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Chen Inequalities for Statistical Submanifolds of Kähler-Like Statistical Manifolds

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Abstract: We consider Kähler-like statistical manifolds, whose curvature tensor field satisfies a natural condition. For their statistical submanifolds, we prove a Chen first inequality and a Chen inequality for the invariant $\delta(2,2)$.

Keywords: statistical manifolds; Kähler-like statistical manifolds; Chen inequalities

MSC: 53C05; 53C40

1. Introduction

In [1], the notion of a statistical manifold was defined by Amari. It has applications in information geometry, which represents one of the main tools for machine learning and evolutionary biology. In 2004, K. Takano [2] defined and investigated Kähler-like statistical manifolds and their statistical submanifolds.

A statistical manifold is an m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$ endowed with a pairing of torsion-free affine connections $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ satisfying:

$$Z\widetilde{g}(X,Y) = \widetilde{g}(\widetilde{\nabla}_Z X, Y) + \widetilde{g}(X, \widetilde{\nabla}^*_Z Y), \tag{1}$$

for any $X, Y, Z \in \Gamma(T\widetilde{M})$. The connections $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ are called dual connections (see [1,3]), and it is easily seen that $(\widetilde{\nabla}^*)^* = \widetilde{\nabla}$. The pairing $(\widetilde{\nabla}, \widetilde{g})$ is called a statistical structure.

Furthermore,

$$\left(\widetilde{\nabla}_X \widetilde{g}\right)(Y,Z) - \left(\widetilde{\nabla}_Y \widetilde{g}\right)(X,Z) = 0 \tag{2}$$

holds for $X, Y, Z \in T\widetilde{M}$ [4]. Formula (2) is also known as the Codazzi equation.

Any torsion-free affine connection $\widetilde{\nabla}$ always has a dual connection given by:

$$\widetilde{\nabla} + \widetilde{\nabla}^* = 2\widetilde{\nabla}^0, \tag{3}$$

where $\widetilde{\nabla}^0$ is the Levi–Civita connection on $\widetilde{M}$.

Similar definitions can be considered for semi-Riemannian manifolds (see also [5]).

A statistical structure is said to be of constant curvature $\epsilon \in \mathbf{R}$ [6] if:

$$\widetilde{R}(X,Y)Z = \epsilon[g(Y,Z)X - g(X,Z)Y],$$

for any vector fields X, Y, Z. The same equation holds for $\widetilde{R}^*(X,Y)Z$.

A statistical structure of null constant curvature is called a Hessian structure.

In [2,7], K. Takano considered a (semi-)Riemannian manifold $\left(\widetilde{M}, \widetilde{g}\right)$ with an almost complex structure $\widetilde{J}$, endowed with another tensor field $\widetilde{J}^*$ of type $(1,1)$ satisfying:

$$\widetilde{g}\left(\widetilde{J}X, Y\right) + \widetilde{g}\left(X, \widetilde{J}^*Y\right) = 0, \tag{4}$$

for vector fields X and Y on $\left(\widetilde{M}, \widetilde{g}\right)$. Then, $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$ is called an almost Hermite-like manifold. It is easy to see that $\left(\widetilde{J}^*\right)^* = \widetilde{J}$, $\left(\widetilde{J}^*\right)^2 = -I$ and $\widetilde{g}\left(\widetilde{J}X, \widetilde{J}^*Y\right) = \widetilde{g}\left(X, Y\right)$. If $\widetilde{J}$ is parallel with respect to $\widetilde{\nabla}$, then $\left(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}, \widetilde{J}\right)$ is called a Kähler-like statistical manifold [7].

One also has:

$$\widetilde{g}\left(\left(\widetilde{\nabla}_X \widetilde{J}\right) Y, Z\right) + \widetilde{g}\left(Y, \left(\widetilde{\nabla}^*_X \widetilde{J}^*\right) Z\right) = 0$$

(see [2,7]).

On the other hand, in 1993, B.-Y. Chen introduced new intrinsic invariants, more precisely curvature invariants, called Chen invariants (or δ-invariants) (see [8] for details). In [9], the author proved the Chen first inequality for arbitrary submanifolds in Riemannian space forms.

The Chen first invariant of a Riemannian manifold $\widetilde{M}$ is given by $\delta_{\widetilde{M}} = \tau - \inf K$, where τ and K represent the scalar and sectional curvatures of $\widetilde{M}$, respectively.

Furthermore, the Chen $\delta(2,2)$ invariant is defined by $\delta(2,2)(p) = \tau(p) - \inf(K(\pi_1) + K(\pi_2))$, where π_1 and π_2 are mutually orthogonal plane sections at $p \in \widetilde{M}$. This is a generalization of the Chen first invariant, but also, a particular case of the $\delta(n_1, n_2, ..., n_k)$ invariant, introduced by B.-Y. Chen, as well (see [8]).

Statistical submanifolds in statistical manifolds were considered by few authors, and the interest in this subject grew in the recent period. Closely related to our research target, we would like to mention the following.

In [5], M. E. Aydin, A. Mihai, and I. Mihai studied statistical submanifolds in statistical manifolds of constant curvature and proved inequalities for the scalar curvature and the Ricci curvature associated with the dual connections. The same authors obtained in [10] a generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature. In their paper, another definition of the sectional curvature, due to Opozda, given in [11], was used (see also [12]).

Recently, in [13], B.-Y. Chen, A. Mihai, and I. Mihai established the Chen first inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature. The study was continued in [14], where the authors obtained a Chen inequality for the $\delta(2,2)$ invariant. Recall that a Hessian manifold of constant Hessian curvature c is a statistical manifold of null curvature and also a Riemannian space form of constant sectional curvature $-c/4$ (with respect to the sectional curvature defined by the Levi–Civita connection) [6].

In the present article, motivated by the above studies, we obtain a Chen first inequality and an inequality for the Chen $\delta(2,2)$ invariant for statistical submanifolds in Kähler-like statistical manifolds.

Furthermore, for our next study, we would like to point out that, by referring to the papers [15–17], the curvature invariants of statistical submanifolds in Kähler-like statistical manifolds will be investigated.

2. Preliminaries

In general, the dual connections are not metric; it follows that one cannot define a sectional curvature with respect to them by the standard definition from Riemannian geometry. B. Opozda proposed two different definitions, in [11,12]. We will work in this article with the definition from [12].

Let $\widetilde{M}$ be a statistical manifold, and consider π a plane in $T\widetilde{M}$, with an orthonormal basis $\{X, Y\}$; the sectional K-curvature was defined by [12]:

$$\widetilde{K}(\pi) = \frac{1}{2}\left[\widetilde{g}(\widetilde{R}(X,Y)Y,X) + \widetilde{g}(\widetilde{R}^*(X,Y)Y,X) - 2\widetilde{g}(\widetilde{R}^0(X,Y)Y,X)\right], \tag{5}$$

where $\widetilde{R}^0$ denotes the curvature tensor field of the Levi–Civita connection $\widetilde{\nabla}^0$ on $T\widetilde{M}$.

Denote by $\widetilde{R}$ and $\widetilde{R}^*$ the curvature tensor fields of $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, respectively. Then, $\widetilde{R}$ and $\widetilde{R}^*$ satisfy:

$$\widetilde{g}\left(\widetilde{R}(X,Y)Z,W\right) = -\widetilde{g}\left(\widetilde{R}^*(X,Y)W,Z\right) \tag{6}$$

(see [4]).

Let $\left(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}\right)$ be a statistical manifold and $f : M \longrightarrow \widetilde{M}$ an immersion. One defines a pair g and ∇ on M by:

$$g = f^*\widetilde{g}, \quad g(\nabla_X Y, Z) = \widetilde{g}\left(\widetilde{\nabla}_{f_*X} f_*Y, Z\right),$$

for any $X, Y, Z \in TM$, where the connection induced from $\widetilde{\nabla}$ by f on the induced bundle $f^* : T\widetilde{M} \longrightarrow TM$ is denoted by the same symbol $\widetilde{\nabla}$. Then, the pair (∇, g) is a statistical structure on M, which is called the induced statistical structure by f from $\left(\widetilde{\nabla}, \widetilde{g}\right)$ [4].

Let (M, g, ∇) and $\left(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}\right)$ be two statistical manifolds. An immersion $f : M \longrightarrow \widetilde{M}$ is called a statistical immersion if (∇, g) coincides with the induced statistical structure.

Let M be an n-dimensional submanifold of $\widetilde{M}$. Then, we have the Gauss formulae:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\widetilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X,Y),$$

where h and h^* are symmetric and bilinear, called the imbedding curvature tensors of M in $\widetilde{M}$ for $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, respectively. In this case, ∇ and ∇^* are called the induced connections of $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, respectively.

Since h and h^* are bilinear, there exist linear transformations A_ξ and A^*_ξ on TM defined by:

$$g(A_\xi X, Y) = g(h(X,Y), \xi),$$

$$g(A^*_\xi X, Y) = g(h^*(X,Y), \xi),$$

for any $\xi \in \Gamma(T^\perp M)$ and $X, Y \in \Gamma(TM)$. Further (see [3]), the corresponding Weingarten formulas are:

$$\widetilde{\nabla}_X \xi = -A^*_\xi X + D_X \xi,$$

$$\widetilde{\nabla}^*_X \xi = -A_\xi X + D^*_X \xi,$$

for any $\xi \in \Gamma(T^\perp M)$ and $X \in \Gamma(TM)$. The connections D and D^* are Riemannian dual connections with respect to the induced metric on $\Gamma(T^\perp M)$.

Let $\widetilde{R}$ and R be the Riemannian curvature tensors of $\widetilde{\nabla}$ and ∇, respectively. Then, the Gauss equation is given by:

$$\widetilde{g}\left(\widetilde{R}(X,Y)Z,W\right) = g(R(X,Y)Z,W) \tag{7}$$

$$+\widetilde{g}\left(h\left(X,Z\right),h^{*}\left(Y,W\right)\right)-\widetilde{g}\left(h^{*}\left(X,W\right),h\left(Y,Z\right)\right),$$

where $X, Y, Z, W \in TM$ (see [3]).

3. An Example of a Submanifold of a Kähler-Like Statistical Manifold

We start this section by recalling an example of a Kähler-like statistical manifold, given by K. Takano in [2].

Example 1. *Let $\mathbb{R}^{2n}_n$ be a $2n$-dimensional semi-Euclidean space with a local coordinate system $(x_1, ..., x_n, y_1, ..., y_n)$, which admits the following almost complex structure $\widetilde{J}$ and the metric $\widetilde{g}$:*

$$\widetilde{J}=\begin{pmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{pmatrix}, \qquad \widetilde{g}=\begin{pmatrix} 2\delta_{ij} & 0 \\ 0 & -\delta_{ij} \end{pmatrix}.$$

Denote the flat affine connection by $\widetilde{\nabla}$. Then, $\left(\mathbb{R}^{2n}_n, \widetilde{\nabla}, \widetilde{g}, \widetilde{J}\right)$ is a Kähler-like statistical manifold. The conjugate connection $\widetilde{\nabla}^*$ is flat and

$$\widetilde{J}^{*}=\frac{1}{2}\begin{pmatrix} 0 & -\delta_{ij} \\ 4\delta_{ij} & 0 \end{pmatrix}.$$

Next, we will present another example of a Kähler-like statistical manifold and construct a submanifold.

Example 2. *We consider the half upper space:*

$$\widetilde{M}^{n+1}_{\nu}=\{\,(x_1,\ldots,x_n,\,x_{n+1})\mid x_{n+1}>0\,\}$$

admitting components of the metric $\widetilde{g}$ as follows:

$$\widetilde{g}_{ij}=\frac{\varepsilon_i}{x_{n+1}^2}\,\delta_{ij}, \qquad \widetilde{g}_{in+1}=\widetilde{g}_{n+1i}=0, \qquad \widetilde{g}_{n+1n+1}=\frac{\omega^2}{x_{n+1}^2},$$

where ω is a positive constant and ε_i is -1 or $+1$. The signature of $\widetilde{g}$ is $(\nu, n+1-\nu)$.

We consider the following two connections:

$$\begin{aligned}
&\widetilde{\nabla}^{(1)}_{\partial_i}\partial_j=0,\\
&\widetilde{\nabla}^{(1)}_{\partial_i}\partial_{n+1}=\widetilde{\nabla}^{(1)}_{\partial_{n+1}}\partial_i=-\frac{2}{x_{n+1}}\,\partial_i,\\
&\widetilde{\nabla}^{(1)}_{\partial_{n+1}}\partial_{n+1}=-\frac{3}{x_{n+1}}\,\partial_{n+1}
\end{aligned}$$

and:

$$\begin{aligned}
&\widetilde{\nabla}^{(-1)}_{\partial_i}\partial_j=\frac{2}{\omega^2 x_{n+1}}\,\varepsilon_i\delta_{ij}\,\partial_{n+1},\\
&\widetilde{\nabla}^{(-1)}_{\partial_i}\partial_{n+1}=\widetilde{\nabla}^{(-1)}_{\partial_{n+1}}\partial_i=0,\\
&\widetilde{\nabla}^{(-1)}_{\partial_{n+1}}\partial_{n+1}=\frac{1}{x_{n+1}}\,\partial_{n+1},
\end{aligned}$$

where $\partial_i = \partial/\partial x_i$.

We define the α-connection $\widetilde{\nabla}^{(\alpha)}$ (see [1]) by:

$$\widetilde{\nabla}^{(\alpha)} = \frac{1+\alpha}{2}\widetilde{\nabla}^{(1)} + \frac{1-\alpha}{2}\widetilde{\nabla}^{(-1)}.$$

It follows that:

$$\widetilde{\nabla}^{(\alpha)}_{\partial_i}\partial_j = \frac{1-\alpha}{\omega^2 x_{n+1}}\,\varepsilon_i\delta_{ij}\,\partial_{n+1},$$
$$\widetilde{\nabla}^{(\alpha)}_{\partial_i}\partial_{n+1} = \widetilde{\nabla}^{(\alpha)}_{\partial_{n+1}}\partial_i = -\frac{1+\alpha}{x_{n+1}}\,\partial_i,$$
$$\widetilde{\nabla}^{(\alpha)}_{\partial_{n+1}}\partial_{n+1} = -\frac{1+2\alpha}{x_{n+1}}\,\partial_{n+1}.$$

Then, $\left(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}^{(\alpha)}\right)$ is a statistical manifold.

Furthermore, the curvature tensors $\widetilde{R}^{(\alpha)}$ with respect to the α-connection $\widetilde{\nabla}^{(\alpha)}$ are:

$$\widetilde{R}^{(\alpha)}(\partial_i,\partial_j)\partial_k = -\frac{c(\alpha)}{\omega^2 x_{n+1}^2}\left(\varepsilon_j\delta_{jk}\partial_i - \varepsilon_i\delta_{ik}\partial_j\right) \qquad (i,\, j,\, k \neq n+1)$$
$$\widetilde{R}^{(\alpha)}(\partial_i,\partial_j)\partial_{n+1} = 0 \qquad (i,\, j \neq n+1)$$
$$\widetilde{R}^{(\alpha)}(\partial_i,\partial_{n+1})\partial_k = \frac{c(\alpha)}{\omega^2 x_{n+1}^2}\,\varepsilon_i\delta_{ik}\partial_{n+1} \qquad (i,\, k \neq n+1)$$
$$\widetilde{R}^{(\alpha)}(\partial_i,\partial_{n+1})\partial_{n+1} = -\frac{c(\alpha)}{x_{n+1}^2}\,\partial_i \qquad (i \neq n+1),$$

where $c(\alpha) = (1-\alpha)(1+\alpha)$.

Therefore, $\left(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}^{(\alpha)}\right)$ is of constant curvature $-\frac{c(\alpha)}{\omega^2}$. Especially, $\left(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}^{(\pm 1)}\right)$ is flat, respectively.

We put:

$$e_i = x_{n+1}\,\partial_i \;\; (i = 1, 2, \ldots, n), \qquad\qquad e_{n+1} = \frac{x_{n+1}}{\omega}\,\partial_{n+1}.$$

From $g(e_i, e_j) = \varepsilon_i\delta_{ij}$ and $g(e_{n+1}, e_{n+1}) = 1$, it follows that the set $\{e_1, \ldots, e_n, e_{n+1}\}$ is an orthonormal base.

Then, the α-connection can be rewritten as follows:

$$\widetilde{\nabla}^{(\alpha)}_{e_i}e_j = \frac{1-\alpha}{\omega}\,\varepsilon_i\delta_{ij}\,e_{n+1} \qquad (i, j \neq n+1)$$
$$\widetilde{\nabla}^{(\alpha)}_{e_i}e_{n+1} = -\frac{1+\alpha}{\omega}\,e_i \qquad (i \neq n+1)$$
$$\widetilde{\nabla}^{(\alpha)}_{e_{n+1}}e_i = -\frac{\alpha}{\omega}\,e_i \qquad (i \neq n+1)$$
$$\widetilde{\nabla}^{(\alpha)}_{e_{n+1}}e_{n+1} = -\frac{2\alpha}{\omega}\,e_{n+1}.$$

• Almost complex structures:

We will construct almost complex structures $\widetilde{J}^{(1)}$ and $\widetilde{J}^{(-1)}$ satisfying $\widetilde{\nabla}^{(\alpha)}\widetilde{J}^{(\alpha)} = 0$. We get:

$$
\begin{aligned}
&e_i\widetilde{J}_j^{(\alpha)\,k} - \frac{1+\alpha}{\omega}\,\delta_i^k\widetilde{J}_j^{(\alpha)\,n+1} - \frac{(1-\alpha)\varepsilon_i}{\omega}\,\delta_{ij}\widetilde{J}_{n+1}^{(\alpha)\,k} = 0,\\
&e_i\widetilde{J}_j^{(\alpha)\,n+1} + \frac{(1-\alpha)\varepsilon_i}{\omega}\left(\widetilde{J}_j^{(\alpha)\,i} - \delta_{ij}\widetilde{J}_{n+1}^{(\alpha)\,n+1}\right) = 0,\\
&e_i\widetilde{J}_{n+1}^{(\alpha)\,k} + \frac{1+\alpha}{\omega}\left(\widetilde{J}_i^{(\alpha)\,k} - \delta_i^k\widetilde{J}_{n+1}^{(\alpha)\,n+1}\right) = 0,\\
&e_i\widetilde{J}_{n+1}^{(\alpha)\,n+1} + \frac{1+\alpha}{\omega}\,\widetilde{J}_i^{(\alpha)\,n+1} + \frac{(1-\alpha)\varepsilon_i}{\omega}\,\widetilde{J}_{n+1}^{(\alpha)\,i} = 0,\\
&e_{n+1}\widetilde{J}_i^{(\alpha)\,k} = 0,\\
&e_{n+1}\widetilde{J}_i^{(\alpha)\,n+1} - \frac{\alpha}{\omega}\,\widetilde{J}_i^{(\alpha)\,n+1} = 0,\\
&e_{n+1}\widetilde{J}_{n+1}^{(\alpha)\,k} + \frac{\alpha}{\omega}\,\widetilde{J}_{n+1}^{(\alpha)\,k} = 0,\\
&e_{n+1}\widetilde{J}_{n+1}^{(\alpha)\,n+1} = 0.
\end{aligned}
$$

Because $\widetilde{M}$ is of constant curvature, we have that $\widetilde{M}$ is flat, $n+1 \geq 4$, that is $\alpha = \pm 1$ [18]. When $\alpha = 1$, we find:

$$
\begin{aligned}
&e_i\widetilde{J}_j^{(1)\,k} - \frac{2}{\omega}\,\delta_i^k\widetilde{J}_j^{(1)\,n+1} = 0,\\
&e_i\widetilde{J}_j^{(1)\,n+1} = 0,\\
&e_i\widetilde{J}_{n+1}^{(1)\,k} + \frac{2}{\omega}\left(\widetilde{J}_i^{(1)\,k} - \delta_i^k\widetilde{J}_{n+1}^{(1)\,n+1}\right) = 0,\\
&e_i\widetilde{J}_{n+1}^{(1)\,n+1} + \frac{2}{\omega}\,\widetilde{J}_i^{(1)\,n+1} = 0,\\
&e_{n+1}\widetilde{J}_i^{(1)\,k} = 0,\\
&e_{n+1}\widetilde{J}_i^{(1)\,n+1} - \frac{1}{\omega}\,\widetilde{J}_i^{(1)\,n+1} = 0,\\
&e_{n+1}\widetilde{J}_{n+1}^{(1)\,k} + \frac{1}{\omega}\,\widetilde{J}_{n+1}^{(1)\,k} = 0,\\
&e_{n+1}\widetilde{J}_{n+1}^{(1)\,n+1} = 0.
\end{aligned}
$$

Thus, we obtain:

$$
\begin{aligned}
&\widetilde{J}_j^{(1)\,k} = \frac{2}{\omega}\,C_j\,x_k + A_j^k,\\
&\widetilde{J}_j^{(1)\,n+1} = C_j\,x_{n+1},\\
&\widetilde{J}_{n+1}^{(1)\,k} = -\frac{2}{\omega\,x_{n+1}}\left\{\left(\frac{2}{\omega}\,C_s x_s + D\right)x_k + A_s^k x_s + B^k\right\},\\
&\widetilde{J}_{n+1}^{(1)\,n+1} = -\frac{2}{\omega}\,C_s x_s - D,
\end{aligned}
$$

where we used the Einstein summation convention.

If we put:

$$\Sigma = \begin{pmatrix} A_i^j & B^j \\ -\frac{2}{\omega} C_i & -D \end{pmatrix},$$

then $(\widetilde{J}^{(1)})^2 = -id$ if and only if the constants A_i^j, B^j, C_i and D satisfy $\Sigma^2 = -id$.

We remark that:

$$\mathrm{trace}\, \widetilde{J}^{(1)} = \sum A_s^s - D = \mathrm{trace}\, \Sigma.$$

Then, $\left(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}^{(1)}, \widetilde{J}^{(1)}\right)$ is a Kähler-like statistical manifold.

Furthermore, we set:

$$\begin{aligned}
\widetilde{J}_j^{(-1)\,k} &= -\varepsilon_j \varepsilon_k \left(\frac{2}{\omega} C_k x_j + A_k^j \right), \\
\widetilde{J}_j^{(-1)\,n+1} &= \frac{2\varepsilon_j}{\omega\, x_{n+1}} \left\{ \left(\frac{2}{\omega} C_s x_s + D \right) x_j + A_s^j x_s + B^j \right\}, \\
\widetilde{J}_{n+1}^{(-1)\,k} &= -\varepsilon_k C_k\, x_{n+1}, \\
\widetilde{J}_{n+1}^{(-1)\,n+1} &= \frac{2}{\omega} C_s x_s + D.
\end{aligned}$$

Then, $\widetilde{g}\left(\widetilde{J}^{(1)} e_i, e_j\right) + g\left(e_i, \widetilde{J}^{(-1)} e_j\right) = 0$ and $\widetilde{\nabla}^{(-1)} \widetilde{J}^{(-1)} = 0$ hold.

- A submanifold of $\widetilde{M}_\nu^{n+1}$:

We consider a submanifold of $\widetilde{M}_\nu^{n+1}$:

$$M_\xi^\ell = \{(x_1, x_2, \ldots, x_\ell, 0, \ldots, 0) \mid -\infty < x_i < \infty \ (i = 1, \ldots, \ell)\} = \mathbb{R}_\xi^\ell,$$

where $\xi \le \nu$. Let $\{e_1, \ldots, e_\ell\}$ and $\{e_{\ell+1}, \ldots, e_{n+1}\}$ be orthonormal bases of $T_p M$ and $T_p^\perp M$, respectively. We set $i, j, s \in \{1, \ldots, \ell\}$ and $a, b \in \{\ell+1, \ldots, n\}$.

We have:

$$\begin{aligned}
\widetilde{\nabla}_{e_i}^{(\alpha)} e_j &= \frac{1-\alpha}{\omega}\, \varepsilon_i \delta_{ij}\, e_{n+1}, \\
\widetilde{\nabla}_{e_i}^{(\alpha)} e_b &= 0, \\
\widetilde{\nabla}_{e_i}^{(\alpha)} e_{n+1} &= -\frac{1+\alpha}{\omega}\, e_i, \\
\widetilde{\nabla}_{e_a}^{(\alpha)} e_j &= 0, \\
\widetilde{\nabla}_{e_a}^{(\alpha)} e_b &= \frac{1-\alpha}{\omega}\, \varepsilon_a \delta_{ab}\, e_{n+1}, \\
\widetilde{\nabla}_{e_a}^{(\alpha)} e_{n+1} &= -\frac{1+\alpha}{\omega}\, e_a, \\
\widetilde{\nabla}_{e_{n+1}}^{(\alpha)} e_j &= -\frac{\alpha}{\omega}\, e_j, \\
\widetilde{\nabla}_{e_{n+1}}^{(\alpha)} e_b &= -\frac{\alpha}{\omega}\, e_b, \\
\widetilde{\nabla}_{e_{n+1}}^{(\alpha)} e_{n+1} &= -\frac{2\alpha}{\omega}\, e_{n+1}.
\end{aligned}$$

It follows that:

$$\begin{aligned}
&\nabla^{(\alpha)}_{e_i} e_j = 0,\\
&h^{(\alpha)}(e_i, e_j) = \frac{1-\alpha}{\omega}\, \varepsilon_i \delta_{ij}\, e_{n+1},\\
&A^{(\alpha)}_{e_b} e_i = 0,\\
&A^{(\alpha)}_{e_{n+1}} e_i = \frac{1+\alpha}{\omega}\, e_i,\\
&D^{(\alpha)}_{e_i} e_b = 0,\\
&D^{(\alpha)}_{e_i} e_{n+1} = 0.
\end{aligned}$$

Moreover, the mean curvature vector $H^{(\alpha)}$ with respect to $\widetilde{\nabla}^{(\alpha)}$ satisfies:

$$\begin{aligned}
H^{(\alpha)} &= \frac{1-\alpha}{\omega}\, e_{n+1},\\
\widetilde{\nabla}^{(\alpha)}_{e_i} H^{(\alpha)} &= -\frac{c(\alpha)}{\omega^2}\, e_i,\\
\widetilde{\nabla}^{(\alpha)}_{e_a} H^{(\alpha)} &= -\frac{c(\alpha)}{\omega^2}\, e_a,\\
\widetilde{\nabla}^{(\alpha)}_{e_{n+1}} H^{(\alpha)} &= -\frac{2\alpha}{\omega}\, H^{(\alpha)}.
\end{aligned}$$

Next, we consider $\left(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}^{(1)}, \widetilde{J}^{(1)}\right)$ a Kähler-like statistical manifold.
We have:

$$\begin{aligned}
\widetilde{J}^{(1)} e_i &= \left(\frac{2}{\omega}\, C_i x_s + A_i^s\right) e_s + \left(\frac{2}{\omega}\, C_i x_a + A_i^a\right) e_a + C_i x_{n+1}\, e_{n+1},\\
\widetilde{J}^{(1)} e_a &= \left(\frac{2}{\omega}\, C_a x_s + A_a^s\right) e_s + \left(\frac{2}{\omega}\, C_a x_b + A_a^b\right) e_b + C_a x_{n+1}\, e_{n+1},\\
\widetilde{J}^{(1)} e_{n+1} &= -\frac{2}{\omega\, x_{n+1}} \left\{ \left(\frac{2}{\omega}\, C_S x_S + D\right) x_s + A_S^s x_S + B^s \right\} e_s\\
&\quad -\frac{2}{\omega\, x_{n+1}} \left\{ \left(\frac{2}{\omega}\, C_S x_S + D\right) x_a + A_S^a x_S + B^a \right\} e_a\\
&\quad -\left(\frac{2}{\omega}\, C_S x_S + D\right) e_{n+1},
\end{aligned}$$

where $S \in \{1, \ldots, n\}$.

Let $X \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$. We decompose $\widetilde{J}^{(1)} X = P^{(1)} X + F^{(1)} X$ and $\widetilde{J}^{(1)} \xi = t^{(1)} \xi + f^{(1)} \xi$, respectively, where $P^{(1)} X, t^{(1)} \xi \in \Gamma(TM)$ and $F^{(1)} X, f^{(1)} \xi \in \Gamma(T^\perp M)$.

Therefore, we find:

$$P^{(1)}e_i = \left(\frac{2}{\omega}\,C_i x_s + A_i^s\right) e_s,$$

$$F^{(1)}e_i = \left(\frac{2}{\omega}\,C_i x_a + A_i^a\right) e_a + C_i x_{n+1}\, e_{n+1},$$

$$t^{(1)}e_a = \left(\frac{2}{\omega}\,C_a x_s + A_a^s\right) e_s,$$

$$f^{(1)}e_a = \left(\frac{2}{\omega}\,C_a x_b + A_a^b\right) e_b + C_a x_{n+1}\, e_{n+1},$$

$$t^{(1)}e_{n+1} = -\frac{2}{\omega\, x_{n+1}} \left\{ \left(\frac{2}{\omega}\,C_S x_S + D\right) x_s + A_S^s x_S + B^s \right\} e_s,$$

$$f^{(1)}e_{n+1} = -\frac{2}{\omega\, x_{n+1}} \left\{ \left(\frac{2}{\omega}\,C_S x_S + D\right) x_a + A_S^a x_S + B^a \right\} e_a - \left(\frac{2}{\omega}\,C_S x_S + D\right) e_{n+1}$$

and

$$\left(\nabla^{(1)}_{e_i} P^{(1)}\right) e_j = \frac{2}{\omega}\,C_j x_{n+1}\, e_i,$$

$$\left(\nabla^{(1)}_{e_i} F^{(1)}\right) e_j = 0,$$

$$\left(\nabla^{(1)}_{e_i} t^{(1)}\right) e_a = \frac{2}{\omega}\,C_a\, e_i,$$

$$\left(\nabla^{(1)}_{e_i} t^{(1)}\right) e_{n+1} = -\frac{2}{\omega\, x_{n+1}} \left\{ \left(\frac{2}{\omega}\,C_S x_S + D\right) \delta_i^s + \frac{2}{\omega}\,C_i x_s + A_i^s \right\} e_s,$$

$$\left(\nabla^{(1)}_{e_i} f^{(1)}\right) e_a = 0,$$

$$\left(\nabla^{(1)}_{e_i} f^{(1)}\right) e_{n+1} = -\frac{2}{\omega\, x_{n+1}} \left\{ \left(\frac{2}{\omega}\,C_i x_a + A_i^a\right) e_a + C_i\, x_{n+1}\, e_{n+1} \right\} = -\frac{2}{\omega\, x_{n+1}}\, F^{(1)} e_i.$$

4. A Chen First Inequality

On a Kähler-like statistical manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}, \widetilde{J}\right)$, K. Takano [7] considered the curvature tensor $\widetilde{R}$ of $\widetilde{\nabla}$ such that:

$$\widetilde{R}\,(X,Y)\,Z = \frac{c}{4}\left\{ \widetilde{g}\,(Y,Z)\,X - \widetilde{g}\,(X,Z)\,Y - \widetilde{g}\left(Y,\widetilde{J}Z\right)\widetilde{J}X + \widetilde{g}\left(X,\widetilde{J}Z\right)\widetilde{J}Y + \left[\widetilde{g}\left(X,\widetilde{J}Y\right) - \widetilde{g}\left(Y,\widetilde{J}X\right)\right]\widetilde{J}Z \right\}. \tag{8}$$

We point out that a Kähler manifold satisfying (8) is a space of constant holomorphic sectional curvature (complex space form), which gives sense to this condition.

In the same paper [7], the following Lemma was proven:

Lemma 1. *On a Kähler-like statistical manifold whose curvature tensor $\widetilde{R}$ is of the form of (8), one has $c = 0$ or* $\mathrm{trace}(AB) = \mathrm{trace}(AB)^2$, *where $A = (\widetilde{g}_{\alpha\beta})$ and $B = (\widetilde{g}^{\alpha\beta})$ (as defined in [7]).*

More precisely, one denotes by $\widetilde{g}_{\alpha\beta} = \widetilde{g}\left(\frac{\partial}{\partial z^{\alpha}}, \frac{\partial}{\partial z^{\beta}}\right)$, $\widetilde{g}_{\alpha\bar{\beta}} = \widetilde{g}\left(\frac{\partial}{\partial z^{\alpha}}, \frac{\partial}{\partial \bar{z}^{\beta}}\right)$, etc., and $\widetilde{g}^{CD}$ the components of the inverse matrix of $\widetilde{g}$, where $C, D \in \{\alpha, \bar{\alpha} | \alpha \in \{1, ..., n\}\}$.

As a consequence, such a manifold is not a trivial one.

Let $\left(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}, \widetilde{J}\right)$ (for simplicity, we will write next g and J instead of $\widetilde{g}$, respectively $\widetilde{J}$) be a $2m$-dimensional Kähler-like statistical manifold whose curvature tensor $\widetilde{R}$ is of the form of (8) and M an n-dimensional statistical submanifold of $\widetilde{M}$, $p \in M$ and π a plane section at p. We consider an orthonormal basis $\{e_1, e_2\}$ of π and $\{e_1, ..., e_n\}$, $\{e_{n+1}, ..., e_{2m}\}$ orthonormal bases of T_pM and $T_p^{\perp}M$, respectively.

The mean curvature vectors are given by:

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) = \frac{1}{n}\sum_{\alpha=n+1}^{2m}\left(\sum_{i=1}^{n} h_{ii}^{\alpha}\right)e_{\alpha}, \quad h_{ij}^{\alpha} = \widetilde{g}\left(h\left(e_i, e_j\right), e_{\alpha}\right),$$

and:

$$H^* = \frac{1}{n}\sum_{i=1}^{n} h^*(e_i, e_i) = \frac{1}{n}\sum_{\alpha=n+1}^{2m}\left(\sum_{i=1}^{n} h_{ii}^{*\alpha}\right)e_{\alpha}, \quad h_{ij}^{*\alpha} = \widetilde{g}\left(h^*\left(e_i, e_j\right), e_{\alpha}\right).$$

We denote by K_0 the sectional curvature of the Levi–Civita connection ∇^0 on M and by h^0 the second fundamental form of M w.r.t. the Levi–Civita connection.

From (5), the sectional K-curvature $K(\pi)$ of the plane section π is:

$$K(\pi) = \frac{1}{2}\left[g\left(R(e_1, e_2)e_2, e_1\right) + g\left(R^*(e_1, e_2)e_2, e_1\right) - 2g\left(R^0(e_1, e_2)e_2, e_1\right)\right].$$

From (6)–(8), we have:

$$\begin{aligned} g\left(R(e_1, e_2)e_2, e_1\right) &= \frac{c}{4}\Big\{1 + 2g^2(e_1, Je_2) - g(e_2, Je_2)g(e_1, Je_1) \\ &\quad - g(Je_1, e_2)g(e_1, Je_2)\Big\} + \sum_{\alpha=n+1}^{2m}\left(h_{11}^{*\alpha}h_{22}^{\alpha} - h_{12}^{*\alpha}h_{12}^{\alpha}\right), \\ g\left(R^*(e_1, e_2)e_2, e_1\right) = -g\left(R(e_1, e_2)e_1, e_2\right) &= \frac{c}{4}\Big\{-1 - 2g^2(Je_1, e_2) + g(e_2, Je_2)g(e_1, Je_1) \\ &\quad + g(Je_1, e_2)g(e_1, Je_2)\Big\} + \sum_{\alpha=n+1}^{2m}\left(h_{12}^{*\alpha}h_{12}^{\alpha} - h_{11}^{\alpha}h_{22}^{*\alpha}\right). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} K(\pi) &= \frac{c}{4}\Big\{1 + g^2(e_1, Pe_2) + g^2(Pe_1, e_2) - g(e_2, Pe_2)g(e_1, Pe_1) \\ &\quad - g(Pe_1, e_2)g(e_1, Pe_2)\Big\} - K_0(\pi) \\ &\quad + \frac{1}{2}\sum_{\alpha=n+1}^{2m}\left[h_{11}^{\alpha}h_{22}^{*\alpha} + h_{11}^{*\alpha}h_{22}^{\alpha} - 2h_{12}^{*\alpha}h_{12}^{\alpha}\right], \end{aligned}$$

where JX decomposes into its tangent and normal parts, i.e., $JX = PX + FX$.

By using $h + h^* = 2h^0$, the last equality can be written as (see [13]):

$$\begin{aligned} K(\pi) &= \frac{c}{4}\Big\{1 + g^2(e_1, Pe_2) + g^2(Pe_1, e_2) - g(e_2, Pe_2)g(e_1, Pe_1) \\ &\quad - g(Pe_1, e_2)g(e_1, Pe_2)\Big\} - K_0(\pi) \\ &\quad + 2\sum_{\alpha=n+1}^{2m}\left[h_{11}^{0\alpha}h_{22}^{0\alpha} - \left(h_{12}^{0\alpha}\right)^2\right] - \frac{1}{2}\sum_{\alpha=n+1}^{2m}\left\{\left[h_{11}^{\alpha}h_{22}^{\alpha} - \left(h_{12}^{\alpha}\right)^2\right] + \left[h_{11}^{*\alpha}h_{22}^{*\alpha} - \left(h_{12}^{*\alpha}\right)^2\right]\right\}. \end{aligned}$$

By using the Gauss equation with respect to the Levi–Civita connection, we find:

$$K(\pi) = K_0(\pi) + \frac{c}{4}\left\{1 + g^2(e_1, Pe_2) + g^2(Pe_1, e_2) - g(e_2, Pe_2)\,g(e_1, Pe_1) - g(Pe_1, e_2)\,g(e_1, Pe_2)\right\} - 2\widetilde{K}_0(\pi)$$
$$- \frac{1}{2}\sum_{\alpha=n+1}^{2m}\left[h_{11}^{\alpha}h_{22}^{\alpha} - (h_{12}^{\alpha})^2\right] - \frac{1}{2}\sum_{\alpha=n+1}^{2m}\left[h_{11}^{*\alpha}h_{22}^{*\alpha} - (h_{12}^{*\alpha})^2\right], \tag{9}$$

where $\widetilde{K}_0$ is the sectional curvature of the Levi–Civita connection $\widetilde{\nabla}^0$ on $\widetilde{M}^{2m}$.

Next, we will calculate τ, the scalar curvature of M, corresponding to the sectional K-curvature. Then, using (5) and (6), we get:

$$\tau = \frac{1}{2}\sum_{1\le i<j\le n}\left[g(R(e_i, e_j)e_j, e_i) + g(R^*(e_i, e_j)e_j, e_i) - 2g\left(R^0(e_i, e_j)e_j, e_i\right)\right]$$
$$= \frac{1}{2}\sum_{1\le i<j\le n}\left[g(R(e_i, e_j)e_j, e_i) - g(R(e_i, e_j)e_i, e_j)\right] - \tau_0, \tag{10}$$

where τ_0 is the scalar curvature of the Levi–Civita connection ∇^0 on M^n.

By the use of (7) and (8), we obtain:

$$\sum_{1\le i<j\le n} g(R(e_i, e_j)e_j, e_i) = \frac{c}{4}\sum_{1\le i<j\le n}\big[\{g(e_j, e_j)\,g(e_i, e_i) - g(e_i, e_j)\,g(e_i, e_j) - g(e_j, Je_j)\,g(Je_i, e_i) + g(e_i, Je_j)\,g(e_i, Je_j) + [g(e_i, Je_j) - g(e_j, Je_i)]\,g(e_i, Je_j)\} + g(h^*(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h^*(e_i, e_j))\big].$$

Then, we have:

$$\sum_{1\le i<j\le n} g(R(e_i, e_j)e_j, e_i) = \frac{c}{8}n(n-1) + \frac{c}{4}\sum_{1\le i<j\le n}\{g(e_i, Pe_j)\,g(Pe_j, e_i) - g(e_j, Pe_j)\,g(Pe_i, e_i) + [g(e_i, Pe_j) - g(Pe_i, e_j)]\,g(e_i, Pe_j)\}$$
$$+ \sum_{1\le i<j\le n}\left[g(h^*(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h^*(e_i, e_j))\right].$$

Similar calculations will give:

$$\sum_{1\le i<j\le n} g(R(e_i, e_j)e_i, e_j) = -\frac{c}{8}n(n-1) + \frac{c}{4}\sum_{1\le i<j\le n}\{g(e_j, Pe_j)\,g(Pe_i, e_i) - g(Pe_i, e_j)\,g(e_j, Pe_i) + [g(e_i, Pe_j) - g(Pe_i, e_j)]\,g(Pe_i, e_j)\}$$
$$+ \sum_{1\le i<j\le n}\left[g(h^*(e_i, e_j), h(e_i, e_j)) - g(h(e_i, e_i), h^*(e_j, e_j))\right].$$

If we insert the last two equalities in (10), we obtain:

$$\tau = \frac{c}{8}n(n-1) + \frac{c}{4}\sum_{1\le i<j\le n}\{g(e_i, Pe_j)\,g(Pe_j, e_i) - g(e_j, Pe_j)\,g(Pe_i, e_i)$$
$$-g(e_i, Pe_j)\,g(Pe_i, e_j) + g(Pe_i, e_j)\,g(Pe_i, e_j)\} - \tau_0$$
$$+\frac{1}{2}\sum_{\alpha=n+1}^{2m}\sum_{1\le i<j\le n}\left[h_{ii}^{*\alpha}h_{jj}^{\alpha} + h_{ii}^{\alpha}h_{jj}^{*\alpha} - 2h_{ij}^{*\alpha}h_{ij}^{\alpha}\right]. \tag{11}$$

By using the the following standard notations:

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j) = \sum_{i,j=1}^{n} g(Pe_i, e_j)\,g(Pe_i, e_j),$$

$$\mathrm{trace}P = \sum_{i=1}^{n} g(Pe_i, e_i),$$

$$\mathrm{trace}P^2 = \sum_{i=1}^{n} g\left(P^2e_i, e_i\right)$$

and formula (4), we find:

$$\sum_{1\le i<j\le n}\{g(e_i, Pe_j)\,g(Pe_j, e_i) - g(e_j, Pe_j)\,g(Pe_i, e_i) - g(e_i, Pe_j)\,g(Pe_i, e_j)$$
$$+g(Pe_i, e_j)\,g(Pe_i, e_j)\} = \|P\|^2 - \frac{(\mathrm{trace}P)^2}{2} + \frac{1}{2}\sum_{i=1}^{n} g(Pe_i, P^*e_i).$$

The equality (11) becomes:

$$\tau = \frac{c}{8}n(n-1) + \frac{c}{4}\left\{\|P\|^2 - \frac{1}{2}(\mathrm{trace}P)^2 - \frac{1}{2}\mathrm{trace}P^2\right\}$$
$$+\frac{1}{2}\sum_{\alpha=n+1}^{2m}\sum_{1\le i<j\le n}\left[h_{ii}^{*\alpha}h_{jj}^{\alpha} + h_{ii}^{\alpha}h_{jj}^{*\alpha} - 2h_{ij}^{*\alpha}h_{ij}^{\alpha}\right] - \tau_0.$$

The above equality can be written as (see also [13]):

$$\tau = \frac{c}{8}n(n-1) + \frac{c}{4}\left\{\|P\|^2 - \frac{1}{2}(\mathrm{trace}P)^2 - \frac{1}{2}\mathrm{trace}P^2\right\}$$
$$+2\sum_{\alpha=n+1}^{2m}\sum_{1\le i<j\le n}\left[h_{ii}^{0\alpha}h_{jj}^{0\alpha} - \left(h_{ij}^{0\alpha}\right)^2\right]$$
$$-\frac{1}{2}\sum_{\alpha=n+1}^{2m}\sum_{1\le i<j\le n}\left[h_{ii}^{\alpha}h_{jj}^{\alpha} - \left(h_{ij}^{\alpha}\right)^2\right] - \frac{1}{2}\sum_{\alpha=n+1}^{2m}\sum_{1\le i<j\le n}\left[h_{ii}^{*\alpha}h_{jj}^{*\alpha} - \left(h_{ij}^{*\alpha}\right)^2\right] - \tau_0.$$

By using the Gauss equation for the Levi–Civita connection, we have:

$$\tau = \tau_0 + \frac{c}{8}n(n-1) + \frac{c}{4}\left\{\|P\|^2 - \frac{1}{2}(\mathrm{trace}P)^2 - \frac{1}{2}\mathrm{trace}P^2\right\} - 2\tilde{\tau}_0$$
$$-\frac{1}{2}\sum_{\alpha=n+1}^{2m}\sum_{1\le i<j\le n}\left[h_{ii}^{\alpha}h_{jj}^{\alpha} - \left(h_{ij}^{\alpha}\right)^2\right] - \frac{1}{2}\sum_{\alpha=n+1}^{2m}\sum_{1\le i<j\le n}\left[h_{ii}^{*\alpha}h_{jj}^{*\alpha} - \left(h_{ij}^{*\alpha}\right)^2\right]. \tag{12}$$

By subtracting (9) from (12), we obtain:

$$(\tau - K(\pi)) - (\tau_0 - K_0(\pi)) = \frac{c}{8}(n-2)(n+1) + \frac{c}{4}\Big\{\|P\|^2 - \frac{1}{2}(\mathrm{trace}P)^2 - \frac{1}{2}\mathrm{trace}P^2$$

$$-g^2(e_1, Pe_2) - g^2(Pe_1, e_2) + g(e_2, Pe_2)\, g(e_1, Pe_1) + g(Pe_1, e_2)\, g(e_1, Pe_2)\Big\}$$

$$-\frac{1}{2}\sum_{\alpha=n+1}^{2m}\sum_{1\le i<j\le n}\left\{\left[h_{ii}^{\alpha}h_{jj}^{\alpha} - \left(h_{ij}^{\alpha}\right)^2\right] + \left[h_{ii}^{*\alpha}h_{jj}^{*\alpha} - \left(h_{ij}^{*\alpha}\right)^2\right]\right\}$$

$$+\frac{1}{2}\sum_{\alpha=n+1}^{2m}\left\{\left[h_{11}^{\alpha}h_{22}^{\alpha} - (h_{12}^{\alpha})^2\right] + \left[h_{11}^{*\alpha}h_{22}^{*\alpha} - (h_{12}^{*\alpha})^2\right]\right\} + 2\widetilde{K}_0(\pi) - 2\widetilde{\tau}_0.$$

Furthermore, let H and H^* denote the mean curvature vectors with respect to the dual connections ∇ and ∇^*, respectively.

We recall the following algebraic lemma from [13], which is essential for the proof of the Chen first inequality.

Lemma 2. *Let $n \geq 3$ be an integer and $a_1, \cdots, a_n$ n real numbers. Then, we have:*

$$\sum_{1\le i<j\le n}^{n} a_i a_j - a_1 a_2 \le \frac{n-2}{2(n-1)}\left(\sum_{i=1}^{n} a_i\right)^2.$$

Furthermore, the equality case of the above inequality holds if and only if $a_1 + a_2 = a_3 = \ldots = a_n$.

Applying Lemma 2 (see also [13]), we have:

$$\sum_{1\le i<j\le n} h_{ii}^{\alpha}h_{jj}^{\alpha} - h_{11}^{\alpha}h_{22}^{\alpha} \le \frac{(n-2)}{2(n-1)}\left(\sum_{i=1}^{n} h_{ii}^{\alpha}\right)^2 = \frac{n^2(n-2)}{2(n-1)}(H^{\alpha})^2,$$

$$\sum_{1\le i<j\le n} h_{ii}^{*\alpha}h_{jj}^{*\alpha} - h_{11}^{*\alpha}h_{22}^{*\alpha} \le \frac{(n-2)}{2(n-1)}\left(\sum_{i=1}^{n} h_{ii}^{*\alpha}\right)^2 = \frac{n^2(n-2)}{2(n-1)}(H^{*\alpha})^2.$$

Using the above inequalities, we continue our calculations, and we get:

$$(\tau - K(\pi)) - (\tau_0 - K_0(\pi)) \ge \frac{c}{8}(n-2)(n+1) + \frac{c}{4}\Big\{\|P\|^2 - \frac{1}{2}(\mathrm{trace}P)^2 - \frac{1}{2}\mathrm{trace}P^2$$

$$-g^2(e_1, Pe_2) - g^2(Pe_1, e_2) + g(e_2, Pe_2)\, g(e_1, Pe_1) + g(Pe_1, e_2)\, g(e_1, Pe_2)\Big\}$$

$$-\frac{n^2(n-2)}{4(n-1)}\left[\|H\|^2 + \|H^*\|^2\right] - 2\left(\widetilde{\tau}_0 - \widetilde{K}_0(\pi)\right),$$

which represents the Chen first inequality for arbitrary statistical submanifolds in a Kähler-like statistical manifold whose curvature tensor $\widetilde{R}$ is of the form of (8).

Recall that a submanifold M of an almost Hermitian manifold $\widetilde{M}$ is called holomorphic (resp. totally real) if each tangent space of M is mapped into itself (resp. the normal space) by the almost complex structure $\widetilde{J}$ of $\widetilde{M}$ (see [19,20]). A totally real submanifold of maximum dimension is a Lagrangian submanifold.

We can now state the following main theorem of this section:

Theorem 1. *Let $\left(\widetilde{M}, g, \widetilde{\nabla}, J\right)$ be a $2m$-dimensional Kähler-like statistical manifold whose curvature tensor $\widetilde{R}$ is of the form (8) and M an n-dimensional statistical submanifold of $\widetilde{M}$.*

(*a*) *If M is holomorphic, then:*

$$(\tau - \tau_0) - (K(\pi) - K_0(\pi)) \geq \frac{c}{8}\left(n^2 + 2n - 2\right) - \frac{c}{4}\left\{\frac{1}{2}(\mathrm{trace}J)^2\right.$$

$$\left. + g^2(e_1, Je_2) + g^2(Je_1, e_2) - g(e_2, Je_2)\, g(e_1, Je_1) - g(Je_1, e_2)\, g(e_1, Je_2)\right\}$$

$$-\frac{n^2(n-2)}{4(n-1)}\left[\|H\|^2 + \|H^*\|^2\right] - 2\left(\widetilde{\tau}_0 - \widetilde{K}_0(\pi)\right).$$

(*b*) *If M is totally real, then:*

$$(\tau - \tau_0) - (K(\pi) - K_0(\pi)) \geq \frac{c}{8}(n-2)(n+1)$$

$$-\frac{n^2(n-2)}{4(n-1)}\left[\|H\|^2 + \|H^*\|^2\right] - 2\left(\widetilde{\tau}_0 - \widetilde{K}_0(\pi)\right).$$

Moreover, one of the equalities holds in the all cases if and only if:

$$\begin{gathered} h_{11}^{\alpha} + h_{22}^{\alpha} = h_{33}^{\alpha} = \cdots = h_{nn}^{\alpha} \\ h_{11}^{*\alpha} + h_{22}^{*\alpha} = h_{33}^{*\alpha} = .\cdots = h_{nn}^{*\alpha} \\ h_{ij}^{\alpha} = h_{ij}^{*\alpha} = 0, \;\; i \neq j, \;\; (i,j) \neq (1,2),(2,1), \end{gathered}$$

for any $\alpha \in \{n+1, ..., 2m\}$.

Remark 1. *(a) If* $\widetilde{\nabla}$ *is the Levi–Civita connection,* $K = 0$*, and consequently,* $\tau = 0$*. Then, we refind Chen first inequality for submanifolds in complex space forms.*

(b) The difference $K(\pi) - K_0(\pi)$ *is the sectional curvature of* π *defined by B. Opozda in [11]. We used the sectional K-curvature K because it is the most known sectional curvature on statistical manifolds (see, for instance, [6]). The sectional curvature* $K - K_0$ *was used only by a few authors.*

Corollary 1. *Let* $\left(\widetilde{M}, g, \widetilde{\nabla}, J\right)$ *be a* $2m$*-dimensional Kähler-like statistical manifold whose curvature tensor* $\widetilde{R}$ *is of the form* (8) *and M an n-dimensional totally real statistical submanifold of* $\widetilde{M}$*. If there exists a point* $p \in M$ *and* $\pi \subset T_pM$ *a plane such that:*

$$\tau - \tau_0 < K(\pi) - K_0(\pi) + (n-2)(n-1)\frac{c}{8} - 2\left[\widetilde{\tau}_0 - \widetilde{K}_0(\pi)\right],$$

then M is non-minimal, i.e., $H \neq 0$ *or* $H^* \neq 0$*.*

We also obtain the following characterization of a Lagrangian submanifold, which satisfies the equality case of the inequality in Theorem 1.

Theorem 2. *Let* $\left(\widetilde{M}, g, \widetilde{\nabla}, J\right)$ *be a* $2n$*-dimensional Kähler-like statistical manifold whose curvature tensor* $\widetilde{R}$ *is of the form* (8) *and M an n-dimensional Lagrangian statistical submanifold of* $\widetilde{M}$*. If* $n \geq 4$ *and M satisfies the equality case of the Chen first inequality, identically, then it is minimal, i.e.,* $H = H^* = 0$*.*

Proof. We will give two alternative (equivalent) proofs of this theorem.

Proof 1. For $X, Y \in \Gamma(TM)$, we have:

$$0 = (\widetilde{\nabla}_X J)Y = \widetilde{\nabla}_X JY - J\widetilde{\nabla}_X Y$$

$$= -A_{JY}X + D_X JY - J\nabla_X Y - Jh(X,Y).$$

It follows that the tangent component vanishes, i.e.,

$$-A_{JY}X - Jh(X,Y) = 0,$$

and then:

$$A_{JY}X = -Jh(X,Y) = -Jh(Y,X) = A_{JX}Y,$$

which implies $h_{ij}^{n+k} = h_{ki}^{n+j} = h_{jk}^{n+i}$, for all $i,j,k \in \{1,...,n\}$.

Applying this in the relations that characterize the equality case of the Chen first inequality, we obtain:

(i) For $\alpha \in \{1,2\}$, $h_{33}^{n+\alpha} = h_{\alpha 3}^{n+3} = 0$, which implies $h_{11}^{n+\alpha} + h_{22}^{n+\alpha} = 0$, and then:

$$h_{11}^{n+\alpha} + h_{22}^{n+\alpha} + \cdots + h_{nn}^{n+\alpha} = 0.$$

(ii) For $\alpha \in \{3,\cdots,n\}$, let $i \in \{3,\cdots,n\}$, $i \neq \alpha$. Then, $h_{ii}^{n\alpha} = h_{i\alpha}^{n+i} = 0$ and:

$$h_{11}^{n+\alpha} + h_{22}^{n+\alpha} + \cdots + h_{nn}^{n+\alpha} = 0.$$

It follows that $H = 0$ and, in a similar way, $H^* = 0$, and then, M is minimal.

Proof 2. M being Lagrangian, we have $P = 0$, and by similar arguments as in the first proof, $A_{FX}Y = A_{FY}X$. We consider the basis $\{Fe_1,\cdots,Fe_n\} \in T^{\perp}M$ (rank$F = n$).

For $i \geq 3$, $h_{ii}^{Fe_j} = g(A_{Fe_j}e_i, e_i) = g(A_{Fe_i}e_j, e_i) = 0$, for $j \neq i$.

From the characterization of the equality case of the Chen first inequality, we have:

$$h_{11}^{\alpha} + h_{22}^{\alpha} = h_{33}^{\alpha} = \cdots = h_{nn}^{\alpha} = 0,$$

for any $\alpha \in \{n+1,...,2n\}$.

Moreover, $h_{12}^{Fe_i} = 0$, for $i \geq 3$.

Therefore, $h_{ij}^{\alpha} = 0$, for $\alpha \geq n+3$ and any $i,j \in \{1,...,n\}$.

We obtain:

$$h_{12}^{Fe_1} = g(A_{Fe_1}e_2, e_1) = g(A_{Fe_2}e_1, e_1) = h_{11}^{n+2} = -h_{22}^{n+2},$$
$$h_{12}^{Fe_2} = g(A_{Fe_2}e_1, e_2) = g(A_{Fe_1}e_2, e_2) = h_{22}^{n+1} = -h_{11}^{n+1}$$

and $h_{22}^{\alpha} = -h_{11}^{\alpha}$, for $\alpha \geq n+3$.

Similar calculations hold for h^* and A^*.

Then, M is minimal.

We remark that from the second proof, we also obtain:

$$h_{ij}^{\alpha} = h_{ij}^{*\alpha} = 0,$$

for $\alpha \geq n+3$ and any $i,j \in \{1,...,n\}$.

□

5. A Chen $\delta(2,2)$ Inequality

We will use the same notations as in the previous sections.

The following algebraic lemma from [14] has the key role in the proof of the main result of this section.

Lemma 3. *Let $n \geq 4$ be an integer and $\{a_1, ..., a_n\}$ n real numbers. Then, we have:*

$$\sum_{1 \leq i < j \leq n} a_i a_j - a_1 a_2 - a_3 a_4 \leq \frac{n-3}{2(n-2)} \left(\sum_{i=1}^{n} a_i \right)^2 .$$

Equality holds if and only if $a_1 + a_2 = a_3 + a_4 = a_5 = \cdots = a_n$.

Let $p \in M$, $\pi_1, \pi_2 \subset T_p M$, mutually orthogonal planes spanned respectively by $\mathrm{sp}\{e_1, e_2\} = \pi_1$, $\mathrm{sp}\{e_3, e_4\} = \pi_2$. Consider $\{e_1, ..., e_n\} \subset T_p M$, $\{e_{n+1}, ..., e_{2m}\} \subset T_p^{\perp} M$ orthonormal bases. Then, by Formula (9), we have:

$$\begin{aligned} K(\pi_1) &= K_0(\pi_1) + \frac{c}{4} \Big\{ 1 + g^2(e_1, Pe_2) + g^2(Pe_1, e_2) \\ &- g(Pe_1, e_2) g(e_1, Pe_2) - g(Pe_1, e_1) g(Pe_2, e_2) \Big\} - 2\widetilde{K}_0(\pi_1) \\ &- \frac{1}{2} \sum_{\alpha=n+1}^{2m} \left[h_{11}^{\alpha} h_{22}^{\alpha} - (h_{12}^{\alpha})^2 \right] - \frac{1}{2} \sum_{\alpha=n+1}^{2m} \left[h_{11}^{*\alpha} h_{22}^{*\alpha} - (h_{12}^{*\alpha})^2 \right] \end{aligned} \tag{13}$$

and:

$$\begin{aligned} K(\pi_2) &= K_0(\pi_2) + \frac{c}{4} \Big\{ 1 + g^2(e_3, Pe_4) + g^2(Pe_3, e_4) \\ &- g(Pe_3, e_4) g(e_3, Pe_4) - g(Pe_3, e_3) g(Pe_4, e_4) \Big\} - 2\widetilde{K}_0(\pi_2) \\ &- \frac{1}{2} \sum_{\alpha=n+1}^{2m} \left[h_{33}^{\alpha} h_{44}^{\alpha} - (h_{34}^{\alpha})^2 \right] - \frac{1}{2} \sum_{\alpha=n+1}^{2m} \left[h_{33}^{*\alpha} h_{44}^{*\alpha} - (h_{34}^{*\alpha})^2 \right] . \end{aligned} \tag{14}$$

From (13), (14), and (12) we have:

$$\begin{aligned} &(\tau - K(\pi_1) - K(\pi_2)) - (\tau_0 - K_0(\pi_1) - K_0(\pi_2)) \geq \\ &\left(n^2 - n - 4 \right) \frac{c}{8} + \frac{c}{4} \Big[\|P\|^2 - \frac{1}{2} (\mathrm{trace} P)^2 - \frac{1}{2} \mathrm{trace} P^2 \\ &- g^2(e_1, Pe_2) - g^2(Pe_1, e_2) + g(Pe_1, e_2) g(e_1, Pe_2) + g(Pe_1, e_1) g(Pe_2, e_2) \\ &- g^2(e_3, Pe_4) - g^2(Pe_3, e_4) + g(Pe_3, e_4) g(e_3, Pe_4) + g(Pe_3, e_3) g(Pe_4, e_4) \Big] \\ &- \frac{1}{2} \sum_{\alpha=n+1}^{2m} \sum_{1 \leq i < j \leq n} \Big\{ \left[h_{ii}^{\alpha} h_{jj}^{\alpha} - h_{11}^{\alpha} h_{22}^{\alpha} - h_{33}^{\alpha} h_{44}^{\alpha} \right] \\ &+ \left[h_{ii}^{*\alpha} h_{jj}^{*\alpha} - h_{11}^{*\alpha} h_{22}^{*\alpha} - h_{33}^{*\alpha} h_{44}^{*\alpha} \right] \Big\} \\ &+ 2\widetilde{K}_0(\pi_1) + 2\widetilde{K}_0(\pi_2) - 2\widetilde{\tau}_0. \end{aligned}$$

Lemma 3 implies:

$$\begin{aligned} &\sum_{1 \leq i < j \leq n} \left[h_{ii}^{\alpha} h_{jj}^{\alpha} - h_{11}^{\alpha} h_{22}^{\alpha} - h_{33}^{\alpha} h_{44}^{\alpha} \right] \\ &\leq \frac{n-3}{2(n-2)} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} \right)^2 = \frac{n^2(n-3)}{2(n-2)} (H^{\alpha})^2 \end{aligned}$$

and similarly for h^*.

Summing, we get:

$$\sum_{\alpha=n+1}^{2m} \sum_{1 \leq i < j \leq n} \left[h_{ii}^{\alpha} h_{jj}^{\alpha} - h_{11}^{\alpha} h_{22}^{\alpha} - h_{33}^{\alpha} h_{44}^{\alpha} \right] \leq \frac{n^2(n-3)}{2(n-2)} \|H\|^2$$

and similarly for H^*.

We obtain the following inequality:

$$\begin{aligned}
&(\tau - K(\pi_1) - K(\pi_2)) - (\tau_0 - K_0(\pi_1) - K_0(\pi_2)) \\
&\geq \left(n^2 - n - 4\right)\frac{c}{8} - \frac{n^2(n-3)}{4(n-2)}\left[\|H\|^2 + \|H^*\|^2\right] \\
&+\frac{c}{4}\left[\|P\|^2 - \frac{1}{2}(\mathrm{trace}P)^2 - \frac{1}{2}\mathrm{trace}P^2\right. \\
&-g^2(e_1, Pe_2) - g^2(Pe_1, e_2) + g(Pe_1, e_2)g(e_1, Pe_2) + g(Pe_1, e_1)g(Pe_2, e_2) \\
&\left.-g^2(e_3, Pe_4) - g^2(Pe_3, e_4) + g(Pe_3, e_4)g(e_3, Pe_4) + g(Pe_3, e_3)g(Pe_4, e_4)\right] \\
&-2\left[\widetilde{\tau}_0 - \widetilde{K}_0(\pi_1) - \widetilde{K}_0(\pi_2)\right],
\end{aligned}$$

which represents the Chen $\delta(2,2)$ inequality for an arbitrary statistical submanifold in a Kähler-like statistical manifold.

We can state now the following theorem:

Theorem 3. *Let $\left(\widetilde{M}, g, \widetilde{\nabla}, J\right)$ be a $2m$-dimensional Kähler-like statistical manifold whose curvature tensor $\widetilde{R}$ is of the form (8) and M an n-dimensional statistical submanifold of $\widetilde{M}$.*

(a) If M is holomorphic, then:

$$\begin{aligned}
&(\tau - \tau_0) - (K(\pi_1) - K_0(\pi_1) + K(\pi_2) - K_0(\pi_2)) \\
&\geq \left(n^2 + 2n - 4\right)\frac{c}{8} - \frac{n^2(n-3)}{4(n-2)}[\|H\|^2 + \|H^*\|^2] \\
&-\frac{c}{4}[\frac{1}{2}(\mathrm{trace}J)^2 + g^2(e_1, Je_2) + g^2(Je_1, e_2) \\
&-g(Je_1, e_2)g(e_1, Je_2) - g(Je_1, e_1)g(Je_2, e_2) \\
&+g^2(e_3, Je_4) + g^2(Je_3, e_4) - g(Je_3, e_4)g(e_3, Je_4) - g(Je_3, e_3)g(Je_4, e_4)] \\
&-2\left[\widetilde{\tau}_0 - \widetilde{K}_0(\pi_1) - \widetilde{K}_0(\pi_2)\right].
\end{aligned}$$

(b) If M is totally real, then:

$$\begin{aligned}
&(\tau - \tau_0) - (K(\pi_1) - K_0(\pi_1) + K(\pi_2) - K_0(\pi_2)) \\
&\geq \left(n^2 - n - 4\right)\frac{c}{8} - \frac{n^2(n-3)}{4(n-2)}\left[\|H\|^2 + \|H^*\|^2\right] \\
&-2\left[\widetilde{\tau}_0 - \widetilde{K}_0(\pi_1) - \widetilde{K}_0(\pi_2)\right].
\end{aligned}$$

Moreover, one of the equalities holds if and only if:

$$h^\alpha_{11} + h^\alpha_{22} = h^\alpha_{33} + h^\alpha_{44} = h^\alpha_{55} = ... = h^\alpha_{nn},$$

$$h^{*\alpha}_{11} + h^{*\alpha}_{22} = h^{*\alpha}_{33} + h^{*\alpha}_{44} = h^{*\alpha}_{55} = ... = h^{*\alpha}_{nn},$$

$$h^\alpha_{ij} = h^{*\alpha}_{ij} = 0, \;\; i \neq j, \;\; (i,j) \neq (1,2), (2,1), (3,4), (4,3),$$

where $\alpha \in \{n+1, ..., 2m\}$, $1 \leq i < j \leq n$.

Corollary 2. *Let* $\left(\widetilde{M}, g, \widetilde{\nabla}, J\right)$ *be a* $2m$*-dimensional Kähler-like statistical manifold whose curvature tensor* $\widetilde{R}$ *is of the form* (8) *and* M *an* n*-dimensional totally real statistical submanifold of* $\widetilde{M}$*. If there exists a point* $p \in M$, $\pi_1, \pi_2 \subset T_pM$ *mutually orthogonal planes such that:*

$$\tau - \tau_0 < K(\pi_1) - K_0(\pi_1) + K(\pi_2) - K_0(\pi_2)$$
$$+ \left(n^2 - n - 4\right) \frac{c}{8} - 2\left[\widetilde{\tau}_0 - \widetilde{K}_0(\pi_1) - \widetilde{K}_0(\pi_2)\right],$$

then M *is non-minimal, i.e.,* $H \neq 0$ *or* $H^* \neq 0$.

The following theorem represents a characterization of a Lagrangian submanifold that satisfies the equality case of the inequality in Theorem 3.

Theorem 4. *Let* $\left(\widetilde{M}, g, \widetilde{\nabla}, J\right)$ *be a* $2n$*-dimensional Kähler-like statistical manifold whose curvature tensor* $\widetilde{R}$ *is of the form* (8) *and* M *an* n*-dimensional Lagrangian statistical submanifold of* $\widetilde{M}$*.*

If $n \geq 6$ *and* M *satisfies the equality case of the Chen* $\delta(2,2)$ *inequality, identically, then it is minimal.*

The proof follows the same idea as in the proof of Theorem 2.

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