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Examples of conditional SIC-POVMs

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Abstract The state of a quantum system is a density matrix with several parameters. The concern herein is how to recover the parameters. Several possibilities exist for the optimal recovery method, and we consider some special cases. We assume that a few parameters are known and that the others are to be recovered. The optimal positive-operator-valued measure (POVM) for recovering unknown parameters with an additional condition is called a conditional symmetric informationally complete POVM (SIC-POVM). In this paper, we study the existence or nonexistence of conditional SIC-POVMs. We provide a necessary condition for existence and some examples.

Keywords Quantum state tomography · SIC-POVM · Conditional SIC-POVM

1 Introduction

Positive-operator-valued measures (POVMs) are motivated by quantum information theory. A POVM is a set $\{F_i\}_{i=1}^k \subset M_n(\mathbb{C})$ of positive operators such that $\sum_i F_i = I$. A density matrix $\rho \in M_n(\mathbb{C})$ can be (partially) informed by the probability distribution $\{\mathrm{Tr}\, \rho F_i\}_{i=1}^k$. A density matrix $\rho \in M_n(\mathbb{C})$ has $n^2 - 1$ real parameters. To recover all

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of the parameters, $k \geq n^2$ must hold for the POVM. We can take rank-one projections P_i, $1 \leq i \leq n^2$, such that

$$\sum_{i=1}^{n^2} P_i = nI, \qquad \mathrm{Tr}\, P_i P_j = \frac{1}{n+1} \quad (i \neq j), \qquad F_i = \frac{1}{n} P_i$$

comprise a symmetric informationally complete POVM (SIC-POVM), also known as an equiangular tight frame [8,25–29]. This currently popular idea [1,2,4,10,20,33] was defined by Zauner [31,32]. Zauner proved the existence when $n \leq 5$, and there has been further mathematical and numerical discussions on the same [5,23]. Whether a SIC-POVM exists for every dimension is not known. We can also consider fewer than n^2 projections with similar properties.

A SIC-POVM $\{F_i\}_{i=1}^{n^2}$ of an n-level system is relevant for quantum state tomography [2,15,17,19,21] and is optimal for several arguments; for example, SIC-POVMs are optimal for linear quantum state tomography. The minimization of the determinant of the average covariance matrix was studied in [17], and the minimization of the square of the Hilbert–Schmidt distance between the estimation and the true density was investigated in [21].

However, if some of the $n^2 - 1$ parameters of a density matrix ρ are known, a SIC-POVM is not the optimal POVM for linear quantum state tomography for such ρ. It is obvious that the optimal POVM depends on the known parameters. The optimal POVM in such a case is studied in [18,22]. A set of projectors P_i, $1 \leq i \leq N$ satisfying

$$\sum_{i=1}^{N} P_i = \frac{N}{n} I, \qquad \mathrm{Tr}\, P_i P_j = \frac{N-n}{n(N-1)} \quad (i \neq j), \qquad F_i = \frac{n}{N} P_i,$$

where N is the dimension of the subspace corresponding to the unknown parameter, comprises a conditional SIC-POVM [18]. A conditional SIC-POVM is the optimal POVM for linear quantum state tomography in this case.

The existence of a conditional SIC-POVM is a fundamental question. SIC-POVMs and mutually unbiased bases (MUBs) involve similar problem. Neither the existence of SIC-POVMs in higher-dimensional Hilbert spaces nor the nonexistence of MUBs in a six-dimensional Hilbert space has been proved. On the other hand, conditional SIC-POVMs are considered as equiangular tight frames. The existence of equiangular tight frames is also studied in many literature (e.g., [8,26–28]).

The main result of this paper, Theorem 2, presents a necessary condition for the existence of a conditional SIC-POVM. Using the necessary condition, we show some examples of nonexistence of conditional SIC-POVMs. Some other examples of conditional SIC-POVMs are also considered in Sect. 4.

2 The optimality of conditional SIC-POVMs

In this section, we survey the optimal POVMs for linear quantum state tomography and conditional SIC-POVMs according to [18,21,22].

A quantum state (or a density matrix) $\rho \in M_n(\mathbb{C})$ satisfies the conditions $\mathrm{Tr}\,\rho = 1$ and $\rho \geq 0$. We decompose $M_n(\mathbb{C})$ into three orthogonal self-adjoint subspaces:

$$M_n(\mathbb{C}) = A \oplus B \oplus C, \tag{1}$$

where $A := \{\lambda I : \lambda \in \mathbb{C}\}$ is one dimensional. Denote the orthogonal projections onto the subspaces A, B, C by $\mathbf{A}, \mathbf{B}, \mathbf{C}$. A density matrix $\rho \in M_n(\mathbb{C})$ has the form

$$\rho = \frac{1}{n} I + \mathbf{B}\rho + \mathbf{C}\rho.$$

Assume that $\mathbf{B}\rho$ and $\mathbf{C}\rho$ are the known and unknown traceless parts of ρ, respectively, and that $\mathbf{C}\rho$ is to be estimated.

We use the notation $\rho_* = \rho - \mathbf{B}\rho$. The aim of quantum state tomography is to recover ρ_*. Though there exist several methods of quantum state tomography, we only consider linear quantum state tomography (see Remark 1 or [21] for details).

If the dimension of B is m, then the dimension of C is $n^2 - m - 1$. To reconstruct ρ_*, we must use a POVM with at least $N = n^2 - m$ elements. Additionally, we assume that a POVM $\{F_i\}_{i=1}^k$ is in $A \oplus C$ according to [18].

An informationally complete POVM is a POVM with the property that each quantum state is uniquely determined by its measurement statistics.

Definition 1 A POVM $\{F_i\}_{i=1}^k \subset A \oplus C$ is informationally complete with respect to $A \oplus C$, if for each pair of states $\rho, \sigma \in M_n(\mathbb{C})$ with $\rho_* \neq \sigma_*$ there exists i such that $\mathrm{Tr}\,(\rho_* F_i) \neq \mathrm{Tr}\,(\sigma_* F_i)$.

We remark $\mathrm{Tr}\,(\rho F_i) = \mathrm{Tr}\,(\rho_* F_i) + \mathrm{Tr}\,((\mathbf{B}\rho) F_i) = \mathrm{Tr}\,(\rho_* F_i)$.

The next proposition follows easily.

Proposition 1 *A POVM $\{F_i\}_{i=1}^k \subset A \oplus C$ is informationally complete with respect to $A \oplus C$ if and only if* $\mathrm{span}\{F_i\}_{i=1}^k = A \oplus C$.

If a POVM $\{F_i\}_{i=1}^k \subset A \oplus C$ is informationally complete w.r.t. $A \oplus C$, then there exists a set of self-adjoint operators $\{Q_i\}_{i=1}^k \subset M_n(\mathbb{C})$ satisfying

$$\rho_* = \sum_{i=1}^k \mathrm{Tr}\,(\rho F_i) Q_i \tag{2}$$

for any state $\rho \in M_n(\mathbb{C})$. We call $\{Q_i\}_{i=1}^k$ a dual frame of $\{F_i\}_{i=1}^k$, though this is slightly different from the original definition. Define the superoperator $\mathbf{F}$ on $M_n(\mathbb{C})$ by

$$\mathbf{F} = \sum_{i=1}^k \frac{1}{\mathrm{Tr}\,F_i} |F_i\rangle\langle F_i|,$$

which is invertible on $\mathrm{supp}\mathbf{F} = A \oplus C$, where $|F_i\rangle\langle F_i|X = \mathrm{Tr}\,(F_i X) F_i$ for all $X \in M_n(\mathbb{C})$. Then, the set of

$$Q_i = \mathbf{F}^{-1} \frac{1}{\mathrm{Tr}\, F_i} F_i$$

is a dual frame of $\{F_i\}_{i=1}^k$, and is called the canonical dual frame of $\{F_i\}_{i=1}^k$.

If a measurement corresponding to $\{F_i\}_{i=1}^k$ is performed on a system in a state corresponding to $\rho \in M_n(\mathbb{C})$, then the probability of obtaining a measurement value i is

$$p_i = \mathrm{Tr}\,(\rho F_i).$$

Assume that $y_1, \ldots, y_M$ are outcomes of measurements on M identical copies of the state ρ, with the result that $y_i \in \{1, 2, \ldots, k\}$. Let x_j be the number of outcomes j, i.e., $x_j = |\{i \,:\, y_i = j\}|$. An approximate value of p_i is

$$\hat{p}_i = \frac{1}{M} x_i. \tag{3}$$

Then, our estimate of ρ_* is

$$\hat{\rho}_* = \sum_{i=1}^{k} \hat{p}_i Q_i. \tag{4}$$

The error measured by the square of the Hilbert–Schmidt norm is

$$\|\rho_* - \hat{\rho}_*\|_2^2 = \left\| \sum_{i=1}^{k} \left(p_i - \hat{p}_i\right) Q_i \right\|_2^2.$$

Since $\hat{p}_i$ is considered as a random variable, we can take the expectation $E\left(\|\rho_* - \hat{\rho}_*\|_2^2\right)$, of the above error. We assume that the choice of an unknown state $\rho \in M_n(\mathbb{C})$ depends on a probability measure μ on the set of all states in $M_n(\mathbb{C})$. Additionally, we assume $\int \rho \mathrm{d}\mu(\rho) = \frac{1}{n} I$. For a POVM $\{F_i\}_{i=1}^k$ and a dual frame $\{Q_i\}_{i=1}^k$ of $\{F_i\}_{i=1}^k$, we denote the expected value of the error by

$$\begin{aligned} e(F, Q) &:= \int E\left(\|\rho_* - \hat{\rho}_*\|_2^2\right) \mathrm{d}\mu(\rho) \\ &= \frac{1}{nM} \sum_{i=1}^{k} \mathrm{Tr}\,(F_i) \mathrm{Tr}\left(Q_i^2\right) - \frac{1}{M} \int \mathrm{Tr}\left(\rho_*^2\right) d\mu(\rho) \end{aligned}$$

(see, e.g., [18]). We would like to minimize $e(F, Q)$.

Proposition 2 [18,21,22] *For a fixed informationally complete POVM $\{F_i\}_{i=1}^k \subset A \oplus C$ w.r.t. $A \oplus C$, the expected value of the error $e(F, Q)$ is minimized when $\{Q_i\}_{i=1}^k$ is the canonical dual frame of $\{F_i\}_{i=1}^k$.*

A POVM $\{F_i\}_{i=1}^k$ (with the canonical dual frame $\{Q_i\}_{i=1}^k$) is optimal, if the average of the expected values of the error $e(F, Q)$ is the minimum among all POVMs and all dual frames. If a POVM is optimal, then it satisfies the following condition.

Theorem 1 [18,22] *An informationally complete POVM $\{F_i\}_{i=1}^k \subset A \oplus C$ w.r.t. $A \oplus C$ is optimal if and only if*

$$\mathbf{F} = \mathbf{A} + \frac{n-1}{N-1}\mathbf{C}. \tag{5}$$

In addition, let $k = N$ and $P_i = \frac{N}{n}F_i$. Then, the POVM $\{F_i\}_{i=1}^N$ is optimal, or equivalently the equality (5) holds if and only if

$$\sum_{i=1}^N P_i = \frac{N}{n}I, \qquad \mathrm{Tr}\, P_i P_j = \frac{N-n}{n(N-1)} \quad (i \neq j), \qquad \mathrm{Tr}\, X P_i = 0 \quad (X \in B).$$

When $\{F_i\}_{i=1}^N$ satisfies (5), then P_i is a rank-one projection and $\{F_i\}_{i=1}^N$ (and also $\{P_i\}_{i=1}^N$) is called a conditional SIC-POVM.

Remark 1 There exist several methods of quantum state tomography. In this paper, we only consider an estimate $\hat{\rho}_*$ given by (3) and (4). Such a method is called linear quantum state tomography [21]. Remark that there exist better estimation methods of quantum state tomography (see, e.g., [19,30]).

Remark 2 The square root of the value

$$\mathrm{Tr}\, P_i P_j = \frac{N-n}{n(N-1)}$$

in Theorem 1 is called Welch bound [25,29]. A set of unit vectors $\{\xi_i\}_{i=1}^N \subset \mathbb{C}^n$ is called an equiangular tight frame if

$$|\langle \xi_i, \xi_j \rangle| = \sqrt{\frac{N-n}{n(N-1)}}$$

for all $1 \leq i, j \leq N$, and $i \neq j$. Therefore, a conditional SIC-POVM is actually an equiangular tight frame. However, the existence of conditional SIC-POVM also depends on the subspace B which corresponds to the known space. Hence, the existence of conditional SIC-POVMs and the existence of equiangular tight frames are slightly different (see Remark 4 for details).

3 Necessary condition for existence of conditional SIC-POVMs

In this section, we present a necessary condition for the existence of a conditional SIC-POVM. The existence of conditional SIC-POVMs is a fundamental question. It is often not easy to prove nonexistence, as in the case of MUBs in a six-dimensional Hilbert space, but we can prove nonexistence in some cases using the necessary condition.

Lemma 1 *Let a set of rank-one projections* $\{P_i\}_{i=1}^N$ *be a conditional SIC-POVM in* $A \oplus C$*, and let*

$$S_i = \sqrt{\frac{n(N-1)}{N(n-1)}}\left(P_i - \frac{1}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right)I\right). \tag{6}$$

Then, $\{S_i\}_{i=1}^N$ *is an orthonormal basis of* $A \oplus C$*.*

Proof The normality of $\{S_i\}_{i=1}^N$ is proved by simple calculation:

$$\begin{aligned}
&\mathrm{Tr}\left(\left(P_i - \frac{1}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right)I\right)^2\right)\\
&\quad = \mathrm{Tr}\left(P_i - \frac{2}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right)P_i + \frac{1}{n^2}\left(1+\sqrt{\frac{n-1}{N-1}}\right)^2 I\right)\\
&\quad = 1 - \frac{2}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right) + \frac{1}{n}\left(1+2\sqrt{\frac{n-1}{N-1}}+\frac{n-1}{N-1}\right)\\
&\quad = 1 - \frac{1}{n} + \frac{n-1}{n(N-1)} = \frac{N(n-1)}{n(N-1)}
\end{aligned}$$

for any $1 \le i \le N$. The orthogonality of $\{S_i\}_{i=1}^N$ is shown as follows:

$$\begin{aligned}
&\mathrm{Tr}\left(\left(P_i - \frac{1}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right)I\right)\left(P_j - \frac{1}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right)I\right)\right)\\
&= \mathrm{Tr}\left(P_iP_j - \frac{1}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right)(P_i+P_j) + \frac{1}{n^2}\left(1+\sqrt{\frac{n-1}{N-1}}\right)^2 I\right)\\
&= \frac{N-n}{n(N-1)} - \frac{2}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right) + \frac{1}{n}\left(1+2\sqrt{\frac{n-1}{N-1}}+\frac{n-1}{N-1}\right)\\
&= \frac{N-n}{n(N-1)} - \frac{1}{n} + \frac{n-1}{n(N-1)} = 0
\end{aligned}$$

for any $1 \le i < j \le N$. □

Theorem 2 *If there exists a conditional SIC-POVM in* $A \oplus C$*, then for any* $X \in B$ *and any orthonormal basis* $\{R_i\}_{i=1}^m$ *of* B*, the following equation holds:*

$$\sum_{i=1}^m R_i^* X R_i = \frac{N-n}{n(n-1)}X. \tag{7}$$

Proof Let $\{P_i\}_{i=1}^N$ be a conditional SIC-POVM in $A \oplus C$ and define $\{S_i\}_{i=1}^N$ by (6). From the previous lemma, $\{S_1, \ldots, S_N, R_1, \ldots R_m\}$ is an orthonormal basis of $M_n(\mathbb{C})$. It is well known that

$$\sum_{i=1}^{N} S_i^* X S_i + \sum_{i=1}^{m} R_i^* X R_i = \mathrm{Tr}\,(X). \tag{8}$$

B is orthogonal to $A = \mathbb{C}I$, with the result that $\mathrm{Tr}\,(X) = 0$. Hence, we will calculate $\sum_{i=1}^N S_i^* X S_i$. Since P_i is a rank-one projection, $P_i X P_i = t P_i$ for some $t \in \mathbb{C}$. However, $\mathrm{Tr}\,(P_i X P_i) = \langle P_i, X \rangle = 0$ implies that $t = 0$; therefore, $P_i X P_i = 0$. From the equation

$$\sum_{i=1}^{N} P_i = \frac{N}{n} I,$$

we have

$$\begin{aligned}
&\frac{N(n-1)}{n(N-1)} \sum_{i=1}^{N} S_i^* X S_i \\
&= \sum_{i=1}^{N} \left(P_i - \frac{1}{n}\left(1 + \sqrt{\frac{n-1}{N-1}}\right)\right) X \left(P_i - \frac{1}{n}\left(1 + \sqrt{\frac{n-1}{N-1}}\right)\right) \\
&= \sum_{i=1}^{N} \left(P_i X P_i - \frac{1}{n}\left(1 + \sqrt{\frac{n-1}{N-1}}\right)(X P_i + P_i X) + \frac{1}{n^2}\left(1 + \sqrt{\frac{n-1}{N-1}}\right)^2 X \right) \\
&= -\frac{1}{n}\left(1 + \sqrt{\frac{n-1}{N-1}}\right)\left(X \sum_{i=1}^{N} P_i + \sum_{i=1}^{N} P_i X \right) + \frac{N}{n^2}\left(1 + \sqrt{\frac{n-1}{N-1}}\right)^2 X \\
&= \left(-\frac{2N}{n^2}\left(1 + \sqrt{\frac{n-1}{N-1}}\right) + \frac{N}{n^2}\left(1 + 2\sqrt{\frac{n-1}{N-1}} + \frac{n-1}{N-1}\right)\right) X \\
&= \frac{N}{n^2}\left(-1 + \frac{n-1}{N-1}\right) X = \frac{N(n-N)}{n^2(N-1)} X.
\end{aligned}$$

This implies the assertion. □

If Eq. (7) in Theorem 2 holds for an orthonormal basis $\{R_i\}_{i=1}^m$ of B, the equation also holds for any other orthonormal basis of B. Indeed, let $\{T_i\}_{i=1}^m$ be another orthonormal basis of B. Then, we have the equation

$$\sum_{i=1}^{N} S_i^* X S_i + \sum_{i=1}^{m} T_i^* X T_i = \mathrm{Tr}\,(X)$$

for any $X \in M_n(\mathbb{C})$. Therefore, combining (8), we obtain

$$\sum_{i=1}^{m} R_i^* X R_i = \sum_{i=1}^{m} T_i^* X T_i$$

so that

$$\sum_{i=1}^{m} T_i^* X T_i = \frac{N-n}{n(n-1)} X.$$

Hence, it is enough to check Eq. (7) for only one orthonormal basis of B.

4 Examples of conditional SIC-POVMs

In this section, we show some examples of the existence or nonexistence of conditional SIC-POVMs. Some open problems are also discussed.

Example 1 If we have no information regarding the state ($m = 0$, $N = n^2$), then

$$\mathrm{Tr}\, P_i P_j = \frac{1}{n+1} \quad (i \neq j).$$

Hence, this well-known SIC-POVM is a conditional SIC-POVM (if it exists [20]). □

One of the most important POVMs is a POVM which corresponds to an orthonomal basis. Example 2 says that such a POVM is a conditional POVM. In Examples 3, 4, and 5, we consider a conditional SIC-POVM which complements such a POVM.

Example 2 If we know the off-diagonal elements of the state, and we want to estimate the diagonal entries ($m = n^2 - n$, $N = n$), then it follows from Theorem 1 that a conditional SIC-POVM has the properties

$$\mathrm{Tr}\, P_i P_j = 0 \quad (i \neq j), \quad \sum_{i=1}^{n} P_i = I, \quad \text{and} \quad P_i \text{ is diagonal.}$$

Hence, the diagonal matrix units form a conditional SIC-POVM. □

Example 3 If we know the diagonal elements of the state, and we want to estimate the off-diagonal entries ($m = n - 1$, $N = n^2 - n + 1$), then it follows from Theorem 1 that a conditional SIC-POVM has the properties

$$\mathrm{Tr}\, P_i P_j = \frac{n-1}{n^2} \quad (i \neq j), \quad \sum_{i=1}^{n} P_i = \frac{n^2 - n + 1}{n} I$$

and that P_i has a constant diagonal.

Existence is not clear generally, but if $n-1$ is a prime power, then it can be constructed on the basis of the prime power conjecture [6,24], and details are provided in [18]. We provide examples in $M_2(\mathbb{C})$, $M_3(\mathbb{C})$, and $M_4(\mathbb{C})$. The case of a two-dimensional space is very simple, and it will be presented in the next example.

On the other hand, we cannot prove nonexistence using Theorem 2. Let $\lambda = \exp(2\pi \mathrm{i}/n)$, where $\mathrm{i} = \sqrt{-1}$, and $W = \mathrm{Diag}\,(1, \lambda, \lambda^2, \ldots, \lambda^{n-1}) \in M_n(\mathbb{C})$. Then, the subspace B is $\mathrm{span}\{W^k\}_{k=1}^{n-1}$ and $\{\frac{1}{\sqrt{n}} W^k\}_{k=1}^{n-1}$ is an orthonormal basis of B. Here, we have

$$\sum_{k=1}^{n-1} \frac{1}{n} W^{*k} W^j W^k = \frac{n-1}{n} W^j$$

for any $1 \leq j \leq n-1$. Therefore, the condition in Theorem 2 holds. □

Example 4 Assume that for a density matrix $\rho \in M_2(\mathbb{C})$ the diagonal entries are known. To recover the other parameters, we use a POVM $\{F_1, F_2, F_3\}$. A conditional SIC-POVM is described by projections $P_i = 3F_i/2\,(1 \leq i \leq 3)$ such that the diagonal terms are the same and

$$\sum_i P_i = \frac{3}{2} I, \qquad \mathrm{Tr}\, P_i P_j = \frac{1}{4} \quad (i \neq j).$$

Concretely,

$$P_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad P_2 = \frac{1}{2}\begin{bmatrix} 1 & \lambda \\ \lambda^2 & 1 \end{bmatrix}, \quad P_3 = \frac{1}{2}\begin{bmatrix} 1 & \lambda^2 \\ \lambda & 1 \end{bmatrix},$$

where $\lambda = \exp(2\pi \mathrm{i}/3)$. □

Example 5 Assume that for a density matrix $\rho \in M_3(\mathbb{C})$ the diagonal entries are known, and the other parameters are to be found from a POVM $\{F_i\}_{i=0}^{6}$. A conditional SIC-POVM is described by projections $P_i = 7F_i/3\,(0 \leq i \leq 6)$ such that the diagonal terms are the same and

$$\sum_i P_i = \frac{7}{3} I, \qquad \mathrm{Tr}\, P_i P_j = \frac{2}{9} \quad (i \neq j).$$

We have

$$P_0 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

and the other projection matrices are parameterized by $\alpha = \exp(2\pi \mathrm{i}/7)$:

$$P_1 = \frac{1}{3}\begin{bmatrix} 1 & \alpha^1 & \alpha^3 \\ \alpha^6 & 1 & \alpha^2 \\ \alpha^4 & \alpha^5 & 1 \end{bmatrix}, \quad P_2 = \frac{1}{3}\begin{bmatrix} 1 & \alpha^2 & \alpha^6 \\ \alpha^5 & 1 & \alpha^4 \\ \alpha^1 & \alpha^3 & 1 \end{bmatrix}, \quad P_3 = \frac{1}{3}\begin{bmatrix} 1 & \alpha^3 & \alpha^2 \\ \alpha^4 & 1 & \alpha^6 \\ \alpha^5 & \alpha^1 & 1 \end{bmatrix},$$
$$P_4 = \frac{1}{3}\begin{bmatrix} 1 & \alpha^4 & \alpha^5 \\ \alpha^3 & 1 & \alpha^1 \\ \alpha^2 & \alpha^6 & 1 \end{bmatrix}, \quad P_5 = \frac{1}{3}\begin{bmatrix} 1 & \alpha^5 & \alpha^1 \\ \alpha^2 & 1 & \alpha^3 \\ \alpha^6 & \alpha^4 & 1 \end{bmatrix}, \quad P_6 = \frac{1}{3}\begin{bmatrix} 1 & \alpha^6 & \alpha^4 \\ \alpha^1 & 1 & \alpha^5 \\ \alpha^3 & \alpha^2 & 1 \end{bmatrix}.$$

We can see that in all of the matrices, the numbers $\{\alpha^i : 1 \le i \le 6\}$ appear exactly once; therefore, $\mathrm{Tr}\, P_0 P_i$ is constant for $1 \le i \le 6$. Since $\sum_{i=0}^{6} \mathrm{Tr}\, P_0 P_i = 7/3$, we have $\mathrm{Tr}\, P_0 P_i = 2/9$. If we use the notation $P_i = |x_i\rangle\langle x_i|$, then

$$|x_0\rangle = \frac{1}{\sqrt{3}}(1, 1, 1) \qquad \text{and} \qquad |x_{i+1}\rangle = U|x_i\rangle$$

with a unitary $U = \mathrm{Diag}\,(1, \alpha^6, \alpha^4)$. This implies that $\mathrm{Tr}\, P_i P_{i+j} = \mathrm{Tr}\, P_0 P_j$; hence, $\mathrm{Tr}\, P_i P_k = 2/9$ holds for $i \ne k$. Consequently, the operators $F_i = 3P_i/7$ $(0 \le i \le 6)$ form a conditional SIC-POVM.

It is easy to provide a similar example for $M_4(\mathbb{C})$. Let

$$|x_0\rangle = \frac{1}{2}(1, 1, 1, 1) \qquad \text{and} \qquad |x_{i+1}\rangle = U|x_i\rangle$$

with a unitary $U = \mathrm{Diag}\,(1, \alpha, \alpha^4, \alpha^6)$ and $\alpha = \exp(2\pi \mathrm{i}/13)$. Then, the operators $F_i = 4|x_i\rangle\langle x_i|/13$ $(0 \le i \le 12)$ form a conditional SIC-POVM. □

Remark 3 We consider a similar example for $M_7(\mathbb{C})$. Assume that for a density matrix $\rho \in M_7(\mathbb{C})$, the diagonal entries are known, and the other parameters are to be found from a POVM $\{F_i\}_{i=1}^{43}$. The question is the existence of the projections $\{P_k : 1 \le k \le 43\}$ such that the diagonal terms are the same and

$$\mathrm{Tr}\, P_k P_\ell = \frac{6}{49} \quad (k \ne \ell), \quad \sum_{k=1}^{43} P_k = \frac{43}{7} I.$$

This appears to be a complicated situation. Assume that $\{\xi_i : 1 \le i \le 7\}$ is an orthonormal basis in the Hilbert space. The unit vectors $\{p_k : 1 \le k \le 43\}$ provide the projections

$$\{P_k := |p_k\rangle\langle p_k| : 1 \le k \le 43\}.$$

The formulation $p_k = \sum_{i=1}^{7} p_{ki}\xi_i$ gives

$$P_k = \sum_{i,j} p_{ki}\overline{p}_{kj}|\xi_i\rangle\langle\xi_j|, \qquad \sum_{i=1}^{7} |p_{ki}|^2 = 1,$$

so the (i, j)-entry of the matrix P_k is $p_{ki}\overline{p}_{kj}$. Then, $\mathrm{Tr}\, P_k P_\ell = 6/49$ implies that

$$\mathrm{Tr}\, P_k P_\ell = |\langle p_k, p_\ell\rangle|^2 = \left|\left\langle \sum_{i=1}^{7} p_{ki}\xi_i, \sum_{j=1}^{7} p_{\ell j}\xi_j \right\rangle\right|^2 = \left|\sum_{i=1}^{7} \overline{p}_{ki} p_{\ell i}\right|^2 = \frac{6}{49} \quad (k \neq \ell). \tag{9}$$

The condition

$$\sum_{k=1}^{43} \langle \xi_i, P_k \xi_j\rangle = \frac{43}{7}\langle \xi_i, \xi_j\rangle$$

which is equivalent to $\sum_k P_k = 43I/7$ can be reformulated as

$$\sum_{k=1}^{43} p_{ki}\overline{p}_{kj} = \frac{43}{7}\langle \xi_i, \xi_j\rangle = \frac{43}{7}\delta_{ij} \qquad (1 \leq i, j \leq 7). \tag{10}$$

The condition that the diagonal entries are the same implies

$$|p_{ki}|^2 = \frac{1}{7}. \tag{11}$$

The essential problem is to construct an example that satisfies (9), (10), and (11).

Since $7 - 1 = 6$ is not a prime power, the existence of the projections $\{P_k : 1 \leq k \leq 43\}$ does not follow from Example 3. Moreover, we cannot prove nonexistence using Theorem 2. It would be interesting to know the relation to existence.

A POVM which corresponds to MUBs is important in quantum information; for example, better estimation methods of quantum state tomography stated in Remark 1 use MUBs. It is known that $n + 1$ MUBs exist in $\mathbb{C}^n$ when n is a prime power (see, e.g., [3,9,30]). In the case of $n = 6$, the existence of 7 MUBs is open problem and it was conjectured that 4 MUBs do not exist [31,32]. We consider a conditional POVM which complements 3 MUBs in $\mathbb{C}^6$.

Example 6 Consider MUBs in $M_6(\mathbb{C})$. It is known that there are three mutually unbiased orthonormal bases in $\mathbb{C}^6$. Let $\lambda = \exp(2\pi \mathrm{i}/6)$, and let

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^5 \end{pmatrix}.$$

Then, maximal abelian subalgebras corresponding to the three bases are

$$\mathrm{span}\{S^k : 0 \leq k \leq 5\}, \quad \mathrm{span}\{W^k : 0 \leq k \leq 5\}, \quad \mathrm{span}\{S^k W^k : 0 \leq k \leq 5\}.$$

Assume that the known space is $B = \mathrm{span}\{S^k, W^k, S^k W^k : 1 \le k \le 5\}$ ($m = 15$, $N = 21$), and consider a conditional SIC-POVM. An orthonormal basis of B is

$$\left\{ \frac{1}{\sqrt{6}} S^k, \frac{1}{\sqrt{6}} W^k, \frac{1}{\sqrt{6}} S^k W^k : 1 \le k \le 5 \right\}.$$

The equation $SW = \lambda WS$ implies that

$$\sum_{k=1}^{5} \left(S^{*k} S^j S^k + W^{*k} S^j W^k + W^{*k} S^{*k} S^j S^k W^k \right) = 5S^j + 2 \sum_{k=1}^{5} \lambda^{jk} S^j = 3S^j$$

for any $1 \le j \le 5$. Similarly,

$$\sum_{k=1}^{5} \left(S^{*k} W^j S^k + W^{*k} W^j W^k + W^{*k} S^{*k} W^j S^k W^k \right) = 3W^j$$

$$\sum_{k=1}^{5} \left(S^{*k} S^j W^j S^k + W^{*k} S^j W^j W^k + W^{*k} S^{*k} S^j W^j S^k W^k \right) = 3S^j W^j.$$

Therefore, the condition in Theorem 2 holds, and we cannot prove the nonexistence of a conditional SIC-POVM using Theorem 2. The existence of a conditional SIC-POVM in this case remains an open problem. □

States on the coupled quantum system have recently been studied from many points of view. Here, we focus on density matrices in coupled quantum system $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ whose restrictions are the same or I/n. A set of such density matrices is considered in many literature [7,11–14,16]. A density matrix whose restrictions are I/n corresponds to a unital completely positive trace-preserving map. Moreover, if such a density matrix has rank one, the density matrix (or the state) is called a maximally entangled state. In Examples 7 and 8, we consider states whose restrictions are the same and show examples of nonexistence of conditional SIC-POVMs using Theorem 2.

Example 7 We consider a density matrix $\rho \in M_4(\mathbb{C}) = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$. The density matrix

$$\rho = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

has reduced densities:

$$\rho_1 = \begin{bmatrix} a_{11} + a_{22} & a_{13} + a_{24} \\ a_{31} + a_{42} & a_{33} + a_{44} \end{bmatrix}, \qquad \rho_2 = \begin{bmatrix} a_{11} + a_{33} & a_{12} + a_{34} \\ a_{21} + a_{43} & a_{22} + a_{44} \end{bmatrix}.$$

Note that $a_{ij} = \overline{a}_{ji}$ for $1 \le i, j \le 4$. We assume that ρ satisfies the condition $\rho_1 = \rho_2$. This condition implies that

$$a_{22} = a_{33}, \quad a_{13} + a_{24} = a_{12} + a_{34}, \quad \text{and} \quad a_{31} + a_{42} = a_{21} + a_{43}$$

$(m = 3, N = 13)$. Let

$$\begin{aligned} R_1 &= \frac{1}{\sqrt{2}}(e_{22} - e_{33}), \; R_2 = \frac{1}{2}(e_{12} - e_{13} - e_{24} + e_{34}), \\ R_3 &= \frac{1}{2}(e_{21} - e_{31} - e_{42} + e_{43}), \end{aligned}$$

where $\{e_{ij}\}_{i,j=1}^n$ is a set of matrix units. Then, an orthonormal basis of the known space B is $\{R_1, R_2, R_3\}$. Furthermore, ρ has the form

$$\rho = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12}^* & b & a_{23} & c - a_{13} \\ a_{13}^* & a_{23}^* & b & c - a_{12} \\ a_{14}^* & c^* - a_{13}^* & c^* - a_{12}^* & a_{44} \end{bmatrix},$$

which is orthogonal to B. In this case, a conditional SIC-POVM does not exist. Indeed, the equations

$$R_1^* R_1 R_1 = \frac{1}{2} R_1, \quad R_2^* R_1 R_2 = 0, \quad R_3^* R_1 R_3 = 0$$

imply $\sum_{i=1}^3 R_i^* R_1 R_i = R_1/2$, and this contradicts the condition in Theorem 2. □

Remark 4 It is known that an equiangular tight frame exists if $n = 4$ and $N = 13$ (see, e.g., [27]), where n is the dimension of the Hilbert space and N is the number of vectors. The conditional SIC-POVM shown in Example 5 is an example of such an equiangular tight frame. On the other hand, Example 7 is also in the case $n = 4$ and $N = 13$. However, a conditional SIC-POVM does not exist in Example 7. This says that the existence of conditional SIC-POVMs depends on the known space B.

Example 8 We extend Example 7 to the case $M_{n^2}(\mathbb{C}) = M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. The density matrix

$$\rho = \sum_{i,j,k,\ell=1}^{n} a_{i,j,k,\ell} e_{ij} \otimes e_{k\ell}$$

has reduced densities:

$$\rho_1 = \sum_{i,j=1}^{n} \sum_{k=1}^{n} a_{i,j,k,k} e_{ij}, \quad \rho_2 = \sum_{i,j=1}^{n} \sum_{k=1}^{n} a_{k,k,i,j} e_{ij}.$$

We assume that ρ satisfies the condition $\rho_1 = \rho_2$. This condition implies that

$$\sum_{k=1}^{n} a_{i,j,k,k} = \sum_{k=1}^{n} a_{k,k,i,j}$$

for all $1 \le i, j \le n$. Let

$$Q_{i,j} = \sum_{k=1}^{n} (e_{kk} \otimes e_{ij} - e_{ij} \otimes e_{kk}) = (I \otimes e_{ij} - e_{ij} \otimes I),$$

and let

$$R_{i,j} = \begin{cases} \frac{1}{\sqrt{2n}} \sum_{k=1}^{n} \lambda^{ik} Q_{k,k} & (i = j) \\ \frac{1}{\sqrt{2n}} Q_{i,j} & (i \neq j) \end{cases}$$

for $1 \le i, j \le n$, where $\lambda = \exp(2\pi \mathrm{i}/n)$. Then, the known space B is $\mathrm{span}\{Q_{i,j}\}_{i,j=1}^{n}$, and $\{R_{i,j} : 1 \le i, j \le n\} \backslash \{R_{n,n}\}$ is an orthonomal basis of B ($m = n^2 - 1$, $N = n^4 - n^2 + 1$). Note that $R_{n,n} = 0$.

For $1 \le i \le n-1$, $1 \le j, k \le n$ and $j \neq k$, we have

$$\begin{aligned} & R_{j,k}^{*} R_{i,i} R_{j,k} \\ &= \frac{1}{2\sqrt{2}n^2} \sum_{\ell=1}^{n} \lambda^{i\ell} (I \otimes e_{kj} - e_{kj} \otimes I)(I \otimes e_{\ell\ell} - e_{\ell\ell} \otimes I)(I \otimes e_{jk} - e_{jk} \otimes I) \\ &= \frac{1}{2\sqrt{2}n^2} \left(\lambda^{ij} (I \otimes e_{kk} - e_{kk} \otimes I) + \sum_{\ell=1}^{n} \lambda^{i\ell} (e_{kk} \otimes e_{\ell\ell} - e_{\ell\ell} \otimes e_{kk}) \right) \end{aligned}$$

so that

$$\begin{aligned} & \sum_{j \neq k} R_{j,k}^{*} R_{i,i} R_{j,k} \\ &= \frac{1}{2\sqrt{2}n^2} \left(\sum_{j=1}^{n} \lambda^{ij} (-I \otimes e_{jj} + e_{jj} \otimes I) + (n-1) \sum_{\ell=1}^{n} \lambda^{i\ell} (I \otimes e_{\ell\ell} - e_{\ell\ell} \otimes I) \right) \\ &= \frac{n-2}{2n} R_{i,i}. \end{aligned}$$

Furthermore, for $1 \le i, j \le n-1$,

$$\begin{aligned} & R_{j,j}^{*} R_{i,i} R_{j,j} \\ &= \frac{1}{2\sqrt{2}n^3} \sum_{p,q,r} \lambda^{iq + j(r-p)} (I \otimes e_{pp} - e_{pp} \otimes I)(I \otimes e_{qq} - e_{qq} \otimes I)(I \otimes e_{rr} - e_{rr} \otimes I) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\sqrt{2}n^3}\left(\sum_{p=1}^{n}\lambda^{ip}(I\otimes e_{pp}-e_{pp}\otimes I)+\sum_{p,r}\lambda^{ip+j(r-p)}(e_{pp}\otimes e_{rr}-e_{rr}\otimes e_{pp})\right.\\
&\quad\left.+\sum_{p,r}\lambda^{ir+j(r-p)}(e_{rr}\otimes e_{pp}-e_{pp}\otimes e_{rr})+\sum_{p,q}\lambda^{iq}(e_{pp}\otimes e_{qq}-e_{qq}\otimes e_{pp})\right)\\
&= \frac{1}{n^2}R_{i,i}+\frac{1}{2\sqrt{2}n^3}\sum_{p,r}\lambda^{j(r-p)}\Big(\lambda^{ip}(e_{pp}\otimes e_{rr}-e_{rr}\otimes e_{pp})\\
&\quad+\lambda^{ir}(e_{rr}\otimes e_{pp}-e_{pp}\otimes e_{rr})\Big)\\
&= \frac{1}{n^2}R_{i,i}+\frac{1}{2\sqrt{2}n^3}\sum_{p\neq r}\lambda^{j(r-p)}\Big(\lambda^{ip}(e_{pp}\otimes e_{rr}-e_{rr}\otimes e_{pp})\\
&\quad+\lambda^{ir}(e_{rr}\otimes e_{pp}-e_{pp}\otimes e_{rr})\Big)
\end{aligned}
$$

so that

$$
\begin{aligned}
&\sum_{j=1}^{n-1}R_{j,j}^{*}R_{i,i}R_{j,j}\\
&\quad= \frac{n-1}{n^2}R_{i,i}-\frac{1}{2\sqrt{2}n^3}\sum_{p\neq r}\Big(\lambda^{ip}(e_{pp}\otimes e_{rr}-e_{rr}\otimes e_{pp})\\
&\qquad+\lambda^{ir}(e_{rr}\otimes e_{pp}-e_{pp}\otimes e_{rr})\Big)\\
&\quad= \frac{n-1}{n^2}R_{i,i}-\frac{1}{2\sqrt{2}n^3}\sum_{p,r}\Big(\lambda^{ip}(e_{pp}\otimes e_{rr}-e_{rr}\otimes e_{pp})\\
&\qquad+\lambda^{ir}(e_{rr}\otimes e_{pp}-e_{pp}\otimes e_{rr})\Big)\\
&\quad= \frac{n-1}{n^2}R_{i,i}-\frac{1}{2\sqrt{2}n^3}\left(\sum_{p}\lambda^{ip}(e_{pp}\otimes I-I\otimes e_{pp})\right.\\
&\qquad\left.+\sum_{r}\lambda^{ir}(e_{rr}\otimes I-I\otimes e_{rr})\right)\\
&\quad= \frac{1}{n}R_{i,i}.
\end{aligned}
$$

Therefore, for $1\leq i\leq n-1$, we obtain

$$
\sum_{j=1}^{n-1}R_{j,j}^{*}R_{i,i}R_{j,j}+\sum_{j\neq k}R_{j,k}^{*}R_{i,i}R_{j,k}=\frac{1}{2}R_{i,i},
$$

and this contradicts the condition in Theorem 2. Consequently, a conditional SIC-POVM does not exist. □

Example 9 We consider another example similar to Example 7. We assume that $\rho \in M_4(\mathbb{C}) = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ satisfies the condition $\rho_1 = \rho_2 = \frac{1}{2}I$. This condition implies that

$$a_{11} = a_{44},\ a_{22} = a_{33},\ a_{34} = -a_{12},\ a_{24} = -a_{13},\ a_{43} = -a_{21},\ a_{42} = -a_{31}$$

$(m = 6, N = 10)$. Let

$$R_1 = \frac{1}{\sqrt{2}}(e_{11} - e_{44}),\quad R_2 = \frac{1}{\sqrt{2}}(e_{22} - e_{33}),\quad R_3 = \frac{1}{\sqrt{2}}(e_{12} + e_{34}),$$
$$R_4 = \frac{1}{\sqrt{2}}(e_{21} + e_{43}),\quad R_5 = \frac{1}{\sqrt{2}}(e_{13} + e_{24}),\quad R_6 = \frac{1}{\sqrt{2}}(e_{31} + e_{42}).$$

The known space B is $B = \mathrm{span}\{R_i\}_{i=1}^6$. Moreover, ρ has the form

$$\rho = \begin{bmatrix} b & a_{12} & a_{13} & a_{14} \\ a_{12}^* & \frac{1}{2} - b & a_{23} & -a_{13} \\ a_{13}^* & a_{23}^* & \frac{1}{2} - b & -a_{12} \\ a_{14}^* & -a_{13}^* & -a_{12}^* & b \end{bmatrix}, \tag{12}$$

which is orthogonal to B. In this case, it is easy to see that for all $X \in B$,

$$\sum_{i=1}^{6} R_i^* X R_i = \frac{1}{2}X.$$

This is the condition in Theorem 2. Hence, we cannot prove the nonexistence of a conditional SIC-POVM in this case using Theorem 2. The existence of a conditional SIC-POVM in this case remains an open problem. □

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