E_2[p](t)\} + \|\sqrt{b}\nabla p_{tt}(t)\|_{L^2}^2 + \|\sqrt{b}\Delta p_t(t)\|_{L^2}^2 + \|\sqrt{r(t)}\nabla\Delta p(t)\|_{L^2}^2$$
$$\lesssim (1 + \|\nabla r(t)\|_{L^4} + \|\nabla r(t)\|_{L^4}^2 + \|r_t\|_{L^2} + \|r_t\|_{L^2}^2)\{E_1[p](t) + E_2[p](t)\}$$
$$+ \|\partial_t f_1(t)\|_{H^{-1}}^2 + \|f_1(t)\|_{H^1}^2.$$

By collecting the above estimates, we arrive at a bound that involves the combined acoustic energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[p](t) + \mathcal{D}[p](t) \lesssim (1 + \Lambda(t))\mathcal{E}[p](t) + \mathbb{F}(t), \tag{3.17}$$

where $\Lambda(t)$ and $\mathbb{F}(t)$ are defined in (3.5) and (3.6), respectively. By Gronwall's inequality, we then immediately have

$$\mathcal{E}[p](t) + \int_0^t \mathcal{D}[p](s)\mathrm{d}s \lesssim \mathcal{E}[p](0)\exp\left(\int_0^t (1 + \Lambda(s))\mathrm{d}s\right)$$
$$+ \int_0^t \exp\left(\int_s^t (1 + \Lambda(\sigma))\mathrm{d}\sigma\right)\mathbb{F}(s)\mathrm{d}s. \tag{3.18}$$

Additional bootstrap arguments. We can obtain more information on the pressure field by relying on the (semi-discrete) PDE. Indeed, by the acoustic PDE we have

$$\|\Delta p_t(t)\|_{L^2}^2 \lesssim \alpha_1^2\|p_{tt}(t)\|_{L^2}^2 + r_1^2\|\Delta p(t)\|_{L^2}^2 + \|f_1(t)\|_{L^2}^2. \tag{3.19}$$

We can then further estimate the right-hand side of (3.19) by employing the acoustic energy:

$$\|\Delta p_t(t)\|_{L^2}^2 \lesssim \mathcal{E}[p](t) + \|f_1(t)\|_{L^2}^2$$
$$\lesssim \mathcal{E}[p](0)\exp\left(\int_0^t (1 + \Lambda(s))\mathrm{d}s\right) + \int_0^t \exp\left(\int_s^t (1 + \Lambda(\sigma))\mathrm{d}\sigma\right)\mathbb{F}(s)\mathrm{d}s,$$

where we have also used estimate (3.2) to bound the $\|f_1(t)\|_{L^2}^2$ term. Adding this bound to (3.18) yields

$$\mathcal{E}[p](t) + \|\Delta p_t(t)\|_{L^2}^2 + \int_0^t \mathcal{D}[p](s)\mathrm{d}s$$
$$\lesssim \mathcal{E}[p](0)\exp\left(\int_0^t (1 + \Lambda(s))\mathrm{d}s\right) + \int_0^t \exp\left(\int_s^t (1 + \Lambda(\sigma))\mathrm{d}\sigma\right)\mathbb{F}(s)\mathrm{d}s.$$

Similarly,

$$\|\sqrt{b}\nabla\Delta p_t(t)\|_{L^2}^2 \lesssim \alpha_1^2\|\nabla p_{tt}(t)\|_{L^2}^2 + \|\nabla\alpha(t)\|_{L^3}\|p_{tt}(t)\|_{L^6} + r_1$$
$$\times\|\sqrt{r}\nabla\Delta p(t)\|_{L^2}^2 + \|\nabla r(t)\|_{L^4}^2\|\Delta p(t)\|_{L^4}^2 + \|\nabla f_1(t)\|_{L^2}^2. \tag{3.20}$$

Adding γ (3.20) to (3.17) with small enough $\gamma > 0$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[p](t) + \mathcal{D}[p](t) + \|\sqrt{b}\nabla\Delta p_t(t)\|_{L^2}^2 \lesssim (1+\Lambda(t))\mathcal{E}[p](t) + \mathbb{F}(t),$$

on which we can apply Gronwall's inequality.

Additionally, from the time-differentiated equation (3.10), standard arguments (see, e.g., [9, chapter 7, p 383]) give the following bound in the dual space $H^{-1}(\Omega)$:

$$\begin{aligned}\|\partial_t(\alpha(t)p_{tt})(t)\|_{H^{-1}} &\leqslant \|r(t)\Delta p_t(t)\|_{H^{-1}} + \|r_t(t)\Delta p(t)\|_{H^{-1}} + \|b\Delta p_{tt}(t)\|_{H^{-1}} + \|\partial_t f_1(t)\|_{H^{-1}} \\ &\lesssim \|r(t)\|_{L^\infty}\|\Delta p_t(t)\|_{L^2} + \|r_t(t)\|_{L^2}\|\nabla\Delta p(t)\|_{L^2} + \|\nabla p_{tt}(t)\|_{L^2} + \|\partial_t f_1(t)\|_{H^{-1}},\end{aligned}$$

where we have used the embedding $L^{6/5}(\Omega) \hookrightarrow H^{-1}(\Omega)$ together with Hölder's inequality to get

$$\|r_t\Delta p\|_{H^{-1}} \lesssim \|r_t\Delta p\|_{L^{6/5}} \lesssim \|r_t\|_{L^2}\|\Delta p\|_{L^3} \lesssim \|r_t\|_{L^2}\|\nabla\Delta p\|_{L^2}.$$

Thus, we have

$$p_{tt} \in L^2(0,T;H_0^1(\Omega)), \qquad \partial_t(\alpha(\cdot)p_{tt}) \in L^2(0,T;H^{-1}(\Omega))$$

with a uniform bound

$$\begin{aligned}\|p_{ttt}\|_{H^{-1}} &\lesssim \|\alpha p_{ttt}\|_{H^{-1}}\left(\left\|\alpha^{-1}\right\|_{L^\infty} + \left\|\nabla(\alpha^{-1})\right\|_{L^3}\right) \\ &\lesssim \left(\|\partial_t(\alpha p_{tt})\|_{H^{-1}} + \|\alpha_t p_{tt}\|_{H^{-1}}\right)\left(\alpha_1^{-1} + \alpha_1^{-2}\|\nabla\alpha\|_{L^3}\right).\end{aligned}$$

By using again the embedding $L^{6/5}(\Omega) \hookrightarrow H^{-1}(\Omega)$ and Hölder's inequality, we have, similarly to before,

$$\|\alpha_t p_{tt}\|_{H^{-1}} \lesssim \|\alpha_t p_{tt}\|_{L^{6/5}} \lesssim \|\alpha_t\|_{L^3}\|p_{tt}\|_{L^2},$$

and thus

$$\begin{aligned}\|p_{ttt}\|_{H^{-1}}^2 &\lesssim (1+\|\nabla\alpha\|_{L^3}^2)\big(\|r\|_{L^\infty}^2\|\Delta p_t\|_{L^2}^2 + \|r_t\|_{L^2}^2\|r^{-1}\|_{L^\infty}\|\sqrt{r}\nabla\Delta p\|_{L^2}^2 \\ &\quad + \|\nabla p_{tt}\|_{L^2}^2 + \|\partial_t f_1\|_{H^{-1}}^2 + \|\alpha_t\|_{L^3}^2\|p_{tt}\|_{L^2}^2\big).\end{aligned} \tag{3.21}$$

Then adding γ (3.21) to (3.17) with $\gamma > 0$ small enough, and using Gronwall's inequality yields

$$\mathcal{E}[p](t) + \|p_{ttt}\|_{L^2(H^{-1})}^2 + \int_0^t \mathcal{D}[p](s)\mathrm{d}s \lesssim \mathcal{E}[p](0)\exp\left(\int_0^t(1+\Lambda(s))\mathrm{d}s\right) + \int_0^t \exp\left(\int_s^t(1+\Lambda(\sigma))\mathrm{d}\sigma\right)\mathbb{F}(s)\mathrm{d}s.$$

Combining the three derived estimates yields (3.4), at first in a semi-discrete setting. The obtained uniform bound allows us to employ standard compactness arguments and prove existence of a solution $p \in X_p$ to the pressure equation; see, e.g., [9, chapter 7] for similar arguments. By the weak/weak-$\star$ lower semi-continuity of norms, p satisfies the same energy bound (3.4). Note that $p \in X_p$ implies

$$p \in C([0,T];H_\diamondsuit^3(\Omega)), \qquad p_t \in C_w([0,T];H_\diamondsuit^2(\Omega));$$

cf [36, lemma 3.3].

Uniqueness. Uniqueness in the pressure equation follows by showing that the only solution of the homogeneous problem is zero. To this end, let $p \in X_p$ solve

$$\alpha(x,t)p_{tt} - r(x,t)\Delta p - b\Delta p_t = 0, \qquad p(x,0) = p_t(x,0) = 0, \qquad p|_{\partial\Omega} = 0.$$

We can repeat our previous energy analysis up to (3.17), where instead of testing with $\Delta^2 p$ (which is not a valid test function), we take the gradient of the equation and test with $\nabla\Delta p \in L^\infty(L^2(\Omega))$. In this manner, from (3.4) we obtain $\mathcal{E}[p](t) = 0$, which immediately yields $p = 0$.

Analysis of the heat equation. We next rewrite the heat equation as

$$\Theta_t - \frac{\kappa_\mathrm{a}}{\rho_\mathrm{a} C_\mathrm{a}}\Delta\Theta + \frac{\rho_\mathrm{b} C_\mathrm{b} W}{\rho_\mathrm{a} C_\mathrm{a}}\Theta = \tilde{f}$$

with

$$\tilde{f} = \frac{1}{\rho_\mathrm{a} C_\mathrm{a}}\mathcal{Q}(p_t) + \frac{1}{\rho_\mathrm{a} C_\mathrm{a}} f_2(x,t) + \frac{\rho_\mathrm{b} C_\mathrm{b} W \Theta_\mathrm{a}}{\rho_\mathrm{a} C_\mathrm{a}}.$$

According to, e.g., [40, chapter 1, theorem 1.3.2], the unique solution $\Theta \in X_\Theta$ of this problem satisfies

$$\begin{aligned} &\|\Theta(t)\|^2_{H^2_\diamondsuit(\Omega)} + \|\Theta_t(t)\|^2_{L^2} + \int_0^t (\|\Theta_{tt}\|^2_{H^{-1}} + \|\Theta_t\|^2_{H^1})\mathrm{d}s \\ &\quad \leqslant C_T\left(\|\Theta_0\|^2_{H^2_\diamondsuit(\Omega)} + \|\tilde{f}(0)\|^2_{L^2} + \int_0^t \|\tilde{f}_t\|^2_{L^2}\mathrm{d}s\right) \end{aligned} \tag{3.22}$$

for all $t \in [0,T]$; see also [36, chapter 2, theorem 3.2]. Thanks to the assumed properties of the mapping $\mathcal{Q}$, we have

$$\|\tilde{f}\|_{L^2(L^2)} \lesssim \|f_2\|_{L^2(L^2)} + \|p_t\|_{L^\infty(L^\infty)}\|p_t\|_{L^2(L^2)} + C(T,\Omega,\Theta_\mathrm{a}).$$

Further,

$$\|\tilde{f}_t\|_{L^2(L^2)} \lesssim \|\partial_t f_2\|_{L^2(L^2)} + \|p_t\|_{L^2(L^\infty)}\|p_{tt}\|_{L^\infty(L^2)}.$$

Thus, by the embedding $H^1(0,T) \hookrightarrow C[0,T]$, from (3.22) we have

$$\begin{aligned} &\|\Theta(t)\|^2_{H^2_\diamondsuit(\Omega)} + \|\Theta_t(t)\|^2_{L^2} + \int_0^t (\|\Theta_{tt}\|^2_{H^{-1}} + \|\Theta_t\|^2_{H^1})\mathrm{d}s \\ &\quad \leqslant C_T\Big(\|\Theta_0\|^2_{H^2} + \|f_2\|^2_{H^1(L^2)} + \|p_t\|^2_{L^\infty(L^\infty)}\|p_t\|^2_{L^2(L^2)} + \|p_t\|^2_{L^2(L^\infty)}\|p_{tt}\|^2_{L^\infty(L^2)} + 1\Big), \end{aligned}$$

as claimed. This finishes the proof of proposition 3.1. □

4. Local well-posedness of the nonlinear problem

To prove local well-posedness of the coupled Westervelt–Pennes model, we intend to rely on Banach's fixed point theorem. To this end, let us introduce the fixed-point mapping $\mathcal{T} : (p_*, \Theta_*) \mapsto (p, \Theta)$, which associates

$$(p_*, \Theta_*) \in B \subset X_T := X_p \times X_\Theta,$$

where B will be a suitably chosen ball in X_T, with the solution $(p, \Theta) \in X_p \times X_\Theta$ of

$$\begin{cases} (1-2k(\Theta_*)p_*)p_{tt} - q(\Theta_*)\Delta p - b\Delta p_t = 2k(\Theta_*)p_{*t}^2, & \text{in } \Omega \times (0,T), \\ \rho_a C_a \Theta_t - \kappa_a \Delta\Theta + \rho_b C_b W(\Theta - \Theta_a) = \mathcal{Q}(p_t), & \text{in } \Omega \times (0,T), \end{cases} \tag{4.1}$$

with the boundary (2.1b) and initial (2.1c) conditions. Our main results reads as follows.

Theorem 4.1. *Let $T > 0$ and*

$$(p_0, p_1) \in H^3_\diamondsuit(\Omega) \times H^2_\diamondsuit(\Omega), \quad \Theta_0 \in H^2_\diamondsuit(\Omega).$$

There exists $\delta = \delta(T) > 0$, such that if

$$\mathcal{E}[p](0) \leqslant \delta, \tag{4.2}$$

then there exist a unique solution (p, Θ) of (2.1) in X_T. Furthermore, the solution depends continuously on the data with respect to $\|\cdot\|_{X_T}$.

Proof. As already announced, we intend to rely on Banach's fixed-point theorem to arrive at the claim. To facilitate the fixed-point argument, we define the pressure and temperature norms:

$$\begin{aligned} \|p\|_{X_p} = &\|p\|_{L^\infty(H^3)} + \|p_t\|_{L^\infty(H^2)} + \|\nabla\Delta p_t\|_{L^2(L^2)} + \|p_{tt}\|_{L^\infty(L^2)} \\ &+ \|p_{tt}\|_{L^2(H^1(\Omega))} + \|p_{ttt}\|_{L^2(H^{-1}(\Omega))} \end{aligned}$$

and

$$\|\Theta\|_{X_\Theta} = \|\Theta\|_{L^\infty(H^2)} + \|\Theta_t\|_{L^\infty(L^2)} + \|\Theta_t\|_{L^2(H^1)} + \|\Theta_{tt}\|_{L^2(H^{-1})}.$$

We can then also define the combined norm as follows:

$$\|(p,\Theta)\|_{X_T} = \|p\|_{X_p} + \|\Theta\|_{X_\Theta}.$$

To have an equivalence between this norm and the energies, we introduce the total pressure energy $\mathbb{E}[p]$ as

$$\mathbb{E}[p](T) = \sup_{t\in(0,T)} \mathcal{E}[p](t) + \sup_{t\in(0,T)} \|\Delta p_t(t)\|^2_{L^2}$$

and the associated dissipation rate as

$$\mathbb{D}(t) = \mathcal{D}[p](t) + \int_0^t (\|p_{ttt}(s)\|^2_{H^{-1}} + \|\nabla\Delta p_t(s)\|^2_{L^2})\mathrm{d}s.$$

Then on account of assumption 3, there exist positive constants $C_1, \dots, C_4$, such that

$$C_1(\mathbb{E}[p](T) + \mathbb{D}[p](T)) \leqslant \|p\|^2_{X_p} \leqslant C_2(\mathbb{E}[p](T) + \mathbb{D}[p](T)) \tag{4.3}$$

and

$$C_3\left(\sup_{t\in(0,T)} \mathcal{E}[\Theta](t) + \mathcal{D}[\Theta](T)\right) \leqslant \|\Theta\|^2_{X_\Theta} \leqslant C_4\left(\sup_{t\in(0,T)} \mathcal{E}[\Theta](t) + \mathcal{D}[\Theta](T)\right). \tag{4.4}$$

We next introduce a ball in X_T:

$$B = \{(p_*, \Theta_*) \in X_T : \|p_*\|_{L^\infty(L^\infty)} \leqslant \gamma < \frac{1}{2k_1}, \quad \|p_*\|_{X_p} \leqslant R_1,$$
$$\|\Theta_*\|_{X_\Theta} \leqslant R_2, \quad (p_*, p_{*t}, \Theta_*)_{|t=0} = (p_0, p_1, \Theta_0)\big\},$$

where the radii $R_1 > 0$ and $R_2 > 0$ are to be determined by the proof. The constant $k_1 > 0$ is such that

$$|k(\Theta)| \leqslant k_1;$$

cf assumption (2.3). In the course of the proof we will impose a smallness condition on the pressure, but not on the temperature data, which is why we have introduced two different radii here.

Note that the solution of the linear problem with $\alpha = r = 1$ and $f_1 = f_2 = 0$, belongs to this ball if $\delta > 0$ is small enough and R_2 large enough, so that

$$R_1^2 \geqslant C_T\delta \geqslant C_T\mathcal{E}[p](0), \qquad R_2^2 \geqslant \tilde{C}_T(\|\Theta_0\|^2_{H^2_\diamondsuit(\Omega)} + \delta^2 + 1),$$

so this set is non-empty. We consider the ball to be equipped with the distance

$$d[(p_1, p_2), (\Theta_1, \Theta_2)] = \|p_1 - p_2\|_{X_p} + \|\Theta_1 - \Theta_2\|_{X_\Theta}.$$

Then (B, d) is a complete metric space. We first prove that $\mathcal{T}$ is a self-mapping.

Lemma 4.1. *For sufficiently small R_1 and δ, it holds that $\mathcal{T}(B) \subset B$.*

Proof. We wish to rely on the well-posedness result from the previous section. To this end, we set

$$\alpha(x,t) = 1 - 2k(\Theta_*)p_*, \qquad r(x,t) = q(\Theta_*), \qquad f_1(x,t) = 2k(\Theta_*)p_{*t}^2, \qquad f_2(x,t) = 0.$$

to fit problem (4.1) into the framework of proposition 3.1. We next verify assumption 3 on these functions. Since

$$\|2k(\Theta_*)p_*\|_{L^\infty(L^\infty)} \leqslant 2k_1\|p_*\|_{L^\infty(L^\infty)} \leqslant 2k_1\gamma$$

we have

$$0 < \alpha_0 = 1 - 2k_1\gamma \leqslant \alpha(x,t) = 1 - 2k(\Theta_*)p_* \leqslant 1 + 2k_1\gamma = \alpha_1$$

and so the non-degeneracy condition is fulfilled. Further, by the embeddings $H^1(\Omega) \hookrightarrow L^3(\Omega)$ and $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we have

$$\begin{aligned}\|\alpha\|_{L^\infty(W^{1,3})} \lesssim\ & \|1 - 2k(\Theta_*)p_*\|_{L^\infty(L^3)} + \|\nabla(k(\Theta_*))p_*\|_{L^\infty(L^3)} \\ & + \|k(\Theta_*)\nabla p_*\|_{L^\infty(L^3)} \\ \lesssim\ & 1 + k_1\|p_*\|_{L^\infty(H^1)} + \|k'(\Theta_*)\|_{L^\infty(L^\infty)}\|\nabla\Theta_*\|_{L^\infty(L^3)} \\ & \times \|p_*\|_{L^\infty(H^2)} + k_1\|p_*\|_{L^\infty(H^1)}.\end{aligned}$$

From here and properties (2.4) of the function k, it follows that

$$\|\alpha\|_{L^\infty(W^{1,3})} \lesssim 1 + R_1 + (1 + R_2^{\gamma_2+1})R_1R_2.$$

Again by the embedding $H^1(\Omega) \hookrightarrow L^3(\Omega)$ and properties of the function k, it holds that

$$\begin{aligned}\|\alpha_t\|_{L^2(L^3)} &= \| -2k(\Theta_*)p_{*t} - 2k'(\Theta_*)\Theta_{*t}p_*\|_{L^2(L^3)}\\ &\lesssim q_0^{-1}\|\nabla p_{*t}\|_{L^2(L^2)} + q_0^{-2}(1+\|\Theta_*\|^{\gamma_2+1}_{L^\infty(L^\infty)})\|\nabla\Theta_{*t}\|_{L^2(L^2)} \times \|p_*\|_{L^\infty(L^\infty)},\end{aligned}$$

which implies

$$\|\alpha_t\|_{L^2(L^3)} \lesssim R_1 + (1+R_2^{\gamma_2+1})R_1R_2.$$

Similarly,

$$\begin{aligned}\|\alpha_t\|_{L^\infty(L^2)} &= \| -2k(\Theta_*)p_{*t} - 2k'(\Theta_*)\Theta_{*t}p_*\|_{L^\infty(L^2)}\\ &\lesssim q_0^{-1}\|\nabla p_{*t}\|_{L^\infty(L^2)} + q_0^{-2}(1+\|\Theta_*\|^{\gamma_2+1}_{L^\infty(L^\infty)}) \times \|\Theta_{*t}\|_{L^\infty(L^2)}\|p_*\|_{L^\infty(L^\infty)}\\ &\lesssim R_1 + (1+R_2^{\gamma_2+1})R_1R_2.\end{aligned}$$

We can analogously estimate the function r:

$$\begin{aligned}\|r_t\|_{L^\infty(L^2)} &\lesssim \|q'(\Theta_*)\|_{L^\infty(L^\infty)}\|\Theta_{t*}\|_{L^\infty(L^2)},\\ \|\nabla r\|_{L^\infty(L^4)} = \|q'(\Theta_*)\nabla\Theta_*\|_{L^\infty(L^4)} &\lesssim \|q'(\Theta_*)\|_{L^\infty(L^\infty)}\|\Theta_*\|_{L^\infty(H^2)},\end{aligned}$$

and thus

$$\|r_t\|_{L^\infty(L^2)} \lesssim 1 + (1+R_2^{\gamma_1+1})R_2, \quad \|r\|_{L^\infty(W^{1,4})} \lesssim 1 + (1+R_2^{\gamma_1+1})R_2.$$

We can further estimate the source term in the pressure equation as follows:

$$\begin{aligned}\|f_1\|_{L^2(H^1)} + \|\partial_t f_1\|_{L^2(H^{-1})} &\lesssim \|k(\Theta_*)p_{*t}^2\|_{L^2(H^1)} + \|\partial_t(k(\Theta_*)p_{*t}^2)\|_{L^2(H^{-1})}\\ &\lesssim \|k'(\Theta_*)\nabla\Theta_* p_{*t}^2\|_{L^2(L^2)} + \|k(\Theta_*)p_{*t}\nabla p_{*t}\|_{L^2(L^2)} + \|k(\Theta_*)p_{*t}^2\|_{L^2(L^2)}\\ &\quad + \|k'(\Theta_*)\Theta_{t*}p_{*t}^2\|_{L^2(H^{-1})} + \|k(\Theta_*)p_{*t}p_{*tt}\|_{L^2(H^{-1})}.\end{aligned}$$

By using the embedding $L^{6/5}(\Omega) \hookrightarrow H^{-1}(\Omega)$ and the inequality

$$\|uvw\|_{L^{6/5}} \leqslant \|u\|_{L^2}\|v\|_{L^3}\|w\|_{L^\infty}$$

we then further have

$$\begin{aligned}\|f_1\|_{L^2(H^1)} + \|\partial_t f_1\|_{L^2(H^{-1})} &\lesssim \|k'(\Theta_*)\|_{L^\infty(L^\infty)}\|\nabla\Theta_*\|_{L^\infty(L^6)}\|p_{*t}^2\|_{L^2(L^3)} + k_1\|p_{*t}\|_{L^\infty(L^4)}\\ &\quad \times \|\nabla p_{*t}\|_{L^2(L^4)} + k_1\|p_{*t}^2\|_{L^2(L^2)} + \|k'(\Theta_*)\|_{L^\infty(L^\infty)}\|\Theta_{t*}\|_{L^2(L^3)}\\ &\quad \times \|p_{*t}^2\|_{L^\infty(L^2)} + k_1\|p_{*t}\|_{L^\infty(L^3)}\|p_{*tt}\|_{L^2(L^2)}.\end{aligned} \tag{4.5}$$

Thus,

$$\|f_1\|_{L^2(H^1)} + \|\partial_t f_1\|_{L^2(H^{-1})} \lesssim (1+R_2^{\gamma_2+1})R_2R_1^2 + R_1^2.$$

On account of proposition 3.1, the mapping $\mathcal{T}$ is well-defined, and, furthermore,

$$\mathcal{E}[p](t) + \|\Delta p_t(t)\|_{L^2}^2 + \int_0^t \mathcal{D}[p](s)\mathrm{d}s + \int_0^t \|p_{ttt}(s)\|_{H^{-1}}^2 \mathrm{d}s$$
$$\lesssim \mathcal{E}[p](0)\exp\left(\int_0^t (1+\Lambda(s))\mathrm{d}s\right) + \int_0^t \exp\left(\int_s^t (1+\Lambda(\sigma))\mathrm{d}\sigma\right)\mathbb{F}(s)\mathrm{d}s \tag{4.6}$$

a.e. in time, with $\Lambda(t)$ and $\mathbb{F}(t)$ defined in (3.5) and (3.6), respectively; that is,

$\Lambda(t) = \| r_t(t) \|_{L^2} + \| r_t(t) \|_{L^2}^2 + \| \nabla r(t)\|_{L^4} + \| \nabla r(t)\|_{L^4} + \| \alpha_t(t) \|_{L^2} + \| \alpha_t(t)\|_{L^3}^2 + \| \nabla\alpha(t) \|_{L^3}^2$

and

$$\mathbb{F}(t) = \|f_1(t)\|_{H^1}^2 + (1+\|\nabla\alpha(t)\|_{L^3}^2)\|\partial_t f_1(t)\|_{H^{-1}}^2.$$

By our calculations above, we immediately have

$$\|\Lambda\|_{L^1(0,t)} \leqslant C_1(R_1, R_2, T),$$

where $C_1 = C_1(T, R_1, R_2)$ is a positive constant that depends on T, R_1, and R_2. Furthermore, by relying on (4.5), we obtain

$$\|\mathbb{F}\|_{L^1(0,t)} \lesssim (1+\|\nabla\alpha\|_{L^\infty(L^3)}^2)(\|f_1\|_{L^2(H^1)}^2 + \|\partial_t f_1\|_{L^2(H^{-1})}^2)$$
$$\lesssim (1+R_1^2 + (1+R_2^{2\gamma_2+2})R_1^2R_2^2)\left\{(1+R_2^{2\gamma_2+2})R_2^2R_1^4 + R_1^4\right\}.$$

Altogether, from (4.6) and the above bounds, we have

$$\|p\|_{X_p}^2 \lesssim \delta\ \exp(C_1(R_1,R_2,T)T) + \exp(C_1(R_1,R_2,T)T)R_1^4C_2(R_1,R_2). \tag{4.7}$$

Thus, from (4.7), by decreasing R_1 and δ, we can achieve that

$$\|p\|_{X_p}^2 \leqslant R_1^2.$$

Further, by the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we know that

$$\|p\|_{L^\infty(L^\infty)}^2 \lesssim \|\Delta p\|_{L^\infty(L^2)}^2 \lesssim \|p\|_{X_p}^2,$$

which we can then bound by $\gamma \in (0, 1/(2k))$ by possibly additionally reducing δ and R_1. It remains to show that $\|\Theta\|_{X_\Theta} \leqslant R_2$. Proposition 3.1 with $f_2 = 0$ implies that

$$\mathcal{E}[\Theta](t) + \int_0^t \mathcal{D}[\Theta](s)\mathrm{d}s \leqslant C_T\Big(\|\Theta_0\|_{H_\diamondsuit^2(\Omega)}^2 + \|p_t\|_{L^\infty(L^\infty)}^2\|p_t\|_{L^2(L^2)}^2$$
$$+ \|p_t\|_{L^2(L^\infty)}^2\|p_{tt}\|_{L^\infty(L^2)}^2 + 1\Big).$$

With the equivalence of the temperature norm and energy (4.4), we have

$$\|\Theta\|_{X_\Theta}^2 \leqslant C_T\Big(\|\Theta_0\|_{H_\diamondsuit^2(\Omega)}^2 + \|p_t\|_{L^\infty(L^\infty)}^2\|p_t\|_{L^2(L^2)}^2$$
$$+ \|p_t\|_{L^2(L^\infty)}^2\|p_{tt}\|_{L^\infty(L^2)}^2 + 1\Big)$$
$$\leqslant \tilde{C}_T\Big(\|\Theta_0\|_{H_\diamondsuit^2(\Omega)}^2 + 2R_1^4 + 1\Big).$$

Thus, if we additionally choose R_2 large enough, so that

$$R_2^2 \geqslant \tilde{C}_T\Big(\|\Theta_0\|^2_{H^2_\diamondsuit(\Omega)} + 2R_1^4 + 1\Big),$$

we have $(p, \Theta) \in B$. □

Lemma 4.2. *For sufficiently small R_1 and δ, the mapping $\mathcal{T}$ is strictly contractive in the topology induced by $\|\cdot\|_{X_T}$.*

Proof. To prove contractivity, take any $(p_*^{(1)}, \Theta_*^{(1)})$ and $(p_*^{(2)}, \Theta_*^{(2)})$ from B. Denote their images by $(p^{(1)}, \Theta^{(1)}) = \mathcal{T}(p_*^{(1)}, \Theta_*^{(1)})$ and $(p^{(2)}, \Theta^{(2)}) = \mathcal{T}(p_*^{(2)}, \Theta_*^{(2)})$. We introduce the differences

$$\begin{aligned} \overline{p} &= p^{(1)} - p^{(2)}, \qquad & \overline{p}_* &= p_*^{(1)} - p_*^{(2)}, \\ \overline{\Theta} &= \Theta^{(1)} - \Theta^{(2)}, \qquad & \overline{\Theta}^* &= \Theta_*^{(1)} - \Theta_*^{(2)}. \end{aligned}$$

Our goal now is to prove that

$$\|\mathcal{T}(p_*^{(1)}, \Theta_*^{(1)}) - \mathcal{T}(p_*^{(2)}, \Theta_*^{(2)})\|_{X_T} \leqslant R_1 C(T, R_1, R_2)\|(p_*^{(1)} - p_*^{(2)}, \Theta_*^{(1)} - \Theta_*^{(2)})\|_{X_T},$$

where C is a positive constant that depends on T, R_1, and R_2. Observe that $(\overline{p}, \overline{\Theta})$ solves the following problem:

$$\begin{cases} (1 - 2k(\Theta_*^{(1)})p_*^{(1)})\overline{p}_{tt} - q(\Theta_*^{(1)})\Delta\overline{p} - b\Delta\overline{p}_t = \overline{f}_1 & \text{in } \Omega \times (0, T), \\ \rho_{\mathrm{a}} C_{\mathrm{a}} \overline{\Theta}_t - \kappa_{\mathrm{a}}\Delta\overline{\Theta} + \rho_{\mathrm{b}} C_{\mathrm{b}} W \overline{\Theta} = \overline{f}_2 & \text{in } \Omega \times (0, T), \\ \overline{p} = \overline{\Theta} = 0, & \text{on } \partial\Omega \times (0, T), \\ \overline{p}(x, 0) = \overline{p}_t(x, 0) = \overline{\Theta}(x, 0) = 0, & \text{in } \Omega, \end{cases} \tag{4.8}$$

with the right-hand sides

$$\begin{aligned} \overline{f}_1 = {} & \big\{2k(\Theta_*^{(1)})p_*^{(1)} - 2k(\Theta_*^{(2)})p_*^{(2)}\big\}p_{*tt}^{(2)} + \big\{q(\Theta_*^{(1)}) - q(\Theta_*^{(2)})\big\}\Delta p_*^{(2)} \\ & + 2k(\Theta_*^{(1)})(p_{*t}^{(1)})^2 - 2k(\Theta_*^{(2)})(p_{*t}^{(2)})^2 \end{aligned} \tag{4.9}$$

and

$$\overline{f}_2 = \mathcal{Q}(p_{*t}^{(1)}) - \mathcal{Q}(p_{*t}^{(2)}). \tag{4.10}$$

We can rearrange the acoustic source term $\overline{f}_1$ as follows:

$$\begin{aligned} \overline{f}_1 &= 2\big\{k(\Theta_*^{(1)}) - k(\Theta_*^{(2)})\big\}p_*^{(1)}p_{*tt}^{(2)} + 2k(\Theta_*^{(2)})\overline{p}_* p_{*tt}^{(2)} + \big\{q(\Theta_*^{(1)}) - q(\Theta_*^{(2)})\big\} \\ &\quad \times \Delta p_*^{(2)} + 2\big\{k(\Theta_*^{(1)}) - k(\Theta_*^{(2)})\big\}(p_{*t}^{(1)})^2 + 2k(\Theta_*^{(2)})\overline{p}_{*t}(p_{*t}^{(1)} + p_{*t}^{(2)}) \\ &= 2\big\{k(\Theta_*^{(1)}) - k(\Theta_*^{(2)})\big\}\big(p_*^{(1)}p_{*tt}^{(2)} + (p_{*t}^{(1)})^2\big) + \big\{q(\Theta_*^{(1)}) - q(\Theta_*^{(2)})\big\} \\ &\quad \times \Delta p_*^{(2)} + 2k(\Theta_*^{(2)})\big(\overline{p}_* p_{*tt}^{(2)} + \overline{p}_{*t}(p_{*t}^{(1)} + p_{*t}^{(2)})\big) \\ &:= \overline{f}_{11} + \overline{f}_{12} + \overline{f}_{13} \end{aligned}$$

and next wish to show that it satisfies assumption 3.

The estimate of $\|\overline{f}_1\|_{L^2(H^1)}$. Note that since $\overline{f}_1 = 0$ on $\partial\Omega$, it is sufficient to estimate $\|\nabla\overline{f}_1\|_{L^2(L^2)}$. We first estimate the $\overline{f}_{11}$ contribution, that is

$$\overline{f}_{11} = 2\{k(\Theta_*^{(1)}) - k(\Theta_*^{(2)})\}\left(p_*^{(1)}p_{*tt}^{(2)} + (p_{*t}^{(1)})^2\right).$$

By Hölder's inequality, we have

$$\begin{aligned}\|\nabla\overline{f}_{11}\|_{L^2(L^2)} \lesssim &\|k(\Theta_*^{(1)}) - k(\Theta_*^{(2)})\|_{L^\infty(L^\infty)}\|\nabla(p_*^{(1)}p_{*tt}^{(2)} + (p_{*t}^{(1)})^2)\|_{L^2(L^2)} \\ &+ \|\nabla(k(\Theta_*^{(1)}) - k(\Theta_*^{(2)}))\|_{L^\infty(L^4)}\|p_*^{(1)}p_{*tt}^{(2)} + (p_{*t}^{(1)})^2\|_{L^2(L^4)}.\end{aligned} \tag{4.11}$$

Recalling properties (2.3) and (2.4) of the function k, and using the algebraic inequality:

$$(A+B)^\nu \leqslant \max\{1, 2^\nu\}(A^\nu + B^\nu), \quad \text{for } A, B \geqslant 0,\ \nu > 0,$$

we have

$$\begin{aligned}\|k(\Theta_*^{(1)}) - k(\Theta_*^{(2)})\|_{L^\infty(L^\infty)} &= \left\|(\Theta_*^{(1)} - \Theta_*^{(2)})\int_0^1 k'(\Theta_*^{(1)} + \tau(\Theta_*^{(1)} - \Theta_*^{(2)}))\mathrm{d}\tau\right\|_{L^\infty(L^\infty)} \\ &\lesssim \|\Theta_*^{(1)} - \Theta_*^{(2)}\|_{L^\infty(L^\infty)}\left(1 + \|\Theta_*^{(1)} + \tau(\Theta_*^{(1)} - \Theta_*^{(2)})\|_{L^\infty(L^\infty)}^{\gamma_2+1}\right) \\ &\lesssim \|(p_*^{(1)} - p_*^{(2)}, \Theta_*^{(1)} - \Theta_*^{(2)})\|_{X_T}\left\{1 + \|\Theta_*^{(1)}\|_{X_\Theta}^{\gamma_2+1} + \|\Theta_*^{(2)}\|_{X_\Theta}^{\gamma_2+1}\right\}.\end{aligned} \tag{4.12}$$

We also have, by using the embeddings $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, the following estimate:

$$\begin{aligned}\|\nabla(p_*^{(1)}p_{*tt}^{(2)} + (p_{*t}^{(1)})^2)\|_{L^2(L^2)} \lesssim\ &\|\nabla p_*^{(1)}\|_{L^\infty(L^4)}\|p_{*tt}^{(2)}\|_{L^2(L^4)} + \|p_*^{(1)}\|_{L^\infty(L^\infty)}\|\nabla p_{*tt}^{(2)}\|_{L^2(L^2)} \\ &+ \|p_{*t}^{(1)}\|_{L^2(L^\infty)}\|\nabla p_{*t}^{(1)}\|_{L^\infty(L^2)} \\ \lesssim\ &\|\Delta p_*^{(1)}\|_{L^\infty(L^2)}\|\nabla p_{*tt}^{(2)}\|_{L^2(L^2)} + \|\Delta p_*^{(1)}\|_{L^\infty(L^2)}\|\nabla p_{*tt}^{(2)}\|_{L^2(L^2)} \\ &+ \|\Delta p_{*t}^{(1)}\|_{L^2(L^2)}\|\nabla p_{*t}^{(1)}\|_{L^\infty(L^2)}.\end{aligned} \tag{4.13}$$

Thus, from (4.13) it follows that

$$\|\nabla(p_*^{(1)}p_{*tt}^{(2)} + (p_{*t}^{(1)})^2)\|_{L^2(L^2)} \lesssim \|p_*^{(1)}\|_{X_p}^2.$$

Further, we know that

$$\begin{aligned}\nabla(k(\Theta_*^{(1)}) - k(\Theta_*^{(2)})) &= k'(\Theta_*^{(1)})\nabla\Theta_*^{(1)} - k'(\Theta_*^{(2)})\nabla\Theta_*^{(2)} \\ &= k'(\Theta_*^{(1)})\nabla(\Theta_*^{(1)} - \Theta_*^{(2)}) + \nabla\Theta_*^{(2)}(k'(\Theta_*^{(1)}) - k'(\Theta_*^{(2)}))\end{aligned}$$

and

$$k'(\Theta_*^{(1)}) - k'(\Theta_*^{(2)}) = (\Theta_*^{(1)} - \Theta_*^{(2)})\int_0^1 k''(\Theta_*^{(1)} + \tau(\Theta_*^{(1)} - \Theta_*^{(2)}))\mathrm{d}\tau.$$

By keeping in mind properties (2.3) and (2.4) of the function k, this implies that

$$\|\nabla(k(\Theta_*^{(1)}) - k(\Theta_*^{(2)}))\|_{L^\infty(L^4)} \lesssim \left(1 + \|\Theta_*^{(1)}\|^{\gamma_2+1}_{L^\infty(L^\infty)}\right)\|\nabla(\Theta_*^{(1)} - \Theta_*^{(2)})\|_{L^\infty(L^4)} + \big(1 + \|\Theta_*^{(1)}\|^{\gamma_2}_{L^\infty(L^\infty)} + \|\Theta_*^{(2)}\|^{\gamma_2}_{L^\infty(L^\infty)}\big)\|\nabla\Theta_*^{(2)}\|_{L^\infty(L^4)}\|\Theta_*^{(1)} - \Theta_*^{(2)}\|_{L^\infty(L^\infty)}$$

and thus

$$\|\nabla(k(\Theta_*^{(1)}) - k(\Theta_*^{(2)}))\|_{L^\infty(L^4)} \lesssim \left\{1 + \|\Theta_*^{(1)}\|^{\gamma_2}_{X_\Theta} + \|\Theta_*^{(2)}\|^{\gamma_2}_{X_\Theta} + \|\Theta_*^{(1)}\|^{\gamma_2+1}_{X_\Theta}\right\} \times \|(p_*^{(1)} - p_*^{(2)}, \Theta_*^{(1)} - \Theta_*^{(2)})\|_{X_T}. \tag{4.14}$$

To obtain a bound on $\nabla\overline{f}_{11}$, we note that

$$\begin{aligned}\|p_*^{(1)}p_{*tt}^{(2)} + (p_{*t}^{(1)})^2\|_{L^2(L^4)} &\lesssim \|p_*^{(1)}\|_{L^2(L^\infty)}\|p_{*tt}^{(2)}\|_{L^2(L^4)} + \|p_{*t}^{(2)}\|_{L^\infty(L^4)}\|p_{*t}^{(2)}\|_{L^2(L^\infty)} \\ &\lesssim \sqrt{T}\|\Delta p_*^{(1)}\|_{L^\infty(L^2)}\|\nabla p_{*tt}^{(2)}\|_{L^2(L^2)} + \|\nabla p_{*t}^{(1)}\|_{L^\infty(L^2)}\|\Delta p_{*t}^{(1)}\|_{L^2(L^2)} \\ &\lesssim (1 + \sqrt{T})\left(\|p_*^{(1)}\|^2_{X_p} + \|p_*^{(2)}\|^2_{X_p}\right).\end{aligned}$$

Plugging the derived estimates into (4.11) yields

$$\|\nabla\overline{f}_{11}\|_{L^2(L^2)} \lesssim (1 + \sqrt{T})R_1^2\left(1 + R_2^{\gamma_2} + R_2^{\gamma_2+1}\right) \times \|(p_*^{(1)} - p_*^{(2)}, \Theta_*^{(1)} - \Theta_*^{(2)})\|_{X_T}.$$

We can similarly estimate $\overline{f}_{12} = \left\{q(\Theta_*^{(1)}) - q(\Theta_*^{(2)})\right\}\Delta p_*^{(2)}$ as follows:

$$\|\nabla\overline{f}_{12}\|_{L^2(L^2)} \lesssim \left\|\nabla(q(\Theta_*^{(1)}) - q(\Theta_*^{(2)}))\right\|_{L^\infty(L^4)}\|\Delta p_*^{(2)}\|_{L^2(L^4)} + \|q(\Theta_*^{(1)}) - q(\Theta_*^{(2)})\|_{L^\infty(L^\infty)}\|\nabla\Delta p_*^{(2)}\|_{L^2(L^2)}. \tag{4.15}$$

The first term on the right-hand side of (4.15) can be estimated analogously to (4.14). Thus we have by recalling assumption 1,

$$\left\|\nabla(q(\Theta_*^{(1)}) - q(\Theta_*^{(2)}))\right\|_{L^\infty(L^4)} \lesssim \left\{1 + \|\Theta_*^{(1)}\|^{\gamma_1+1}_{X_\Theta} + \|\Theta_*^{(2)}\|^{\gamma_1+1}_{X_\Theta}\right\}\|(p_*^{(1)} - p_*^{(2)}, \Theta_*^{(1)} - \Theta_*^{(2)})\|_{X_T}.$$

By using the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$, we obtain

$$\|\Delta p_*^{(2)}\|_{L^2(L^4)} \lesssim \|\Delta p_*^{(2)}\|_{L^2(L^2)} + \|\Delta\nabla p_*^{(2)}\|_{L^2(L^2)} \lesssim \sqrt{T}\|p_*^{(2)}\|_{X_p}.$$

We also have as in (4.12),

$$\left\|q(\Theta_*^{(1)}) - q(\Theta_*^{(2)})\right\|_{L^\infty(L^\infty)} \lesssim \left\{1 + \|\Theta_*^{(1)}\|^{\gamma_1+1}_{X_\Theta} + \|\Theta_*^{(2)}\|^{\gamma_1+1}_{X_\Theta}\right\}\|(p_*^{(1)} - p_*^{(2)}, \Theta_*^{(1)} - \Theta_*^{(2)})\|_{X_T}.$$

Consequently, we obtain from above the following estimate:

$$\|\nabla\overline{f}_{12}\|_{L^2(L^2)} \lesssim (1 + \sqrt{T})R_1\left(1 + R_2^{\gamma_1} + R_2^{\gamma_1+1} + R_2^{2\gamma_1+2}\right) \times \|(p_*^{(1)} - p_*^{(2)}, \Theta_*^{(1)} - \Theta_*^{(2)})\|_{X_T}.$$

Next we estimate $\overline{f}_{13} = 2k(\Theta_*^{(2)})\Big(\overline{p}_* p_{*tt}^{(2)} + \overline{p}_{*t}(p_{*t}^{(1)} + p_{*t}^{(2)})\Big)$. We note that

$$\begin{aligned}\|\nabla \overline{f}_{13}\|_{L^2(L^2)} \lesssim\ & \|k'(\Theta_*^{(2)})\nabla\Theta_*^{(2)}\|_{L^\infty(L^4)}\big(\|\overline{p}\|_{L^\infty(L^\infty)}\|p_{*tt}^{(2)}\|_{L^2(L^4)} \\ & + \|\overline{p}_t\|_{L^\infty(L^4)}(\|p_{*t}^{(1)}\|_{L^2(L^\infty)} + \|p_{*t}^{(2)}\|_{L^2(L^\infty)})\big) \\ & + \|k(\Theta_*^{(2)})\|_{L^\infty(L^\infty)}\|\nabla(\overline{p}_* p_{*tt}^{(2)} + \overline{p}_{*t}(p_{*t}^{(1)} + p_{*t}^{(2)}))\|_{L^2(L^2)}.\end{aligned}$$

Using properties (2.4) of the function k, we can bound the first term on the right:

$$\begin{aligned}\|k'(\Theta_*^{(2)})\nabla\Theta_*^{(2)}\|_{L^\infty(L^4)} &\lesssim \|k'(\Theta_*^{(2)})\|_{L^\infty(L^\infty)}\|\nabla\Theta_*^{(2)}\|_{L^\infty(L^4)} \\ &\lesssim (1 + \|\Theta_*^{(2)}\|_{L^\infty(L^\infty)}^{\gamma_2+1})\|\Theta_*^{(2)}\|_{L^\infty(H^2_\diamondsuit(\Omega))} \\ &\lesssim (1 + R_2^{\gamma_2+1})R_2.\end{aligned}$$

Further, we have

$$\begin{aligned}&\|\overline{p}\|_{L^\infty(L^\infty)}\|p_{*tt}^{(2)}\|_{L^2(L^4)} + \|\overline{p}_t\|_{L^\infty(L^4)}(\|p_{*t}^{(1)}\|_{L^2(L^\infty)} + \|p_{*t}^{(2)}\|_{L^2(L^\infty)}) \\ &\lesssim \|\Delta\overline{p}\|_{L^\infty(L^2)}\|\nabla p_{*tt}^{(2)}\|_{L^\infty(L^2)} + \|\nabla\overline{p}_t\|_{L^\infty(L^2)} \\ &\quad \times (\|\Delta p_{*t}^{(1)}\|_{L^2(L^2)} + \|\Delta p_{*t}^{(2)}\|_{L^2(L^2)}) \\ &\lesssim R_1\|(p_*^{(1)} - p_*^{(2)}, \Theta_*^{(1)} - \Theta_*^{(2)})\|_{X_T}.\end{aligned}$$

By using the fact that $|k(s)| \lesssim \frac{1}{q_0}$, we find

$$\begin{aligned}&\|k(\Theta_*^{(2)})\|_{L^\infty(L^\infty)}\|\nabla(\overline{p}_* p_{*tt}^{(2)} + \overline{p}_{*t}(p_{*t}^{(1)} + p_{*t}^{(2)}))\|_{L^2(L^2)} \\ &\lesssim \|\nabla\overline{p}_*\|_{L^\infty(L^4)}\|p_{*tt}^{(2)}\|_{L^2(L^4)} + \|\overline{p}_*\|_{L^\infty(L^\infty)}\|\nabla p_{*tt}^{(2)}\|_{L^2(L^2)} + \|\nabla\overline{p}_{*t}\|_{L^2(L^4)} \\ &\quad \times \|p_{*t}^{(1)} + p_{*t}^{(2)}\|_{L^\infty(L^4)} + \|\overline{p}_{*t}\|_{L^2(L^\infty)}\|\nabla p_{*t}^{(1)} + \nabla p_{*t}^{(2)}\|_{L^\infty(L^2)} \\ &\lesssim \|\Delta\overline{p}_*\|_{L^\infty(L^2)}\|\nabla p_{*tt}^{(2)}\|_{L^2(L^2)} + \|\Delta\overline{p}_*\|_{L^\infty(L^2)}\|\nabla p_{*tt}^{(2)}\|_{L^2(L^2)} \\ &\quad + \|\Delta\overline{p}_{*t}\|_{L^2(L^2)}\|\nabla p_{*t}^{(1)} + \nabla p_{*t}^{(2)}\|_{L^\infty(L^2)} \\ &\quad + \|\Delta\overline{p}_{*t}\|_{L^2(L^2)}\|\nabla p_{*t}^{(1)} + \nabla p_{*t}^{(2)}\|_{L^\infty(L^2)}.\end{aligned}$$

Hence,

$$\begin{aligned}&\|k(\Theta_*^{(2)})\|_{L^\infty(L^\infty)}\|\nabla(\overline{p}_* p_{*tt}^{(2)} + \overline{p}_{*t}(p_{*t}^{(1)} + p_{*t}^{(2)}))\|_{L^2(L^2)} \\ &\lesssim R_1\|(p_*^{(1)} - p_*^{(2)}, \Theta_*^{(1)} - \Theta_*^{(2)})\|_{X_T}.\end{aligned}$$

Consequently, from the derived bounds we infer

$$\|\nabla\overline{f}_{13}\|_{L^2(L^2)} \lesssim C_T R_1(1 + R_2 + R_2^{\gamma_2+2})\|(p_*^{(1)} - p_*^{(2)}, \Theta_*^{(1)} - \Theta_*^{(2)})\|_{X_T}.$$

By collecting the derived estimates of separate contributions to $\overline{f}_1$, we arrive at

$$\begin{aligned}\|\nabla\overline{f}_1\|_{L^2(L^2)} \lesssim\ & C_T(R_1 + R_1^2)\Big(1 + R_2^{\gamma_1} + R_2^{\gamma_1+1}\Big) \\ & \times \|(p_*^{(1)} - p_*^{(2)}, \Theta_*^{(1)} - \Theta_*^{(2)})\|_{X_T}.\end{aligned} \tag{4.16}$$

The estimate of $\|\partial_t\overline{f}_1\|_{L^2(H^{-1})}$. Our next task is to estimate $\|\partial_t\overline{f}_1\|_{L^2(H^{-1})}$. As above, we estimate the contributions $\|\partial_t\overline{f}_{1j}\|_{L^2(H^{-1})}$ for $j=1,2,3$ separately. We start by noting that

$$\begin{aligned}\partial_t\overline{f}_{11} =&2\big\{k(\Theta_*^{(1)})-k(\Theta_*^{(2)})\big\}\big(p_*^{(1)}p_{*ttt}^{(2)}+p_{*t}^{(1)}p_{*tt}^{(2)}+2p_{*t}^{(1)}p_{*tt}^{(1)}\big)\\ &+2\partial_t\big\{k(\Theta_*^{(1)})-k(\Theta_*^{(2)})\big\}\big(p_*^{(1)}p_{*tt}^{(2)}+(p_{*t}^{(1)})^2\big).\end{aligned}$$

By employing the H^{-1} estimate stated in (2.10), we then find that

$$\begin{aligned}\|\partial_t\overline{f}_{11}\|_{H^{-1}} \lesssim\ &\big(\|k(\Theta^{(1)})_*-k(\Theta_*^{(2)})\|_{L^\infty}+\|\nabla(k(\Theta^{(1)})_*-k(\Theta_*^{(2)}))\|_{L^3}\big)\\ &\times\|p_*^{(1)}p_{*ttt}^{(2)}\|_{H^{-1}}+\|k(\Theta^{(1)})_*-k(\Theta_*^{(2)})\|_{L^\infty})(\|p_{*t}^{(1)}p_{*tt}^{(2)}\|_{L^2}+\|p_{*t}^{(1)}p_{*tt}^{(1)}\|_{L^2})\\ &+\|\partial_t(k(\Theta_*^{(1)})-k(\Theta_*^{(2)}))\|_{L^6}\|(p_{*t}^{(1)})^2\|_{L^3}\\ &+\big\|\partial_t\big\{k(\Theta_*^{(1)})-k(\Theta_*^{(2)})\big\}p_*^{(1)}p_{*tt}^{(2)}\big\|_{H^{-1}}.\end{aligned}\tag{4.17}$$

Hence, we obtain from above

$$\begin{aligned}\|\partial_t\overline{f}_{11}\|_{L^2(H^{-1})} \lesssim\ &\big(\|k(\Theta^{(1)})_*-k(\Theta_*^{(2)})\|_{L^\infty(L^\infty)}+\|\nabla(k(\Theta^{(1)})_*-k(\Theta_*^{(2)}))\|_{L^\infty(L^3)}\big)\\ &\times\|p_*^{(1)}p_{*ttt}^{(2)}\|_{L^2(H^{-1})}+\|k(\Theta^{(1)})_*-k(\Theta_*^{(2)})\|_{L^\infty(L^\infty)}\\ &\times(\|p_{*t}^{(1)}p_{*tt}^{(2)}\|_{L^2(L^2)}+\|p_{*t}^{(1)}p_{*tt}^{(1)}\|_{L^2(L^2)})+\|\partial_t(k(\Theta_*^{(1)})-k(\Theta_*^{(2)}))\\ &\times\|_{L^2(L^6)}\|(p_{*t}^{(1)})^2\|_{L^\infty(L^3)}+\big\|\partial_t\big\{k(\Theta_*^{(1)})-k(\Theta_*^{(2)})\big\}p_*^{(1)}p_{*tt}^{(2)}\big\|_{L^2(H^{-1})}.\end{aligned}$$

We estimate the second term by using the H^{-1} inequality (2.10) as follows:

$$\begin{aligned}\|p_*^{(1)}p_{*ttt}^{(2)}\|_{L^2(H^{-1})} &\lesssim \|p_{*ttt}^{(2)}\|_{L^2(H^{-1})}(\|\nabla p_*^{(1)}\|_{L^\infty(L^3)}+\|p_*^{(1)}\|_{L^\infty(L^\infty)})\\ &\lesssim \|p_{*ttt}^{(2)}\|_{L^2(H^{-1})}\|\Delta p_*^{(1)}\|_{L^\infty(L^2)}\\ &\lesssim R_1^2,\end{aligned}$$

where we have also used the embeddings $H^1(\Omega)\hookrightarrow L^3(\Omega)$, $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, and elliptic regularity. Next, as in (4.12), we have

$$\|k(\Theta^{(1)})_*-k(\Theta_*^{(2)})\|_{L^\infty(L^\infty)}\lesssim(1+R_2^{\gamma_2+1})\|(\overline{p}_*,\overline{\Theta}_*)\|_{X_T}.$$

Further,

$$\begin{aligned}&\|p_{*t}^{(1)}p_{*tt}^{(2)}\|_{L^2(L^2)}+\|\partial_t(p_{*t}^{(1)})^2\|_{L^2(L^2)}\\ &\quad\lesssim \|p_{*t}^{(1)}\|_{L^2(L^\infty)}\|p_{*tt}^{(2)}\|_{L^\infty(L^2)}+\|p_{*t}^{(1)}\|_{L^2(L^\infty)}\|p_{*tt}^{(1)}\|_{L^\infty(L^2)}\\ &\quad\lesssim \|\Delta p_{*t}^{(1)}\|_{L^2(L^2)}\|p_{*tt}^{(2)}\|_{L^\infty(L^2)}+\|\Delta p_{*t}^{(1)}\|_{L^2(L^2)}\|p_{*tt}^{(1)}\|_{L^\infty(L^2)}\ \lesssim\ R_1^2.\end{aligned}$$

Now, we can use the following re-arrangement:

$$\begin{aligned}\partial_t(k(\Theta_*^{(1)})-k(\Theta_*^{(2)})) &= k'(\Theta_*^{(1)})\Theta_{*t}^{(1)}-k'(\Theta_*^{(2)})\Theta_{*t}^{(2)}\\ &= k'(\Theta_*^{(1)})(\Theta_{*t}^{(1)}-\Theta_{*t}^{(2)})+\Theta_{*t}^{(2)}(k'(\Theta_*^{(1)})-k'(\Theta_*^{(2)})).\end{aligned}$$

Hence, by the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$,

$$\begin{aligned}\|\partial_t(k(\Theta_*^{(1)}) - k(\Theta_*^{(2)}))\|_{L^2(L^6)} &\lesssim \|k'(\Theta_*^{(1)})\|_{L^\infty(L^\infty)}\|\Theta_{*t}^{(1)} - \Theta_{*t}^{(2)}\|_{L^2(L^6)} \\ &\quad + \|\Theta_{*t}^{(2)}\|_{L^2(L^6)}\|k'(\Theta_*^{(1)}) - k'(\Theta_*^{(2)})\|_{L^\infty(L^\infty)} \\ &\lesssim \|\Theta_{*t}^{(1)} - \Theta_{*t}^{(2)}\|_{L^2(H^1)}\left(1 + \|\Theta_*^{(1)}\|_{L^\infty(L^\infty)}^{\gamma_2+1}\right) \\ &\quad + \|\Theta_{*t}^{(2)}\|_{L^2(H^1)}\|\Theta_*^{(1)} - \Theta_*^{(2)}\|_{L^\infty(L^\infty)}\left(1 + \|\Theta_*^{(1)}\|_{L^\infty(L^\infty)}^{\gamma_2}\right) \\ &\lesssim (1 + R_2 + R_2^{\gamma_2+1})\|(\overline{p}_*, \overline{\Theta}_*)\|_{X_T}. \end{aligned} \tag{4.18}$$

Furthermore, we have

$$\|(p_{*t}^{(1)})^2\|_{L^\infty(L^3)} \lesssim \|p_{*t}^{(1)}\|_{L^\infty(L^6)}^2 \lesssim \|\nabla p_{*t}^{(1)}\|_{L^\infty(L^2)}^2 \lesssim R_1^2.$$

Next by using the embedding $L^{6/5}(\Omega) \hookrightarrow H^{-1}(\Omega)$ and Hölder's inequality, we infer

$$\begin{aligned}&\left\|\partial_t\left\{k(\Theta_*^{(1)}) - k(\Theta_*^{(2)})\right\}p_*^{(1)}p_{*tt}^{(2)}\right\|_{L^2(H^{-1})} \\ &\quad \lesssim \|\partial_t(k(\Theta_*^{(1)}) - k(\Theta_*^{(2)}))\|_{L^2(L^3)}\|p_*^{(1)}p_{*tt}^{(2)}\|_{L^\infty(L^2)}.\end{aligned}$$

As in (4.18), using the embedding $H^1(\Omega) \hookrightarrow L^3(\Omega)$ yields

$$\|\partial_t(k(\Theta_*^{(1)}) - k(\Theta_*^{(2)}))\|_{L^2(L^3)} \lesssim (1 + R_2 + R_2^{\gamma_2+1})\|(\overline{p}_*, \overline{\Theta}_*)\|_{X_T},$$

whereas

$$\|p_*^{(1)}p_{*tt}^{(2)}\|_{L^\infty(L^2)} \lesssim \|\Delta p_*^{(1)}\|_{L^\infty(L^2)}\|p_{*tt}^{(2)}\|_{L^\infty(L^2)} \lesssim R_1^2.$$

Consequently, by collecting the derived estimates, we obtain from (4.17),

$$\|\partial_t\overline{f}_{11}\|_{L^2(H^{-1})} \leqslant CR_1^2(1 + R_2^{\gamma_2} + R_2^{\gamma_2+1} + R_2^{2\gamma_2+2})\|(\overline{p}_*, \overline{\Theta}_*)\|_{X_T}.$$

Next, we estimate $\overline{f}_{12} = \left\{q(\Theta_*^{(1)}) - q(\Theta_*^{(2)})\right\}\Delta p_*^{(2)}$. We have $\|\partial_t\overline{f}_{12}\|_{L^2(H^{-1})} \lesssim \|\partial_t\overline{f}_{12}\|_{L^2(L^2)}$ and further

$$\begin{aligned}\|\partial_t\overline{f}_{12}\|_{L^2(L^2)} &\lesssim \|q(\Theta_*^{(1)}) - q(\Theta_*^{(2)})\|_{L^\infty(L^\infty)}\|\Delta p_{*t}^{(2)}\|_{L^2(L^2)} \\ &\quad + \|\partial_t(q(\Theta_*^{(1)}) - q(\Theta_*^{(2)}))\|_{L^2(L^4)}\|\Delta p_*^{(2)}\|_{L^\infty(L^4)}.\end{aligned}$$

Similarly to the estimate of $\|\partial_t\overline{f}_{11}\|_{L^2(L^2)}$ and by using the fact that

$$\|\Delta p_*^{(2)}\|_{L^\infty(L^4)} \lesssim \|\Delta p_*^{(2)}\|_{L^\infty(L^2)} + \|\Delta\nabla p_*^{(2)}\|_{L^\infty(L^2)} \lesssim \|p_*^{(2)}\|_{X_p},$$

we obtain

$$\|\partial_t\overline{f}_{12}\|_{L^2(L^2)} \leqslant C_TR_1^2(1 + R_2 + R_2^{\gamma_1+1})\|(\overline{p}_*, \overline{\Theta}_*)\|_{X_T}.$$

It remains to estimate $\|\partial_t\overline{f}_{13}\|_{L^2(H^{-1})}$. Indeed, recalling that

$$\overline{f}_{13} = 2k(\Theta_*^{(2)})\left(\overline{p}_*p_{*tt}^{(2)} + \overline{p}_{*t}(p_{*t}^{(1)} + p_{*t}^{(2)})\right)$$

we have

$$\begin{aligned}\|\partial_t\overline{f}_{13}\|_{L^2(H^{-1})} \lesssim\ &\|\partial_t(\overline{p}_* p^{(2)}_{*tt} + \overline{p}_{*t}(p^{(1)}_{*t} + p^{(2)}_{*t}))\|_{L^2(H^{-1})}\\ &+ \|k'(\Theta^{(2)}_*)\Theta^{(2)}_{*t}\|_{L^2(L^4)}\|\overline{p}_* p^{(2)}_{*tt} + \overline{p}_{*t}(p^{(1)}_{*t} + p^{(2)}_{*t})\|_{L^\infty(L^4)}.\end{aligned} \tag{4.19}$$

We estimate the first term in (4.19) as follows:

$$\begin{aligned}&\|\partial_t(\overline{p}_* p^{(2)}_{*tt} + \overline{p}_{*t}(p^{(1)}_{*t} + p^{(2)}_{*t}))\|_{L^2(H^{-1})}\\ &\lesssim \|\overline{p}_{*t}\|_{L^2(L^4)}\|p^{(2)}_{*tt}\|_{L^\infty(L^4)} + (\|\overline{p}_*\|_{L^\infty(L^\infty)} + \|\nabla\overline{p}_*\|_{L^\infty(L^3)})\\ &\quad\times\|p^{(2)}_{*ttt}\|_{L^2(H^{-1})} + \|\overline{p}_{*tt}\|_{L^\infty(L^2)}(\|p^{(1)}_{*t}\|_{L^2(L^\infty)} + \|p^{(2)}_{*t}\|_{L^2(L^\infty)})\\ &\quad+ \|\overline{p}_{*t}\|_{L^2(L^4)}(\|p^{(1)}_{*tt}\|_{L^\infty(L^4)} + \|p^{(2)}_{*tt}\|_{L^\infty(L^4)})\\ &\lesssim \|\nabla\overline{p}_{*t}\|_{L^2(L^2)}\|\nabla p^{(2)}_{*tt}\|_{L^\infty(L^2)} + \|\Delta\overline{p}_*\|_{L^\infty(L^2)}\|p^{(2)}_{*ttt}\|_{L^2(H^{-1})}\\ &\quad+ \|\overline{p}_{*tt}\|_{L^\infty(L^2)}(\|\Delta p^{(1)}_{*t}\|_{L^2(L^2)} + \|\Delta p^{(2)}_{*t}\|_{L^2(L^2)})\\ &\quad+ \|\nabla\overline{p}_{*t}\|_{L^2(L^2)}(\|\nabla p^{(1)}_{*tt}\|_{L^\infty(L^2)} + \|\nabla p^{(2)}_{*tt}\|_{L^\infty(L^2)}).\end{aligned}$$

Hence, we obtain

$$\|\partial_t(\overline{p}_* p^{(2)}_{*tt} + \overline{p}_{*t}(p^{(1)}_{*t} + p^{(2)}_{*t}))\|_{L^2(H^{-1})} \lesssim R_1\|(\overline{p}_*, \overline{\Theta}_*)\|_{X_T}. \tag{4.20}$$

Next, we estimate the second term on the right-hand side of (4.19) as:

$$\begin{aligned}\|k'(\Theta^{(2)}_*)\Theta^{(2)}_{*t}\|_{L^2(L^4)} &\lesssim \|k'(\Theta^{(2)}_*)\|_{L^\infty(L^\infty)}\|\Theta^{(2)}_{*t}\|_{L^2(L^4)}\\ &\lesssim \left(1 + \|\Theta^{(2)}_*\|^{\gamma_2+1}_{L^\infty(L^\infty)}\right)\|\Theta^{(2)}_{*t}\|_{L^2(H^1)}\\ &\lesssim R_2(1 + R_2^{\gamma_2+1}).\end{aligned} \tag{4.21}$$

Finally, we estimate the last term on the right-hand side of (4.19) as

$$\begin{aligned}\|\overline{p}_* p^{(2)}_{*tt} + \overline{p}_{*t}(p^{(1)}_{*t} + p^{(2)}_{*t})\|_{L^\infty(L^4)} &\lesssim \|\overline{p}_*\|_{L^\infty(L^\infty)}\|p^{(2)}_{*tt}\|_{L^\infty(L^4)}\\ &\quad+ \|\overline{p}_{*t}\|_{L^\infty(L^4)}(\|p^{(1)}_{*t}\|_{L^\infty(L^\infty)} + \|p^{(2)}_{*t}\|_{L^\infty(L^\infty)})\\ &\lesssim \|\Delta\overline{p}_*\|_{L^\infty(L^2)}\|\nabla p^{(2)}_{*tt}\|_{L^\infty(L^2)}\\ &\quad+ \|\nabla\overline{p}_{*t}\|_{L^\infty(L^2)}(\|\Delta p^{(1)}_{*t}\|_{L^\infty(L^2)} + \|\Delta p^{(2)}_{*t}\|_{L^\infty(L^2)}).\end{aligned} \tag{4.22}$$

Using the embedding $H^1(0,t) \hookrightarrow C[0,t]$, we find that

$$\|\nabla\overline{p}_{*t}\|_{L^\infty(L^2)} \lesssim \|\nabla\overline{p}_{*t}\|_{L^2(L^2)} + \|\nabla\overline{p}_{*tt}\|_{L^2(L^2)} \lesssim \|\overline{p}_*\|_{X_p}.$$

Consequently, we obtain from (4.22),

$$\|\overline{p}_* p^{(2)}_{*tt} + \overline{p}_{*t}(p^{(1)}_{*t} + p^{(2)}_{*t})\|_{L^\infty(L^4)} \lesssim R_1\|(\overline{p}_*, \overline{\Theta}_*)\|_{X_T}. \tag{4.23}$$

Collecting (4.20), (4.21) and (4.23) results in

$$\|\partial_t\overline{f}_{13}\|_{L^2(H^{-1})} \lesssim R_1(1 + R_2 + R_2^{\gamma_2+2})\|(\overline{p}_*, \overline{\Theta}_*)\|_{X_T}.$$

Finally, by collecting the bounds of separate contributions, we infer that

$$\|\partial_t\overline{f}_1\|_{L^2(H^{-1})} \leqslant C_T(R_1 + R_1^2)(1 + R_2 + R_2^{\gamma_2+1} + R_2^{\gamma_2+2})\|(\overline{p}_*, \overline{\Theta}_*)\|_{X_T}. \tag{4.24}$$

The estimate of $\|\overline{f}_2\|_{H^1(L^2)}$. We can bound the source term in the heat equation as follows:

$$\|\overline{f}_2\|_{H^1(L^2)} \lesssim \|\mathcal{Q}(p_{*t}^{(1)}) - \mathcal{Q}(p_{*t}^{(2)})\|_{L^2(L^2)} + \|\partial_t(\mathcal{Q}(p_{*t}^{(1)}) - \mathcal{Q}(p_{*t}^{(2)}))\|_{L^2(L^2)}. \tag{4.25}$$

Since $p_{*t}^{(j)} \in B$ for $j = 1, 2$, we have by the Sobolev embedding

$$\|p_{*t}^{(j)}\|_{L^\infty(L^\infty)} \lesssim \|\Delta p_{*t}^{(j)}\|_{L^\infty(L^2)} \lesssim \|p_{*t}^{(j)}\|_{L^\infty(X_p)} \lesssim R_1.$$

Hence, in view of the assumption (2.5), this yields

$$\|\mathcal{Q}(p_{*t}^{(1)}) - \mathcal{Q}(p_{*t}^{(2)})\|_{L^2(L^2)} \lesssim R_1\|p_{*t}^{(1)} - p_{*t}^{(2)}\|_{L^2(L^2)} \lesssim R_1\|p_{*t}^{(1)} - p_{*t}^{(2)}\|_{X_p}. \tag{4.26}$$

Similarly, using (2.6), we have

$$\begin{aligned}
\|\partial_t(\mathcal{Q}(p_{*t}^{(1)}) - \mathcal{Q}(p_{*t}^{(2)}))\|_{L^2(L^2)} &\lesssim \|p_{*t}^{(1)}\|_{L^2(L^\infty)}\|p_{*tt}^{(1)} - p_{*tt}^{(2)}\|_{L^\infty(L^2)} \\
&\quad + \|p_{*tt}^{(2)}\|_{L^\infty(L^2)}\|p_{*t}^{(1)} - p_{*t}^{(2)}\|_{L^2(L^\infty)} \\
&\lesssim \|\Delta p_{*t}^{(1)}\|_{L^2(L^2)}\|p_{*tt}^{(1)} - p_{*tt}^{(2)}\|_{L^\infty(L^2)} \\
&\quad + \|p_{*tt}^{(2)}\|_{L^\infty(L^2)}\|\Delta(p_{*t}^{(1)} - p_{*t}^{(2)})\|_{L^2(L^2)} \\
&\lesssim R_1\|p_{*t}^{(1)} - p_{*t}^{(2)}\|_{X_p}.
\end{aligned} \tag{4.27}$$

Plugging (4.26) and (4.27) into (4.25), we obtain

$$\|\overline{f}_2\|_{H^1(L^2)} \lesssim R_1\|(\overline{p}_*, \overline{\Theta}_*)\|_{X_T}. \tag{4.28}$$

The energy bound for the difference equations. Now we can apply the energy results of proposition 3.1 to system (4.8) by setting

$$\alpha = 1 - 2k(\Theta_*^{(1)})p_*^{(1)}, \qquad r = q(\Theta_*^{(1)}), \qquad f_1 = \overline{f}_1, \qquad f_2 = \overline{f}_2.$$

Adding the energy estimate for the pressure to the energy bound (3.22) for the temperature (where now $\tilde{f} = f_2 = \overline{f}_2$), we obtain

$$\begin{aligned}
\|(\overline{p}, \overline{\Theta})\|_{X_T}^2 &= \|\mathcal{T}(p_*^{(1)}, \Theta_*^{(1)}) - \mathcal{T}(p_*^{(2)}, \Theta_*^{(2)})\|_{X_T}^2 \\
&\lesssim \int_0^t \exp\left(\int_s^t (1 + \Lambda(\sigma))\mathrm{d}\sigma\right)\left(\|\overline{f}_1(t)\|_{H^1}^2 + (1 + \|\nabla\alpha(t)\|_{L^3}^2)\|\partial_t\overline{f}_1(t)\|_{H^{-1}}^2\right)\mathrm{d}s + \|\partial_t\overline{f}_2\|_{L^2(L^2)}^2
\end{aligned}$$

with $\Lambda = \Lambda(t)$ defined in (3.5). We have

$$\begin{aligned}
\|\Lambda\|_{L^1(0,T)} &\lesssim \|r_t\|_{L^1(L^2)} + \|\alpha_t\|_{L^1(L^2)} + \|\nabla r\|_{L^1(L^4)} + \|\alpha_t\|_{L^2(L^4)}^2 + \|r_t\|_{L^2(L^4)}^2 \\
&\lesssim \|q'(\Theta_*^{(1)})\Theta_{*t}^{(1)}\|_{L^1(L^2)} + \|k'(\Theta_*^{(1)})\Theta_{*t}^{(1)}p_*^{(1)}\|_{L^1(L^2)} + \|k(\Theta_*^{(1)})p_{*t}^{(1)}\|_{L^1(L^2)} \\
&\quad + \|q'(\Theta_*^{(1)})\nabla\Theta_*^{(1)}\|_{L^1(L^4)} + \|k'(\Theta_*^{(1)})\Theta_{*t}^{(1)}p_*^{(1)}\|_{L^2(L^4)}^2 \\
&\quad + \|k(\Theta_*^{(1)})p_{*t}^{(1)}\|_{L^2(L^4)}^2 + \|q'(\Theta_*^{(1)})\Theta_{*t}^{(1)}\|_{L^2(L^4)}^2.
\end{aligned} \tag{4.29}$$

We estimate the terms on the right-hand side of (4.29) as follows: using (2.2), we have

$$\begin{aligned}\|q'(\Theta_*^{(1)})\Theta_{*t}^{(1)}\|_{L^1(L^2)} &\lesssim \sqrt{T}\|q'(\Theta_*^{(1)})\|_{L^\infty(L^\infty)}\|\Theta_{*t}^{(1)}\|_{L^2(L^2)}\\ &\lesssim \sqrt{T}(1+\|\Theta_*^{(1)}\|_{L^\infty(L^\infty)}^{\gamma_1+1})\|\Theta_{*t}^{(1)}\|_{L^2(L^2)}\\ &\lesssim \sqrt{T}R_2(1+R_2^{\gamma_1+1}).\end{aligned}$$

Further, by using assumption (2.4) we have

$$\begin{aligned}\|k'(\Theta_*^{(1)})\Theta_{*t}^{(1)}p_*^{(1)}\|_{L^1(L^2)} &\lesssim \|k'(\Theta_*^{(1)})\|_{L^\infty(L^\infty)}\|\Theta_{*t}^{(1)}\|_{L^2(L^2)}\|p_*^{(1)}\|_{L^2(L^\infty)}\\ &\lesssim \sqrt{T}(1+\|\Theta_*^{(1)}\|_{L^\infty(L^\infty)}^{\gamma_2+1})\|\Theta_{*t}^{(1)}\|_{L^2(L^2)}\\ &\quad\times\|\Delta p_*^{(1)}\|_{L^\infty(L^2)}\\ &\lesssim \sqrt{T}R_1(R_2+R_2^{\gamma_2+2}).\end{aligned}$$

Next we find that

$$\|k(\Theta_*^{(1)})p_{*t}^{(1)}\|_{L^1(L^2)} \lesssim T\|k(\Theta_*^{(1)})\|_{L^\infty(L^\infty)}\|p_{*t}^{(1)}\|_{L^\infty(L^2)} \lesssim TR_1,$$

where we have used (2.3) in the last estimate. Using the bound $\|\nabla\Theta_*^{(1)}\|_{L^4} \lesssim \|\Theta_*^{(1)}\|_{H^2_\diamondsuit(\Omega)}$, we also have

$$\begin{aligned}\|q'(\Theta_*^{(1)})\nabla\Theta_*^{(1)}\|_{L^1(L^4)} &\lesssim \|q'(\Theta_*^{(1)})\|_{L^\infty(L^\infty)}\|\nabla\Theta_*^{(1)}\|_{L^1(L^4)}\\ &\lesssim T(1+\|\Theta_*^{(1)}\|_{L^\infty(L^\infty)}^{\gamma_1+1})\|\nabla\Theta_*^{(1)}\|_{L^\infty(H^2_\diamondsuit(\Omega))}\\ &\lesssim T(R_2+R_2^{\gamma_1+2}).\end{aligned}$$

Also, we have as above

$$\begin{aligned}\|k'(\Theta_*^{(1)})\Theta_{*t}^{(1)}p_*^{(1)}\|_{L^2(L^4)}^2 &\lesssim \|k'(\Theta_*^{(1)})\|_{L^\infty(L^\infty)}^2\|\Theta_{*t}^{(1)}\|_{L^2(L^4)}^2\|p_*^{(1)}\|_{L^\infty(L^\infty)}^2\\ &\lesssim (1+\|\Theta_*^{(1)}\|_{L^\infty(L^\infty)}^{2\gamma_2+2})\|\Theta_{*t}^{(1)}\|_{L^2(H^1)}^2\|\Delta p_*^{(1)}\|_{L^\infty(L^2)}^2\\ &\lesssim R_1^2R_2^2(1+R_2^{2\gamma_2+2}).\end{aligned}$$

Further, we have the estimate

$$\|k(\Theta_*^{(1)})p_{*t}^{(1)}\|_{L^2(L^4)}^2 \lesssim \|p_{*t}^{(1)}\|_{L^2(L^4)}^2 \lesssim \|\nabla p_{*t}^{(1)}\|_{L^2(L^2)}^2 \lesssim R_1^2.$$

Finally, we have

$$\begin{aligned}\|q'(\Theta_*^{(1)})\Theta_{*t}^{(1)}\|_{L^2(L^4)}^2 &\lesssim \|q'(\Theta_*^{(1)})\|_{L^\infty(L^\infty)}^2\|\Theta_{*t}^{(1)}\|_{L^2(L^4)}^2\\ &\lesssim (1+\|\Theta_*^{(1)}\|_{L^\infty(L^\infty)}^{2\gamma_1+2})\|\Theta_{*t}^{(1)}\|_{L^2(H^1)}^2\\ &\lesssim R_2^2(1+R_2^{2\gamma_1+2}).\end{aligned}$$

Collecting the above estimates leads to

$$\|\Lambda\|_{L^1(0,T)} \leqslant C(T,R_1,R_2),$$

where $C=C(T,R_1,R_2)$ is a positive constant that depends on T, R_1, and R_2.

Finally, taking into account (4.29) and recalling (4.16), (4.24), (4.26) and (4.28) we obtain

$$\|\mathcal{T}(p_*^{(1)},\Theta_*^{(1)}) - \mathcal{T}(p_*^{(2)},\Theta_*^{(2)})\|_{X_T} \lesssim \mathrm{e}^{C(T,R_1,R_2)}(R_1+R_1^2)C(T,R_2)\|(p_*^{(1)}-p_*^{(2)},\Theta_*^{(1)}-\Theta_*^{(2)})\|_{X_T}.$$

Thus, by selecting the radius $R_1 > 0$ sufficiently small, we can guarantee that $\mathcal{T}$ is a strict contraction in B. □

On account of lemmas 4.1 and 4.2, an application of the contraction mapping theorem implies that there exists a unique $(p,\Theta) = \mathcal{T}(p,\Theta)$ in B which solves the coupled problem.

Continuous dependence on the data. To prove continuous dependence on the data, take $(p^{(1)},\Theta^{(1)})$ and $(p^{(2)},\Theta^{(2)})$ to be two solutions of (2.1a) that correspond to the initial data $(p_0^{(1)},p_1^{(1)},\Theta_0^{(1)})$ and $(p_0^{(2)},p_1^{(2)},\Theta_0^{(2)})$, respectively. Similarly to the proof of contractivity, we have the following energy bound:

$$\begin{aligned}\|(p^{(1)}-p^{(2)},\Theta^{(1)}-\Theta^{(2)})\|_{X_T}^2 \lesssim\ & \mathcal{E}[p^{(1)}-p^{(2)}](0) + \mathcal{E}[\Theta^{(1)}-\Theta^{(2)}](0)\\ &+ \int_0^t \exp\left(\int_s^t (1+\Lambda(\sigma))\mathrm{d}\sigma\right)\Big(\|\overline{f}_1(t)\|_{H^1}^2 + (1+\|\nabla\alpha(t)\|_{L^3}^2)\\ &\times \|\partial_t\overline{f}_1(t)\|_{H^{-1}}^2\Big)\mathrm{d}s + C_T\|\partial_t\overline{f}_2\|_{L^2(L^2)}^2.\end{aligned}$$

Here $\overline{f}_1$ and $\overline{f}_2$ are functions of $p^{(1)} = p_*^{(1)}$ and $p^{(2)} = p_*^{(2)}$; see (4.9) and (4.10) for their definitions. Following the same steps as in the proof of contractivity, we can deduce that there exists a function Ψ that depends on $\|p^{(j)}\|_{X_p}$ and $\|\Theta^{(j)}\|_{X_p}$ with $j = 1, 2$, such that

$$\begin{aligned}\|(p^{(1)}-p^{(2)},\Theta^{(1)}-\Theta^{(2)})\|_{X_T}^2 \lesssim\ & \|(p_0^{(1)}-p_0^{(2)},\Theta_0^{(1)}-\Theta_0^{(2)})\|_{X_T}^2 + \int_0^t \Psi\big(\|p^{(1)}\|_{X_p},\|p^{(2)}\|_{X_p},\|\Theta^{(1)}\|_{X_\Theta},\\ &\times \|\Theta^{(2)}\|_{X_\Theta}\big)\|(p^{(1)}-p^{(2)},\Theta^{(1)}-\Theta^{(2)})\|_{X_T}^2\mathrm{d}s.\end{aligned}$$

An application of Gronwall's inequality leads to

$$\|(p^{(1)}-p^{(2)},\Theta^{(1)}-\Theta^{(2)})\|_{X_T}^2 \lesssim \|(p_0^{(1)}-p_0^{(2)},\Theta_0^{(1)}-\Theta_0^{(2)})\|_{X_T}^2 \exp\left\{\int_0^T \Psi(t)\mathrm{d}t\right\}.$$

This last inequality yields the desired result, which also implies uniqueness in X_T by taking the data to be the same. □

5. Conclusion and outlook

In this work, we have analysed the coupled Westervelt–Pennes model of HIFU-induced heating. By relying on the energy analysis of a linearised problem and a subsequent fixed-point argument, we proved the local-in-time well-posedness of this model under the assumption of smooth and (with respect to pressure) small data. Physically, the results imply that, for a given final propagation time, if the initial acoustic pressure is chosen to be small enough in the sense of (4.2), one can guarantee that a unique (and smooth) pressure-temperature field exists up to this time. The smallness condition imposed on the data is in practice mitigated by the fact that the weighting factor $k(\Theta)$ in the involved nonlinearities is quite small as it is proportional to the inverse of speed of sound squared (1.4). In addition to establishing sufficient conditions for the validity of the Westervelt–Pennes model, the present theoretical work also provides a rigorous foundation for devising accurate and reliable numerical simulation strategies for the models of HIFU-induced heating. These can help practitioners set up lab experiments for HIFU treatments and reduce the need to repeat them unnecessarily.

We note that in the energy estimates in section 3, b must be a positive constant, independent of Θ. To permit more realistic modelling scenarios in complex propagation media, future analysis will involve studying the case $b = b(\Theta)$ together with allowing for (time- or space-) fractional damping in the model.

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