The Detour Monophonic Convexity Number of a Graph

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Abstract

A set S is detour monophonic convexif$J_{dm}[S\} = S$. The detour monophonic convexity number is denoted by $C_{dm}(G)$, is the cardinality of a maximum proper detour monophonic convex subset of V.Some general properties satisfied by this concept are studied. The detour monophonic convexity number of certain classes of graphs are determined. It is shown that for every pair of integers a and b with $3 \leq a < b,$ there exists a connected graph G such that $C_m(G) = a$ and $C_{dm}(G) = 2(b+1)$, where$C_m(G)$ is the monophonic convexity number of G.

Keywords: convex, detour, chord, detour monophonic path, monophonic convexity number, detour monophonic, convexity number.

AMS subject classification: 05C12,05C38§.

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§ Received on June 12 th, 2022. Accepted on Sep 1st, 2022. Published on Nov 30th, 2022. doi: 10.23755/rm.v44i0.918. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to [1]. A vertex v is adjacent to another vertex u if and only if there exists an edge $e = uv \in E(G)$. If $uv \in E(G)$, we say that u is a *neighbor* of v and denote by $N_G(v)$, the set of neighbors of v. A vertex v is said to be *universal vertex* if $\deg_G(v) = p - 1$. A vertex v is called an *extreme vertex* if the subgraph induced by v iscomplete.

The length of a path is the number of its edges. Let u and v be vertices of a connected graph G. A shortest u-v path is also called a u-vgeodesic. The (shortest path) distance is defined as the length of a u-v geodesic in G and is denoted by$d_G(u, v)$ or $d(u, v)$ for short if the graph is clear from the context. For a set S of vertices, let $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set $S \subset V$ is called a *convex set* of G if $I[S] = S$. These concepts were studied in [1, 3]

A chord of a path P is an edge which connects two non-adjacent vertices of P. A u-v path is called a monophonic path if it is a chordless path. For two vertices u and v, the closed interval $\mathrm{J[u, v]}$ consists of all the vertices lying in a $\mathrm{u - v}$ monophonic path including the vertices u and v. If u and v are adjacent, then $\mathrm{J[u, v] = \{u, v\}}$. For a set M of vertices, let $\mathrm{J[M] = \cup_{u,v \in M} J[u, v]}$. Then certainly $\mathrm{M \subseteq J[M]}$. A set $\mathrm{M \subseteq V(G)}$ is called a *monophonic se*t of G if $J[M] = V$. The monophonic number $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is called a m-set of G. A set $M \subseteq V(G)$ is called a monophonic convex set of G if $J(M) = M$. The monophonic convexity number $C_m(G)$ of G is the cardinality of a maximum proper monophonic convex subset of V. These concepts were studied in [5-10].

The detour distance $D(u, v)$ between two vertices u and v in a connected graph G from u to v is defined as the length of a longest u-v path in G. An u-v path of length $D(u, v)$ is called an u-v detour. The detour monophonic distance dm(u,v) between two vertices u and v is the length of a longest u-v monophonic path in G, Any monophonic path of length dm(u,v) is called u-v detour monophonic path. For two vertices $u, v \in V$, let $J_{dm}[u, v]$ denotes the set of all vertices that lies in u-$v$detour monophonic path including u and v, and $J_{dm}(u, v)$denotes the set of all internal vertices that lies in $u - v$detour monophonic path . For $M \subseteq V$, let $J_{dm}[M] = \bigcup_{u,v \in M} J_{dm}[u, v]$.A set $M \subseteq V$ is a *detour monophonic set* if $J_{dm}[M] = V$.The minimum cardinality of a detour monophonic set of G is the *detour monophonic number* of G and is denoted by $dm(G)$. The detour monophonic set of cardinality $dm(G)$ is called dm-set. These concepts were studied in [2, 4, 11].

2.The detour monophonic convexity number of a Graph

Definition 2.1. A set S is *detour monophonic convex* if$J_{dm}[S\} = S$. Clearly $S = \{v\}$ or $S = V$ then S is detour monophonic convex. The detour monophonic convexity number is denoted by $C_{dm}(G)$, is the cardinality of a maximum proper detour mono-phonic convex subset of V.

Example 2.2. For the graph Gin Figure 2.1, $M_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$is a C_{dm}-set of G so that $C_{dm}(G)$ = 8. Also $M_2 = \{v_1, v_2\}$is a C_m-set of G so that $C_m(G) = 2$.

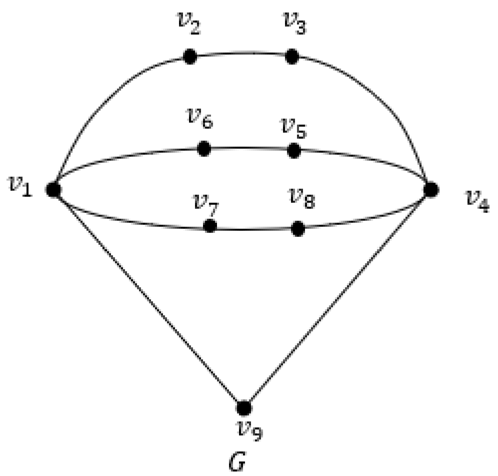

Figure 2.1

Observation 2.3. Let G be a connected graph of order $p \geq 3$. Then $2 \leq C_{dm}(G) \leq p - 1$.

Theorem 2.4. Let G be a connected graph of order p and G contains an extreme vertex. Then $C_{dm}(G) = p - 1$.
Proof. Let G contain an extreme vertex, say v. Then $S = V(G) - \{v\}$ is a dm-convex set of G so that $C_{dm}(G) = p - 1$.

Theorem 2.5. Let G be a connected graph of order $p \geq 3$. Then$2 \leq \omega(G) \leq C_{dm}(G) \leq p - 1$, where $\omega(G)$ is the clique number of G.
Proof. Since G is a connected graph of order $p \geq 3, \omega(G) \geq 2$. Let H be a subgraph of G such that $\langle V(H) \rangle$ is a maximal complete subgraph of G so that $C_{dm}(G) \geq |v(H)| = \omega(G)$. Let S be a dm-convex set of G. Then S is a convex set of G so that $C_{dm}(G) \leq C(G)$. Since every convex set of G is a proper subset of $G, C(G) \leq p - 1$. Therefore $2 \leq \omega(G) \leq C_{dm}(G) \leq p - 1$..

Corollary 2.6. (i) For the complete graph $G = K_p$ $(p \geq 3)$, $C_{dm}(G) = p - 1$.
(ii) For a trivial tree G of order $p \geq 3$, $C_{dm}(G) = p - 1$.
(iii) For the fan graph $G = K_1 + P_{p-1}$ $(p \geq 4)$, $C_{dm}(G) = p - 1$.

Theorem 2.7. For the cycle $G = C_p, (p \geq 3), C_{dm}(G) = 2$.

Proof. Let$S = \{x, y\}$be a set of two adjacent vertices of G. Then $J_{dm}[S] = S$, it follows that S is a dm-convex set of G so that $C_{dm}(G) \geq 2$. We prove that $C_{dm}(G) = 2$. Suppose that $C_{dm}(G) \geq 3$. Then there exists a dm-convex set S_1 such that $|S_1| \geq 3$. Hence it follows that S_1 contains two independent vertices of G. Then $J_{dm}[S_1] \neq S_1$. Therefore $C_{dm}(G) = 2$.■

Theorem 2.8. For the complete bipartite graph $G = K_{m,n}, C_{dm}(G) = 2$.
Proof: Let (V_1, V_2) be a partition of G. Since $\omega(G) = 2, C_{dm}(G) \geq 2$. We prove that $C_{dm}(G) = 2$. Suppose that $C_{dm}(G) \geq 3$. Then there exists two vertices x and y belong to the same partite V_1 (or V_2). Since $d(x, y) = 2$ in G, every vertex in V_1 (or V_2) lie on $x - y$ detour monophonic. Hence it follows that $J_{dm}[S] \neq S$. Therefore $C_{dm}(G) = 2$.
■

Theorem 2.9. For the wheel graph G= $W_p = K_1 + C_{p-1}(p \geq 4), C_{dm}(G) = 3$.
Proof. Let$V(K_1) = x$and$V(C_{p-1}) = \{v_1, v_2, ... v_{p-1}\}$. Then $S = \{x, v_1, v_2\}$ is a detour monophonic convex set of G so that $C_{dm}(G) \geq 3$.We prove that $C_{dm}(G) = 3$. Suppose that $C_{dm}(G) \geq 4$. Then there exists dm-convex set S_1 such that $|S_1| \geq 4$. Hence it follows that S_1 contains two independent vertices of G. Then $J_{dm}[S_1] \neq S_1$. Therefore$C_{dm}(G) = 3$.
■

Theorem 2.10. For any two positive integers such that $2 \leq a \leq b$, there exists a connected graph G such that $\omega(G) = a$ and $C_{dm}(G) = b$.
Proof. For $a = b$,let $G = K_{a+1} - \{e\}$. Then $\omega(G) = C_{dm}(G) = a$.For $a < b$, let K_a be the complete graph with vertices $v_1, v_2, \dots, v_a$. Let $P: u_1, u_2, \dots, u_{b-a}, u_{b-a+1}, \dots, u_c$ where $c > b - a$ a path on c vertices. Let G be the graph obtained from K_a and P by joining u_1 with $v_{a-1}$and v_a each$u_i (2 \leq i \leq b - a)$ with v_{a-1} and u_c with v_{a-1}. The graph G is shown in Figure 2.2.
First, we prove that $\omega(G) = a$. Let $S = \{v_1, v_2, \dots, v_a\}$. It is clear that S is a maximal complete subgraph of G such that $\omega(G) = a$.
Next, we prove that $C_{dm}(G) = b$. Let $W = \{v_1, v_2, \dots, v_a, u_1, u_2, \dots, u_{b-a}\}$. It is clear that W is a dm-convex set of G so that$C_{dm}(G) \geq G$. We prove that $C_{dm}(G) = b$. Suppose that $C_{dm}(G) > b$. Let S_1 be a dm-convex set with $|S_1| \geq b + 1$.Then there exists a vertex$u_i (b - a + 1 \leq i \leq c)$such that $u_i \in S_1$. Then $J_{dm}[S_1] \neq S_1$.Therefore$C_{dm}(G) = b$. ■

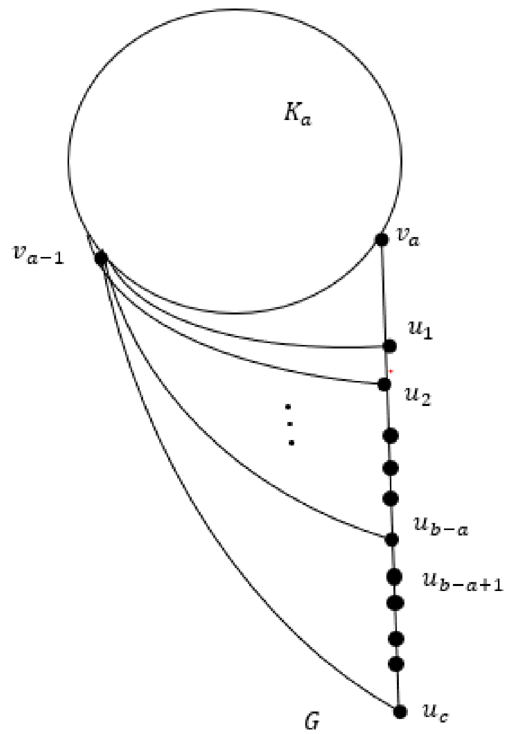

Figure 2.2

Theorem 2.11. For every pair of integers a and b with $3 \leq a < b$, there exists a connected graph G such that $C_m(G) = a$ and $C_{dm}(G) = 2(b+1)$.

Proof. Let $V(\overline{K}_2) = \{x, y\}$. Let $P_i: u_i, v_i (1 \leq i \leq b)$be a copy of path of order two. Let G be the graph obtained from $\overline{K}_2, P_i(1 \leq i \leq b)$ and K_{a-1} by joining x with each $u_i(1 \leq i \leq b)$ and y with each $v_i(1 \leq i \leq b)$and x and y with each vertex of K_a. The graph G is shown in Figure 2.3.

First, we prove that $C_m(G) = a$. Let $M = V(K_a) \cup \{x\}$. Then M is a monophonic convex set of G and so $C_m(G) \geq a$. We prove that $C_m(G) = a$. Suppose that $C_m(G) \geq a + 1$. Let M_1 be m-convex set with $|S| \geq a + 1$. Then there exists at least one vertex, say x such that $x \in M_1$ and $x \notin M$. Hence it follows that $x = u_i$ or v_i or y for some i $(1 \leq i \leq b)$. Then $J_m[M_1] \neq M_1$, which is a contradiction. Therefore $C_m(G) = a$.

Next we prove that $C_{dm}(G) = 2(b+1)$. Let $S = V(G) - V(K_a)$. Then S is a detour monophonic convex set of G and so $C_{dm}(G) \geq 2(b+1)$. We prove that $C_{dm}(G) = 2(b+1)$. On the contrary $C_{dm}(G) > 2(b+1)$. Let S_1 be a dm-convex set with $|S_1| \geq 2(b+1)+1$. Then there exists a vertex $x \in S_1$ such that $x \notin S$. Hence it follows that $x \in K_a$. Then $J_{dm}[S_1] \neq S_1$. Therefore $C_{dm}(G) = 2(b+1)$.■

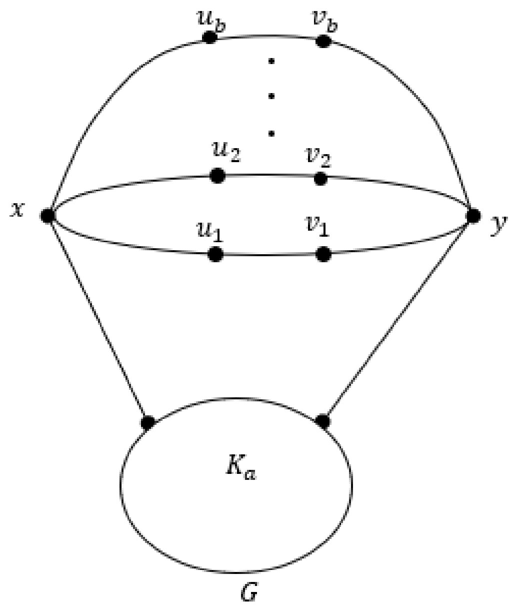

Figure 2.3

3. Conclusions

In this paper, we investigated the detour monophonic convexity number of some standard graphs. Also, we proved for every pair of integers a and b with $3 \leq a < b$, there exists a connected graph G such that $C_m(G) = a$ and $C_{dm}(G) = 2(b+1)$.

Acknowledgements

The author would like to express her gratitude to the referees for their valuable comments and suggestions.

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