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Spectral characterizations of signed lollipop graphs

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ABSTRACT

Let $\Gamma = (G, \sigma)$ be a signed graph, where G is the underlying simple graph and $\sigma : E(G) \to \{+, -\}$ is the sign function on the edges of G. In this paper we consider the spectral characterization problem extended to the adjacency matrix and Laplacian matrix of signed graphs. After giving some basic results, we study the spectral determination of signed lollipop graphs, and we show that any signed lollipop graph is determined by the spectrum of its Laplacian matrix.

1. Introduction

Let $G = (V(G), E(G))$ be a graph of order $n = |V(G)| = |G|$ and size $m = |E(G)|$, and let $\sigma : E(G) \to \{+, -\}$ be a mapping defined on the edge set of G. Then $\Gamma = (G, \sigma)$ is a signed graph (sometimes called also sigraph). The graph G is its underlying graph, while σ its sign function (or signature). It is common to interpret the signs as the integers

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$\{1, -1\}$. An edge e is positive (negative) if $\sigma(e) = 1$ (resp. $\sigma(e) = -1$). If $\sigma(e) = 1$ (resp. $\sigma(e) = -1$) for all edges in $E(G)$ then we write $(G, +)$ (resp. $(G, -)$). A cycle of Γ is said to be balanced, or positive, if it contains an even number of negative edges, otherwise the cycle is unbalanced, or negative. A signed graph is said to be balanced if all its cycles are balanced; otherwise, it is unbalanced. By $\sigma(\Gamma)$ we denote the product of signs of all cycles in Γ. Most of the concepts defined for graphs are directly extended to signed graphs. For example, the degree of a vertex v in G (denoted by $\deg(v)$) is also its degree in Γ. So $\Delta(G)$, the maximum (vertex) degree in G, also stands for $\Delta(\Gamma)$, interchangeably. Furthermore, if some subgraph of the underlying graph is observed, then the sign function for the subgraph is the restriction of the previous one. Thus, if $v \in V(G)$, then $\Gamma - v$ denotes the signed subgraph having $G - v$ as the underlying graph, while its signature is the restriction from $E(G)$ to $E(G - v)$ (all edges incident to v are deleted). Similar considerations hold for the disjoint union of signed graphs. If $U \subset V(G)$ then $\Gamma[U]$ denotes the signed induced subgraph of U, while $\Gamma - U = \Gamma[V(G) \setminus U]$. For $\Gamma = (G, \sigma)$ and $U \subset V(G)$, let Γ^U be the signed graph obtained from Γ by reversing the signature of the edges in the cut $[U, V(G) \setminus U]$, namely $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G) \setminus U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be (signature) switching equivalent to Γ. In fact, switching equivalent signed graphs can be considered as (switching) isomorphic graphs and their signatures are said to be equivalent. Switching equivalent graphs have the same set of positive cycles.

In the literature, simple graphs are studied by means of the eigenvalues of several matrices associated to graphs. The adjacency matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ whenever vertices i and j are adjacent and $a_{ij} = 0$ otherwise, is one of the most studied together with the Laplacian, or Kirchhoff, matrix $L(G) = D(G) - A(G)$, where $D(G) = \mathrm{diag}(\deg(v_1), \deg(v_2), \dots, \deg(v_n))$ is the diagonal matrix of vertex degrees. In the last 10 years another graph matrix has attracted the attention of many researchers, the so-called signless Laplacian matrix defined as $Q(G) = A(G) + D(G)$. Matrices can be associated to signed graphs, as well. The adjacency matrix $A(\Gamma) = (a^\sigma_{ij})$ with $a^\sigma_{ij} = \sigma(ij)a_{ij}$ is called the (signed) adjacency matrix and $L(\Gamma) = D(G) - A(\Gamma)$ is the corresponding Laplacian matrix. Both the adjacency and Laplacian matrices are real symmetric matrices, so the eigenvalues are real.

In this paper we shall consider both the characteristic polynomial of the adjacency matrix and of the Laplacian matrix of a signed graph Γ. Hence to avoid confusion we denote by

$$\phi(\Gamma, x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n,$$

the adjacency characteristic polynomial (or A-polynomial) whose roots, namely the adjacency eigenvalues (A-eigenvalues), are denoted by $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma)$. Similarly, for the Laplacian matrix, we denote by

$$\psi(\Gamma, x) = x^n + b_1 x^{n-1} + \cdots + b_{n-1}x + b_n,$$

the Laplacian polynomial (or L-polynomial), and $\mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \cdots \geq \mu_n(\Gamma) \geq 0$ are the Laplacian eigenvalues (L-eigenvalues). Suffix and variables will be omitted if clear from the context (so $\phi(\Gamma, x) = \phi(\Gamma)$). A connected signed graph is balanced if and only if $\mu_n = 0$ (see [20]). If Γ is disconnected, then its polynomial is the product of the components polynomials.

Finally, it is important to observe that switching equivalent signed graphs will have similar adjacency and Laplacian matrices. In fact, any switching on U can be interpreted as a diagonal matrix $S_U = \mathrm{diag}(s_i)$ having $s_i = 1$ for any $i \in U$ and $s_i = -1$ otherwise. S_U is usually called the state matrix. Hence, $A(\Gamma) = S_U A(\Gamma^U) S_U$ and $L(\Gamma) = S_U L(\Gamma^U) S_U$. Similar effect features with eigenvectors. When we consider a signed graph Γ, from a spectral viewpoint, we are considering its switching isomorphism class $[\Gamma]$. For a (possibly) complete bibliography on signed graphs, the reader is referred to [22]. For all notations not given here the reader is referred to [23] for signed graphs, to [8] for graph spectra, and to [5,16] for some basic results on the Laplacian spectral theory of signed graphs.

Recently in [5] the authors have considered an expression for the coefficient of the L-polynomial of a signed graph and they gave some formulas which express the L-polynomial by means of the A-polynomial of some related signed graphs. In this article, we consider those results to define some spectral invariants and study their effect on the combinatorial structure of the signed graphs. Hence, we consider (as in [4]) the problem of spectral determination (up to switching isomorphism), and we give some basic results for this kind of investigations. As an application, we study the Laplacian spectral determination problem for a class of signed graphs known as signed lollipop graphs, and we show that any signed lollipop graph is determined by the eigenvalues of its Laplacian matrix.

Here is the remainder of the paper. In Section 2, we consider the properties that can be deduced by the coefficients of the Laplacian polynomial of a signed graph. In Section 3, we give some results useful for the spectral determination problems. Finally, in Section 4 we study the Laplacian spectral determination of signed lollipop graphs.

2. Preliminaries

In this section we recall some basic results which will be useful for the study of signed graphs from a spectral viewpoint. More details on these results can be found in [5].

We first recall a formula useful to compute the coefficients of Laplacian polynomial of signed graphs. We need first to introduce some additional notation. A signed TU-subgraph H of a signed graph Γ is a subgraph whose components are trees or unbalanced unicyclic graphs. If H is a signed TU-subgraph, then $H = \bigcup_{i=1}^{t} T_i \bigcup_{j=1}^{c} U_j$, where, if any, the T_i's are trees and the U_j's are unbalanced unicyclic graphs. The weight of the signed TU-subgraph H is defined as $w(H) = 4^c \prod_{i=1}^{t} |T_i|$. For the special case $\Gamma = (G, -)$, we get the formula for the signless Laplacian of simple graphs, where instead of signed TU-subgraphs we have the TU-subgraphs, namely subgraphs whose components are trees or odd unicyclic graphs [10].

Theorem 2.1. *(See [5,7].) Let Γ be a signed graph and $\psi(\Gamma, x) = x^n + b_1 x^{n-1} + \cdots + b_{n-1}x + b_n$ be the Laplacian polynomial of Γ. Then we have*

$$b_i = (-1)^i \sum_{H \in \mathcal{H}_i} w(H), \quad i = 1, 2, \ldots, n, \tag{1}$$

where $\mathcal{H}_i$ denotes the set of the signed TU-subgraphs of Γ built on i edges.

From the above formula it is (again) evident that the L-polynomial is invariant under switching isomorphisms, since switching preserves the sign of the cycles. Furthermore, it is important to observe that the signature is relevant only on the edges that are not bridges, hence we will always consider the all-positive signature for trees. In the sequel signed trees and unsigned trees will be considered as the same object. For the same reason, the edges which do not lie on some cycle are not relevant for the signature and they will be always considered as positive. Another straight consequence of the above formula is described in the following corollary.

Corollary 2.2. *Let (G, σ) and (G, σ') be two signed graphs, on the same underlying graph G. Let $\psi(G, \sigma) = \sum_{i=1}^n b_i x^{n-i}$ and $\psi(G, \sigma') = \sum_{i=1}^n b'_i x^{n-i}$. If the girth of G is g then $b_i = b'_i$ for $i = 0, 1, \ldots, g-1$.*

Now, we recall some useful formulas, given in [5], which relate the Laplacian polynomial of a signed graph to the adjacency polynomials of its opportunely defined signed subdivision graph and signed line graph. In order to do so, we need to introduce a special oriented vertex-edge incidence matrix B_η of a signed graph $\Gamma = (G, \sigma)$ with n vertices and m edges. Assign any random orientation η on the positive edges of Γ. Then, the $n \times m$ matrix $B_\eta = (b^\eta_{ij})$ is defined as

$$b^\eta_{ij} = \begin{cases} +1 & \text{if } e_j \text{ is incident } v_i \text{ and } \sigma(e_j) = -1, \\ +1 & \text{if } v_i \text{ is the head of } e_j \text{ and } \sigma(e_j) = 1, \\ -1 & \text{if } v_i \text{ is the tail of } e_j \text{ and } \sigma(e_j) = 1, \\ 0 & \text{if } e_j \text{ is not incident } v_i. \end{cases}$$

It is not difficult to see that $L(\Gamma) = B_\eta B_\eta^\top$, which implies that $L(\Gamma)$ is a positive semidefinite matrix. From the above matrix we define two signed graphs, one of order $n + m$ and the other of order m, corresponding to the signed subdivision graph and the signed line graph, respectively. Recall that the subdivision of a simple graph G is the graph $\mathcal{S}(G)$ obtained from G by inserting in each edge a vertex of degree 2. In fact, $\mathcal{S}(G)$ is a graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if and only if they

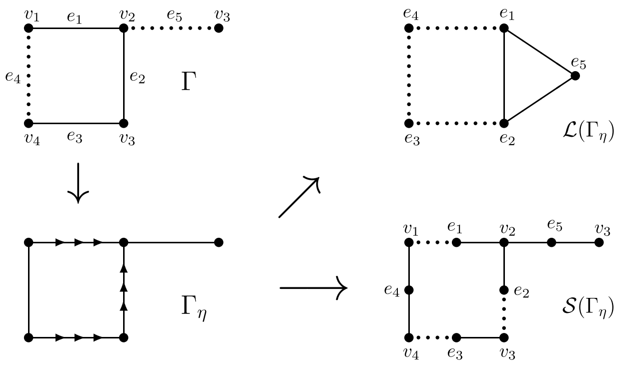

Fig. 1. A signed graph and the corresponding signed subdivision and line graphs.

are incident in G. Now, let us assign an orientation η to the positive edges and consider the corresponding incidence matrix $B_\eta = (b_{ij})$. The signed subdivision graph, associated to B_η, is the signed graph $\mathcal{S}(\Gamma_\eta) = (\mathcal{S}(G), \sigma_\eta^{\mathcal{S}})$, where

$$\sigma_\eta^{\mathcal{S}}(v_i e_j) = b_{ij}^\eta$$

It is worth to observe that a signed subdivision graph is balanced if and only if each cycle in the signed root graph contains an even number of positive edges.

Next we define the signed line graph associated to B_η. The signed line graph of $\Gamma = (G, \sigma)$ is the signed graph $(\mathcal{L}(G), \sigma_\eta^{\mathcal{L}})$, where $\mathcal{L}(G)$ is the (usual) line graph and

$$\sigma_\eta^{\mathcal{L}}(e_i e_j) = \begin{cases} b_{ki}^\eta b_{kj}^\eta & \text{if } e_i \text{ is incident } e_j \text{ at } v_k; \\ 0 & \text{otherwise.} \end{cases}$$

Note that both $\mathcal{S}(\Gamma_\eta)$ and $\mathcal{L}(\Gamma_\eta)$ depend on the chosen edge orientation η, but it is not difficult to see that a different orientation η' gives rise to a, respectively, switching equivalent signed subdivision graph and signed line graph. For example, reverting the orientation of some (positive) edge corresponds to having the value -1 in the state matrix entry related to the vertex subdividing the edge. Hence, $\mathcal{S}(\Gamma_\eta)$ and $\mathcal{L}(\Gamma_\eta)$ are uniquely defined up to switching isomorphisms, and for this reason the index η will be not anymore specified. For further details, the interested reader is referred to [5].

An example of subdivision and line graphs of a signed graph are depicted in Fig. 1, where positive edges are bold lines, while negative edges are dotted lines.

The following result holds

Theorem 2.3. *(See [5].) Let Γ be a signed graph of order n and size m, and $\phi(\Gamma)$ and $\psi(\Gamma)$ its adjacency and Laplacian polynomials, respectively. Then*

(i) $\phi(\mathcal{L}(\Gamma), x) = (x + 2)^{m-n} \psi(\Gamma, x + 2)$,
(ii) $\phi(\mathcal{S}(\Gamma), x) = x^{m-n} \psi(\Gamma, x^2)$.

Remark 2.4. In [5] the authors gave an interpretation of the oriented incidence matrix B_η in terms of bi-oriented graphs, here we have considered a slightly different (but equivalent) interpretation that is analogous to the Laplacian theory of mixed graphs, for which the edges can be either oriented or unoriented. It is clear that the Laplacian theory of mixed graphs is the same as that of signed graphs, e.g., [11,25]. Finally, it is necessary to observe that in the literature we have also different definitions of signed line graphs for a signed graph (see, for example, [2,21]).

The following result is the interlacing theorem in the edge variant. It can be deduced from the ordinary vertex variant interlacing theorem for the adjacency matrix combined with Theorem 2.3 (ii).

Theorem 2.5. *Let $\Gamma = (G, \sigma)$ be a signed graph and $\Gamma - e$ be the signed graph obtained from Γ by deleting the edge e. Then*

$$\mu_1(\Gamma) \geq \mu_1(\Gamma - e) \geq \mu_2(\Gamma) \geq \mu_2(\Gamma - e) \geq \cdots \geq \mu_n(\Gamma) \geq \mu_n(\Gamma - e).$$

From the above theorem, we can characterize the signed graphs whose Laplacian spectral radius does not exceed 4. Recall that the signatures of trees are omitted. Also, for the sake of readability, for signed unicyclic graphs, the signature denoted by $\bar{\sigma}$ means that the unique cycle is unbalanced. Note that signed unicyclic graphs have just two non-switching equivalent signatures: the all-positive edges, denoted by $\sigma = +$, and the unique cycle is unbalanced, denoted by $\bar{\sigma}$. Under the above notation we have the following results (cf. also [11]).

Lemma 2.6. *Let $\Gamma = (C_{2n}, \bar{\sigma})$ be the unbalanced cycle on $2n$ vertices. Then $\mu_1(C_{2n}, \bar{\sigma}) < 4$.*

Proof. In view of Corollary 2.2, $\psi(C_{2n}, +)$ and $\psi(C_{2n}, \bar{\sigma})$ have all coefficients but one equal. In fact, it is not difficult to see that $\psi((C_{2n}, +), x) - \psi((C_{2n}, \bar{\sigma}), x) = -4 < 0$ for every $x \in \mathbb{R}$. Since the spectral radius of $\psi(C_{2n}, +)$ is 4, then $\psi((C_{2n}, \bar{\sigma}), x) > 0$ for all $x \geq 4$. The latter implies that the corresponding spectral radius is less than 4. □

Theorem 2.7. *Let μ be the largest eigenvalue, or spectral radius, of the Laplacian of a connected signed graph $\Gamma = (G, \sigma)$. The following statements hold:*

(i) $\mu(\Gamma) = 0$ *iff* $\Gamma = K_1$;
(ii) $\mu(\Gamma) = 2$ *iff* $\Gamma = K_2$;
(iii) $\mu(\Gamma) = 3$ *iff* $\Gamma \in \{P_3, (K_3, +)\}$;
(iv) $3 < \mu(\Gamma) < 4$ *iff* $\Gamma \in \{P_n\ (n \geq 4), (C_{2n}, \bar{\sigma}), (C_{2n+1}, +)\ (n \geq 2)\}$;
(v) $\mu(\Gamma) = 4$ *iff* $\Gamma \in \{(C_{2n}, +), (C_{2n+1}, \bar{\sigma})\ \ (n \geq 2), K_{1,3}, (K_{1,3}^{+}, +), (K_4^{-}, +), (K_4, +)\}$,

where $K_{1,3}^{+}$ (K_4^{-}) is obtained from $K_{1,3}$ (resp., K_4) by adding (resp., deleting) an edge.

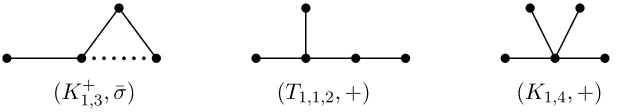

Fig. 2. Forbidden subgraphs for $\mu_1 \leq 4$.

Proof. Most of the above values of the Laplacian spectral radii of signed graphs can be deduced from the ordinary (signless) Laplacian theory of simple graphs (e.g., [18]). Let us denote by $\varphi(G)$ the characteristic polynomial of the signless Laplacian $D(G) + A(G)$, and by $\kappa(G)$ the corresponding spectral radius.

Clearly, $\Delta(\Gamma) < 4$, otherwise $K_{1,4}$ appears and $\mu(\Gamma) \geq 5$ by Theorem 2.5. Items (i), (ii) and (iii) are trivial and they can be easily verified.

Regarding the graphs in Items (iv) and (v), we have the following considerations. From the Laplacian theory of unsigned graph we get that $\mu(P_n) < 4$, $\mu(C_{2n+1}, +) < 4$, $\mu(K_{1,3}) = \mu(K_{1,3}^{+}, +) = \mu(K_4^{-}, +) = \mu(K_4, +) = 4$. From the signless Laplacian theory of graphs we get that $\psi(C_{2n}, +\sigma) = \varphi(C_{2n})$ with $\kappa(C_{2n}) = 4$, $\psi(C_{2n+1}, \bar{\sigma}) = \varphi(C_{2n+1})$ with $\kappa(C_{2n+1}) = 4$, and $\psi(K_{1,3}^{+}, \bar{\sigma}) = \varphi(K_{1,3}^{+})$ with $\kappa(K_{1,3}^{+}) > 4$.

We now consider those spectra which cannot be deduced from the theory of unsigned graphs. One graph is $(C_{2n}, \bar{\sigma})$, for which we have that $\mu(C_{2n}, \bar{\sigma}) < 4$ by Lemma 2.6. Any other (connected) graph different from the previous ones will contain a vertex of degree 3, and at least one of the two following graphs: $(K_{1,3}^{+}, \bar{\sigma})$ or the tree $T_{1,1,2}$ (see Fig. 2) which, according to Theorem 2.5, lead to signed graphs with spectral radius greater than 4. $\square$

In general, we can give the following upper bound for the largest Laplacian eigenvalue of a signed graph. Other similar bounds can be found in [16].

Lemma 2.8. *Let $\Gamma = (G, \sigma)$ be a signed graph with Δ_1 and Δ_2 being the first and second largest vertex degrees in G, and let $\mu(\Gamma)$ be its Laplacian spectral radius. Then $\mu(\Gamma) \leq \Delta_1 + \Delta_2$, with equality if and only if $\Gamma = K_{1,n}$ or $\Gamma = (K_n, -)$.*

Proof. For any given matrix A, let $|A|$ be the absolute value matrix whose entries are obtained from A by replacing each entry with the corresponding absolute value. Recall that the largest eigenvalue of a square matrix A is less than or equal to the largest eigenvalue of $|A|$ (due to the Perron–Frobenius theorem). Hence, we have that $\mu(\Gamma) = \mu(BB^\top) \leq \mu(|BB^\top|) = \mu(G, -)$, namely the largest L-eigenvalue of a signed graph is bounded by the largest L-eigenvalue of the corresponding all-negative edges signed graph. On the other hand, $L(G, -)$ is the signless Laplacian of the underlying simple graph G, for which the inequality $\mu(G, -) \leq \Delta_1 + \Delta_2$ holds (e.g., Theorem 4.2 in [9]). The equality holds if and only if G is either the n-star $K_{1,n}$ or the complete graph K_n. This completes the proof. $\square$

We conclude this section with two formulas (see Theorems 3.2 and 3.4 in [3]) useful for the computation of the A-polynomial of any weighted non-oriented graph.

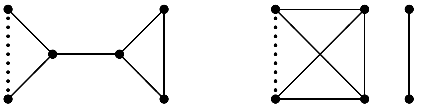

Fig. 3. A pair of A-cospectral signed graphs.

Theorem 2.9. *Let $A = (a_{ij})$ be the adjacency matrix of a weighted graph G. Let $v \in G$ be any vertex. Then we have*

$$\phi(G, x) = (x - a_{vv})\phi(G - v, x) - \sum_{u\sim v} a_{uv}^2 \phi(G - u - v, x) - 2\sum_{C\in\mathcal{C}_v} \omega(C)\phi(G \setminus V(C), x),$$

$$\phi(G, x) = \phi(G - uv, x) - a_{uv}^2 \phi(G - u - v, x) - 2\sum_{C\in\mathcal{C}_{uv}} \omega(C)\phi(G \setminus V(C), x),$$

where $\mathcal{C}_a$ is the set of cycles passing through a and $\omega(C) = \prod_{uw\in C} a_{uw}$.

The formulas in Theorem 2.9 have a natural use in the context of the adjacency matrix. However, they can be used for the Laplacian of signed graphs by mapping the Laplacian matrix of a signed graph as the adjacency matrix of a weighted multigraph. In fact, by doing so, any positive edge becomes a negative edge and vice versa, while the vertex degrees are expressed as weighted loops. The weight of an n-cycle C will be $+1$ if the cycle contains an even number of positive edges, and -1 if it contains an odd number of positive edges, that is $(-1)^n\sigma(C)$.

3. Spectral determination of signed graphs

In this section we give some results which will be useful for the study of spectral determination of signed graphs. This problem was, possibly, first introduced by Acharya in [1] in the context of the adjacency matrix.

Definition 3.1. A signed graph $\Gamma = (G, \sigma)$ is determined by the spectrum, or the eigenvalues, of its matrix $M(\Gamma)$ if and only if any signed graph $\Lambda = (H, \sigma')$ such that $M(\Lambda)$ has the same spectrum of $M(\Gamma)$ implies that Γ and Λ are two switching isomorphic graphs. In the latter case, Γ is said to be determined by the spectrum of the matrix M, or Γ is a DMS graph for short. If Λ is not switching isomorphic to Γ but $M(\Lambda)$ has the same spectrum of $M(\Gamma)$, then the two graphs are said to be M-cospectral, or Λ is an M-cospectral mate of Γ.

As shown in Fig. 3, there are pairs of cospectral signed graphs. However, cospectral mates share the spectral invariants. Let us first consider some spectral invariants which can be deduced from the powers of the matrices of signed graphs. For this purpose, we need to introduce some additional notation. A walk of length k in a signed graph Γ is a sequence $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ of vertices $v_1, v_2, \ldots, v_{k+1}$ and edges $e_1, e_2, \ldots, e_k$

such that $v_i \neq v_{i+1}$ for each $i = 1, 2, \ldots, k$; a walk is said to be positive if it contains an even number of positive edges, otherwise it is said to be negative. Let $w^+_{v_iv_j}(k)$ (resp. $w^-_{v_iv_j}(k)$) denote the number of positive (resp., negative) walks of length k from the vertex v_i to the vertex v_j. Finally let t^+_Γ (resp., t^-_Γ) denote the number of balanced (resp., unbalanced) triangles in Γ (the suffix is omitted if clear from the context). The following fact is well known (see, for example, [21]):

Lemma 3.2. *Let Γ be a signed graph and A its adjacency matrix. Then the (i, j)-entry of the matrix A^k is $w^+_{v_iv_j}(k) - w^-_{v_iv_j}(k)$.*

Corollary 3.3. *Let Γ be a signed graph, A its adjacency matrix, D the diagonal matrix of vertex degrees and t^+ (resp., t^-) the number of balanced (resp., unbalanced) triangles.*

Then $\operatorname{tr}(A^2) = \operatorname{tr}(D)$, and $\operatorname{tr}(A^3) = 6(t^+ - t^-)$.

Let $T_k = \sum_{i=1}^n \mu_i^k$ $(k = 0, 1, 2, \ldots)$ be the k-th spectral moment for the Laplacian spectrum of a signed graph Γ.

Theorem 3.4. *Let $\Gamma = (G, \sigma)$ be a signed graph with n vertices, m edges, t^+ balanced triangles, t^- unbalanced triangles, and degree sequence $(d_1, d_2, \ldots, d_n)$. We have*

$$T_0 = n, \qquad T_1 = \sum_{i=1}^n d_i = 2m, \qquad T_2 = 2m + \sum_{i=1}^n d_i^2,$$

$$T_3 = 6\,(t^- - t^+) + 3\sum_{i=1}^n d_i^2 + \sum_{i=1}^n d_i^3.$$

Proof. Recall that $\operatorname{tr} MN = \operatorname{tr} NM$ for any two feasible matrices M and N. The formulas for T_0 and T_1 are obvious. The formula for T_2 follows from $\operatorname{tr} L^2 = \operatorname{tr}(D - A)^2 = \operatorname{tr} D^2 + \operatorname{tr} A^2$, since $\operatorname{tr} AD = \operatorname{tr} DA = 0$ and, by Corollary 3.3, we have $\operatorname{tr} A^2 = \operatorname{tr} D = 2m$. Finally, $T_3 = \operatorname{tr}(D - A)^3 = \operatorname{tr} D^3 + 3\operatorname{tr} A^2D - 3\operatorname{tr} AD^2 - \operatorname{tr} A^3$. Since $\operatorname{tr} AD^2 = 0$, and, by Corollary 3.3, $\operatorname{tr}(A^3) = 6(t^+ - t^-)$ we get the assertion. □

It is well-known that the multiplicity of the eigenvalue 0 counts the number of balanced components (see, for example, [20]). The below result synthesizes the considerations so far made.

Theorem 3.5. *Let $\Gamma = (G, \sigma)$ and $\Lambda = (H, \sigma')$ be two L-cospectral signed graphs. Then,*

(i) *Γ and Λ have the same number of vertices and edges;*
(ii) *Γ and Λ have the same number of balanced components;*
(iii) *Γ and Λ have the same Laplacian spectral moments;*
(iv) *Γ and Λ have the same sum of squares of degrees, $\sum_{i=1}^n d_G(v_i)^2 = \sum_{i=1}^n d_H(v_i)^2$;*
(v) *$6(t^-_\Gamma - t^+_\Gamma) + \sum_{i=1}^n d_G(v_i)^3 = 6(t^-_\Lambda - t^+_\Lambda) + \sum_{i=1}^n d_H(v_i)^3$.*

The following theorem can be useful in those situations in which a signed graph Γ and its signed subdivision graph $\mathcal{S}(\Gamma)$ maintain the same structure (e.g., lollipop graphs).

Theorem 3.6. *Let $\Gamma = (G, \sigma)$ be a signed graph of order n and size m, and $\mathcal{S}(\Gamma)$ the subdivision graph of Γ.*

(i) *The signed graphs Γ and Λ are L-cospectral iff $\mathcal{S}(\Gamma)$ and $\mathcal{S}(\Lambda)$ are A-cospectral;*
(ii) *Let Γ be a signed graph and $\mathcal{S}(\Gamma)$ a DAS-graph. Then Γ is a DLS-graph;*
(iii) *Let Γ be a DLS-graph. If any graph A-cospectral to $\mathcal{S}(\Gamma)$ is a subdivision of some graph, then $\mathcal{S}(\Gamma)$ is a DAS-graph.*

Proof. (i) Since Γ and Λ are L-cospectral, then $\psi(\Gamma, x) = \psi(\Lambda, x)$, and Γ and Λ have the same order and size which implies that $m(\Gamma) - n(\Gamma) = m(\Lambda) - n(\Lambda)$. Thus,

$$x^{m(\Gamma)-n(\Gamma)}\psi(\Gamma, x^2) = x^{m(\Lambda)-n(\Lambda)}\psi(\Lambda, x^2),$$

which implies by Lemma 2.3 (i) that $\phi(\mathcal{S}(\Gamma), x) = \phi(\mathcal{S}(\Lambda), x)$. This ends the necessity.

Conversely, since $\mathcal{S}(\Gamma)$ and $\mathcal{S}(\Lambda)$ are A-cospectral, then

$$\phi(\mathcal{S}(\Gamma), x) = \phi(\mathcal{S}(\Lambda), x), \quad n(\mathcal{S}(\Gamma)) = n(\mathcal{S}(\Lambda)), \quad m(\mathcal{S}(\Gamma)) = m(\mathcal{S}(\Lambda)).$$

Note that

$$m(\mathcal{S}(\Gamma)) = 2m(\Gamma), \quad m(\mathcal{S}(\Lambda)) = 2m(\Lambda),$$
$$n(\mathcal{S}(\Gamma)) = m(\Gamma) + n(\Gamma), \quad n(\mathcal{S}(\Lambda)) = m(\Lambda) + n(\Lambda).$$

From the above equalities, we obtain that $m(\Gamma) = m(\Lambda)$ and $n(\Gamma) = n(\Lambda)$, and so

$$(\sqrt{x})^{n(\Gamma)-m(\Gamma)}\phi(\mathcal{S}(\Gamma), \sqrt{x}) = (\sqrt{x})^{n(\Lambda)-m(\Lambda)}\phi(\mathcal{S}(\Lambda), \sqrt{x}),$$

which shows from that $\psi(\Gamma, x) = \psi(\Lambda, x)$.

(ii) Assume that $\psi(\Lambda, x) = \psi(\Gamma, x)$. Then by (i) we get $\phi(\mathcal{S}(\Lambda), x) = \phi(\mathcal{S}(\Gamma), x)$. Since $\mathcal{S}(\Gamma)$ is a DAS-graph, then $\mathcal{S}(\Lambda)$ is switching isomorphic to $\mathcal{S}(\Gamma)$, that implies Λ being switching isomorphic to Γ.

(iii) Assume that Λ and Λ' are two signed graphs such that $\Lambda = \mathcal{S}(\Lambda')$ and $\phi(\lambda, x) = \phi(\mathcal{S}(\Lambda'), x) = \phi(\mathcal{S}(\Gamma), x)$, which implies from (i) that $\psi(\Lambda', x) = \psi(\Gamma, x)$. Since Γ is a DLS-graph, then Λ' is switching isomorphic to Γ, and so $\Lambda = \mathcal{S}(\Lambda')$ is switching isomorphic to $\mathcal{S}(\Gamma)$ which shows that $\mathcal{S}(\Gamma)$ is indeed a DAS-graph. $\square$

In view of Theorem 3.6 the following problem naturally arises: under which conditions a signed graph $\Gamma = (G, \sigma)$ can be seen as a signed subdivision graph. The answer is the same as that for unsigned graphs, and the signature can be easily deduced.

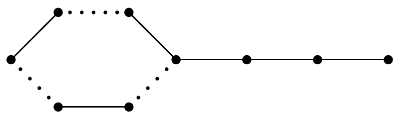

Fig. 4. The signed lollipop graph $(L_{6,9}, \bar{\sigma})$.

Theorem 3.7. *A signed graph $\Gamma = (G, \sigma)$ is the signed subdivision graph of Λ if and only if the following items hold:*

(i) *G is bipartite;*
(ii) *One of the two color classes, say S_2, consists of exactly $m(G)/2$ vertices of degree 2;*
(iii) *G does not contain C_4 as its subgraph.*

Then Λ is obtained from Γ by replacing each vertex from S_2 with an edge. The signature of the edge will be: a) positive, if the two deleted edges were of different sign; b) negative, if both deleted edges had the same sign.

4. Spectral determination of signed lollipop graphs

In this section we study the Laplacian spectral determination of signed lollipop graphs. Another spectral determination problem is considered in [4] for the signed graphs whose second largest L-eigenvalue does not exceed 3, in which signed friendship graphs are included. A lollipop graph is the coalescence between a cycle and a path for which the end vertex of the path is identified with a vertex from the cycle. By $L_{g,n}$ we denote the lollipop graph whose girth is g and the order is n. Since the lollipop is a unicyclic graph, then it admits only two different non-equivalent signatures: the all positive edges $\sigma = +$, and $\bar{\sigma}$ for which the unique cycle is unbalanced. In Fig. 4 we depicted an example of signed lollipop graph.

In the literature, the spectral determination of (unsigned) lollipop graphs, and related graphs, has been already considered in the papers [6,14,13,24]. Here we continue such investigations by extending the problem to the wider settings of signed graphs. Clearly, the main results from the above cited papers must be taken into account. We restate such results in terms of signed graphs.

Theorem 4.1. *Let $(L_{g,n}, \sigma)$ be a signed lollipop graph of order n and girth g. We have:*

- *$(L_{g,n}, +)$ has no A-cospectral mates with only positive edges [6];*
- *$(L_{g,n}, +)$ has no L-cospectral mates with only positive edges [14];*
- *$(L_{g,n}, -)$ has no L-cospectral mates with only negative edges [15,24].*

Now we spectrally characterize the signed lollipop graphs and extend the result of Theorem 4.1 to all signed lollipop graphs. The following lemma gives two bounds on the first and second largest eigenvalues of any signed lollipop graph.

Lemma 4.2. *Let $(L_{g,n}, \sigma)$ be a lollipop graph. Then we have $4 < \mu_1(L_{g,n}, \sigma) < 5$ and $\mu_2(L_{g,n}, \sigma) < 4$.*

Proof. The upper bound for $\mu_1(L_{g,n}, \sigma)$ comes from the fact that largest and second largest vertex degrees of $(L_{g,n}, \sigma)$ are 3 and 2, and from Lemma 2.8 we obtain $\mu_1(L_{g,n}, \sigma) < 5$. The lower bound for $\mu_1(L_{g,n}, \sigma)$ comes from $K_{1,3}$ being a subgraph (interlacing theorem). Finally, from the interlacing theorem applied to the edge in the cycle incident with the vertex of degree 3, we obtain the path P_n. Hence, in view of Theorem 2.7 (iii), we have $\mu_1(L_{g,n}, \sigma) \geq 4 > \mu_1(P_n) \geq \mu_2(L_{g,n}, \sigma)$. Finally, it is also easy to see that 4 cannot be an eigenvalue of $(L_{g,n}, \sigma)$ (see for example, Lemma 4.15). □

In the following lemma we determine the degree sequence of any L-cospectral mate of a signed lollipop graph.

Lemma 4.3. *Let $\Gamma = (G, \sigma)$ be L-cospectral with $(L_{g,n}, \sigma)$, then Γ has the same degree sequence of $(L_{g,n}, \sigma)$.*

Proof. Let $\Gamma = (G, \sigma)$ be L-cospectral with $(L_{g,n}, \sigma)$. Since $\mu_1(L_{g,n}, \sigma) < 5$, then Γ cannot have vertices whose degree is greater than 3, otherwise $K_{1,4}$ appears as a subgraph of Γ and $\mu_1(\Gamma) \geq 5$. Let n_i be the number of vertices whose degree is i, where $0 \leq i \leq 3$. From Theorem 3.5 (i) and (iv) we deduce the following linear system of equations:

$$\begin{cases} n_0 + n_1 + n_2 + n_3 = n, \\ \quad n_1 + 2n_2 + 3n_3 = 2n, \\ \quad n_1 + 4n_2 + 9n_3 = 4n + 2, \end{cases}$$

whose unique (acceptable) solution is indeed $n_0 = 0$, $n_1 = 1$, $n_2 = n - 2$ and $n_3 = 1$. Hence the underlying graph of Γ consists of a lollipop graph with possibly one or more cycles as connected components. □

Now we have restricted the structure of a tentative L-cospectral mate of a signed lollipop graph. Let us denote by Γ a signed graph cospectral with $(L_{g,n}, \sigma)$. We have proved that Γ is a signed lollipop graph with possibly one or more cycles as components. However, Γ cannot have any kind of cycles as a component. In fact, $(C_{2r+1}, \bar{\sigma})$ and $(C_{2r}, +)$ are not acceptable since 4 would appear as an eigenvalue. Also, the eigenvalue 0 appears at most once, so Γ can have no more than one balanced cycle as a component.

The following lemma lists the spectra of signed cycles and paths (see [8] for the balanced ones, those unbalanced can be deduced from the balanced by Theorem 2.3), and it will be useful to the reader. For the sake of readability, the suffices and the polynomials variables will be omitted if clear from the context.

Lemma 4.4. *Let P_n and C_n be the path and the cycle on n vertices, respectively. Let $\mathrm{Spec}_M(\Gamma)$ denote the multiset of eigenvalues of $M(\Gamma)$.*

$$\begin{aligned}
\mathrm{Spec}_A\,(C_n,+) &= \{2\cos\frac{2k}{n}\pi,\ k=0,1,\dots,n-1\};\\
\mathrm{Spec}_A\,(C_n,\bar\sigma) &= \{2\cos\frac{2k+1}{n}\pi,\ k=0,1,\dots,n-1\};\\
\mathrm{Spec}_A\,(P_n) &= \{2\cos\frac{k}{n+1}\pi,\ k=1,2,\dots,n\};\\
\mathrm{Spec}_L\,(C_{2n},+) &= \{2+2\cos\frac{2k}{2n}\pi,\ k=0,1,\dots,2n-1\};\\
\mathrm{Spec}_L\,(C_{2n+1},+) &= \{2+2\cos\frac{2k+1}{2n+1}\pi,\ k=0,1,\dots,2n\};\\
\mathrm{Spec}_L\,(C_{2n},\bar\sigma) &= \{2+2\cos\frac{2k+1}{2n}\pi,\ k=0,1,\dots,2n-1\};\\
\mathrm{Spec}_L\,(C_{2n+1},\bar\sigma) &= \{2+2\cos\frac{2k}{2n+1}\pi,\ k=0,1,\dots,2n\};\\
\mathrm{Spec}_L\,(P_n) &= \{2+2\cos\frac{k}{n}\pi,\ k=1,2,\dots,n\}.
\end{aligned}$$

Remark 4.5. In view of the above lemma we get that $(C_{2n},+)$ is L-cospectral with $(C_n,+)\cup(C_n,\bar\sigma)$, and in view of Lemma 2.3 (i) the same applies to the spectrum of their corresponding adjacency matrices. Moreover, the L-spectrum of $(C_{2n+1},+)$ (resp., $(C_{2n+1},\bar\sigma)$) contains the L-spectrum of $(C_d,+)$ (resp., $(C_d,\bar\sigma)$) for any d divisor of $2n+1$. The L-spectrum of $(C_{2n},\bar\sigma)$ contains the L-spectrum of $(C_d,\bar\sigma)$ provided that $\frac{2n}{d}$ is an odd number. For example, the L-spectrum $(C_{120},\bar\sigma)$ contains the L-spectrum of $(C_d,\bar\sigma)$ only for $d\in\{8,24,40\}$. Similarly, the L-spectrum of $(C_{2n,+})$ contains the L-spectrum of $(C_d,+)$ for all divisors d of $2n$, while it also contains the L-spectrum of $(C_d,\bar\sigma)$ when $\frac{2n}{d}$ is an even number.

The lemma below stems from the above observations.

Lemma 4.6. *Let $(C_{2n},+)$ be an even balanced cycle and let $2n = 2^{t+1}r$, where t and r are positive integer and r is odd. If $r\geq 3$, then $(C_{2^{t+1}r},+)$ is L-cospectral with $(C_{2^s r},+)\bigcup_{i=s}^{t}(C_{2^i r},\bar\sigma)$, with $0\leq s\leq t$. If $r=1$ then $(C_{2^{t+1}},+)$ is L-cospectral with $(C_{2^s},+)\bigcup_{i=s}^{t}(C_{2^i},\bar\sigma)$, with $2\leq s\leq t$.*

Let $\mu(n) = 2+2\cos\frac{\pi}{n}$ be the Laplacian spectral radius of the path P_n; in view of Lemma 4.4 for n odd $\mu(n)$ is $\mu_1(C_n,+)$, while for n even it is $\mu_1(C_n,\bar\sigma)$. The observations of Remark 4.5 play a crucial role in the following theorem. Let $\mathrm{GCD}(a,b)$ be the greatest common divisor between the integers a and b. Also, let $[c(n),\sigma]$ be the set of the L-eigenvalues of multiplicity two of the cycle (C_n,σ).

Theorem 4.7. *The signed lollipop graph* $(L_{g,n},\sigma) = \Lambda$ *has simple* L*-eigenvalues if* $\mathrm{GCD}(g,n) = 1$*. If* $\mathrm{GCD}(g,n) = d \geq 2$*, then we have the following possibilities*

- *if* g *is odd, then the eigenvalues of* Λ *of multiplicity two are those of* $[c(d),\sigma]$*;*
- *if* g *is even,* $\frac{d}{g}$ *odd (resp., even), and* $\sigma = +$*, then the eigenvalues of* Λ *of multiplicity two are those of* $[c(d),+]$ *(resp.,* $[c(2d),+]$*);*
- *if* g *is even and* $\sigma = \bar{\sigma}$*, then for* $\frac{g}{d}$ *odd the eigenvalues of* Λ *of multiplicity two are those of* $[c(d),\bar{\sigma}]$*, while for* $\frac{g}{d}$ *even,* Λ *has just simple eigenvalues.*

Proof. Recall that the L-eigenvalues of signed cycles, other than 0 and 4, have multiplicity two.

First, note that P_n is an edge-deleted subgraph of $(L_{g,n},\sigma)$. Since P_n has only simple eigenvalues, by Theorem 2.5 each L-eigenvalue has at most multiplicity two (and it must be an L-eigenvalue for P_n). Similarly, for $(C_g,\sigma) \cup P_{n-g}$ we have:

$$\begin{aligned}\mu_1(L_{g,n},\sigma) &\geq \mu_1((C_g,\sigma)\cup P_{n-g}) \geq \mu_2(L_{g,n},\sigma) \geq \mu_2((C_g,\sigma)\cup P_{n-g})\\ &\geq \cdots \geq \mu_n(L_{g,n},\sigma) \geq \mu_n((C_g,\sigma)\cup P_{n-g}).\end{aligned} \tag{2}$$

Let μ be, if any, an L-eigenvalue of multiplicity two, then μ is an L-eigenvalue of $(C_g,\sigma)\cup P_{n-g}$. Consider the subdivision graph $\mathcal{S}(L_{g,n},\sigma) = (L_{2g,2n},\sigma')$. By applying Theorem 2.9 at the hanging path edge that is incident to the vertex of degree 3, we have:

$$\phi(L_{2g,2n},\sigma') = \phi(C_{2g},\sigma')\phi(P_{2n-2g}) - \phi(P_{2g-1})\phi(P_{2n-2g-1}). \tag{3}$$

Let $\lambda = \sqrt{\mu}$, in view of Theorem 2.3, λ is an A-eigenvalue of multiplicity two, as well. From μ being an L-eigenvalue of $(C_g,\sigma)\cup P_{n-g}$, we deduce that λ is an A-eigenvalue of $(C_{2g},\sigma')\cup P_{2n-2g-1}$. Since λ is a root of $\phi(C_{2g},\sigma')$ or a root of $\phi(P_{2n-2g-1})$, then in (3) we have that λ is of multiplicity two if and only if λ is a root of both $\phi(C_{2g},\sigma')$ and $\phi(P_{2n-2g-1})$ (note, if $\lambda \neq 4$ is an A-eigenvalue of (C_{2g},σ'), then it is an A-eigenvalue of P_{2g-1}). The latter implies that μ is an L-eigenvalue of both (C_g,σ) and P_{n-g}. Clearly, if $d = GCD(g,n-g) = 1$, then such a μ cannot exist and the L-eigenvalues of Λ have multiplicity 1. So let $d \geq 2$ in the sequel.

Assume first that g is odd, then also d is odd. Since d divides both g and $n-g$, we have that $[c(g),\sigma] \cap \mathrm{Spec}_L(P_{n-g}) = [c(d),\sigma]$.

Assume next that g is even and $\sigma = +$. Since g is even then also the L-eigenvalues of $(C_r,\bar{\sigma})$ appear in $\mathrm{Spec}_L(C_{g,n},+)$ for any divisor r of g such that $\frac{g}{r}$ is even (note, r is a proper divisor). Hence, if $\frac{g}{d}$ is even, then d divides both g and $n-g$, and we have that $[c(g),+] \cap \mathrm{Spec}_L(P_{n-g}) = [c(2d),+]$. If instead we have $\frac{g}{d}$ odd, then we get $[c(g),+]\cap\mathrm{Spec}_L(P_{n-g}) = [c(d),+]$. In particular, if $g = d$ or $g = 2d$, then the eigenvalues of multiplicity two are those of $[c(g),+]$.

Finally, assume that g is even and $\sigma = \bar{\sigma}$, then $[c(g),\bar{\sigma}] \cap \mathrm{Spec}_L(P_{n-g})$ is non-empty if and only if $\frac{g}{d}$ is odd, and in the latter case we get $[c(d),\bar{\sigma}]$; if $\frac{g}{d}$ is even, then $\frac{g}{r}$ is even for all the divisors r of d, hence $[c(g),\bar{\sigma}] \cap \mathrm{Spec}_L(P_{n-g}) = \emptyset$. □

Corollary 4.8. *Let Γ be a signed graph L-cospectral with $(L_{g,n}, \sigma)$, with $GCD(g,n) = d \geq 3$. If (C_r, σ) is a component of Γ, then r divides d.*

Proof. Since (C_r, σ) is a component of Γ, then $\mu(r)$ is in the spectrum of Λ with multiplicity two. Hence, according to Theorem 4.7, $\mu(r)$ is in the spectrum of (C_g, σ) and of P_{n-g}. Consequently, r divides both g and $n-g$, that is, r divides d as well. □

Corollary 4.9. *Let Γ be a signed graph L-cospectral with $(L_{g,n}, \sigma)$. If $GCD(g,n) = d \leq 2$, then Γ is connected.*

Proof. Recall that in view of Lemma 4.3, any L-cospectral mate of a signed lollipop graph consists of a lollipop graph with possibly one or more cycles as components. From Theorem 4.7, we deduce that $(L_{g,n}, \sigma)$ has simple eigenvalues when $\mathrm{GCD}(g,n) = 1$, consequently any L-cospectral mate Γ cannot have cycles as components, as cycles carry eigenvalues of multiplicity two. If $\mathrm{GCD}(g,n) = 2$ there could be eigenvalues of multiplicity two but they belong to, say, degenerate cycle (C_2, σ), which are not allowed. Hence, when $\mathrm{GCD}(g,n) \leq 2$, Γ must be connected. □

Lemma 4.10. *Let $\psi(\Gamma, x) = \sum_{i=0}^{n} (-1)^n \, b_i(\Gamma) \, x^{n-1}$. Then we have:*

$$b_n(C_n, +) = 0, \quad b_n(C_n, \bar{\sigma}) = 4, \quad b_n(L_{n,g}, +) = 0, \quad b_n(L_{n,g}, \bar{\sigma}) = 4,$$
$$b_{n-1}(C_n, \sigma) = n^2, \quad b_{n-1}(L_{n,g}, +) = gn, \quad b_{n-1}(L_{n,g}, \bar{\sigma}) = gn + 2(n-g)(n-g+1).$$

Proof. The proof is a straightforward application of Theorem 2.1. □

Theorem 4.11. *Let Γ be an L-cospectral mate of $(L_{g,n}, \bar{\sigma})$. Then Γ is connected.*

Proof. By Lemma 4.3, Γ is a disjoint union of a signed lollipop graph with possibly one or more signed cycles. In view of Theorem 3.5, since $(L_{g,n}, \bar{\sigma})$ is unbalanced then Γ cannot have any balanced component, which implies that Γ can possibly have just unbalanced cycles as components. However if Γ consists of $t \geq 2$ components, all of them unicyclic and unbalanced, then $b_n(\Gamma) = 4^t > 4 = b_n(L_{g,n}, \bar{\sigma})$, that is a contradiction. Hence, Γ is a connected graph. □

Theorem 4.12. *Let Γ be an L-cospectral mate of $\Lambda = (L_{g,n}, +)$, with $d = GCD(g,n)$ an odd number. If $g \neq d$ and $n \neq 4d$, then Γ is connected.*

Proof. If $GCD(g,n) = d \leq 2$ the assertion is obviously true, so let $GCD(g,n) = d \geq 3$ for the remainder of the proof. If Γ is disconnected then Γ has one or more cycles as components. Let Λ' be the lollipop component of Γ.

Assume first that Γ has an unbalanced cycle component, say, $(C_s, \bar{\sigma})$ with s even. If so, $\mu(s)$ is in the spectrum of Λ with multiplicity two, and s divides g and $n-g$, and consequently also d. But d is odd, so s cannot divide d, and Γ cannot have unbalanced cycles as component. So Γ has a positive cycle and Λ' is an unbalanced component. Assume next that $(C_r, +)$ is the positive cycle of Γ, then $\mu(r)$ is an eigenvalue of Γ with multiplicity two and the same applies to Λ. Similarly to above, r must divide d, so the only possibility is that $d = kr$ with k odd.

Let $k > 1$. Hence, Λ contains the eigenvalue $\mu(kr)$, which cannot belong to the component $(C_r, +)$. The latter implies that $\mu(kr)$ is in Λ', but Λ' has an unbalanced cycle $(C_{g'}, \bar{\sigma})$ as subgraph and $\mu(kr)$ cannot be an eigenvalue of $(C_{g'}, \bar{\sigma})$. So it is $k = 1$ and $r = d$.

By Lemma 4.10, we have that $b_{n-1}(\Gamma) = 4d^2 = gn = b_{n-1}(\Lambda)$. Since d divides both g and n, the latter equality implies that either $g = d$ and $n = 4d$, or $g = 2d$ and $n = 2d$, or $g = 4d$ and $n = d$. Clearly, it is $g < n$ and the only acceptable values are $g = d$ and $n = 4d$. Also, Λ' has order $n' = n - r = 3d$. The latter special case requires additional investigation, so we will consider it separately in a subsequent lemma. In all other cases, Γ must be a connected signed graph, hence it reduces to the lollipop component Λ'.

This completes the proof. □

For the case d even, we need a more involved analysis due to the fact that $(C_{2n}, +)$ is not a DLS graphs.

Let B_n be the matrix of order n obtained from $L(P_{n+1})$ by deleting the row and column corresponding to some end-vertex of P_{n+1}. Let H_n be the matrix of order n obtained from $L(P_{n+2})$ by deleting the rows and columns corresponding to both the end vertices of P_{n+2} respectively. Both matrices represent augmented paths so their spectrum is not depending on the signature of the edges. The first two of the following equalities were given by Guo in [12], the third is proved in [19].

Lemma 4.13. *Let P_n be the path of order n and H_n, B_n defined as above. Then*

(i) $x\,\psi(B_n) = \psi(P_{n+1}) + \psi(P_n)$,
(ii) $\psi(P_n) = x\,\psi(H_{n-1})$,
(iii) $\psi(P_n) = (x - 2)\psi(P_{n-1}) - \psi(P_{n-2})$.

We now express the L-polynomials of signed cycles and signed lollipop graphs in terms of those of paths. For a signed unicyclic graph Γ of girth g, let $\varsigma(\Gamma) = (-1)^{g+1}\sigma(\Gamma)$.

Lemma 4.14. *We have the following equalities*

$$\psi(C_n, \sigma) = \frac{\psi(P_{n+1})}{x} - \frac{\psi(P_{n-1})}{x} + 2\varsigma(C_n, \sigma),$$

and

$$\psi(L_{g,n}\bar{\sigma}) = \frac{1}{x}(\psi(P_{n-g+1}) + \psi(P_{n-g}))\Big[\frac{x-3}{x}\psi(P_g) - \frac{2}{x}\psi(P_{g-1}) + 2\varsigma(\Lambda)\Big] - \frac{1}{x^2}(\psi(P_{n-g}) + \psi(P_{n-g-1}))\psi(P_g).$$

Proof. The results can be easily obtained by iterated use of Theorem 2.9, combined with Lemma 4.13.

In fact, for $\psi(C_n, \sigma)$, in view of Theorem 2.9, we have the following decomposition whose result depends on the parity of n and the value $\sigma(C_n, \sigma)$:

$$\begin{aligned}\psi(C_n, \sigma) &= \psi(H_n) - \psi(H_{n-2}) - 2(-1)^n \sigma(C_n, \sigma) \\ &= \frac{\psi(P_{n+1})}{x} - \frac{\psi(P_{n-1})}{x} + 2\varsigma(C_n, \sigma).\end{aligned}$$

A similar computation holds for $\Lambda = (L_{g,n}, \sigma)$:

$$\begin{aligned}\psi(\Lambda) &= (x-3)\psi(H_{g-1})\psi(B_{n-g}) - 2\psi(H_{g-2})\psi(B_{n-g}) - \psi(H_{g-1})\psi(B_{n-g-1}) \\ &\quad - 2(-1)^g \sigma(\Lambda)\psi(B_{n-g}) \\ &= \psi(B_{n-g})\Big[\frac{x-3}{x}\psi(P_g) - \frac{2}{x}\psi(P_{g-1}) + 2\varsigma(\Lambda)\Big] - \frac{1}{x}\psi(B_{n-g-1})\psi(P_g) \\ &= \frac{1}{x}(\psi(P_{n-g+1}) + \psi(P_{n-g}))\Big[\frac{x-3}{x}\psi(P_g) - \frac{2}{x}\psi(P_{g-1}) + 2\varsigma(\Lambda)\Big] \\ &\quad - \frac{1}{x^2}(\psi(P_{n-g}) + \psi(P_{n-g-1}))\psi(P_g).\end{aligned}$$

This completes the proof. $\square$

Lemma 4.15. *We have*

$$\psi(P_n, 4) = 4n; \quad \psi((C_{2n}, +), 4) = \psi((C_{2n+1}, \bar{\sigma}), 4) = 0;$$
$$\psi((C_{2n+1}, +), 4) = \psi((C_{2n}, \bar{\sigma}), 4) = 4;$$
$$\psi((L_{g,n}, +), 4) = \begin{cases} -4g(n-g), & g \text{ is even}, \\ -4[g(n-g) - (2n-2g+1)], & g \text{ is odd}; \end{cases}$$
$$\psi((L_{g,n}\bar{\sigma}), 4) = \begin{cases} -4[g(n-g) - (2n-2g+1)], & g \text{ is even}, \\ -4g(n-g), & g \text{ is odd}. \end{cases}$$

Proof. The results can be easily obtained by Lemma 4.14. In fact, $\psi(P_1, 4) = 4$ and by induction $\psi(P_n, 4) = (4-2)\psi(P_{n-1}, 4) - \psi(P_{n-2}, 4) = 2(4n-4) - (4n-8) = 4n$.

$$\begin{aligned}\psi((C_n, \sigma), 4) &= \frac{\psi(P_{n+1}, 4)}{4} - \frac{\psi(P_{n-1}, 4)}{4} + 2\varsigma(C_n, \sigma) \\ &= n + 1 - n + 1 + 2\varsigma(C_n, \sigma) = 2 + 2\varsigma(C_n, \sigma).\end{aligned}$$

A similar computation holds for $\Lambda = (L_{g,n}, \sigma)$:

$$\begin{aligned}\psi(\Lambda,4) &= \frac{1}{4}(\psi(P_{n-g+1},4)+\psi(P_{n-g},4))\Big[\frac{4-3}{4}\psi(P_g,4)-\frac{2}{4}\psi(P_{g-1},4)+2\varsigma(\Lambda)\Big]\\ &\quad -\frac{1}{4^2}(\psi(P_{n-g},4)+\psi(P_{n-g-1},4))\psi(P_g,4)\\ &= -4[g(n-g)+(2+2\varsigma(\Lambda))(2n-2g+1)].\end{aligned}$$

This completes the proof. □

Lemma 4.16. *Let $(L_{g,n},\sigma)=\Lambda$ and $(L_{g',n'},\sigma')=\Lambda'$ be two signed lollipop graphs such that $2n'\leq n$. Then $\mu_2(\Lambda)>\mu_2(\Lambda')$.*

Proof. Since the L-eigenvalues of Λ are interlaced by those of P_n, we have that $\mu_2(\Lambda)\geq\mu_2(P_n)\geq\mu_1(P_{n'})\geq\mu_2(\Lambda')$. We next prove that it is $\mu_1(P_n)\neq\mu_2(\Lambda)$, so that the last inequality is strict.

Assume first that either g odd and $\sigma=+$, or g even and $\sigma=\bar{\sigma}$, then by Lemma 4.4 and (2) for $n>g$ it is $\mu_1(P_n)>\mu_1((C_g,\sigma)\cup P_{n-g})\geq\mu_2(\Lambda)$.

Consider next the case when either g is even and $\sigma=+$, or g odd and $\sigma=\bar{\sigma}$; in both cases the subdivision graph of $(L_{g,n},\sigma)$ is $(L_{2g,2n},+)$, since they both contain an even number of positive edges in the cycle. We show that $(L_{2g,2n},+)$ does not have $\lambda_1(P_{2n-1})=\sqrt{\mu_1(P_n)}=\underline{\lambda}$ as an A-eigenvalue, which implies that $(L_{g,n},\sigma)$ cannot have $\mu_1(P_n)$ as an L-eigenvalue. Apply Theorem 2.9 to one vertex in the cycle of degree 2 adjacent to the vertex of degree 3. We have:

$$\phi((L_{2g,2n},+),x)=x\phi(P_{2n-1})-\phi(P_{2n-2},x)-\phi(P_{2g-2},x)[2+\phi(P_{2n-2g},x)],$$

from which we deduce that

$$\phi((L_{2g,2n},+),\underline{\lambda})=-\phi(P_{2n-2},\underline{\lambda})-\phi(P_{2g-2},\underline{\lambda})[2+\phi(P_{2n-2g},\underline{\lambda})]<0.$$

Consequently, $\mu_2(\Lambda)\geq\mu_1(P_{n'})>\mu_2(\Lambda')$. This completes the proof. □

Lemma 4.17. *Let Γ be a disconnected L-cospectral mate of $\Lambda=(L_{g,n},+)$. If either $n\neq 2g$, or $n=2g$ with g even, then $\mu_2(\Gamma)$ is not an eigenvalue of the cycle components of Γ.*

Proof. Since Γ is disconnected than Γ has at least one cycle as component. Let $GCD(g,n)=d$, if $d\leq 2$, Γ must be connected, so we consider $d\geq 3$. Recall that d divides both g and $n-g$, hence $n=kd$ with $k\geq 2$. According to Corollary 4.8, if (C_r,σ) is a cycle of Γ, then r divides d; recall that $\mu_1(C_r,\sigma)<4$. So r is at most $\frac{n}{2}$, and the latter equality is possible if and only if $g=n-g=d$, that is $n=2g$. Assume that $n\neq 2g$, then $r<\frac{n}{2}$. Consequently, by interlacing, we have $\mu_2(\Gamma)=\mu_2(\Lambda)\geq\mu_2(P_n)>\mu_1(C_r,\sigma)$.

To complete the proof we need to consider the case $n=2g$ and g even. So assume that Γ has one cycle of order g. Clearly, $\mu_1(C_g,+)=4$, so we need to consider only $(C_g,\bar{\sigma})$. Observe that $\mu_1(C_g,\bar{\sigma})=\mu_1P_g=\mu_2(P_n)$. We will use a similar strategy to the one used in Lemma 4.16, in fact we will show that $\mu_1(P_g)$ is not an L-eigenvalue of $(L_{g,2g},+)$ by showing that $\lambda_1(P_{2g-1})$ is not an A-eigenvalue of $(L_{2g,4g},+)$. Let us use Theorem 2.9 at the vertex of degree 3, we then obtain

$$\phi(L_{2g,4g},+) = x\phi(P_{2g})\phi(P_{2g-1}) - 2\phi(P_{2g})\phi(P_{2g-2}) - \phi^2(P_{2g-1}) - 2\phi(P_{2g}).$$

By computing the polynomials in $\lambda_1(P_{2g-1}) = \underline{\lambda}$ we have

$$\phi((L_{2g,4g},+),\underline{\lambda}) = -2\phi(P_{2g},\underline{\lambda})[\phi(P_{2g-2},\underline{\lambda}) + 1] > 0,$$

since $\lambda_2(P_{2g}) < \underline{\lambda} < \lambda_1(P_{2g})$ and $\lambda_1(P_{2g-2}) < \underline{\lambda}$. The latter shows that indeed $\lambda_2(L_{g,2g},+) > \mu_2(P_n) = \mu_1(P_g) = \mu_1(C_g,\bar{\sigma})$. This completes the proof. $\square$

We can finally prove the result below.

Theorem 4.18. *Let Γ be an L-cospectral mate of $\Lambda = (L_{g,n},+)$, with $d = GCD(g,n)$ an even number. Then Γ is connected.*

Proof. Since Λ is a balanced lollipop whose $GCD(g,n) = d$ is even, then by Theorem 4.7 the eigenvalues of multiplicity two for Λ are those of $[c(2k),+]$, for some number k equal to either d or $2d$. The even number $2k$ can be written in the form $2^{t+1}r$, where r is a positive odd number. We give the proof for $r \geq 3$, the case $r = 1$ can be solved similarly. By Lemma 4.6 we have that $(C_{2^{t+1}r},+)$ is cospectral with $(C_{2^s r},+)\bigcup_{i=s}^{t}(C_{2^i r},\bar{\sigma})$ for any $0 \leq s \leq t$.

Let Γ be a disconnected tentative cospectral mate of Λ, and denote by $\Lambda' = (L_{g',n'},\sigma)$ the lollipop component of Γ. In the sequel we show that Γ should be one of the two following signed graphs:

(i) $\Gamma = \Lambda' \cup (C_r,+)\bigcup_{i=0}^{t}(C_{2^i r},\bar{\sigma})$;
(ii) $\Gamma = \Lambda'\bigcup_{i=s}^{t}(C_{2^i r},\bar{\sigma})$.

From Γ being disconnected, we have that Γ has at least one cycle.

Assume that there is a balanced cycle $(C_q,+)$ among its components, then Λ' is unbalanced. The value q divides g and $n-g$ (see Corollary 4.8), but then it divides r as well, due to r being the greatest odd factor of $GCD(g,n-g)$. Evidently, $q = r$ otherwise if $q < r$, $\mu(r)$ cannot be an eigenvalue of $(C_q,+)$ or of Λ', and thus of Γ, while it appears in Λ. Also, Λ' must contain the eigenvalues of $[c(r),\bar{\sigma}]$ with the same multiplicity, since these eigenvalues cannot appear in some cycle component, and the latter implies that $GCD(g',n') = kr$, with k odd. However it is $k = 1$, otherwise Λ' contains the eigenvalues of multiplicity two of a longer odd unbalanced cycle whose eigenvalues do not appear in Λ. In addition g' is odd, otherwise g' is even, $\frac{g'}{r}$ is also even, and the eigenvalues of $[c(r),\bar{\sigma}]$ cannot appear in Λ' with multiplicity two. Now, since $GCD(g',n')$ is odd, then Λ' cannot have the eigenvalues of unbalanced even cycles, necessary to complete the spectrum of $(C_{2^t r}, +)$, the latter implies that Γ must have an unbalanced even cycle for each necessary even multiple of r. Consequently, Γ is of type (i).

Assume next that Γ has not any balanced cycle as a component. In this case Λ' is balanced and it contains the eigenvalues of both $[c(r), +]$ and $[c(r), \bar{\sigma}]$ with multiplicity two, which implies that Λ' has the eigenvalues of $[c(2r), +]$ with multiplicity two. The latter implies that $2r$ divides g', and g' must be even. Let $(C_{q'}, \bar{\sigma})$ be the shortest unbalanced even cycle component of Γ. Clearly, q' must divide $2^t r$, but it must be of the form $2^s r$, where $s \geq 1$. In fact, let $q' = 2^s$ for some $2 \leq s \leq t$. If so, for any $s' \geq s$, neither Γ can have some cycle component $C(2^{s'} r, \bar{\sigma})$, as it would lead to common eigenvalues of multiplicity two among the cycle components, nor Λ' can have as eigenvalues of multiplicity two those of $[c(2^{s'} r), \bar{\sigma}]$, because then those of $[c(2^{s'+1} r), +]$ are in Λ' with multiplicity two and, due to $\frac{2^{s'+1} r}{2^{s'}} = 2r$ being even, we get that also the eigenvalues of $[c(2^{s'}), \bar{\sigma}]$ are eigenvalues for Λ' with multiplicity two, leading again to eigenvalues of multiplicity greater than two. Hence, q' must be of the form $2^s r$. If $s > 1$, the eigenvalue $\mu(2^{s'} r)$, with $0 \leq s' \leq s - 1$ is in Λ and it must appear in Γ as well. Since $(C_{2^s r}, \bar{\sigma})$ does not contain the eigenvalues of $(C_{2^{s'} r}, \bar{\sigma})$ for $s' < s$, it implies that $\mu(2^{s'} r)$ cannot appear in some cycle component $(C_{2^{s'} r}, \bar{\sigma})$ (due to the minimality of s), so it must appear in Λ'. The latter implies that the eigenvalues of $[c(2^s r), +]$ appear with multiplicity two for Λ'. Now, for every $s \leq s' \leq t$ we have the cycle $(C_{2^{s'} r}, \bar{\sigma})$ is a component of Γ. If not, then some $\mu(2^{s'} r)$, with $s' > s$, appears in Λ' with multiplicity two, together with the eigenvalues $[c(2^{s'+1} r), +]$. Then $\mu(2^s r)$ appears in both Λ' and $(C_{2^s r}, \bar{\sigma})$, and the multiplicity of $\mu(2^s r)$ jumps to four, a contradiction. Hence, Γ is of the form (ii).

The next step is to show that both forms (i) and (ii) are not admissible for Γ, by comparing the spectral invariants b_{n-1} (cf. Lemma 4.10) and the polynomial computed at 4 (cf. Lemma 4.15). For Λ we have that $b_{n-1}(\Lambda) = gn$ and $\psi(\Lambda, 4) = -4g(n - g)$.

Assume first that Γ is of type (i). Recall that Λ' is unbalanced and g' is odd. In this case, we have that $b_{n-1}(\Gamma) = 4^{t+1} r^2$ and $\psi(\Gamma, 4) = -4^{t+2}(g'(n' - g')$. So we get the system

$$\begin{cases} gn = 4^{t+1} r^2; \\ -4g(n - g) = -4^{t+2} g'(n' - g'), \end{cases}$$

from which we get that $g^2 = 4^{t+1}(r^2 - g'(n' - g'))$ (recall that $n' > g'$). Clearly, the quantity $r^2 - g'(n' - g')$ must be positive, that is $r^2 > g'(n' - g')$ but the latter inequality has no solutions since r divides both g' and $n' - g'$. Hence Γ is not of type (i).

Assume now that Γ is of type (ii). Recall that Λ' is balanced and g' is even. In this case we have that $b_{n-1}(\Gamma) = 4^{t-s+1} g' n'$ and $\psi(\Gamma, 4) = -4^{t-s+2} g'(n' - g')$. Now we obtain the system

$$\begin{cases} gn = 4^{t-s+1} g' n'; \\ -4g(n - g) = -4^{t-s+2} g'(n' - g'), \end{cases}$$

whose solutions are $g = 2^{t-s+1}g'$ and $n = 2^{t-s+1}n'$. The latter equality implies that $n \geq 2n'$ and by Lemmas 4.16 and 4.17, we obtain that $\mu_2(\Lambda) > \mu_2(\Gamma)$. Hence, Γ is not of type (ii).

If $r = 1$, an analogous proof holds, in which $\Gamma = (L_{g',n'}, +) \bigcup_{i=s}^{t} (C_{2^i}, \bar{\sigma})$, with $s \geq 2$ being the shortest length of the unbalanced cycle component of Γ. We leave the details to the reader.

This completes the proof. □

Theorem 4.19. *No two non-switching isomorphic signed lollipop graphs are L-cospectral.*

Proof. Let $\Lambda = (L_{g,n}, \sigma)$ be a signed lollipop graph. In Lemma 4.15 we have decomposed the L-polynomial of Λ in the combination of paths polynomials.

$$\psi(\Lambda, x) = \frac{1}{x}(\psi(P_{n-g+1}) + \psi(P_{n-g}))\Big[\frac{x-3}{x}\psi(P_g) - \frac{2}{x}\psi(P_{g-1}) + 2\varsigma\Big] - \frac{1}{x^2}\psi(P_g)(\psi(P_{n-g}) + \psi(P_{n-g-1})). \tag{4}$$

Consider Lemma 4.13 (iii), the formula $\psi(P_n) = (x-2)\psi(P_{n-1}) - \psi(P_{n-2})$ can be seen as a homogeneous second order recurrence equation

$$p_n = (x-2)p_{n-1} - p_{n-2},$$

with $p_0 = 0$ and $p_1 = x$ as boundary conditions. It is a matter of computation (cf. [19] for the details) to check that the solution is

$$p_n = \frac{(y^{2n} - 1)(y+1)}{y^n(y-1)},$$

where y is the solution of the characteristic equation $y^2 - (x-2)y + 1 = 0$.

For any signed graph Γ, let

$$\Phi(\Gamma) = y^n\,(y-1)^2\,\psi(\Gamma, y) - (y^{2n+2} - 2y^{2n+1} - 2y + 1),$$

then, by applying the above described transformation to (4), we get

$$\Phi(L_{g,n}, \sigma) = 2\varsigma y^{2n-g+2} - 2\varsigma y^{2n-g+1} + y^{2n-2g+2} + y^{2g} - 2\varsigma y^{g+1} + 2\varsigma y^g. \tag{5}$$

From the above polynomial, it is evident that two signed lollipop graphs are L-cospectral if and only if both g and $\sigma(\Lambda)$ are the same, namely, the two signed lollipop graphs are also switching equivalent. This completes the proof. □

By using the comparison technique of the above theorem, we now deal with the last case, left by Theorem 4.12.

Lemma 4.20. *The signed graphs $(L_{d,4d}, +)$ and $(L_{g',3d}, \bar{\sigma}) \cup (C_d, +)$, with d odd, are not L-cospectral.*

Proof. We compare the polynomials in order to obtain compatible values for g'.

From Lemma 4.14, we have the polynomial of the odd balanced cycle $(C_d, +)$, that is

$$\psi(C_d, +) = \frac{\psi(P_{d+1})}{x} - \frac{\psi(P_{d-1})}{x} + 2.$$

Let $\varsigma' = \varsigma(L_{g',3d})$. After some computations we get for $\Gamma = (L_{g',3d}, \bar{\sigma}) \cup (C_d, +)$ the below polynomial

$$\begin{aligned}\Phi(\Gamma) = {} & 2\varsigma' y^{8d-g'+2} - 2\varsigma' y^{8d-g'+1} + y^{8d-2g+2} + 4\varsigma' y^{7d-g'+2} - 4\varsigma' y^{7d-2g'+2} \\ & + 2\varsigma' y^{6d-g'+2} - 2\varsigma' y^{6d-g'+1} + y^{6d-2g'+2} + y^{6d+2} - 2y^{6d+1} + y^{2d+2g} \\ & - 2\varsigma' y^{2d+g'+1} + 2\varsigma' y^{2d+g'} - 2y^{2d+1} + y^{2d} + 2y^{d+2g} - 4\varsigma' y^{d+g'+1} \\ & + 4\varsigma' y^{d+g'+1} + 4\varsigma' y^{d+g'} + y^{2g'} - 2\varsigma' y^{g'+1} + 2\varsigma' y^{g} - 4y^{d+1} + 2y^{d}.\end{aligned}$$

For the ease of comparison, we also write the polynomial corresponding to $\Lambda = (L_{d,4d}, +)$. Recall that d is odd and $\sigma = +$, hence $\varsigma = 1$.

$$\Phi(L_{d,4d}, +) = 2y^{7d+2} - 2y^{7d+1} + y^{6d+2} + y^{2d} - 2y^{d+1} + 2y^{d}.$$

We next compare the lowest degree monomials of both the above polynomials. For $\Phi(\Lambda)$ it is $2y^d$, while for $\Phi(\Gamma)$ we have three candidates, namely $y^{6d-2g'+2}$, $2\varsigma y^{g'}$ and $2y^d$. Since the polynomial must be the same, we deduce that $g' > d$ and $g' < \frac{1}{2}(5d + 2)$. If we look at the monomials of degree $d + 1$, we have for $\Phi(\Lambda)$ that it is $2y^{d+1}$. So $\Phi(\Gamma)$ should have the same monomial, and the only possibility is that $g' = d + 1$ and $\varsigma' = 1$. But with the latter substitution the two polynomials do not coincide. Hence, Γ cannot be cospectral with Λ. $\square$

We can finally state the main result of this section.

Theorem 4.21. *The signed lollipop graph $(L_{g,n}, \sigma)$ is determined by the spectrum of its Laplacian matrix.*

Proof. Let Γ be a tentative L-cospectral mate of $(L_{g,n}, \sigma) = \Lambda$. According to Lemma 4.3, Γ is a signed lollipop graph with possibly one or more signed cycles. If $\sigma(\Lambda) = \bar{\sigma}$, by Theorem 4.11 we get that Γ is connected, and it reduces to a signed lollipop graph. If $\sigma(\Lambda) = +$ by Theorems 4.12 and 4.18, we get that, excluding the special case $n = 4g$ and $\sigma = +$, the tentative cospectral mate Γ is connected, and it reduces to a signed lollipop graph. By Theorem 4.19, if Γ is a signed lollipop graph, then it is switching isomorphic to $(L_{g,n}, \sigma)$. The remaining special case is considered in Lemma 4.20 and it leads to non-cospectral graphs. $\square$

From Theorem 3.6 (iii), we deduce that the (signed) subdivisions of signed lollipop graphs that A-cospectral mates cannot be subdivision graphs. Since the subdivision of lollipop graph is a lollipop graph with even order and even girth not less than 6, the following corollary holds:

Corollary 4.22. *Let Γ be A-cospectral with a signed lollipop graph $(L_{2g,2n}, \sigma)$, where $g \geq 3$. If Γ is a subdivision graph, then Γ is switching isomorphic to $(L_{g,2n}, \sigma)$.*

Clearly, it is interesting to solve the spectral determination problem of signed lollipop graph with respect to the adjacency spectrum, using the results of this paper together with those from [17,18] and others. However, we shall not attempt to do it within this article.

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References

[1] B.D. Acharya, Spectral criterion for cycle balance in networks, J. Graph Theory 4 (1) (1980) 1–11.
[2] B.D. Acharya, M. Acharya, D. Sinha, Characterization of a signed graph whose signed line graph is S-consistent, Bull. Malays. Math. Sci. Soc. 32 (3) (2009) 335–341.
[3] F. Belardo, E.M. Li Marzi, S.K. Simić, Combinatorial approach for computing the characteristic polynomial of a matrix, Linear Algebra Appl. 433 (2010) 1513–1523.
[4] F. Belardo, P. Petecki, J.F. Wang, On signed graphs whose second largest L-eigenvalue does not exceed 3, submitted for publication.
[5] F. Belardo, S.K. Simić, On the Laplacian coefficients of signed graphs, Linear Algebra Appl. 475 (2015) 94–113.
[6] R. Boulet, B. Jouve, The lollipop graph is determined by its spectrum, Electron. J. Combin. 15 (2008), research paper R#74.
[7] S. Chaiken, A combinatorial proof of the all minors matrix tree theorem, SIAM J. Algebr. Discrete Methods 3 (3) (1982) 319–329.
[8] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs – Theory and Applications, 3rd revised and enlarged edition, Johan Ambrosius Bart. Verlag, Heidelberg–Leipzig, 1995.
[9] D. Cvetković, P. Rowlinson, S.K. Simić, Eigenvalue bounds for the signless Laplacian, Publ. Inst. Math. (Beograd) (N.S.) 81 (95) (2007) 11–27.
[10] D. Cvetković, P. Rowlinson, S.K. Simić, Signless Laplacian of finite graphs, Linear Algebra Appl. 423 (2007) 155–171.
[11] Y.-Z. Fan, Largest eigenvalue of a unicyclic mixed graph, Appl. Math. J. Chinese Univ. Ser. B 19 (2) (2004) 140–148.
[12] J.-M. Guo, A conjecture on the algebraic connectivity of connected graphs with fixed girth, Discrete Math. 308 (2008) 5702–5711.
[13] G.Q. Guo, G. Wang, On the (signless) Laplacian spectral characterization of the line graphs of lollipop graphs, Linear Algebra Appl. 438 (12) (2013) 4595–4605.

[14] W.H. Haemers, X.-G. Liu, Y.-P. Zhang, Spectral characterizations of lollipop graphs, Linear Algebra Appl. 428 (2008) 2415–2423.
[15] H. Hamidzade, D. Kiani, Erratum to "The lollipop graph is determined by its Q-spectrum", Discrete Math. 310 (2010) 1649.
[16] Y. Hou, J. Li, Y. Pan, On the Laplacian eigenvalues of signed graphs, Linear Multilinear Algebra 51 (2003) 21–30.
[17] J.F. Wang, Q.X. Huang, F. Belardo, E.M. Li Marzi, A note on the spectral characterization of dumbbell graphs, Linear Algebra Appl. 431 (10) (2009) 1707–1714.
[18] J.F. Wang, F. Belardo, Q.X. Huang, E.M. Li Marzi, Spectral characterizations of dumbbell graphs, Electron. J. Combin. 17 (2010), #R42.
[19] J.F. Wang, F. Belardo, Q.-L. Zhang, Signless Laplacian spectral characterization of line graphs of T-shape trees, Linear Multilinear Algebra 62 (11) (2014) 1529–1545.
[20] T. Zaslavsky, Signed graphs, Discrete Appl. Math. 4 (1982) 47–74.
[21] T. Zaslavsky, Matrices in the theory of signed simple graphs, in: Advances in Discrete Mathematics and Applications, Mysore, 2008, in: Ramanujan Math. Soc. Lect. Notes Ser., vol. 13, Ramanujan Math. Soc., 2010, pp. 207–229.
[22] T. Zaslavsky, A mathematical bibliography of signed and gain graphs and allied areas, Electron. J. Combin., Dynamic Survey #DS8, http://www.combinatorics.org/ojs/index.php/eljc/article/view/DS8/pdf.
[23] T. Zaslavsky, Glossary of signed and gain graphs, Electron. J. Combin., Dynamic Survey #DS9, http://www.combinatorics.org/ojs/index.php/eljc/article/view/DS9/pdf.
[24] Y.P. Zhang, X.G. Liu, B.Y. Zhang, X.R. Yong, The lollipop graph is determined by its Q-spectrum, Discrete Math. 309 (2009) 3364–3369.
[25] X.-D. Zhang, J.-S. Li, The Laplacian spectrum of a mixed graph, Linear Algebra Appl. 353 (2002) 11–20.