BORDISM AND SEMI-FREE S^1-ACTION

Masato NAKAMURA

Let X be a topological space with $A \subset X$ a subspace, and let $\tau : (X, A) \to (X, A)$ be a S^1 action, such that $\tau A \subset A$.

We consider the semi-free equivalent bordism group $\mathfrak{R}_*$ (X, A, τ) of s^1-action (X, A, τ) by the analogue of R. E. Stong.

A semi-free (free) equivariant bordism class of (X, A, τ) is an equivariant class of triples (M, μ, f) with M a compact differentiable manifold with boundary. μ: $M \to M$ a differentiable semi-free (fixed point free) s^1-action on M and $f:(M, \partial M) \to (X, A)$ a continuous equivariant map $[\tau f = f\mu]$ sending ∂M into A. Two triples (M, μ, f) and (M', μ', f') are equivalent, or bordant, if there is a 4 tuple (W, V, ν, g) such that W and V are compact differentiable manifolds with boundary, $\partial V = \partial M \cup \partial M'$ and $\partial W = M \cup M' \cup V / \partial M \cup \partial M' \equiv \partial V$, $\nu:(W, V) \to (W, V)$ is a differentiable semi-free (fixed point free) s^1-action restricting to μ on M and μ' on M', and g: $(W, V) \to (X, A)$ is a continuous equivariant map $[\tau g = g\nu]$ restricting to f on M and f' on M'.

The disjoint union of triples induces an operation on the set of semi-free (free) equivariant bordism class of (X, A, τ) making this set into an abelian group. This is a graded group, where the grading is given by the dimension of the manifold M, and lets $\mathfrak{R}_*$ (X, A, τ) be the group of the semi-free equivariant bordism classes of (X, A, τ). and we let $\widehat{\mathfrak{R}}_* (X, A, \tau)$ be the group of free equivariant bordism classes of (X, A, τ). If A is empty, we write $\mathfrak{R}_* (X, \tau)$ and $\widehat{\mathfrak{R}}_*$ (X, τ) for these groups.

In this paper we obtain the following result.

Proposition 1. There is an exact sequence

$$\begin{tikzcd}
\widehat{\mathfrak{R}}_* (X, A, \tau) \arrow[rr, "k^*"] & & \mathfrak{R}_* (X, A, \tau) \arrow[dl, "F"] \\
& \bigoplus_{k=0}^{[*/2]} \mathfrak{R}_{*-2k} (F_\tau \times B\mathrm{U}(k), (A \cap F_\tau) \times B\mathrm{U}(k)) \arrow[ul, "S"] &
\end{tikzcd}$$

where k_* forgets freeness, F is obtained from analysis of the fixed-poit set, and S is obtained from a sphere bundle construction.

We consider the bordism classification of 4-tuples $(M, \sigma, \xi, \sigma^*)$ where M is a closed differentiable manifold, $\sigma : M \to M$ is a differentiable semi-free s^1-action, ξ is a complex n-plane bundle over M, and σ^* is a s^1-action of ξ by a complex bundle map covering σ. Since complex n-bundle of this type are classified by equiavriant map into BU(n) with appropriate s^1-action τ, the problem is to analyze the semi-free equivariant bordism group $\mathfrak{R}_* (B\mathrm{U}(n), \tau)$. We obtain the following proposition.

Proposition 2. The correspondence

$$(M,\ \sigma,\ \xi,\ \sigma^*)\longrightarrow(M/\sigma\xrightarrow{f}B\mathrm{U}(1)\times B\mathrm{U}(n)),$$

where f classifies the complex line bundle associated to the principle s^1-bundle $M \rightarrow M/\sigma$ and the complex n-plane bundle $\xi/\sigma^* \rightarrow M/\sigma$, defines an isomorhpism q: $\widehat{\mathfrak{R}}_*\ (B\mathrm{U}(n),\ \tau)\xrightarrow{\cong}\mathfrak{R}_*(B\mathrm{U}(1)\times B\mathrm{U}(n))$.

Proposition 3. The fixed point set of the s^1-action $\tau: B\mathrm{U}(n)\rightarrow B\mathrm{U}(n)$ is the disjoint union of the spaces $B\mathrm{U}(i_1)\times B\mathrm{U}(i_2)\times\cdots B\mathrm{U}(i_j)$, $i_1+i_2++i_j=$n and the sequence of proposition becomes a split exact sequence

$$0\longrightarrow\mathfrak{R}_*\ (B\mathrm{U}(n),\ \tau)\xrightarrow{\mathrm{F}}\bigoplus_{k=0}^{[*/2]}\bigoplus_{(i_1,i_2,\cdots,i_j\mid i_1+i_2\cdots+i_j=n)}\mathfrak{R}_{*-2k}(B\mathrm{U}(k)\times B\mathrm{U}(i_1)\times B\mathrm{U}(i_2)\times\cdots$$

$$\times B\mathrm{U}(i_j)\xrightarrow{S}\mathfrak{R}_*(B\mathrm{U}(1)\times B\mathrm{U}(n)\longrightarrow 0.$$

2. Nature of the fixed-point set.

If $\tau: X\rightarrow X$ is the s^1-action, the fixed-point set of τ, is the set of points $x\in X$ with $\tau(x)=x$.

Being given a class $\alpha\in\mathfrak{R}_n(X,\ A,\ \tau)$ represented by $(M^n,\ \mu,\ f)$, we have the equivariant map $f:(M,\ \partial M)\rightarrow(X,\ A)$. If $m\in F\mu$, $\tau f(m)=f(\mu m)=f(m)$, so $f(m)\in F_\tau$, defining a map $f\mid F_\mu:(F_\mu,\ F_\mu\cup\partial M)\rightarrow(F_\tau, F_\tau\cap A)$. As in Conner and Floyd [1, §22], F_μ is a manifold with boundary, ∂F_μ being $F_\mu\cap\partial M$, and we let $F_\mu{}^{n-2k}$ be the union of the (n-2k)-dimensional components of F_μ. The normal bundle of $F_\mu{}^{n-2k}$, ν_k, is an k-dimensional complex plane bundle classified by a map $\nu_k: F_\mu{}^{u-2k}\rightarrow B\mathrm{U}(k)$. We then have a map

$$\varphi: \cup_{k=0}^{[n/2]}(f\mid F_\mu{}^{n-2k}\times\nu_k): \cup_{k=0}^{[n/2]}F_\mu{}^{n-2k}\longrightarrow\cup_{k=2}^{[n/2]}F_\tau\times B\mathrm{U}(k)$$

and sending $\cup_{k=0}^{[n/2]}\partial(F_\mu{}^{n-2k})$ into $\cup_{k=0}^{[n/2]}(F_\tau\cap A)\times B\mathrm{U}(k)$.

Assingning to $(M^n,\ {}_\mu,\ f)$ the class of this map defines an element of

$$\bigoplus_{k=0}^{[n/2]}\mathfrak{R}_{n-2k}\ (F_\tau\times B\mathrm{U}(k),\ \ F_\tau\cap A\times B\mathrm{U}(k)).$$

which may be seen depend only α by making the same construction on a bordism. This defines the homomophism

$$F:\ \mathfrak{R}_*(X,\ A,\ \tau)\longrightarrow\bigoplus_{k=0}^{[*/2]}\mathfrak{R}_{*-2k}(F_\tau\times B\mathrm{U}(k),\ \ (F_\tau\cap A)\times B\mathrm{U}(k)).$$

Specifically, there is a map g: $(M,\ \partial M)\rightarrow(X,\ A)$ equivariantly homotopic (through such maps) of f, with $g\mid F_\mu=f\mid F_\mu$ and such that on a tubular neiborhood V_k of $F_\mu{}^{n-2k}$ $g\mid V_k=f\mid F_\mu{}^{n-2k}\circ\pi$: $V_k\cong D(\nu_k)\xrightarrow{\pi}F_\mu{}^{n-2k}\xrightarrow{f\mid F_\mu{}^{n-2k}}F_\tau$, π being the projection of the disc bundle of ν_k, $D(\nu_k)$ on $F_\mu{}^{n-2k}$, V_k being identyfied with $D\ (\nu_k)$ in the standard fasion so that $\mu\mid V_k$ coincides with the scaler multiplication on the fibers of $D(\nu_k)$.

To see that this is possible, we identifies some tuber neiborhood V'_k of F^{n-2k} with $D'(\nu_k)$ equivariantly and applies a fiberwise deformation in $D'(\nu_k)$ fixed on a neiborhood of the sphere bundle $S'(\nu_k)$ and shrinking a smaller neiborhood $D(\nu_k)$ to its zore section, $F_\mu{}^{n-2k}$.

Being given $\beta\in\mathfrak{R}_{n-2k}$ $(F_\tau\times BU(k),\ (F_\tau\cap A)\times BU(k))$ represented by a map g: $H^{n-2k}\to F_\tau\times BU(k)$, let ρ be the complex k-plane bundle over H^{n-2k} induced by $\pi_2\circ g$ from the canonical bundle over $BU(k)$. We then have f: $D(\rho)\to F_\tau$ defined by $\pi_1\circ g\circ\pi$, with π the projection of $D(\rho)$. Using the semi-free s^1-action of $D(\rho)$ given by the scaler multiplication in the fibers, f gives an equivariant map of $D(\rho)$ into X. The restriction of the semi-free s^1-action to $S(\rho)$, the sphere bundle of ρ, is fixed point free, and sends the boundary of $S(\rho)$ (the invers image of ∂H^{n-2k} under π, or $S(\rho|\partial H^{n-2k})$) into A, so $f|S(\rho){:}S(\rho)\to A$ defines a class $S(\beta)\in\widehat{\mathfrak{R}}_{n-1}(X,\ A,\ \tau)$.

Letting κ_*: $\widehat{\mathfrak{R}}_*(X,\ A,\ \tau)\to\mathfrak{R}_*(X,\ A,\ \tau)$ be the homomorphism induced by forgetting the free condition, we then have

Proposition 1. The sequence

$$\begin{tikzcd}
\widehat{\mathfrak{R}}_*(X,\ A,\ \tau) \arrow[rr, "\kappa_*"] & & \mathfrak{R}_*(X,\ A,\ \tau) \arrow[dl, "F"] \\
& \oplus_{k=0}^{[*/2]}\mathfrak{R}_{*-2k}(F_\tau\times BU(k),\ F_\tau\cap A)\times BU(k) \arrow[ul, "S"] &
\end{tikzcd}$$

is exact.

Proof. The proof is an obvious repetition of the proof given by R. E. Stong [2].

(1) $F\kappa_*=0$. If $\gamma=[M,\ \mu,\ f]\in\mathfrak{R}(X,\ A,\ \tau)$, F_μ is empty so $F\kappa_*\ (\gamma)=0$.

(2) $SF=0$. If $\alpha=[M,\ \mu,\ f]\in\mathfrak{R}_*\ (X,\ A,\ \tau)$, with fixed set F_μ and tubular neiborhoods $V_k=D(\nu_k)$ on which $f|V_k=f|F_\mu{}^{n-2k}\circ\pi$, then $SF(\alpha)$ is represented by $\cup_{k=0}^{[n/2]}f|S(\nu_k)$: $\cup_{k=0}^{[n/2]}S(\nu_k)\to X$, and

$$(M-\cup V_k{}^0,\ \partial M-\cup(V_k\cap\partial M)^0,\ \mu,\ f)$$

is a bordism of this to zero (where $\circ$ denotes interior).

(3) $\kappa_*\ S=0$. If

$$\beta=[H,\ g]\in\mathfrak{R}_{n-2k}(F_\tau\times BU(k),\ (F_\tau\cap A)\times BU(k)),$$

$\kappa_*\ S(\beta)$ is represented by $(S(\rho),\ m,\ f|s(\rho))$, where m is the scaler multiplication, and $(D(\rho),\ D(\rho|\partial H),\ m,\ f)$ is a bordism of this to zero in $\mathfrak{R}_{n-1}(X,\ A,\ \tau)$.

(4) *ker* $F\subset$*im* κ_*. If $\alpha=[M,\ \mu,\ f]\in\mathfrak{R}_n\ (X,\ A,\ \tau)$ with f being $f|F_\mu{}^{n-2k}\circ\pi$ on $V_k=D(\nu_k)$ and $F(\alpha)=0$, there are manifolds W_k with boundary $F_\mu{}^{n-2k}\cup W'_k$ and maps g_k: $(W_k,\ W'_k)\to(F_\tau, F_\tau\cup A)$ extending $f|F_\mu{}^{n-2k}$ and complex k plane bundles ρ_k over W_k restricting to ν_k on $F_\mu{}^{n-2k}$. Let W be formed from $M\times I\cup\cup_{k=0}^{[n/2]}D(\rho_k)$ by identifying $V_k\times 1$ with $D(\rho_k)|F_\mu{}^{n-2k}\cong D(\nu_k)$. W has an semi-free s^1-action ω given by $\mu\times 1$ on $M\times I$ and m in the fibers of $D(\rho_k)$, and there is an equivariant map h: $W\to X$ given by $f\circ\ \pi_1$ on $M\times I$ and by $g_k\circ\pi$ on $D(\rho_k)$. Then $(W,\ \partial M\times I\cup\cup_{k=0}^{[n/2]} D(\rho_k|W'_k), \omega, h)$ is a bordism of $(M,\ \mu,\ f)$ and the free s^1-action induced on $\{M-$

$\cup_{k=0}^{[n/2]} V_k{}^\circ\} \cup \{S(\rho_k)\}/\partial V_k \equiv S(\rho_k | F_\mu{}^{n-2k})$, and so $\alpha \in im\ \kappa_*$.

(5) $ker\ S \subset imF$. If $\beta = [\cup_{k=0}^{[n/2]} (H^{n-2k}, g_k)]$ with $S(\beta)=0$, there is a free manifold W^n with $\partial W = \{\cup S(\rho_k)\} \cup W'/\partial W' \equiv \cup S(\rho_k|_{\partial H})$ and an equivariant map $h:(W, W') \rightarrow (X, A)$ extending $\cup f_k | S(\rho_k)$. Let M^n be formed from $W \cup \cup D(\rho_k)$ by identifying the two copies of $\cup S\ (\rho_k)$, with f: $M \rightarrow X$ given by h on W and $\cup f_k$ on $\cup D(\rho_k)$, and with semi-free s^1-action ω given by that of W and m on the fibers of $D(\rho_k)$. The fixed point set of M is precisely the union of the zero sections in the $D(\rho_k)$; i. e., $\cup H^{n-2k}$, with normal bundles ρ_k classified by $\pi_2 \circ g_k$ and maps to $F\tau$ given by $f_k | H^{n-2k} = \pi_1 \circ g_k$. thus $\beta = F[M, \mu, f]$.

(6) $ker \kappa_* \subset imS$. If $\gamma = [M, \mu, f] \in \mathfrak{R}^n(X, A, \tau)$ and $\kappa_*(\gamma)=0$, there is a manifold W with $\partial W = M \cup W'$ with semi-free s^1-action ω extending μ and an equivariant map h: $(W, W') \rightarrow (X, A)$ extending f. The fixed-point set of ω is a submanifold with boundary, with boundary contained in the interior of W', and not meeting M. One may then deform h, supporting it kept the same on M so that $h | V_k = h | F_\omega{}^{n-2k} \circ \pi$ on tubular neiborhoods V_k of $F_\omega{}^{n-2k}$, identifying V_k with $D(\nu_k)$. Then $(W - \cup V_k{}^0,\ W' - \cup (W' \cap V_j)^\circ,\ \omega,\ h)$ is a free bordism of (M, μ, f) with $(\cup_k S(\nu_k)$, m, $\cup_k (h | F_\omega{}^{n-2k} \circ \pi))$ and hence γ is in the image of S.

3. Bordism of bundles with semi-free s^1-action.

As an application of the equivaliant bordism technique, one may consider the classification of bundles with semi-free s^1-action over manifolds with semi-free s^1-action. Specifically one considers 4-tuples $(M, \sigma, \xi, \sigma^*)$ with M a closed differentiable manifold, σ: $M \rightarrow M$ a differentiable semi-free s^1-action, ξ a complex n-plane bundle over M, and σ^*: $E(\xi) \rightarrow E(\xi)$ a semi-free s^1-action given by a complex vector bundle map covering σ. Two such 4 tuples are bordant if there is a similar 4 tuple (V, ρ, η, ρ^*) with base V being a manifold with bondary and with $(\partial V, \rho | \partial V, \eta | \partial V, \rho^*)$ being the disjont union of the two given 4 tuples.

Taking the operation induced by disjoint union, the equivalence classes of such 4 tuples form an abelian group given by $\mathfrak{R}_*(B\mathrm{U}(n), \tau)$ where $(B\mathrm{U}(n), \tau)$ is described as follows.

Let $B\mathrm{U}(n)$ be the grassmannian of n-dimensional complex subspace of infinite dimensional spaces $C^\infty \times C^\infty \times \cdots\cdots$. The linear transformation t_θ: $C^\infty \times C^\infty \times \cdots\cdots \rightarrow C^\infty \times C^\infty \cdots\cdots$: $(Z, Z_1, Z_2, \cdots) \rightarrow (Z, \theta Z_1, \theta^2 Z_2, \cdots)$ $\theta \in S^1$ induces an semi-free s^1-action τ: $B\mathrm{U}(n) \rightarrow B\mathrm{U}(n)$ by sending an n-dimensional subspace into its image under t.

Over $B\mathrm{U}(n)$ one has the universal n-plane bundle γ^n consisting of pairs (α, a) with α an n-plane in $C^\infty \times C^\infty \times \cdots\cdots$ and a vector in α, with t inducing an semi-free s^1-action τ^* on the total space of γ^n by $\tau^*\ (\alpha, a) = (\tau\alpha, ta)$. By taking the induced bundle and semi-free s^1-action we estabish a one to one correspondence between the equivariant homotopy classes of maps of (X, ρ) into $(B\mathrm{U}(n), \tau)$, with

X a compact hausdorff space and ρ an semi-free s^1-action of X, and the isomorphism classes of n-plane bundles over X with bundle semi-free s^1-action covering ρ.

In order to analyze the group $\mathfrak{R}^*$ $(BU(n),\ \tau)$ we consider first the free equivariant bordism $\widehat{\mathfrak{R}}_*(BU(n),\ \tau)$. This is clearly equivalent to the study of manifold-bundle 4 tuples $(M,\ \sigma,\ \xi,\ \sigma^*)$ in which σ is fixed point free, and we have

Proposition 2. The homorphism $g{:}\widehat{\mathfrak{R}}_*(BU(n),\ \tau)\to\mathfrak{R}_*(BU(1)\times BU(\text{n}))$ induced by sending $(M,\ \sigma,\ \xi,\ \sigma^*)$ to the class of the map $f\times g$: $M/\sigma\to BU(1)\times BU(n)$, with f inducing the principle S^1 bundle $M\to M/\sigma$ and g classifying the complex n-plane bundle $E(\xi)/\sigma^*\to M/\sigma$ is an isomorphism.

Proof. Being given h: $N\to BU(1)\times BU(n)$ we has the principle S^1 bundle $\widetilde{N}\to N$ induced from $\pi_1\circ h$, with semi-free S^1 action given by scaler multiplication and letting ξ be the complex n-plane bundle over N induced by $\pi_2\circ h$, $\pi^*\xi$ has an semi-free s^1-action ν^* induced by restriction of $\nu\times 1$ on $\widetilde{N}\times E(\xi)$ to $E(\pi^*\xi)$. The homorphism q': $\mathfrak{R}_*(BU(1)\times BU(n))\to\widehat{\mathfrak{R}}_*(BU(n),\ \tau)$ defined by q' $([\widehat{N}, h])=[(N, \nu,\ \pi^*\xi,\ \nu^*)]$ is clearly inverse to q.

The fixed point set of the semi-free s^1-action τ: $BU(n)\to BU(n)$ is the union of components $BU(i_1)\times BU(i_2)\times\cdots\times BU(i_j)$, $\imath_1+i_2+\cdots i_j=n$. Applying proposition 2 we have an exact sequence

$$\begin{tikzcd}
\widehat{\mathfrak{R}}_*\ (BU(n),\ \tau) \arrow[r, "k_*"] & \mathfrak{R}_*(BU(n),\ \tau) \arrow[r, "F"] & \bigoplus_{k=0}^{[*/2]} \bigoplus_{(i_1,\, i_2,\, \cdots,\, i_j)\,|\, i_1+i_2\cdots+i_j=n)} \mathfrak{R}_{*-2k}(BU(k)\times BU(i_1)\times \cdots\times BU(i_j)). \arrow[ll, bend left, "S"]
\end{tikzcd}$$

Fixing the attention on the summand $\mathfrak{R}_{m-2}$ $(BU(1)\times BU(n))$ we consider a map h: $N^{m-2}\to BU(1)\times BU(n)$ which maps under S to the class of $((s(\eta),\ m,\ i\circ(\pi_2\circ h)\circ\pi)$ where η is induced by $\pi\ \circ$h, m is the scaler multiplication on the sphere bundle $S(\eta)$, and

$$S(\eta)\xrightarrow{\pi} N\xrightarrow{\pi_2\circ h} BU(n)\xrightarrow{i} BU(n)$$

which i being the inclusuin of $BU(n)=BU(o)\times\cdots BU(o)\times BU(n)\times BU(o)\times\cdots BU(o)$ in $BU(n)$ *as* part of the fixed point set of τ. Since $BU(o)\times\cdots BU(o)\times BU(n)\times BU(o)\times\cdots BU(o)$ is covered by the universal bundle with trivial s^1-action, this is presicely $(S(\eta),\ m,\ \pi^*\xi,\ m^*)$ where ξ is induced by $i\circ(\pi_2\circ h)$ and m_* by $m\times 1$ on $S(\eta)\times E(\xi)$. Then q' coinsides with S: $\mathfrak{R}_{m-2}$ $(BU(1)\times BU(n))\to\widehat{\mathfrak{R}}_{m-2}$ $(BU(n),\ \tau)$ and q defines a spliting of the above sequence, which proves

Prposition 4. The homomorphism q defines a splitting of the sequence of proposition 1 for the semi-free s^1-action $(BU(n),\ \tau)$ and hence $\mathfrak{R}_m(BU(n),\ \tau)\ \overset{F}{\cong}\ \bigoplus_{k=0}^{[m/2]}{}_{(k\fallingdotseq 1,\, i\fallingdotseq n)}\mathfrak{R}_{m-2k}$ $(BU(k)\times BU(i_1)\times\cdots BU(i_j))$. The groups involved are of course well known since $\mathfrak{R}_*(BU(k)$ is known and the kunneth theorem holds for ordinary bordism.

A splliting homomorphism

$$\rho\colon \oplus_{k=0}^{[*/2]} \oplus_{(i_1,\cdots,i_j\,|\,i_1+\cdots i_j=n)} \mathfrak{N}_{*-2k}(BU(k)\times BU(i_1)\times\cdots BU(i_j)\to\mathfrak{N}_*(BU(n),\ \tau)$$

may be constructed by sending a map $h\colon N^{m-2k}\to BU(k)\times BU(i_1)\times\cdots BU(i_j)$, $i_1+\cdots+i_j=n$, to the class of $(CP(\xi_1+1^c),\ \sigma,\ \pi^*\xi_{i_1}\otimes\lambda\underbrace{\otimes\cdots\otimes}_{n_{i_1}}\lambda\oplus\cdots\pi^*\xi_{i_j}\otimes\lambda\underbrace{\otimes\cdots\otimes}_{n_{i_j}}\lambda)$ where ξ_i is the bundle induced by $\pi_i\circ h$, $CP(\xi_1+1^c)$ is the complex projective space bundle of the complex lines on the fibers of the whitney sum ξ_1 with a trivial bundle, σ is the semi-free s^1-action induced on the space of lines by s=scaler multiplication $\times$ 1 on $E(\xi_1)\times C=E(\xi_1\otimes 1^c)$, λ is the canonical line bundle over $CP(\xi_1+1^c)$ with total space consisting of pairs $(\alpha,\ a)$ with α a complex line in a fiber and a a vector in that line, $\pi\colon CP(\xi_1+1^c)\to N$ is the projection, n_i is related to semi-free s^1-action induced by scaler multiplication θ^{n_i}, $\theta\in S^1\subset C$, and σ^* is the semi-free s^1-action described as follows. On $\lambda\otimes\cdots\lambda\otimes\pi^*\xi_{i_k}$, σ^* is the n_{i_k}-th tensor product of the semi-free s^1-action S^* on λ given by $S^*(\alpha,\ a)=(\sigma\alpha,\ sa)$ and $\sigma\times\theta^{n}{}_{i_k}$ on $\pi^*\xi_{i_k}$.

To verify that this is a splitting we simply look at the fixed point set, formed as the disjoint union of $CP(\xi_1)$ and $CP(1^c)$ contained in $CP(\xi_1+1^c)$.
For the $CP(1^c)$ component, we idetifies $CP(1^c)$ with N by π, and the normal bundle is identified with ξ_1, while λ is the trivial bundle with $S^*=1$, so the complex n-plane bundle is $\xi_{i_1}\oplus\cdots\oplus\xi_{i_j}$ with semi-free s^1-action $\theta^{n}{}_{i_1}\oplus\cdots\oplus\theta^{n}{}_{i_j}$ and this fixed component recovers $(N,\ h)$. The component $CP(\xi_1)$ is of codimension 2, with normal bundle $\lambda'=\lambda|CP(\xi_1)$ with S^* acting as conjugate scaler multiplication in the fiber of λ', so that σ^* is the trivial s^1-action given by the identity on the bundle $\pi^*\xi_{i_1}\otimes\lambda'\underbrace{\otimes\cdots\otimes}_{n_{i_1}}\lambda'\oplus\cdots\oplus\pi^*\xi_{i_j}\otimes\lambda'\underbrace{\otimes\cdots\otimes}_{n_{i_j}}\lambda'$ and thus we obtain a component in the summand $\mathfrak{N}_{m-2}(BU(1)\times BU(i_1)\times\times BU(i_j))$ which is the image of the splitting q as defined above. In particular, $F\rho F(M,\ \sigma,\ \xi,\ \sigma^*)$ and $F(M,\ \sigma,\ \xi,\ \sigma^*)$ differ only in $(k=2,\ i=n)$ term, so $\rho F=1$.

References

[1] P. E. Conner and E. E. Floyd, Differentiable Periodic Maps, Springer-Verlag, Berlin.

[2] R. E. Stong, Bordism and involutions, Ann of Math. 90 (1969.)