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# Equilibrium price and advertisement distributions[☆]

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ABSTRACT

We consider an economy where many sellers sell identical goods to many buyers. Each seller has a unit supply and each buyer has a unit demand. The only possible information flow about prices is through costly advertising. We show that in equilibrium the sellers use mixed strategies in pricing which leads to price and advertisement distributions. With convex advertising costs each seller sends only one advertisement in the market. We also delineate a class of advertising costs which ensures that sellers may send multiple advertisements in equilibrium. Higher prices are advertised more than lower prices.

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## 1. Introduction

Butters's (1977) article on informative advertising is seminal in at least two respects. First, it is an equilibrium analysis of firms that compete both by prices and advertising. Secondly, the urn–ball meeting technology, which has become widely used in many fields in economics (in particular directed search models), is introduced. In the model there are multiple firms that produce a homogeneous good at a constant marginal cost. There are many consumers each with a unit demand and identical valuations. The consumers neither know where the goods are available, nor at which prices, unless they receive advertisements (ads, hereafter) from the firms. The firms send multiple ads at a constant unit cost, and the ads are randomly allocated amongst the consumers. Consumers who do not receive any ads cannot consume (there is no search by uninformed consumers). If a consumer receives multiple ads, she contacts the firm with the lowest price. In equilibrium the firms mix over prices which leads to price dispersion. All the firms send the same number of ads.

In the empirical advertising literature there are many papers which suggest that heavily advertised brands are more expensive than are less-advertised goods within the same class of goods (see Bagwell, 2007 for a comprehensive review of the literature). This phenomenon has been usually explained by persuasive advertising that alters consumers' tastes and brand loyalty. We apply a version of Butters's model, and show that this positive relationship between prices and advertising is a natural feature of informative advertising, too.[1] To achieve this result we deviate from Butters's model in two respects.

First, we assume that the firms are capacity constrained each firm possessing just one unit of an indivisible good. In Butters (1977) the firms have unlimited capacity which is in stark contrast with the more recent directed search literature: sellers have just one unit for sale (e.g. Burdett, Shi, and Wright, 2001), or firms have just one vacancy (e.g. Pissarides, 2000 and Shimer, 2005).

Second, we generalise the advertising cost scheme by considering a large class of cost functions which plays a crucial role in our set-up.

If the cost function is convex, as in Butters (1977), we show that each firm sends only one ad in equilibrium. Pricing is in mixed strategies, and the equilibrium price distribution of our model coincides with Butters (1977) once the parametrisation between the papers is harmonised (the number of consumers and their valuations of the good are normalised to unity and the cost of production to zero). This is a surprising result as the capacity constraint seems to play no role in pricing. The explanation hinges on the linear cost function in Butters (1977): sending $k$ more ads is equivalent to adding $k$ more firms who send one ad each. Hence, the equilibrium of Butters (1977) can be interpreted as a case in which each firm sends a single ad and there is a free entry.[2]

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1 We point out that interpreting the low prices that result from the mixed pricing strategy as 'sale'-prices is misleading. The concept of a sale would require a multiperiod model and it does not make sense in our static model.

2 Butters (1977) studies the limit case in which the number of firms is taken to infinity which makes each firm's profits zero. This limit case resembles free entry of firms. Moreover, as the number of the firms goes to infinity, there are

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Our main contribution is to delineate a class of cost functions such that in equilibrium multiple ads are sent. Pricing is still in mixed strategies. In equilibrium the support of the mixed strategy is divided into intervals such that in each interval the firms send the same number of ads, and *the number of ads increases with the price*. To the best of our knowledge, this is a somewhat novel equilibrium in the theoretical advertising literature.

In a multiple-ad equilibrium the advertising costs must be sufficiently concave. The positive relationship between the prices and the number of ads arises as the firms that price low do not face much competition, while those who price high are likely to be undercut if they send only one ad. Sending more ads increases the probability of a sale and the expected revenue. If the increase in revenue is greater than the increase in advertising costs, then the firm can also ask a higher price (in equilibrium these two effects must be equal). Since in our model the firms have a limited capacity and there is competition for the potential consumers, advertising has diminishing marginal revenue. Hence, price-increasing advertising necessitates, indeed, that the advertising expenditure per ad must fall as more ads are sent in a multiple-ad equilibrium.

On the other hand, the advertising costs cannot grow too slowly. The construction of the equilibrium presupposes that the consumers always contact the firm with the lowest price. This is obvious if there are no capacity constraints (i.e., in Butters, 1977); a consumer who contacts a firm always gets an object. However, if the firms are capacity constrained, not every consumer who receives an ad gets an object. This implies that a consumer who receives multiple ads may find it profitable to choose a higher price offer if it is more probable that she gets the object. To guarantee that the consumers contact the firm with the lowest price there is a minimum speed at which the costs have to increase. It is somewhat surprising that making the lowest priced good the most desirable for the consumers restricts the possible cost functions of the advertisers; this emphasises that the logic of the model with capacity constrained firms is different from that of unlimited capacity.

We delineate a class of cost functions that supports an equilibrium with multiple ads by using *functional equations*. The functional equations determine the upper and lower bounds for the changes in advertising costs. In this class, the equilibrium can be determined simply by examining the successive differences of the cost function. Moreover, the cost of the first ad (which can also include the entry or capacity costs) immediately fixes the highest possible number of ads sent by a single firm in equilibrium. It turns out that this maximum is decreasing in the cost of the first ad.

The theoretical contribution of our model stems from highlighting the issues that arise once we give up the assumption of unlimited capacity. In particular, constructing an equilibrium where the consumers regard low prices as more attractive than high prices turns out to restrict the growth of the cost function from below, while, more expectedly, the firms are willing to send multiple ads only if the growth rate is restricted from above. One would expect the same issues to arise if capacity were allowed to be at any finite level.

The positive association between the price and the number of ads requires the capacity constraint. We elaborate this in Section 5.

This paper is organised as follows: In Section 2 we relate our analysis to the literature. In Section 3 we list the set of assumptions and build the model. In Section 4 we define and construct so-called *configurations* for different amounts of ads sent in the market. After that we study which conditions are needed for a configuration to be an equilibrium. In Section 5 we discuss our findings, and in Section 6 we conclude the paper. We relegate all the proofs to the Appendix to improve readability.

some firms that do not send any ads, which can be interpreted as free exit of firms.

## 2. Related literature

Our analysis contributes to two different fields. The first consists of directed search models originated by Peters (1991) and Montgomery (1991). A typical application consists of buyers and sellers, the latter ones posting prices. These models aim to depict markets with frictions. The frictions are of coordination type, and they arise as each seller has only one good but in equilibrium she may be contacted by several buyers, or no buyers at all. The frictions, however, arise in a symmetric equilibrium; depending on the details of the model there may be asymmetric equilibria which do not give rise to frictions. Instead of price posting the sellers in our model send ads, and only those who receive the ads get informed about the offers in the market. In this set-up there is a unique equilibrium that gives rise to frictions.

Our results, in particular the price distribution, is reminiscent of what happens in models of noisy search. In these models there are features of directed search but the agents have only partial or noisy information about some aspects of the environment. For instance, in Shi (2018) the buyers enter a submarket based on the maximum price the sellers commit not to exceed. In the submarket the sellers contact the buyers making offers without knowing how many other sellers contact the same buyer. The optimal behaviour in the price offer subgame is mixing, and this results in a price distribution. In a similar vein Bethune et al. (2020) in a model of money and credit, and Acemoglu and Shimer (2000) in a model of labour market, assume that the contacting parties choose how much information they acquire about the deals available. In equilibrium they have only partial information which leads the parties who offer the deals to use mixed strategies as the buyers' partial information gives the offerers some monopoly power but at the same time exposes them to some competitive pressures. In our set-up the buyers have only partial information, albeit endogenously determined, about the available deals, while the sellers still face some competition as a buyer may get ads from several sellers. It is worth noticing that unlike in Bethune et al. (2020) and Acemoglu and Shimer (2000) it is the party that offers the deals, i.e., the sellers, who are responsible for the noisy environment.

The second field naturally deals with the economics of advertising. This is a very large area covered for instance in Bagwell (2007). We only mention a couple of models that are directly related to Butters (1977).

Robert and Stahl (1993) allow the consumers who remain uninformed to search. The model still exhibits price dispersion but a mass of sellers charge the highest price that is paid only by the searchers. Roberts and Stahl assume strictly convex advertising costs, and find that firms advertise lower prices more intensively which is just the opposite of our result. Convex costs and uninformed searchers imply that firms advertise "sale" prices more than high prices in equilibrium.[3]

In McAfee (1994) the firms choose a continuous advertising intensity instead of physical ads as in Butters (1977). McAfee shows that when the firms first choose the intensity and only after that the price, there is one high-intensity high-price firm in equilibrium, while the other firms advertise at lower intensity and mix in prices.

[3] Search makes the marginal benefit of sending an offer with a high price smaller than with a low price. In equilibrium the marginal benefit must be equal to the marginal cost, and then the convexity of advertising costs implies that low prices are advertised more than high prices.

Gomis-Porqueras, Julien, and Wang (2017) study a model which differs from Butters (1977) in two respects. Firms are capacity constrained, and advertising takes place by choosing intensity continuously. The cost of intensity is assumed convex and increasing, while we study a broader class of advertising cost schemes. Moreover, in our model each firm sends a finite number of ads, and there is a discrete jump in the cost between zero and one ads. This means that in our set-up it is natural to assume free entry and exit. In Gomis-Porqueras et al. (2017) the intensity of advertising can be continuously adjusted, and the market tightness is taken as a parameter. They study trading by both posted prices and auction, and in both cases find a unique equilibrium in pure strategies. As the number of firms is fixed the advertising intensity is non-monotonic in the number of buyers. If there are relatively many buyers there is little competition, and a low intensity in advertising results in trade with high probability. If there are relatively few buyers then there is a lot of competition, and the marginal pay-off from advertising is low. Consequently, the equilibrium advertising intensity is low. Between these extremes there is some competition and a need to make sure that advertising reaches the buyers. As a result there is more advertising than at the extremes. If, in our model, the number of firms were fixed we would expect same kind of non-monotonicity; with relatively few buyers some firms would not advertise at all.

## 3. Model

In the spirit of Butters (1977), we assume the following:

(i) There is a large economy with $S$ sellers and $B$ buyers. Denote the ratio of sellers to buyers by $\theta = \frac{S}{B}$. Since this ratio is the only relevant magnitude in the sequel, we normalise $B = 1$. Then the number of sellers is $S = \theta$.

(ii) All the sellers are risk neutral and have a unit supply of an identical good. They value the good at zero. Also the buyers are risk neutral and have a unit demand. The buyers value the good at unity.

(iii) There is free entry and exit of sellers.

(iv) The sellers can sell their goods only via sending ads. An ad contains the location and price of the good. Buyers who do not receive any ads cannot shop at all.

(v) The price and the number of ads are the choice variables of a seller.

(vi) The cost of sending $k$ ads is given by function $c(k)$ where $c : \mathbb{N}_0 \to \mathbb{R}_+$ such that $c(0) = 0$ and for all $k \in \mathbb{N}_0$, $\Delta c(k+1) \equiv c(k+1) - c(k) > 0$.

(vii) Buyers receive ads randomly and independently of all other ads and each buyer has an equal probability of receiving each offer. Receiving an ad and sending orders (i.e. contacting a seller) are costless for a buyer. Buyers can contact exactly one seller.

The timing of the static game is as follows. First, sellers set prices and send ads. Second, each buyer who has received an ad (or ads) makes an order. Lastly, the orders are executed by sellers. If a seller receives many orders, she chooses randomly with equal probabilities one buyer with whom to trade. There is no discounting between stages.

Since we consider a large economy where there is an infinite number of buyers and sellers, the random process by which the ads are allocated follows the Poisson distribution.[4]

In equilibrium, the sellers use mixed strategies in pricing. This can be seen by assuming the opposite. Suppose that all the sellers ask the same price and send a single ad. Then lowering the price a little leads to a discrete increase in the selling probability when a potential buyer receives ads from multiple sellers. This further increases the profits which is a contradiction. The same logic shows that there are no mass points or gaps which means that the support of the mixed strategy is some interval $[l, U] \subset \mathbb{R}$. The highest price is the value of buyers, $U = 1$, by two reasons. First, it is clear that it is never profitable to ask price greater than the value of the buyers. Second, if $U$ were less than 1, then it would be profitable to increase the price since it does not change the probability of sale (a buyer contacts a seller with the highest price only if she does not receive ads from any other seller). For the seller who asks the lowest price in the support, $l$, it is optimal to send only one ad; an ad always reaches a buyer and the probability of a sale with the lowest price is 1. The free entry and exit assumption implies that we must have $l = c(1)$.

A seller who asks the lowest price need not send more than one ad since this always leads to a sale. If each buyer chooses the lowest price offer she receives, then a seller who asks the highest price only sells if a buyer who receives his ad does not receive any other ads. In this light, we construct the equilibrium of the following type. Depending on the price, sellers send different numbers of ads such that the higher the price, the higher the number of ads sent. Denote a partition of a unit interval by $\mathcal{P}_n = \{p_i\}_{i=-1}^{n}$ where $p_{i-1} < p_i$ for all $i \in \{0, 1, \ldots, n\}$, $p_{-1} = 0$, and $p_n = 1$. A seller who advertises a price $p \in [p_{i-1}, p_i)$ sends $i$ ads, and a seller with the highest price, $p_n$, sends $n$ ads. The partition of the unit interval with a maximum of $n$ ads is illustrated in Fig. 1.

The equilibrium mixed strategy $F$ is a probability distribution over $[p_0, p_n]$ defined piecewise for each subinterval of the partition. The corresponding probability density function of $F$ is denoted by $f$ and called a *price distribution*.[5] In equilibrium, each seller makes zero profits (due to free entry). Here we assume that the buyers who receive multiple ads always contact the seller with the lowest price; we return to this point in Section 4.2.

## 4. Results

In this section we define configurations related to the partition of the unit interval; these are used to construct an equilibrium. In a 1-configuration all the sellers send exactly one ad, in a 2-configuration some sellers (low pricing ones) send one ad, and the others two ads, and in an $n$-configuration the sellers send different numbers of ads between one and $n$ as depicted in Fig. 1. In other words, configurations are indexed by their maximum number of ads that are sent. If no one wants to send more ads in a configuration and each buyer chooses the lowest price offer that she receives, then the configuration constitutes an equilibrium. That is why we start the analysis with careful derivation of configurations.

After defining configurations, we determine the conditions on the advertising costs that guarantee that a configuration constitutes an equilibrium. In particular, we determine a class of cost functions under which an $n$-configuration constitutes an equilibrium.

### 4.1. Configurations

For the formal definition of a configuration we need the following components:

[4] The idea is basically based on the following argument. Assume that there are $N$ discrete buyers, and each buyer has an equal probability of getting an ad — that is, $\frac{1}{N}$. Then for any total amount of ads $\theta N$, each buyer receives zero ads with probability $\left(1 - \frac{1}{N}\right)^{\theta N}$. This converges to $e^{-\theta}$ as $N$ goes to infinity. This is the Poisson probability with parameter $\theta > 0$.

[5] Note that by a price distribution we refer to a density function, not the distribution function.

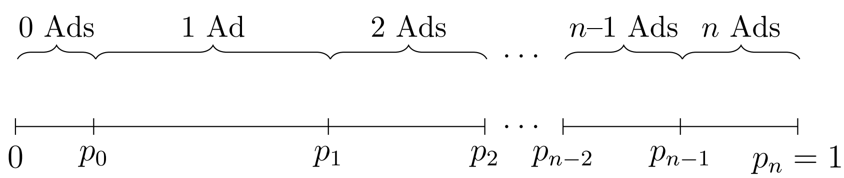


**Fig. 1.** The partition of the unit interval with a maximum of $n$ ads.

1. The partition of the unit interval $\mathcal{P}_n = \{p_i\}_{i=-1}^{n}$ which assigns prices to the number of ads such that sellers with prices in $[p_{i-1}, p_i)$ sends $i$ ads for all $i \in \{0, 1, \ldots, n\}$ and a seller with the highest price $p_n$ sends $n$ ads.
2. The mixed strategy $F_n$ over $[p_0, p_n]$ defined piecewise for each subinterval of the partition:
$$F_n(p) = \begin{cases} F_n^{(1)}(p) & \text{for } p \in [p_0, p_1) \\ F_n^{(2)}(p) & \text{for } p \in [p_1, p_2) \\ \vdots & \\ F_n^{(n)}(p) & \text{for } p \in [p_{n-1}, p_n]. \end{cases}$$
Since there are no gaps or mass points in the support, we have $F_n^{(i)}(p_i) = F_n^{(i+1)}(p_i)$ for all $i \in \{1, 2, \ldots, n-1\}$.
3. The number of sellers $\theta_n$.

Since the partition, the mixed strategy, and the number of sellers vary with configurations, we use the subscripts in each component to refer to the index of a configuration.

Using these components we define the expected number of ads with price less than $p \in [p_{i-1}, p_i]$ as follows:

$$\lambda_n(p) = \sum_{j=1}^{i-1} j \cdot \left(F_n^{(j)}(p_j) - F_n^{(j)}(p_{j-1})\right) \cdot \theta_n + i \cdot \left(F_n^{(i)}(p) - F_n^{(i)}(p_{i-1})\right) \cdot \theta_n, \tag{1}$$

where term $j \cdot \left(F_n^{(j)}(p_j) - F_n^{(j)}(p_{j-1})\right) \cdot \theta_n$ is the number of ads times the expected number of sellers who send $j$ ads. In particular, the total (expected) number of ads is

$$\lambda_n(p_n) = \sum_{i=1}^{n} i \cdot \left(F_n^{(i)}(p_i) - F_n^{(i)}(p_{i-1})\right) \cdot \theta_n.$$

Next, consider a seller who sends $k$ ads with price $p \in [p_0, p_n]$, and a buyer who receives her ad. Assume that the buyer chooses the lowest price offer that she receives.[6] The number of ads with a price lower than $p$ is distributed as $Poisson(\lambda_n(p))$.[7] Hence, the buyer who receives the seller's ad contacts the seller with probability $e^{-\lambda_n(p)}$, which is the probability that the buyer receives zero ads from price range of $[p_0, p)$. The probability that the buyer does *not* contact the seller is $1-e^{-\lambda_n(p)}$. Consequently, the seller's expected profit by sending $k$ ads at price $p$ is given by

$$\pi_n(p, k) = \left(1 - \left(1 - e^{-\lambda_n(p)}\right)^k\right) p - c(k), \tag{3}$$

where $\left(1 - \left(1 - e^{-\lambda_n(p)}\right)^k\right)$ is the probability that a seller who sends $k$ ads at price $p$ is contacted by at least one buyer.

Using this notation we can give the formal definition of an $n$-configuration.

**Definition 1.** An $n$-configuration is a triplet $(\mathcal{P}_n, F_n, \theta_n)$ which solves the following system of equations for $i \in \{1, 2, \ldots, n\}$:

$$\pi_n(p, i) = 0 \qquad \text{for all } p \in [p_{i-1}, p_i] \tag{$ZP_i$}$$

$$\pi_n(p_{i-1}, i-1) = \pi_n(p_{i-1}, i). \tag{$I_i$}$$

Condition ($ZP_i$) is the zero profit condition which says that each seller has to make zero profits by setting any price $p \in [p_{i-1}, p_i]$ and sending $i$ ads. Conditions ($I_i$) for all $i \in \{1, 2, \ldots, n\}$ are indifference conditions which require that a seller who sets price $p_{i-1}$ must be indifferent between sending $i-1$ and $i$ ads for all $i \in \{1, 2, \ldots, n\}$. Note that a configuration is not necessarily an equilibrium, but an equilibrium is a configuration. Before we go in more detail into this, we prove that if we find a partition for a configuration, then it is unique. Then given partition $\mathcal{P}_n$, it is always possible to uniquely determine mixed strategies $F_n$ and the number of sellers $\theta_n$.

**Proposition 1.** *If an n-configuration exists, then it is unique.*

Proposition 1 is a technical result which shows that if an $n$-configuration exists, it has a unique partition $\mathcal{P}_n$, and given that partition, the total number of ads with price less than $p \in [p_{i-1}, p_i]$ is given by

$$\lambda_n(p) = -\log\left(1 - \sqrt[i]{1 - \frac{c(i)}{p}}\right), \tag{4}$$

the mixed strategy over $[p_{i-1}, p_i]$ for the $i$th subinterval of the partition by

$$F_n^{(i)}(p) = \frac{1}{\theta_n}\left[\frac{\lambda_n(p)}{i} + \sum_{j=1}^{i-1} \frac{\lambda_n(p_j)}{j(j+1)}\right], \tag{5}$$

and the number of sellers by

$$\theta_n = \left[\frac{\lambda_n(p_n)}{n} + \sum_{j=1}^{n-1} \frac{\lambda_n(p_j)}{j(j+1)}\right]. \tag{6}$$

In words, a configuration is completely pinned down by the maximum number of ads and advertising costs $c(\cdot)$.

Notice that an $(n-1)$-configuration and an $n$-configuration satisfy the same zero profit and indifference conditions up until price $p_{n-2}$. Consequently, the partitions in both configurations are the same except that the last subinterval is divided in two in the $n$-configuration. The number of sellers changes, but for any price $p \le p_{n-2}$ in any subinterval the number of ads remains the same. The mixed strategy in each subinterval has a logarithmic form, and the price distribution is decreasing and convex in each subinterval of the partition.

[6] In Section 4.2 we determine conditions when this is, indeed, optimal behaviour.

[7] This is due to the properties of the Poisson distribution. Buyers receive $n_T \sim Poisson(\lambda_n(1))$ ads in total. The probability of each of these ads has a price less than $p \in [p_{i-1}, p_i]$ is $\frac{\lambda_n(p)}{\lambda_n(1)}$. The probability for a buyer to receive $n_L$ ads with price offer less than $p$ therefore equals

$$\sum_{n_T=n_L}^{\infty} e^{-\lambda_n(1)} \frac{\lambda_n(1)^{n_T}}{n_T!} \binom{n_T}{n_L} \left(\frac{\lambda_n(p)}{\lambda_n(1)}\right)^{n_L} \left(1 - \frac{\lambda_n(p)}{\lambda_n(1)}\right)^{n_T - n_L} = e^{-\lambda_n(p)} \frac{\lambda_n(p)^{n_L}}{n_L!}. \tag{2}$$

That is, $n_L \sim Poisson(\lambda_n(p))$ (see, e.g., Lester et al. (2015)). We thank the referee for suggesting this clarification.

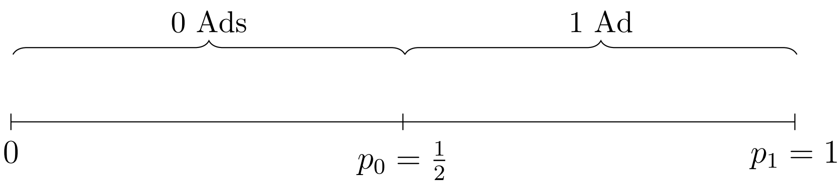


**Fig. 2.** The partition of the unit interval in a 1-configuration.

Next we give an example of a 1-configuration where each seller sends a single ad. In the Appendix we derive 2- and 3-configurations (Examples 2 and 3). Solving the 1- and 2-configurations is pretty simple, but determining the partition of the unit interval for the 3-configuration requires solving of a cubic equation and gets somewhat arduous. Constructing higher-indexed configurations is probably possible only numerically.

**Example 1.** Consider a market in which each seller sends only one ad at maximum. A seller who sets the lowest price, $p_0$, sells her good for sure and earns $p_0 - c(1)$. Since free entry implies zero profits, we must have $p_0 = c(1)$, and the partition of the unit interval becomes $\mathcal{P}_1 = \{0, c(1), 1\}$.

Since in the 1-configuration each seller sends a single ad, the total number of ads is the same as the number of sellers $\theta$. A seller who asks the highest price, 1, sells only if the buyer who receives her ad does not get any other ads; this happens with probability $e^{-\theta}$. The zero profit condition requires $e^{-\theta} - c(1) = 0$, which implies that the number of sellers in the market is $\theta = -\log c(1)$.

We know that all the sellers have to get the same revenue from sending a single ad and setting a price according to the mixed strategy, $F$. Consider a seller who sends an ad with price $p \in (c(1), 1)$. Her expected revenue is $e^{-F(p)\theta}p - c(1)$, where $e^{-F(p)\theta}$ is the probability that a buyer who receives the seller's ad does not receive any other ads with price less than $p$. Then we can use the zero profit condition and substitute $\theta = -\log c(1)$ into this and obtain

$$F(p) = 1 - \frac{\log p}{\log c(1)}.$$

We have thus found a unique 1-configuration that consists of the following *three* elements: (i) partition of the unit interval $\mathcal{P}_1 = \{0, c(1), 1\}$, (ii) mixed strategy $F(p) = 1 - \frac{\log p}{\log c(1)}$, for $p \in [c(1), 1]$, and (iii) number of sellers $\theta = -\log c(1)$.

The partition of the unit interval and an example of a price distribution with $c(1) = \frac{1}{2}$ are given in Figs. 2 and 3.

The 1-configuration constitutes an equilibrium if no seller finds it profitable to send more than 1 ad. It turns out that if the advertising cost function is convex, then each seller sends exactly one ad in equilibrium. This is the case in Butters (1977) where the advertising costs are linear. We postpone the proof of this for later analysis where we have the sufficient tools and notation.

*4.2. Equilibrium*

In this section we construct an equilibrium. There are two things that could go wrong with an $n$-configuration to be an equilibrium. The first one is that some of the sellers might want to deviate and send more than $n$ ads. The second one is that we have implicitly assumed so far that each buyer always chooses the lowest price offer that she receives. Basically, these two problems occur if the advertising costs are not increasing fast enough. On the other hand, if the costs are increasing too fast, then it is not possible to construct an $n$-configuration. To tackle these issues we need to determine a class of cost functions under which no seller wants to deviate and all the buyers choose the lowest price offer that they receive.

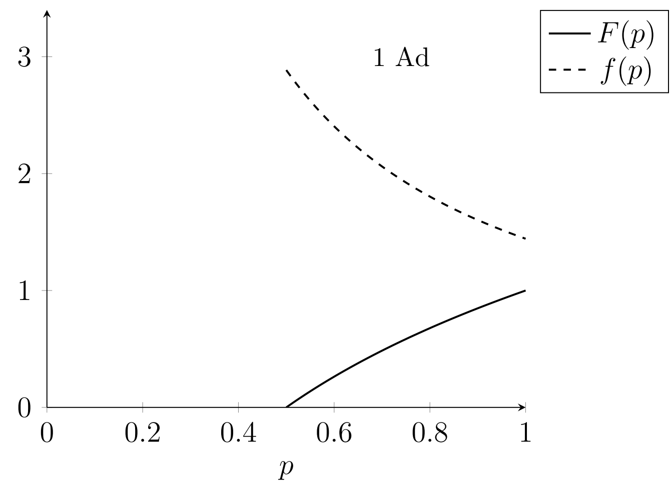


**Fig. 3.** The price distribution, $f(p)$, and the mixed strategy, $F(p)$, of the 1-configuration with $c(1) = \frac{1}{2}$.

We start by proving two lemmas. In the first lemma we derive a class of advertising costs under which an optimal behaviour for buyers is to choose the lowest price offer. In the second lemma we show that a seller who asks the highest price, has the highest incentive to send more ads. This result eases the construction of an equilibrium; once we have found a configuration, we only need to check that a seller with the highest price does not find it profitable to send more ads.

First, let us define the following class of advertising cost functions.

**Definition 2.** For $n \in \mathbb{N}\backslash\{1\}$ and $\gamma \in (0, 1)$, let $\underline{\mathcal{C}}_n(\gamma)$ be the set of advertising costs defined on $\mathbb{N}_0$ with the following two properties:

1. Any $c \in \underline{\mathcal{C}}_n(\gamma)$ is strictly increasing on $\mathbb{N}_0$ such that $c(0) = 0$ and $c(1) = \gamma$.
2. Let $\underline{c}(k) = \frac{k\gamma}{1+(k-1)\gamma}$. Any $c \in \underline{\mathcal{C}}_n(\gamma)$ satisfies $\Delta\underline{c}(k) < \Delta c(k)$ for all $k \in \{2, 3, \ldots, n\}$.[8]

In equilibrium the buyers know the sellers' pricing and advertising strategies, and they must best-respond to them. In particular, choosing the lowest price offer received has to be optimal. It turns out that this is the case when the advertising costs belong to the class $\underline{\mathcal{C}}_n(\gamma)$ given in Definition 2, and the construction of an $n$-configuration is correct.

**Lemma 1.** *If $c \in \underline{\mathcal{C}}_n(\gamma)$ for any $\gamma \in (0, 1)$, then in an $n$-configuration a buyer always chooses the lowest price offer that she receives.*

The intuition of Lemma 1 is the following. If the advertising costs do not increase fast enough, the proportions of sellers who

8 Recall that $\Delta c(k + 1) \equiv c(k + 1) - c(k) > 0$.

send multiple ads are relatively large. This means that a buyer who receives an ad with a low price from a seller who has sent many ads is in competition with the other buyers who have received this seller's ads. For all these buyers, the seller's offer is likely to be the lowest one. But then contacting the lowest price compromises the probability of getting a good.

Next consider the $i$th subinterval of the unit partition and the sellers who send $i$ ads and ask prices between $p_{i-1}$ and $p_i$. By construction, any seller with price $p < p_i$ makes negative profit by sending $i+1$ ads, while a seller with price $p = p_i$ is just indifferent; both $i$ and $i+1$ ads generate zero profits.

**Lemma 2.** *Assume $c \in \underline{\mathcal{C}}_n(\gamma)$. In an $n$-configuration for any $i \in \{1, 2, \dots, n\}$ a seller who asks the highest price $p_i \in [p_{i-1}, p_i]$ has the highest incentive to send more than $i$ ads. Furthermore, a seller who asks the lowest price $p_{i-1} \in [p_{i-1}, p_i]$ has the highest incentive to send fewer than $i$ ads.*

Using Lemmas 1 and 2 we get the following proposition.

**Proposition 2.** *Assume $c \in \underline{\mathcal{C}}_n(\gamma)$. In an $n$-configuration, a seller who sets price $p \in [p_{i-1}, p_i]$ cannot increase her profits by sending $k \in \{1, \dots, i-1, i+1, \dots, n\}$.*

Although Proposition 2 is not surprising, it provides an easy test for an equilibrium: if in an $n$-configuration a seller with the highest price does not want to deviate and send more than $n$ ads, then the $n$-configuration constitutes an equilibrium.[9]

**Corollary 1.** *Assume $c \in \underline{\mathcal{C}}_n(\gamma)$. An $n$-configuration constitutes an equilibrium if the seller who asks the highest price does not find it profitable to send more than $n$ ads.*

By these results, the Butters's (1977) model with capacity constrained sellers features each seller sending just one ad in equilibrium as the cost function is linear. This follows because the marginal return of the second ad is always lower than that of the first ad; the second ad is useless if the first ad leads to a sale. Therefore, for linear advertising costs, if a seller finds it profitable to send a second ad, it gets surplus from the first one, which violates the zero profit condition. Consequently, if the second ad is sufficiently more expensive than the first ad, sellers send only one ad in a free entry equilibrium. We state the result as follows.

**Proposition 3.** *If the advertising cost function is convex, then each seller sends exactly one ad in equilibrium.*

The characterisation of the single-ad equilibrium is given in Example 1. The equilibrium price distribution coincides with Butters (1977) by setting the cost of production to zero, normalising the number of buyers to unity, and assuming that each buyer values the good at unity in the Butters's model.[10] This is due to the free entry and exit assumption and convex advertising costs.

### 4.3. Multi-advertisement equilibria

In this section we study a class of advertising costs under which an $n$-configuration constitutes an equilibrium. It turns out that even the concavity of advertising costs is not enough to guarantee that some sellers send more than 1 ad in equilibrium. Next we construct a class of cost functions that allows an $n$-configuration to constitute an equilibrium for some $n > 1$. The idea is to determine an upper bound for advertising costs such that if the advertising costs increase faster than the upper bound after $k+n$ ads ($k \in \{1, 2, \dots\}$), then a seller with the highest price does not find it profitable to send more than $n$ ads. Then, if the advertising costs belong to the intersection of the class of costs given in Definition 2 and the class defined by the upper bound, an $n$-configuration constitutes an equilibrium.

Consider an $n$-configuration and a seller who sets price at 1. She does not want to send more than $n$ ads if $\pi_n(1, n+k) \le \pi_n(1, n)$ for all $k > 1$ — that is,

$$\left(1-\left(1-e^{-\lambda_n(1)}\right)^{n+k}\right)-c(n+k) \le \left(1-\left(1-e^{-\lambda_n(1)}\right)^{n}\right)-c(n). \tag{7}$$

From the zero profit condition $ZP_n$ we get that $1 - e^{-\lambda_n(1)} = (1-c(n))^{\frac{1}{n}}$ and so (7) becomes

$$c(n) \le 1-(1-c(n+k))^{\frac{n}{n+k}}, \tag{8}$$

which gives us the upper bound for the advertising costs. As with the proof of Lemma 1, let us treat the upper bound in (8) as a functional equation and denote it as $\bar{c}(x) = 1-(1-\bar{c}(x+k))^{\frac{x}{x+k}}$ such that $\bar{c} : \mathbb{R}_+ \to \mathbb{R}$. This functional equation has a solution of $\bar{c}(x) = 1-\phi(x)^x$ such that $\phi(x) = \phi(x+1)$ for all $x \in \mathbb{R}$. Since the advertising costs are assumed to be increasing, we must have $\phi(x) = \phi \in (0, 1)$ for all $x$, which makes $\bar{c}$ an increasing concave function. Furthermore, from Example 1 we know that the upper bound for $\Delta c(2)$ is $c(1)(1-c(1))$. This is the initial value for the upper bound from which we can solve $\phi = 1-c(1)$. The upper bound becomes

$$\bar{c}(x) = 1-\phi^x \text{ for all } x \in \mathbb{R}_+,$$

where $\phi = 1-c(1)$. Simple algebra shows that the upper bound is greater than the lower bound in Definition 2 — that is, $\bar{c}(x) > \underline{c}(x)$ for all $x > 1$.[11] If the costs increase as fast as the upper bound, i.e. if $\Delta c(n+k) = \Delta\bar{c}(n+k)$, then a seller who asks the highest price is indifferent between sending $n$ and $n+k > n$ ads.

Using the upper bound we define the following class of advertising costs.

**Definition 3.** For $n \in \mathbb{N}\backslash\{1\}$ and $\gamma \in (0, 1)$, let $\bar{\mathcal{C}}_n(\gamma)$ be the set of advertising costs defined on $\mathbb{N}_0$ with the following two properties:

1. Any $c \in \bar{\mathcal{C}}_n(\gamma)$ is strictly increasing on $\mathbb{N}_0$ such that $c(0) = 0$ and $c(1) = \gamma$.
2. Let $\bar{c}(k) = 1-(1-\gamma)^k$. Any $c \in \bar{\mathcal{C}}_n(\gamma)$ satisfies $\Delta c(k) < \Delta\bar{c}(k)$ for all $k \in \{2, 3, \dots, n-1\}$ and $\Delta c(k) \ge \Delta\bar{c}(k)$ for all $k \in \{n, n+1, \dots\}$.

Our aim is to determine when an $n$-configuration constitutes an equilibrium. To that end, let the intersection of the classes of advertising costs in Definitions 2 and 3 be denoted by $\mathcal{C}_n(\gamma) = \underline{\mathcal{C}}_n(\gamma) \cap \bar{\mathcal{C}}_n(\gamma)$. The class of advertising costs we study is then

[9] We impose that each seller sends all the ads with the same price in equilibrium. Butters (1977) does not use this premise. However, as the proof of Lemma 2 indicates, we can relax the assumption and allow sellers to post different prices in each ad. In equilibrium, sellers who choose a price in interval $[p_{i-1}, p_i]$ use a mixed strategy $F_n^{(i)}$ and send $i$ ads. They could equally well choose $i$ different prices by making $i$ independent draws from $F_n^{(i)}$; they would still make zero profits. If a seller deviates and chooses a price $p' \notin [p_{i-1}, p_i]$ and sends $i$ ads, she makes losses. Assuming that each seller advertises just one price simplifies the analysis; otherwise there would be buyers with different price offers approaching the seller.

[10] Note that Butters's advertising price density $a(p)$ equals $f_1(p)\theta_1$ in our case.

[11] See the proof of Proposition 4.

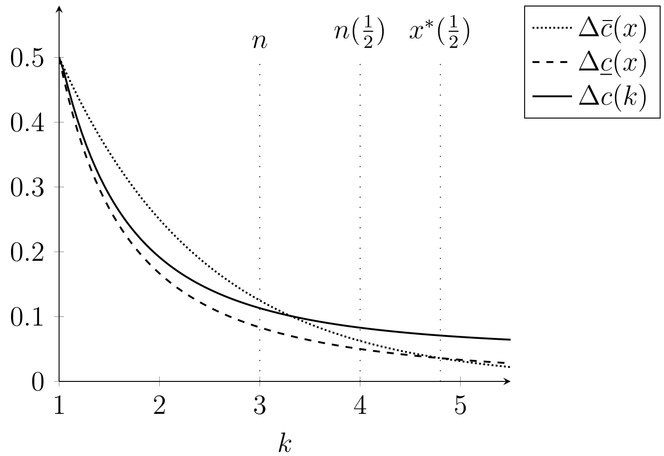


**Fig. 4.** An example of advertising costs $c \in \mathcal{C}_3(\frac{1}{2})$.

defined as $\mathcal{C}(\gamma) = \bigcup_{i=2}^{\infty} \mathcal{C}_i(\gamma)$.[12] Then let the advertising costs be $c(k) = \underline{c}(k) + a_k = \frac{k\gamma}{1+(k-1)\gamma} + a_k$ for all $k \geq 1$.

We still need to check under which conditions $\mathcal{C}_n(\gamma)$ is not an empty set to guarantee than an $n$-configuration exists and is a equilibrium for $c \in \mathcal{C}_n(\gamma)$. This result is given by the following proposition.

**Proposition 4.** *There exists a unique $n(\gamma) \in \mathbb{N}$ which gives the highest possible configuration under costs $c \in \mathcal{C}(\gamma)$. Moreover, $n(\gamma)$ is decreasing in $\gamma$ and $\mathcal{C}_n(\gamma) \neq \emptyset$ for all $n \leq n(\gamma)$.*

In Fig. 4 we illustrate the relationship between $n$, $n(\gamma)$, and $x^*(\gamma)$, which are used in the proof of Proposition 4.

Next we give our last result which is a direct implication of the construction of $\mathcal{C}(\gamma)$, Proposition 1, Corollary 1, and Proposition 4.

**Proposition 5.** *Assume that $c \in \mathcal{C}_n(\gamma)$ such that $n \leq n(\gamma)$. Then there exists a unique $n$-configuration which constitutes an equilibrium.*

If the advertising costs coincide with the upper bound, then a 1-configuration is an equilibrium. This can be easily seen by considering $p_1 = \frac{c(1)^2}{2c(1)-c(2)}$ derived in Example 1. If we substitute $\bar{c}(k)$ into this formula we get that $p_1 = 1$ for all $\gamma \in (0, 1)$.

Proposition 5 gives us a simple test to find an equilibrium: if the advertising costs belong to class $\mathcal{C}(\gamma)$, find an $n \leq n(\gamma)$ such that $c \in \mathcal{C}_n(\gamma)$. Then the $n$-configuration is an equilibrium.

## 5. Discussion

In this section we point out, on the one hand, some limitations of our analysis, and, on the other hand, possible applications and interpretations of the model.

Comparative statics in a multiple-ad equilibrium is complicated. For instance, changing advertising costs affects not only the equilibrium strategies, but also the number of sellers (free entry and exit). It is just hard to keep track of both effects.

Nevertheless, some comparative statics can still be conducted. First, the equilibrium configuration with the highest index is determined by the cost of the first ad. The higher it is, the smaller the maximum index (Proposition 4). This has the following economic intuition. Let us interpret the cost of the first ad as the sum of an entry cost and the marginal cost of the first ad. Then the higher the entry cost, the less there can be potential entrants. This implies that the probability of a sale is greater or competition is less severe. Therefore fewer ads are sent with a higher entry cost in equilibrium.

Furthermore, comparative statics can be done *within* the equilibrium configuration. Suppose that an $n$-configuration forms the equilibrium. If we decrease the advertisements costs such that the same configuration is still an equilibrium, we have the following effects: (i) the total number of ads sent is greater (see Eq. (4)), and (ii) the number of sellers is higher (a consequence of the first effect).

Capacity constrained sellers is a crucial feature of our model. It guarantees the positive association of prices and the number of ads.

A simple example demonstrates this. Assume that the sellers have unlimited capacity, and assume a cost function such that each seller sends a finite number of ads. By the standard arguments pricing is in mixed strategies on some interval $[p_0, 1]$.

Consider a seller with price $p_0$, and assume that she sends $k$ ads. Her pay-off is given by $kp_0 - c(k)$. Then consider a seller with price unity, and assume provisionally that she sends $k$ ads, too. The number of buyers she attracts is given by a binomial distribution with success probability $e^{-\theta k}$, where $\theta$ is the number of sellers and $\theta k$ the total number of ads sent assuming that each seller sends $k$ ads. Consequently, the seller's pay-off is given by $ke^{-\theta k} - c(k)$. Under mixed strategy the pay-offs have to be equal, and this condition allows solving $p_0 = e^{-\theta k}$. This implies that the pay-offs of these two sellers are identical, and hence the optimality condition for the sellers is $c(k+h) - c(k) > hp_0$ for $h \in \{-k, -(k-1), \ldots, -1, 0, 1, \ldots\}$ (no profitable deviations to send $k+h$ ads). This shows that in equilibrium all the sellers send the same number of ads.

This leaves open the possibility that there are equilibria where the sellers send different numbers of ads. Let us consider this next. Assume temporarily that low pricing sellers with $p \in [p_0, p_1)$ send $k$ ads, and high pricing sellers with $p \in [p_1, 1]$ send $k+1$ ads. The pay-off of the lowest pricing seller is given by $kp_0 - c(k) = 0$ by free entry. We can thus solve $p_0 = \frac{c(k)}{k}$. Denote the total number of ads with price lower than $p$ by $\lambda(p)$. The pay-off of the highest pricing seller is given by $(k+1)e^{-\lambda(1)} - c(k+1) = 0$, and we can solve $e^{-\lambda(1)} = \frac{c(k+1)}{k+1}$.

In equilibrium the lowest pricing seller does not find it profitable to send $k+1$ ads, and the highest pricing seller does not find it profitable to send $k$ ads, or

$$kp_0 - c(k) > (k+1)p_0 - c(k+1) \tag{9}$$

$$(k+1)e^{-\lambda(1)} - c(k+1) > ke^{-\lambda(1)} - c(k). \tag{10}$$

Substituting $p_0 = \frac{c(k)}{k}$ in the first condition, and $e^{-\lambda(1)} = \frac{c(k+1)}{k+1}$ in the latter condition, and manipulating a little yields

$$c(k+1) > \frac{k+1}{k} c(k) \tag{11}$$

and

$$c(k+1) < \frac{k+1}{k} c(k) \tag{12}$$

which is a contradiction. This demonstrates that we lose the positive association with prices and the number of ads in general, if we allow unlimited capacity.

We assume unit capacity, but relaxing this to some finite capacity $k > 1$ does not affect the basic message of our model. In equilibrium pricing is still in mixed strategies, and the seller with the lowest price sends $k$ ads, while higher pricing sellers send more than $k$ ads if the cost function is concave enough.

[12] An example of an advertising cost function that belongs to class $\mathcal{C}_n(\gamma)$ can be constructed by using the lower bound. Consider a sequence $(a_k)$ that increases up to $a_n$ sufficiently slowly, and after that sufficiently fast with $a_1 = 0$.

Although our model is highly stylised in the sense that the capacity constrained sellers are assumed to possess just one unit of a good, it may be applicable to some settings where capacity constraints are salient. For instance, suppose that each seller has a room for rent, and the rooms are more or less equal in quality (distance, ratings, etc.). The sellers use internet platforms to advertise their items. Posting the offer onto a platform is costly, but it is reasonable to argue that the costs are marginally decreasing in the number of platforms chosen since the first offer involves costs, such as taking pictures and composing the ad, that are not incurred for the succeeding offers. Based on this, our model suggests that the sellers who use multiple platforms should ask higher prices.

An alternative interpretation of our theoretical framework is as follows. There are a large number of agents divided into two different types. To produce a unit surplus a member of both types has to form a pair. One party can commit to the division of the surplus in the sense that it sends take-it-or-leave-it offers to the other party. The senders are, however, subject to competition by other senders as the receiving party accepts the best offer. Sending offers is costly, and in equilibrium the senders mix over the offers and number of offers sent. This is the typical setting of a decentralised model of a job market. Low wage offers would then be advertised more as they correspond to high prices of goods offered for sale.

## 6. Conclusion

We study a version of Butters's seminal model of informative advertising with a large number of buyers and sellers, assuming that the sellers are capacity constrained each with one unit of a good. In order to trade a buyer has to receive an ad. Pricing is in mixed strategies, and we establish an equilibrium where high pricing sellers send more ads than low pricing sellers, i.e., a positive relationship between prices and the number of ads. In equilibrium, the buyers who received multiple ads contact the seller with lowest price.

The key ingredient in our analysis is the cost of advertising. If the cost function is convex, then each seller sends exactly one ad in equilibrium. The reason is diminishing marginal returns of advertising that stem from the capacity constraint. If the marginal cost of advertising is decreasing there may be equilibria where multiple ads are sent.

We delineate a class of advertising costs which permits an equilibrium where sellers send multiple ads, and each buyer contacts the seller who offers the lowest price. The first property requires that the marginal cost of advertising is decreasing, and we determine an upper bound for the advertising costs. The second property requires that the cost is increasing sufficiently fast; otherwise there could be too many sellers who price high, and send many ads, such that the buyers could profitably trade-off low price and the probability of getting a good. We determine a lower bound for the advertising costs.

## Appendix

*Proof of Proposition 1*

**Proof.** We start by showing that an $n$-configuration has a unique partition. Assume that there is an $n$-configuration with two different partitions $\mathcal{P}_n$ and $\mathcal{P}_n'$ such that the elements of the partitions are denoted by $p_i \in \mathcal{P}_n$ and $p_i' \in \mathcal{P}_n'$. Let index $k$ be the first one where $p_k \neq p_k'$ and let $p_k < p_k'$. The numbers of ads less than $p_k$ and $p_k'$ with different partitions are denoted by $\lambda_n(p_k)$ and $\lambda_n'(p_k')$, respectively. Up until price $p_k$ we necessarily have $\lambda_n(p_k) = \lambda_n'(p_k)$ since in both cases the sellers make zero profits.

Now we have the following two equalities

$$\left(1-\left(1-e^{-\lambda_n(p_k)}\right)^k\right)p_k - c(k) = \left(1-\left(1-e^{-\lambda_n'(p_k')}\right)^k\right)p_k' - c(k)$$

$$\left(1-\left(1-e^{-\lambda_n(p_k)}\right)^{k-1}\right)p_k - c(k-1) = \left(1-\left(1-e^{-\lambda_n'(p_k')}\right)^{k-1}\right)p_k' - c(k-1).$$

This system of equations can be rewritten as

$$\frac{1-\left(1-e^{-\lambda_n'(p_k')}\right)^{k-1}}{1-\left(1-e^{-\lambda_n'(p_k')}\right)^{k}} = \frac{1-\left(1-e^{-\lambda_n(p_k)}\right)^{k-1}}{1-\left(1-e^{-\lambda_n(p_k)}\right)^{k}}. \tag{13}$$

Both sides have the same functional form of $f(z) = \frac{1-z^{k-1}}{1-z^k}$ such that

$$\frac{\partial}{\partial z} f(z) = \frac{z^{k-2}(1-z)\left(k-\frac{1-z^k}{1-z}\right)}{\left(1-z^k\right)^2} > 0 \tag{14}$$

since $z \in (0,1)$ and $k - \frac{1-z^k}{1-z} = k - \sum_{i=1}^k z^{i-1} > 0$ by the sum of a geometric progression where each element $z^{i-1} \in (0,1)$. In other words, $f$ is a strictly increasing function and so $f(x) = f(y)$ only if $x = y$. Hence, the system of equations in (13) has a solution of $\lambda_n(p_k) = \lambda_n'(p_k')$. This is a contradiction and therefore the partition must be unique.

Next we show that given partition $\mathcal{P}_n$, there is a unique mixed strategy $F_n$ that solves the zero profit conditions. Consider a seller who sets price $p \in [p_{i-1}, p_i]$ and sends $i$ ads. From zero profit condition ($ZP_i$) we get

$$\lambda_n(p) = -\log\left(1-\sqrt[i]{1-\frac{c(i)}{p}}\right). \tag{15}$$

On the other hand, from expression (1) we know the total number of ads with a price less than $p$ is

$$\lambda_n(p) = \lambda_n(p_{i-1}) + i\left[F_n^{(i)}(p) - F_n^{(i-1)}(p_{i-1})\right]\theta_n.$$

Combining these two we have the following expression for mixed strategy $F_n^{(i)}(p)$ for prices $p \in [p_{i-1}, p_i]$:

$$F_n^{(i)}(p) = F_n^{(i-1)}(p_{i-1}) + \frac{1}{i\theta_n}\left[\lambda_n(p) - \lambda_n(p_{i-1})\right].$$

This recursive formula can be rewritten as follows by substituting in the mixed strategies from the earlier subintervals:

$$F_n^{(i)}(p) = \frac{1}{\theta_n}\left[\frac{\lambda_n(p)}{i} + \sum_{j=1}^{i-1}\frac{\lambda_n(p_j)}{j(j+1)}\right], \tag{16}$$

where $\lambda_n(\cdot)$ is given in Eq. (15).

Finally, using the fact that $F_n^{(n)}(1) = 1$ we can solve the number of sellers:

$$\theta_n = \left[\frac{\lambda_n(p_n)}{n} + \sum_{j=1}^{n-1}\frac{\lambda_n(p_j)}{j(j+1)}\right]. \tag{17}$$

Eqs. (4), (5), and (6) uniquely determine the mixed strategies $F_n$ and the number of the sellers in the market $\theta_n$. □

*Proof of Lemma 1*

**Proof.** Let us state the obvious case first: if a buyer receives an ad at price $p \in [p_0, p_1)$, she knows that the seller has sent only a single ad and therefore by contacting the seller she always gets the good.

Then, consider a buyer who receives an ad at price $p \in [p_{i-1}, p_i)$ for some $i > 1$. She knows that the seller who has sent this offer has sent $i$ ads. If the buyer contacts this seller, the probability that she gets the object is

$$Q_i(p) \equiv \sum_{k=0}^{i-1} \frac{1}{k+1}\binom{i-1}{k}\left(e^{-\lambda_n(p)}\right)^k\left(1-e^{-\lambda_n(p)}\right)^{i-1-k}. \tag{18}$$

This can be written as

$$\frac{e^{\lambda_n(p)}}{i}\sum_{k=1}^{i}\binom{i}{k}\left(e^{-\lambda_n(p)}\right)^k\left(1-e^{-\lambda_n(p)}\right)^{i-k}$$

and using the binomial theorem it becomes

$$Q_i(p) = \frac{e^{\lambda_n(p)}}{i}\left(1-\left(1-e^{-\lambda_n(p)}\right)^i\right).$$

Further, from zero profit condition ($ZP_i$) we get

$$e^{-\lambda_n(p)} = 1-\left(1-\frac{c(i)}{p}\right)^{\frac{1}{i}}.$$

Substituting this into the formula of $Q_i(p)$ we get

$$Q_i(p) = \frac{c(i)}{ip\left(1-\sqrt[i]{1-\frac{c(i)}{p}}\right)}. \tag{19}$$

A buyer's utility of getting an object at price $p$ is $1-p$. Then, the expected utility of a buyer who contacts a seller with price $p \in [p_{i-1}, p_i)$ is $U(p) = Q_i(p)(1-p)$. The derivative of this with respect to $p$ is

$$U'(p) = -\frac{1}{p}\frac{c(i)}{ip\left(1-\sqrt[i]{1-\frac{c(i)}{p}}\right)} + \frac{1-p}{p}\frac{c(i)^2\left(1-\frac{c(i)}{p}\right)^{\frac{1}{i}-1}}{\left(ip\left(1-\sqrt[i]{1-\frac{c(i)}{p}}\right)\right)^2}. \tag{20}$$

After some multiplications and rearrangements that retain the sign, this expression becomes

$$-1+U(p)\left(1-\frac{c(i)}{p}\right)^{\frac{1}{i}-1}. \tag{21}$$

First we prove that (21) is strictly negative at $p = p_{i-1}$. After that we show that $U(p)$ is an inverted-U-shaped function ($\cap$-shaped) and thus if $U'(p_{i-1}) < 0$, then also $U'(p) < 0$ for all $p \in [p_{i-1}, p_i)$.

Expression (21) is strictly negative at $p = p_{i-1}$ if

$$(1-p_{i-1})\frac{c(i)}{ip_{i-1}} < \left(1-\frac{c(i)}{p_{i-1}}\right)^{1-\frac{1}{i}} - 1 + \frac{c(i)}{p_{i-1}}. \tag{22}$$

We know that a seller who asks price $p_{i-1}$ is indifferent between sending $i$ and $i-1$ ads. We hence get from indifference condition ($I_i$) and zero profit condition ($ZP_i$) that

$$\left(1-\frac{c(i)}{p_{i-1}}\right)^{1-\frac{1}{i}} = 1-\frac{c(i-1)}{p_{i-1}}.$$

By substituting this into (22) and rearranging the terms we get

$$p_{i-1} > 1 - i\frac{\Delta c(i)}{c(i)}. \tag{23}$$

We know that $p_{i-1} > c(i-1)$. So, if $c(i-1) > 1 - i\frac{\Delta c(i)}{c(i)}$, then also (23) is satisfied. This is equivalent to

$$c(i) \geq \frac{ic(i-1)}{i-1+c(i-1)}. \tag{24}$$

From this we get lower bound $\underline{c} : \mathbb{R}_+ \to \mathbb{R}_+$ for the differences of advertising costs under which each buyer chooses the lowest price offer she receives. The lower bound $\underline{c}$ must satisfy

$$\underline{c}(x-1) = 1 - x\frac{\Delta\underline{c}(x)}{\underline{c}(x)}, \tag{25}$$

for all $x \in \mathbb{R}_+$. This is a functional equation which has the following increasing solution:

$$\underline{c}(x) = \frac{x}{x+a}$$

for some $a > 0$. The initial value for advertising costs is given by $\underline{c}(1) = c(1)$ from where we can solve $a = \frac{1-c(1)}{c(1)}$. The lower bound for the advertising cost function thus is

$$\underline{c}(x) = \frac{x}{x+\frac{1-c(1)}{c(1)}} = \frac{xc(1)}{1+(x-1)c(1)},$$

which is exactly the function given in Definition 2 with $\gamma = c(1)$. So we know that if $\Delta c(i) \geq \Delta\underline{c}(i)$ then $U'(p_{i-1}) < 0$.

Let us consider the left-hand side and the right-hand side of expression (22) as functions of $p$ instead of $p_{i-1}$. The left-hand side is a strictly convex strictly decreasing function of $p$ for all $p \in [p_{i-1}, p_i)$. The right-hand side is a strictly concave and strictly increasing function of $p$ for all $p \in [p_{i-1}, p_i)$. Moreover, the left-hand side is strictly positive at $p = c(i)$, whereas the right-hand side is 0. If $p = 1$, then the left-hand side is 0 and the right-hand side is strictly positive. Hence, there is a unique intersection which implies that $U(p)$ is a $\cap$-shaped function of $p$. Since $U'(p_{i-1}) < 0$, then also $U'(p) < 0$ for all $p \in [p_{i-1}, p_i]$.

Finally, we need to show that a buyer wants to contact a seller who has sent an ad at price $p \in [p_{i-1}, p_i)$ rather than a seller who has sent an ad at price $p' \in [p_{k-1}, p_k)$ for some $k > i$. We know that $Q_i(p)(1-p) \geq Q_i(p_i)(1-p_i)$. By the similar arguments, we know that $Q_{i+1}(p_i)(1-p_i) \geq Q_{i+1}(p')(1-p')$ for all $p' \in [p_{i-1}, p_i)$. It is easy to verify that $Q_i(p_i) \geq Q_{i+1}(p_i)$, and hence we have $Q_i(p)(1-p) \geq Q_{i+1}(p')(1-p')$ for all $p' \in [p_i, p_{i+1})$. This implies that $Q_i(p)(1-p) \geq Q_k(p')(1-p')$ for all $k \geq i$ and $p' \geq p$. □

*Proof of Lemma 2*

**Proof.** The derivative of $\pi_n(p,k)$ with respect to $p \in [p_{i-1}, p_i]$ and for any $k \in \mathbb{N}_0$ is

$$\pi_n'(p,k) = 1-\left(1-\frac{c(i)}{p}\right)^{k/i} - \frac{k}{i}\left(1-\frac{c(i)}{p}\right)^{\frac{k}{i}-1}\frac{c(i)}{p}. \tag{26}$$

Let us simplify the notation and denote $x = \frac{c(i)}{p} \in (0,1)$ and $z = \frac{k}{i}$. By rearranging terms we get

$$\pi_n'(x,z) = 1-(1-x)^{z-1}\left(1-(1-z)x\right). \tag{27}$$

This is non-negative if

$$(1-x)^{1-z} \geq 1-(1-z)x. \tag{28}$$

and negative if

$$(1-x)^{1-z} < 1-(1-z)x. \tag{29}$$

The derivative in (27) is zero if $z = 0$ or $z = 1$. Moreover, since $(1-x)^{1-z}$ is a strictly increasing convex function of $z$ and $1-(1-z)x$ is a strictly increasing linear function of $z$, we have that $\pi_n'(p,k) \geq 0$ for all $z \geq 1$ and $\pi_n'(x,z) \leq 0$ for all $z \in [0,1]$. In other words, we have that for all $p \in [p_{i-1}, p_i]$, $\pi_n'(p,k) = 0$ if $k = i$, $\pi_n'(p,k) < 0$ if $k \in \{1, 2, \ldots, i-1\}$, and $\pi_n'(p,k) > 0$ if $k > i$. □

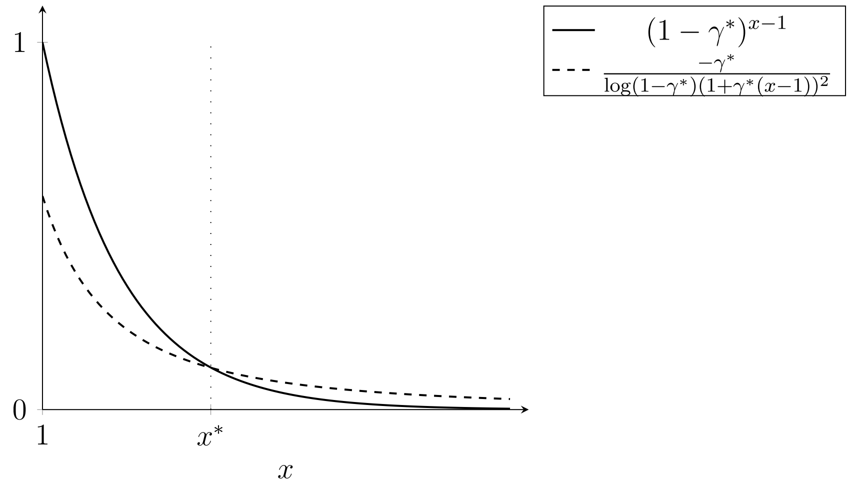


**Fig. 5.** Determination of $x^*$.

*Proof of Proposition 2*

**Proof.** Lemma 2 shows that $\pi_n(p,k) \le \pi_n(p',k)$ for some $p \le p'$ and $k \in \{i+1,\dots,n\}$. So, if a seller sends $k \in \{i+1,\dots,n\}$ ads at price $p \in [p_{i-1}, p_i]$, her profits are $\pi_n(p,k) \le \pi_n(p_k,k) = 0$ for $p_k \ge p$.

Analogously, if a seller sets price at $p \in [p_{i-1}, p_i]$ and sends $k \in \{1,\dots,i-1\}$ ads her profits are $\pi_n(p,k) \le \pi_n(p_k,k) = 0$ for $p_k \le p$ by Lemma 2. □

*Proof of Corollary 1*

**Proof.** Assume that the cost function is linear $c(k) = k\alpha$, for some $\alpha \in (0,1)$. Then consider a 1-configuration and a seller who asks price 1, but deviates and sends $k$ ads. Then her expected profits $\pi_1(k,1)$ are

$$\left(1-\left(1-e^{-\lambda_1(1)}\right)^k\right) - c(k) = \left(1-(1-c(1))^k\right) - c(k) \qquad (30)$$

since $\lambda_1(1) = -\log c(1)$. This is decreasing for all $k \ge 0$ or

$$\left(1-(1-\alpha)^k\right) - k\alpha \ge \left(1-(1-\alpha)^{k+1}\right) - (k+1)\alpha,$$

which can be simplified to

$$1 \ge (1-\alpha)^k. \qquad (31)$$

So, in a 1-configuration it is not profitable to send more than one ad if the advertising costs are linear. This clearly holds good also for costs that increase faster, i.e. for convex cost functions. □

*Proof of Proposition 4*

**Proof.** The goal of this proof is to show that there exists a unique and decreasing $n(\gamma) \in \mathbb{N}$ such that $\Delta\bar{c}(k) > \Delta\underline{c}(k)$ for all $k = 2,3,\dots,n(\gamma)$ and $\Delta\bar{c}(k) \le \Delta\underline{c}(k)$ for all $k > n(\gamma)$. Then the set $\mathcal{C}_n(\gamma) = \underline{\mathcal{C}}_n(\gamma) \cap \bar{\mathcal{C}}_n(\gamma)$ is non-empty for all $n \le n(\gamma)$. In order to do that, we first prove that there exists a unique $x^*(\gamma) \in (1,\infty)$ such that $\underline{c}'(x^*(\gamma)) = \bar{c}'(x^*(\gamma))$ and $\frac{\partial}{\partial\gamma}x^*(\gamma) < 0$. Once we have shown this, we can show that this applies to integer values as well.

One can show by induction that $\underline{c}(x) < \bar{c}(x)$ holds for all $x \in \mathbb{N}\backslash\{1\}$, and since the functions are concave and continuous it holds for all real numbers $x > 1$. Moreover, we have that $\underline{c}(1) = \bar{c}(1) = \gamma$ and $\lim_{x\to\infty}\underline{c}(x) = \lim_{x\to\infty}\bar{c}(x) = 1$. The derivatives of the upper and lower bounds are $\bar{c}'(x) = -(1-\gamma)^x\log(1-\gamma)$ and $\underline{c}'(x) = \frac{\gamma(1-\gamma)}{(1+\gamma(x-1))^2}$. These are equal if

$$(1-\gamma)^{x-1} = \frac{-\gamma}{\log(1-\gamma)(1+\gamma(x-1))^2}. \qquad (32)$$

From here we can see that the left-hand side equals 1 when $x = 1$ and the right-hand side is less unity since $-\log(1-\gamma) = \gamma + \frac{\gamma^2}{2} + \cdots > \gamma$ for all $\gamma \in (0,1)$. Both sides are strictly decreasing functions of $x$ and they both converge to zero as $x$ goes to infinity. However, since the left-hand side decreases exponentially and the right-hand side decreases slower than exponentially, there exists a unique $x^*(\gamma) \in (0,\infty)$ such that $\underline{c}'(x^*(\gamma)) = \bar{c}'(x^*(\gamma))$. In other words, $(1-\gamma)^{x-1} \ge \frac{-\gamma}{\log(1-\gamma)(1+\gamma(x-1))^2}$ for all $x \le x^*(\gamma)$ and $(1-\gamma)^{x-1} < \frac{-\gamma}{\log(1-\gamma)(1+\gamma(x-1))^2}$ for all $x > x^*(\gamma)$. This is depicted in Fig. 5.

Next we show that $\frac{\partial}{\partial\gamma}x^*(\gamma) < 0$. However, it turns out that the proof is not straightforward and we must do it inversely. We show that $\gamma^*(x)$ which solves (32) is strictly decreasing in $x$. Then its inverse $x^*(\gamma)$ is decreasing in $\gamma$.

Let us denote $h(x,\gamma) = (1-\gamma)^{x-1}(1+\gamma(x-1))^2$ and $g(\gamma) = \frac{\gamma}{-\log(1-\gamma)}$. Clearly, $h(x,0) = 1$ and $h(x,1) = 0$, whereas $\lim_{\gamma\to 0} g(\gamma) = 1$ and $\lim_{\gamma\to 1} g(\gamma) = 0$. One can show that $g$ is strictly decreasing in $\gamma$ while

$$\frac{\partial}{\partial\gamma}h(x,\gamma) = (x-1)(1-\gamma)^{x-2}(1+\gamma(x-1))(1-\gamma(x+1)), \qquad (33)$$

which is non-negative for all $\gamma \le \frac{1}{x+1}$ and negative for all $\gamma < \frac{1}{x+1}$. Moreover, when $\gamma$ goes to 0, the derivative in (33) approaches $x-1 \ge 0$. Thus, $h(x,\gamma)$ is a single-peaked function of $\gamma$ with a global maximum at $\frac{1}{x+1}$. We know that $g(\gamma)$ and $h(x,\gamma)$ intersect once at some $\gamma^* \in (0,1)$, and hence the intersection must be on the decreasing part of $h(x,\gamma)$. This is depicted in Fig. 6. Next we show that increasing $x$ shifts $h(x,\gamma)$ to the left and, consequently, $\gamma$ under which $h(x,\gamma) = g(\gamma)$ decreases.

Let us fix arbitrary $\gamma^*$ and $x^*$ such that Eq. (32) is satisfied or $h(x^*,\gamma^*) = g(\gamma^*)$. By the uniqueness of $x^*$ we know that for all $x > x^*$ the expression in (32) holds as inequality:

$$(1-\gamma^*)^{x-1} < \frac{-\gamma^*}{\log(1-\gamma^*)(1+\gamma^*(x-1))^2}. \qquad (34)$$

This is equivalent to

$$(1-\gamma^*)^{x-1}(1+\gamma^*(x-1))^2 < \frac{\gamma^*}{-\log(1-\gamma^*)}. \qquad (35)$$

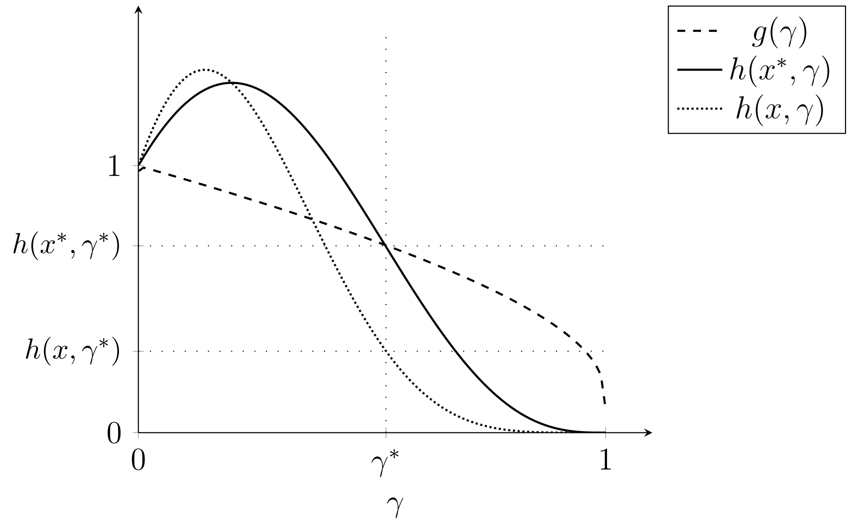


**Fig. 6.** Determination of $\gamma^*$.

On the left-hand side we have now $h(x, \gamma^*)$ and on the right-hand side $g(\gamma^*)$. However, since $g(\gamma^*) = h(x^*, \gamma^*)$ we have that $h(x, \gamma^*) < h(x^*, \gamma^*)$ for all $x > x^*$. This implies that if $h(x, \gamma) = g(\gamma)$ and $x > x^*$ then $\gamma < \gamma^*$. This is depicted in Fig. 6.

We have thus shown that $\gamma^*$ which solves (32) exists and is unique for all $x > 1$. It is also strictly decreasing in $x$ and therefore its inverse $x^*(\gamma)$ is strictly decreasing in $\gamma$ for all $\gamma \in (0, 1)$.

Finally, let $n(\gamma) = \lceil x^*(\gamma)\rceil - 1$ (where $\lceil\cdot\rceil$ is the ceiling function). By the properties of $x^*(\gamma)$ it directly implies that $\Delta\bar{c}(k) > \Delta\underline{c}(k)$ for all $k \in \{2, 3, \dots, n(\gamma)\}$ and $\Delta\bar{c}(k) \leq \Delta\underline{c}(k)$ for all $k > n(\gamma)$. We have thus shown that $n(\gamma)$ gives us the highest possible configuration under costs $c \in \mathcal{C}(\gamma)$ such that $\mathcal{C}_n(\gamma) \neq \emptyset$ for all $n \leq n(\gamma)$, and $n(\gamma)$ is decreasing in $\gamma$. □

**Example 2** (*2-configuration*)**.** From the example of a 1-configuration we get that $p_0 = c(0)$ and thus the partition in a 2-configuration is $\mathcal{P}_2 = \{0, c(0), p_1, 1\}$. So we are left with solving $p_1$.

Consider a seller who sets a price at $p_1$. She must make zero profits, and she must be indifferent between sending 1 and 2 ads. More precisely, it means that the following two conditions must hold

$$\begin{cases} e^{-\lambda_2(p_1)}p_1 - c(1) &= 0 \\ e^{-\lambda_2(p_1)}p_1 - c(1) &= \left(1 - \left(1 - e^{-\lambda_2(p_1)}\right)^2\right)p_1 - c(2). \end{cases}$$

From this set of equations we solve that $p_1 = \frac{c(1)}{1-\frac{\Delta c(2)}{c(1)}}$. Since we must have $p_1 \in (p_0, 1)$, it requires that $\Delta c(2) < c(1)(1 - c(1))$ which necessitates the strict concavity of advertising costs.

The mixed strategies are solved by using Eq. (5). For $p \in [p_0, p_1]$ we have

$$F_2^{(1)}(p) = \frac{\log p - \log c(1)}{\theta_2},$$

and for $p \in [p_1, 1]$

$$F_2^{(2)}(p) = \frac{1}{2\theta_2}\left[\lambda_2(p_1) + \lambda_2(p)\right],$$

where $\lambda_2(p_1) = \log \frac{p_1}{c(1)}$ and

$$\lambda_2(p) = -\log\left(1 - \sqrt[2]{1 - \frac{c(2)}{p}}\right).$$

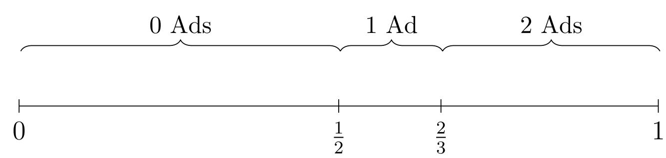


**Fig. 7.** The partition of the unit interval in the 2-configuration with $c(1) = \frac{1}{2}$ and $c(2) = \frac{5}{8}$.

The number of sellers in the market is given by Eq. (6):

$$\theta_2 = \frac{1}{2}\log\left[\frac{c(1)}{(2c(1) - c(2))\left(1 - \sqrt{1 - c(2)}\right)}\right].$$

We have thus solved the unique 2-configuration $(\mathcal{P}_2, F_2, \theta_2)$, where $\mathcal{P}_2 = \{0, c(1), \frac{c(1)}{1-\frac{\Delta c(2)}{c(1)}}, 1\}$,

$$F_2(p) = \begin{cases} \frac{1}{\theta_2}\left[\log p - \log c(1)\right] & p \in [p_0, p_1) \\ \frac{1}{2\theta_2}\log\left[\frac{c(1)}{(2c(1)-c(2))\left(1-\sqrt{1-\frac{c(2)}{p}}\right)}\right] & p \in [p_1, 1] \end{cases}$$

and

$$\theta_2 = \frac{1}{2}\log\left[\frac{c(1)}{(2c(1) - c(2))\left(1 - \sqrt{1 - c(2)}\right)}\right].$$

With $c(1) = \frac{1}{2}$ and $c(2) = \frac{5}{8}$ we have $\theta_2 \approx 0.617$, which is less than the number of sellers in the 1-configuration $\theta_1 = -\log(2) \approx 0.69$ with the same $c(1) = \frac{1}{2}$. The partition is $\mathcal{P}_2 = \{0, \frac{1}{2}, \frac{2}{3}, 1\}$. This and the price distribution are given in Figs. 7 and 8.

**Example 3** (*3-configuration*)**.** From the 2-configuration we know that $p_0 = c(1)$ and $p_1 = \frac{c(1)}{1-\frac{\Delta c(2)}{c(1)}}$. Combining indifference condition $I_2$ and zero profit condition ($ZP_2$) we get

$$\left(1 - \frac{c(3)}{p_2}\right)^2 = \left(1 - \frac{c(2)}{p_2}\right)^3.$$

There is a unique solution to this cubic equation which satisfies $p_2 > c(3)$ and the advertising costs remain concave:

$$p_2 = \frac{c(3)^2 - 3c(2)^2 + (c(3) - c(2))^{\frac{3}{2}}\sqrt{c(3) + 3c(2)}}{2(2c(3) - 3c(2))}.$$

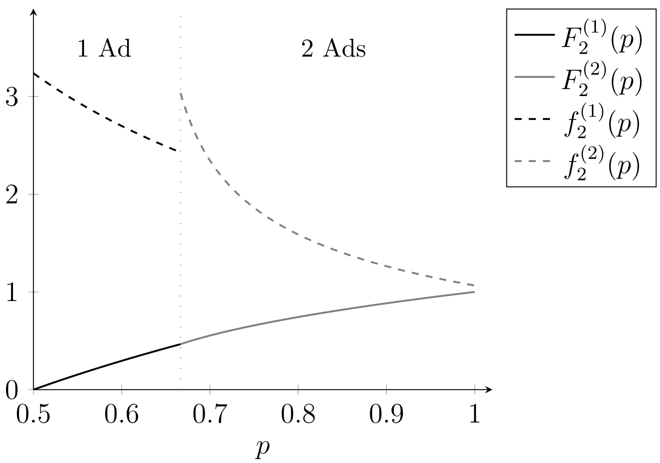


**Fig. 8.** The price distribution of the 2-configuration with $c(1) = \frac{1}{2}$ and $c(2) = \frac{5}{8}$.

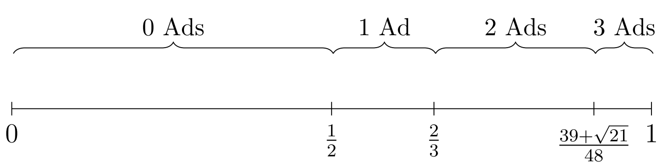


**Fig. 9.** The partition of the unit interval in the 3-configuration with $c(1) = \frac{1}{2}, c(2) = \frac{5}{8}$, and $c(3) = \frac{3}{4}$.

The mixed strategies are derived by the similar steps as in the 2-configuration:

$$F_3(p) = \begin{cases} \frac{1}{\theta_3}\lambda_3(p), & p \in [p_0, p_1) \\ \frac{1}{2\theta_3}\left(\lambda_3(p_1) + \lambda_3(p)\right), & p \in [p_1, p_2) \\ \frac{1}{\theta_3}\left(\frac{1}{2}\lambda_3(p_1) + \frac{1}{6}\lambda_3(p_2) + \frac{1}{3}\lambda_3(p)\right), & p \in [p_2, 1] \end{cases}$$

where $\lambda_3(\cdot)$ is given in (4). $\theta_3$ is given by (6):

$$\theta_3 = \frac{1}{2}\lambda_3(p_1) + \frac{1}{6}\lambda_3(p_2) + \frac{1}{3}\lambda_3(1)$$

With $c(1) = \frac{1}{2}, c(2) = \frac{5}{8}$, and $c(3) = \frac{3}{4}$ we have $\theta_3 \approx 0.611$ which is less than the number of sellers in the 2-configuration with the same advertising costs ($\theta_2 \approx 0.617$). The unit partition

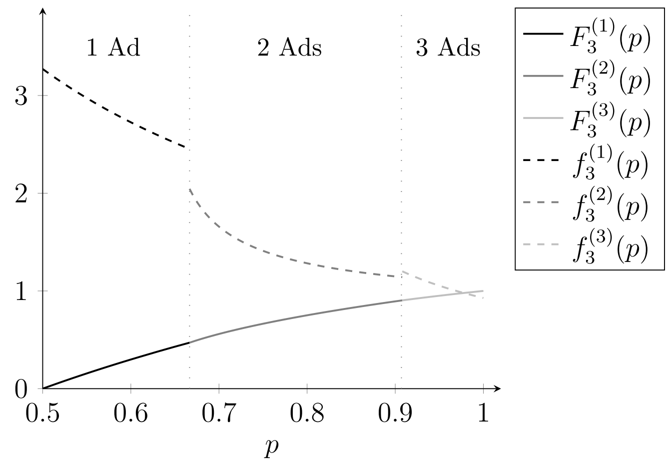


**Fig. 10.** The price distribution of the 3-configuration with $p_2 = \frac{39+\sqrt{21}}{48}$ and $c(1) = \frac{1}{2}$, $c(2) = \frac{5}{8}$, and $c(3) = \frac{6}{8}$.

is $\mathcal{P}_3 = \{0, \frac{1}{2}, \frac{2}{3}, \frac{39+\sqrt{21}}{48}, 1\}$. This and the price distribution are given in Figs. 9 and 10.

## References


Acemoglu, D., Shimer, R., 2000. Wage and technology dispersion. Rev. Econom. Stud. 67 (4), 585–607.
Bagwell, K., 2007. The economic analysis of advertising. In: Handbook of Industrial Organization, Vol. 3. Elsevier, pp. 1701–1844.
Bethune, Z., Choi, M., Wright, R., 2020. Frictional goods markets: Theory and applications. Rev. Econom. Stud. 87 (2), 691–720.
Burdett, K., Shi, S., Wright, R., 2001. Pricing and matching with frictions. J. Political Econ. 109 (5), 1060–1085.
Butters, G.R., 1977. Equilibrium distributions of sales and advertising prices. Rev. Econom. Stud. 44 (3), 465–491.
Gomis-Porqueras, P., Julien, B., Wang, C., 2017. Strategic advertising and directed search. Internat. Econom. Rev. 58 (3), 783–806.
Lester, B., Visschers, L., Wolthoff, R., 2015. Meeting technologies and optimal trading mechanisms in competitive search markets. J. Econom. Theory 155, 1–15.
McAfee, R.P., 1994. Endogenous availability, cartels, and merger in an equilibrium price dispersion. J. Econom. Theory 62 (1), 24–47.
Montgomery, J.D., 1991. Equilibrium wage dispersion and interindustry wage differentials. Q. J. Econ. 106 (1), 163–179.
Peters, M., 1991. Ex ante price offers in matching games non-steady states. Econometrica 59 (5), 1425.
Pissarides, C.A., 2000. Equilibrium Unemployment Theory. MIT Press.
Robert, J., Stahl, D.O., 1993. Informative price advertising in a sequential search model. Econometrica 61 (3), 657–686.
Shi, S., 2018. Sequentially mixed search and equilibrium price dispersion. Available at SSRN 3262935.
Shimer, R., 2005. The assignment of workers to jobs in an economy with coordination frictions. J. Political Econ. 113 (5), 996–1025.