On the spectral characterizations of 3-rose graphs

Jianfeng Wang[a,b], Qiongxiang Huang[a], Francesco Belardo[c,*]

[a]Department of Mathematics and Information Science, Qinghai Normal University, Xining, Qinghai 810008, P.R. China

[b]College of Mathematics and System Science, Xinjiang University, Urumqi 830046, P.R. China

[c]Department of Mathematics, University of Messina, 98166 Sant'Agata, Messina, Italy

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Abstract

A rose graph with p petals (or p-rose graph) is a graph obtained by taking p cycles with just a vertex in common. In this paper, we prove that all 3-rose graphs, having at least one triangle, are determined by their Laplacian spectra and all 3-rose graphs are determined by their signless Laplacian spectra.

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1 Introduction

All graphs considered here are undirected and simple (i.e., loops and multiple edges are not allowed). Let $G = G(V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n(G) = n$ and $|E(G)| = m(G) = m$. For a graph G, let $M = M(G)$ be a corresponding *graph matrix* defined in a prescribed way. The M-*polynomial* of G is defined as $\det(\lambda I - M)$, where I is the identity matrix. The M-*eigenvalues*

*Corresponding author.

Email addresses: jfwang4@yahoo.com.cn (J. Wang), huangqx@xju.edu.cn(Q. Huang), fbelardo@gmail.com(F. Belardo)

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of G are the roots of its M-polynomial. The *M-spectrum* of G, denoted by $\mathrm{Spec}_M(G)$, is the multiset consisting of the M-eigenvalues. The *M-spectral radius* (or *M-index*) of G is the largest M-eigenvalue of G. It is well-known that there are several graph matrices including *adjacency matrix* $A = A(G)$, *Laplacian matrix* $L = L(G)$, *signless Laplacian matrix* $Q = Q(G)$, etc.(for general remarks on this topic see [4, 5, 6]).

We now introduce the standard terminology of spectral graph theory (see [1]). Graphs with the same spectrum of a graph matrix M are called *M-cospectral graphs* or *M-cospectral mates*. A graph G is said to be *determined by its M-spectrum* (or *G is a* DMS-*graph* for short) if there is no other non-isomorphic graph with the same spectrum, that is, $\mathrm{Spec}_M(H) = \mathrm{Spec}_M(G)$ implies $H \cong G$ for any graph H. Clearly two M-cospectral graphs share the same M-polynomial.

In the last few years, many researchers tried to determine the graphs that are DMS-graphs, specially when M is one among the matrices $A(G)$, $L(G)$ and $Q(G)$. While the general problem was firstly posed in Chemistry about 50 years ago (see [11]), only recently the mathematicians devoted their attention to such a problem. For general results and remarks on this topic we refer the readers to see the excellent surveys [7, 8]. Some very recent contributions are the papers [10, 14, 18, 19, 20]. In [21] the authors studied the spectral characterization of the ∞-graphs, that have, in fact, a similar but simpler structure with respect to the graphs considered in this paper.

Recall that the signless Laplacian matrix $Q(G) = D(G) + A(G)$, and the Laplacian matrix $L(G) = D(G) - A(G)$, where $D(G) = \mathrm{diag}(d_1, d_2, \cdots, d_n)$ with $d_i = d(v_i) = d_G(v_i)$ being the degree of vertex v_i of G $(1 \leq i \leq n)$ and $d_1 \geq d_2 \geq \cdots \geq d_n$. The A-polynomial and the A-spectrum of a graph G are, respectively, denoted by $\phi(G) = \phi(G, \lambda)$ and $\mathrm{Spec}_A(G) = \{\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G)\}$ with a non-increasing order. Let us also denote by $\psi(G)$ ($\varphi(G)$) the L-polynomial (resp. Q-polynomial) of the graph G.

As usual, let P_n and C_n be respectively the path and the cycle on n vertices. Let $S(G)$, $n_G(H)$, $\omega(G)$, $\tau(G)$ and $\deg(G) = (d_1, d_2, \cdots, d_n)$ be the subdivision, the number of subgraphs (not necessarily induced) isomorphic to H in G, the number of connected component, the number of spanning trees, and the degree sequence of G, respectively. $G \cup H$ stands for the disjoint union of two graphs G and H. A connected graph G is said to be a *k-cyclic graph* if $m(G) = n(G) + k - 1$, if $k = 0$, then the graph is a tree, if $k = 1$ then the graph is unicyclic, if $k = 2$ it is bicyclic and so on.

In this paper we will consider a special type of connected graphs called rose graphs. The *rose graph* with *p petals* (or *p-rose graph* for short), denoted by $R_{a_1, a_2, \cdots, a_p}$, is the graph obtained by taking p cycles of order $a_1, a_2, \cdots, a_p$ with just a vertex in common (see Fig. 1). Clearly, the fa-

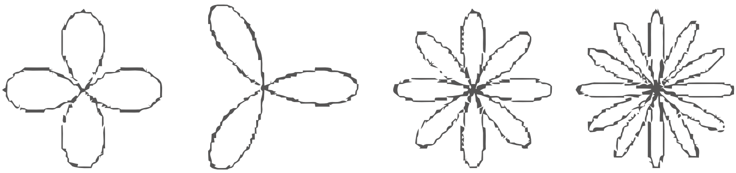

Figure 1: Rose graphs with four, three, eight, twelve petals.

mous graph known as the *Friendship graph* is a special rose graph whose petals are triangles C_3 [9]. Indeed, the name "rose graph" is derived from the *rose curve* (see [22], for example) with polar equation $\rho = a\cos(b\theta)$, where b is a positive integer. The graphs in Figure 1 are, respectively, obtained from $b = 2, 3, 4, 6$. For convenience, let $R_{r,s,t}$ denote the rose graph with three petals of size r, s and t.

The paper is organized as follows. In Section 2 we give some useful results used in the rest of the paper. In Section 3 we recall a recent method to compute the degree sequence of $\{L, Q\}$-cospectral mates of a graph. In Section 4 we show that a 3-rose graph with at least one triangle is a DLS-graph. In Section 5 we show that all 3-rose graphs are DQS-graphs. Finally, in Section 6 we give some concluding remarks.

2 Basic tools

Let G and H be two graphs. A property is called an *M-invariant for M-cospectral graphs* if $\mathrm{Spec}_M(G) = \mathrm{Spec}_M(H)$ implies that G and H shares that property. In order to study the M-spectral characterization of graphs, it is necessary to determine as much as possible M-invariants.

In the following, the L-polynomial and the Q-polynomial of a graph G are respectively reported as:

$$\psi(G) = \det(\lambda I - L(G)) = q_0(G)\lambda^n + q_1(G)\lambda^{n-1} + \cdots + q_{n-1}(G)\lambda + q_n(G)$$

and

$$\varphi(G) = \det(\lambda I - Q(G)) = p_0(G)\lambda^n + p_1(G)\lambda^{n-1} + \cdots + p_{n-1}(G)\lambda + p_n(G).$$

Cvetković et al. [3] showed that $p_0(G) = 1$, $p_1(G) = -2m$ and $p_2(G) = a + \frac{3}{2}m(m-1)$, where a is the number of pairs of non-adjacent edges in G. It is easy to see that $a = \binom{m}{2} - \sum_{i=1}^{n}\binom{d_i}{2}$, then $p_2(G) = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2$.

On the other hand, Oliveira et al. [13] determined the first four coefficients of $\psi(G)$. The following lemma follows by combining their results:

Lemma 2.1. *Let G be a graph with $\deg(G) = (d_1, d_2, \cdots, d_n)$, order n and size m. Then*

(i) $p_0(G) = q_0(G) = 1$, $p_1(G) = q_1(G) = -2m$, $p_2(G) = q_2(G) = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2$.

(ii) $q_3(G) = \frac{1}{3}\Big(-4m^3 + 6m^2 + 3m\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 - 3\sum_{i=1}^{n} d_i^2 + 6n_G(C_3)\Big)$.

Cvetković et al. [3] defined the *semi-edge walks* of a graph G in the following way: a semi-edge walk (of length k) in an (undirected) graph G is a sequence $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ of vertices v_1, v_2, ... , v_{k+1} and edges e_1, e_2, ... , e_k such that for any $i = 1, 2, \ldots, k$ the vertices v_i and v_{i+1} are end-vertices (not necessarily distinct) of the edge e_i. They also defined the *k-th spectral moment* $T_k(G) = \sum_{i=1}^{n} \eta_i^k$ $(k = 0, 1, 2, \cdots)$ for $\mathrm{Spec}_Q(G) = \{\eta_1, \eta_2, \cdots, \eta_n\}$. Clearly, $T_k(G) = \mathrm{tr}(Q^k)$. They proved the following result:

Lemma 2.2. *Let G be a graph with $\deg(G) = (d_1, d_2, \cdots, d_n)$, order n and size m. Then*

(i) *The k-th spectral moment T_k is equal to the number of closed semi-edge walks of length k;*

(ii) $T_0 = n$, $T_1 = \sum_{i=1}^{n} d_i = 2m$, $T_2 = 2m + \sum_{i=1}^{n} d_i^2$, $T_3 = 6n_G(C_3) + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3$;

(iii) *If H is a Q-cospectral mate of G, $T_k(G) = T_k(H)$, for any k.*

van Dam and Haemers in [7] surveyed several invariants for cospectral graphs. By the results given in [7] and by the previous lemmas, we have the following result.

Lemma 2.3. *Let G and H be two graphs.*

(i) *If the two graphs are L-cospectral, then*

$n(G) = n(H)$, $m(G) = m(H)$, $\omega(G) = \omega(H)$, $\tau(G) = \tau(H)$, $\sum_{i=1}^{n} d_G(v_i)^2 = \sum_{i=1}^{n} d_H(v_i)^2$, $6n_G(C_3) - \sum_{i=1}^{n} d_G(v_i)^3 = 6n_H(C_3) - \sum_{i=1}^{n} d_H(v_i)^3$.

(ii) *If the two graphs are Q-cospectral, then*

$n(G) = n(H)$, $m(G) = m(H)$, $\sum_{i=1}^{n} d_G(v_i)^2 = \sum_{i=1}^{n} d_H(v_i)^2$, $6n_G(C_3) + \sum_{i=1}^{n} d_G(v_i)^3 = 6n_H(C_3) + \sum_{i=1}^{n} d_H(v_i)^3$.

Remark 2.1. *By Lemma 2.3, the sum of the squares of vertex degrees is a L-invariant and a Q-invariant for cospectral graphs.*

We now consider the relation between the Q-polynomial of a graph G and the A-polynomial of its subdivision graph $S(G)$. The following lemma can be found in many references, see [2, 17] for example.

Lemma 2.4. *Let G be a graph of order n and size m, and $S(G)$ be the subdivision graph of G. Then*

$$\phi(S(G), \lambda) = \lambda^{m-n}\varphi(G, \lambda^2).$$

The following lemma was given in [21] (see also [5]).

Lemma 2.5. *Let G and H be two graphs. Then G and H are Q-cospectral if and only if $S(G)$ and $S(H)$ are A-cospectral.*

To conclude this section, we give a lower and upper bound for the Q-index of p-rose graphs. The following two lemmas are respectively cited in [16] and [17]. Note that in Lemma 2.6, the upper bound is different from the one in [16] due to a typo (the number $\frac{1}{2}$ has been replaced by a square root).

Lemma 2.6. *Let R be a p-rose graph. Then*

$$\frac{2p}{\sqrt{2p-1}} < \lambda_1(R) \leq \frac{\sqrt{8p+1}+1}{2},$$

where the lower bound is the best possible (it is the limit point for the A-index when the lengths of all cycles tend to infinity), while the upper bound is attained with the friendship graph.

Before stating the analogous result with respect to the Q-index, we need an additional result. The well-known Hoffman-Smith lemma about *internal paths* (see [12]) has been proved with respect to the Q-index. Recall, an *internal path* of a graph G is a walk $v_0, v_1, \cdots, v_k$ ($k \geq 1$) where the vertices $v_1, \cdots, v_k$ are distinct (v_0, v_k need not be distinct), $d(v_0) > 2$, $d(v_k) > 2$ and $d(v_i) = 2$ whenever $0 < i < k$.

Lemma 2.7 ([5, 17])**.** *Let uv be an edge of the connected graph G. Let G_{uv} be obtained from G by subdividing the edge uv of G and $\kappa(G)$ the Q-index of G.*

(i) *If $G = C_n$, then $\kappa(G_{uv}) = \kappa(G) = 4$.*

(ii) *If uv is not in an internal path of $G \neq C_n$, then $\kappa(G_{uv}) > \kappa(G)$.*

(iii) *If uv belongs to an internal path of G, then $\kappa(G_{uv}) < \kappa(G)$.*

Corollary 2.1. *Let R be a p-rose graph and $\kappa(R)$ its Q-index. Then*

$$\frac{4p^2}{2p-1} < \kappa(R) \leq \frac{3+2p+\sqrt{4p^2-4p+9}}{2},$$

where the lower bound is the best possible (it is the limit point for the Q-index), while the upper bound is attained with the friendship graph.

Proof. The left inequality follows from Lemma 2.4 and from the lower bound of Lemma 2.6 (note, the subdivision of R is still a p-rose graph). From Lemma 2.7(iii), we have that the upper bound is attained with the friendship graph. By Lemma 2.4 and the upper bound of Lemma 2.6, we have that the Q-index of the friendship graph is equal to the square of the A-index of its subdivision graph. Finally, by the eigenvalue equations, we obtain that the Q-index of the friendship graph is indeed $\frac{3+2p+\sqrt{4p^2-4p+9}}{2}$. □

3 Degree sequences of $\{L, Q\}$-cospectral mates to 3-rose graphs

Let $M = L$ or $M = Q$. The main result of this section is that the degree sequence of a tentative M-cospectral graph to $R_{r,s,t}$ is determined. The following lemma is proved in [21].

Lemma 3.1. *Let G be a graph with $\deg(G) = (d_1, d_2, \cdots, d_n)$, order n and size m. Let H be a graph such that $\mathrm{Spec}_M(H) = \mathrm{Spec}_M(G)$ and $\deg(H) = (d_1+t_1, d_2+t_2, \cdots, d_n+t_n)$. Then, for $i = 1, 2, \cdots, n$, t_i is an integer such that*

$$\sum_{i=1}^{n} t_i = 0 \quad \textit{and} \quad \sum_{i=1}^{n}(t_i^2 + 2d_it_i) = 0.$$

Lemma 3.2. *Let H be a graph M-cospectral to a 3-rose graph $R_{r,s,t}$ of order n. Then $\deg(H)$ belongs to the set*

$$\mathcal{S} = \bigcup_{i=1}^{9} \mathcal{S}_i,$$

where $\mathcal{S}_1 = \{(6, 2^{n-1})\}$, $\mathcal{S}_2 = \{(5, 4, 3, 2^{n-5}, 1^2)\}$, $\mathcal{S}_3 = \{(5, 3^4, 2^{n-8}, 1^3)\}$, $\mathcal{S}_4 = \{(5, 3^3, 2^{n-5}, 0), (4^3, 3, 2^{n-7}, 1^3)\}$, $\mathcal{S}_5 = \{(4^3, 2^{n-4}, 0), (4^2, 3^4, 2^{n-10}, 1^4)\}$, $\mathcal{S}_6 = \{(4^2, 3^3, 2^{n-7}, 1, 0), (4, 3^7, 2^{n-13}, 1^5)\}$, $\mathcal{S}_7 = \{(4, 3^6, 2^{n-10}, 1^2, 0), (3^{10}, 2^{n-16}, 1^6)\}$, $\mathcal{S}_8 = \{(3^9, 2^{n-13}, 1^3, 0)\}$ and $\mathcal{S}_9 = \{(3^8, 2^{n-10}, 0^2)\}$.

Proof. Note that $\deg(R_{r,s,t}) = (6, 2^{n-1})$ and set $\deg(H) = (6 + t_1, 2 + t_2, \cdots, 2 + t_n)$. Then $t_1 \geq -6$ and $t_i \geq -2$ $(2 \leq i \leq n)$. By Lemma 3.1 we have that t_i $(i = 1, 2, \cdots, n)$ is an integer such that

$$\sum_{i=1}^{n} t_i = 0 \quad and \quad \sum_{i=1}^{n} t_i^2 + 12t_1 + 4\sum_{i=2}^{n} t_i = 0. \tag{1}$$

Thus, the first equality of (1) leads to $t_1 = -\sum_{i=2}^{n} t_i$. By substituting the latter into the second equality of (1) we get

$$t_1^2 + 8t_1 + a = 0, \quad where\ a = \sum_{i=2}^{n} t_i^2 \geq 0.$$

Solving the above quadratic equation we obtain $t_1 = -4 \pm \sqrt{16 - a}$. Recall that t_1 is integer. Hence $0 \leq a \leq 16$ and $16 - a$ is a square number, which implies that $a = 0, 7, 12, 15, 16$. We will consider several cases depending on a.

If $a = 0$, then $t_1 = 0$. From $\sum_{i=2}^{n} t_i^2 = 0$ we get $t_i = 0$ $(2 \leq i \leq n)$ and so $\deg(H) = (6, 2^{n-1})$.

If $a = 7$, then $t_1 = -1$. From $\sum_{i=2}^{n} t_i^2 = 7$ and $\sum_{i=2}^{n} t_i = 1$, we get that one among $t_2, \ldots, t_n$ is 2, one is 1 and two are -1; or one of them is -2 and three of them are 1; or four of them are 1 and three of them are -1. Thus, we obtain $\deg(H)$ is $(5, 4, 3, 2^{n-5}, 1^2)$ or $(5, 3^3, 2^{n-5}, 0)$ or $(5, 3^4, 2^{n-8}, 1^3)$.

If $a = 12$, then $t_1 = -2$ or $t_1 = -6$.

For $t_1 = -2$, from $a = \sum_{i=2}^{n} t_i^2 = 12$ and $\sum_{i=2}^{n} t_i = 2$ we get that one among $t_2, \ldots, t_n$ is 3, one is 1 and two are -1. Consequently, the degree sequence is $(5, 4, 3, 2^{n-5}, 1^2)$. Other possibilities are that one of them is 2, four of them are 1 and four of them are -1 and so $(4^2, 3^4, 2^{n-10}, 1^4)$ appears; or two of them are 2, one of them is 1 and three of them are -1 and so $(4^3, 3, 2^{n-7}, 1^3)$ appears; or one of them is 2, one of them is -2, three of them are 1 and one of them is -1, so $(4^2, 3^3, 2^{n-7}, 1, 0)$ appears; or two of them are 2, one of them is -2 and thus $(4^3, 2^{n-4}, 0)$ appears; or one of them is -2, six of them are is 1 and two of them are -1 and thus $(4, 3^6, 2^{n-10}, 1^2, 0)$ appears; or seven of them are 1 and five of them are -1 and thus $(4, 3^7, 2^{n-13}, 1^5)$ appears.

For $t_1 = -6$, in the same way as above, we get $\deg(H)$ is $(5, 3^3, 2^{n-5}, 0)$ or $(4, 3^6, 2^{n-10}, 1^2, 0)$ or $(4^2, 3^3, 2^{n-7}, 1, 0)$ or $(4^3, 2^{n-4}, 0)$ or $(3^8, 2^{n-10}, 0^2)$ or $(3^9, 2^{n-13}, 1^3, 0)$.

If $a = 15$, then $t_1 = -3$ or $t_1 = -5$. Similarly, $\deg(H)$ is $(5, 3^4, 2^{n-8}, 1^3)$ or $(5, 3^3, 2^{n-5}, 0)$ or $(5, 4, 3, 2^{n-5}, 1^2)$ or $(4, 3^7, 2^{n-13}, 1^5)$ or $(4^2, 3^4, 2^{n-10}, 1^4)$

or $(4^2,3^3,2^{n-7},1,0)$ or $(4^3,3,2^{n-7},1^3)$ or $(4,3^6,2^{n-10},1^2,0)$ or $(3^{10},2^{n-16},1^6)$ or $(3^9,2^{n-13},1^3,0)$ or $(3^8,2^{n-10},0^2)$.

If $a=16$, then $t_1=-4$. Similarly, $\deg(H)$ is $(6,2^{n-1})$ or $(5,4,3,2^{n-5},1^2)$ or $(5,3^4,2^{n-8},1^3)$ or $(5,3^3,2^{n-5},0)$ or $(4^3,3,2^{n-7},1^3)$ or $(4^3,2^{n-4},0)$ or $(4^2,3^4,2^{n-10},1^4)$ or $(4^2,3^3,2^{n-7},1,0)$ or $(4,3^7,2^{n-13},1^5)$ or $(4,3^6,2^{n-10},1^2,0)$ or $(3^{10},2^{n-16},1^6)$ or $(3^9,2^{n-13},1^3,0)$ or $(3^8,2^{n-10},0^2)$. □

4 L-spectral characterization of 3-rose graphs

Lemma 4.1. *If a graph H is L-cospectral to a 3-rose graph $G=R_{r,s,t}$, then* $\deg(H)=(6,2^{n-1})$.

Proof. By Lemma 3.2 we have $\deg(H)\in\bigcup_{i=1}^{9}\mathcal{S}_i$. Note that $n_G(C_3)\le 3$. By Lemma 2.3(i) we have

$$6n_G(C_3)-\sum_{i=1}^{n}d_G(v_i)^3=6n_G(C_3)-8n-208=6n_H(C_3)-\sum_{i=1}^{n}d_H(v_i)^3.$$

Thus, $n_H(C_3)=n_G(C_3)-5<0$ if $\deg(H)\in\mathcal{S}_2$, that is impossible. By reasoning in a similar way, we get that if $\deg(H)$ belongs to $\mathcal{S}_i$, then $n_H(C_3)=n_G(C_3)-(i+3)<0$, $i=3,4,\ldots,9$. Thus, the only possibility is that $\deg(H)=(6,2^{n-1})$. □

Theorem 4.1. *All 3-rose graph having at least one triangle are DLS-graphs.*

Proof. Let H be a graph L-cospectral to a 3-rose graph $R_{r,s,t}$. Then $n(H)=n(R_{r,s,t})=n$. By Lemma 2.3(i) we get $\omega(H)=1$, i.e. H is a connected graph. From Lemma 4.1 it follows that $\deg(H)=(6,2^{n-1})$. Thus, H must be a 3-rose graph denoted by R_{r_1,s_1,t_1} and $n_{R_{r_1,s_1,t_1}}(C_3)=n_{R_{r,s,t}}(C_3)$ by Lemma 2.3(i). Recall that a 3-rose graph contains at most three triangles.

If $n_{R_{r,s,t}}(C_3)=3$, then $n_{R_{r_1,s_1,t_1}}(C_3)=3$ and so $R_{r,s,t}=R_{3,3,3}\cong R_{r_1,s_1,t_1}$.

If $n_{R_{r,s,t}}(C_3)=2$, then $n_{R_{r_1,s_1,t_1}}(C_3)=2$ and thus $R_{r,s,t}=R_{r,3,3}$ and $R_{r_1,s_1,t_1}=R_{r_1,3,3}$. Since $r+4=n(R_{r,s,t})=n(R_{r_1,s_1,t_1})=r_1+4$ we get $r=r_1$ and so $R_{r_1,s_1,t_1}\cong R_{r,s,t}$.

If $n_{R_{r,s,t}}(C_3)=1$, then $n_{R_{r_1,s_1,t_1}}(C_3)=1$ and thus $R_{r,s,t}=R_{r,s,3}$ and $R_{r_1,s_1,t_1}=R_{r_1,s_1,3}$. Since $3rs=\tau(R_{r,s,3})=\tau(R_{r_1,s_1,3})=3r_1s_1$ and $r+s=r_1+s_1$ we obtain $r=r_1,s=s_1$ or $r=s_1,s=r_1$. Therefore $R_{r_1,s_1,t_1}\cong R_{r,s,t}$. □

5 Q-spectral characterization of 3-rose graphs

We will study the Q-spectral characterization of 3-rose graphs through two subsections. In the first one we restrict the structure of Q-cospectral mates of 3-rose graphs. In the second one we finally discuss the Q-spectral characterization of 3-rose graphs.

5.1 A Q-cospectral graph to a 3-rose graph is a 3-rose graph

In the following, set $R_{a_1,a_2,\cdots,a_p} = R$ and let $m_G(0)$ denote the multiplicity of the eigenvalue 0 of the Q-spectrum of a graph G. The following lemma is taken from [3].

Lemma 5.1. *For any graph G, $m_G(0)$ equals the number of bipartite components.*

Lemma 5.2. *Let R be a p-rose graph of order n. Then $\lambda_2(S(R)) < 2$.*

Proof. Let u be the vertex of degree $2p$ in R. By interlacing theorem for the A-spectrum, we get

$$\lambda_2(S(R)) \leq \lambda_1(S(R) - u) = \lambda_1(\bigcup_{i=1}^{p} P_{2a_i-1}) < 2,$$

since the A-index of a path is less than 2. □

Lemma 5.3. *Let H be a graph Q-cospectral to a p-rose graph R. Then either H is a (connected) p-cyclic graph or $H = H_1 \cup H_2$, where H_1 is a (connected) $(p+1)$-cyclic graph and H_2 is a tree.*

Proof. Note that since R is connected then, by Lemma 5.1, $m_R(0) \leq 1$, more precisely $m_R(0) = 1$ if and only if R is bipartite, i.e. its cycles are all even. It is easy to see that H cannot have more than one tree as component, otherwise, $m_H(0) \geq 2$ which contradicts $m_R(0) \leq 1$.

Now assume, by way of contradiction, that $H = H_1 \cup H_2$ where both H_1 and H_2 contain at least one cycle as subgraph. By Lemma 2.5 we get

$$\phi(S(H)) = \phi(S(H_1))\phi(S(H_2)).$$

If so $\lambda_2(S(H)) \geq \min\{\lambda_1(S(H_1)), \lambda_1(S(H_2))\} \geq 2$ which contradicts (cf. Lemma 5.2) $\lambda_2(S(R)) < 2$.

Since H is Q-cospectral to R, from $m(R) = n(R) + p - 1$ we have that $m(H) = n(H) + p - 1$. So if R is not bipartite then H must be a (connected) p-cyclic graph. Otherwise, if R is bipartite, then either H is a p-cyclic bipartite graph or H is a disjoint union of a $(p+1)$-cyclic graph and a tree. This completes the proof. □

Lemma 5.4. *Let H be a graph Q-cospectral to a 3-rose graph $G = R_{r,s,t}$. Then*

(i) $n_H(C_3) = n_G(C_3)$ *if and only if* $\deg(H) \in \mathcal{S}_1 = \{(6, 2^{n-1})\}$.

(ii) $n_H(C_3) \geq n_G(C_3) + 5$ *if and only if* $\deg(H) \in \bigcup_{i=2}^{9} \mathcal{S}_i$. *Moreover, the equality holds if and only if* $\deg(H) \in \mathcal{S}_2 = \{(5, 4, 3, 2^{n-5}, 1^2)\}$, *where* $\mathcal{S}_i$ $(1 \leq i \leq 9)$ *is defined in Lemma 3.2.*

Proof. Since $\mathrm{Spec}_Q(H) = \mathrm{Spec}_Q(G)$, by Lemma 2.3(ii) we get that

$$6n_H(C_3) + \sum_{i=1}^{n} d_H(v_i)^3 = 6n_G(C_3) + 8n + 208. \tag{2}$$

From Lemma 3.2 it follows that $\deg(H) \in \bigcup_{i=1}^{9} \mathcal{S}_i$. By (2) we have $n_H(C_3) = n_G(C_3)$ if and only if $\deg(H) \in \mathcal{S}_1$ and so (i) holds.

For $i = 2, 3, \cdots, 9$, $n_H(C_3) = n_G(C_3) + (i+3)$ iff $\deg(H) \in \mathcal{S}_i$ and so (ii) holds. □

The next lemma follows from simple observations.

Lemma 5.5. *Let H be a connected graph and let K_4 be the complete graph on 4 vertices. Then:*

(i) *If $m(H) = n(H) + 2$, then $n_H(C_3) \leq 4$ with equality iff H contains K_4 as its subgraph;*

(ii) *If $m(H) = n(H) + 3$, then $n_H(C_3) \leq 5$ with equality iff H contains K_4 as its subgraph.*

Theorem 5.1. *Let H be a graph Q-cospectral to a 3-rose graph. Then H is also a 3-rose graph.*

Proof. By Lemma 3.2 we get that $\deg(H) \in \bigcup_{i=1}^{9} \mathcal{S}_i$. Assume by way of contradiction that $\deg(H) \in \bigcup_{i=2}^{9} \mathcal{S}_i$ and so, by Lemma 5.4(ii), $n_H(C_3) \geq 5$. By Lemma 5.3 we know that H contains at most one tree as component. Now we consider the cases below:

If H contains no trees as its components, by Lemma 5.3, H is a connected graph with $m(H) = m(H) + 2$ which implies, by Lemma 5.5(i), $n_H(C_3) \leq 4$. The latter is a contradiction.

If H contains one tree T as its component, by Lemma 5.3, $H = H_1 \bigcup T$, where H_1 is a connected graph with $m(H_1) = n(H_1) + 3$. By Lemma 5.5(ii) we get that $n_{H_1}(C_3) \leq 5$. Recall $n_H(C_3) \geq 5$. Then $n_H(C_3) = n_{H_1}(C_3) = 5$ which implies that, by Lemma 5.5(ii), H contains K_4 as its subgraph and, by Lemma 5.4(ii), $\deg(H) \in \mathcal{S}_2 = \{(5,4,3,2^{n-5},1^2)\}$. It is easy to see that such a graph does not exist.

Therefore $\deg(H) \in \mathcal{S}_1 = \{(6,2^{n-1})\}$ which yields that H is a 3-rose graph. □

5.2 No two non-isomorphic 3-rose graphs are Q-cospectral

In this section, we show that no two non-isomorphic 3-rose graphs are Q-cospectral. To prove this, we need the following lemmas:

Lemma 5.6. [10] *$\phi(P_n, 2) = n+1$ for $n \geq 1$.*

Lemma 5.7. [1, 15] *Let v be a vertex of G and $\mathscr{C}(v)$ the set of all cycles containing v. Then*

$$\phi(G) = \lambda\phi(G-v) - \sum_{w\sim v} \phi(G-v-w) - 2\sum_{C\in\mathscr{C}(v)} \phi(G-V(C))$$

We assume that $\phi(G) = 1$ if G is the null graph (i.e. with zero vertices).

In the next lemmas we will follow the ideas of [10, 14]. To keep the notation easier to read, let us set $\phi(P_a, \lambda) = p_a$, and, by convention, let $p_0 = 1$, $p_{-1} = 0$ and $p_{-2} = -1$.

Lemma 5.8. *No two non-isomorphic 3-rose graphs of order n are A-cospectral.*

Proof. Assume that $\phi(R_{r,s,t}) = \phi(R_{r_1,s_1,t_1})$. Without loss of generality, we can assume that $r \geq s \geq t$ and $r_1 \geq s_1 \geq t_1$. Since $n(R_{r,s,t}) = n(R_{r_1,s_1,t-1})$, then

$$r+s+t = r_1+s_1+t_1 = n+2. \tag{3}$$

By applying Lemma 5.7 to the vertex v of degree 6, we have

$$\begin{aligned}\phi(R_{r,s,t},\lambda) = \lambda p_{r-1}p_{s-1}p_{t-1} - 2(p_{r-2}p_{s-1}p_{t-1} + p_{r-1}p_{s-2}p_{t-1} \\ + p_{r-1}p_{s-1}p_{t-2}) - 2(p_{r-1}p_{s-1} + p_{r-1}p_{t-1} + p_{s-1}p_{t-1}),\end{aligned}$$

which shows by Lemma 5.6 that $-4rst = \phi(R_{r,s,t},2) = \phi(R_{r_1,s_1,t_1},2) = -4r_1s_1t_1$, so

$$rst = r_1s_1t_1. \tag{4}$$

By Lemma 5.7 again we get $p_a = \lambda p_{a-1} - p_{a-2}$ for $a \geq 0$. Solving the above recurrence equation we obtain that

$$p_a = \frac{x^{2a+2}-1}{x^{a+2}-x^a},$$

where x satisfies $x^2 - \lambda x + 1 = 0$. Then we get

$$f(r,s,t;x) = (x^2-1)^3 x^n \phi(R_{r,s,t},\lambda) + 1 - 5x^2 - x^{2n+4}(x^2-5), \qquad (5)$$

where $n+2 = r+s+t$ and

$$\begin{aligned} f(r,s,t;x) = {} & 2x^t + 2x^s + 2x^r - 2x^{t+2} - x^{2r} - 2x^{r+2} - x^{2t} - x^{2s} - 3x^{2s+2} \\ & - 2x^{s+2} - 3x^{2r+2} - 2x^{s+2t} - 2x^{2s+t} - 2x^{r+2t} - 2x^{r+2s} - 2x^{2r+t} \\ & + 2x^{2r+s+2} + 2x^{s+2t+2} + 2x^{2s+t+2} + 2x^{r+2t+2} + 2x^{r+2s+2} \\ & - 2x^{2r+s} + 3x^{2s+2t} + 3x^{2r+2t} + 3x^{2r+2s} + x^{2s+2t+2} + x^{2r+2t+2} \\ & + x^{2r+2s+2} + 2x^{r+2s+2t} + 2x^{2r+s+2t} + 2x^{2r+2s+t} - 2x^{r+2s+2t+2} \\ & + 2x^{2r+t+2} - 3x^{2t+2} - 2x^{2r+s+2t+2} - 2x^{2r+2s+t+2}. \end{aligned}$$

and by (5),

$$f(r,s,t;x) = f(r_1,s_1,t_1;x). \qquad (6)$$

The smallest exponent of x in $f(r,s,t;x)$ and $f(r_1,s_1,t_1;x)$ is respectively equal to t and t_1. Hence, by (6) we get $t = t_1$ and, by (3) and (4),

$$r+s = r_1+s_1 \quad \text{and} \quad rs = r_1 s_1. \qquad (7)$$

By solving (7), we get $r = r_1, s = s_1$ or $r = s_1, s = r_1$.

Therefore $R_{r,s,t} \cong R_{r_1,s_1,t_1}$. □

Theorem 5.2. *No two non-isomorphic 3-rose graphs of order n are Q-cospectral.*

Proof. Assume that $\varphi(R_{r,s,t}) = \varphi(R_{r_1,s_1,t_1})$. By Lemma 2.5 we get that $\phi(S(R_{r,s,t})) = \phi(S(R_{r_1,s_1,t_1}))$, so $\phi(R_{2r,2s,2t}) = \phi(R_{2r_1,2s_1,2t_1})$ and, by Lemma 5.8, $R_{2r,2s,2t} \cong R_{2r_1,2s_1,2t_1}$. Hence $R_{r,s,t} \cong R_{r_1,s_1,t_1}$. □

5.3 The main result

The main result of this section follows from Theorems 5.1 and 5.2.

Theorem 5.3. *All 3-rose graphs are DQS-graphs.*

6 Conclusion

In this paper we showed that all 3-rose graphs having at least one triangle are DLS-graphs and all 3-rose graphs are DQS-graphs. The key point is that we restrict the degree sequences of its tentative $\{L,Q\}$-cospectral mates. Using this method (Lemma 3.1) we can determine the degree sequence of a tentative $\{L,Q\}$-cospectral mate to a 4-rose graph and even go further.

All 2-rose graphs $R_{r,s}$, with the exception of $R_{r,r+1}$, have been proved in [21] to be a DQS-graph, and all triangle-free 2-rose graphs have been proved to be DLS-graphs.

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References

[1] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs - Theory and Applications, III revised and enlarged edition, Johan Ambrosius Bart Verlag, Heidelberg - Leipzig, 1995.

[2] D. Cvetković, P. Rowlinson, S.K. Simić, Eigenvalue bounds for the signless Laplacians, Publ. Inst. Math. (Beograd) 81(95) (2007) 11–27.

[3] D.M. Cvetković, P. Rowlinson, S.K. Simić, Signless Laplacian of finite graphs, Linear Algebra Appl. 423 (2007) 155–171.

[4] D. Cvetković, S.K. Simić, Towards a spectral theory of graphs based on signless Laplacian, I, Publ. Inst. Math. (Beograd) 85 (99) (2009) 19–33.

[5] D. Cvetković, S.K. Simić, Towards a spectral theory of Graphs based on signless Laplacian, II, Linear Algebra Appl. 432 (2010) 2257–2272.

[6] D. Cvetković, S.K. Simić, Towards a spectral theory of Graphs based on signless Laplacian, III, Appl. Anal. Discrete Math. 4 (2010) 156–166.

[7] E.R. van Dam, W.H. Haemers, Which graphs are determined by their spectra?, Linear Algebra Appl. 373 (2003) 241–272.

[8] E.R. van Dam, W.H. Haemers, Developments on spectral characterizations of graphs, Discrete Math. 309 (2009) 576–586.

[9] P. Erdös, A. Rényi, V. Sós, On a problem of graph theory, Studia Sci. Math. 1 (1966) 215–235.

[10] N. Ghareghani, G.R. Omidi, B. Tayfeh-Rezaie, Spectral characterization of graphs with index at most $\sqrt{2+\sqrt{5}}$, Linear Algebra Appl. 420 (2007) 483–489.

[11] Hs.H. Günthard, H.Primas, Zusammenhang von Graphtheorie und Mo-Theotie von Molekeln mit Systemen konjugierter Bindungen, Helv. Chim. Acta 39 (1956) 1645-1653.

[12] A.J. Hoffman, J.H. Smith, On the spectral radii of topological equivalent graphs, in: Recent Advances in Graph Theory (ed. M. Fiedker), Academia Praha 1975, 273–281.

[13] C.S. Oliveira, N.M. Maia de Abreu, S. Jurkiewicz, The characteristic polynomial of the Laplacian of graphs in (a, b)-linear classes, Linear Algebra Appl. 356 (2002) 113–121.

[14] F. Ramezani, N. Broojerdian, B. Tayfeh-Rezaie, A note on the spectral characterization of θ-graphs, Linear Algebra Appl. 431 (2009) 626–632.

[15] A.J. Schwenk, Computing the characteristic polynomial of a graph, in: Graphs and Combinatorics, Lect. Notes in Math. 406 (1973) 275–307.

[16] S.K. Simić, V. Lj. Kocić, On the Largest eigenvalue of some homeomorphic graphs, Publ. Inst. Math. (Beograd) 40(54) (1986) 3–9.

[17] J.F. Wang, Q.X. Huang, F. Belardo, E.M. Li Marzi, On graphs whose signless Laplacian index does not exceed 4.5, Linear Algebra Appl. 431 (2009) 162–178.

[18] J.F. Wang, Q.X. Huang, F. Belardo, E.M. Li Marzi, On the Spectral Characterization of Theta Graphs, MATCH Commun. Math. Comput. Chem. Vol. 62, No. 3 (2009) 581–598.

[19] J.F. Wang, Q.X. Huang, F. Belardo, E.M. Li Marzi, A note on the spectral characterization of dumbbell graphs, Linear Algebra Appl. 431 (2009) 1707–1714.

[20] J.F. Wang, Q.X. Huang, F. Belardo, E.M. Li Marzi, Spectral characterizations of dumbbell graphs, Electron. J. Comb. 17 (2010), #R42.

[21] J.F. Wang, Q.X. Huang, F. Belardo, E.M. Li Marzi, On the spectral characterizations of ∞-graphs, Discrete Math. (2010), doi:10.1016/j.disc.2010.01.021, in press.

[22] http://en.wikipedia.org/wiki/Rose_(mathematics).