The Impedance Matrix of the Induction Machine Having a General Stator-winding Axis and its Torque Formula

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1. Inductory Remarks

Toward the end of the Second World War, many researches on the analysis of the three-phase induction motor operated on the single-phase electric source or the analysis of the induction machin having an asymmetrical stator-winding axis have been reported in Japanese. But the research works dealing with the induction machine having a general stator-winding axis are comparatively few. The difference between 'an asymmetrical stator-winding axis' and 'a general asymmetrical stator-winding axis' is as follows:

In the former the effective turus of some stator phase are not always equal to others, but their axes are placed apart symmetrically in space. In the latter their axes are not always placed symmetrically, their effective turns are not always equal either.

Dr. Takeuchi,(1) by his poli-axis matrix method, analized the three-phase induction machine of the latter. Dr. Takegami and his group,(2) by two revolutions theory of the magnetic field, analized two or three-phase induction machine of the latter. Already I(3) have reported the analytical method on the three-phase induction machine of the former. In this paper I developed and generalized the basic theory used in my research described above so that it might be adaptable to any phase induction machine of the latter.

2. Derivation of Impedance Matrix by Solving Differential Equations

i) Notations relating to the stator-winding

Now let the number of phases of the stator-winding be n, each phase be indicated by 1. 2. 3. ………… n orderly in counterclockwise, and the constants of any phase α (α=1. 2. 3. ………n) be as mentioned below:

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$r_{\alpha\alpha}$=ohmic resistance

$l_{\alpha\alpha}$=leakage inductance

$L_{\alpha\alpha}$=effective inductance

w_α=number of efective turns

The relations within the effective inductances are established as follows.

$$\left.\begin{array}{l}\frac{L_{11}}{w_1^2}=\frac{L_{22}}{w_2^2}=\frac{L_{33}}{w_3^2}=\cdots\cdots=\frac{L\alpha\alpha}{w_\alpha^2}=\cdots\cdots=\frac{L_{nn}}{w_n^2}\equiv L \\ L=\text{effective inductance/one effective turn}\end{array}\right\}\cdots\cdots(1)$$

θ_{12}, θ_{13}, θ_{23}, ········are defined as are shown in Fig. 1, and the suffix of θ is decided as is shown in the following examples.

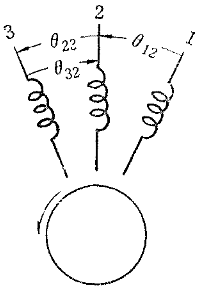


Fig. 1

$$\begin{array}{l}\theta_{13}=\theta_{12}+\theta_{23},\ \ \theta_{32}=-\theta_{23} \\ \theta_{11}=\theta_{22}=\theta_{33}=\cdots\cdots=0\end{array}\cdots\cdots(2)$$

Consequently $\theta_{\alpha\beta}$ is explained as follows:

$\theta_{\alpha\beta}$ is not only the angle between axis of phase α and that of phase β, but also has a direction from α to β. When the direction of this angle is equal or opposite to the running direction of the rotor, the value of $\theta_{\alpha\beta}$ is positive or negative. In Fig. 1 the rotor is rotating counterclockwise, and so θ_{12} $\theta_{23}$are positive and θ_{32} is negative.

The mutual inductance between phase α and phase β, $M_{\alpha\beta}$ is:

$$M_{\alpha\beta}=M_{\beta\alpha}=w_\alpha w_\beta L\cos\theta_{\alpha\beta} \cdots\cdots(3)$$

ii) Notations relating to the rotor-winding

The rotor is assumed to have a symmetrical winding axis. And the constants of each phase are:

r_{uu}=ohmic resistance

l_u=leakage inductance

L_u=effective inductance

Let the number of phases be m, then the angle between some phase and its adjacent phase becomes $\gamma=2\pi/m$ in electrical degree. Each phase is indicated by 1. 2. 3 ·············· m oderly in counterclockwise as I did in stator-winding. The mutual inductance between phase k and r of rotor-winding, M_{2-kr} is :

$$M_{2-kr}=L_u\cos(k-r)\gamma \cdots\cdots(4)$$

iii) Mutual inductance between stator phase and rotor phase

Now let the mutual inductance between phase α of stator winding and any phase of rotor winding in such a position that its winding axis coincides with that of phase α be $M_{12-\alpha}$, then we have the following relations.

$$\frac{M_{12-1}}{w_1}=\frac{M_{12-2}}{w_2}=\frac{M_{12-3}}{w_3}=\cdots\cdots=\frac{M_{12-\alpha}}{w_\alpha}=\cdots\cdots=\frac{M_{12-n}}{w_n}\equiv M \cdots\cdots(5)$$

M=max. value of mutual inductance between primary phase of which the effective turns is supposed to be 1 and any phase of rotor

$$\left.\begin{array}{l} M^2_{12-1}=L_{11}L_u, \quad M^2_{12-2}=L_{22}L_u, \quad \cdots\cdots \\ M^2=L_u L \end{array}\right\} \cdots\cdots (6)$$

iv) Differential equations on the secondary winding and their transformations

When the rotor is running at the slip s, the relative angular velocity between stator and rotor is $(1-s)\omega$. Now let the origin of time be the instant when the axis of stator phase 1 coincides with that of rotor phase 1, then at any instant t every stator phase and every rotor phase are placed as shown in Fig 2. In Fig 2 $i_1\ i_2\ i_3 \cdots\cdots i_n$ or $i_{2-1}\ i_{2-2}\ i_{2-3} \cdots\cdots\cdots i_{2-m}$ are primary or secondary currents in instantanuous value, and we can establish the following equation on the rotor phase k.

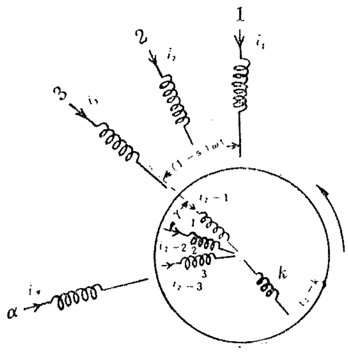


Fig. 2

$$(r_u+Pl_u)\,i_{2-k}+PL_u\sum_{r=1}^{m} i_{2-r}\cos(k-r)\gamma$$

$$+P\sum_{\alpha=1}^{n}\left[M_{12-\alpha}\,i_\alpha\cos\left\{(1-s)\,\omega t+(k-1)\gamma-\theta_{1\alpha}\right\}\right]=0 \quad \cdots\cdots (7)_k$$

Next multiply $\varepsilon^{-j(k-1)\gamma}$ or $\varepsilon^{j(k-1)\gamma}$ to each term of the above equation, put 1. 2. 3. $\cdots\cdots$ m for k, and sum up, then we have following two equations.

$$\frac{1}{m}\sum_{k=1}^{m}\left[(7)_k\,\varepsilon^{-j(k-1)\gamma}\right]=0 \quad \cdots\cdots (8\text{—a})$$

$$\frac{1}{m}\sum_{k=1}^{m}\left[(7)_k\,\varepsilon^{j(k-1)\gamma}\right]=0 \quad \cdots\cdots (8\text{—b})$$

Eq. (8—a) is transformed to the following.

$$(r_u+Pl_u)\frac{1}{m}\sum_{k=1}^{m} i_{2-k}\,\varepsilon^{-j(k-1)\gamma}+\frac{1}{2m}L_uP\sum_{k=1}^{m}\sum_{r=1}^{m}\Big(i_{2-r}\,\varepsilon^{j(k-r)\gamma}\,\varepsilon^{-j(k-1)\gamma}+$$

$$i_{2-r}\,\varepsilon^{-j(k-r)\gamma}\,\varepsilon^{-j(k-\gamma)\gamma}\Big)+\frac{1}{2m}MP\sum_{k=1}^{m}\sum_{\alpha=1}^{n}\Big[W_\alpha i_\alpha\Big(\varepsilon^{j\{(1-s)\omega t+(k-1)\gamma-\theta_{1\alpha}\}}\cdot\varepsilon^{-j(k-1)\gamma}$$

$$+\varepsilon^{-j\{(1-s)\omega t+(k-1)\gamma-\theta_{1\alpha}\}}\cdot\varepsilon^{-j(k-1)\gamma}\Big)\Big\}=0 \quad \cdots\cdots (9)$$

Here the subscript notation is introduced.

$$i_{u2}=\frac{1}{m}\sum_{k=1}^{m} i_{2-k}\,\varepsilon^{-j(k-1)\gamma} \quad \cdots\cdots (10)$$

Substituting Eq. (10) in Eq. (9), each term becomes as follows.

1 st term $=(r_u+Pl_u)\,i_{u2}$

$$2\text{nd term}=\frac{1}{2m}L_uP\sum_{k=1}^{m}\left\{\varepsilon^{jk\gamma}\varepsilon^{-jk\gamma}\sum_{r=1}^{m}\left(i_{2-r}\,\varepsilon^{-j(k-1)\gamma}\right)\right\}$$

$$+\frac{1}{2m}L_uP\sum_{k=1}^{m}\left\{\varepsilon^{-j\,2k\gamma}\sum_{r=1}^{m}\left(i_{2-r}\,\varepsilon^{j(k+1)\gamma}\right)\right\}=\frac{m}{2}L_uP\,i_{u2}$$

$$3\text{rd term}=\frac{1}{2}MP\,\varepsilon^{j(1-s)\,\omega t}\sum_{\alpha=1}^{n}w_\alpha\,i_\alpha\,\varepsilon^{-j\,\theta_{1\alpha}}$$

Hence Eq. (9) becomes:

$$\left(r_u+l_uP+\frac{m}{2}L_u\,P\right)i_{u2}+\frac{1}{2}MP\varepsilon^{j(1-s)\,\omega t}\sum_{\alpha=1}^{n}w_\alpha\,i_\alpha\,\varepsilon^{-j\,\theta_{1\alpha}}=0 \quad\cdots\cdots(11)$$

Applying Heaviside's shifting theorem to Eq. (11), we have

$$i_{u2}=-\varepsilon^{-j(1-s)\,\omega t}\,\frac{{}^1\!/_2\,M\{P+j(1-s)\omega\}}{r_u+(l_u+m/2\,L_u)\ \{P+j(1-s)\omega\}}\sum_{\alpha=1}^{n}w_\alpha\,i_\alpha\,\varepsilon^{-j\,\theta_{1\alpha}} \quad\cdots\cdots(12)$$

By entirely the same procedure as before, from Eq. (8–b)

$$i_{u1}=-\varepsilon^{j(1-s)\,\omega t}\,\frac{{}^1\!/_2\,M\{P-j(1-s)\omega\}}{r_u+(l_u+m/2\,L_u)\ \{P-j(1-s)\omega\}}\sum_{\alpha=1}^{n}w_\alpha\,i_\alpha\,\varepsilon^{j\theta_{1\alpha}} \quad\cdots\cdots(13)$$

where

$$i_{u1}=\frac{1}{m}\sum_{k=1}^{m}i_{2-k}\,\varepsilon^{j(k-1)\gamma} \quad\cdots\cdots(14)$$

v) Differential Equation on Stator–winding and its Solution

In Fig. (2) we have the following differential equation on the stator phase α

$$(r_{\alpha\alpha}+Pl_{\alpha\alpha})i_\alpha+P\sum_{\beta=1}^{n}i_\beta M_{\alpha\beta}+P\sum_{k=1}^{m}M_{12-\alpha}\,i_{2-k}\cdot$$

$$\cos\left\{(1-s)\ \omega t-\theta_{1\alpha}+(k-1)\gamma\right\}=e_\alpha \quad\cdots\cdots(15)$$

where e_α is the instantanuous value of phase volage of phase α. The second and third terms of the above equation become the following respectively.

$$\frac{1}{2}PLw_\alpha\left\{\sum_{\beta=1}^{n}w_\beta\,i_\beta\,\varepsilon^{j\theta\alpha\beta}+\sum_{\beta=1}^{n}w_\beta\,i_\beta\,\varepsilon^{-j\,\theta\alpha\beta}\right\},$$

$$\frac{1}{2}w_\alpha\,PM\sum_{k=1}^{m}\left\{\varepsilon^{j\{(1-s)\omega t-\theta_{1\alpha}+(k-1)\gamma\}}\cdot i_{2-k}+\varepsilon^{-j\{(1-s)\omega t-\theta_{1\alpha}+(k-1)\gamma\}}\cdot i_{2-k}\right\}$$

$$=\frac{1}{2}mw_\alpha\,PM\,\varepsilon^{j\{(1-s)\,\omega t-\theta_{1\alpha}\}}\cdot i_{u1}+\frac{1}{2}mw_\alpha\,PM\,\varepsilon^{-j\{(1-s)\,\omega t-\theta_{1\alpha}\}}\cdot i_{u2}$$

substituting Eq. (12) and (13), and vanishing i_{u1} and i_{u2},

$$=-\frac{1}{2}m\,w_\alpha\,PM\frac{{}^1\!/_2\,M\{P-j(1-s)\omega\}}{r_u+(l_u+m/2\,L_u)\ \{P-j(1-s)\omega\}}\sum_{\beta=1}^{n}w_\beta\,i_\beta\,\varepsilon^{j\theta\alpha\beta}$$

$$-\frac{1}{2} m w_\alpha PM \frac{{}^1/_2 M\{P+j(1-s)\omega\}}{r_u+(l_u+m/2L_u)\ \{P+j(1-s)w\}} \sum_{\beta=1}^{n} w_\beta i_\beta \varepsilon^{-j\theta_{\alpha\beta}}$$

Therefore, Eq. (15) becomes:

$$(r_{\alpha\alpha}+Pl_{\alpha\alpha})\, i_\alpha$$

$$+\left\{\frac{1}{2} w_\alpha PL-\frac{m}{4} w_\alpha PM^2 \frac{P-j(1-s)\omega}{r_u+(l_u+m/2L_u)\ \{P-j\ (1-s)\omega\}}\right\} \sum_{\beta=1}^{n} w_\beta i_\beta \varepsilon^{j\theta_{\alpha\beta}}$$

$$+\left\{\frac{1}{2} w_\alpha PL-\frac{m}{4} w_\alpha PM^2 \frac{P+j(1-s)\omega}{r_u+(l_u+m/2L_u)\ \{P+j(1-s)\omega\}} \sum_{\beta=1}^{n} w_\beta i_\beta \varepsilon^{-j\theta_{\alpha\beta}}\right.$$

$$=e_\alpha \cdots\cdots, \quad \cdots\cdots (16)$$

The solution of the above equation only for the electrical stationary state is obtained, as is well-known, by putting $j\omega$ in place of P, effective value E_α in place of e_α, vector I_α of effective value in place of i_α, etc. That is,

$$(r_{\alpha\alpha}+j\omega l_{\alpha\alpha})\, I_\alpha + w_\alpha\left\{\frac{1}{2} j\omega\ L-\frac{1}{4}(j\omega M)^2\cdot m \frac{s}{r_u+(l_u+m/2\ L_u)j\omega s}\right\} \sum_{\beta=1}^{n} w_\beta I_\beta \varepsilon^{j\theta_{\alpha\beta}}$$

$$+w_\alpha\left\{\frac{1}{2} j\omega\ L-\frac{1}{4}(j\omega\ M)^2\cdot m\cdot \frac{2-s}{r_u+(l_u+m/2L_u)\ j\omega(2-s)}\right\} \sum_{\beta=1}^{n} w_\beta I_\beta \varepsilon^{-j\theta_{\alpha\beta}}$$

$$=E_\alpha \quad \cdots\cdots (17)$$

Now, let { } in the second term of Eq. (17) and that in the third term be Z_p, Z_n respectively, that is :

$$\left.\begin{aligned} Z_p&=\frac{1}{2} j\omega L\left\{1-\frac{{}^1/_2\cdot j\omega L}{(r_u/s+j\omega l_u)\,{}^1/_m\cdot L/L_u+{}^1/_2\cdot j\omega L}\right\}=\frac{jX_m(r_2/s+jx_2)}{r_2/s+j(x_2+X_m)} \\ Z_n&=\frac{1}{2} j\omega L\left\{1-\frac{{}^1/_2\cdot j\omega L}{(r_2/(2-s)+j\omega l_u)\,{}^1/_m\cdot L/L_u+{}^1/_2\cdot j\omega L}\right\}=\frac{jX_m(r_2/(2-s)+jx_2)}{r_2/(2-s)+j(x_2+X_m)} \end{aligned}\right\} \cdots (18)$$

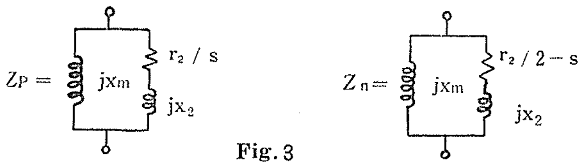


Fig. 3

Therefore Z_p, Z_n can be expressed in electrical networks as shown in Fig. 3, and r_2+jx_2, X_m are given in the following equation.

$$\left.\begin{aligned} X_m&={}^1/_2\cdot\omega L \\ r_2+jx_2&=\frac{1}{m}\frac{L}{L_u}(r_u+j\omega l_u) \end{aligned}\right\} \cdots\cdots (19)$$

From Eq. (19) we can notice that r_2+jx_2 is the leakage impedance reduced to the stator phase of which effective turn constitute 1 and X_m is effective inductance reduced to that as above.
Accordingly Eq. (17) becomes:

$$Z_{\alpha\alpha} I_\alpha + w_\alpha Z_p \sum_{\beta=1}^{n} w_\beta I_\beta \varepsilon^{j\theta_{\alpha\beta}} + w_\alpha Z_n \sum_{\beta=1}^{n} w_\beta I_\beta \varepsilon^{-j\theta_{\alpha\beta}} = E_\alpha \quad \cdots\cdots\cdots\cdots(20)$$

where $Z_{\alpha\alpha} = r_{\alpha\alpha} + j\omega l_{\alpha\alpha}$

Here replace suffix α in Eq.(20) by 1. 2. 3. ············n respectively, and express these results in a matrix formula, then we have:

$$[Z]\,[I] = [E] \cdots\cdots\cdots\cdots\cdots\cdots(21)$$

where

$$[Z] =$$

$$\begin{pmatrix} Z_{11}+w_1^2(Z_p+Z_n), & w_1 w_2(Z_p\varepsilon^{j\theta_{12}}+Z_n\varepsilon^{-j\theta_{12}}), & \cdots\cdots, & w_1 w_n(Z_p\varepsilon^{j\theta_{1n}}+Z_n\varepsilon^{-j\theta_{1n}}) \\ w_2 w_1(Z_p\varepsilon^{j\theta_{21}}+Z_n\varepsilon^{-j\theta_{21}}), & Z_{22}+w_2^2(Z_p+Z_n), & \cdots\cdots\ \cdots\cdots, & w_2 w_n(Z_p\varepsilon^{j\theta_{2n}}+Z_n\varepsilon^{-j\theta_{2n}}) \\ w_3 w_1 Z(_p\varepsilon^{j\theta_{31}}+Z_n\varepsilon^{-j\theta_{31}}), & w_3 w_2(Z_p\varepsilon^{j\theta_{32}}+Z_n\varepsilon^{-j\theta_{32}}), & \cdots\cdots, & w_3 w_n(Z_p\varepsilon^{j\theta_{3n}}+Z_n\varepsilon^{-j\theta_{3n}}) \\ \vdots & \vdots & & \vdots \\ w_n w_1(Z_p\varepsilon^{j\theta_{n1}}+Z_n\varepsilon^{-j\theta_{n1}}), & w_n w_2(Z_p\varepsilon^{j\theta_{n2}}+Z_n\varepsilon^{-j\theta_{n2}}), & & Z_{nn}+w_n^2(Z_p+Z_n) \end{pmatrix}$$

$$\cdots\cdots\cdots\cdots(22)$$

$$[I] = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_n \end{pmatrix} \qquad [E] = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_n \end{pmatrix}$$

The physical meanings of Eq.(22) will be described later.

Thus we can construct the impedauee matrix $[Z]$ of which axes are in accordance with actual phases of stator-winding. Combining the condition of connecting stator phases, terminal supply woltage, and Eq (21), we can obtion I_1 I_2 $I_3 \cdots\cdots\cdots I_n$, and I suppose the readers don't need the details about this process.

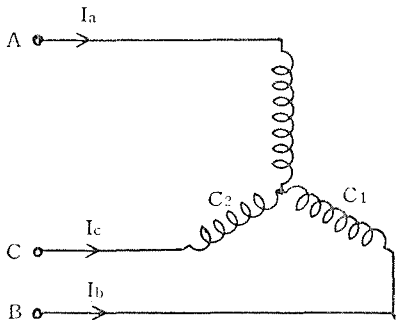


Fig. 4

Next, an example of constructing $[Z]$ by applying what was described hitherto in this section will be shown. Fig.(4) shows the three-phase induction motor having the asymmetrical stator-winding axis, and the rotating direction of the rotor is assumed to be clockwise. In Fig. 4 the suffix 1.2.3. in Eq.(22) must be replaced by a.b.c. respectively, and we have:

$$\varepsilon^{j\theta ab} = \varepsilon^{j120°} \equiv \alpha, \qquad \varepsilon^{j\theta ac} = \varepsilon^{j240°} = \alpha^2$$

$$\varepsilon^{-jba} = \varepsilon^{j\theta ab} = \alpha, \qquad \varepsilon^{j\theta ba} = \varepsilon^{-j\theta ab} = \alpha^2, \qquad \text{etc.}$$

As the effective turn w_a, w_b, w_c and impedance Z_p, Z_n can' t be measured by experiments, we had better replace each elements of $[Z]$ by measurable quantities $w_a^2Z_p \equiv Z_{pa}$, $w_a^2Z_n \equiv Z_{na}$, $w_b/w_a = C_1$, $w_c/w_a = C_2$, and then we have:

$$[Z] = \begin{array}{c|c|c|c|} & a & b & c \\ \hline a & Z_{aa}+Z_{pa}+Z_{na} & C_1(\alpha Z_{pa}+\alpha^2 Z_{na}) & C_2(\alpha^2 Z_{pa}+\alpha Z_{na}) \\ \hline b & C_1(\alpha^2 Z_{pa}+\alpha Z_{na}) & Z_{bb}+C^2(Z_{pa}+Z_{na}) & C_1C_2(\alpha Z_{pa}+\alpha^2 Z_{na}) \\ \hline c & C_2(\alpha Z_{pa}+\alpha^2 Z_{na}) & C_2C_1(\alpha^2 Z_{pa}+\alpha Z_{na}) & Z_{cc}+C_2^2(Z_{pa}+Z_{na}) \\ \hline \end{array} \quad \cdots\cdots\cdots\cdots (23)$$

3. Torque Formula

The instantanuous torque occured betewen any two currents i_1, i_2 is given as follows, as well known:

$$T_i = i_1\, i_2 \frac{dM}{d\theta} \quad \text{(Newton-m)} \quad \cdots\cdots\cdots\cdots (24)$$

where M is the mutual inductance between two coils in which i_1 or i_2 flow and is a function of θ. θ is a mechanical angle in radian. On the motor in question, mutual inductace M between phase k of rotor and phase α of stator is, as is already described,

$$M = w_\alpha M \cos\{(1-s)\omega t + (k-1)\gamma - \theta_{1\alpha}\} \quad \cdots\cdots\cdots\cdots (25)$$

M in the right term of above equation is identified to that defined by Eq. (5), and { } in the same term is the angle between two coils axes in electrical degree, so that { }/p is correspond to θ in Ep. (24). Therefore $dM/d\theta$ in question is:

$$dM/d\theta = -p w_\alpha M \sin\{(1-s)\omega t + (k-1)\gamma - \theta_{1\alpha}\}$$

where p is the number of pair of poles of stator or rotor-winding.

Hence, the instantanuous torque produced by the said motor Ti is :

$$Ti = pM \sum_{\alpha=1}^{n} \sum_{k=1}^{m} i_\alpha\, i_{2-k}\, w_\alpha \sin\left\{(1-s)\omega t + (k-1)\gamma - \theta_{1\alpha}\right\}$$

$$= p\frac{jM}{2} \varepsilon^{j(1-s)\omega t} \sum_{\alpha=1}^{n} (i_\alpha w_\alpha \varepsilon^{-j\theta_{1\alpha}}) \cdot \sum_{k=1}^{m} (i_{2-k} \varepsilon^{-j(k-1)\gamma})$$

$$- p\frac{jM}{2} \varepsilon^{-j(1-s)\omega t} \sum_{\alpha=1}^{n} (i_\alpha w_\alpha \varepsilon^{j\theta_{1\alpha}}) \sum_{k=1}^{m} (i_{2-k} \varepsilon^{-j(k-1)\gamma})$$

Substituting Eq. (10), Eq. (14)

$$= p\frac{mjM}{2} \sum_{\alpha=1}^{n} (i_\alpha w_\alpha \varepsilon^{-j\theta_{1\alpha}}) \varepsilon^{j(1-s)\omega t} i_{u1}$$

$$- p\frac{mjM}{2} \sum_{\alpha=1}^{n} (i_\alpha w_\alpha \varepsilon^{j\theta_{1\alpha}}) \cdot \varepsilon^{-j(1-s)\omega t} i_{u2}$$

Substituting Eq. (12), Eq. (13) in i_{u1}, i_{u2} of the above equation

$$=-p\frac{jmM}{2}\sum_{\alpha=1}^{n}(w_\alpha i_\alpha \varepsilon^{-j\theta_{1\alpha}})\cdot\frac{{}^1\!/_2 M\{P-j(1-s)\omega\}}{r_u+(l_u+m/2L_u)\{P-j(1-s)\omega\}}\sum_{\alpha=1}^{n}(w_\alpha i_\alpha \varepsilon^{j\theta_{1\alpha}}$$

$$+p\frac{jmM}{2}\sum_{\alpha=1}^{n}(w_\alpha i_\alpha \varepsilon^{j\theta_{1\alpha}})\frac{{}^1\!/_2 M\{P+j(1-s)\omega\}}{r_u+(l_u+m/2\,2\cdot L_u)\{P+j(1-s)\omega\}}\sum_{\alpha=1}^{n}w_\alpha i_\alpha \varepsilon^{-j\theta_{1\alpha}})\cdots(26)$$

The relation between the iustanuous current i_α and its effective value I_α is as follows:

$$i_\alpha=\frac{\sqrt{2}}{2}(I_\alpha\varepsilon^{j\omega t}+I_\alpha^*\varepsilon^{-j\omega t})$$

Substituting the above equation into Eq. (26) and transforming it, the result is:

$$T_t=\frac{p}{\omega}\frac{X_m^2 r_2/s}{(r_2/s)^2+(x_2+X_m)^2}|I_p|^2-\frac{p}{w}\frac{X_m^2 r_2/2-s}{(r_2/2-s)^2+(x_2+X_m)^2}|I_n|^2$$

$$+\frac{p}{\omega}\Big[\frac{X_m^2(r_2/s-r_2/2-s)}{\{r_2/s+j(x_2+X_m)\}\{r_2/2-s+j(x_2+X_m)\}}I_p I_n\varepsilon^{j2\omega t}\Big]\text{real part},\ \cdots\cdots(26)'$$

where

$$I_p=\sum_{\alpha=1}^{n}(w_\alpha I_\alpha\varepsilon^{j\theta_{1\alpha}}),\qquad I_n=\sum_{\alpha=1}^{n}(w_\alpha I_\alpha\varepsilon^{-j\theta_{1\alpha}})\cdots\cdots(27)$$

From Eq. (26)′ we have the average value of instantanuous torque T and pulsating torque T_p in synchrouous watt:

$$T=\frac{X_m^2}{(r_2/s)^2+(x_2+X_m)^2}|I_p|^2\frac{r_2}{s}-\frac{X_m^2}{(r_2/2-s)^2+(X_2+X_m)^2}|I_n|^2\frac{r_2}{2-s}\cdots\cdots(28)$$

$$T_p=\Big[\frac{X_m^2(r_2/s-r_2/2-s)}{\{r_2/s+j(x_2+X_m)\}\{r_2/2-s+j(x_2+X_m)\}}I_p I_n\varepsilon^{j2\omega t}\Big]\text{real part}\ \cdots\cdots(29)$$

Usually it is average value T that is useful to the user of a motor, as is well known. The pulsating torque becomes the cause of vibration and noise of the motor. Combining Eq. (27) and Eq. (28), and replacing a, b, c·········in place of suffix 1. 2. 3. ····················Eq. (28) becomes:

$$T=\frac{X_{ma}^2}{(r_{2a}/s)^2+(x_{2a}+X_{ma})^2}\Big|I_a+\frac{w_b}{w_a}I_b\varepsilon^{j\theta ab}+\frac{w_c}{w_a}I_c\varepsilon^{j\theta ac}+\cdots\cdots\Big|^2\frac{r_{2a}}{s}$$

$$-\frac{X_{ma}^2}{(r_{2a}/2-s)^2+(x_{2a}+X_{ma})^2}\Big|I_a+\frac{w_b}{w_a}I_b\varepsilon^{-j\theta ab}+\frac{w_c}{w_a}I_c\varepsilon^{-j\theta ac}+\cdots\cdots\Big|^2\frac{r_{2a}}{2-s}\ \cdots\cdots(30)$$

where $X_{ma}=w_a^2X_m,\quad r_{2a}=w_a^2r_2,\quad x_2=w_a^2\,x_2$ ·····································(31)

Example: For illustration of the method of using Eq. (30) or of its correctiveness, we show the torque of the usual three-phase induction motor by virtue of Eq. (30). When the rotor rotates in oder of phase A→B→C, we have:

$$\varepsilon^{j\theta ab}=\alpha,\qquad \varepsilon^{j\theta ac}=\alpha^2,$$

$$w_b/w_a=1,\qquad w_b/w_a=1$$

And when this motor is impressed by three-phase balanced voltage, the results are:

$$I_b=\alpha^2 I_a, \qquad I_c=\alpha I_a$$

Consequently substituting above conditions in Eq. (30),

| | in the 1st term of Eq. (30)$=3I_a$

| | in the 2nd term of Eq. (30)$=0$

Therefore we have:

$$T=3\frac{(3X_{ma})^2}{(3r_{2a}/s)^2+(3x_{2a}+3X_{ma})^2}|I_a|^2\frac{3r_{2a}}{s} \qquad (32)$$

This result is identified to the torque obtained from the equvivalent circuit of the usual three-phase induction motor.

3. Reconsidering [Z] in Combining Two Revolutions Theory of Magnetic Field & Thory of Equivalent Circuit

I have described the procedure of obtaining the impeadance matrix by solving the differential equations, and there will be no question in this procedure and results. But we had better reconsider in combinning two revolutions theory of the magnetic field and the theory of the equivalent circuit of the pure single-phase induction machine in order to comprehend the physical meanings of the elements of [Z]. And provided these meannings are well understood, we can construct [Z] for any induction machine easily even if we can't bring such a general formula of [Z] as Eq. (22) to our minds.

The equivalent circuit where the said motor is running on the single-phase of α can be shown in Fig. 5, as is already well known by the readers.

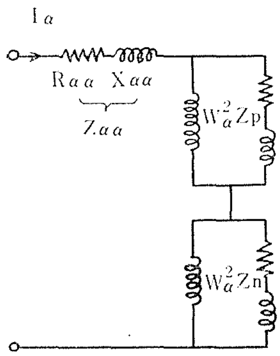


Fig. 5

The alternating m. m. fs. produced by either primary or secondary current can be divided respectively into two rotating m. m. fs. by two revolutions theory, that is, one is positive or forward in direction and the other is negative or back-ward. Both are equal in magnitude and in angular velocity. The group of all positive m. m. fs. are combined to produce a positive rotating magnetic field in air gap likewise all negative ones, a negative rotating field. And these two rotating fields are not always equal in magnitude.

In Fig. 5 $I_\alpha w_\alpha{}^2 Z_p$ is the impressed voltage necessary to overcome the induced e. m. f. in phase α by the positive field, and $I_\alpha w_\alpha{}^2 Z_n$ is that by the negative field. The positive or the negative magnetic field in air gap, therefore, can be indicated by $I_\alpha w_\alpha{}^2 Z_p$ or $I_\alpha w_\alpha{}^2 Z_n$ as far as the relative magnitude and relative phase are concerned.

In the motor in question there are other phases in the stator, and positive or negative magnetic fields induce e. m. fs. in these phases as well as phase α.

And e. m. fs. produced in stator phase β (β=1. 2. 3. ………n) by the positive or negative field are lagged or led in phase by $\theta_{\alpha\beta}$ to e. m. fs. produced in stator phase α by the same fields, and they are proportional in magnitude to the effective turn ratio of phase β and α. Accordingly the e. m. f. in phase β is :

$$\frac{w_\beta}{w_\alpha}(-w_\alpha^2 Z_p I_\alpha)\varepsilon^{-j\theta\alpha\beta}+\frac{w_\beta}{w_\alpha}(-w_\alpha^2 Z_n I_\alpha)\varepsilon^{j\theta\alpha\beta} \cdots\cdots\cdots (33)$$

In the above equation the first term is due to the positive field and the second to the negative.

In formula (33) putting α=1, β=1. 2. 3. …………n, we have the first line of [Z] expressed in Eq. (22), and putting α=2 β=1. 2. 3. ………n, we have the second line of [Z], etc.

Reversely putting α=1. 2. 3. …………n, β=1, we have the first column of [Z], putting α=1. 2. 3. …………n, β=2, the second column, etc.

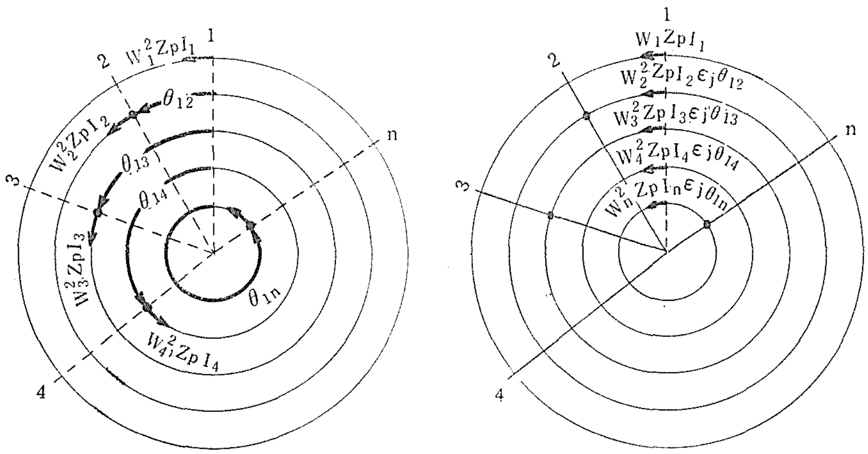
According to the above description, we can understand the following: When the currents I_1, I_2, I_3, ………, I_n flow in each phase simultanuously, we have positive fields of $w_1^2 Z_p I_1$, $w_2^2 Z_p I_2$, …………, $w_n^2 Z_p I_n$ or negative fields of $w_1^2 Z_n I_1$, $w_2^2 Z_n I_2$, …………, $w_n^2 Z_n I_n$ in air gap. And vector sum of all positive fields reduced to phase α is:

$$\frac{w_\alpha}{w_1}w_1^2 Z_p I_1 \varepsilon^{-j\theta_{1\alpha}}+\frac{w_\alpha}{w_2}w_2^2 Z_p I_2 \varepsilon^{-j\theta_{2\alpha}}+\cdots\cdots\cdots +\frac{w_\alpha}{w_n}w_n^2 Z_p I_n \varepsilon^{-j\theta_{n\alpha}}$$

$$= w_\alpha \sum_{\beta=1}^{n} w_\beta I_\beta \varepsilon^{j\theta\alpha\beta} \quad \text{(see Fig. 6)}$$

By a similar procedure to the negative fields, we have

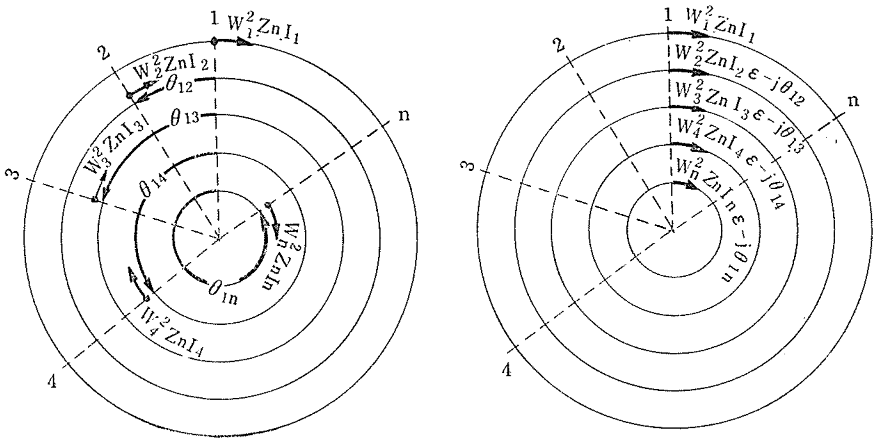
$$w_\alpha \sum_{\beta=1}^{n} w_\beta I_\beta \varepsilon^{-j\theta\alpha\beta} \quad \text{(see Fig. 7)}$$



(a) Distribution of all positive fields

(b) All positive fields reduced to axis of phase 1.

Fig. 6



(a) Distribution of all negative fields

(b) All negative fields reduced to axis of phase 1.

Fig. 7

Therefore the second and the third terms in Eq. (20) are the impressced voltages to overcome the induced e. m. fs. in phase α due to the positive and negative fields in air gap respectively.

4. Symmetrical Coordinate Method for Asymmetrical Three-phase Winding Axis

The difference between' general asymmetrical' and' asymmetrical' has been already described in the inductory remarks. That is, in the latter the stater phases are placed symmetrically in space. In this section the three-phase symmetrical coordinate method adaptable to the three-phase induction machine having an asymmetrical winding axis will be indicated., and its physical meaning will be explained.

From Eq. (23)

$$\left.\begin{aligned} V_a &= Z_{aa}I_a + Z_{pa}(I_a + \alpha c_1 I_b + \alpha^2 c_2 I_c) + Z_{na}(I_a + \alpha^2 c_1 I_b + \alpha c_2 I_c) \\ V_b &= Z_{bb}I_b + \alpha^2 Z_{pa}(I_a + \alpha c_1 I_b + \alpha^2 c_2 I_c) + \alpha Z_{na}(I_a + \alpha^2 c_1 I_b + \alpha c_2 I_c) \\ V_c &= Z_{cc}I_c + \alpha Z_{pa}(I_a + \alpha c_1 I_b + \alpha^2 c_2 I_c) + \alpha^2 Z_{na}(I_a + \alpha^2 c_1 I_b + \alpha c_2 I_c) \end{aligned}\right\} \cdots\cdots (34)$$

At this case according to nearly the same method as the usual three-phase symmetrical coordinate method, the following are introduced.

$$\left.\begin{aligned} &{}^1/_3(I_a + c_1 I_b + c_2 I_c) \equiv I_0 && \text{(zero phase-sequence current)} \\ &{}^1/_3(I_a + \alpha c_1 I_b + \alpha^2 c_2 I_c) \equiv I_1 && \text{(positive 〃 〃)} \\ &{}^1/_3(I_a + \alpha^2 c_1 I_b + \alpha c_2 I_c) \equiv I_2 && \text{(negative 〃 〃)} \end{aligned}\right\} \cdots\cdots (35)$$

$$\left.\begin{array}{lll} Z_{aa}+3Z_{pa}\equiv Z_{01} & \text{(poaitive phase-sequence impedance of the motor)} \\ Z_{aa}+3Z_{na}\equiv Z_{02} & \text{(negative} \quad // \quad // \quad) \\ Z_{aa} & \text{(zero} \quad // \quad // \quad) \end{array}\right\} \cdots (36)$$

$$\left.\begin{array}{lll} Z_{aa}I_0\equiv V_0 & \text{(zero phase-sequence voltage of the motor)} \\ Z_{01}I_1\equiv V_1 & \text{(positive} \quad // \quad // \quad) \\ Z_{02}I_2\equiv V_2 & \text{(negative} \quad // \quad // \quad) \end{array}\right\} \cdots\cdots (37)$$

Substituting Eq. (35), (36), (37) in Eq. (34), we have the following equation of the same form as that appeared in the usual three-phase symmetrical coordinate method.

$$\left.\begin{array}{l} V_a=V_0+V_1+V_2 \\ V_b'=V_0+\alpha^2V_1+\alpha V_2 \\ V_c'=V_0+\alpha V_1+\alpha^2 V_2 \end{array}\right\} \cdots\cdots (38)$$

where

$$\left.\begin{array}{ll} V_b'=1/c_1(V_b-\Delta Z_b I_b), & V_c'=1/c_2(V_c-\Delta Z_c I_c) \\ \Delta Z_b=Z_{bb}-c_1^2Z_{aa}, & \Delta Z_c=Z_{cc}-c_2^2Z_{aa} \end{array}\right\} \cdots\cdots (39)$$

Therefore on the asymmetrical three-phase winding axis, I_a, c_1I_b, c_2I_c, V_a, V_b', V_c' must be regarded as the elements of three phases. This reason will be described physically below.

Now let the effective turns of phase B or C indentify to those of phase A, then the phase currents must be multiplied by C_1 or C_2, the phase voltages by $1/c_1$ or $1/C_2$, and the leakage impedances by c_1^2 or c_2^2, just as is done in the transformer. If Z_{bb}/c_1^2 or Z_{cc}/c_2^2 are equal to Z_{aa}, —in other words, Z_{bb} or Z_{cc} are equal to $c_1^2Z_{aa}$ or $c_2^2Z_{aa}$—, these three phases are symmetrical windings. So the usual three-phase symmetrical coordinate method is applicable to these windings. If it is not so, $\Delta Z_b=Z_{bb}-c_1^2Z_{aa}$ or $\Delta Z_c=Z_{cc}-c_2^2Z_{aa}$ are regarded as the inserted impedances, and consequently we must regard $1/c_1(V_b-\Delta Z_b I_b)$ or $1/c_2(V_c-\Delta Z_c I_c)$as repective phase voltage.

The two-phase symmetrical coordinate method adaptable to the two-phase induction machine having an asymmetrical stator winding axis can be established by entirely the same procedure as before, but as this had been already researched by Mr. Kondo(4), here this is omitted.

5. Conclusion

Main items described in this paper are as follows:

1. The impedance matrix of any phase induction motor having the general asymmetrical stator winding axis is derived by solving the differential equations. And as this is reconsidered by two revolutions theory of the magnetic field and by the theory of equivalent circuit of single-phase induction motor, the physical meanings of the elements of $[Z]$ becomes obvious. The readers of this paper will already notice that the definition

of $\theta_{\alpha\beta}$ introduced by me is very convenient in constructing $[Z]$.

2. Torque formula is given, and thereby the pulsating torque becomes obvious. This pulsating torque will be unable to be obtained by Dr. Takegami and his group's method.
   I have explained hitherto various things on the assumption that the rotating direction is known to the analizer. When it is unknown, we assume it be counterclockwise or clockwise, and advance the analysis. And if this results show the negative starting torque, the motor runs contrarily to the direction as has been assumed first. Thus we can know the rotating direction.
3. I have explained in regard to the three-phase symmetrical coordinate method how it be changed to the three-phse asymmetrical winding axis and its physical meanings.

In conclusion, I should like to express my cordial thanks to Mr. Morrill, from whose excellent paper I have obtained a number of valuable informations.

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(3) S. Miyairi : J. I. E. E. Oct. 1948
(4) H. Kondo : J. I. E. E. Oct. 1947
(5) W. J. Morrill : T. A. I. E. E. p.614 1929

Summary

Such a stator-winding as every phase of it is not always placed apart symmetrically in space and its effective turns are not necessarily equal to others is named here as 'a general stator-winding axis'. In this paper primarily the impeadance matrix $[Z]$ and torque formula of any phase induction machine having a general asymmetrical stator-winding axis are derived by solving the differential equations. To calculate the various characteristics of the above machine from these results is well-known, so I have omitted its description here. Therefore I believe this paper gives an analytical method on the above machine.

Secondly $[Z]$ is reconsidered by two revolutions theory of the magnetic field and the theory of equivalent circuit of a single-phase induction motor so that the physical meanings of the elements of $[Z]$ may be understood well.

Thirdly it is described how the three-phase symmetrical coordinate method be extended so as to be adaptable to three-phase asymmetrical winding axis.