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On the Ricci tensor and the generalized Tanaka-Webster connection of real hypersurfaces in a complex space form

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Abstract. We prove that the Ricci tensor $\hat{S}$ with respect to the generalized Tanaka-Webster connection of a real hypersurface with the almost contact structure (η, ϕ, ξ, g) in a complex space form of complex dimension $n \geq 3$ satisfies $\hat{S}(X, \phi Y) = \lambda g(X, \phi Y)$ for any vector field X and Y, λ being a function, if and only if the real hypersurface is locally congruent to some type (A) hypersurface.

1. Introduction

Tanaka-Webster connection is a unique affine connection on a non-degenerate, pseudo-Hermitian CR manifold which associated with the almost contact structure ([12], [14]). Tanno [13] gave the *generalized Tanaka-Webster connection* (*g-Tanaka-Webster connection*) for contact metric manifolds, which coincides with Tanaka-Webster connection if the associated CR-structure is integrable. For a real hypersurface in a Kählerian manifold with an almost contact metric structure (η, ϕ, ξ, g), in [3] and [4], Cho defined the g-Tanaka-Webster

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connection $\hat{\nabla}^{(k)}$ for a non-zero real number k. Then we can see that $\hat{\nabla}^{(k)}\eta = 0$, $\hat{\nabla}^{(k)}\xi = 0$, $\hat{\nabla}^{(k)}g = 0$, $\hat{\nabla}^{(k)}\phi = 0$. Moreover, if the shape operator A of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection.

For real hypersurfaces in a complex space form $M^n(c)$ of constant holomorphic sectional curvature $4c \neq 0$, one of the major problem is to determine real hypersurfaces satisfying certain geometrical assumptions. Cho [5] determined flat Hopf hypersurfaces in a non-flat complex space form with respect to the g-Tanaka-Webster connection. Besides, he classified Hopf hypersurfaces in a non-flat complex space form which admits a pseudo-Einstein CR-structure for the g-Tanaka-Webster connection.

The purpose of this paper is to study real hypersurfaces in a complex space form whose Ricci tensor $\hat{S}$ with respect to the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ satisfies $\hat{S}(X, \phi Y) = \lambda g(X, \phi Y)$ for any vector fields X and Y.

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2. Preliminaries

Let $M^n(c)$ denote the complex space from of complex dimension n (real dimension $2n$) of constant holomorphic sectional curvature $4c$. For the sake of simplicity, if $c > 0$, we only use $c = +1$ and call it the complex projective space $\mathbb{C}P^n$, and if $c < 0$, we just consider $c = -1$, so that we call it the complex hyperbolic space $\mathbb{C}H^n$. We denote by J the almost complex structure of $M^n(c)$. The Hermitian metric of $M^n(c)$ will be denoted by G.

Let M be a real $(2n-1)$-dimensional hypersurface immersed in $M^n(c)$. We denote by g the Riemannian metric induced on M from G. We take the unit normal vector field V of M in $M^n(c)$. For any vector field X tangent to M, we define ϕ, η and ξ by

$$JX = \phi X + \eta(X)V, \qquad JV = -\xi,$$

where ϕX is the tangential part of JX, ϕ is a tensor field of type (1,1), η is a 1-form, and ξ is the unit vector field on M. Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0,$$
$$\eta(X) = g(X, \xi), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Thus (ϕ, ξ, η, g) defines an almost contact metric structure on M. Let H_0 denote the holomorphic distribution on M defined by $H_0(x) = \{X \in T_x(M) | \eta(X) = 0\}$.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation in $M^n(c)$, and by ∇ the one in M determined by the induced metric. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)V, \qquad \tilde{\nabla}_X V = -AX$$

for any vector fields X and Y tangent to M. We call A the *shape operator* of M.

From the Gauss and Weingarten formulas, we have

$$\nabla_X \xi = \phi AX, \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi.$$

We denote by R the Riemannian curvature tensor field of M. Then the *equation of Gauss* is given by

$$\begin{aligned} R(X,Y)Z &= c\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &\quad -2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

and the *equation of Codazzi* by

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}.$$

If $A\xi = \lambda\xi$, λ being a function, then M is called a *Hopf hypersurface*. There are many results for real hypersurfaces in complex space forms under the assumption that they are Hopf hypersurfaces. By the Codazzi equation, we have the following result (c.f. [8]).

Proposition A. *Let M be a Hopf hypersurface in $M^n(c)$, $n \geq 2$, If $X \perp \xi$ and $AX = \beta X$, then $\alpha = g(A\xi, \xi)$ is constant and*

$$(2\beta - \alpha)A\phi X = (\beta\alpha + 2c)\phi X.$$

We use the following results for the proof of the main theorem.

Theorem B ([7]). *Let M be a Hopf hypersurface in $\mathbb{C}P^n$. Then M has constant principal curvatures if and only if M is locally congruent to one of the following*:

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$,*

(A_2) *a tube over a totally geodesic $\mathbb{C}P^l$ $(1 \leq l \leq n-2)$, where $0 < r < \pi/2$,*

(B) *a tube of radius r over a complex quadric Q^{n-1} and $\mathbb{R}P^n$, where $0 < r < \pi/4$.*

(C) *a tube of radius r over $\mathbb{C}P^1 \times \mathbb{C}P^{\frac{n-1}{2}}$, where $0 < r < \pi/4$ and n (≥ 5) is odd,*

(D) *a tube of radius r over a complex Grassmann $\mathbb{C}G_{2,5}$, where $0 < r < \pi/4$ and $n = 9$,*

(E) *a tube of radius r over a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/4$ and $n = 15$.*

Theorem C ([1]). *Let M be a Hopf hypersurface in $\mathbb{C}H^n$. Then M has constant principal curvatures if and only if M is locally congruent to one of the following*:

(A_0) *a horosphere,*

(A_1) *a tube over a complex hyperbolic hyperplane $\mathbb{C}H^k$ $(k = 0, n-1)$,*

(A_2) *a tube over a totally geodesic $\mathbb{C}H^l$ $(1 \leq l \leq n-2)$,*

(B) *a tube over a totally real hyperbolic space $\mathbb{R}H^n$.*

Next we introduce the notion of Tanaka-Webster connection and its generalization. Tanaka [12] defined the canonical affine connection on a non-degenerate, pseudo-Hermitian CR manifold. As a generalization of Tanaka-Webster connection, Tanno [13] defined the g-Tanaka-Webster connection for contact metric manifolds by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y,$$

where (η, ϕ, ξ, g) is a contact metric structure. Using the naturally extended affine connection of Tanno's g-Tanaka-Webster connection, the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for real hypersurfaces in Kähler manifold is given by,

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$$

for a non-zero real number k (see Cho [3], [4]). Then we see that

$$\hat{\nabla}^{(k)}\eta = 0,\ \hat{\nabla}^{(k)}\xi = 0,\ \hat{\nabla}^{(k)} g = 0,\ \hat{\nabla}^{(k)}\phi = 0.$$

In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then the g-Tanaka-Webster connection coincides with the Tanaka-Webster connection. Next we define the g-Tanaka-Webster curvature tensor $\hat{R}$ with respect to $\hat{\nabla}^{(k)}$ by

$$\hat{R}(X,Y)Z = \hat{\nabla}_X(\hat{\nabla}_Y Z) - \hat{\nabla}_Y(\hat{\nabla}_X Z) - \hat{\nabla}_{[X,Y]}Z$$

for all vector fields X, Y, Z in M. We denote by $\hat{S}$ the g-Tanaka Webster Ricci tensor, which is defined by

$$\hat{S}(Y,Z) = \text{trace of } \{X \mapsto \hat{R}(X,Y)Z\}.$$

3. The Ricci tensor of real hypersurfaces in a complex space form

To prove the theorem, we prepare the following lemma.

Lemma 3.1. *Let M be a real hypersurface in a complex space form $M^n(c)$, $n \geq 3$, $c \neq 0$. If there exists an orthonormal frame $\{e_1, \cdots, e_{2n-2}, \xi\}$ on a sufficiently small neighborhood $\mathcal{N}$ of $x \in M$ such that the shape operator A can be represented as*

$$A = \left(\begin{array}{cccc|c} a_1 & & & 0 & h_1 \\ & \ddots & & \vdots & 0 \\ & & \ddots & & \vdots \\ 0 & & & a_{2n-2} & 0 \\ \hline h_1 & 0 & \cdots & 0 & \alpha \end{array}\right),$$

then we have

$$(a_1 - a_j)g(\nabla_{e_i}e_1, e_j) + (a_j - a_i)g(\nabla_{e_1}e_i, e_j) + a_i h_1 g(\phi e_i, e_j) = 0, \tag{3.1}$$

$$(a_j - a_1)g(\nabla_{e_i}e_j, e_1) - (a_i - a_1)g(\nabla_{e_j}e_i, e_1) + h_1(a_i + a_j)g(\phi e_i, e_j) = 0, \tag{3.2}$$

$$\{2c - 2a_i a_j + \alpha(a_i + a_j)\}g(\phi e_i, e_j) - h_1 g(\nabla_{e_i}e_j, e_1) + h_1 g(\nabla_{e_j}e_i, e_1) = 0, \tag{3.3}$$

$$(a_1 - a_i)g(\nabla_{e_i}e_1, e_i) - (e_1 a_i) = 0, \tag{3.4}$$

$$h_1(2a_i + a_1)g(\phi e_i, e_1) + (a_1 - a_i)g(\nabla_{e_1}e_i, e_1) + (e_i a_1) = 0, \tag{3.5}$$

$$(c + a_1\alpha - a_1 a_i - h_1^2)g(\phi e_1, e_i) - (a_1 - a_i)g(\nabla_\xi e_1, e_i) + h_1 g(\nabla_{e_1}e_1, e_i) = 0 \tag{3.6}$$

for any $i, j \geq 2$, $i \neq j$.

PROOF. By the equation of Codazzi, we have

$$g((\nabla_{e_i}A)e_1 - (\nabla_{e_1}A)e_i, e_j) = 0,$$

where $i, j = 2, \cdots, 2n-2$. On the other hand, we have

$$\begin{aligned} &g((\nabla_{e_i}A)e_1 - (\nabla_{e_1}A)e_i, e_j) \\ &= g(\nabla_{e_i}(Ae_1) - A\nabla_{e_i}e_1 - \nabla_{e_1}(Ae_i) + A\nabla_{e_1}e_i, e_j) \\ &= (a_1 - a_j)g(\nabla_{e_i}e_1, e_j) + (a_j - a_i)g(\nabla_{e_1}e_i, e_j) + a_i h_1 g(\phi e_i, e_j). \end{aligned}$$

Thus we obtain (3.1). By the similar computation, we have our results. □

Theorem 3.2. *Let M be a real hypersurface in a complex space form $M^n(c)$, $n \geq 3$, $c \neq 0$. We suppose that the Ricci tensor $\hat{S}$ of the generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$ satisfies $\hat{S}(X, \phi Y) = \lambda g(X, \phi Y)$ for any vector fields X and Y, λ being a function.*
(1) *If $c > 0$ and $k^2 \neq 4c$, then M is a Hopf hypersurface.*
(2) *If $c < 0$, then M is a Hopf hypersurface.*

Proof. By the definition of the g-Tanaka-Webster connection, we have (see [5])

$$\begin{aligned} \hat{R}(X,Y)Z \;=\; & R(X,Y)Z + g\big(\phi((\nabla_X A)Y - (\nabla_Y A)X), Z\big)\xi \\ & +2g(\phi AY, Z)\phi AX - 2g(\phi AX, Z)\phi AY \\ & +g\big((\nabla_X \phi)AY - (\nabla_Y \phi)AX, Z\big)\xi \\ & -\eta(Z)\Big(\phi\big((\nabla_X A)Y - (\nabla_Y A)X\big) + (\nabla_X \phi)AY - (\nabla_Y \phi)AX\Big) \\ & -k\Big(g\big((\phi A + A\phi)X, Y\big)\phi Z + \eta(Y)(\nabla_X \phi)Z - \eta(X)(\nabla_Y \phi)Z\Big) \\ & +g(\phi AX, F_Y Z)\xi - \eta(F_Y Z)\phi AX - k\eta(X)\phi F_Y Z \\ & -g(\phi AY, F_X Z)\xi + \eta(F_X Z)\phi AY + k\eta(Y)\phi F_X Z, \end{aligned} \tag{3.7}$$

where F is given by

$$F_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y.$$

By the definition of g-Tanaka-Webster Ricci tensor, equation of Gauss and Codazzi, direct calculation shows that

$$\begin{aligned} \hat{S}(Y,Z) \;=\; & 2ncg(Y,Z) + (\mathrm{tr}A - \eta(A\xi) + k)g(AY,Z) \\ & -g(A^2Y,Z) - g(\phi A\phi AY, Z) - kg(\phi A\phi Y, Z) + \eta(AY)g(A\xi, Z) \\ & +\eta(Z)\big(-2nc\eta(Y) - \eta(AY)\mathrm{tr}A + \eta(A^2Y) - k\eta(AY)\big). \end{aligned}$$

Now we use the following lemma of Ryan [10].

Lemma D. Let A be a symmetric tensor field of type (1,1) on a connected Riemannian manifold M^n. Then there exists $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ such that for each point x, $\{\lambda_i(x)\}(i = 1, \cdots, n)$ are the eigenvalues of A_x.

For the shape operator A of a real hypersurface M, we consider the symmetric tensor field $\phi A\phi$ of type (1,1). By the above lemma, we can take an ortonormal frame $\{v_1, ..., v_{2n-2}, \xi\}$ in a neighborhood of a point x such that $\phi A\phi\xi = 0$, $\phi A\phi v_1 = -a_1 v_1, \cdots, \phi A\phi v_{2n-2} = -a_{2n-2}v_{2n-2}$. Then we have

$$g(A\phi v_i, \phi v_j) = -g(\phi A\phi v_i, v_j) = 0 \ (i \neq j),$$
$$g(A\phi v_i, \phi v_i) = -g(\phi A\phi v_i, v_i) = a_i.$$

We take an orthonormal frame $\{e_1 = \phi v_1, ..., e_{2n-2} = \phi v_{2n-2}, \xi\}$ in a neighborhood $\mathcal{N}$ of a point x. Then, in the neighborhood, A is of the form

$$A = \left(\begin{array}{ccc|c} a_1 & \cdots & 0 & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{2n-2} & h_{2n-2} \\ \hline h_1 & \cdots & h_{2n-2} & \alpha \end{array}\right),$$

where we have put $h_i = g(Ae_i, \xi)$, $i = 1, \cdots, 2n-2$, and $\alpha = g(A\xi, \xi)$.

The condition $\hat{S}(X, \phi Y) = \lambda g(X, \phi Y)$ for any vector fields X and Y is equivalent to $\hat{S}(X, Y) = \lambda g(X, Y)$ for any vector field X and any vector field Y orthogonal to ξ. By the direct computation using the previous equation, we have

$$\hat{S}(\xi, \xi) = 0, \ \hat{S}(e_i, \xi) = 0,$$
$$\hat{S}(\xi, e_i) = (\mathrm{tr}A - \alpha + k - a_i)h_i - g(\phi A\phi A\xi, e_i) = 0, \tag{3.8}$$
$$\hat{S}(e_i, e_i) \tag{3.9}$$
$$= 2nc + (\mathrm{tr}A)a_i - a_i^2 - \alpha a_i + ka_i + (a_i + k)g(A\phi e_i, \phi e_i) = \lambda,$$
$$\hat{S}(e_i, e_j) = (a_i + k)g(A\phi e_i, \phi e_j) = 0 \qquad (i \neq j). \tag{3.10}$$

In the following, we suppose that M is not a Hopf hypersurface. Then there is a point x and hence an open neighborhood $\mathcal{N}$ of x where $A\xi \neq \alpha\xi$ on $\mathcal{N}$. Then $h_i \neq 0$ for some i.

If $a_i = -k$ for all i at some $x \in \mathcal{N}$, then (3.9) and $\mathrm{tr}A = -(2n-2)k + \alpha$ imply that

$$2nc + (2n-4)k^2 = \lambda.$$

By (3.8),

$$(\mathrm{tr}A - \alpha + 2k)h_i + g(\phi A\xi, A\phi e_i) = 0.$$

Since $g(\phi A\xi, A\phi e_i) = -kh_i, \mathrm{tr}A - \alpha = -(2n-2)k$, we have

$$(2n-3)kh_i = 0.$$

for all i. Thus we have $k = 0$. This contradicts to our assumption. Therefore, $a_i \neq -k$ for some i. From (3.10), if $a_i \neq -k$, then $g(A\phi e_i, \phi e_j) = 0$ for all $j \neq i$. Thus we set

$$A\phi e_i = \bar{a}_i \phi e_i + \bar{h}_i \xi,$$

where we have put $\bar{a}_i = g(A\phi e_i, \phi e_i)$ and $\bar{h}_i = g(A\phi e_i, \xi)$. We also have

$$\hat{S}(\phi e_i, \phi e_i) = 2nc + (\mathrm{tr}A)\bar{a}_i - \bar{a}_i^2 - \alpha\bar{a}_i + k\bar{a}_i + (\bar{a}_i + k)a_i = \lambda. \quad (3.11)$$

Using (3.9) and (3.11), we obtain

$$(a_i - \bar{a}_i)(\mathrm{tr}A - \alpha - a_i - \bar{a}_i) = 0.$$

When $a_i = \bar{a}_i$, (3.9) implies

$$2nc - \lambda = a_i(\alpha - 2k - \mathrm{tr}A).$$

Otherwise, if $a_i \neq \bar{a}_i$, then $\mathrm{tr}A - \alpha = a_i + \bar{a}_i$. Using (3.9), we obtain

$$2a_i^2 - 2(\mathrm{tr}A - \alpha)a_i - k(\mathrm{tr}A - \alpha) - 2nc + \lambda = 0,$$

from which

$$(a_i - a_j)(\mathrm{tr}A - \alpha - a_i - a_j) = 0$$

for a_j that satisfies $a_j \neq k$ and $a_j \neq \bar{a}_j$. If $a_i \neq a_j$, then $\mathrm{tr}A - \alpha = a_i + a_j = a_i + \bar{a}_i$. Hence we have $a_j = \bar{a}_i$. We put $b = a_i$ and $\bar{b} = \bar{a}_i$. They satisfy

$$b + \bar{b} = \mathrm{tr}A - \alpha, \tag{3.12}$$

$$b\bar{b} = -\frac{k}{2}(\mathrm{tr}A - \alpha) - nc + \frac{\lambda}{2}. \tag{3.13}$$

We remark that $b \neq -k$ or $\bar{b} \neq -k$.

From these, in $\mathcal{N}$, we have

$$A = \left(\begin{array}{ccccccccccc|c}
b & & & & & & & & & & & h_1 \\
& \ddots & & & & & & & & & & \\
& & b & & & & & & & & & \\
& & & \bar{b} & & & & & & & & \vdots \\
& & & & \ddots & & & & & & & \\
& & & & & \bar{b} & & & & & & \\
& & & & & & d & & & & & \\
& & & & & & & \ddots & & & & \\
& & & & & & & & d & & & \\
& & & & & & & & & -k & & \\
& & & & & & & & & & \ddots & \\
& & & & & & & & & & -k & h_{2n-2} \\
\hline
h_1 & & & & \cdots & & & & & & h_{2n-2} & \alpha
\end{array}\right),$$

where

$$d = g(Ae_s, e_s) = g(A\phi e_s, \phi e_s) \neq -k,$$
$$2nc - \lambda = d(\alpha - 2k - \mathrm{tr}A). \tag{3.14}$$

In the following, we use integers $y, z, \cdots$ for $Ae_y = be_y + h_y\xi$, $s \cdots$ for $Ae_s = de_s + h_s\xi$ and $v \cdots$ for $Ae_v = -ke_v$. We denote by $H_1(x)$, $H_2(x)$, $H_3(x)$ and $H_4(x)$ the subspaces of a tangential space at x spanned by $\{e_y\}$, $\{\phi e_y\}$, $\{e_s\}$ and $\{e_v\}$, respectively.

We suppose that $\dim H_3(x) \neq 0$ and $\dim H_4(x) \neq 0$ at some $x \in \mathcal{N}$. Taking $e_s \in H_3(x)$ and $e_v \in H_4(x)$, (3.9) implies

$$\hat{S}(e_v, e_v) = 2nc - k(\mathrm{tr}A) - 2k^2 + \alpha k = \lambda.$$

From this and (3.14), we have

$$(d + k)(\alpha - 2k - \mathrm{tr}A) = 0.$$

Since $d \neq -k$, then we have $\mathrm{tr}A - \alpha = -2k$ and $2nc - \lambda = 0$.

Moreover, if $\dim H_1(x) = \dim H_2(x) \neq 0$, taking $e_y \in H_1(x)$, (3.12), (3.13) and (3.14) imply $a_y = b = -k$ and $\bar{a_y} = \bar{b} = -k$. This case cannot be occured. Hence we have $\dim H_1(x) = \dim H_2(x) = 0$. Then, by $\phi e_s \in H_3(x)$ and $\phi e_v \in H_4(x)$, we have $a_i = \bar{a_i}$ for any $i \in \{1 \cdots, 2n-2\}$. Thus, by (3.8) and $\mathrm{tr}A - \alpha = -2k$,

$$(-k - a_i)h_i - g(\phi A \phi A \xi, e_i) = -kh_i = 0$$

for all i. This implies $k = 0$. This contradicts to our assumption.

So, we see that $\dim H_3(x) = 0$ or $\dim H_4(x) = 0$ at any point $x \in \mathcal{N}$, that is,

$$A = \left(\begin{array}{ccccccccc|c} b & & & & & & & & & h_1 \\ & \ddots & & & & & & & & \\ & & b & & & & & & & \\ & & & \bar{b} & & & & & & \vdots \\ & & & & \ddots & & & & & \\ & & & & & \bar{b} & & & & \\ & & & & & & f & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & f & h_{2n-2} \\ \hline h_1 & & & & \cdots & & & & h_{2n-2} & \alpha \end{array}\right),$$

When $\dim H_4 = 0$, f denotes $a_s = d$. We remark that $f = d$ satisfies (3.14). Otherwise, when $\dim H_3 = 0$, f denotes $a_v = -k$. In this case, we see that $\bar{a_v} = -k$ by the definition of b and $\bar{b}$. Thus, using (3.9), $f = -k$ also satisfies

$$2nc - \lambda = -k(\alpha - 2k - \mathrm{tr}A).$$

Hence, $f=\bar{f}$ and f satisfies

$$2nc-\lambda=f(\alpha-2k-\mathrm{tr}A) \tag{3.15}$$

in both cases.

In the following, we use integers $s\cdots$ for $Ae_s=fe_s+h_s\xi$ and redefine $H_3(x)$ as the subspaces of a tangential space at x spanned by $\{e_s\}$.

By a direct computation using (3.8),

$$(\mathrm{tr}A-\alpha+k-b+\bar{b})h_y=0, \tag{3.16}$$
$$(\mathrm{tr}A-\alpha+k+b-\bar{b})\bar{h}_y=0, \tag{3.17}$$
$$(\mathrm{tr}A-\alpha+k)h_s=0. \tag{3.18}$$

Lemma 3.3. *We have $h_s=0$ for all $e_s\in H_3$.*

PROOF. If there exists $e_s\in H_3$ that satisfies $h_s\neq 0$ at some x, and hence on some neighborhood $\mathcal{N}'\subset\mathcal{N}$, then

$$\mathrm{tr}A-\alpha+k=0.$$

From (3.16) and (3.17), we have

$$(-b+\bar{b})h_y=0, \quad (b-\bar{b})\bar{h}_y=0.$$

Since $b\neq\bar{b}$, we have $h_y=0$ and $\bar{h}_y=0$ for all y. The direct computation shows that

$$|tE-A|=(t-b)^p(t-\bar{b})^p(t-f)^{q-1}\{(t-f)(t-\alpha)-\sum_{s=1}^{q}h_s^2\},$$

where p and q are the multiplicities of b and f, respectively. We remark that $2p+q=2n-2$.

Suppose $Ae'=fe'$ is satisfied by $e'=X+\beta\xi$, where $X\in H_3$. Since $AX=fX+h\xi$ for some h, we obtain

$$Ae'=fX+h\xi+\beta(\sum h_se_s+\alpha\xi).$$

On the other hand, we have

$$Ae' = f(X + \beta\xi) = fX + f\beta\xi.$$

From these equations, we obtain

$$\beta \sum h_s e_s + (h + \alpha\beta - f\beta)\xi = 0.$$

Since $h_s \neq 0$ for some e_s, we have $\beta = 0$, that is, $g(e', \xi) = 0$. Thus, in $\mathcal{N}'$, we can represent the shape operator A by a following matrix with respect to a local orthonormal frame $\{e_1, \cdots, e_p, \phi e_1, \cdots, \phi e_p, e_{2p+1}, \cdots, e_{2n-2}, \xi\}$:

$$A = \left(\begin{array}{ccccccccc|c}
b & & & & & & & & & 0 \\
& \ddots & & & & & & & & \\
& & b & & & & & & & \\
& & & \bar{b} & & & & & & \\
& & & & \ddots & & & & & \vdots \\
& & & & & \bar{b} & & & & \\
& & & & & & f & & & \\
& & & & & & & \ddots & & 0 \\
& & & & & & & & f & h_{2n-2} \\
\hline
0 & & & \cdots & & & & 0 & h_{2n-2} & \alpha
\end{array}\right).$$

From (3.15) and (3.18) we obtain

$$2nc - \lambda = -fk, \qquad \mathrm{tr}A - \alpha = -k.$$

We now suppose that there is a point x in $\mathcal{N}'$ where $p \neq 0$. Then (3.12) implies

$$-(p-1)k + qf = 0.$$

By (3.13), we also have

$$b\bar{b} = \frac{1}{2}(k^2 + fk).$$

Using $b + \bar{b} = \mathrm{tr}A - \alpha = -k$, we see

$$(b + \frac{k}{2})^2 + \frac{1}{4}(k + 2f)k = 0.$$

Since $(p-1)k = qf$, we see $fk \geq 0$. This implies that $k + 2f = 0$ and hence $(2p-2+q)k = 0$. Thus we have $k = 0$. This contradicts to our assumption.

Let us suppose that $p = 0$ on $\mathcal{N}'$ of x. Then $\mathrm{tr}A - \alpha = (2n-2)f = -k$ shows that f is non-zero constant on $\mathcal{N}'$ of x. By (3.5), we see that $h_{2n-2}f = 0$. This is also a contradiction. This proves our lemma. □

If there exist $e_y \in H_1$ and $\phi e_z \in H_2$ that satisfy $h_y \neq 0$ and $\bar{h}_z \neq 0$, (3.16) and (3.17) implies $b = \bar{b}$. This case cannot be occured. So it is sufficient to consider the case that $\bar{h}_y = 0$ for any $\phi e_y \in H_2$. Using (3.12) and (3.16), we have

$$b = \mathrm{tr}A - \alpha + \frac{k}{2}, \quad \bar{b} = -\frac{k}{2}. \tag{3.19}$$

By the similar calculation as Lemma 3.3, in $\mathcal{N}$, we can represent the shape operator A by a following matrix with respect to an orthonormal frame $\{e_1, \cdots, e_p, \phi e_1, \cdots, \phi e_p, e_{2p+1}, \cdots, e_{2n-2}, \xi\}$:

$$A = \left(\begin{array}{ccccccccc|c} b & & & & & & & & & h_1 \\ & \ddots & & & & & & & & 0 \\ & & b & & & & & & & \\ & & & \bar{b} & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & \bar{b} & & & & \vdots \\ & & & & & & f & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & f & 0 \\ \hline h_1 & 0 & & & \cdots & & & & 0 & \alpha \end{array}\right).$$

Then we have

$$\mathrm{tr}A = p(b + \bar{b}) + qf + \alpha.$$

Using (3.12),

$$(p-1)(b + \bar{b}) + qf = 0. \tag{3.20}$$

First, we suppose that $\mathrm{tr}A - \alpha = b + \bar{b} \neq 0$ at a point x and hence an open neighborhood $\mathcal{N}'' \subset \mathcal{N}$ of x. Then (3.20) implies that $q \neq 0$

on $\mathcal{N}''$. Because, if $q = 0$ at some point $x \in \mathcal{N}''$, then $p - 1 = 0$ and hence $n = 2$. This contradicts to $n \geq 3$. From (3.13) and (3.19), we have

$$-\frac{k^2}{4} = -nc + \frac{\lambda}{2}, \tag{3.21}$$

from which we see that $-nc + (\lambda/2) \neq 0$ and λ is constant on $\mathcal{N}''$. Thus, by (3.15) and (3.20), we obtain $f \neq 0$ and $p \neq 1$. So we have $p \geq 2$. Using (3.15) and (3.19),

$$2nc - \lambda = f(\alpha - 2k - \mathrm{tr}A) = f\Big(-b - \frac{3}{2}k\Big). \tag{3.22}$$

From (3.19), (3.20), (3.22) and $2p + q = 2n - 2$, we obtain

$$b^2 + kb - \frac{3}{4}k^2 - \frac{(2nc - \lambda)(2n - 2p - 2)}{p - 1} = 0.$$

Since b is continuous and p is positive integer, we see that b is constant. So (3.22) implies that f is also constant on $\mathcal{N}''$.

We put $AU = bU + h_1\xi$ and $AZ = fZ$. By the equation of Codazzi, computing $g((\nabla_Z A)U - (\nabla_U A)Z, \phi Z)$, we have

$$(b - f)g(\nabla_Z U, \phi Z) + fh_1 = 0$$

on $\mathcal{N}''$. Similarly, computing $g((\nabla_Z A)\phi U - (\nabla_{\phi U} A)Z, Z)$,

$$(\bar{b} - f)g(\nabla_Z \phi U, Z) = 0.$$

If $\bar{b} = f$, then (3.21) and (3.22) imply that $b = \bar{b} = -k/2$. This case cannot be occured. So we have $g(\nabla_Z \phi U, Z) = 0$. On the other hand, we obtain

$$\begin{aligned} g(\nabla_Z U, \phi Z) &= -g(U, (\nabla_Z \phi)Z) - g(U, \phi \nabla_Z Z) \\ &= g(\phi U, \nabla_Z Z) = -g(\nabla_Z \phi U, Z) = 0. \end{aligned}$$

From these we have $fh_1 = 0$. This contradicts to $f \neq 0$.

Finally, we consider the case $\mathrm{tr}A - \alpha = b + \bar{b} = 0$ on $\mathcal{N}''$. Then (3.20) implies that $qf = 0$. If $f = 0$, then (3.15) gives $2nc - \lambda = 0$ and hence, by (3.13), we see

$$b\bar{b} = -\frac{k^2}{4} = 0,$$

which contradicts to $k \neq 0$. So we have $q = 0$ on $\mathcal{N}''$.

From (3.13), (3.19) and (3.20),

$$b = -\bar{b} = \frac{k}{2}, \quad b\bar{b} = -nc + \frac{\lambda}{2}.$$

We can choose an orthonormal frame $\{e_1, e_2, \cdots, e_{n-1}, e_n, \cdots, e_{2n-2}, \xi\}$ on M which satisfies $Ae_1 = be_1 + h_1\xi$, $Ae_y = be_y$ for $y = 2, \cdots, n-1$ and $A\phi e_y = \bar{b}\phi e_y$ for $y = 1, \cdots, n-1$. Then, in $\mathcal{N}''$, the shape operator A is represented by the following

$$A = \left(\begin{array}{ccccccc|c} b & & & & & & & h_1 \\ & \ddots & & & & & & 0 \\ & & b & & & & & \\ & & & \bar{b} & & & & \vdots \\ & & & & \ddots & & & \\ & & & & & & \bar{b} & 0 \\ \hline h_1 & 0 & \cdots & & & & 0 & \alpha \end{array}\right).$$

Using Lemma 3.1, we have

Lemma 3.4. *Let $\phi e_y \in H_2$ be perpendicular to ϕe_1. Then,*

$$\nabla_{e_1} e_1 = \frac{h_1}{2}\phi e_1, \tag{3.23}$$

$$\nabla_{\phi e_y} e_1 = \frac{2c + 2nc - \lambda}{h_1} e_y. \tag{3.24}$$

PROOF. Using (3.5), we have $g(\nabla_{e_1}\phi e_y, e_1) = -g(\nabla_{e_1}e_1, \phi e_y) = 0$. On the other hand, putting $e_i = \phi e_1$ in (3.5),

$$h_1(2\bar{b} + b)g(\phi^2 e_1, e_1) + (b - \bar{b})g(\nabla_{e_1}\phi e_1, e_1) = 0,$$

from which we obtain

$$g(\nabla_{e_1}e_1, \phi e_1) = \frac{h_1}{2}.$$

By (3.6), we see that $g(\nabla_{e_1}e_1, e_y) = 0$ for any $e_y \in H_1$. Since $g(\nabla_{e_1}e_1, \xi) = -g(e_1, \phi A e_1) = 0$, we have (3.23).

Next, putting $e_i = \phi e_y$ and $e_j = \phi e_z$ in (3.1), we have $g(\nabla_{\phi e_y} e_1, \phi e_z) = 0$ for any $\phi e_y, \phi e_z \in H_2$, $y \neq z$. Moreover, we have $g(\nabla_{\phi e_y} e_1, \phi e_y) = 0$ by (3.4). On the other hand, using (3.2), we see that

$$g(\nabla_{e_z} \phi e_y, e_1) = 0 \tag{3.25}$$

for any $e_z \in H_1$. Thus, putting $e_i = e_z$ and $e_j = \phi e_y$ in (3.3), direct calculation shows that

$$g(\nabla_{\phi e_y} e_1, e_z) = \frac{2c + 2nc - \lambda}{h_1} g(\phi e_z, \phi e_y).$$

Since $g(\nabla_{\phi e_y} e_1, \xi) = 0$ and $g(\nabla_{\phi e_y} e_1, e_1) = 0$, we have (3.24).

□

Using this lemma, we compute the sectional curvature spanned by e_1 and $\phi e_y \perp \phi e_1$. From (3.23), we have

$$g(\nabla_{\phi e_y} \nabla_{e_1} e_1, \phi e_y) = -\frac{h_1}{2} g(\phi e_1, \nabla_{\phi e_y} \phi e_y).$$

Since $g(\phi e_1, \phi e_y) = 0$, we have

$$\begin{aligned} & g(\phi e_1, \nabla_{\phi e_y} \phi e_y) = -g(\nabla_{\phi e_y} \phi e_1, \phi e_y) = -g(\phi \nabla_{\phi e_y} e_1, \phi e_y) \\ & = -g(\nabla_{\phi e_y} e_1, e_y) = \frac{-2c - 2nc + \lambda}{h_1}. \end{aligned}$$

Thus we obtain

$$g(\nabla_{\phi e_y} \nabla_{e_1} e_1, \phi e_y) = c + nc - \frac{\lambda}{2}.$$

On the other hand, by (3.24),

$$\begin{aligned} g(\nabla_{e_1} \nabla_{\phi e_y} e_1, \phi e_y) &= \nabla_{e_1} g(\nabla_{\phi e_y} e_1, \phi e_y) - g(\nabla_{\phi e_y} e_1, \nabla_{e_1} \phi e_y) \\ &= \frac{-2c - 2nc + \lambda}{h_1} g(e_y, \nabla_{e_1} \phi e_y). \end{aligned}$$

Putting $e_i = \phi e_y$ and $e_j = e_y$ in (3.1), we have $g(\nabla_{e_1} \phi e_y, e_y) = -h_1/2$. From these equations, we obtain

$$g(\nabla_{e_1} \nabla_{\phi e_y} e_1, \phi e_y) = c + nc - \frac{\lambda}{2}.$$

Next, we see that

$$\begin{aligned} & g(\nabla_{[\phi e_y, e_1]} e_1, \phi e_y) \\ & = g(\nabla_\xi e_1, \phi e_y) g(\xi, [\phi e_y, e_1]) + g(\nabla_{e_1} e_1, \phi e_y) g(e_1, [\phi e_y, e_1]) \\ & \quad + \sum_{z \geq 2} g(\nabla_{e_z} e_1, \phi e_y) g(e_z, [\phi e_y, e_1]) + \sum_{z \geq 1} g(\nabla_{\phi e_z} e_1, \phi e_y) g(\phi e_z, [\phi e_y, e_1]) \\ & = 0. \end{aligned}$$

Here we note that we have $g(\nabla_{\phi e_z} \phi e_y, e_1) = 0$ for $z \neq y$ from (3.1) and $g(\nabla_{\phi e_y} \phi e_y, e_1) = 0$ from (3.4).

From these equations, we see that

$$\begin{aligned} & g(R(\phi e_y, e_1) e_1, \phi e_y) \\ & = g(\nabla_{\phi e_y} \nabla_{e_1} e_1, \phi e_y) - g(\nabla_{e_1} \nabla_{\phi e_y} e_1, \phi e_y) - g(\nabla_{[\phi e_y, e_1]} e_1, \phi e_y) \\ & = 0. \end{aligned}$$

On the other hand, the equation of Gauss implies that

$$g(R(\phi e_y, e_1) e_1, \phi e_y) = c + b\bar{b} = c - nc + \frac{\lambda}{2}.$$

So we have $nc - \lambda/2 = c$. Since $b\bar{b} = -c$ and $b = -\bar{b} = k/2$, we see that $c > 0$, $b^2 = c$ and $k^2 = 4c$. This contradicts to our assumption $k^2 \neq 4c$.

From these considerations we see that M has no point x where $A\xi \neq \alpha\xi$, and hence M is a Hopf hypersurface. This proves our theorem. $\square$

Using Theorem 3.2 and Theorem B-C, we have our main result.

Theorem 3.5. *Let M be a real hypersurface in a complex space form $M^n(c)$, $n \geq 3$, $c \neq 0$. We suppose that the Ricci tensor $\hat{S}$ of the generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$ satisfies $\hat{S}(X, \phi Y) = \lambda g(X, \phi Y)$ for any vector fields X and Y, λ being a function.*
(1) *If M is a real hypersurface in $\mathbb{C}P^n$ and $k^2 \neq 4$, then M is locally congruent to one of the following*:
(a) *a geodesic hypersphere with $k^2 \geq (2n-2)(2n-\lambda)$,*
(b) *a tube over a totally geodesic $\mathbb{C}P^l$ $(1 \leq l \leq n-2)$ with $\lambda = 2n$.*

(2) *If M is a real hypersurface in $\mathbb{C}H^n$, then M is locally congruent to one of the following*:

(a) *a geodesic hypersphere with* $k^2 \geq (-2n-2)(2n-\lambda)$,

(b) *a tube over a complex hyperbolic hyperplane with* $k^2 \geq (-2n-2)(2n-\lambda)$,

(c) *a horosphere with* $\lambda = 2k-2$,

(d) *a tube over a totally geodesic* $\mathbb{C}H^l$ $(1 \leq l \leq n-2)$ *with* $\lambda = -2n$.

PROOF. From Theorem 3.2, M is a Hopf hypersurface of $M^n(c)$. Then Proposition A shows

$$(2\beta-\alpha)A\phi X = (\beta\alpha + 2c)\phi X,$$

where $AX = \beta X,\ g(X,\xi) = 0$ and $\alpha = g(A\xi,\xi)$. We notice that α is constant. If $2\beta-\alpha = 0$, then $\beta\alpha+2c = 0$, and hence $\alpha^2+4c = 0$. Thus we have $c < 0$ and M has two distinct constant principal curvatures α and b with multiplicities 1 and $2n-2$ respectively. Moreover b is constant and M is a horosphere of principal curvatures 2 and 1 with multiplicities 1 and $2n-2$, respectively (see Berndt [1]). By (3.9) and $c = -1$, we have $\lambda = 2k-2$.

In the following, we assume that $2\beta-\alpha \neq 0$. Then

$$A\phi X = \frac{\beta\alpha + 2c}{2\beta-\alpha}\phi X.$$

We put $\bar{\beta} = (\beta\alpha + 2c)/(2\beta-\alpha)$. Then, by the assumption on $\hat{S}$, we obtain

$$\begin{aligned} \lambda &= 2nc + (\mathrm{tr}A - \alpha + k)\beta - \beta^2 + \beta\bar{\beta} + k\bar{\beta}, \\ \lambda &= 2nc + (\mathrm{tr}A - \alpha + k)\bar{\beta} - \bar{\beta}^2 + \bar{\beta}\beta + k\beta. \end{aligned} \tag{3.26}$$

These imply

$$0 = (\beta-\bar{\beta})(\mathrm{tr}A - \alpha - \beta - \bar{\beta}).$$

Suppose $\beta \neq \bar{\beta}$. Then $\mathrm{tr}A-\alpha-\beta-\bar{\beta} = 0$. Substituting $\bar{\beta} = \mathrm{tr}A-\alpha-\beta$ into the equation above, we obtain

$$2\beta^2 - 2(\mathrm{tr}A-\alpha)\beta - k(\mathrm{tr}A-\alpha) - 2nc + \lambda = 0. \tag{3.27}$$

Therefore, β satisfies the quadratic equation

$$2t^2 - 2(\mathrm{tr}A - \alpha)t - k(\mathrm{tr}A - \alpha) - 2nc + \lambda = 0.$$

From this we see that at most two distinct β satisfies the above equation. But $\bar{\beta}$ also satisfies the above quadratic equation, and M has two principal curvatures b and $\bar{b}$ with multiplicities p and p, $0 \leq p \leq n-1$, that satisfies $b \neq \bar{b}$.

We next suppose that $\beta = \bar{\beta}$. Then $\beta^2 - \alpha\beta - c = 0$. Therefore, M has at most two non-zero distinct constant principal curvatures d and f such that $d = \bar{d}$, $f = \bar{f}$ with multiplicities q and r, respectively, where $2p + q + r = 2n - 2$. On the other hand, from (3.26), we have

$$\begin{aligned} &2nc - \lambda + (\mathrm{tr}A - \alpha + 2k)d = 0, \\ &2nc - \lambda + (\mathrm{tr}A - \alpha + 2k)f = 0. \end{aligned} \tag{3.28}$$

If M has 5 distinct principal curvatures $b \neq \bar{b}$, d, f and α, then the above equations show that $\mathrm{tr}A - \alpha + 2k = 0$ and $2nc - \lambda = 0$ since $d \neq f$. Moreover, from (3.27), we have $2b^2 + 4kb + 2k^2 = 2(b+k)^2 = 0$ and $(\bar{b} + k)^2 = 0$. Hence we obtain $b = \bar{b} = -k$. This contradicts to the assumption $b \neq \bar{b}$.

We now suppose that M has 4 distinct principal curvatures $b \neq \bar{b}$, d, α. Then we have

$$\mathrm{tr}A - \alpha = b + \bar{b} = p(b + \bar{b}) + qd.$$

From this and $2p + q = 2n - 2$,

$$(p - 1)(b + \bar{b}) + (2n - 2p - 2)d = 0.$$

We notice that b and $\bar{b}$ is continuous. Since p is positive integer and d is non-zero constant, we see that $p \neq 1$ and $b + \bar{b}$ is constant. Moreover, $\mathrm{tr}A - \alpha$ is constant. So (3.28) shows that λ is constant. Hence, from (3.27), b and $\bar{b}$ are also constant. But there is no Hopf hypersurface with constant four principal curvatures.

If M has two constant principal curvatures d and α, then $\mathrm{tr}A - \alpha = (2n - 2)d$. From (3.26),

$$(2n - 2)d^2 + 2kd + 2nc - \lambda = 0.$$

This gives a root when

$$k^2 - (2n-2)(2nc - \lambda) \geq 0.$$

Next, if M has three distinct principal curvatures b, $\bar{b}$ and α, then

$$\mathrm{tr}A - \alpha = b + \bar{b} = (n-1)(b + \bar{b}).$$

Hence we have $b + \bar{b} = \mathrm{tr}A - \alpha = 0$. On the other hand, b and $\bar{b}$ satisfy

$$b + \bar{b} = \frac{2b^2 + 2c}{2b - \alpha} = 0.$$

Thus we have $c < 0$. But the condition $c < 0$ implies that the principal curvatures b and $\bar{b}$ are positive. This contradicts to $b + \bar{b} = 0$.

Finally we consider the case that M has three constant principal curvatures d, f, α, where $d = \bar{d}, f = \bar{f}$. Since $d \neq f$, we have

$$\mathrm{tr}A - \alpha = -2k, \quad 2nc - \lambda = 0.$$

From these considerations and Thereoms B, C we have our assertion. □

References

1. J. Berndt, Real hypersurfaces with constant principal curvatures in complex hyperbolic space, J. Reine Angew. Math. 395 (1989), 132-141.
2. T. E. Cecil and P. J. Ryan, Focal sets and real hypersurfaces in complex projective space, Trans. Amer. Math. Soc. 269 (1982), 481-499.
3. J. T. Cho, CR structures on real hypersurfaces of a complex space form, Publ. Math. Debrecen 54 (1999), 473-487.
4. J. T. Cho, Levi-parallel hypersurfaces in a complex space form, Tsukuba J. Math. 30 (2006), 329-344.
5. J. T. Cho, Pseudo-Einstein CR-structures on real hypersurfaces in a complex space form, Hokkaido Math. J. 37 (2008), 1-17.
6. J. T. Cho and M. Kimura, Pseudo-holomorphic sectional curvatures of real hypersurfaces in a complex space form, Kyushu J. Math. 62 (2008), 75-87.
7. M. Kimura, Real hypersurfaces and complex submanifolds in complex projective space. Trans. Amer. Math. Soc. 296 (1986), 137-149.

8. Y. Maeda, On real hypersurfaces of a complex projective space. J. Math. Soc. Japan 28 (1976), 529-540.
9. M. Ortega, Classifications of real hypersurfaces in complex space forms by means of curvature conditions, Bull. Belg. Math. Soc. Simon Stevin 9 (2002), 351-360.
10. P. J. Ryan, Homogeneity and some curvature conditions for hypersurfaces , Tohoku Math. J. 21 (1969), 363-388.
11. R. Takagi, Real hypersurfaces in a complex projective space with constant principal curvatures, J. Math. Soc. Japan 27 (1975), 43-53.
12. N. Tanaka, On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, Japan. J. Math. 2 (1976), 131-190.
13. S. Tanno, Variational problems on contact Riemannian manifolds, Trans. Amer. Math. Soc. 314 (1989), 349-379.
14. S. M. Webster, Pseudohermitian structures on a real hypersurface, J. Differential Geom. 13 (1978), 25-41.

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