A Method of Treating Two-Dimensional Elasticity by Means of Integral Equation

By

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Synopsis. This paper will give one method for attacking Airy's stress-function in two-dimensional problems of elasticity by means of integral equation, when any distribution of external forces are given on a specified bounding curve. The example here given is very simple, but the method will suggest how to find the proper form of stress-function for more complicated boundary-value problems.

§ 1. General expression of boundary conditions

Stress-components in two-dimensional elasticity are given under no body forces by

$$\sigma_x = \frac{\partial^2 \chi}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 \chi}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}, \quad \cdots\cdots(1)$$

where χ is Airy's stress-function which satisfies the equation

$$\nabla^4 \chi = \frac{\partial^4 \chi}{\partial x^4} + 2\frac{\partial^4 \chi}{\partial x^2 \partial y^2} + \frac{\partial^4 \chi}{\partial y^4} = 0. \quad \cdots\cdots(2)$$

We suppose that one of external forces, P say, is acting on the bounding curve of the elastic body, whose components are denoted by P_x and P_y in directions of x and y respectively. Then the conditions of equilibrium on the bounding curve are

$$\left.\begin{aligned} P_x &= \sigma_x \cos\alpha + \tau_{xy} \sin\alpha, \\ P_y &= \tau_{xy} \cos\alpha + \sigma_y \sin\alpha. \end{aligned}\right\} \cdots\cdots(3)$$

Let s be denoted by the length along the bounding curve, and n by the direction of an outward normal, then we get

$$\left.\begin{aligned} \frac{\partial y}{\partial s} &= \cos\alpha = \frac{\partial x}{\partial n}, \\ \frac{\partial x}{\partial s} &= -\sin\alpha = \frac{\partial y}{\partial n}. \end{aligned}\right\} \cdots\cdots(4)$$

Substituting (1) and (4) into (3), we obtain the equations

$$\left.\begin{aligned} P_x &= \frac{\partial^2 \chi}{\partial y^2}\frac{\partial y}{\partial s} + \frac{\partial^2 \chi}{\partial x \partial y}\frac{\partial x}{\partial s} = \frac{d}{ds}\left(\frac{\partial \chi}{\partial y}\right), \\ P_y &= -\frac{\partial^2 \chi}{\partial x \partial y}\frac{\partial y}{\partial s} - \frac{\partial^2 \chi}{\partial x^2}\frac{\partial x}{\partial s} = -\frac{d}{ds}\left(\frac{\partial \chi}{\partial x}\right). \end{aligned}\right\} \cdots\cdots(5)$$

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When we integrate (5) from 0 to s, we have

$$\left.\begin{aligned}\int_o^s P_x\, ds &= \left[\frac{\partial\chi}{\partial y}\right]_o^s = \left[\frac{\partial\chi}{\partial y}\right]_s - \beta, \quad \beta = \left[\frac{\partial\chi}{\partial y}\right]_o, \\ \int_o^s P_y\, ds &= -\left[\frac{\partial\chi}{\partial x}\right]_o^s = -\left[\frac{\partial\chi}{\partial x}\right]_s + \alpha, \quad \alpha = \left[\frac{\partial\chi}{\partial x}\right]_o;\end{aligned}\right\} \quad \cdots\cdots(6)$$

and also as we find

$$d\chi = \left(\frac{\partial\chi}{\partial x}\frac{dx}{ds} + \frac{\partial\chi}{\partial y}\frac{dy}{ds}\right)ds, \quad \cdots\cdots(7)$$

$$\frac{\partial\chi}{\partial n} = \frac{\partial\chi}{\partial x}\frac{dx}{dn} + \frac{\partial\chi}{\partial y}\frac{dy}{dn} = \frac{\partial\chi}{\partial x}\frac{dy}{ds} - \frac{\partial\chi}{\partial y}\frac{dx}{ds}, \quad \cdots\cdots(8)$$

with (6), (7), and (8), we obtain

$$\left.\begin{aligned}\chi_s &= -\int_o^s \left\{\frac{dx}{ds}\int_o^s P_y\, ds - \frac{dy}{ds}\int_o^s P_x\, ds\right\} ds + \alpha x + \beta y + \gamma, \\ \left[\frac{\partial\chi}{\partial n}\right]_s &= -\frac{dy}{ds}\int_o^s P_y\, ds - \frac{dx}{ds}\int_o^s P_x\, ds - \alpha\frac{dy}{ds} + \beta\frac{dx}{ds}.\end{aligned}\right\} \quad \cdots\cdots(9)$$

When a distribution of loads on the bounding curve of the elastic body is given, equations (9) afford the boundary value of the stress-function. Hence, the problem is reduced to finding Airy's stress-function χ which satisfies the compatibility equation (2), and takes specified values χ_s and $\left(\frac{\partial\chi}{\partial n}\right)_s$ on the bounding curve.

§ 2. General solution satisfying given boundary conditions

Now let χ be composed of two parts, *i.e.*,

$$\chi = \omega_1 + \omega_2, \quad \cdots\cdots(10)$$

of which the boundary values are 0, θ, and φ;

$$(\omega_1)_s = 0, \quad (\omega_2)_s = \theta, \quad \left(\frac{\partial\omega_1}{\partial n}\right)_s = \varphi, \quad \left(\frac{\partial\omega_2}{\partial n}\right)_s = 0. \quad \cdots\cdots(11)$$

Moreover, ω_2 can be written

$$\omega_2 = \omega'_2 + \Phi(\theta). \quad \cdots\cdots(12)$$

In this expression, ω'_2 is a biharmonic function and $\Phi(\theta)$ is a harmonic function, which take on the bounding curve respectively

$$(\omega'_2)_s = 0, \quad \left(\frac{\partial\omega'_2}{\partial n}\right)_s = -\left[\frac{d}{dn}\{\Phi(\theta)\}\right]_s, \quad \{\Phi(\theta)\}_s = \theta. \quad \cdots\cdots(13)$$

Now, in order to find ω_1, let us refer to equation (2), which at once gives

$$\nabla^4\omega_1 = \nabla^2(\nabla^2\omega_1) = 0, \quad \nabla^2\omega_1 = U_1, \quad \nabla^2 U_1 = 0. \quad \cdots\cdots(14)$$

In (14), U_1 is a harmonic function and can be expressed in terms of displacement-components. When the boundary value of U_1 is definite, we can find U_1 within the domain as solution for Dirichlet's problem in Potential theory. The problem is to search the function that, as an inner point tends to a point on the bounding curve, its value also tends to the boundary value, and is harmonic in the domain as well. In order to solve this problem, let us use the equation for logarithmic potential of a double sheet, which is written

$$U_1 = \int_l \mu_1(s) \frac{\partial}{\partial n} \log \frac{1}{r} ds, \quad r = \sqrt{(x-\xi)^2 + (y-\eta)^2}, \quad \cdots\cdots\cdots\cdots(15)$$

$\mu_1(s)$ being a solution of the equation

$$\mu_1(t) = \frac{U_{1t}}{\pi} - \frac{1}{\pi} \int_l \mu_1(s) \frac{\partial}{\partial n} \log \frac{1}{r} ds. \quad \cdots\cdots\cdots\cdots(16)$$

U_{1t} is the limiting value as U_1 tends to a point on the bounding curve. The equation (16) is a Fredholm's integral equation of 2nd kind, and we obtain U_1 by substituting the solution $\mu_1(t)$ of (16) into (15).

Next, the solution of Poisson's differential equation $\nabla^2 \omega_1 = U_1$ is

$$\omega_1 = -\iint_D G(\xi, \eta, x, y) \, U_1(\xi, \eta) \, d\xi d\eta + C_1. \quad \cdots\cdots\cdots\cdots(17)$$

In this equation (17), $G(\xi, \eta, x, y)$ is Green's function, $U_1(\xi, \eta)$ is the solution of (15) and C_1 is a constant of integration.

Similarly, we obtain

$$\left.\begin{aligned} U_2 &= \int_l \mu_2(s) \frac{\partial}{\partial n} \log \frac{1}{r} ds, \\ \mu_2(t) &= \frac{U_{2t}}{\pi} - \frac{1}{\pi} \int_l \mu_2(s) \frac{\partial}{\partial n} \log \frac{1}{r} ds, \\ \omega'_2 &= -\iint_D G(\xi, \eta, x, y) \, U_2(\xi, \eta) \, d\xi d\eta + C_2. \end{aligned}\right\} \cdots\cdots\cdots\cdots(18)$$

Lastly, $\Phi(\theta)$ may be solved as a Dirichlet's problem in Potential theory; that is

$$\left.\begin{aligned} \Phi(\theta) &= \int_l \mu_3(s) \frac{\partial}{\partial n} \log \frac{1}{r} ds, \\ \mu_3(t) &= \frac{\Phi(\theta)_s}{\pi} - \frac{1}{\pi} \int_l \mu_3(s) \frac{\partial}{\partial n} \log \frac{1}{r} ds. \end{aligned}\right\} \cdots\cdots\cdots\cdots(19)$$

The kernel, $K(s,t)$ say, of these integral equations is

$$\left.\begin{aligned} K(s,t) &= \frac{1}{\pi} \frac{\partial}{\partial n} \log \frac{1}{r} = \frac{1}{\pi} \frac{(y-\eta)\xi' - (x-\xi)\eta'}{(x-\xi)^2 + (y-\eta)^2}, \\ \lambda &= -1. \end{aligned}\right\} \cdots\cdots\cdots\cdots(20)$$

In general, Fredholm's integral equation can be solved by means of the reciprocal function of $K(s,t)$, which is

$$\frac{D(s,t,\lambda)}{D(\lambda)} = \frac{\lambda K(s,t) + \lambda^2 d_1(s,t) + \lambda^3 d_2(s,t) + \cdots}{1 + \lambda d_1 + \lambda^2 d_2 + \cdots},$$

where

$$\left.\begin{aligned} d_1(s,t) &= K(s,t), \\ d_{m+1} &= -\frac{1}{m+1} \int_l d_m(\xi, \xi) d\xi, \end{aligned}\right\} \cdots\cdots\cdots\cdots(21)$$

$$d_m(s,t) = K(s,t)d_m + \int_l K(s,\xi)d_{m-1}(\xi,t)d\xi,$$

and $\mu(t)$ is written

$$\mu(t) = \frac{U_i}{\pi} + \int_l \frac{D(s,t,\lambda)}{D(\lambda)} \frac{U_i}{\pi} ds. \quad \cdots\cdots(22)$$

The aggregation of ω_1, ω'_2, and $\Phi(\theta)$ is the Airy's stress-function χ.

§ 3. Example

For example, let us take a circle with unit thickness and uniform load on the bounding curve. The equation of the circle is

$$\xi = a\cos\frac{s}{a}, \quad \eta = a\sin\frac{s}{a}, \quad \cdots\cdots(23)$$

where a denotes the radius of the circle and s the curve length along the circle. The external force per unit area is denoted by P, and then its components become respectively

$$P_x = P\cos\frac{s}{a}, \quad P_y = P\sin\frac{s}{a}. \quad \cdots\cdots(24)$$

Let us express a point in the domain with x and y, *i.e.*,

$$x = \rho\cos\frac{t}{\rho}, \ y = \rho\sin\frac{t}{\rho}, \quad \cdots\cdots(25)$$

ρ being the distance from the center to the point, and t the curve length of the circle with the radius ρ. Substituting expressions (23) and (24) into (9), we obtain the boundary values of χ and $\frac{\partial\chi}{\partial n}$ in this case, which are

$$\left.\begin{aligned} \chi_s &= Pa^2\left(1-\cos\frac{s}{a}\right) + \alpha a\cos\frac{s}{a} - \beta a\sin\frac{s}{a} + \gamma, \\ \left(\frac{\partial\chi}{\partial n}\right)_s &= Pa\left(1-\cos\frac{s}{a}\right) + \alpha\cos\frac{s}{a} + \beta\sin\frac{s}{a}. \end{aligned}\right\} \cdots\cdots(26)$$

The kernel $K(s,t)$ is

$$K(s,t) = \frac{1}{\pi}\,\frac{a-\rho\cos\left(\frac{t}{\rho}-\frac{s}{a}\right)}{a^2+\rho^2-2a\rho\cos\left(\frac{t}{\rho}-\frac{s}{a}\right)}. \quad \cdots\cdots(27)$$

Consider (14); in the case of plane strain, we can put the term in the displacement U_1 as constant without loss of generality, so that

$$\frac{\partial^2\omega_1}{\partial\xi^2} + \frac{\partial^2\omega_1}{\partial\eta^2} = U_1(\xi,\eta), \quad \cdots\cdots(28)$$

$U_{1s}(\xi,\eta)$ is the boundary value of $U_1(\xi,\eta)$, that is

$$U_{1s}(\xi, \eta) = A_1. \quad \cdots\cdots(29)$$

Substituting (29) into (16), and considering the limit as an inner point has tended to the bounding curve, we have

$$\mu_1(t) = \frac{1}{\pi}A_1 - \frac{1}{\pi}\int_o^{2\pi a}\mu(s)\,\frac{1}{2a}ds. \quad \cdots\cdots(30)$$

From (21) the reciprocal function of $K(s,t)$ is

$$\frac{D(s,t,-1)}{D(-1)}=-\frac{1}{2(a+1)\pi},\quad \left(\begin{array}{l}\lambda=-1,\ K(s,t)=\frac{1}{2a\pi},\ d_1=-\frac{1}{a},\\ d_1(s,t)=0,\ d_2=\cdots\cdots=d_2(s,t)=\cdots\cdots=0.\end{array}\right) \tag{31}$$

Hence (30) becomes

$$\mu_1(t)=\frac{1}{\pi}A_1+\int_o^{2\pi a}\frac{-1}{2(a+1)\pi}\frac{1}{\pi}A_1\,ds=\frac{1}{a+1}\frac{A_1}{\pi}. \quad\cdots\cdots\cdots\cdots(32)$$

Substituting (32) into (15), we have

$$U_1=\int_o^{2\pi a}\frac{1}{a+1}\frac{A_1}{\pi}\frac{a-\rho\cos\left(\theta-\frac{s}{a}\right)}{a^2+\rho^2-2a\rho\cos\left(\theta-\frac{s}{a}\right)}ds=\frac{2a}{a+1}A_1. \quad\cdots\cdots(33)$$

Green's function in (17) is in this case

$$G(x,y,\xi,\eta)=\frac{1}{4\pi}\{\log(\xi^2+\eta^2)-\log(x^2+y^2)\}. \quad\cdots\cdots\cdots\cdots(34)$$

From the expressions (33), (34), and (17), we obtain

$$\omega_1=-\iint_D\frac{1}{4\pi}\{\log(\xi^2+\eta^2)-\log(x^2+y^2)\}\frac{2a}{a+1}A_1\,d\xi d\eta+C_1$$
$$=\frac{4a^2}{a+1}A_1\{\log(x^2+y^2)-\log a^2+1\}+C_1. \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(35)$$

Similarly, the expression of ω_2 becomes

$$\omega_2=\frac{4a^2}{a+1}A_2\{\log(x^2+y^2)-\log a^2+1\}+C_2. \quad\cdots\cdots\cdots\cdots(36)$$

Now, the boundary value of $\Phi(\theta)$ is

$$[\Phi(\theta)]_s=Pa^2\left(1-\cos\frac{s}{a}\right)+\alpha a\cos\frac{s}{a}-\beta a\sin\frac{s}{a}+\gamma.$$

Hence, (19) is written

$$\mu_3(t)=\frac{1}{\pi}\{\Phi(\theta)\}_s-\int_o^{2\pi a}\frac{1}{2(a+1)\pi}\frac{1}{\pi}\{\Phi(\theta)\}_s\,ds$$
$$=\frac{1}{\pi}\left\{Pa^2\left(1-\cos\frac{s}{a}\right)+\alpha a\cos\frac{s}{a}-\beta a\sin\frac{s}{a}+\gamma\right\}$$
$$-\frac{1}{\pi^2}\frac{a}{a+1}\{(Pa^2+\gamma)\pi-2a\beta\}. \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(37)$$

Substituting (37) into (19), we have

$$\Phi(\theta)=\int_o^{2\pi a}\left[\frac{1}{\pi}\left\{Pa^2\left(1-\cos\frac{s}{a}\right)+\alpha a\cos\frac{s}{a}-\beta a\sin\frac{s}{a}+\gamma\right\}\right.$$
$$\left.-\frac{1}{\pi^2}\frac{a}{a+1}\{(Pa^2+\gamma)\pi-2a\beta)\right]\frac{a-\rho\cos\left(\theta-\frac{s}{a}\right)}{a^2+\rho^2-2a\rho\cos\left(\theta-\frac{s}{a}\right)}ds$$
$$=\frac{1}{\pi}\left[\{(Pa-\alpha)\cos\theta-\beta\sin\theta\}\left\{-\frac{2\rho}{a}\sin\theta-\frac{2}{3}\left(\frac{\rho}{a}+\frac{\rho^2}{a^2}\right)\sin 3\theta-\cdots\right\}\right.$$
$$\left.+\{-(Pa-\alpha)\sin\theta+\beta\cos\theta\}\left\{\frac{2\rho}{a}\cos\theta-\frac{2}{3}\left(\frac{\rho}{a}-\frac{\rho^2}{a^2}\right)\cos 3\theta-\cdots\right\}\right]$$

$$+\frac{2}{a+1}\left\{\pi^2(Pa^2+\gamma)-\frac{a\beta}{\pi}(3a+1)\right\}, \quad \cdots\cdots(38)$$

and accordingly

$$\left[\frac{\partial\Phi(\theta)}{\partial n}\right]_s=\left[\frac{\partial\Phi(\theta)}{\partial\rho}\right]_{\rho=a}=\frac{8}{a\pi}(Pa-\alpha)\left(\frac{1^2}{1\cdot3}\sin 2\theta+\frac{2^2}{3\cdot5}\sin 4\theta+\cdots\right)$$

$$-\frac{4\beta}{\pi}\left(\frac{1}{1\cdot3}\cos 2\theta+\frac{2}{3\cdot5}\cos 4\theta+\frac{3}{5\cdot7}\cos 6\theta+\cdots\right). \quad \cdots\cdots(39)$$

Considering the boundary values of respective functions, we obtain

$$\omega_1=a\{Pa(1-\cos\theta)+\alpha\sin\theta+\beta\cos\theta\}\log\frac{\rho}{a}, \quad \cdots\cdots(40)$$

$$\omega'_2=a\left[\frac{8}{a\pi}(Pa-\alpha)\left(\frac{1^2}{1\cdot3}\sin 2\theta+\frac{2^2}{3\cdot5}\sin 4\theta+\cdots\right)\right.$$

$$\left.-\frac{4\beta}{\pi}\left(\frac{1}{1\cdot3}\cos 2\theta+\frac{2}{3\cdot5}\cos 4\theta+\cdots\right)\right]\log\frac{\rho}{a}. \quad \cdots\cdots(41)$$

Hence it follows that

$$\chi=\omega_1+\omega'_2+\Phi(\theta)$$

$$=a\{Pa(1-\cos\theta)+\alpha\cos\theta+\beta\sin\theta\}\log\frac{\rho}{a}$$

$$+a\left\{\frac{8}{a\pi}(Pa-\alpha)\left(\frac{1^2}{1\cdot3}\sin 2\theta+\frac{2^2}{3\cdot5}\sin 4\theta+\cdots\right)\right.$$

$$\left.-\frac{4\beta}{\pi}\left(\frac{1}{1\cdot3}\cos 2\theta+\frac{2}{3\cdot5}\cos 4\theta+\cdots\right)\right\}\log\frac{\rho}{a}$$

$$+\frac{1}{\pi}\left[\{(Pa-\alpha)\cos\theta-\beta\sin\theta\}\left\{-\frac{2\rho}{a}\sin\theta-\frac{2}{3}\left(\frac{\rho}{a}+\frac{\rho^2}{a^2}\right)\sin 3\theta-\cdots\right\}\right.$$

$$\left.+\{-(Pa-\alpha)\sin\theta+\beta\cos\theta\}\left\{\frac{2\rho}{a}\cos\theta-\frac{2}{3}\left(\frac{\rho}{a}-\frac{\rho^2}{a^2}\right)\cos 3\theta-\cdots\right\}\right]$$

$$+\frac{2}{a+1}\left\{\pi^2(Pa^2+\gamma)-\frac{a\beta}{\pi}(3a+1)\right\}. \quad \cdots\cdots(42)$$

The boundary value of χ is

$$\chi_{\rho=a}=\frac{2}{\pi}\left[(Pa-\alpha)\left\{\cos\theta\left(-\sin\theta-\frac{2}{3}\sin 3\theta-\frac{2}{5}\sin 5\theta-\cdots\right)\right\}\right.$$

$$\left.+2\beta\sin^2\theta+\cdots+\frac{2}{a+1}\left\{\pi^2(Pa^2+\gamma)-\frac{a\beta}{\pi}(3a+1)\right\}\right]. \quad \cdots\cdots(43)$$

On the other hand, it is

$$\chi_{\rho=a}=Pa^2(1-\cos\theta)+\alpha a\cos\theta-\beta a\sin\theta+\gamma.$$

The two expressions above should be identical regardless of the variable θ, and hence we find

$$\alpha=Pa,\quad \beta=0,\quad \gamma=-Pa^2.$$

Finally, we have

$$\chi=Pa^2\log\frac{\rho}{a}.$$

Thus we have arrived at the well-known Airy's stress-function for the present boundary-value problem, which a priori has been assumed rather by intuition.

Conclusion. The above work presents a new method for finding Airy's stress-function, when the configuration of the elastic system is given. Though the example treated, which is the circular disk under uniform external force, is simple, the method presented suggests the way to find the stress-function compatible with given boundary-value problems of higher complication.

Analytical solution frequently fails in the problem of elasticity; in such a case the present work would be developed further by means of numerical method of integration. Further study on this line is now in progress.

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