Operational Method for Continuous Beams
Second Report
Generalized Continuous Beams

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SYNOPSIS.

This paper is a continuation from the first report by the writers,[6] and it presents the generalized formulation of the operational method[2)4)5)] for bending problems of continuous beams. The powerfulness of the method in philosophy and computation may be observed from the shifting chart and the geometry matrix. The rigorous solution for all kinds of continuous beams can be obtained by systematic shift operations.

The procedures presented herein can be readily developed to problems of the beams on elastic foundation, the beams with axial force, the column buckling, and their vibration analyses.

INTRODUCTION.

Notation. —— The symbols adopted for use in this paper are defined where they first appear and are arranged alphabetically in the Appendix.

The powerful approach to the bending problems of beams and plates has been established by the eigenmatrix method,[1] which is due to the perfect classification of data, so that physical quantities of similar quality are represented by the corresponding matrix, and hence the problem can be treated systematically.

The operational method is the powerful weapon for various complex structural systems. The physical conditions between respective constituent units or groups can be represented by the matrix operator, and the analysis is carried out by simple and systematic operations. Simultaneous equations are of no use by this method.

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Since these few years, the operational method has been developed for various structures quite successfully.[2)4)6)7)] Moreover, it has been found that the classical Clapeyron's three moment theory can also be treated by this method dispensing with simultaneous equations.[5)]

From the above investigations, further effectiveness in the operational procedures for the analysis of all kinds of ordinary continuous beams has been confirmed.

The elastic behavior of any member in a continuous beam is governed by the well-known differential equation, and any two consecutive members are interconnected with each other at their common connection point through due connection conditions. A member may be loaded arbitrarily, and a given external load will be represented by the corresponding load-matrix. From the superposition law, a system of concentrated loads can then be represented only by the summation of respective load-matrices. The partially distributed load can be treated similarly. The assemblage of all the load-matrices should therefore consist of the positions and the magnitude of given loads, and it is perfectly independent of beam configurations. On the other hand, the operational matrices, resulting from connection or supporting conditions, can be determined from the geometry and material properties of the continuous beam.

Corresponding to the possible states of connection or supporting conditions, there may be several kinds of *shift formulas* with which the eigenmatrix can be shifted from one span to the adjacent span, and hence they will be formulated exhaustively. Boundary conditions will permit the determination of the current-matrix.

The above operation can be illustrated schematically by the "shifting chart" which indicates the procedure suitable for a given system.

By such operations, the eigenmatrix of continuous beam can be obtained in the form of the "geometry matrix" postmultiplied by the assembled load-matrix. A derivation of the geometry matrix is given in the Example.

BASIC CONCEPTS.

The deflection at each domain of the constituent span shown in Fig. 1 is given by

$$\begin{vmatrix} w \\ w^i \\ w' \end{vmatrix}_r = \frac{l_r^{\,3}}{6EI_r} \lfloor 1 \quad \rho \quad \rho^2 \quad \rho^3 \rfloor \begin{vmatrix} \boldsymbol{N} \\ \boldsymbol{N}^i \\ \boldsymbol{N}' \end{vmatrix}_r , \tag{1}$$

provided

$$\rho = \frac{x}{l}. \tag{2}$$

The parallel lines denote respective correspondences of w_r to $\boldsymbol{N}_r$, $w^i{}_r$ to $\boldsymbol{N}^i{}_r$, and $w'{}_r$ to $\boldsymbol{N}'{}_r$, in which the subscript r represents the span order

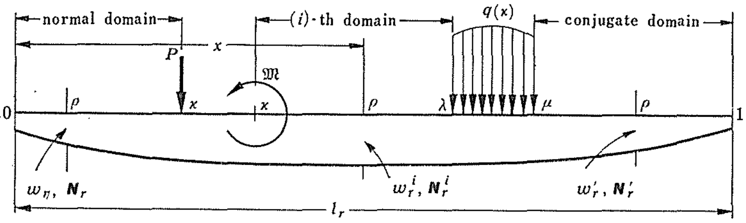


Fig. 1. Constituent Span of Continuous Beam (r-th Span).

and the superscript i the domain order provided the normal and the conjugate domains are denoted by no superscript and prime (′) respectively. w is the deflection, l the span length, EI the rigidity, ρ the non-dimensional current abscissa of coordinates given by Eq. 2. $\boldsymbol{N}$ is the eigenmatrix given for each domain as follows:

$$\boldsymbol{N}_r = \{A \quad B \quad C \quad D\}_r, \tag{3}$$

$$\boldsymbol{N}^i{}_r = \boldsymbol{N}_r + \sum_{\kappa=0}^{\rho} (\boldsymbol{K}_p + \int_{\lambda}^{\mu} d\boldsymbol{K}_p + \boldsymbol{K}_m)_r, \tag{4}$$

$$\boldsymbol{N}'{}_r = \boldsymbol{N}_r + \boldsymbol{K}_r. \tag{5}$$

The second term on the right side of Eq. 4 represents the aggregate matrix for a point where all the load-matrices in the leftward domain are assembled. The respective terms in parentheses are load-matrices for the concentrated load, the partially distributed load, and the external concentrated moment, given by the forms:

$$\boldsymbol{K}_p = P\{-\kappa^3 \quad 3\kappa^2 \quad -3\kappa \quad 1\}, \tag{6}$$

$$\int_{\lambda}^{\mu} d\boldsymbol{K}_p = l\int_{\lambda}^{\mu} q(\kappa)\{-\kappa^3 \quad 3\kappa^2 \quad -3\kappa \quad 1\}d\kappa, \tag{7}$$

$$\boldsymbol{K}_m = \frac{3}{l}\mathfrak{M}\{\kappa^2 \quad -2\kappa \quad 1 \quad 0\}. \tag{8}$$

The term $\boldsymbol{K}_r$ in Eq. 5 is designated as the "load term" of the constituent span considered, which is given by assembling all the load-matrices on the span as follows:

$$\boldsymbol{K}_r = \sum_{\kappa=0}^{1} (\boldsymbol{K}_p + \int_\lambda^\mu d\boldsymbol{K}_p + \boldsymbol{K}_m)_r. \tag{9}$$

Thus, the deflection of each domain of the constituent span is expressed by its eigenmatrix consisting of 4-by-1 elements and the corresponding aggregate load-matrix which can be given by each loading condition. The dimension of each element of the eigenmatrix is the same as that of the concentrated load.

An eigenmatrix is shifted, when possible, to the adjacent span in such a manner that the connection conditions at the common point of the two consecutive spans considered are satisfied. At times the shift operation of the eigenmatrix is prohibited, in which case the degradation of the eigenmatrix is necessary or otherwise. In this way, the boundary conditions for determining the current-matrix may be those at an intermediate junction point as well as those at the extreme ends of the beam.

In the present paper, the continuous beam is considered as an assemblage of members whose deflections are governed by Eq. 1.

First, a member will be subjected to a preliminary treatment when necessary, which will result in a definite restriction upon the eigenmatrix of the member.

After that, the connection conditions remained untreated at an intermediate point between two adjacent members are considered, from which the necessary shift operation can be made.

In the generalized continuous beam, there are several possible states of connection points, and hence they may be classified into two large groups as follows:

(4C)-point: There are four connection conditions but no boundary condition at this point, of which the following four possible cases must be considered:

(a) abruptly-changed cross-section;
(b) abruptly-changed cross-section with resisting moment;
(c) elastic support; and
(d) elastic support with resisting moment.

(2BC)-point: Two respective boundary conditions and connection conditions are considered at the following points:

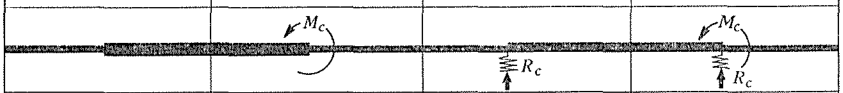
(e) rigid support;
(f) rigid support with resisting moment;
(g) pin joint; and
(h) elastic pin joint.

These conditions are illustrated in Table I with corresponding graphical symbols, in which $\boxed{1}$ represents the number of independent boundary conditions at each end of the constituent span, and ◈ is the number of connection conditions at each connection point. The symbols θ, M, and S denote the slope, bending moment, and shearing force at the left end, respectively. Those symbols primed refer to the right end of the span. M_C is the resisting moment which is proportional to the angle of rotation at the connection point, and R_C is the reaction at the elastic support.

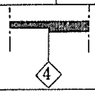
Table I. Physical Characteristics of Connection Point (r-th Connection Point).

(4 C) point

Abruptly-changed cross section	Abruptly-changed cross section with resisting moment	Elastic support	Elastic support with resisting moment
$\begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}'_{r-1} = \begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}_r$	$\begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}'_{r-1} = \begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}_r + \begin{bmatrix} 0 \\ 0 \\ M_c \\ 0 \end{bmatrix}$	$\begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}'_{r-1} = \begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}_r + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -R_c \end{bmatrix}$	$\begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}'_{r-1} = \begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}_r + \begin{bmatrix} 0 \\ 0 \\ M_c \\ -R_c \end{bmatrix}$

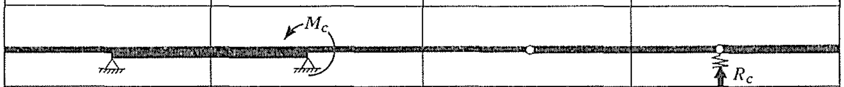


Graphical symbol of (4 C) point

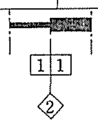


(2 BC) point

Rigid support	Rigid support with resisting moment	Pin joint	Elastic pin joint
$(w')_{r-1}=0$, $(w)_r=0$	$(w')_{r-1}=0$, $(w)_r=0$	$(M')_{r-1}=0$, $(M)_r=0$	$(M')_{r-1}=0$, $(M)_r=0$
$\begin{bmatrix} \theta \\ M \end{bmatrix}'_{r-1} = \begin{bmatrix} \theta \\ M \end{bmatrix}_r$	$\begin{bmatrix} \theta \\ M \end{bmatrix}'_{r-1} = \begin{bmatrix} \theta \\ M \end{bmatrix}_r + \begin{bmatrix} 0 \\ M_c \end{bmatrix}$	$\begin{bmatrix} w \\ S \end{bmatrix}'_{r-1} = \begin{bmatrix} w \\ S \end{bmatrix}_r$	$\begin{bmatrix} w \\ S \end{bmatrix}'_{r-1} = \begin{bmatrix} w \\ S \end{bmatrix}_r + \begin{bmatrix} 0 \\ -R_c \end{bmatrix}$



Graphical symbol of (2BC) point



BOUNDARY CONDITIONS.

The boundary conditions for each constituent span of a continuous beam can be expressed in the following forms:

For the left end: $\mathbf{B}_r \mathbf{N}_r = 0.$ (10)

For the right end: $\mathbf{B}'_r \mathbf{N}'_r = 0.$ (11)

Eqs. 10 and 11 must at times be assumed at intermediate connection points. The matrices $\mathbf{B}_r$ and $\mathbf{B}'_r$ are designated as the "boundary-matrices," consisting of 2-by-4 elements at both extremities of the continuous beam, and of 1-by-4 elements at both ends of an intermediate span.

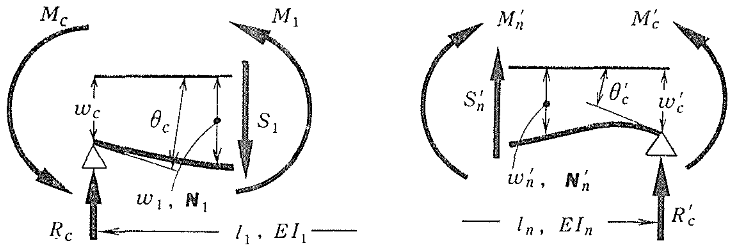
Both Extremities.

Both extremities of the continuous beam are assumed to be of elastic support where the deflection is proportional to its reaction, and in addition, to be subjected to a resisting moment which is again proportional to the angle of rotation at this point. All the possible boundary conditions can be included in such an assumption. The above conditions are written in the forms:

At the left extremity: $w_c = kR_c, \quad \theta_c = mM_c.$ (12)

At the right extremity: $w'_c = k'R'_c, \quad \theta'_c = m'M'_c.$ (13)

Here, k, k', m, and m' are constants to be attached to each supporting point, and other symbols are illustrated in Fig. 2.



(a) Extreme Left Support. (b) Extreme Right Support.

Fig. 2. Boundary Conditions at Both Extremities.

Then Eqs. 10 and 11 become

$$\mathbf{B}_1 \mathbf{N}_1 = \begin{bmatrix} 1 & 0 & 0 & \lambda \\ 0 & 1 & -\mu & 0 \end{bmatrix} \mathbf{N}_1 = 0, \tag{14}$$

and

$$\mathbf{B}'_n \mathbf{N}'_n = \begin{bmatrix} 1 & 1 & 1 & 1-\lambda' \\ 0 & 1 & 2+\mu' & 3+3\mu' \end{bmatrix} \mathbf{N}'_n = 0, \tag{15}$$

in which, for shortness,

$$\begin{bmatrix} \lambda \\ \\ \mu \end{bmatrix} = \frac{2EI_1}{l_1} \begin{bmatrix} \dfrac{3k}{l_1^2} \\ m \end{bmatrix}, \qquad \begin{bmatrix} \lambda' \\ \\ \mu' \end{bmatrix} = \frac{2EI_n}{l_n} \begin{bmatrix} \dfrac{3k'}{l_n^2} \\ m' \end{bmatrix}. \tag{16}$$

Eqs. 14 and 15 are the generalized boundary conditions for both extremities. Every possible condition can be represented by the values of λ, μ, λ', and μ' in the boundary-matrices shown in Table II.

Table II. Values in Extreme Boundary-Matrix for Each Condition.

Left extremity 2		Condition of support			Right extremity 2	
λ	μ				λ'	μ'
λ	μ		elastic support with resisting moment		λ'	μ'
λ	∞		elastic support		λ'	∞
0	μ		rigid support with resisting moment		0	μ'
0	∞		simple support		0	∞
0	0		fixed support		0	0
∞	∞		free end		∞	∞

Intermediate Support.

Eqs. 10 and 11 can also be used for the 2BC-points in Table I. At the rigid support, the following conditions hold regardless of the existence of resisting moment. Then,

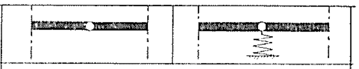
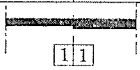
$$w'_{r-1} = w_r = 0. \tag{17}$$

Similarly, the pin joint and the elastic pin joint have common boundary conditions given by

$$M'_{r-1} = M_r = 0. \tag{18}$$

Eqs. 17 and 18 are given in Table III.

Table III. Boundary Conditions at Intermediate Support.

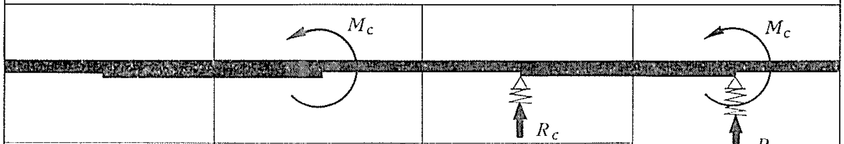
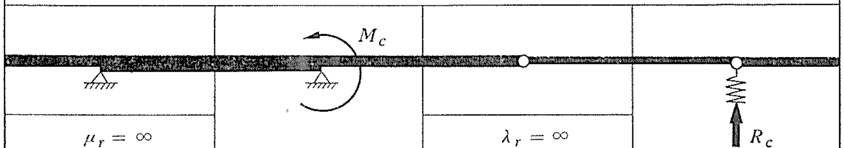
Case		
Condition	$[1\ \ 1\ \ 1\ \ 1]\boldsymbol{N}'_{r-1}=0$ $[1\ \ 0\ \ 0\ \ 0]\boldsymbol{N}_r=0$	$[0\ \ 0\ \ 1\ \ 3]\boldsymbol{N}'_{r-1}=0$ $[0\ \ 0\ \ 1\ \ 0]\boldsymbol{N}_r=0$
Symbol		

CONNECTION CONDITIONS.

The connection conditions at an intermediate connection point of a continuous beam are expressed in the matrix equation

$$\boldsymbol{C}_r\{\boldsymbol{N}'_{r-1}\quad \boldsymbol{N}_r\}=0, \tag{19}$$

Table IV. Connection-Matrix.

(4C) point ④			
			
$\lambda_r=\mu_r=\infty$	$\lambda_r=\infty$	$\mu_r=\infty$	
$\boldsymbol{C}_r=\left[\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&0&1\end{bmatrix},\ -\begin{bmatrix}\frac{\alpha^3}{\beta}&0&0&0\\0&\frac{\alpha^2}{\beta}&0&0\\0&-\frac{\alpha}{\mu}&\alpha&0\\\frac{1}{\lambda}&0&0&1\end{bmatrix}_r\right]$			
(2BC) point ②			
			
$\mu_r=\infty$		$\lambda_r=\infty$	
$\boldsymbol{C}_r=\left[\begin{bmatrix}0&1&2&3\\0&0&1&3\end{bmatrix},\ -\alpha_r\begin{bmatrix}0&\frac{\alpha}{\beta}&0&0\\0&-\frac{1}{\mu}&1&0\end{bmatrix}_r\right]$		$\boldsymbol{C}_r=\left[\begin{bmatrix}1&1&1&1\\0&0&0&1\end{bmatrix},\ -\begin{bmatrix}\frac{\alpha^3}{\beta}&0&0&0\\\frac{1}{\lambda}&0&0&1\end{bmatrix}_r\right]$	

in which $\boldsymbol{C}_r$ is the "connection-matrix," $\boldsymbol{N}'_{r-1}$ and $\boldsymbol{N}_r$ are the eigenmatrices of the two adjacent spans, respectively.

In Table IV is shown the connection-matrix for each possible case. Here, the following symbols have been introduced:

$$\alpha_r = \frac{l_r}{l_{r-1}}, \tag{20}$$

$$\beta_r = \frac{EI_r}{EI_{r-1}}, \tag{21}$$

$$\lambda_r = \frac{6k_r EI_r}{l_r^3}, \tag{22}$$

$$\mu_r = \frac{2m_r EI_r}{l_r}. \tag{23}$$

SHIFT OPERATORS.

From the connection conditions obtained in Eq. 19, we can derive the desired shift formula between eigenmatrices of the two adjacent spans of a continuous beam. The formula is classified into two main groups as follows.

Operators at (4C)-Point.

Using the connection-matrix in Table IV, Eq. 19 yields the shift formulas

$$\boldsymbol{N}_r = \boldsymbol{S}_r \boldsymbol{N}_{r-1} + \boldsymbol{F}_r \boldsymbol{K}_{r-1}, \tag{24}$$

$$\boldsymbol{N}_{r-1} = \boldsymbol{S}'_r \boldsymbol{N}_r + \boldsymbol{F}'_r \boldsymbol{K}_{r-1}, \tag{25}$$

in which $\boldsymbol{S}_r$ or $\boldsymbol{S}'_r$, and $\boldsymbol{F}_r$ or $\boldsymbol{F}'_r$ are the right or leftward shift operators, and feed operators, briefly the "shiftors" and "feeders" repectively, which take the values

$$\boldsymbol{S}_r = \boldsymbol{F}_r = \begin{bmatrix} \frac{\beta}{\alpha^3} & 0 & 0 & 0 \\ 0 & \frac{\beta}{\alpha^2} & 0 & 0 \\ 0 & -\frac{\beta}{\mu\alpha^2} & \frac{1}{\alpha} & 0 \\ -\frac{\beta}{\lambda\alpha^3} & 0 & 0 & 1 \end{bmatrix}_r \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{26}$$

$$\mathbf{S}'_r = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{\alpha^3}{\beta} & 0 & 0 & 0 \\ 0 & \dfrac{\alpha^2}{\beta} & 0 & 0 \\ 0 & \dfrac{\alpha}{\mu} & \alpha & 0 \\ \dfrac{1}{\lambda} & 0 & 0 & 1 \end{bmatrix}_r, \tag{27}$$

and

$$\mathbf{F}'_r = -\mathbf{E} \qquad (\mathbf{E} \text{ being the 4-by-4 unit matrix}), \tag{28}$$

and other symbols have been defined by Eqs. 3 and 9.

All the supporting conditions of (4C)-points can be expressed by the values of λ and μ in the above equations as shown in Table IV.

Operators at (2BC)-Point.

Since the number of connection conditions is deficient in shifting all the elements in the eigenmatrix of a span to the adjacent span, the shift formula results in a particular form in this case. In virtue of Eqs. 3, 5, and 19, and Table IV, the following shift formulas for either direction can be derived. In this case, for the left span with pin connection end, a preliminary treatment must be made before the connection. That is to say, because of the non-square size of the connection-matrix, the eigenmatrix in the left span with pin joint must be reduced to a 3-by-1 eigenmatrix. Thus, the desired shift formulas become

$$\mathbf{A}_r = \mathbf{S}_r\mathbf{A}_{r-1} + \mathbf{F}_r\mathbf{K}_{r-1}, \tag{29}$$

$$\mathbf{A}_{r-1} = \mathbf{S}'_r\mathbf{A}_r + \mathbf{F}'_r\mathbf{K}_{r-1} + \mathbf{R}'_r\boldsymbol{a}_{r-1}. \tag{30}$$

Eq. 29 is the rightward shift formula and Eq. 30 is the leftward one. These equations indicate that the $\mathbf{A}_r$ and $\mathbf{A}_{r-1}$ semi-eigenmatrices on the left sides depend on the $\mathbf{A}_{r-1}$ and $\mathbf{A}_r$ matrices on the right sides, respectively. $\mathbf{S}_r$, $\mathbf{S}'_r$, $\mathbf{F}_r$, and $\mathbf{F}'_r$ are called as before the shiftors and feeders, respectively. A particular term $\mathbf{R}'_r\boldsymbol{a}_{r-1}$ appears in Eq. 30 because of the non-square size of connection-matrix. That is to say, three elements in the left span eigenmatrix participate in the connection equation, but there are only two connection conditions for these unknown elements, and hence one element must be selected as a redundant element and fed for other two elements of its own span. Therefore, $\mathbf{R}'_r$ is called the "feeder for redundant element" providing that $\boldsymbol{a}_{r-1}$ represents the redundant element.

In virtue of the non-square size of the connection-matrix, there also exists an element independent of the rigid support or pin joint condition.

The above redundant and independent elements are always determinable after the shift operations are carried out through the three connection points in question.

In Table V and VI are given operational factors corresponding to intermediate (2BC)-points.

Table V. Operational Factors for Intermediate Rigid Support with Resisting Moment (If $M_c = 0$, then $\mu_r = \infty$).

Left Span	Element	Right Span
$A_{r-1} = -(B+C+D)_{r-1} - \lfloor 1 \quad 1 \quad 1 \quad 1 \rfloor \boldsymbol{K}_{r-1}$	Determinable Element	$A_r = 0$
$\begin{bmatrix} B \\ C \end{bmatrix}_{r-1}$	Dependent Semi-eigenmatrix	$\begin{bmatrix} B \\ C \end{bmatrix}_r$
D_{r-1} (for $\boldsymbol{R}'_r \boldsymbol{a}_{r-1}$)	Redundant Element	none
$\begin{bmatrix} B \\ C \\ D \end{bmatrix}_{r-1}$ (for $\boldsymbol{S}_r \boldsymbol{A}_{r-1}$)	Independent Semi-eigenmatrix	$\begin{bmatrix} B \\ C \end{bmatrix}_r$ (for $\boldsymbol{S}'_r \boldsymbol{A}_r$)
none	Independent Element	D_r
Leftward Operator ⇐	**Operator**	⇒ **Rightward Operator**
$\boldsymbol{S}'_r = \alpha_r \begin{bmatrix} \frac{\alpha}{\beta} + \frac{2}{\mu} & -2 \\ -\frac{1}{\mu} & 1 \end{bmatrix}_r$	Shiftor	$\frac{\beta_r}{\alpha_r^2} \begin{bmatrix} 1 & 2 & 3 \\ \frac{1}{\mu} & \frac{\alpha}{\beta} + \frac{2}{\mu} & 3\left(\frac{\alpha}{\beta} + \frac{1}{\mu}\right) \end{bmatrix}_r$
$\boldsymbol{F}'_r = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & -3 \end{bmatrix}$	Feeder for Load Term	$\frac{\beta_r}{\alpha_r^2} \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & \frac{1}{\mu} & \frac{\alpha}{\beta} + \frac{2}{\mu} & 3\left(\frac{\alpha}{\beta} + \frac{1}{\mu}\right) \end{bmatrix}_r$
$\boldsymbol{R}'_r = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$	Feeder for Redundant Element	none

Table VI. Operational Factors for Intermediate Elastic Pin Joint
(For simple pin joint, $\lambda_r = \infty$).

Left Span	**Element**	**Right Span**
$C_{r-1} = -3D_{r-1} - \lfloor 0\ 0\ 1\ 3 \rfloor \boldsymbol{K}_{r-1}$	Determinable Element	$C_r = 0$
$\begin{bmatrix} A \\ D \end{bmatrix}_{r-1}$	Dependent Semi-eigenmatrix	$\begin{bmatrix} A \\ D \end{bmatrix}_r$
B_{r-1} (for $\boldsymbol{R}'_r \boldsymbol{a}_{r-1}$)	Redundant Element	none
$\begin{bmatrix} A \\ B \\ D \end{bmatrix}_{r-1}$ (for $\boldsymbol{S}_r \boldsymbol{A}_{r-1}$)	Independent Semi-eigenmatrix	$\begin{bmatrix} A \\ D \end{bmatrix}_r$ (for $\boldsymbol{S}'_r \boldsymbol{A}_r$)
none	Independent Element	B_r
Leftward Operator $\Leftarrow$	**Operator**	$\Rightarrow$ **Rightward Operator**
$\boldsymbol{S}'_r = \begin{bmatrix} \frac{\alpha^3}{\beta}+\frac{2}{\lambda} & 2 \\ \frac{1}{\lambda} & 1 \end{bmatrix}_r$	Shiftor	$\boldsymbol{S}_r = \frac{\beta_r}{\alpha_r^3} \begin{bmatrix} 1 & 1 & -2 \\ -\frac{1}{\lambda} & -\frac{1}{\lambda} & \frac{\alpha^3}{\beta}+\frac{2}{\lambda} \end{bmatrix}_r$
$\boldsymbol{F}'_r = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	Feeder for Load Term	$\boldsymbol{F}_r = \frac{\beta_r}{\alpha_r^3} \begin{bmatrix} 1 & 1 & 0 & -2 \\ -\frac{1}{\lambda} & -\frac{1}{\lambda} & 0 & \frac{\alpha^3}{\beta}+\frac{2}{\lambda} \end{bmatrix}_r$
$\boldsymbol{R}'_r = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$	Feeder for Redundant Element	none

Operators for Group Shifting.

Separating the continuous beam into isolated members or member groups at respective (2BC)-points, we designate them as the "constituent groups." In one of these groups the shift operation can always be carried out in either direction by a 4-by-4 shiftor given by Eq. 26 or 27, and in virtue of boundary conditions attached to both ends, the number of unknown elements is reduced to two; that is to say, by the above preliminary operation, the eigenmatrix of each constituent group can be degraded to a 2-by-1 semi-eigenmatrix.

Then the connection conditions at (2BC)-point give the following shift formula between two consecutive groups. This is called the "group shifting," which is effected by the equations

$$\boldsymbol{A}_R = \boldsymbol{S}_{R-1}\boldsymbol{A}_R + \boldsymbol{F}_R\boldsymbol{K}_{R-1} + \boldsymbol{T}_R\boldsymbol{K}_R, \tag{31}$$

$$\boldsymbol{A}_{R-1} = \boldsymbol{S}'_R\boldsymbol{A}_R + \boldsymbol{F}'_R\boldsymbol{K}_R + \boldsymbol{T}'_R\boldsymbol{K}_{R-1}. \tag{32}$$

The subscript R refers to the group number. $\boldsymbol{A}_{R-1}$ and $\boldsymbol{A}_R$ are 2-by-1 semi-eigenmatrices, and $\boldsymbol{K}_{R-1}$ and $\boldsymbol{K}_R$ are 4-by-1 associate load terms of both consecutive constituent groups. $\boldsymbol{S}_R$ and $\boldsymbol{S}'_R$, $\boldsymbol{F}_R$ and $\boldsymbol{F}'_R$, and $\boldsymbol{T}_R$ and $\boldsymbol{T}'_R$ are 2-by-2 shiftors, 2-by-4 feeders for load term of the group with independent eigenmatrix, and 2-by-4 feeders for load term of the group with dependent eigenmatrix, respectively.

This procedure can be applied for continuous plate-girder systems effectively. Although the generalized formulation for group shifting may be made, preference should be given to the numerical treatment for given systems.

SHIFTING CHART.

The operational procedure of continuous beams can be illustrated schematically by the shifting chart expressing the relation between the number of the elements of eigenmatrix and the number of physical conditions at connection points. This chart suggests all possible ways of shift operations.

The following symbols are introduced:

①, ②, ③, ④ ·········number of unknown elements of constituent span,

$\boxed{1}$, $\boxed{2}$ ·····················number of boundary conditions, and

◈, ◈ ·····················number of connection conditions at intermediate supporting point.

The eigenmatrix in each constituent span has four unknown elements. The physical conditions at both ends of the span correspond to the freedom of the eigenmatrix. In a continuous beam, the total number of unknown elements is equal to that of physical conditions. Therefore, the system can be solved by due treatment between the above elements and conditions.

Some typical shifting procedures are shown in Figs. 3 through 7. According to the operation to be adopted, the following arrows are defined:

———► : Operation within its own span. The number of unknown elements can be degraded by boundary conditions attached to each end.

– – –► : Operation for elements in a span with dependent eigen-matrix. Usually, after the connection, these elements can be expressed by current-elements, but occasionally, at (2BC)-point, an exchange of one of the current-elements occurs.

⟹ : Operation for current-elements. The topological array of such arrows shows that the eigenmatrix in each span can be represented by the same current-elements.

⟶ : Final condition.

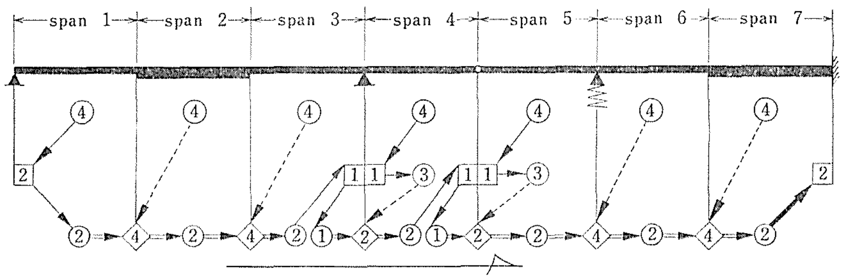


Fig. 3. Shifting Chart for Rightward Operation.

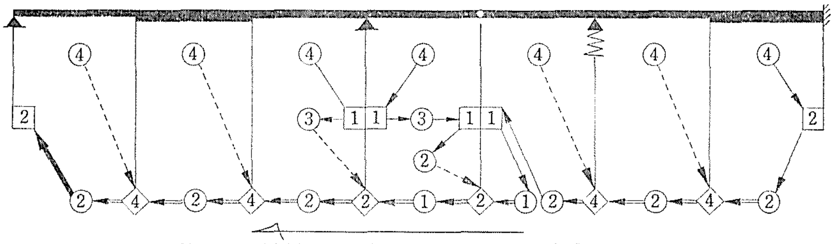


Fig. 4. Shifting Chart for Leftward Operation.

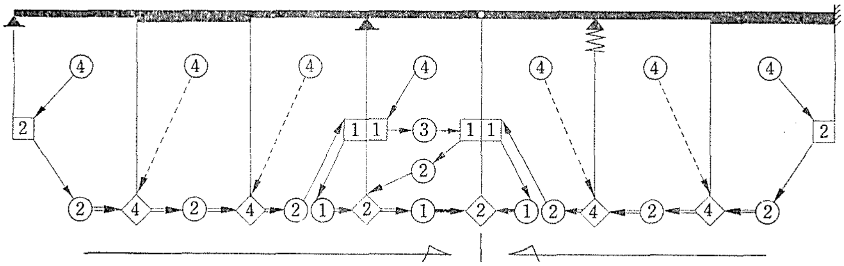


Fig. 5. Shifting Chart for Inward Operation.

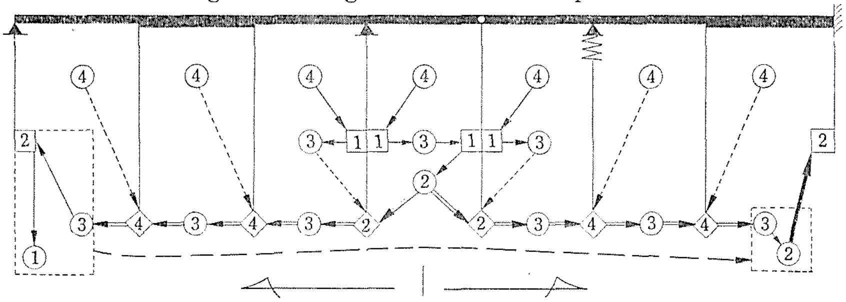


Fig. 6. Shifting Chart for Outward Operation.

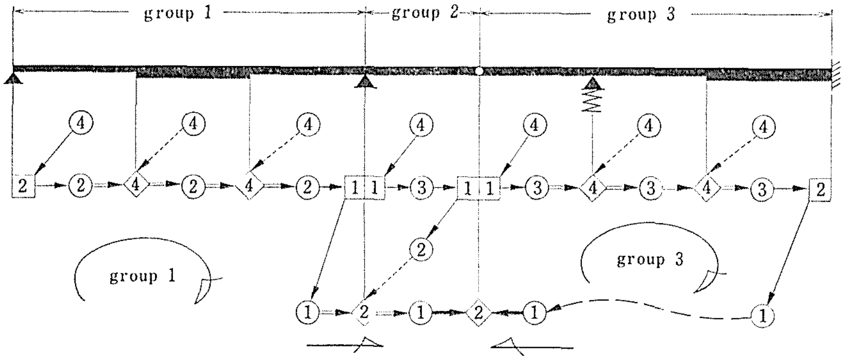


Fig. 7. Shifting Chart for Group Operation.

First, Fig. 3 shows the rightward shifting procedure. At the extreme left end, two unknown elements in the first span are determined. Secondly, in virtue of four connection conditions between the first and second constituent spans, the four unknowns of the second span can be expressed by the two elements of the first span; that is to say, the two elements in the first span are shifted to the second span. Thus, we can shift the elements in the standard span to the adjacent spans successively, so that these standard elements are designated as the "current-elements."

At (2BC)-point, because of its characteristics, several particular treatments must be made. The arrows illustrated in the figure are one of the effective procedures.

At (2BC)-point one of the current-elements in the left span is determined by the boundary condition attached to this point, and from the right span, another element participates in the current-elements.

Proceeding the shiftings on, the eigenmatrix of the extreme right span is expressed by two unknown current-elements. Then these can be readily determined by the extreme boundary conditions as shown in the figure.

Fig. 4 shows the leftward operation. Regardless of the shifting direction, the procedure at (4C)-point is the same as that of the rightward operation, but the treatment at (2BC)-point gives some different aspects.

In the case of the above operations in one direction, the recurrent multiplications by the operators at respective connection point result in the complexity in numerical values treated.

To avoid such an inconvenience, it is recommended to use the inward and outward opertions, or the group shifting operation as follows.

Fig. 5 shows the inward shifting procedure. The operations are carried out from both extreme spans to the interior spans. If a suitable (2BC)-point is taken as the final connection point as shown in the figure, then the two unknowns can be determined, and if a (4C)-point is adopted, then the four elements can be determined.

In Fig. 6 is shown the outward shifting procedure. The starting span of operation can be selected arbitrarily, and the current-elements are shifted to both directions from this span. Finally, the eigenmatrices of respective extreme spans are represented by three unknown elements in which two elements are common current-elements. Therefore, several kinds of treatments may be made between both extremities as shown in the figure.

In the case of continuous plate-girder systems, the group shifting should be adopted as shown in Fig. 7. The preliminary treatment is performed in each constituent group, and the number of current-elements can be reduced to two at the interior group or reduced to one at the both extreme groups.

Next, the complete shifting is carried out by the connection conditions at (2BC)-points, and the final conditions can be given at an arbitrary (2BC)-point.

APPLICATION.

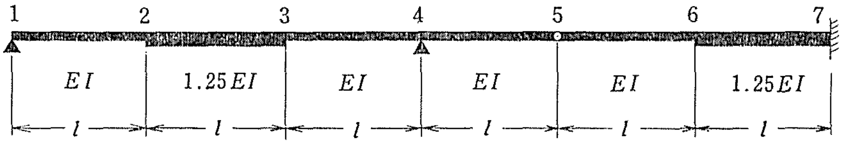
In applying the above described procedures to a given continuous beam, it is advantageous to carry out the shift operations in the form separating the operators from the load terms. By such a treatment, the eigenmatrix of each span can always be expressed in the form

$$\boldsymbol{N}_r = \lfloor \boldsymbol{G}_{r1} \quad \boldsymbol{G}_{r2} \ \cdots \ \boldsymbol{G}_{rn} \rfloor \{ \boldsymbol{K}_1 \quad \boldsymbol{K}_2 \ \cdots \ \boldsymbol{K}_n \}, \tag{33}$$

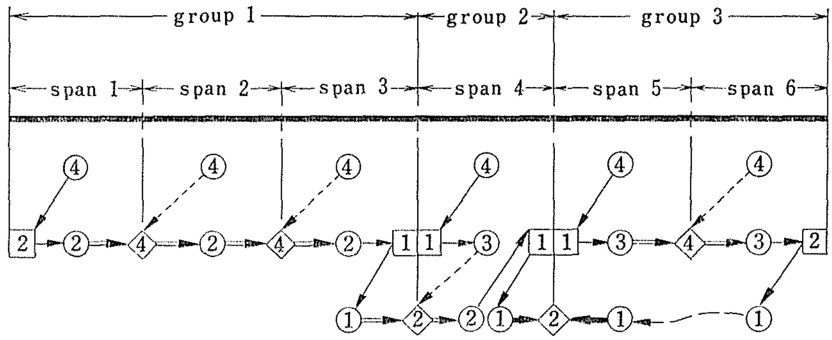
in which $\boldsymbol{G}_{r1}$, $\boldsymbol{G}_{r2}$, $\boldsymbol{G}_{r3}, \cdots$ are designated as the "geometry matrices" consisting of 4-by-4 elements determined by the beam configuration, and $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, $\boldsymbol{K}_3$, $\cdots$ are the load terms of respective constituent spans.

It may be concluded that the geometry matrices are the final solution of the continuous beam, since they can represent all physical properties of the beam for all kinds of loading conditions. That is to say, if the load terms of respective spans are given, the eigenmatrix can be readily obtained by Eq. 33.

As a simple application, let us take the six-span continuous beam shown in Fig. 8, and derive the geometry matrices for the respective spans. The analysis may be carried out numerically by means of the group shifting. Fig. 8b is the desired shifting chart providing that the beam configuration is given in Fig. 8a.



(a) Configuration



(b) Shifting chart

Fig. 8. Configuration and Shifting Chart of Continuous Beam.

——— *Ratios.* ———

The following values of α and β are adopted in the present numerical example:

$$\alpha_r = 1 \qquad (r = 2,\ 3,\ 4,\ 5,\ 6), \tag{34}$$

$$\beta_2 = \beta_6 = 1.25, \quad \beta_3 = 0.8, \quad \beta_4 = \beta_5 = 1. \tag{35}$$

——— *Operators.* ———

Referring to the shifting chart, the necessary operators become as follows:

$$\mathbf{S}_2 = \mathbf{F}_2 = \mathbf{S}_6 = \mathbf{F}_6 = \begin{bmatrix} 1.25 & 1.25 & 1.25 & 1.25 \\ 0 & 1.25 & 2.5 & 3.75 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{36}$$

$$\mathbf{S}_3 = \mathbf{F}_3 = \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.8 \\ 0 & 0.8 & 1.6 & 2.4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{37}$$

$$\mathbf{S}_4=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 3\end{bmatrix}, \tag{38}$$

$$\mathbf{F}_4=\begin{bmatrix}0 & 1 & 2 & 3\\ 0 & 0 & 1 & 3\end{bmatrix}, \tag{39}$$

$$\mathbf{C}_5=\left[\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}, \ -\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\right]. \tag{40}$$

——— *Preliminary Operation in Group 1.* ———

The extreme left boundary conditions are given from Table II as

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\mathbf{N}_1=0, \tag{41}$$

from which $\mathbf{N}_1$ can be reduced to the form

$$\mathbf{N}_1=\{0 \quad B \quad 0 \quad D\}_1. \tag{42}$$

Shifting rightwards, the conjugate $\mathbf{N}'_3$ of the third span becomes

$$\mathbf{N}'_3=\mathbf{S}_3\mathbf{S}_2(\mathbf{N}+\mathbf{K})_1+\mathbf{S}_3\mathbf{K}_2+\mathbf{K}_3, \tag{43}$$

in which

$$\mathbf{S}_3\mathbf{S}_2=\begin{bmatrix}1 & 2 & 3.8 & 7.2\\ 0 & 1 & 3.6 & 10.2\\ 0 & 0 & 1 & 6\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{44}$$

The boundary condition at the right end of the third span is given by

$$\lfloor 1 \quad 1 \quad 1 \quad 1\rfloor\mathbf{N}'_3=0. \tag{45}$$

Substituting from Eq. 43 into Eq. 45,

$$\lfloor 1 \quad 3 \quad 8.4 \quad 24.4\rfloor(\mathbf{N}+\mathbf{K})_1+\lfloor 0.8 \quad 1.6 \quad 3.4 \quad 7.2\rfloor\mathbf{K}_2 + \lfloor 1 \quad 1 \quad 1 \quad 1\rfloor\mathbf{K}_3=0, \tag{46}$$

from which

$$B_1=-\frac{24.4}{3}D_1-\left[\lfloor 1 \quad 3 \quad 8.4 \quad 24.4\rfloor, \quad \lfloor 0.8 \quad 1.6 \quad 3.4 \quad 7.2\rfloor, \quad \lfloor 1 \quad 1 \quad 1 \quad 1\rfloor\right]\{\mathbf{K}_1 \quad \mathbf{K}_2 \quad \mathbf{K}_3\}. \tag{47}$$

Then, in virtue of the preliminary operation, the eigenmatrices $\boldsymbol{N}_1$ and $\boldsymbol{N}'_3$ are given by the forms

$$\boldsymbol{N}_1 = \frac{1}{3}\begin{bmatrix} 0 \\ -24.4 \\ 0 \\ 3 \end{bmatrix} D_1 - \frac{1}{3}\left[\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 3 & 8.4 & 24.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.8 & 1.6 & 3.4 & 7.2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right]\{\boldsymbol{K}_1 \quad \boldsymbol{K}_2 \quad \boldsymbol{K}_3\}, \tag{48}$$

and

$$\boldsymbol{N}'_3 = \frac{1}{3}\begin{bmatrix} -27.2 \\ 6.2 \\ 18 \\ 2 \end{bmatrix} D_1 + \frac{1}{3}\left[\begin{bmatrix} 3 & 6 & 11.4 & 21.6 \\ -1 & 0 & 2.4 & 6.2 \\ 0 & 0 & 3 & 18 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 2.4 & 2.4 & 2.4 & 2.4 \\ -0.8 & 0.8 & 1.4 & 0 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\right]\{\boldsymbol{K}_1 \quad \boldsymbol{K}_2 \quad \boldsymbol{K}_3\}. \tag{49}$$

——— *Preliminary Operation in Group 3.* ———

The left boundary condition of the fifth span is given by

$$\lfloor 0 \quad 0 \quad 1 \quad 0 \rfloor \boldsymbol{N}_5 = 0. \tag{50}$$

Then, $\boldsymbol{N}_5$ becomes

$$\boldsymbol{N}_5 = \{A \quad B \quad 0 \quad D\}_5. \tag{51}$$

Shifting rightwards, the conjugate of the sixth span can be written

$$\boldsymbol{N}'_6 = \boldsymbol{S}_6(\boldsymbol{N} + \boldsymbol{K})_5 + \boldsymbol{K}_6. \tag{52}$$

The extreme right boundary conditions of the present continuous beam are given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \boldsymbol{N}'_6 = 0. \tag{53}$$

Substituting Eq. 52 into the above equation,

$$\begin{bmatrix} A \\ B \end{bmatrix}_5 = \begin{bmatrix} 13.2 \\ -10.2 \end{bmatrix} D_5 + \left[\begin{bmatrix} -1 & 0 & 3.4 & 13.2 \\ 0 & -1 & -3.6 & -10.2 \end{bmatrix}, \begin{bmatrix} -0.8 & 0.8 & 2.4 & 4.0 \\ 0 & -0.8 & -1.6 & -2.4 \end{bmatrix} \right] \{\boldsymbol{K}_5 \quad \boldsymbol{K}_6\}. \tag{54}$$

Then $\boldsymbol{N}_5$ becomes after the above preliminary operation as follows:

$$\boldsymbol{N}_5 = \begin{bmatrix} 13.2 \\ -10.2 \\ 0 \\ 1 \end{bmatrix} D_5 + \left[\begin{bmatrix} -1 & 0 & 3.4 & 13.2 \\ 0 & -1 & -3.6 & -10.2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -0.8 & 0.8 & 2.4 & 4.0 \\ 0 & -0.8 & -1.6 & -2.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right] \times \{\boldsymbol{K}_5 \quad \boldsymbol{K}_6\}. \tag{55}$$

——— *Shift Operation at Intermediate Rigid Support.* ———

Using Table V and Eq. 29, the operation at this point can be performed as follows:

$$A_4 = 0, \tag{56}$$

$$\begin{bmatrix} B \\ C \end{bmatrix}_4 = \frac{1}{3} \begin{bmatrix} 48.2 \\ 24 \end{bmatrix} D_1 + \frac{1}{3} \left[\begin{bmatrix} -1 & 0 & 8.4 & 51.2 \\ 0 & 0 & 3 & 27 \end{bmatrix}, \begin{bmatrix} -0.8 & 0.8 & 7.4 & 27 \\ 0 & 0 & 3 & 18 \end{bmatrix}, \begin{bmatrix} -1 & 2 & 5 & 8 \\ 0 & 0 & 3 & 9 \end{bmatrix} \right] \{\boldsymbol{K}_1 \quad \boldsymbol{K}_2 \quad \boldsymbol{K}_3\}. \tag{57}$$

In virtue of Eq. 5, the conjugate of the fourth span becomes

$$\boldsymbol{N}'_4 = \frac{1}{3} \begin{bmatrix} 0 & 0 \\ 48.2 & 0 \\ 24 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} D_1 \\ D_4 \end{bmatrix} + \frac{1}{3} \left[\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 8.4 & 51.2 \\ 0 & 0 & 3 & 27 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.8 & 0.8 & 7.4 & 27 \\ 0 & 0 & 3 & 18 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 2 & 5 & 8 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\Bigg]\{\boldsymbol{K}_1 \ \ \boldsymbol{K}_2 \ \ \boldsymbol{K}_3 \ \ \boldsymbol{K}_4\}. \tag{58}$$

The right boundary condition of the fourth span is given by

$$\lfloor 0 \quad 0 \quad 1 \quad 3 \rfloor \boldsymbol{N}'_4 = 0, \tag{59}$$

from which

$$D_4 = -\frac{8}{3}D_1 - \frac{1}{3}[\lfloor 0 \quad 0 \quad 1 \quad 9 \rfloor, \lfloor 0 \quad 0 \quad 1 \quad 6 \rfloor, \lfloor 0 \quad 0 \quad 1 \quad 3 \rfloor, \lfloor 0 \quad 0 \quad 1 \quad 3 \rfloor]\{\boldsymbol{K}_1 \ \ \boldsymbol{K}_2 \ \ \boldsymbol{K}_3 \ \ \boldsymbol{K}_4\}. \tag{60}$$

Then the conjugate $\boldsymbol{N}'_4$ can be reduced as follows:

$$\boldsymbol{N}'_4 = \frac{1}{3}\begin{bmatrix} 0 \\ 48.2 \\ 24 \\ -8 \end{bmatrix} D_1 + \frac{1}{3}\Bigg[\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 8.4 & 51.2 \\ 0 & 0 & 3 & 27 \\ 0 & 0 & -1 & -9 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.8 & 0.8 & 7.4 & 27 \\ 0 & 0 & 3 & 18 \\ 0 & 0 & -1 & -6 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 2 & 5 & 8 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & -1 & -3 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}\Bigg]\{\boldsymbol{K}_1 \ \ \boldsymbol{K}_2 \ \ \boldsymbol{K}_3 \ \ \boldsymbol{K}_4\}. \tag{61}$$

——— *Connection at Intermediate Pin Joint.* ———

The connection conditions at this point are given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\boldsymbol{N}'_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\boldsymbol{N}_5. \tag{62}$$

Substitution from Eqs. 55 and 61 into the above equation yields

$$\frac{1}{3}\begin{bmatrix} 64.2 \\ -8 \end{bmatrix} D_1 + \frac{1}{3}\Bigg[\begin{bmatrix} -1 & 0 & 10.4 & 69.2 \\ 0 & 0 & -1 & -9 \end{bmatrix}, \begin{bmatrix} -0.8 & 0.8 & 9.4 & 39 \\ 0 & 0 & -1 & -6 \end{bmatrix},$$

$$\left[\begin{bmatrix} -1 & 2 & 7 & 14 \\ 0 & 0 & -1 & -3 \end{bmatrix}, \begin{bmatrix} 3 & 3 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}\right]\{\boldsymbol{K}_1 \ \boldsymbol{K}_2 \ \boldsymbol{K}_3 \ \boldsymbol{K}_4\}$$

$$= \begin{bmatrix} 13.2 \\ 1 \end{bmatrix} D_5 + \left[\begin{bmatrix} -1 & 0 & 3.4 & 13.2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -0.8 & 0.8 & 2.4 & 4.0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right] \times \{\boldsymbol{K}_5 \ \ \boldsymbol{K}_6\}, \tag{63}$$

from which the final solution becomes

$$\begin{bmatrix} D_1 \\ D_5 \end{bmatrix} = \frac{1}{2\,547}\left[\begin{bmatrix} 15 & 0 & -354 & -2\,820 \\ -40 & 0 & 95 & -121 \end{bmatrix}, \begin{bmatrix} 12 & -12 & -339 & -1\,773 \\ -32 & 32 & 55 & -366 \end{bmatrix},\right.$$
$$\begin{bmatrix} 15 & -30 & -303 & -804 \\ -40 & 80 & -41 & -403 \end{bmatrix}, \begin{bmatrix} -45 & -45 & -228 & 0 \\ 120 & 120 & -241 & 0 \end{bmatrix},$$
$$\left.\begin{bmatrix} -45 & 0 & 153 & 594 \\ 120 & 0 & -408 & -1\,584 \end{bmatrix}, \begin{bmatrix} -36 & 36 & 108 & 180 \\ 96 & -96 & -288 & -48 \end{bmatrix}\right] \times \{\boldsymbol{K}_1 \ \boldsymbol{K}_2 \ \boldsymbol{K}_3 \ \boldsymbol{K}_4 \ \boldsymbol{K}_5 \ \boldsymbol{K}_6\}. \tag{64}$$

——— *Eigenmatrix for Each Constituent Span.* ———

Substituting the above solution into Eqs. 48, 55, and 61, the respective eigenmatrices $\boldsymbol{N}_1$, $\boldsymbol{N}_5$, and $\boldsymbol{N}'_4$ can be obtained in the following form:

$$\begin{bmatrix} \boldsymbol{N}_1 \\ \boldsymbol{N}'_4 \\ \boldsymbol{N}_5 \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} & \boldsymbol{G}_{13} & \boldsymbol{G}_{14} & \boldsymbol{G}_{15} & \boldsymbol{G}_{16} \\ \boldsymbol{G}_{41} & \boldsymbol{G}_{42} & \boldsymbol{G}_{43} & \boldsymbol{G}'_{44} & \boldsymbol{G}_{45} & \boldsymbol{G}_{46} \\ \boldsymbol{G}_{51} & \boldsymbol{G}_{52} & \boldsymbol{G}_{53} & \boldsymbol{G}_{54} & \boldsymbol{G}_{55} & \boldsymbol{G}_{56} \end{bmatrix} \{\boldsymbol{K}_1 \ \boldsymbol{K}_2 \ \boldsymbol{K}_3 \ \boldsymbol{K}_4 \ \boldsymbol{K}_5 \ \boldsymbol{K}_6\}, \tag{65}$$

in which the geometry matrix $\boldsymbol{G}'_{44}$ is the conjugate of $\boldsymbol{G}_{44}$ expressed by

$$\boldsymbol{G}_{44} = \boldsymbol{G}'_{44} - \boldsymbol{E} \qquad (\boldsymbol{E} = \text{the 4-by-4 unit matrix}). \tag{66}$$

The eigenmatrices of other spans are given by the formulas

$$\left.\begin{aligned} \boldsymbol{N}_2 &= \boldsymbol{S}_2(\boldsymbol{N}+\boldsymbol{K})_1, \\ \boldsymbol{N}_3 &= \boldsymbol{S}_3(\boldsymbol{N}+\boldsymbol{K})_2, \\ \boldsymbol{N}_6 &= \boldsymbol{S}_6(\boldsymbol{N}+\boldsymbol{K})_5. \end{aligned}\right\} \tag{67}$$

Then the consolidated final solution of the present continuous beam can be expressed in the form

$$[\boldsymbol{N}_r] = [\boldsymbol{G}_{rj}]\,[\boldsymbol{K}_j], \tag{68}$$

in which, $[\boldsymbol{N}_r]$ is the consolidated eigenmatrix, $[\boldsymbol{G}_{rj}]$ the consolidated geometry matrix, and $[\boldsymbol{K}_j]$ the consolidated load term, respectively. They are given as follows:

$$[\boldsymbol{N}_r]=\{\boldsymbol{N}_1 \quad \boldsymbol{N}_2 \quad \boldsymbol{N}_3 \quad \boldsymbol{N}_4 \quad \boldsymbol{N}_5 \quad \boldsymbol{N}_6\}, \tag{69}$$

$$[\boldsymbol{G}_{rj}]=\begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} & \boldsymbol{G}_{13} & \boldsymbol{G}_{14} & \boldsymbol{G}_{15} & \boldsymbol{G}_{16} \\ \boldsymbol{G}_{21} & \boldsymbol{G}_{22} & \boldsymbol{G}_{23} & \boldsymbol{G}_{24} & \boldsymbol{G}_{25} & \boldsymbol{G}_{26} \\ \boldsymbol{G}_{31} & \boldsymbol{G}_{32} & \boldsymbol{G}_{33} & \boldsymbol{G}_{34} & \boldsymbol{G}_{35} & \boldsymbol{G}_{36} \\ \boldsymbol{G}_{41} & \boldsymbol{G}_{42} & \boldsymbol{G}_{43} & \boldsymbol{G}_{44} & \boldsymbol{G}_{45} & \boldsymbol{G}_{46} \\ \boldsymbol{G}_{51} & \boldsymbol{G}_{52} & \boldsymbol{G}_{53} & \boldsymbol{G}_{54} & \boldsymbol{G}_{55} & \boldsymbol{G}_{56} \\ \boldsymbol{G}_{61} & \boldsymbol{G}_{62} & \boldsymbol{G}_{63} & \boldsymbol{G}_{64} & \boldsymbol{G}_{65} & \boldsymbol{G}_{66} \end{bmatrix}, \tag{70}$$

$$[\boldsymbol{K}_j]=\{\boldsymbol{K}_1 \quad \boldsymbol{K}_2 \quad \boldsymbol{K}_3 \quad \boldsymbol{K}_4 \quad \boldsymbol{K}_5 \quad \boldsymbol{K}_6\}. \tag{71}$$

The geometry matrices for respective eigenmatrices are given in Table VII, in which the check for the result obtained can be readily observed.

——— *Consolidated Formula.* ———

The physical quantities, deflection, slope, bending moment, and shearing force, at any point in the continuous beam can be given by the equation

$$\begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}_r^i = \begin{bmatrix} 1 \\ \dfrac{d}{dx} \\ -EI_r\dfrac{d^2}{dx^2} \\ -EI_r\dfrac{d^3}{dx^3} \end{bmatrix} w^i{}_r, \tag{72}$$

in which $w^i{}_r$ is the deflection at the i-th domain of the r-th constituent span.

For numerical computations, it is convenient to transform the above equation to the following form:

$$\begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}_r^i = \begin{bmatrix} \dfrac{l^3}{6EI} & 0 & 0 & 0 \\ 0 & \dfrac{l^2}{2EI} & 0 & 0 \\ 0 & 0 & l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_r \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \rho & \rho^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times\left[\frac{1}{3}\sum_{1}^{n}\begin{bmatrix}3&0&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&3\\0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&-1&0\\0&0&0&-3\\0&0&0&-3\end{bmatrix}\boldsymbol{G}_{rj}\boldsymbol{K}_j+\sum_{0}^{\rho}\left[P\begin{bmatrix}-\kappa^3\\3\kappa^2\\-3\kappa\\1\\\kappa^2\\-2\kappa\\1\\\kappa\\-1\\-1\end{bmatrix}+l\int_{\lambda}^{\mu}q(\kappa)\begin{bmatrix}-\kappa^3\\3\kappa^2\\-3\kappa\\1\\\kappa^2\\-2\kappa\\1\\\kappa\\-1\\-1\end{bmatrix}d\kappa+\frac{\mathfrak{M}}{l}\begin{bmatrix}3\kappa^2\\-6\kappa\\3\\0\\-2\kappa\\2\\0\\-1\\0\\0\end{bmatrix}\right]_r\right]. \tag{73}$$

This is the consolidated formula for all physical quantities of the continuous beam. The necessary quantities can at once be evaluated by the above equation. For instance, the deflection and the bending moment may be obtained as follows.

Deflection:

$$w^i{}_r=\frac{l_r{}^3}{6EI_r}\lfloor 1 \quad \rho \quad \rho^2 \quad \rho^3\rfloor\left[\sum_{1}^{n}\boldsymbol{G}_{rj}\boldsymbol{K}_j+\sum_{0}^{\rho}\left[P\begin{bmatrix}-\kappa^3\\3\kappa^2\\-3\kappa\\1\end{bmatrix}+l\int_{\lambda}^{\mu}q(\kappa)\begin{bmatrix}-\kappa^3\\3\kappa^2\\-3\kappa\\1\end{bmatrix}d\kappa+\frac{\mathfrak{M}}{l}\begin{bmatrix}3\kappa^2\\-6\kappa\\3\\0\end{bmatrix}\right]\right]. \tag{74}$$

Bending moment:

$$M^i{}_r=l_r\lfloor 1 \quad \rho\rfloor\left[\frac{1}{3}\sum_{1}^{n}\begin{bmatrix}0&0&-1&0\\0&0&0&-3\end{bmatrix}\boldsymbol{G}_{rj}\boldsymbol{K}_j+\sum_{0}^{\rho}\left[P\begin{bmatrix}\kappa\\-1\end{bmatrix}+l\int_{\lambda}^{\mu}q(\kappa)\begin{bmatrix}\kappa\\-1\end{bmatrix}d\kappa+\frac{\mathfrak{M}}{l}\begin{bmatrix}-1\\0\end{bmatrix}\right]_r\right]. \tag{75}$$

Table VII. Geometry Matrix $(\times 1/7\,641)$.

K_j	G_{1j}				G_{2j}			
K_1	0	0	0	0	5 966.25	0	−7 722.75	7 302.75
	−2 913	−7 641	−12 757.2	6 661.2	−3 472.5	0	−826.5	5 255.25
	0	0	0	0	135	0	4 455	−2 457
	45	0	−1 062	−8 460	45	0	−1 062	−819
K_2	0	0	0	0	−2 868	−4 773	−1 756.5	24 504.75
	−2 330.4	−3 782.4	−388.2	24 922.8	−2 778	−4 863	−4 299	11 207.25
	0	0	0	0	108	−108	−3 051	−15 957
	36	−36	−1 017	−5 319	36	−36	−1 017	−5 319
K_3	0	0	0	0	−3 585	−2 381.25	4 921.5	18 323.25
	−2 913	−1 815	4 846.2	17 070.6	−3 472.5	−2 606.25	2 649	12 293.25
	0	0	0	0	135	−270	−2 727	−7 236
	45	−90	−909	−2 412	45	−90	−909	−2 412
K_4	0	0	0	0	1 203.75	1 203.75	6 099	0
	1 098	1 098	5 563.2	0	866.25	866.25	4 389	0
	0	0	0	0	−405	−405	−2 052	0
	−135	−135	−684	0	−135	−135	−684	0
K_5	0	0	0	0	1 203.75	0	−4 092.75	−15 889.5
	1 098	0	−3 733.2	−14 493.6	866.25	0	−2 945.25	−11 434.5
	0	0	0	0	−405	0	1 377	5 346
	−135	0	459	1 782	−135	0	459	1 782
K_6	0	0	0	0	963	−963	−2 889	−4 815
	878.4	−878.4	−2 635.2	−4 392	693	−693	−2 079	−3 465
	0	0	0	0	−324	324	972	1 620
	−108	108	324	540	−108	108	324	540

(Continued.)

Table VII. Geometry Matrix (×1/7 641).

K_j	G_{3j}				G_{4j}			
K_1	2 139	0	−4 125	7 425.6	0	0	0	0
	−2 454	0	3 918	−1 692.6	−1 824	0	4 332	−4 887.6
	270	0	1 269	−4 914	360	0	−855	1 089
	45	0	−1 062	−819	−120	0	285	−363
K_2	1 711.2	−1 711.2	−1 986	17 661.6	0	0	0	0
	−1 963.2	1 963.2	1 464	−10 992.6	−1 459.2	1 459.2	2 448	−16 689.6
	216	−216	1 539	−8 991	288	−288	−495	3 294
	36	−36	−1 017	2 322	−96	96	165	−1 098
K_3	−5 502	−4 278	3 147.6	16 774.8	0	0	0	0
	−2 454	−2 733	−4 425.6	−7 531.8	−1 824	3 648	−1 869.6	−18 376.8
	270	−540	−5 454	−14 472	360	−720	369	3 627
	45	−90	−909	−2 412	−120	240	−123	−1 209
K_4	1 224	1 224	6 201.6	0	0	0	0	0
	−279	−279	−1 413.6	0	−2 169	−2 169	−10 989.6	0
	−810	−810	−4 104	0	−1 080	−1 080	−5 472	0
	−135	−135	−684	0	360	360	−723	−7 641
K_5	1 224	0	−4 161.6	−16 156.8	0	0	0	0
	−279	0	948.6	3 682.8	−2 169	0	7 374.6	28 630.8
	−810	0	2 754	10 692	−1 080	0	3 672	14 256
	−135	0	459	1 782	360	0	−1 224	−4 752
K_6	979.2	−979.2	−2 937.6	−4 896	0	0	0	0
	−223.2	223.2	669.6	1 116	−1 735.2	1 735.2	5 205.6	8 676
	−648	648	1 944	3 240	−864	864	2 592	4 320
	−108	108	324	540	288	−288	−864	−1 440

(Continued.)

Table VII. Geometry Matrix $(\times 1/7\,641)$.

K_j	G_{5j}				G_{6j}			
K_1	−1 584	0	3 762	−4 791.6	−600	0	1 425	−1 815
	1 224	0	−2 907	3 702.6	1 080	0	−2 565	3 267
	0	0	0	0	−360	0	855	−1 089
	−120	0	285	−363	−120	0	285	−363
K_2	−1 267.2	1 267.2	2 178	−14 493.6	−480	480	825	−5 490
	979.2	−979.2	−1 683	11 199.6	864	−864	−1 485	9 882
	0	0	0	0	−288	288	495	−3 294
	−96	96	165	−1 098	−96	96	165	−1 098
K_3	−1 584	3 168	−1 623.6	−15 958.8	−600	1 200	−615	−6 045
	1 224	−2 448	1 254.6	−12 331.8	1 080	−2 160	1 107	10 881
	0	0	0	0	−360	720	−369	−3 627
	−120	240	−123	−1 209	−120	240	−123	−1 209
K_4	4 752	4 752	−9 543.6	0	1 800	1 800	−3 615	0
	−3 672	−3 672	7 374.6	0	−3 240	−3 240	6 507	0
	0	0	0	0	1 080	1 080	−2 169	0
	360	360	−723	0	360	360	−723	0
K_5	−2 889	0	9 822.6	38 134.8	1 800	0	1 521	14 445
	−3 672	−7 641	−15 022.8	−29 467.8	−3 240	0	−4 266	−26 001
	0	0	0	0	1 080	0	3 969	8 667
	360	0	−1 224	−4 752	360	0	−1 224	2 889
K_6	−2 311.2	2 311.2	6 933.6	28 663.2	−6 201	−1 440	3 321	14 562
	−2 937.6	−3 175.2	−3 412.8	−16 869.6	−2 592	−5 049	−7 506	−21 627
	0	0	0	0	864	−864	−2 592	−432
	288	−288	−864	−144	288	−288	−864	−144

CONCLUSIONS.

Regarding the present analysis of bending problems of continuous beams, several concluding words are given as follows:

1. Because the procedure is composed of the treatment of perfectly classified data of beam configurations and external loading conditions, the analysis can be carried out systematically, and the necessary operators are obtained readily from the corresponding tables or equations. The operators at an intermediate connection point can be used independently of that of other point.

2. The eigenmatrix of a domain adjacent to the standard one in a span can be given merely by additions of the corresponding load-matrix at the loaded point.

3. The operational procedures are shown by the shifting charts, from which various ways of processing analyses can be constructed, so that a result obtained by one way can be checked by another one.

4. The geometry matrix is a useful result of the operational method, because it depends only on the configurations and material properties of a given beam, and when multiplied by the load terms of the system, it gives the eigenmatrix for the corresponding loading conditions.

5. The eigenmatrix is the final solution of the beam under given loading conditions. It readily gives the solution of deflection, slope, bending moment, and shearing force of the beam.

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APPENDIX.—NOTATION.

The following symbols have been adopted for use in this paper:

a_r = redundant element see Eq. 30;

$\mathbf{A}_r$ = semi-eigenmatrix of r-th constituent span, see Tables V and VI;

$\mathbf{A}_R$ = semi-eigenmatrix of R-th constituent group, see Eqs. 31 and 32;

$\{A\ B\ C\ D\}_r$ = elements of eigenmatrix of r-th span, see Eq. 3;

$\mathbf{B}_r$, $\mathbf{B}'_r$ = boundary matrices of left or right end of r-th span, see Eqs. 10, 11, and Tables II, III;

$\mathbf{C}_r$ = connection-matrix at r-th connection point, see Eq. 19 and Table IV;

$\mathbf{E}$ = unit mtarix;

EI_r = flexural rigidity of r-th span;

$\mathbf{F}_r$, $\mathbf{F}'_r$ = right or leftward feed operator at r-th connection point, see Eqs. 26, 28, and Tables V, VI;

$\mathbf{F}_R$, $\mathbf{F}'_R$ = right or leftward feed operator at R-th (2BC)-point, see Eq. 31 or 32;

$\mathbf{G}_{r1}$, $\mathbf{G}_{r2}$, $\cdots$ = geometry matrices of r-th constituent span, see Eq. 33 and Table VII;

i = integer representing the order number of domain;

j = integer representing load term of j-th span;

k = constant attached to elastic support, see Eq. 12;

$\mathbf{K}_m$ = load-matrix at loading point of external concentrated momoent, see Eq. 8;

$\mathbf{K}_p$ = load-matrix at loading point of external concentrated load, see Eq. 6;

$\mathbf{K}_r$ = load-term of r-th span, see Eq. 9;

l_r = span length of r-th constituent span;

m = constant between resisting moment and slope at supporting point, see Eq. 12;

M = bending moment;

M_c = resisting moment at supporting point, see Eq. 12;

$\mathfrak{M}$ = external concentrated moment;

n = total number of constituent spans of continuous beam;

$\mathbf{N}_r$ = eigenmatrix of normal domain of r-th span;

$\mathbf{N}^i_r$ = eigenmatrix of i-th domain of r-th span;

$\mathbf{N}'_r$ = eigenmatrix of conjugate domain of r-th span;

P = external concentrated load;

$q(\kappa)$ = intensity of distributed load;

r = order number of constituent span or connection point;

R = order number of constituent group;

R_e = reaction at elastic support;

$\mathbf{R}'_r$ = feeder for redundant element, see Eq. 30 and Table V;

$\mathbf{S}_r, \mathbf{S}'_r$ = right or leftward shiftor at r-th connection point, see Eq. 26 or 27, and Table V or VI;

$\mathbf{S}_R, \mathbf{S}'_R$ = right or leftward shiftor between $R-1$ and R-th constituent groups, see Eqs. 31 and 32;

$\mathbf{T}_R, \mathbf{T}'_R$ = right or leftward feeder for its own group between $R-1$ and R-th constituent groups, see Eqs. 31 and 32;

w_r = beam deflection at normal domain of r-th span;

$w^i{}_r$ = beam deflection at i-th domain of r-th span;

w'_r = beam deflection at conjugate domain of r-th span;

x = current abscissa;

α_r = span ratio, see Eq. 20;

β_r = rigidity ratio, see Eq. 21;

θ = slope angle;

κ = non-dimensional load abscissa, see Fig. 1;

λ = lower boundary of partially distributed load, see Fig. 1, or constant attached to elastic support, see Eq. 22;

μ = upper boundary of partially distributed load, see Fig. 1, or constant attached to elastic support, see Eq. 23;

ρ = non-dimensional current abscissa see Eq. 2;

$\sum$ = summation;

$\int$ = integration;

$\boxed{1}$ = number of boundary conditions in shifting chart;

◇2 = number of connection conditions in shifting chart;

③ = number of unknown elements in shifting chart;

$\longrightarrow$ = reduction operation for eigenmatrix within its own span;

$\dashrightarrow$ = operation for elements in a span with dependent eigenmatrix;

$\Longrightarrow$ = shift operation for current-elememt;

$\Longrightarrow$ = final operation;

$\lfloor \quad \rfloor$ = row vector; and

$\{ \quad \}$ = column vector.