S^1-immersion and imbedding up to cobordism

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1. Introduction.

Being given a class $\alpha \in \Omega_n(S^1)$ in the oriented cobordism group of the oriented manifold with a semi-free S^1-action, we define integers $\phi(\alpha)$, $\psi(\alpha)$ and $\eta(\alpha)$ by:

(a) $\phi(\alpha)$ is the smallest integer r for which there is a representative (M^n, T) of which immerses equivariantly in $R^s \times C^r$ for some s, where $R^s \times C^r$ with the semi-free S^1-action $1 \times (z)$, $z \in S^1$.

(b) $\psi(\alpha)$ is the smallest integer r for which there is a representative (M^n, T) of α which imbeds equivariantly in $R^s \times C^r$ for some s.

Being given α, the fixed data of α consists of classes $\alpha_j \in \Omega_{n-2j}(BU(j))$.

(c) $\eta(\alpha)$ is the smallest integer k such that each α_j lies in the image of $\Omega_{n-2j}(G_{j,k})$, where $G_{j,k}$ is the complex Grassmanian of complex j planes in C^k.

We obtain the following theorem,

Theorem. $\phi(\alpha)=\psi(\alpha)=\eta(\alpha)$.

This is the analogy of the result of R. E. Stong in Z_2 case [1].

2. Proof of the Theorem.

Clearly $\phi(\alpha) \leqq \psi(\alpha)$ since an imbedding is an immersin, so it suffices to show that $\eta(\alpha) \leqq \phi(\alpha)$ and $\psi(\alpha) \leqq \eta(\alpha)$.

To see that $\eta(\alpha) \leqq \phi(\alpha)$, suppose (M^n, T) represents α and $f: (M^n,T) \longrightarrow R^s \times C^r$ is an immersion. Then on the $(n\text{-}2j)$-dimensional component of the fixed set F^{n-2j}, the normal bundle in M^n, ν^j, has a complement ρ of dimension $r\text{-}j$, for F^{n-2j} immerses in $R^s \times O$ and ν^j is a subbundle of the pullback of the normal bundle of $R^s \times O$ in $R^s \times C^r$, which is a trivial complex r-plane bundle. Thus ν^j is classified by a map into $G_{j,r}$, and $\eta(\alpha) \leqq r$.

To see that $\psi(\alpha) \leqq \eta(\alpha)$, let α be given and represent α_j by maps $F^{n-2j} \longrightarrow G_{j,\eta(\alpha)}$. Then the F^{n-2j} may be imbedded in R^s for some larges ($s > 2n+1$ will suffice) and the normal bundle ν^j has a complement ρ of dimension $\eta(\alpha)-j$ so that $D(\nu^j) \subset D(\nu^j \oplus \rho)$ imbeds (fiberwise) in the trivial bundle $R^s \times C^{\eta(\alpha)}$ (in fact in the space $R^s \times D^{2\eta(\alpha)}$). Letting N be the tubular neighorhood of the fixed data given by the union of the $D(\nu^j)$, ∂N imbeds equivariantly in $R^s \times S^{2\eta(\alpha)-1}$ or $\partial N/S^1$ imbeds in $R^s \times CP^{(\eta(\alpha)-1)}$. The map $g: \partial N/S^1 \longrightarrow R^s \times CP^{(\eta(\alpha)-1)}$ bounds in $R^s \times CP(\infty)$, that being the condition that the collection of α_j come from some α, and $\Omega_*(CP(n)) \longrightarrow \Omega_*(CP(\infty))$ is monic, because in the diagram

$$\begin{array}{ccc} \Omega_* \otimes H_*(CP(n); Z) & \xrightarrow[\cong]{\theta_*} & \Omega_*(CP(n)) \\ \downarrow 1\otimes i_* & & \downarrow i_* \\ \Omega_* \otimes H_*(CP(\infty); Z) & \xrightarrow[\cong]{\theta_*} & \Omega_*(CP(\infty)) \end{array}$$

for element $[M]\otimes x$ of $\Omega_*\otimes H_*(CP(n)\ ;\ Z)$, $\theta_*(1\otimes i_*)\ (M\otimes x)=[M]\theta(i_*(x))=[M]i_*(\theta(x))$. $\theta: H_*(X\ ;\ Z)\longrightarrow\Omega_*\ (X)$ is the homomorphism satisfing $\mu\theta=id$ and $\mu: \Omega_*\ (X)\longrightarrow H_*\ (X)$ is Thom homomorphism. i is inclusion. The above diagram commute and $1\otimes\ i_*$ is monic so i_* is monic [Uchida 2]. So g bounds. For s sufficiently large ($s>2n+1$ being sufficient) g bounds an imbedded manifold with boundary h: $W\longrightarrow R^s\times CP(\eta(\alpha)-1)$ and taking induced S^1-bundle $\tilde{h}$: $\tilde{W}\longrightarrow R^s\times S^{\eta(\alpha)-1}$ is an imbedding. Joining N and $\tilde{W}$ along their common boundary gives a closed manifold M^n, with the semi-free S^1-action induced by z in the fibers and the semi-free S^1-action of the pricipal action on $\tilde{W}$, and with an imbedding $f: (M,\ T)\longrightarrow R^s\times C^{\eta(\alpha)}$ induced by $\tilde{h}$ and the imbedding of $D\ (\nu^j)$ in $R^s\times D^{2\eta(\alpha)}$. Since the fixed data of $(M^n,\ T)$ is given by the F^{n-2j} and ν^j, $(M^n,\ T)$ represents α, and so $\psi(\alpha)\leqq\eta(\alpha)$. (Using a tubular neighborhood of $\partial N/S^1$ in W, one make f smooth.)

3. Generator.

Acording N. Shimada [3], the generating set of the bordism group $\Omega_*(S^1)$ is given in the following.

If T_0 is the standard S^1-action on D^2, then for a manifold $(M^n,\ T)$ with a semi-free S^1-action T, we form a manifold $(\tilde{M}^{n+2},\ \tilde{T})$ from $(-D^2\times M^n, T_0\times 1)$ and $(D^2\times M^n, T_0\times T)$ by identifying the boundaries via the equivariant diffeomrphism φ: $(S^1\times M^n,\ T_0\times 1)\longrightarrow (S^1\times M^n,\ T_0\times T)$ which is defined by $\varphi\ (s,\ x)=(s,\ sx)$. We then define Γ by

$$\Gamma(M^n,\ T)=(\tilde{M}^{n+2},\ T)=(-D^2\times M^n,\ T_0\times 1)_1\cup_\varphi (D^2\times M^n,\ T_0\times T)_2.$$

The generators then consists of all classes $\alpha=[\Gamma^j CP(n_1)\times CP(n_2)\times\cdots\cdots CP(n_k)\times M^n]$, with the diagonal action, M having the trivial action and $[M]$ forming a generating set for Ω_*, each $n_j>1$, and $CP(n)$ have the S^1-action $T\ ([x_0,\cdots\cdots,x_n])=[zx_0,\ x_1,\ \cdots,\ x_n]$, $z\in S^1$.

The fixed data of $CP(n)$ is $CP(0)\longrightarrow G_{n,n}$, and $CP(n-1)\longrightarrow G_{1,n}$.
The fixed data of $\Gamma(M^n)$ is obtained by adding a trivial line bundle to the fixed set of M^n, i. e. if $F^{n-2J}\longrightarrow G_{j,k}$, one transformed to $G_{j+1,k+1}$ and by taking M^n with a trivial line bundle in $G_{1,1}$. For a product, the fixed data is classified via the whitney sum maps.

$$G_{j,k}\times G_{m,n}\longrightarrow G_{j+m,k+n}.$$

It is then immediate that for $\alpha=[\Gamma^i CP(n_1)\times\cdots\cdot\times CP(n_k)\times M^n]$, $\eta(\alpha)\leqq i+n_1+\cdots\cdots+n_k$.

The fixed component of least dimension in $\Gamma^i CP(n_1)\times\cdots\cdots\times CP(n_k)\times M^n$ is M^n with trivial normal bundle of dimension $j=i+n_1+\cdots\cdots+n_k$, and if $[M^n]\neq 0$ in Ω_n, this gives a nontrivial class in $\Omega_n(G_{j,j})\subset\Omega_n(BU(j))$, so $\eta(\alpha)\geqq i+n_1+\cdots\cdots+n_k$. Thus, we have the theorem:

If $\alpha=\Gamma^i CP(n_1)\times CP(n_2)\times\cdots\cdots\times CP(n_k)\times M^n]$with[M $\neq$ 0] in Ω_n, then $\eta(\alpha)=i+n_1+\cdots\cdots+n_k$.

Reference

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