REPRESENTATIONS AT FIXED POINTS OF SMOOTH ACTION OF FINITE GROUPS

Masato NAKAMURA

Let Θ be a smooth action of a finite group G on a differentiable manifold M. If $x \in M$ is a stationart point of this action there is an induced representation Θ_x of G on the tangent space to M at x.

In this paper we shall obtain some result which compare the representation Θ_x and Θ_y for differnt stationary poins by the method of G. E. Bredon.

Let E_G be a universal space for G and $B_G = E_G/G$ the corresponding classifying space. we may assume that E_G is a cw-complex with finite skeltons and that G acts cellularly. $R(G)$ denotes the complex represantation ring of G.

The map $E_G \to *$ of E_G to a point induces the homomorphism α: $R(G) \approx K_G(*) \to K_G(E_G) \approx K(B_G)$ in equivariant K-theory. Restricting to the r-skeleton $B^r{}_G$ of B_G we obtain the homomorphism $\alpha^{(r)}: R(G) \to K(B_G{}^{(r)})$.

Now suppose that Θ is a smooth action of G on the simply connected manifold M. Then there is an induced action of G on the homotopy groups of M making $\pi_i(M)$ into G-modules.

Theorem (G. E. Bredon). Let Θ be a smooth action of a finite group G on a simply connected manifold M. Asume that

$$H^i(G;\ \pi_i(M)) = 0 \qquad \textit{for } 1 \leqq i \leqq r.$$

If x and y are stationary points of Θ, then

$$\alpha^{(r)}(\Theta_x - \Theta_y) = 0.$$

Using the above theorem, we obtain the next therem

Theorem. Let Θ be a smooth action of Z_p on a simply connected manifold M, where p is prime. Let n be an integer and let $n = s(p-1) + r\ (0 \leqq r < p-1)$. Assume that

$$H^i(Z_p;\ \pi_i(M) = 0 \qquad \textit{for } 1 \leqq i \leqq 2n+1$$

then $\Theta_x - \Theta_y$ is divisible by p^{s+1}.

Proof. For Z_P the complex representation ring is

$$Z[\eta]/(1-\eta^p)$$

where η is the representation $Z_p \to \mathrm{U}(1)$ taking the generator g into $e^{2\pi i/p}$. $B_{Z_p}^{(2n+1)}$ can be taken to be the lens space $L^n(p)$. By the result of T. Kambe

$$\widetilde{K}(L^n(p)) \cong (Z_{ps+1})^r + (Z_{ps})^{p-r-1}$$

and $(\alpha^{2n+1}(\eta-1))^1, \cdots\cdots, (\alpha^{2n+1}(\eta-1)^r$ generate additively the first r factors and $(\alpha^{2n+1}(\eta-1))^{r+1}, \cdots\cdots, (\alpha^{2n+1}(\eta-1))^{p-1}$ the last p-r-1 factors. Let $I(Z_p)$ be the "augmentation ideal" $(1-\eta)R(Z_p)$. Since $I(Z_p)\approx Z$ additively, this implies that

$$ker\ \alpha^{(2n+1)}\subset p^{s+1}\ I(Z_p).$$

from which the theorem follows.

References

[1] G. E. Bredon, Representations at fixed points of smooth actions of compact groups, Ann. of Math. 89 (1969).

[2] T. Kambe, The structure of K_Λ-ring of the lens space and their application, J. Math. Soc. Japan Vol. 18, No. 2, (1966), 135–146.