A Note on the Dimension Subgroups

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§ 1. Introduction

The n th dimension subgroup $D_n(G)$ of a group G is the subgroup of G consisting of elements x such that $x-1 \in I_G^{n+1}$, where I_G is the augmentation ideal of the group ring ZG.

On the other hand, we write G_n as the n th term in the lower central series of G.

The dimension conjecture is referred to :

$D_n(G)=G_n$ for all groups G and all integers $n \geqq 0$.

The case $n=0$ is trivial, and the case $n=1$ is proved from the fact $G/G_1=I/I^2$, etc. (1).

This conjecture was introduced by MAGNUS.

He established when G is any free group. (5).

Attempts by CORN and LOSEY to establish this conjecture were unsuccessful except in a special case (6).

HIGMANN proved that to establish the dimension conjecture it is enough to give an affirmative answer when G is any finite p-group (1), and from the process of his proof, the following important theorem was induced (1, 6) :

THEOREM 1. If G is a group such that $D_n(G) \neq G_n$, then there is a subquotient group L of G with the properties :

(a) L is a finite p-group of class $\leqq n$.

(b) $D_n(L) \neq \{1\}$.

In this paper the auther tries to give another proof of this theorem, which is successful in a very special case.

REMARKS (1) $D_n(G) \supset G_n$ is an easy exercise.

(2) PASSI proved the following results using THEOREM 1. and various cohomological methods (6) :

(a) $D_2(G)=G_2$ for all groups G.

(b) If G is a group such that none of its subquotients is a 2-group of class 3, then $D_3(G)=G_3$.

(c) If G is a group such that all its subquotients which are 2-groups of class 3 have order 64, then $D_3(G)=G_3$.

(3) CORN and LOSEY proved $D_n(G)=G_n$, for all $n \geqq 0$, when G satisfies :

$x \in G_n,\ x \notin G_{n+1},\ x^k \in G_{n+1} \Rightarrow x^k=1$.

§ 2. Dimension subgroups of a quotient group

Let G be a group, $T=Q/Z$ the additive group of rationals mod 1, and φ a mapping $G\to T$ of the underlying set.

The mapping φ can be extended by liniarity to a homomorphism of the additive group of the group ring ZG to T.

$\varphi: G\to T$ is called a polynomial map of degree$\leqq n$ if φ vanishis on I^{n+1}.

LEMMA 1. If G is a group, then
$D_n(G)=\{x\,|\,x\in G,$ and $\varphi(x)=\varphi(1)$ for all polynomial maps $\varphi: G\to T$ of degree$\leqq n\}$ (6).

LEMMA 2. If G is a group, and N a normal subgroup of G, then
$I_{G/N}^{n+1}\cong(I_G^{n+1}+(ZG)I_N)/(ZG)I_N\cong I_G^{n+1}/(ZG)I_N\cap I_G^{n+1}$.

If N is contained in $D_n(G)$, then $I_{G/N}^{n+1}\cong I_G^{n+1}/(ZG)I_N$.

PROOF. Let $\bar{\pi}: ZG\to Z(G/N)$ be the ring epimorphism induced from the natural group epimorphism $\pi: G\to G/N$, then $Ker\,\bar{\pi}$ is obviously $(ZG)I_N$, the ideal of ZG generated by the augmentation ideal of ZN. $\bar{\pi}$ maps I_G^{n+1} onto $I_{G/N}^{n+1}$, so the upper isomorphisms are established.

Now let N be contained in $D_n(G)$, then $\{g(h-1)\}$, where $g\in G$ and $h\in N$, generates $(ZG)I_N$ over Z. Observing $g(h-1)\in I_G^{n+1}$, we can see $(ZG)I_N\subset I_G^{n+1}$, which leads to the lower isomorphism. Therefore the lemma is proved.

PROPOSITION 1. If N is a normal subgroup of a group G, then
$D_n(G/N)=D_n(G)N/N$.

Especially if N is contained in $D_n(G)$, then $D_n(G/N)$ $D_n(G)/N$.

PROOF. Let $gN\in D_n(G)N/N$, $\psi: G/N\to T$ be any polynomial map of degree $\leqq n$, then we can assume that g belongs to $D_n(G)$. Now $\varphi=\psi\pi$, where π is the natural epimorphism of G onto G/N, is a polynomial map $G\to T$ of degree$\leqq n$ because $\varphi(I_G^{n+1})\subset\psi(I_{G/N}^{n+1})=0$. So from $g\in D_n(G)$ and LEMMA 1, it follows $\psi(gN)=\varphi(g)=\varphi(1)=\psi(N)$. Therefore $gN\in D_n(G/N)$.

Conversely let $gN\in D_n(G/N)$. Then $gN-N\in I_{G/N}^{n+1}$, and observing the upper isomorphisms of LEMMA 2, we can see
$g-1+(ZG)I_N\in(I_G^{n+1}+(ZG)I_N)/(ZG)I_N$, which leads to $gh-1\in I_G^{n+1}$, where h is a suitable element in N. Therefore $gh\in D_n(G)$, i. e. $gN\in D_n(G)N/N$.

Now the first equality is established.

The second equality is naturally induced from the first.

COROLLARY 1. If G is a group such that $D_n(G)\neq G_n$, then there is a quotient group K of G with the properties:

(a) K is a nilpotent group of class$\leqq n$.

(b) $D_n(K)\neq\{1\}$

PROOF. Put $K=G/G_n$, then clearly K is a nilpotent group of class$\leqq n$, and by the proposition,

$D_n(K)=D_n(G)/G_n \neq \{1\}$

This completes the proof.

COROLLARY 2. In the hypothesis of the proposition, there is a one-to-one correspondence between the set of all polynomial maps $G \to T$ of degree$\leqq n$ and the set of all polynomial maps $G/N \to T$ of degree$\leqq n$.

Especially there is a one-to-one correspondence between the set of all polynomial maps $G \to T$ of degree$\leqq n$ and the set of all polynomial maps $G/D_n(G) \to T$ of degree$\leqq n$.

PROOF. For a polynomial map $\psi : G/N \to T$ of degree$\leqq n$, we correspond $\psi\pi$: $G \to G/N \to T$ which, as was seen in the proof of the proposition, is a polynomial map of degree$\leqq n$.

Conversely let $\varphi : G \to T$ be a polynomial map of degree$\leqq n$.

Then observing $h-1 \in I_G^{n+1}$, we see :

$\varphi(gh)-\varphi(g)=\varphi(gh-g)=\varphi(g(h-1)) \in \varphi(I_G^{n+1})=0.$

So the image of gN by φ is uniquely determined and the map $\bar{\varphi} : G/N \to T$ is induced such that $\varphi=\bar{\varphi}\pi$

The fact that $\bar{\varphi}$ is a polynomial map of degree$\leqq n$ is easily seen by $\bar{\varphi}(I_{G/N}^{n+1})=\bar{\varphi}\pi(I_G^{n+1})=\varphi(I_G^{n+1})=0.$

The fact that those correspondences are inverse to each other is trivial.

This completes the proof.

§ 3. A proof of THEOREM 1 in the case G/G_n is finite

LEMMA 3. If $G, H, \ldots, K$ are groups, then

$$(G \times H \times \ldots \times K)_n = G_n \times H_n \times \ldots \times K_n,$$
$$D_n(G \times H \times \ldots \times K) = D_n(G) \times D_n(H) \times \ldots \times D_n(K).$$

PROOF. The upper equaltiy is suitable as an exercise in Group Theory, and we omit its proof.

Let $f : Z(G \times H) \to ZG$ be the ring epimorphism induced from the projection $G \times H \to G$. Then easily $f(I_{G \times H}^{n+1})=I_G$.

Let $\alpha \in D_n(G \times H)$. We write $\alpha=(g, h)$, $g \in G$, $h \in H$.

From the fact $\alpha-1 \in I_{G \times H}^{n+1}$, it follows $g-1=f(\alpha-1) \in I_G^{n+1}$, so $g \in D_n(G)$. Similarly $h \in D_n(H)$.

Now we have $D_n(G \times H) \subset D_n(G) \times D_n(H)$, and the converse is trivial.

Consequently it follows :

$$D_n(G \times H \times \ldots \times K)=D_n(G) \times D_n(H \times \ldots \times K)=\ldots=D_n(G) \times D_n(H) \times \ldots \times D_n(K).$$

This completes the proof.

COROLLARY. If there is a finite nilpotent group G of class$\leqq n$ such that $D_n(G) \neq \{1\}$, then there is a finite p-group L of class$\leqq n$ such that $D_n(L) \neq \{1\}$.

PROOF. As any finite nilpotent group is the direct product of its Sylow subgroups, we find a suitable Sylow subgroup of G as L in the corollary.

This completes the proof.

Combining COROLLARY 1 to PROPOSITION 1 and COROLLARY to LEMMA 3, THEOREM 1 is verified in the case G/G_n is finite.

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Supplementary Note

The auther lectured on the contents of this paper in the 18 *th*. 3, 1974 at the *society for the study of Cohomology Theory of Finite Groups* in The Institute of Mathematical Analysis, Kyoto.