MODULAR REPRESENTATION THEORY OF BIB DESIGNS

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Abstract. Our aim is to study the modular representation theory of coherent configurations. Let p be a prime. We consider structures of modular adjacency algebras of coherent configurations obtained from combinatorial designs. The structures of standard modules of modular adjacency algebras provide more information than p-ranks of incidence matrices of combinatorial designs.

1. Introduction

In this paper, we shall provide an interpretation of the p-ranks of the incidence matrices of symmetric balanced incomplete block (BIB) designs and quasi-symmetric Steiner BIB designs with modular representation theory of coherent configurations.

Some researchers have studied the p-ranks of incidence matrices of combinatorial designs [6, 10, 20]. The p-ranks of incidence matrices of combinatorial designs help us to classify combinatorial designs with the same parameters.

On the other hand, we can construct coherent configurations from some combinatorial designs. Each coherent configuration is accompanied by an algebra. It is called an adjacency algebra. Consequently, we can consider the structures of adjacency algebras of coherent configurations obtained from combinatorial designs. An adjacency algebra of a coherent configuration over a field of characteristic zero is always semisimple. This case was studied by Higman [14, 15] and some researchers studied [7, 8, 16, 21, 24]. The semisimplicity of adjacency algebras of coherent configurations over positive characteristic fields was studied [23]. An adjacency algebra of a coherent configuration over a field of positive characteristic is called a modular adjacency algebra. They are not always semisimple. They have not been sufficiently studied. The first author and Yoshikawa have considered the structures of

2010 *Mathematics Subject Classification.* Primary 05E30, Secondary 05B05.

Key words and phrases. coherent configuration; quasi-symmetric design; symmetric design ;strongly regular graph; modular adjacency algebra; modular standard module.

the standard modules of the modular adjacency algebras of a special coherent configuration, and showed that structures of standard modules of modular adjacency algebras provided more information than p-ranks [12].

In this paper, we consider symmetric BIB designs and quasi-symmetric Steiner BIB designs. We determine the structure of modular adjacency algebras of coherent configurations obtained from these BIB designs over a field of characteristic 2, because 2-ranks of incidence matrices of combinatorial designs play an important role in their classification [6]. We consider the structures of their standard modules. We show that the structures of standard modules determine the 2-ranks of incidence matrices of the designs.

We organize this paper in the following way. In Section 2, we provide definitions of coherent configurations and BIB designs. We discuss a method to determine irreducible representations of coherent configurations in Section 3 and 4. These are applied in Section 5 and 6 to determine the structure of the standard modules of the modular adjacency algebra of coherent configurations obtained from symmetric BIB designs and quasi-symmetric Steiner BIB designs, respectively.

2. Coherent configurations and BIB designs

Let X be a finite nonempty set, S a set of nonempty binary relations on X so that $X \times X = \bigcup_{s \in S} s$ is a disjoint union of $X \times X$. The pair (X, S) is called a *coherent configuration* if the following three axioms hold.

(C1) There is a subset $\{\Delta_1, \dots, \Delta_q\}$ of S such that $\bigcup_{i=1}^{q} \Delta_i = \{(x,\ x) \mid x \in X\}$,
(C2) if $s \in S$, then $s^* = \{(y, x) \mid (x, y) \in s\} \in S$,
(C3) for $s, t, u \in S$ and $(x,\ y) \in u$, a non-negative integer $p_{s,t}^{u} = \sharp\{z \in Z \mid (x,\ z) \in s,\ (z,\ y) \in t\}$ is independent of the choice of x and y.

We put $X_i = \{x \in X \mid (x, x) \in \Delta_i\}$ $(i = 1, \dots, q)$ and call X_i a *fiber*. A coherent configuration (X, S) is said to be *homogeneous* if $q = 1$. A homogeneous coherent configuration is also called an *association scheme* in a sense of [2] and [25]. Let (X, S) be a coherent configuration with fibers $X_1, \dots, X_q$. We denote by $\mathrm{Mat}_{\mathrm{X}}(\mathbb{Z})$ the ring of matrices over $\mathbb{Z}$ whose rows and columns are indexed by X. For $s \in S$, we denote by σ_s the *adjacency matrix* of s, namely

$$(\sigma_s)_{x,y} = \begin{cases} 1 & (x, y) \in s, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of the coherent configuration, $\mathbb{Z}S = \oplus_{s\in S}\mathbb{Z}\sigma_s$ is a subalgebra of $\mathrm{Mat}_{\mathrm{X}}(\mathbb{Z})$ under the usual matrix multiplication. For a commutative ring R with unity, we can define $RS = R \otimes_{\mathbb{Z}} \mathbb{Z}S$ and call this R-algebra the *adjacency algebra* of (X, S) over R. We often use the notation σ_s for the corresponding element in RS. For $s \in S$, there is a unique pair (i, j) such that $\sigma_{\Delta_i}\sigma_s\sigma_{\Delta_j} = \sigma_s$. Subsets $S^{ij} = \{s \in S \mid \sigma_{\Delta_i}\sigma_s\sigma_{\Delta_j} = \sigma_s\}$ of S give a partition of S like $S = \bigcup_i \bigcup_j S^{ij}$. The sub-configuration (X_i, S^{ii}) is homogeneous and $RS^{ii} = \oplus_{s\in S^{ii}} R\sigma_s$ is a subalgebra of RS (with non-common identity).

We consider incidence structures $(X_1, X_2, \mathcal{F})$ consisting of disjoint sets X_1 and X_2, whose elements are called *points* and *blocks* respectively, and a subset $\mathcal{F}$ of the Cartesian product $X_1 \times X_2$, whose elements are called *flags*. A point ω and a block B are *incident* if (ω, B) is a flag. The incidence matrix N of the structure will have rows indexed by the points and columns by the blocks, namely

$$(N)_{\omega,B} = \begin{cases} 1 & (\omega, B) \in \mathcal{F}(\subset X_1 \times X_2), \\ 0 & \text{otherwise.} \end{cases}$$

A *balanced incomplete block design* (BIB design) [4, 5] with parameters v, b, r, ℓ, λ is an arrangement of v points X_1 into b blocks X_2 such that:

(D1) each block is incident with ℓ points (we assume that with $\ell < v$),
(D2) each point is incident with r blocks, and
(D3) two points are incident with λ blocks.

A BIB deign with $\lambda = 1$ is said to be *Steiner system.*

Among parameters v, b, r, ℓ, λ, there are the following relations:

$$vr = bk,\ \ \lambda(v-1) = r(k-1) \text{ and } b \geq v.$$

A BIB design is said to be *symmetric* if

(D4) any two blocks are incident with a constant number μ of points

or it is said to be *quasi-symmetric* if

(D4$'$) the number of points in the intersection of two blocks takes two values μ and ν.

Associated with a symmetric BIB design $(X_1, X_2, \mathcal{F})$ is the configuration $\mathcal{C}_s = (X, \{s_i\}_{i\in I})$ over $I = \{1, \ldots, 8\}$ defined by $X = X_1 \cup X_2$ $(X_1 \cap$

$X_2 = \emptyset$) and

$$s_1 = \{(x,x) \mid x \in X_1\},\ s_2 = \{(x,x) \mid x \in X_2\},\ s_3 = {X_1}^2 - s_1,$$
$$s_4 = {X_2}^2 - s_2,\ s_5 = \mathcal{F},\ s_6 = X_1 \times X_2 - \mathcal{F},$$
$$s_7 = {s_5}^* = \{(y,x) \mid (x,y) \in s_5\},$$
$$s_8 = {s_6}^* = \{(y,x) \mid (x,y) \in s_6\} \text{ (See [16])} .$$

Similarly, associated with a quasi-symmetric BIB design $(X_1, X_2, \mathcal{F})$ is the configuration $\mathcal{C}_{qs} = (X, \{s_i\}_{i\in I})$ over $I = \{1, \dots, 9\}$ defined by $X = X_1 \cup X_2$ $(X_1 \cap X_2 = \emptyset)$ and

$$s_1 = \{(x,x) \mid x \in X_1\},\ s_2 = \{(x,x) \mid x \in X_2\},\ s_3 = {X_1}^2 - s_1,$$
$$s_4 = \{(x,y) \in X_2^2 \mid \sharp(x \cap y) = \mu\},\ s_5 = \{(x,y) \in X_2^2 \mid \sharp(x \cap y) = \nu\},$$
$$s_6 = \mathcal{F},\ s_7 = X_1 \times X_2 - \mathcal{F},$$
$$s_8 = {s_6}^* = \{(y,x) \mid (x,y) \in s_5\},\ s_9 = {s_7}^* = \{(y,x) \mid (x,y) \in s_7\}.$$

We will provide tables of multiplications of algebras obtained by these configurations. Consequently, we can prove that $\mathcal{C}_s$, $\mathcal{C}_{qs}$ are coherent configurations of type $(2,2;2)$, $(2,2;3)$, respectively. On the other hand, coherent configurations of type $(2,2;2)$ and $(2,2;3)$ are equivalent to complementary pairs of symmetric and quasi-symmetric designs, respectively. The types of coherent configurations were considered in [15] for the first time. The *block graph* Γ of BIB design is the graph with the blocks as vertices, to being adjacent if and only if they are incident with μ common points. Since $\mathcal{C}_s$ and $\mathcal{C}_{qs}$ are coherent, the block graph of symmetric BIB design is the complete graph with v vertices and the block graph of quasi-symmetric BIB design is a strongly regular graph.

In this paper, we consider coherent configurations obtained from symmetric BIB designs and quasi-symmetric Steiner BIB designs with $(\mu, \nu) = (1, 0)$.

3. Irreducible representations of coherent configurations

Higman gave a method to compute irreducible ordinary characters of a coherent configuration by characters of its fibers [14]. We generalize them to modular representations. We consider more general situation. In this article, modules are finitely generated right modules.

Let F be an algebraically closed field, and let A be a finite dimensional F-algebra with unity. We denote by $\mathrm{pi}(A)$ the set of primitive idempotents of A. For $e, f \in \mathrm{pi}(A)$, we define $e \sim_A f$ if $eA \cong fA$ as right A-modules. Then this is an equivalence relation on $\mathrm{pi}(A)$. We denote by $[e]_A$ the equivalence class containing e and by $\widetilde{\mathrm{pi}}(A)$ the set of equivalence classes. It is well-known that there is a bijection between $\widetilde{\mathrm{pi}}(A)$ and the set of representatives of isomorphism classes of simple right A-modules $\mathrm{IRR}(A)$ by $[e]_A \mapsto eA/e\mathrm{Rad}(A)$ where $\mathrm{Rad}(A)$ is the Jacobson radical of A.

Suppose that $1_A = \sum_{i=1}^{q} \varepsilon_i$ is an orthogonal idempotent decomposition, namely $\varepsilon_i^2 = \varepsilon$ and $\varepsilon_i\varepsilon_j = 0$ $(i \neq j)$. We put $A^{ij} = \varepsilon_i A \varepsilon_j$. Thereby $A = \bigoplus_{i,j} A^{ij}$, $A^{ij}A^{jk} \subset A^{ik}$, and $A^{ij}A^{k\ell} = 0$ if $j \neq k$. In particular, A^{ii} is a subalgebra of A with non-common identity.

An idempotent e of A^{ii} is primitive in A^{ii} if and only if so is in A. For $e, f \in \mathrm{pi}(A^{ii})$, $e \sim_{A^{ii}} f$ if and only if $e \sim_A f$ (see [19, 22]). This means that we can define a map $\Phi_i : \widetilde{\mathrm{pi}}(A^{ii}) \to \widetilde{\mathrm{pi}}(A)$ by $[e]_{A^{ii}} \mapsto [e]_A$ and this map is injective. We also define $\Phi : \bigcup_{i=1}^{q} \widetilde{\mathrm{pi}}(A^{ii}) \to \widetilde{\mathrm{pi}}(A)$ by $\Phi([e]_{A^{ii}}) = \Phi_i([e]_{A^{ii}})$. Then Φ is surjective.

We define a subalgebra E of A by $E = \bigoplus_{i=1}^{q} A^{ii}$ and consider the restriction of a simple A-module to E. In the subalgebra E, ε_i is a central idempotent. For each A-module V, the restriction $V{\downarrow_E} = \bigoplus_{i=1}^{q} V\varepsilon_i$ is a direct sum of A^{ii}-modules. Let $V = eA/e\mathrm{Rad}(A)$ be a simple right A-module. Since $\mathrm{Rad}(\varepsilon_i A \varepsilon_i) = \varepsilon_i \mathrm{Rad}(A)\varepsilon_i$, $V{\downarrow_E}$ is semisimple. We suppose that $[e]_A \in \Phi(\widetilde{\mathrm{pi}}(A^{ii}))$. There exists $f \in \mathrm{pi}(A^{ii})$ such that $[f]_A = [e]_A$. Then $V\varepsilon_i$ contains $fA^{ii}/f\mathrm{Rad}(A^{ii}) = fE/f\mathrm{Rad}(E)$ as a direct summand. Let $\varepsilon_i = \sum_{j=1}^{q_i} e_{ij}$ be a primitive decomposition of ε_i in A^{ii}. Consequently, we get a primitive decomposition of $1_A = \sum_{i=1}^{q}\sum_{j=1}^{q_i} e_{ij}$ in A and B. We claim that the number of idempotents equivalent to e is the dimension of the corresponding simple module. Therefore, we can see that $V{\downarrow_E} \cong \bigoplus_{[f]\in\Phi^{-1}([e])} fE/f\mathrm{Rad}(E)$.

Proposition 3.1. *In the above situation, there is a surjection*

$$\Phi' : \bigcup_{i=1}^{q} \mathrm{IRR}(A^{ii}) \to \mathrm{IRR}(A)$$

such that the restriction $\Phi'_{\mathrm{IRR}(A^{ii})}$ is injective. Moreover, the restriction is

$$V{\downarrow_E} \cong \bigoplus_{W\in\Phi'^{-1}(V)} W$$

for $V \in \mathrm{IRR}(A)$.

We claim that $\Phi([e]) = \Phi([f])$ if and only if eAf does not contained in $\mathrm{Rad}(A)$ for $[e], [f] \in \bigcup_{i=1}^{q} \widetilde{\mathrm{pi}}(A^{ii})$.

Now we consider a coherent configuration (X, S) with fibers X_i $(i = 1, \dots, q)$. We have a partition $S = \bigcup_{i=1}^{q}\bigcup_{j=1}^{q} S^{ij}$ of the set of relations S. We denote by FS the adjacency algebra of (X, S) over F and by

FS^{ij} the subspace of FS spanned by adjacency matrices of relations in S^{ij}. Let ε_i be the identity element of FS^{ii}. Then $1_{FS} = \sum_{i=1}^{q} \varepsilon_i$ is an orthogonal decomposition of 1_{FS}. We can apply the arguments above. By Proposition 3.1, the restriction of a simple FS-module to the subalgebra $\bigoplus_{i=1}^{q} FS^{ii}$ is a sum of simple FS^{ii}-modules. In [14], Higman provided the same arguments for ordinary representations. Let V be a simple FS-module and $P(V)$ be the projective cover of V. We define the *multiplicity* m_V of V by $m_V = \dim_F \mathrm{Hom}_{FS}(P(V), FX)$, where FX is the *standard module* of (X, S). The multiplicity m_V is the number of V as simple constituents of FX. We can see $FX{\downarrow_B} = \bigoplus_{i=1}^{q} FX_i$, where $B = \bigoplus_{i=1}^{q} FS^{ii}$ and FX_i is the standard module of the sub-configuration (X_i, S^{ii}). We have the following proposition.

Proposition 3.2. *The map in Proposition 3.1 preserves the multiplicities.*

4. Decomposition and Cartan matrices

We introduce decomposition and Cartan matrices of algebras as in representation theory of finite groups. We only summarize the arguments. For more details, see [22]. Let p be a prime number and (K, R, F) a *p-modular system*. Namely, R is a complete discrete valuation ring with the maximal ideal πR, K is the quotient field of R of characteristic 0, and F is the residue field $R/\pi R$ of characteristic p. We assume that K and F are large enough (splitting fields for algebras considered there).

Let (X, S) be a coherent configuration, and let M be a KS-module. Then there exists an R-form $\widetilde{M}$ of M, that is an R-free RS-module such that $K \otimes_R \widetilde{M} \cong M$ [22, II, Section 1]. Then $M^* = \widetilde{M}/\pi\widetilde{M}$ is an FS-module. Remark that M^* is not uniquely determined but composition factors are determined. For FS-modules V and W, we write $V \leftrightarrow W$ if their composition factors are the same.

Let M be a simple KS-module. We consider the composition factors of M^* and write $M^* \leftrightarrow \bigoplus_{V \in \mathrm{IRR}(FS)} \delta_{M,V} V$. The number $\delta_{M,V}$ is called the *decomposition number*. We define a matrix $D = (\delta_{M,V})_{\mathrm{IRR}(KS) \times \mathrm{IRR}(FS)}$ and call this the *decomposition matrix*.

For a simple FS-module V, There is a primitive idempotent e_V of FS such that $e_V FS \cong P(V)$ and $e_V FS/e_V \mathrm{Rad}(FS) \cong V$. For V, $W \in \mathrm{IRR}(FS)$, we define the *Cartan invariant* $\xi_{V,W}$ by $P(V) \leftrightarrow \bigoplus_{W \in \mathrm{IRR}(FS)} \xi_{V,W} W$. Note that $\xi_{V,W} = \dim_F \mathrm{Hom}_{FS}(e_V FS, e_W FS) = \dim_F e_W FS e_V$. We define a matrix $C = (\xi_{V,W})_{\mathrm{IRR}(FS) \times \mathrm{IRR}(FS)}$ and call this the *Cartan matrix*.

The next theorem is a generalization of [22, III, Theorem 6.8] and the proof is exactly the same as the proof.

Theorem 4.1. *For the decomposition and Cartan matrices of adjacency algebra of (X, S), we have $C = {}^tDD$. In particular, the Cartan matrix is symmetric.*

5. Symmetric BIB designs

In the rest of this paper, we assume that (K, R, F) is a 2-modular system. We see that prime fields are splitting field of our algebra. For this reason, we can choose an arbitrary field of characteristic 2 for 2-modular system. We construct a coherent configuration of type $(2, 2; 2)$ by a symmetric BIB design.

In this section, we assume that $(X_1, X_2, \mathcal{F})$ be a symmetric BIB design with parameters v, b, r, ℓ, λ and consider the coherent configuration $\mathcal{C}_s = (X, \{s_i\}_{i\in I})$ over $I = \{1, \ldots, 8\}$ obtained from $(X_1, X_2, \mathcal{F})$.

Proposition 5.1. [15, 9.1] *$\mathcal{C}_s$ is coherent.*

Proof. We now write out explicitly tables of multiplications of adjacency matrices except zeros. Due to the condition (D4), we can assume that

$$v = b,\ \ r = \ell.$$

	σ_1	σ_3	σ_5	σ_6
σ_1	σ_1	σ_3	σ_5	σ_6
σ_3	σ_3	$(v-1)\sigma_1 + (v-2)\sigma_3$	$(\ell-1)\sigma_5 + \ell\sigma_6$	$(v-\ell)\sigma_5 + (v-\ell-1)\sigma_6$
σ_7	σ_7	$(\ell-1)\sigma_7 + \ell\sigma_8$	$\ell\sigma_2 + \lambda\sigma_4$	$(\ell-\lambda)\sigma_4$
σ_8	σ_8	$(v-\ell)\sigma_7 + (v-\ell-1)\sigma_8$	$(\ell-\lambda)\sigma_4$	$(v-\ell)\sigma_2 + (v-2\ell+\lambda)\sigma_4$

Table 1

	σ_2	σ_4	σ_7	σ_8
σ_2	σ_2	σ_4	σ_7	σ_8
σ_4	σ_4	$(v-1)\sigma_2 + (v-2)\sigma_4$	$(\ell-1)\sigma_7 + \ell\sigma_8$	$(v-\ell)\sigma_7 + (v-\ell-1)\sigma_8$
σ_5	σ_5	$(\ell-1)\sigma_5 + \ell\sigma_6$	$\ell\sigma_1 + \lambda\sigma_3$	$(\ell-\lambda)\sigma_3$
σ_6	σ_6	$(v-\ell)\sigma_5 + (v-\ell-1)\sigma_6$	$(\ell-\lambda)\sigma_3$	$(v-\ell)\sigma_1 + (v-2\ell+\lambda)\sigma_3$

Table 2

These tables show that the configuration $\mathcal{C}_s$ is a coherent configuration of type $(2, 2; 2)$. $\square$

5.1. **Types of adjacency algebras of $\mathcal{C}_s$.** The coefficients of linear combinations in Table 1 and 2 are polynomials of $v(=b), \ell(=r)$ and λ. If their parameters are equal in modulo 2 for some designs, algebras corresponding these designs are isomorphic over a field of characteristic 2. We give a list of possible parameters in characteristic 2.

Lemma 5.2. *There are six types of parameters in characteristic* 2*:*

Type	$v(=b)$	$\ell(=r)$	λ	*example* (v, ℓ, λ)
I	0	0	0	$(16, 6, 2)$
II	1	0	0	$(41, 16, 6)$
III	1	0	1	$(45, 12, 3)$
IV	0	1	0	$(36, 15, 6)$
V	1	1	0	$(19, 9, 4)$
VI	1	1	1	$(15, 7, 3)$

Proof. Since symmetric BIB designs with parameters v, ℓ, λ hold

$$\lambda(v-1) = \ell(\ell-1),$$

the symmetric BIB design with parameters $(v, \ell, \lambda) \equiv (0, 0, 1)$ or $(0, 1, 1)$ (mod 2) is non-existent. We can find other designs in [5]. □

5.2. **Character tables in characteristic zero.** We consider the character table of the coherent configuration. Since (X_1, S^{11}) and (X_2, S^{22}) are complete graphs, their character tables are

	σ_1	σ_3	multiplicity
	1	$v-1$	1
	1	-1	$v-1$

,

	σ_2	σ_4	multiplicity
	1	$v-1$	1
	1	-1	$v-1$

.

Hence the character table of the coherent configuration corresponding a symmetric BIB design with parameters v, ℓ, λ is

	σ_1	σ_3	σ_2	σ_4	multiplicity
	1	$v-1$	1	$v-1$	1
	1	-1	1	-1	$v-1$

.

Note that character values of σ_i $(i = 5, 6, 7, 8)$ are zeros and we omit them.

Before we determine the structures of adjacency algebras and standard modules in characteristic 2 for six types of coherent configurations obtained from symmetric BIB designs with parameters v, ℓ, λ, we give the Frame number of symmetric BIB design. If two does not divide this number, the algebra obtained from this design is semisimple. Hence the structures of modular adjacency algebras and modules are same as the algebra over a field of characteristic zero.

Proposition 5.3. [23] *Let $\mathcal{C}_s$ be a coherent configuration corresponding a symmetric BIB design with parameters v, ℓ, λ. Then the Frame number of $\mathcal{C}_s$ is*

$$Fr(\mathcal{C}_s) = \frac{v^6(v-\ell)^2\ell^2}{(v-1)^2}.$$

5.3. **Type** I : $(v, \ell, \lambda) \equiv (0,0,0) \pmod 2$**.** We suppose $(v, \ell, \lambda) \equiv (0,0,0) \pmod 2$. An example of parameters of this type is a BIB design with parameters $(v, \ell, \lambda) = (16, 6, 2)$. Since $Fr(\mathcal{C}_s) = 2^{28}$, FS is not semisimple. By computation, we have

- $\dim_F \mathrm{Rad}(FS) = 6$ with the basis $\sigma_1 + \sigma_3, \sigma_2 + \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8$,
- $\dim_F \mathrm{Rad}^2(FS) = 0$.

We determine the 2-modular character table. The modular character table of (X_1, S^{11}) and (X_2, S^{22}) are

	σ_1	σ_3	multiplicity
	1	-1	v

,

	σ_2	σ_4	multiplicity
	1	-1	v

.

Since FS is not local but basic, the character table of the coherent configuration is

	σ_1	σ_3	σ_2	σ_4	multiplicity
U	1	-1	0	0	v
V	0	0	1	-1	v

.

We put the simple modules U and V. The decomposition and Cartan matrices are

$$D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = {}^tDD = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

We can choose primitive idempotents $e_U = \sigma_1 + \sigma_6 + \sigma_7$ and $e_V = \sigma_2 + \sigma_6 + \sigma_7$. Put $\alpha_1 = \sigma_1 + \sigma_3$, $\alpha_2 = \sigma_5$, $\alpha_3 = \sigma_6$, $\alpha_4 = \sigma_7$, $\alpha_5 = \sigma_8$ and $\alpha_6 = \sigma_2 + \sigma_4$. We have the following theorem.

Theorem 5.4. *The adjacency algebra of Type I is isomorphic to*

$$FQ/(\{\alpha_i\alpha_j \mid 1 \le i, j \le 6\}),$$

where FQ is a path algebra, Q is the following quiver.

$$Q : \begin{tikzcd} \circ \arrow[loop left, "\alpha_1"] \arrow[r, bend left=40, "\alpha_3"] \arrow[r, bend left=15, "\alpha_2"] & \circ \arrow[loop right, "\alpha_6"] \arrow[l, bend left=15, "\alpha_5"] \arrow[l, bend left=40, "\alpha_4"] \end{tikzcd}$$

There are two simple modules U and V with $\dim_F U = dim_F V = 1$ and the Loewy structure of the projective covers are as follows:

$$P(U) = \begin{bmatrix} & U & \\ V & V & U \end{bmatrix}, P(V) = \begin{bmatrix} & V & \\ U & U & V \end{bmatrix}.$$

It is difficult to determine the structure of the standard module and we could not do it.

5.4. **Type** *II* : $(v, \ell, \lambda) \equiv (1, 0, 0) \pmod 2$**.** We suppose $(v, \ell, \lambda) \equiv (1, 0, 0) \pmod 2$. An example of parameters of this type is a symmetric BIB design with parameters $(v, \ell, \lambda) = (41, 16, 6)$. Since $Fr(\mathcal{C}_s) = 2^4 \cdot 3^2 \cdot 41^6$, FS is not semisimple. By computation, we have

- $\dim_F \mathrm{Rad}(FS) = 2$ with the basis σ_5, σ_7.
- $\dim_F \mathrm{Rad}^2(FS) = 0$.

We determine the 2-modular character table. The modular character table of (X_1, S^{11}) and (X_2, S^{22}) are

	σ_1	σ_3	multiplicity
	1	0	1
	1	1	$v-1$

,

	σ_2	σ_4	multiplicity
	1	0	1
	1	1	$v-1$

.

The character table of the coherent configuration is

	σ_1	σ_3	σ_2	σ_4	multiplicity
U	1	0	1	0	1
V	1	1	0	0	$v-1$
W	0	0	1	1	$v-1$

.

We choose the simple modules U, V and W. The decomposition and Cartan matrices are

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \ C = {}^tDD = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We can choose primitive idempotents $e_U = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$, $e_V = \sigma_3$ and $e_W = \sigma_4$. Let us put $\alpha = \sigma_5$ and $\beta = \sigma_7$. Then we have the following theorem.

Theorem 5.5. *The adjacency algebra of Type II is isomorphic to*

$$M_2(F) \oplus FQ/(\alpha\beta, \beta\alpha)$$

where FQ is a path algebra, Q is the following quiver.

$$Q : \circ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \circ$$

There are three simple modules U*,* V *and* W *with* $\dim_F U = 2$*,* $\dim_F V = \dim_F W = 1$ *and the Loewy structure of the projective covers are as follows:*

$$P(U) = [U],\ P(V) = \begin{bmatrix} V \\ W \end{bmatrix},\ P(W) = \begin{bmatrix} W \\ V \end{bmatrix}.$$

The algebra is of finite representation type (see [1, 3]). The structure of the standard module is completely determined. We can write

$$FX \cong [U] \oplus g_1[V] \oplus g_2 \begin{bmatrix} V \\ W \end{bmatrix} \oplus h_1[W] \oplus h_2 \begin{bmatrix} W \\ V \end{bmatrix}$$

for some non-negative g_1, g_2, h_1 and h_2.

By $m_V = m_W = v - 1$, we have

$$g_1 + g_2 + h_2 = v - 1, \tag{1}$$

$$g_2 + h_1 + h_2 = v - 1, \tag{2}$$

$g_2 = rank(\alpha) = rank(\sigma_5)$, $h_2 = rank(\beta) = rank(\sigma_7)$. Since $\sigma_5^* = \sigma_7$, $g_2 = h_2$. We put $w = rank(\sigma_5)$,

$$FX \cong [U] \oplus (v - 2w - 1)[V] \oplus w \begin{bmatrix} V \\ W \end{bmatrix} \oplus (v - 2w - 1)[W] \oplus w \begin{bmatrix} W \\ V \end{bmatrix}$$

5.5. **Type** *III* : $(v, \ell, \lambda) \equiv (1, 0, 1) \pmod 2$**.** We suppose $(v, \ell, \lambda) \equiv (1, 0, 1) \pmod 2$. Each algebra of this type is always semisimple because the numerator of $Fr(\mathcal{C}_s) = v^6(v-\ell)^2\lambda^2/(\ell-1)^2$ is odd.

5.6. **Type** *IV* : $(v, \ell, \lambda) \equiv (0, 1, 0) \pmod 2$**.** We suppose $(v, \ell, \lambda) \equiv (0, 1, 0) \pmod 2$. This algebra FS is not basic algebra. Using direct computing, we have the table of multiplication of these matrices except zeros:

	σ_1	σ_3	σ_5	σ_6
σ_1	σ_1	σ_3	σ_5	σ_6
σ_3	σ_3	σ_1	σ_6	σ_5
σ_7	σ_7	σ_8	σ_2	σ_4
σ_8	σ_8	σ_7	σ_4	σ_2

,

	σ_2	σ_4	σ_7	σ_8
σ_2	σ_2	σ_4	σ_7	σ_8
σ_4	σ_4	σ_2	σ_8	σ_7
σ_5	σ_5	σ_6	σ_1	σ_3
σ_6	σ_6	σ_5	σ_3	σ_1

.

These tables show that the adjacency algebra of $\mathcal{C}_s$ is isomorphic to an adjacency algebra of a thin coherent configuration [13]. The adjacency algebra of this thin coherent configuration is generated by block matrices

$$\sigma_1 \mapsto \begin{bmatrix} I & O \\ O & O \end{bmatrix},\ \sigma_2 \mapsto \begin{bmatrix} O & O \\ O & I \end{bmatrix},\ \sigma_3 \mapsto \begin{bmatrix} I' & O \\ O & O \end{bmatrix},\ \sigma_4 \mapsto \begin{bmatrix} O & O \\ O & I' \end{bmatrix},$$

$$\sigma_5 \mapsto \begin{bmatrix} O & I \\ O & O \end{bmatrix},\ \sigma_6 \mapsto \begin{bmatrix} O & I' \\ O & O \end{bmatrix},\ \sigma_7 \mapsto \begin{bmatrix} O & O \\ I & O \end{bmatrix},\ \sigma_8 \mapsto \begin{bmatrix} O & O \\ I' & O \end{bmatrix},$$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $I' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The fiber scheme of this thin coherent configuration is corresponding to the cyclic group $\mathbb{Z}_2$ because the fiber scheme generated σ_1 and σ_3 (or σ_2 and σ_4). We know the following theorem due to [13, Theorem 3.5].

Theorem 5.6. *The adjacency algebra of Type IV is isomorphic to*

$$M_2(F\mathbb{Z}_2) \cong M_2(F[x]/(x^2)).$$

By Theorem 5.6, the module category of FS is Morita equivalent to the module category of $F[x]/(x^2)$. Hence we know there are two isomorphic classes of indecomposable modules and $\dim_F \mathrm{Rad}(FS) = 4$. We know the fact that $\sigma_1+\sigma_3$, $\sigma_2+\sigma_4$, $\sigma_5+\sigma_6$ and $\sigma_7+\sigma_8$ are the basis of $\mathrm{Rad}(FS)$ and $\dim_F(FX)\mathrm{Rad}(FS) = 2$ by computation. According to these facts, we know the structure of the standard module FX.

$$FX \cong V_0 \oplus (v-2)V_1,$$

where $\dim_F V_0 = 4$ and $\dim_F V_1 = 2$.

5.7. **Type** *V* : $(v, \ell, \lambda) \equiv (1, 1, 0) \pmod 2$**.** We suppose $(v, \ell, \lambda) \equiv (1, 1, 0) \pmod 2$. We assume that FS is not semisimple. $Fr(\mathcal{C}_s)$ must be divided by 2. This means $\mathrm{ord}_2(v-\ell) > \mathrm{ord}_2(v-1)$ where $\mathrm{ord}_2(a)$ is a 2-adic valuation of an integer a. Since $\mathrm{ord}_2(\lambda) > 0$ and $\mathrm{ord}_2(v-1) = \mathrm{ord}_2(l-1)$, the in-equation leads contradict to $\lambda(v-1) = \ell(\ell-1)$. Hence FS is always semisimple.

5.8. **Type** *VI* : $(v, \ell, \lambda) \equiv (1, 1, 1) \pmod 2$**.** We suppose $(v, \ell, \lambda) \equiv (1, 1, 1) \pmod 2$. Examples of parameters of this type are 2-$(15, 7, 3)$ designs. Since $Fr(\mathcal{C}_s) = 2^4 \cdot 3^6 \cdot 5^6$, FS is not semisimple. By computation, we have

- $\dim_F \mathrm{Rad}(FS) = 2$ with the basis σ_6, σ_8,
- $\dim_F \mathrm{Rad}^2(FS) = 0$.

We determine the 2-modular character table. The modular character table of (X_1, S^{11}) and (X_2, S^{22}) are

	σ_1	σ_3	multiplicity
	1	0	1
	1	1	$v-1$

,

	σ_2	σ_4	multiplicity
	1	0	1
	1	1	$v-1$

.

The character table of the coherent configuration is

	σ_1	σ_3	σ_2	σ_4	multiplicity
U	1	0	1	0	1
V	1	1	0	0	$v-1$
W	0	0	1	1	$v-1$

.

We choose the simple modules U, V and W. The decomposition and Cartan matrices are

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \ C = {}^tDD = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We can choose primitive idempotents $e_U = \sigma_1 + \sigma_3 + \sigma_7 + \sigma_8$, $e_V = \sigma_2 + \sigma_7$ and $e_W = \sigma_3 + \sigma_8$. Put $\alpha = \sigma_6$ and $\beta = \sigma_8$. Then we have the following theorem.

Theorem 5.7. *The adjacency algebra of Type VI is isomorphic to*

$$M_2(F) \oplus FQ/(\alpha\beta, \beta\alpha)$$

where FQ is a path algebra, Q is the following quiver.

$$Q : \circ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \circ$$

There are three simple modules U, V and W with $\dim_F U = 2$, $\dim_F V = \dim_F W = 1$ and the Loewy structure of the projective covers are as follows:

$$P(U) = [U], \ P(V) = \begin{bmatrix} V \\ W \end{bmatrix}, \ P(W) = \begin{bmatrix} W \\ V \end{bmatrix}.$$

This algebra is finite representation type. The structure of the standard module is completely determined. We can wire

$$FX \cong [U] \oplus g_1[V] \oplus g_2 \begin{bmatrix} V \\ W \end{bmatrix} \oplus h_1[W] \oplus h_2 \begin{bmatrix} W \\ V \end{bmatrix}$$

for some non-negative g_1, g_2, h_1 and h_2.

By $m_V = m_W = v - 1$, we have

$$g_1 + g_2 + h_2 = v - 1, \tag{3}$$

$$g_2 + h_1 + h_2 = v - 1, \tag{4}$$

$g_2 = rank(\alpha) = rank(\sigma_6) = rank(\sigma_5) - 1$, $h_2 = rank(\beta) = rank(\sigma_8)$. Since $\sigma_6^* = \sigma_8$, $g_2 = h_2$. We put $w = rank(\sigma_6)$,

$$FX \cong [U] \oplus (v - 2w - 1)[V] \oplus w \begin{bmatrix} V \\ W \end{bmatrix} \oplus (v - 2w - 1)[W] \oplus w \begin{bmatrix} W \\ V \end{bmatrix}$$

Example 5.8. We give a list of parameters for 5 isomorphism classes of 2-$(15, 7, 3)$ designs. By Theorem 5.7, structures are determined by one parameter $rank(\sigma_6)$. The usual 2-rank is $rank(\sigma_5)$.

$\sharp$	number in [5, p.33]	$g_1 = h_1$	$g_2 = h_2$	$rank(\sigma_5)$
1	No.1	6	4	5
1	No.2	4	5	6
1	No.3	2	6	7
2	No.4, 5	0	7	8

6. Quasi-Symmetric BIB designs

In this section, we assume that $(X_1, X_2, \mathcal{F})$ is a quasi-symmetric BIB design with parameters v, b, r, ℓ and $\lambda = 1$.

6.1. Types of adjacency algebras of $\mathcal{C}_{qs}$.

By [4, Proposition 5.2], the block graph of this design is an (n, k, a, c)-strongly regular graph. We obtain expressions for the parameters of this graph in terms of the designs parameters v and ℓ:

$$n = b = \frac{v(v-1)}{\ell(\ell-1)}, \tag{5}$$

$$k = \ell(r-1) = \frac{\ell(v-\ell)}{(\ell-1)}, \tag{6}$$

$$a = (r-1) + (\ell-1)^2 = \frac{v - 2\ell + 1}{\ell - 1} + (\ell-1)^2, \tag{7}$$

$$c = \ell^2. \tag{8}$$

The eigenvalues of the adjacency matrix of this strongly regular graph are k and

$$\theta, \tau = \frac{a - c \pm \sqrt{(a-c)^2 - 4(k-c)}}{2}.$$

Note that $a - c = \theta + \tau$. If the block graph is not a conference graph, then it is known that θ and τ are rational integers. Suppose that the block graph is a conference graph. Then $(n, k, a, c) = (n, (n-1)/2, (n-1)/4, (n-5)/4)$ [9, Corollary 10.22]. By $a = c+1$, we have $v = 2\ell^2 - 1$. By the above equation (5), we have $n = 4\ell^2 + 5$. Since ℓ must be an integer, $(v, \ell) = (7, 2)$. In this case, eigenvalues are rational integers. Hence we can consider θ and τ are always rational integers.

Now, we consider the coherent configuration $\mathcal{C}_{qs} = (X, \{s_i\}_{i \in I})$ over $I = \{1, \ldots, 9\}$ obtained from $(X_1, X_2, \mathcal{F})$.

Proposition 6.1. [15, 9.1] *$\mathcal{C}_{qs}$ is coherent.*

Proof. Using direct computing, we have tables of multiplications of adjacency matrices except zeros.

	σ_1	σ_3	σ_6	σ_7
σ_1	σ_1	σ_3	σ_6	σ_7
σ_3	σ_3	zw $(v-1)\sigma_1 + (v-2)\sigma_3$	zw $(\ell-1)\sigma_6 + \ell\sigma_7$	zw $(v-\ell)\sigma_6 + (v-\ell-1)\sigma_7$
σ_8	σ_8	zw $(\ell-1)\sigma_8 + \ell\sigma_9$	zw $\ell\sigma_2 + \lambda\sigma_4$	zw $(\ell-\lambda)\sigma_4 + \ell\sigma_5$
σ_9	σ_9	zw $(v-\ell)\sigma_8 + (v-\ell-1)\sigma_9$	zw $(\ell-1)\sigma_4 + \ell\sigma_5$	zw $(v-\ell)\sigma_2 + (v-2\ell+\lambda)\sigma_4 + (v-2\ell)\sigma_5$

TABLE 3

	σ_2	σ_4	σ_5	σ_8	σ_9
σ_2	σ_2	σ_4	σ_5	σ_8	σ_9
σ_4	σ_4	$k\sigma_2 + \mathrm{zw}a\sigma_4 + \ell^2\sigma_5$	zw $(k-a-1)\sigma_4 + (k-\ell^2)\sigma_5$	zw $(r-1)\sigma_8 + \ell\sigma_9$	zw $(k-r+1)\sigma_8 + (k-\ell)\sigma_9$
σ_5	σ_5	zw $(k-a-1)\sigma_4 + (k-\ell^2)\sigma_5$	zw $(b-k-1)\sigma_2 + (b+a-2k)\sigma_4 + (b-2k-2+\ell^2)\sigma_5$	zw $(r-\ell)\sigma_9$	zw $(b-k-1)\sigma_8 + \mathrm{zw}(b-r-k+\ell-1)\sigma_9$
σ_6	σ_6	zw $(r-1)\sigma_6 + \ell\sigma_7$	zw $(r-\ell)\sigma_7$	$r\sigma_1 + \sigma_3$	$(r-1)\sigma_3$
σ_7	σ_7	zw $(k-r+1)\sigma_6 + (k-\ell)\sigma_7$	zw $(b-k-1)\sigma_6 + \mathrm{zw}(b-r-k+\ell-1)\sigma_7$	zw $(r-1)\sigma_3$	zw $(b-r)\sigma_1 + (b-2r+1)\sigma_3$

TABLE 4

These tables show that the configuration $\mathcal{C}_{qs}$ is a coherent configuration of type $(2, 2; 3)$ □

We remark that all coefficients are polynomials of v, ℓ, b, r, k, a. Hence if they are equal in modulo 2 for this designs, then the algebras are isomorphic over a field of characteristic 2. We give a list of possible parameters in characteristic 2.

Lemma 6.2. *There are six types of parameters in characteristic 2:*

Type	v	ℓ	b	r	n	k	a	c	*example* (v, ℓ)
1	0	0	0	1	0	0	0	0	$(16, 4)$
2	0	0	1	1	1	0	0	0	$(28, 4)$
3	1	0	0	0	0	0	1	0	$(25, 4)$
4	1	0	1	0	1	0	1	0	$(37, 4)$
5	1	1	0	0	0	1	0	1	$(9, 3)$
6	1	1	1	1	1	0	1	1	$(15, 3)$

Proof. We suppose $(v, \ell) \equiv (0, 0) \pmod 2$. Since any BIB design with parameters v, b, r, ℓ, λ holds

$$r = \frac{v-1}{\ell-1}, \tag{9}$$

parameter r must be odd. Hence the parameters k and a should be even due to (6) and (7), respectively. The parameter b is either even or odd. We assumed $(v, \ell) \equiv (0, 1) \pmod 2$. It contradict to an integral condition for r in (9). Similarly, we have the other cases. □

6.2. **Character tables in characteristic zero.** We consider the character table of the coherent configuration. Since (X_1, S^{11}) is the complete graph with v vertices and (X_2, S^{22}) is a strongly regular graph with the parameters (n, k, a, c), their character tables are as follows (see [9, Chapter 10]).

σ_1	σ_3	multiplicity
1	$v-1$	1
1	-1	$v-1$

,

σ_2	σ_4	σ_5	multiplicity
1	k	$b-k-1$	1
1	θ	$-\theta-1$	m_θ
1	τ	$-\tau-1$	m_τ

.

Since $|S| = 9$ and by Proposition 3.1, the degrees of irreducible characters of the coherent configuration must be $2, 2$ and 1. This means that m_θ or m_τ is equal to $v-1$. Thus we can suppose $m_\theta = v-1$.

Hence we obtain the character table of the coherent configuration corresponding a quasi-symmetric BIB design with the parameters v, b, r, ℓ, λ as follows.

σ_1	σ_3	σ_2	σ_4	σ_5	multiplicity
1	$v-1$	1	k	$b-k-1$	1
1	-1	1	θ	$-\theta-1$	$v-1$
0	0	1	τ	$-\tau-1$	$b-v$

Note that character values of σ_i $(i = 6, 7, 8, 9)$ are zeros and we omit them.

¿From the next subsection, we will determine the structures of adjacency algebras and standard modules over a field of characteristic 2 for six types of coherent configurations obtained by quasi-symmetric BIB designs with parameters v, b, r, ℓ and $\lambda = 1$. The decomposition and Cartan matrices are not necessary to compute the structures but they help our computations. We note that the usual 2-rank of the incidence matrix N of this design is equal to the 2-rank of σ_6. By [10, Theorem 2.1], the 2-rank of N is never less than $v-1$ for Type $3, 4$ and 5.

6.3. **Type 1:** $(v, \ell, b) \equiv (0,0,0) \pmod 2$**.** We suppose $(v, \ell, b) \equiv (0,0,0) \pmod 2$. An example of parameters of this type is $(v, \ell) = (16, 4)$. By computation, we have

- $\dim_F \mathrm{Rad}(FS) = 7$ with the basis $\sigma_1+\sigma_3$, $\sigma_2+\sigma_3$, σ_4, σ_6, σ_8, σ_9,
- $\dim_F \mathrm{Rad}^2(FS) = 4$ with the basis $\sigma_1+\sigma_3$, σ_4, $\sigma_6+\sigma_7$, $\sigma_8+\sigma_9$, and
- $\mathrm{Rad}^3(FS) = 0$.

We determine the 2-modular character table. Since σ_4 is in the Jacobson radical of FS, $\theta \equiv \tau \equiv 0 \pmod 2$. Hence the modular character tables of (X_1, S^{11}) and (X_2, S^{22}) are

	σ_1	σ_3	multiplicity
	1	1	v

,

	σ_2	σ_4	σ_5	multiplicity
	1	0	1	b

.

Since $v \neq b$, the character table of the coherent configuration is

	σ_1	σ_3	σ_2	σ_4	σ_5	multiplicity
U	1	1	0	0	0	v
V	0	0	1	0	1	b

We put the simple modules U and V. The decomposition and Cartan matrices are

$$D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \ C = {}^tDD = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}.$$

We can choose primitive idempotents $e_U = \sigma_1$ and $e_v = \sigma_2$. We put $\alpha = \sigma_6$, $\beta = \sigma_8$ and $\gamma = \sigma_2 + \sigma_5$. By direct computation, we have the following theorem.

Theorem 6.3. *The adjacency algebra of Type* 1 *is isomorphic to*

$$FQ/(\gamma^2, L^3),$$

where FQ is a path algebra, Q is the following quiver and L is an ideal generated by paths of positive length.

$$Q : \begin{tikzcd} \circ \arrow[r, bend left, "\alpha"] & \circ \arrow[l, bend left, "\beta"] \arrow[loop right, "\gamma"] \end{tikzcd}$$

There are two simple modules U and V with $\dim_F U = \dim_F V = 1$ and the Loewy structure of the projective covers are as follows:

$$P(U) = \begin{bmatrix} U \\ V \\ V\ U \end{bmatrix}, P(V) = \begin{bmatrix} V \\ U\ V \\ V\ U \end{bmatrix}.$$

It is difficult to determine the structure of the standard module and we could not do that.

6.4. **Type 2:** $(v, \ell, b) \equiv (0, 0, 1) \pmod 2$**.** We suppose $(v, \ell, b) \equiv (0, 0, 1)$ (mod 2). An example of parameters of this type is $(v, \ell) = (28, 4)$ and 2-ranks have been considered, for example, in [20]. By computation, we have

- $\dim_F \mathrm{Rad}(FS) = 6$ with the basis $\sigma_1 + \sigma_3$, σ_4, σ_6, σ_7, σ_8, σ_9,
- $\dim_F \mathrm{Rad}^2(FS) = 2$ with the basis $\sigma_1 + \sigma_3$, σ_4, and
- $\dim_F \mathrm{Rad}^3(FS) = 0$.

Since σ_4 is in the Jacobson radical, the eigenvalues of σ_4 are zeros. By the similar arguments as in Type 1, we obtain the modular character table.

	σ_1	σ_3	σ_2	σ_4	σ_5	multiplicity
U	1	1	0	0	0	v
V	0	0	1	0	0	1
W	0	0	1	0	1	$b-1$

The decomposition and Cartan matrices are

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ C = {}^tDD = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

We can choose primitive idempotents $e_U = \sigma_1$, $e_V = \sigma_2 + \sigma_4 + \sigma_5$ and $e_W = \sigma_4 + \sigma_5$. We put $\alpha_1 = \sigma_6 + \sigma_7$, $\alpha_2 = \sigma_8 + \sigma_9$, $\alpha_3 = \sigma_7$ and $\alpha_4 = \sigma_9$. Then we have the following theorem.

Theorem 6.4. *The adjacency algebra of Type 2 is isomorphic to*

$$FQ/(\alpha_2\alpha_1, \alpha_2\alpha_4, \alpha_3\alpha_1, \alpha_4\alpha_3, L^3),$$

where Q is the following quiver and L is an ideal generated by paths of positive length.

$$Q:\ \circ \underset{\alpha_4}{\overset{\alpha_3}{\rightleftarrows}} \circ \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} \circ$$

There are three simple modules U, V and W with $\dim_F U = \dim_F V = \dim_F W = 1$ and the Loewy structure of the projective covers are as follows:

$$P(U) = \begin{bmatrix} U \\ V\ W \\ U \end{bmatrix}, P(V) = \begin{bmatrix} V \\ U \end{bmatrix}, P(U) = \begin{bmatrix} W \\ U \\ W \end{bmatrix}.$$

It is difficult to determine the structure of the standard modules and we could not do that.

Example 6.5. In a web page [18], we can find a list of BIB $(28, 4, 1)$ designs (see also [17]). According to this list, 2-ranks are 19, 21, 22, 23, 24, 25, 26, and 27. The values of $w_1 = \mathrm{rank}(\sigma_7)$ and $w_2 = \mathrm{rank}(\sigma_4)$ are

$$\begin{aligned}(w_1, w_2) =&(19, 12), (21, 14), (21, 16), (22, 16), (22, 18), (23, 18), (23, 20),\\ &(24, 20), (24, 22), (25, 22), (25, 24), (26, 24), (27, 26).\end{aligned}$$

For these examples, we can see that $\mathrm{rank}(\sigma_6) = \mathrm{rank}(\sigma_7)$. Since w_1 and w_2 are determined by the structure of the standard module, the structure of the standard module contains more information than the 2-ranks.

6.5. **Type 3: $(v, \ell, b) \equiv (1, 0, 0) \pmod 2$.** We suppose $(v, \ell, b) \equiv (1, 0, 0) \pmod 2$. An example of parameters of this type is $(v, \ell) = (25, 4)$. By computation, we have

- $\dim_F \mathrm{Rad}(FS) = 3$ with the basis $\sigma_2 + \sigma_4 + \sigma_5, \sigma_6 + \sigma_7, \sigma_8 + \sigma_9$,
- $\dim_F \mathrm{Rad}^2(FS) = 1$ with the basis $\sigma_2 + \sigma_4 + \sigma_5$, and
- $\mathrm{Rad}^3(FS) = 0$.

By two equations $k + \theta(v - 1) + \tau(b - v) = 0$ and $a - c = \theta + \tau$ for θ and τ, the modular character table of (X_1, S^{11}) and (X_2, S^{22}) are

σ_1	σ_3	multiplicity
1	0	1
1	1	$v-1$

,

σ_2	σ_4	σ_5	multiplicity
1	0	1	$b-v+1$
1	1	0	$v-1$

.

Since $\dim_F \mathrm{Rad}(FS) = 3$, we have the modular character table of the coherent configuration.

	σ_1	σ_3	σ_2	σ_4	σ_5	multiplicity
U	1	0	0	0	0	1
V	0	0	1	0	1	$b-v+1$
W	1	1	1	1	0	$v-1$

We put the simple modules U, V and W. The decomposition and Cartan matrices are

$$D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ C = {}^tDD = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

There are two block ideals $\mathcal{B}_1$ and $\mathcal{B}_2$. We put the simple modules U and V in $\mathcal{B}_1$ and W in $\mathcal{B}_2$. The block ideal $\mathcal{B}_2$ is simple and isomorphic to the matrix algebra $M_2(F)$. We can choose $e_U = \sigma_1 + \sigma_3$ and $e_V = \sigma_2 + \sigma_4$. We put $\alpha = \sigma_6+\sigma_7$ and $\beta = \sigma_8+\sigma_9$. Then $\alpha\beta = 0$ and $\beta\alpha = \sigma_2+\sigma_4+\sigma_5$.

Theorem 6.6. *The adjacency algebra of Type 3 is isomorphic to*

$$FQ/(\alpha\beta) \oplus M_2(F),$$

where Q is the following quiver.

$$Q: \ \circ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \circ$$

There are three simple modules U, V and W with $\dim_F U = \dim_F V = 1$, $\dim_F W = 2$ and the Loewy structure of the projective covers are as follows:

$$P(U) = \begin{bmatrix} U \\ V \end{bmatrix}, \ P(V) = \begin{bmatrix} V \\ U \\ V \end{bmatrix}, \ P(W) = \begin{bmatrix} W \end{bmatrix}.$$

The algebra is finite representation type. Since $m_U = 1$, the structure of the standard module is completely determined.

Theorem 6.7. *The standard module has the following indecomposable decomposition.*

$$FX \cong \begin{bmatrix} V \\ U \\ V \end{bmatrix} \oplus (b-v-1)[V] \oplus (v-1)[W].$$

By [10, Theorem 2.1 (ii)], the 2-rank of the incidence matrix is $v-1$. In fact, σ_6 has rank 1 only on W.

6.6. **Type 4:** $(v, \ell, b) \equiv (1, 0, 1) \pmod 2$**.** We suppose $(v, \ell, b) \equiv (1, 0, 1) \pmod 2$. An example of parameters of this type is $(v, \ell) = (37, 4)$. By computation, the adjacency algebra is semisimple. The structures of standard modules are completely determined by v and ℓ. By [10, Theorem 2.1 (ii)], the 2-rank of the incidence matrix is $v-1$.

6.7. **Type 5: $(v, \ell, b) \equiv (1, 1, 0) \pmod 2$.** We suppose $(v, \ell, b) \equiv (1, 1, 0) \pmod 2$. An example of this type is $(v, \ell) = (9, 3)$. By computation, we have

- $\dim_F \operatorname{Rad}(FS) = 3$ with the basis $\sigma_2+\sigma_4+\sigma_5, \sigma_6+\sigma_7, \sigma_8+\sigma_9$,
- $\dim_F \operatorname{Rad}^2(FS) = 1$ with the basis $\sigma_2+\sigma_4+\sigma_5$, and
- $\operatorname{Rad}^3(FS) = 0$.

This case is quite similar to the case of Type 3. The modular character table is as follows.

	σ_1	σ_3	σ_2	σ_4	σ_5	multiplicity
U	1	0	0	0	0	1
V	0	0	1	1	0	$b-v+1$
W	1	1	1	0	1	$v-1$

We put the simple modules U, V and W. The decomposition and Cartan matrices are

$$D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C = {}^tDD = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can choose $e_U = \sigma_1+\sigma_3$, $e_V = \sigma_2+\sigma_4$ and $e_W = \sigma_3+\sigma_5$. We put $\alpha = \sigma_6+\sigma_7$ and $\beta = \sigma_8+\sigma_9$. Then we have the following theorem.

Theorem 6.8. *The adjacency algebra of Type 5 is isomorphic to*

$$FQ/(\alpha\beta) \oplus M_2(F),$$

where Q is the following quiver.

$$Q : \circ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \circ$$

There are three simple modules U, V and W with $\dim_F U = \dim_F V = 1$, $\dim_F W = 2$ and the Loewy structure of the projective covers are as follows:

$$P(U) = \begin{bmatrix} U \\ V \end{bmatrix}, \quad P(V) = \begin{bmatrix} V \\ U \\ V \end{bmatrix}, \quad P(W) = \begin{bmatrix} W \end{bmatrix}.$$

Theorem 6.9. *The standard module has the following indecomposable decomposition.*

$$FX \cong \begin{bmatrix} V \\ U \\ V \end{bmatrix} \oplus (b-v-1)[V] \oplus (v-1)[W].$$

Since $\sigma_6 = (\sigma_6+\sigma_7)+\sigma_7$, $\operatorname{rank}(\sigma_6) = \operatorname{rank}(\sigma_6+\sigma_7) + \operatorname{rank}(\sigma_7) = 1 + \operatorname{rank}(\sigma_7)$. Hence $\sigma_6+\sigma_7$ has rank 1 on $P(V)$ and σ_7 has rank 1 on W. We have $\operatorname{rank}(\sigma_6) = v$ in this case.

6.8. **Type 6: $(v,\ell,b)\equiv(1,1,1) \pmod 2$.** We suppose $(v,\ell,b)\equiv(1,1,1) \pmod 2$. An example of parameters of this type is $(v,\ell)=(15,3)$. By computation, we have

- $\dim_F \mathrm{Rad}(FS)=3$ with the basis $\sigma_5,\sigma_7,\sigma_9$,
- $\dim_F \mathrm{Rad}^2(FS)=1$ with the basis σ_5, and
- $\mathrm{Rad}^3(FS)=0$.

We can determine the modular character table as well as other types of parameters in Lemma 6.2.

	σ_1	σ_3	σ_2	σ_4	σ_5	multiplicity
U	1	0	1	0	0	1
V	1	1	0	0	1	$v-1$
W	0	0	1	1	0	$b-1$

We put the simple modules U,V and W. The decomposition and Cartan matrices are

$$D=\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix},\ C={}^tDD=\begin{pmatrix}1&0&0\\0&1&1\\0&1&2\end{pmatrix}.$$

The block ideal containing U is isomorphic to $M_2(F)$. We can choose $e_V=\sigma_3$ and $e_W=\sigma_4+\sigma_5$. We put $\alpha=\sigma_7$ and $\beta=\sigma_9$. Then $\alpha\beta=0$ and $\beta\alpha=\sigma_5$.

Theorem 6.10. *The adjacency algebra of Type 6 is isomorphic to*

$$M_2(F)\oplus FQ/(\alpha\beta),$$

where Q is the following quiver.

$$Q:\ \begin{tikzcd} \circ \arrow[r, bend left, "\alpha"] & \circ \arrow[l, bend left, "\beta"] \end{tikzcd}$$

There are three simple modules U, V and W with $\dim_F U=2$, $\dim_F V=\dim_F W=1$ and the Loewy structure of the projective covers are as follows:

$$P(U)=\begin{bmatrix}U\end{bmatrix},\ P(V)=\begin{bmatrix}V\\W\end{bmatrix},\ P(W)=\begin{bmatrix}W\\V\\W\end{bmatrix}.$$

Since the algebra is finite representation type and $m_U=1$, we can determine the structure of the standard module of Type 6.

Theorem 6.11. *The standard module has the following indecomposable decomposition.*

$$FX \cong \begin{bmatrix} U \end{bmatrix} \oplus g_1 \begin{bmatrix} V \end{bmatrix} g_2 \begin{bmatrix} V \\ W \end{bmatrix} h_1 \begin{bmatrix} W \end{bmatrix} h_2 \begin{bmatrix} W \\ V \end{bmatrix} h_3 \begin{bmatrix} W \\ V \\ W \end{bmatrix},$$

where parameters satisfy

$$v - 1 = g_1 + g_2 + h_2 + h_3, \tag{10}$$

$$b - 1 = g_2 + h_1 + h_2 + 2h_3, \tag{11}$$

$$g_2 = h_2. \tag{12}$$

Proof. Since $m_V = v - 1$ and $m_W = b - 1$, we have equations (10) and (11). By $g_2 + h_3 = \mathrm{rank}(\sigma_7) = \mathrm{rank}(\sigma_9) = h_2 + h_3$, we have the equation (12). □

Since $\mathrm{rank}(\sigma_6) = \mathrm{rank}(\sigma_6+\sigma_7)+\mathrm{rank}(\sigma_7) = 1+\mathrm{rank}(\sigma_7)$, the usual 2-rank of the incidence matrix of the design is $\mathrm{rank}(\sigma_6) = 1+g_2+h_3$. The values g_i and h_j are determined by ranks of some adjacency matrices. For example, we put $w_1 = \mathrm{rank}(\alpha) = \mathrm{rank}(\sigma_7)$ and $w_2 = \mathrm{rank}(\beta\alpha) = \mathrm{rank}(\sigma_5)$. Then

$$(g_1, g_2, h_1, h_2, h_3) = (v-1-2w_1+w_2, w_1-w_2, b-1-2w_1, w_1-w_2, w_2).$$

Example 6.12. We give a list of parameters for 80 isomorphism classes of quasi-symmetric BIB design with parameters $v = 15$, $\ell = 3$ and $\lambda = 1$, the Steiner triple systems on 15 points. In Theorem 6.11, the structure is determined by two parameters $\mathrm{rank}(\sigma_7)$ and $\mathrm{rank}(\sigma_5)$.

♯	number in [5, p.30]	g_1	g_2	h_1	h_2	h_3	$\mathrm{rank}(\sigma_7)$	$\mathrm{rank}(\sigma_5)$	$\mathrm{rank}(\sigma_6)$
1	No.1	0	4	14	4	6	10	6	11
1	$No.2$	0	3	12	3	8	11	8	12
5	$No.3-7$	0	2	10	1	10	12	10	13
15	$No.8-22$	0	1	8	1	12	13	12	14
58	$No.23-80$	0	0	6	0	14	14	14	15

7. Acknowledgment

The authors would like to thank the referees for their valuable comments which helped to improve the manuscript. This work was partially supported by JSPS KAKENHI Grant nuumber JP25400011.

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