Competitive Analysis for the Flat-Rate Problem*

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SUMMARY We consider a problem of the choice of price plans offered by a telecommunications company: a "pay-as-you-go" plan and a "flat-rate" plan. This problem is formulated as an online optimization problem extending the ski-rental problem, and analyzed using the competitive ratio. We give a lemma for easily calculating the competitive ratio. Based on the lemma, we derive a family of optimal strategies for a realistic class of instances.
key words: *online algorithm, competitive analysis, online optimization, ski-rental problems*

1. Introduction

Suppose that you are going to start using a mobile device with a telecommunications company. The simplest price plan offered by the company is of course a "pay-as-you-go" plan. Namely, you are charged a fee proportional to the amount of your data usage. A typical alternative plan is as follows:

- You are charged an initial minimum fee even if you use little data.
- If you use more data than a pre-defined limit, your fee increases linearly with the amount of your data usage.
- There is a maximum fee that may be charged.

Interestingly, it seems that they often call such a plan a "flat-rate" plan, even though the fee is not always completely flat, as above. Is such a "flat-rate" plan good for saving cost? One who thinks oneself to be a light user would not choose the "flat-rate" plan. But he/she will regret in the case where he/she consequently uses more data than he/she thinks. Needless to say, that is a bad choice. On the other hand, it is also a sad ending that one who has chosen the "flat-rate" plan does not use so much data after all.

These observations let us think of a strategy like: start by the "pay-as-you-go" plan, keep it for a while, and then take the "flat-rate" plan. With this strategy we can avoid big

risks seen above. Then, our next question is what timing is good for changing plans. In this paper we consider a best timing via the analysis of the *competitive ratio* [1].

The competitive ratio is a performance measure that indicates how many times fee one may be charged at most compared with an ideal player who knows its data usage in the future. That is to say, the smaller the competitive ratio is, the smaller the regret in the worst case is.

1.1 Our Contribution

Our results are summarized as follows:

(A) We give an easy way to calculate the competitive ratio of a strategy. A naive approach involves evaluation for all possible amount of data usage. We show that a correct competitive ratio is obtained by calculating a maximum value just for three cases.

(B) Based on the result of (A), we analyze a typical class of instances in the real world that "the initial minimum fee is appropriated for data communication." A family of optimal strategies is given in a closed form with parameters of the instance.

1.2 Related Work

Our problem is an extension of the famous ski-rental problem [2], which is often cited as an example of online optimization problems. In the ski-rental problem, the player is asked choose either to rent or to buy a ski gear without information about how many times he/she skis in the future. It seems that the mostly related problem to ours is the parking permit problem [3], which is another extension of the ski-rental problem. The task is to keep buying some parking tickets, each valid for a fixed duration, so that they cover an uncertain period. The multislope ski-rental problem [4]–[7] is yet another extension in which the player is offered three or more plans.

2. Problem Statement

We would like to introduce our problem by first formulating a mathematical program and then explaining its details.

Input: An instance $(a, b, r) \in \mathbb{Q}^3$ with $0 < a$, $0 \le b < 1$, and $0 < r$.
Output: A strategy $x \in \mathbb{R}$ with $0 \le x$.
Objective: Minimize the competitive ratio

$$R_x := \inf \{c \mid ON(x,t) \le c \cdot OPT(t) \text{ for all } t \ge 0\},$$
$$(1)$$

where

$$ON(x,t) :=$$

$$\begin{cases} t, & 0 \le t < x; \\ x + b, & x \le t < x + a; \\ r(t - (x+a)) + x + b, & x + a \le t < x + a + \frac{1-b}{r}; \\ x + 1, & x + a + \frac{1-b}{r} \le t, \end{cases} \quad (2)$$

and

$$OPT(t) := \inf \{ON(x,t) \mid x \ge 0\}. \quad (3)$$

In *instance* (a, b, r), if you choose the "flat-rate" plan, then you are charged as follows: You have to pay at least a fee of b. In the case where your data usage exceeds a, you need to pay r times the excess. But, you are not charged over one. There is of course a "pay-as-you-go" plan. If you choose it, you are charged as the same amount of fee as your data usage.

In this way, the maximum fee in the "flat-rate" plan and the rate in the "pay-as-you-go" plan are both fixed to one, which is a normalization for ease of calculation. This normalization is done without lose of generality. At the end of this section, we mention some price plans offered by telecommunications companies in the real world.

A *strategy x* is to switch to the "flat-rate" plan after having used x amount of data with the "pay-as-you-go" plan, that is, after having paid x. For strategy x, $ON(x,t)$ in (2) is the total cost paid so far when you have used t amount of data. If $t < x$, you will use just the "pay-as-you-go" plan. Otherwise, you will have switched the "flat-rate" plan after using x amount of data.

To measure the performance of a strategy, we assume an ideal player, sometimes called the *offline player*, who knows how much data it uses in the future. An *optimal offline strategy* is an strategy optimized making use of the future knowledge. $OPT(t)$ in (3), called the *optimal offline cost*, is the cost paid along an optimal offline strategy. The definition (3) is translated as follows: With the amount of data usage t known in advance, a strategy x is chosen so that the cost $ON(x,t)$ is minimized.

The *competitive ratio R_x* in (1) is the objective function to be minimized. One can think that for any strategy with $x > 0$, R_x is the maximum value of $\frac{ON(x,t)}{OPT(x)}$ over all $t > 0$. That is to say, the competitive ratio is the maximum ratio of the cost incurred by strategy x to the cost incurred by an optimal offline strategy, when the same data usage happens for the both.

Throughout this paper we analyze the problem for general a, b, and r. We sometimes employ an instance of $(a, b, r) = (\frac{1}{3}, \frac{2}{9}, \frac{2}{3}) \approx (0.33, 0.22, 0.67)$ as a numerical example. The reason why we use this is simply its easiness of explanation using figures. Figure 1 depicts the cost incurred by taking the "pay-as-you-go" plan from the beginning, and
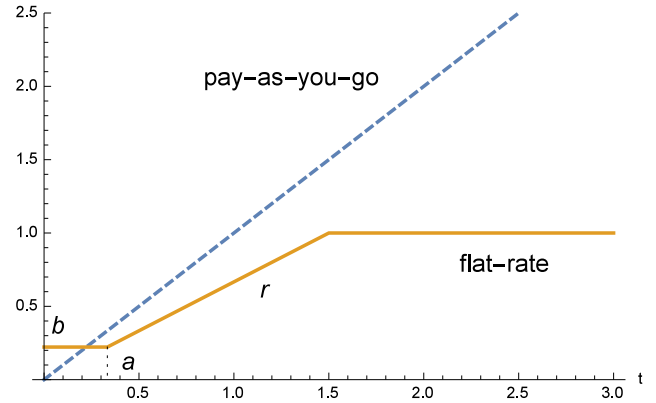

Fig. 1 The cost incurred by taking the "pay-as-you-go" plan from the beginning $(= t)$, and the cost incurred by taking the "flat-rate" plan from the beginning $(= ON(0,t))$, both for instance $(a, b, r) = (\frac{1}{3}, \frac{2}{9}, \frac{2}{3})$, when the amount of data usage is t. For this instance, the optimal offline cost $OPT(t)$ is the lower envelope of these two costs. Compare Fig. 4.

the cost incurred by taking the "flat-rate" plan from the beginning, as the data usage grows. These costs are described as functions $t \mapsto t$ and $t \mapsto ON(0,t)$, respectively.

Although our instance above is somewhat artificial, it is not very far from real ones. We consulted the websites of Japanese telecommunications companies NTT docomo and KDDI, and found some price plans there. With normalization of values, each plan can be translated into an instance of our problem. The range of each parameter is as follows: $0.08 < a < 0.24$, $0.07 < b < 0.24$, and $0.4 < r < 0.8$.

3. Optimal Offline Strategy

In the ski-rental problem [2], the optimal offline strategy is either "to buy skis at the beginning" or "to rent skis forever." Once we know which one is cheaper, we are done. In contrast, to do the same for our problem may lead us to a suboptimal strategy. Unlike the ski-rental problem, "to rent skis for a while and then buy skis" can be an optimal offline strategy. Thus, the derivation of an optimal offline strategy needs a bit care.

The following lemma gives an optimal offline strategy for our problem. Here strategies $x_{\text{off}} = t + \varepsilon$ and $x_{\text{off}} = 0$ mean "to take the pay-as-you-go plan forever" and "to take the flat-rate plan from the beginning," respectively.

Lemma 1: For instance (a, b, r), the following x_{off} is an optimal offline strategy. The optimal offline cost $OPT(t)$ is also given as follows. Let ε be an arbitrary positive number.
Case 1: if $0 < a \le b < 1$ and $0 < r < \frac{1-b}{1-a}$, then

$$x_{\text{off}} = \begin{cases} t + \varepsilon, & 0 \le t < \frac{b-ar}{1-r}; \\ 0, & \frac{b-ar}{1-r} \le t \end{cases}$$

and

$$OPT(t) = \begin{cases} t, & 0 \le t < \frac{b-ar}{1-r}; \\ r(t-a) + b, & \frac{b-ar}{1-r} \le t < a + \frac{1-b}{r}; \\ 1, & a + \frac{1-b}{r} \le t. \end{cases}$$

Case 2: if $0 < a \le b < 1$ and $\frac{1-b}{1-a} \le r$, then

$$x_{\text{off}} = \begin{cases} t + \varepsilon, & 0 \le t < 1; \\ 0, & 1 \le t \end{cases}$$

and

$$OPT(t) = \begin{cases} t, & 0 \le t < 1; \\ 1, & 1 \le t. \end{cases}$$

Case 3: if $0 \le b < a$ and $r < 1$, then

$$x_{\text{off}} = \begin{cases} t + \varepsilon, & 0 \le t < b; \\ 0, & b \le t \end{cases}$$

and

$$OPT(t) = \begin{cases} t, & 0 \le t < b; \\ b, & b \le t < a; \\ r(t-a) + b, & a \le t < a + \frac{1-b}{r}; \\ 1, & a + \frac{1-b}{r} \le t. \end{cases}$$

Case 4: if $0 \le b < a$ and $1 \le r$, then

$$x_{\text{off}} = \begin{cases} t + \varepsilon, & 0 \le t < b; \\ 0, & b \le t < a; \\ t - a, & a \le t < a - b + 1; \\ 0, & a - b + 1 \le t \end{cases}$$

and

$$OPT(t) = \begin{cases} t, & 0 \le t < b; \\ b, & b \le t < a; \\ t - a + b, & a \le t < a + 1 - b; \\ 1, & a + 1 - b \le t. \end{cases}$$

As a result, we can write $OPT(t)$ in a simple form.

Corollary 1: For any instance (a, b, r), it holds that

$$OPT(t) =$$
$$\begin{cases} \min(t, b), & 0 \le t < a; \\ \min(t, r(t-a) + b, t - a + b, 1), & a \le t. \end{cases}$$

Figures 2, 3, 4, and 5 show function $t \mapsto OPT(t)$ for instances belonging to Case 1, 2, 3, and 4 of Lemma 1, respectively, where the dashed lines are the cost of the "pay-as-you-go" plan from the beginning ($= t$), and the cost of "flat-rate" plan from the beginning ($= ON(0, t)$). Note that for Case 4 of Lemma 1, $OPT(t)$ is not the lower envelope of functions $t \mapsto t$ and $t \mapsto ON(0, t)$ (See Fig. 5). The reason is that it is an optimal offline strategy to take the "pay-as-you-go" plan for a while and then take the "flat-rate" plan, when the amount of data usage is between a and $a - b + 1$. This is a remarkable difference from the ski-rental problem.

Case 3 of Lemma 1 applies to instance $(a, b, r) = (\frac{1}{3}, \frac{2}{9}, \frac{2}{3})$. The lemma says that: If you know that you will
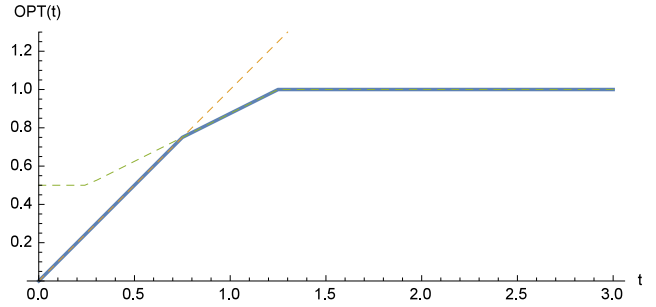

Fig. 2 Graph of $t \mapsto OPT(t)$ for instance $(a, b, r) = (\frac{1}{4}, \frac{1}{2}, \frac{1}{2})$ belonging to Case 1.
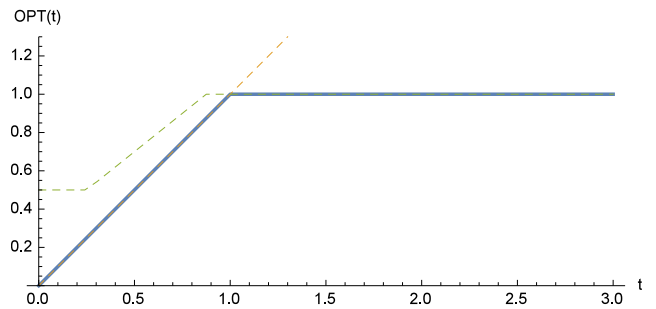

Fig. 3 Graph of $t \mapsto OPT(t)$ for instance $(a, b, r) = (\frac{1}{4}, \frac{1}{2}, \frac{4}{5})$ belonging to Case 2.

Fig. 4 Graph of $t \mapsto OPT(t)$ for instance $(a, b, r) = (\frac{1}{3}, \frac{2}{9}, \frac{2}{3})$ belonging to Case 3.

Fig. 5 Graph of $t \mapsto OPT(t)$ for instance $(a, b, r) = (\frac{1}{3}, \frac{2}{9}, \frac{3}{2})$ belonging to Case 4.

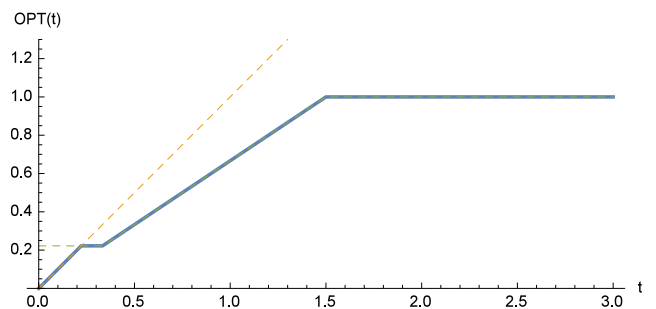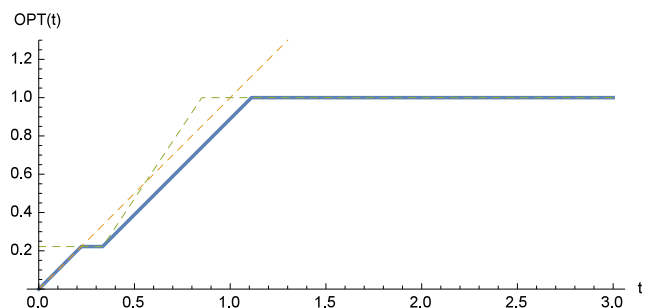use less than $\frac{2}{9}$ amount of data, you should take the "pay-as-you-go" plan and keep on it. If you know otherwise, you should take the "flat-rate" plan from the beginning. For this case, the optimal offline cost $OPT(t)$ is the lower envelope

Table 1 Behavior of function $x \mapsto ON(x,t)$ on $(-\infty, \infty)$ for an instance with $0 < r < 1$.

x			$t - a - \frac{1-b}{r}$		$t - a$	$\cdots$
$ON(x,t)$		$\nearrow$	$t - a - \frac{1-b}{r} + 1$	$\nearrow$	$t - a + b$	$\nearrow$

	$\cdots$	t	$t + \varepsilon$	
	$\nearrow$	$t + b$	t	$\rightarrow$

of $t \mapsto t$ and $t \mapsto ON(0,t)$. See Figs. 1 and 4.

Proof of Lemma 1: We find a strategy x_{off} that minimizes $ON(x,t)$ with t as a parameter. To this aim, we formally regard $ON(x,t)$ as a function of $x \in (-\infty, \infty)$ for a fixed t, not a function of t. The behavior of the function is roughly classified into two cases depending on the range of r. We focus on the interval $[0, \infty) \ni x$ and see what x achieves a minimum.

(I) For an instance with $0 < r < 1$, we know the behavior of $x \mapsto ON(x,t)$ on $(-\infty, \infty)$ as Table 1. Please recall (2). Apparently, the function is piece-wise linear and has a discontinuous point at $x = t$.

Here, ε is an arbitrary positive number. Note that $ON(x,t)$ is a constant on $x \in (t, \infty)$. The fact that $t + b \geq t$ leads us that the minimum value over $[0, \infty)$ is either $ON(0,t)$ for $x = 0$, or t for $x = t + \varepsilon$. We next see which case happens, by investigating what x satisfies $ON(x,t) = t$.

Case (I-a): $0 \leq b < a$. It holds that $t - a + b < t \leq t + b$. Hence, $ON(x,t)$ takes a value of t when $x = t - b \in (t - a, t]$, which means that the minimum value over $[0, \infty)$ varies depending on whether $t - b$ is contained in $[0, \infty)$ or not. We thus obtain

$$x_{\text{off}} = \begin{cases} t + \varepsilon, & 0 \leq t < b; \\ 0, & b \leq t. \end{cases}$$

(This case applies to instance $(a, b, r) = (\frac{1}{3}, \frac{2}{9}, \frac{2}{3})$. Figure 6 illustrates how to derive $x_{\text{off}} = 0$ when $t = \frac{3}{2} \geq b$.)

Case (I-b): $0 < a \leq b < 1$ and $0 < r < \frac{1-b}{1-a}$. Since $t - a - \frac{1-b}{r} + 1 < t \leq t - a + b$, the equality $ON(x,t) = t$ holds for $x = t - \frac{b-ar}{1-r} \in (t - a - \frac{1-b}{r}, t - a]$. The minimum value is determined by whether $t - \frac{b-ar}{1-r}$ is in $[0, \infty)$ or not. We get

$$x_{\text{off}} = \begin{cases} t + \varepsilon, & 0 \leq t < \frac{b-ar}{1-r}; \\ 0, & \frac{b-ar}{1-r} \leq t. \end{cases}$$

Case (I-c): $0 < a \leq b < 1$ and $\frac{1-b}{1-a} \leq r < 1$. By the inequality $t \leq t - a - \frac{1-b}{r} + 1$, we know that $ON(x,t) = t$ occurs when $x = t - 1 \in (-\infty, t - a - \frac{1-b}{r}]$. Thus,

$$x_{\text{off}} = \begin{cases} t + \varepsilon, & 0 \leq t < 1; \\ 0, & 1 \leq t. \end{cases}$$

(II) For an instance with $1 \leq r$, the behavior of $x \mapsto ON(x,t)$ on $(-\infty, \infty)$ is a bit more complicated as shown in Table 2. The function has a minimal point of $t - a + b$ at $x = t - a$. In addition to a similar analysis to (I), we have to
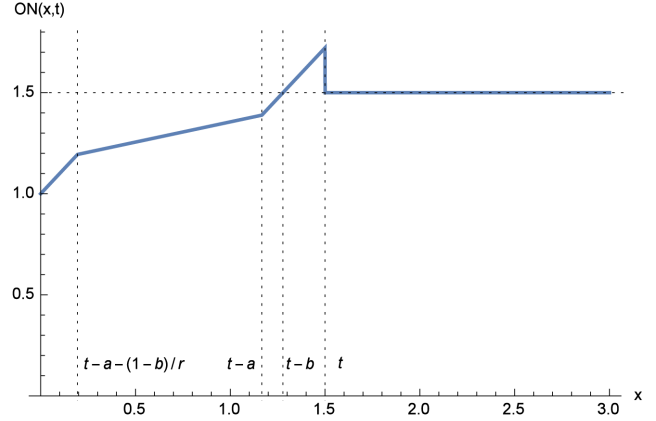

Fig. 6 Graph of $x \mapsto ON(x,t)$ for instance $(a, b, r) = (\frac{1}{3}, \frac{2}{9}, \frac{2}{3})$. Here we set $t = \frac{3}{2} (\geq b)$. Then, the function is minimized at $x = 0$. (Please note that this is a graph of a function of x, not of t.)

Table 2 Behavior of function $x \mapsto ON(x,t)$ on $(-\infty, \infty)$ for an instance with $1 \leq r$.

x			$t - a - \frac{1-b}{r}$		$t - a$	$\cdots$
$ON(x,t)$		$\nearrow$	$t - a - \frac{1-b}{r} + 1$	$\searrow$	$t - a + b$	$\nearrow$

	$\cdots$	t	$t + \varepsilon$	
	$\nearrow$	$t + b$	t	$\rightarrow$

compare this minimal value with t.

Case (II-a): $0 \leq b < a$. Since $t - a + b < t$, it happens that the minimal value $t - a + b$ is a minimum. We should note that the equality $ON(x,t) = t - a + b$ holds also for $x = t - a + b - 1$. By focusing on the interval $[0, \infty)$, we obtain

$$x_{\text{off}} = \begin{cases} t + \varepsilon, & 0 \leq t < b; \\ 0, & b \leq t < a; \\ t - a, & a \leq t < a - b + 1; \\ 0, & a - b + 1 \leq t. \end{cases}$$

Case (II-b): $0 < a \leq b < 1$. The minimal value $t - a + b$ cannot be a unique minimum because $t \leq t - a + b$. Function $ON(x,t)$ takes a value of t for $x = t - 1 \in (-\infty, t - a - \frac{1-b}{r}]$. Therefore,

$$x_{\text{off}} = \begin{cases} t + \varepsilon, & 0 \leq t < 1; \\ 0, & 1 \leq t. \end{cases}$$

The statement of the lemma is derived as follows: Case 1 is from Case (I-b). Case 2 is from Case (I-c) together with Case (II-b). Case 3 is from Case (I-a). And, Case 4 is from Case (II-a). Each $OPT(t)$ is obtained immediately by applying the strategy. ∎

4. A Lemma for Easily Calculating the Competitive Ratio

As mentioned before, for any strategy with $x > 0$, the competitive ratio R_x is the maximum value of $\frac{ON(x,t)}{OPT(x)}$ over all

$t > 0$. A straightforward approach is to check the value of function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ for all possible values of t. The following lemma states that it suffices to evaluate the function values just for $t = x, x + a + \frac{1-b}{r}$, and a.

Lemma 2: For any strategy with $x > 0$, it holds that

$$R_x =$$
$$\max \left\{ \frac{ON(x,x)}{OPT(x)}, \frac{ON(x, x + a + \frac{1-b}{r})}{OPT(x + a + \frac{1-b}{r})}, \frac{ON(x,a)}{OPT(a)} \right\}.$$

Proof: We give a sketch of the proof. We first show that function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ is piece-wise monotone, which implies that the maximum occurs at an endpoint of some interval. We then conclude that only the three points $t = x, x + a + \frac{1-b}{r}$, and a can be such an endpoint, through checking all possible endpoints. In this proof we drop the case $t = 0$, since for strategy $x > 0$, any c satisfies $0 = ON(x,0) \le c \cdot OPT(0) = 0$.

(I) We look at the domain on which function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ is defined. We start from function $t \mapsto OPT(t)$. Suppose that the given instance corresponds to Case i of Lemma 1, where i is either of 1, 2, 3, or 4. Lemma 1 says that function $t \mapsto OPT(t)$ is a linear function, specifically, either an increasing linear function or a constant, on each interval in V_i, where

$$V_1 = \{(0, \frac{b-ar}{1-r}), [\frac{b-ar}{1-r}, 1), [1, \infty)\},$$

$$V_2 = \{(0, 1), [1, \infty)\},$$

$$V_3 = \{(0, b), [b, a), [a, a + \frac{1-b}{r}), [a + \frac{1-b}{r}, \infty)\},$$

and

$$V_4 = \{(0, b), [b, a), [a, a + 1 - b), [a + 1 - b, \infty)\}.$$

It is immediately derived that for any case, function $t \mapsto OPT(t)$ is continuous everywhere on $(0, \infty)$.

On the other hand, we know from (2) that function $t \mapsto ON(x,t)$ is either an increasing linear function or a constant on each interval in

$$U = \{(0, x), [x, x + a), [x + a, x + a + \frac{1-b}{r}),$$
$$[x + a + \frac{1-b}{r}, \infty)\}.$$

We know that function $t \mapsto ON(x,t)$ is discontinuous at $t = x$ and continuous elsewhere.

Consequently, function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ is explicitly expressed for each interval $A \cap B$ with $A \in U$, $B \in V_i$, and $A \cap B \ne \emptyset$. We claim that on any of such interval, $t \mapsto \frac{ON(x,t)}{OPT(t)}$ is monotone. In what follows we deal comprehensively with Cases 1, 2, 3, and 4. Please notice that involved intervals vary according to the case.

It is obvious that for $A = [x, x + a)$ or $A = [x + a + \frac{1-b}{r}, \infty)$, function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ is monotone on $A \cap B$; whereas $t \mapsto ON(x,t)$ is a constant, $t \mapsto OPT(t)$ is an increasing function or a constant. The rest to be checked are $A = (0, x)$

and $A = [x + a, x + a + \frac{1-b}{r})$. If B is an interval on which $t \mapsto OPT(t)$ is a constant, then function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ is monotone. Otherwise, we can write, using constants α, β, γ, and δ, as

$$\frac{ON(x,t)}{OPT(t)} = \frac{\alpha t + \beta}{\gamma t + \delta},$$

where $\gamma t + \delta > 0$ for $t \in A \cap B$. This turns out be monotone, since

$$\frac{\alpha t + \beta}{\gamma t + \delta} = \frac{\alpha}{\gamma} + \frac{\beta - \frac{\alpha \delta}{\gamma}}{\gamma t + \delta}.$$

(II) We have known that a maximum is never achieved at an inner point of some interval $A \cap B$. We are going to check each endpoint of intervals. Obvious candidates for a maximizer are $t = x$ and $t = x + a + \frac{1-b}{r}$. Since function $t \mapsto ON(x,t)$ is discontinuous at $t = x$, so is function $t \mapsto \frac{ON(x,t)}{OPT(t)}$. See that for any case, $\frac{ON(x,t)}{OPT(t)}$ is a constant on $[x + a + \frac{1-b}{r}, \infty)$. Although $\frac{ON(x,t)}{OPT(t)}$ is also a constant on the left most interval, that is $(0, x) \cap (0, a) \cap (0, \frac{b-ar}{1-r})$, the interval cannot be a candidate since the constant is one.

(III) We show that function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ can be maximum at $t = a$ for some cases. For Case 3 or 4, function $t \mapsto OPT(t)$ is a constant on the left neighborhood of $t = a$, and an increasing function on the right neighborhood. Therefore, if $t \mapsto ON(x,t)$ is a constant around $t = a$, which occurs when $A = [x, x + a) \ni a$, then function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ is a constant on the left neighborhood of $t = a$ and an decreasing function on the right neighborhood. This fact implies that $\frac{ON(x,t)}{OPT(t)}$ can be maximum at $t = a$.

(IV) The last task is to state that function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ cannot achieve a maximum at any other endpoint: $t = \frac{b-ar}{1-r}, b, 1, a + 1 - b, a + \frac{1-b}{r}$, and $x + a$. We first discuss $t = \frac{b-ar}{1-r}$ which appears in Case 1. Function $t \mapsto OPT(t)$ is an increasing linear function on both of the left and right neighborhoods of $t = \frac{b-ar}{1-r}$. For $A = [x, x + a)$, which lets $ON(x,t)$ be a constant, function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ decreases on the left neighborhood. Hence, $t = \frac{b-ar}{1-r}$ cannot be a maximizer, since there is t in the left neighborhood such that the value of $\frac{ON(x,t)}{OPT(t)}$ becomes larger. For $A = (0, x) \ni \frac{b-ar}{1-r}$, we have for the right neighborhood

$$\frac{ON(x,t)}{OPT(t)} = \frac{t}{r(t-a)+b} = \frac{1}{r} - \frac{b-ar}{r} \cdot \frac{1}{r(t-a)+b},$$

which is an increasing function. This implies that $t = \frac{b-ar}{1-r}$ cannot be a maximizer. For $A = [x+a, x+a+\frac{1-b}{r}) \ni \frac{b-ar}{1-r}$, we have a decreasing function for the left neighborhood:

$$\frac{ON(x,t)}{OPT(t)} = \frac{r(t-(x+a))+x+b}{t} = r + \frac{b-ar+(1-r)x}{t},$$

since $b - ar > 0$ for Case 1. Thus, $t = \frac{b-ar}{1-r}$ cannot yield a maximum. Note that $A = [x + a + \frac{1-b}{r}, \infty) \ni \frac{b-ar}{1-r}$ does not happen because $\frac{b-ar}{1-r} < a + \frac{1-b}{r} < x + a + \frac{1-b}{r}$.

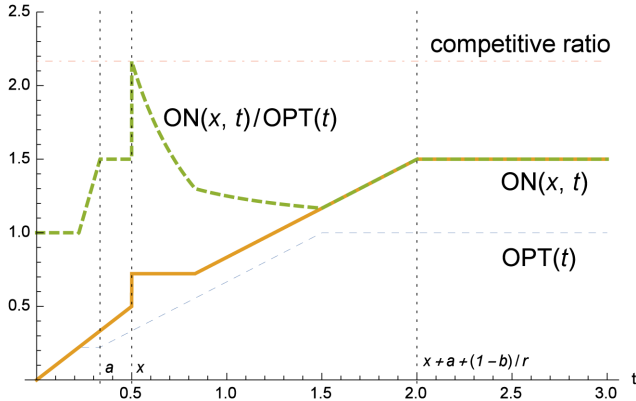We next see $t = b, 1, a + 1 - b$, and $a + \frac{1-b}{r}$ together.

Fig. 7 The cost incurred by strategy $x = \frac{1}{2}$, denoted as $ON(x,t)$, for instance $(a,b,r) = (\frac{1}{3}, \frac{2}{9}, \frac{2}{3})$, when the amount of data usage is t. Three vertical lines are drawn at $t = x$, $x + a + \frac{1-b}{r}$, and a, which are candidates for a maximizer of $\frac{ON(x,t)}{OPT(t)}$.

(Please think that the corresponding case applies to the instance.) Consider each one of these points and denote it by $\hat{t}$ below. Function $t \mapsto OPT(t)$ is an increasing function on the left neighborhood of $\hat{t}$ and a constant on the right neighborhood. For $A = (0, x) \ni \hat{t}$ or $A = [x + a, x + a + \frac{1-b}{r}) \ni \hat{t}$, we immediately know that function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ increases on the right neighborhood. For $A = [x, x + a) \ni \hat{t}$, function $t \mapsto ON(t)$ is a constant. Then, function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ turns out to be a decreasing function on the left neighborhood of $\hat{t}$. Therefore, function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ does not take a maximum at either of $t = b, 1, a + 1 - b$, or $a + \frac{1-b}{r}$.

We check $t = x + a$ similarly. If B is an interval on which function $t \mapsto OPT(t)$ is a constant, function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ increases on the right neighborhood. Otherwise, function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ decreases on the right neighborhood. In this way we confirm that $t \mapsto \frac{ON(x,t)}{OPT(t)}$ never takes a maximum at $t = x + a$.

We add that $t = x + a$ may coincide with some of $t = \frac{b-ar}{1-r}, b, 1, a+1-b$, or $a + \frac{1-b}{r}$. For any of these cases, it holds that function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ decreases on the left neighborhood, or increases on the right neighborhood. Therefore, function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ does not achieve a maximum at $t = x + a$. □

We demonstrate how to apply Lemma 2 for instance $(a,b,r) = (\frac{1}{3}, \frac{2}{9}, \frac{2}{3})$. Consider the competitive ratio of a strategy $x = \frac{1}{2}$ for example. We calculate $(\frac{ON(x,x)}{OPT(x)}, \frac{ON(x,x+a+\frac{1-b}{r})}{OPT(x+a+\frac{1-b}{r})}, \frac{ON(x,a)}{OPT(a)}) = (\frac{13}{6}, \frac{3}{2}, \frac{3}{2})$. Applying Lemma 2, we know the competitive ratio $R_x = \frac{13}{6} \approx 2.16667$. Figure 7 graphically explains that the three values are the candidates of the maximum of function $t \mapsto \frac{ON(x,t)}{OPT(t)}$ for this example.

5. Optimal Online Strategy for a Typical Class of Instances

In this section we consider a class of instances (a,b,r) satisfying $b = ar$. We believe that such an instance is typical in

Table 3 Optimal strategy for instance (a,b,r) with $b = ar$, and the competitive ratio.

range of r and a	optimal strategy $\bar{x}$	competitive ratio R_x
$0 < r < 1$ $0 < a \leq 2 - \frac{1}{r}$	$\frac{1-r+\sqrt{(1-r)^2+4ar^2}}{2r}$	$1 + \frac{1-r+\sqrt{(1-r)^2+4ar^2}}{2r}$
$0 < r < 1$ $2 - \frac{1}{r} < a < \frac{1}{r}$	ar	2
$1 \leq r$ $0 < a \leq \frac{(1-r)^2}{r^3}$	$1 - \frac{1}{r}$	$2 - \frac{1}{r}$
$1 \leq r$ $\frac{(1-r)^2}{r^3} < a < \frac{1}{r}$	$\sqrt{ar}$	$1 + \sqrt{ar}$

the real world, because equality $b = ar$ is quite reasonable for us in a sense that "the initial minimum fee b is later appropriated for fee of data communication at rate r." In other words, the initial minimum fee is worth paying as long as you are going to use at least a amount of data. As far as we consulted websites, all the "flat-rate" plans in fact belong to this class.

We obtain a family of optimal strategies for this class of instances with the help of Lemma 2. Since the definition of the problem requires $b < 1$, an implicit condition of $a < \frac{1}{r}$ is imposed.

Theorem 1: Strategy $\bar{x}$ in Table 3 is optimal for instance (a,b,r) with $b = ar$, depending on the range of a and r. Its competitive ratio is as given in Table 3.

Proof: We show the theorem by finding a strategy x of minimum competitive ratio, with the help of Lemma 2. Note that for instance (a,b,r) with $b = ar$, it holds that $\frac{ON(x,x+a+\frac{1-ar}{r})}{OPT(x+a+\frac{1-ar}{r})} = \frac{ON(x,x+\frac{1}{r})}{OPT(x+\frac{1}{r})}$. Thus, the function to be minimized is now $x \mapsto \max\{\frac{ON(x,x)}{OPT(x)}, \frac{ON(x,x+\frac{1}{r})}{OPT(x+\frac{1}{r})}, \frac{ON(x,a)}{OPT(a)}\}$, which we denote by $f(x)$. We consider the two different cases of $0 < r < 1$ and $1 \leq r$.

(I) For an instance with $0 < r < 1$, we analyze function $f(x)$ separately on domains $(0, a)$ and $[a, \infty)$. (I-a) By definition of ON and OPT, we have for domain $(0, a)$,

$$\frac{ON(x,x)}{OPT(x)} = \begin{cases} \frac{x+ar}{x}, & 0 < x < ar; \\ \frac{x+ar}{ar}, & ar \leq x < a, \end{cases}$$

$$\frac{ON(x, x + \frac{1}{r})}{OPT(x + \frac{1}{r})} = x + 1,$$

and

$$\frac{ON(x,a)}{OPT(a)} = \frac{x + ar}{ar}.$$

Since $\frac{x+ar}{ar} = \frac{x}{ar} + 1 > x + 1$ for $0 < x < a$ and $\frac{x+ar}{ar} > \frac{x+ar}{x}$ for $0 < x < ar$, $\frac{ON(x,x)}{OPT(x)}$ is the largest among the three functions everywhere on $(0, a)$. (See Fig. 8 for instance $(a,b,r) = (\frac{1}{3}, \frac{2}{9}, \frac{2}{3})$, which satisfies $b = ar$.) Therefore, it suffices to consider only $\frac{ON(x,x)}{OPT(x)}$. Obviously, function $x \mapsto \frac{ON(x,x)}{OPT(x)}$ is a decreasing function on $(0, ar)$ and an increasing function on (ar, a). Hence, the minimum value of
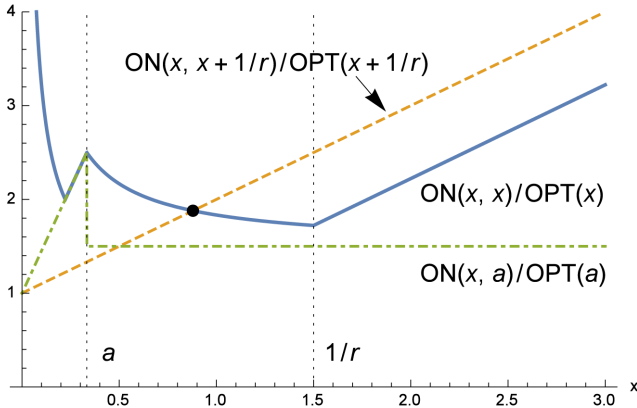
Fig. 8 Graphs of $\frac{ON(x,x)}{OPT(x)}$, $\frac{ON(x,x+\frac{1}{r})}{OPT(x+\frac{1}{r})}$, and $\frac{ON(x,a)}{OPT(a)}$ for instance $(a,b,r) = (\frac{1}{3}, \frac{2}{9}, \frac{2}{3})$. The point indicates the minimum value of the three functions: $1 + \frac{1-r+\sqrt{(1-r)^2+4ar^2}}{2r}$ at $x = \frac{1-r+\sqrt{(1-r)^2+4ar^2}}{2r}$.

$f(x)$ over domain $(0,a)$ is 2 at $x = ar$.

(I-b) On domain $[a, \infty)$, we derive

$$\frac{ON(x,x)}{OPT(x)} = \begin{cases} \frac{x+ar}{rx}, & a \le x < \frac{1}{r}; \\ x + ar, & \frac{1}{r} \le x, \end{cases}$$

$$\frac{ON(x, x + \frac{1}{r})}{OPT(x + \frac{1}{r})} = x + 1,$$

and

$$\frac{ON(x,a)}{OPT(a)} = \frac{1}{r}.$$

We first find a minimizer of function $x \mapsto \max\{\frac{ON(x,x)}{OPT(x)}, \frac{ON(x,x+\frac{1}{r})}{OPT(x+\frac{1}{r})}\}$ on domain $[a, \infty)$. We calculate an intersection of the graphs of $x \mapsto \frac{x+ar}{rx}$ and $x \mapsto x + 1$. By formally solving equation $\frac{x+ar}{rx} = x + 1$, we get a positive root of $x = \frac{1-r+\sqrt{(1-r)^2+4ar^2}}{2r}$. The following observation confirms that this root is on domain $[a, \frac{1}{r})$. When $x = a$, $x \mapsto \frac{x+ar}{rx}$ is above $x \mapsto x + 1$, since $\frac{a+ar}{ar} = 1 + \frac{1}{r} > 1 + a$. On the other hand, when $x = \frac{1}{r}$, $x \mapsto \frac{x+ar}{rx}$ is below $x \mapsto x + 1$, since $\frac{\frac{1}{r}+ar}{r \cdot \frac{1}{r}} = \frac{1}{r} + ar < \frac{1}{r} + 1$. (See Fig. 8.) Besides, $x \mapsto \frac{x+ar}{rx}$ is decreasing whereas $x \mapsto x + 1$ is increasing. Thus, function $x \mapsto \max\{\frac{ON(x,x)}{OPT(x)}, \frac{ON(x,x+\frac{1}{r})}{OPT(x+\frac{1}{r})}\}$ achieves a minimum of $1 + \frac{1-r+\sqrt{(1-r)^2+4ar^2}}{2r}$ at $x = \frac{1-r+\sqrt{(1-r)^2+4ar^2}}{2r}$.

Our task is to find a minimum of $f(x)$. Inequality $1 + \frac{1-r+\sqrt{(1-r)^2+4ar^2}}{2r} > 1 + \frac{1-r+\sqrt{(1-r)^2}}{2r} = 1 + \frac{2(1-r)}{2r} = \frac{1}{r}$ tells that the value of $\frac{ON(x,a)}{OPT(a)}$ for $x = \frac{1-r+\sqrt{(1-r)^2+4ar^2}}{2r}$ is smaller than the obtained minimum of $\max\{\frac{ON(x,x)}{OPT(x)}, \frac{ON(x,x+\frac{1}{r})}{OPT(x+\frac{1}{r})}\}$. Hence, it follows that the minimum value of $f(x)$ over $[a, \infty)$ domain is also $1 + \frac{1-r+\sqrt{(1-r)^2+4ar^2}}{2r}$ at $x = \frac{1-r+\sqrt{(1-r)^2+4ar^2}}{2r}$.

From (I-a) and (I-b), we know that the minimum value of $f(x)$ is achieved on either of the domains, depending on

the values of a and r. Some basic calculation leads us that the value of $1 + \frac{1-r+\sqrt{(1-r)^2+4ar^2}}{2r}$ is larger than 2 if and only if $a > 2 - \frac{1}{r}$. The optimal strategy for $0 < r < 1$ in Theorem 1 has been obtained in this way.

(II) For an instance with $1 \le r$, we have

$$\frac{ON(x,x)}{OPT(x)} = \begin{cases} \frac{x+ar}{x}, & 0 < x < 1; \\ x + ar, & 1 \le x, \end{cases}$$

$$\frac{ON(x, x + \frac{1}{r})}{OPT(x + \frac{1}{r})} = \begin{cases} \frac{x+1}{x}, & 0 < x < 1 - \frac{1}{r}; \\ x + 1, & 1 - \frac{1}{r} \le x, \end{cases}$$

and

$$\frac{ON(x,a)}{OPT(a)} = \begin{cases} \frac{x+ar}{a}, & 0 < x \le a; \\ 1, & a < x. \end{cases}$$

We first state that $\frac{ON(x,x)}{OPT(x)} \ge \frac{ON(x,a)}{OPT(a)}$ holds for every x, which allows us to forget function $x \mapsto \frac{ON(x,a)}{OPT(a)}$. This statement is justified as follows. On domain $(0, a)$, $x \mapsto \frac{ON(x,x)}{OPT(x)}$ decreases due to $ar > 0$, while $x \mapsto \frac{ON(x,a)}{OPT(a)}$ increases. Only at $x = a$, the function values coincide. For domain (a, ∞), the minimum value of $\frac{ON(x,x)}{OPT(x)}$ is $1 + ar > 1 = \frac{ON(x,a)}{OPT(a)}$.

We thus consider only $\frac{ON(x,x)}{OPT(x)}$ and $\frac{ON(x,x+\frac{1}{r})}{OPT(x+\frac{1}{r})}$. It is immediately derived that the minimum value of $x \mapsto \frac{ON(x,x+\frac{1}{r})}{OPT(x+\frac{1}{r})}$ is $2 - \frac{1}{r}$ at $x = 1 - \frac{1}{r}$. Hence, if the value of $\frac{ON(x,x)}{OPT(x)}$ for $x = 1 - \frac{1}{r}$ does not exceed $2 - \frac{1}{r}$, then the minimum is a minimum of $f(x)$ as well. Solving inequality

$$2 - \frac{1}{r} \ge \frac{ON(1 - \frac{1}{r}, 1 - \frac{1}{r})}{OPT(1 - \frac{1}{r})} = 1 - \frac{ar^2}{1-r}, \qquad (4)$$

we know that this occurs if and only if $a \le \frac{(1-r)^2}{r^3}$.

What remains is to analyze the case where $a > \frac{(1-r)^2}{r^3}$. We show that $x \mapsto \frac{ON(x,x+\frac{1}{r})}{OPT(x+\frac{1}{r})}$ and $x \mapsto \frac{ON(x,x)}{OPT(x)}$ have a unique intersection on interval $(1 - \frac{1}{r}, 1)$ and that $f(x)$ achieves a minimum at that point. We have already known that when $a > \frac{(1-r)^2}{r^3}$, inequality (4) does not hold. That is to say, $\frac{ON(x,x+\frac{1}{r})}{OPT(x+\frac{1}{r})} < \frac{ON(x,x)}{OPT(x)}$ for $x = 1 - \frac{1}{r}$. On the other hand, it holds that for $x = 1$, $\frac{ON(x,x+\frac{1}{r})}{OPT(x+\frac{1}{r})} = 2 > 1 + ar = \frac{ON(x,x)}{OPT(x)}$. Therefore, the two functions have a unique intersection on interval $(1 - \frac{1}{r}, 1)$.

Function $x \mapsto \frac{ON(x,x)}{OPT(x)}$ decreases on $(0, 1 - \frac{1}{r})$, while $x \mapsto \frac{ON(x,x+\frac{1}{r})}{OPT(x+\frac{1}{r})}$ increases on $(1 - \frac{1}{r}, \infty)$. Hence, the obtained intersection is indeed a minimizer of $f(x)$. Solving equation $\frac{x+ar}{a} = x + 1$, we get a root of $x = \sqrt{ar}$, for which $f(x)$ takes a value of $1 + \sqrt{ar}$. □

The proof may not seem short. However, one should note that a naive analysis is much more exhausting. Lemma 2 enables us to skip many steps of classification.
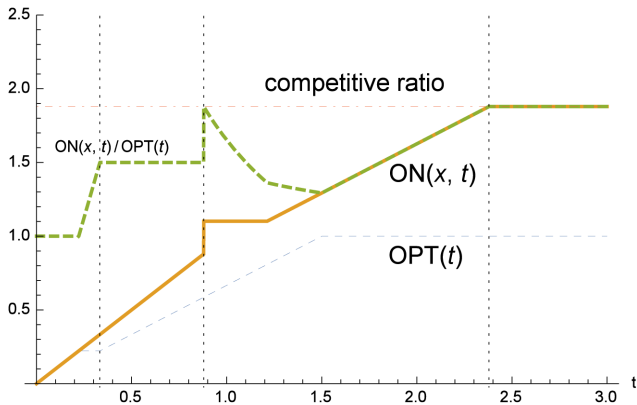
Fig. 9 Optimal strategy $\overline{x} = \frac{3+\sqrt{57}}{12} \approx 0.87915$ for instance $(a, b, r) = (\frac{1}{3}, \frac{2}{9}, \frac{2}{3})$. The competitive ratio $R_{\overline{x}} = 1 + \frac{3+\sqrt{57}}{12} \approx 1.87915$.

Evaluating the values of the competitive ratios in Table 3, we can immediately have the following corollary. The upper bound of two here comes from the competitive ratio of the 2-competitive optimal strategy for the ski-rental problem: "buy skis when the player has so far spent a cost of the price of skis for renting skis."

Corollary 2: For any instance (a, b, r) with $b = ar$, the competitive ratio of any optimal strategy is no larger than two.

The numerical example $(a, b, r) = (\frac{1}{3}, \frac{2}{9}, \frac{2}{3})$ which we have used so far in fact belongs to this class because $\frac{2}{9} = \frac{1}{3} \cdot \frac{2}{3}$. Since $0 < r < 1$ and $0 < a \leq 2 - \frac{1}{r}$, we have $\overline{x} = \frac{1-r+\sqrt{(1-r)^2+4ar^2}}{2r} = \frac{3+\sqrt{57}}{12} \approx 0.87915$ and $R_{\overline{x}} = 1 + \frac{3+\sqrt{57}}{12} \approx 1.87915$. It is confirmed that $(\frac{ON(\overline{x},\overline{x})}{OPT(\overline{x})}, \frac{ON(\overline{x},\overline{x}+a+\frac{1-b}{r})}{OPT(\overline{x}+a+\frac{1-b}{r})}, \frac{ON(\overline{x},a)}{OPT(a)}) = (1 + \frac{3+\sqrt{57}}{12}, 1 + \frac{3+\sqrt{57}}{12}, \frac{3}{2})$. See Fig. 9.

6. Discussion

The analysis in this paper finds an optimal strategy among those such as "start by the pay-as-you-go plan, keep it for a while, and then take the flat-rate plan." We should remark that this setting is fairly strong. Consider an instance with $a > b$, $b < \frac{1}{2}$, and $b < ar$. If it is allowed to switch plans arbitrary times, an optimal offline strategy is to repeatedly take the "flat-rate". More specifically, the offline player will take the "flat-rate" while using a amount of data, take the "pay-as-you-go" just a short while, and then take the "flat-rate" again. For example, when the offline player has used $2a$ amount of data, it pays about $2b$, which is below our $OPT(2a) = \min(b + ar, 1)$. For such a setting, we will need to carry out an analysis more like the parking permit problem [3].

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