Operational Method for Continuous Beams
First Report
Rigid Support and Pin Joint

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SYNOPSIS.

A further development of the eigenmatrix method[1] is presented for continuous beams involving pin joints as well as rigid supports. The approach of the paper is based on the perfect classification of physical quantities, so that the behavior of the beams can be completely represented by the eigenmatrix.

The analysis is carried out by systematic shift operation of 2-by-2 or 4-by-4 operational matrices. The concept of such as statically indeterminate system or simultaneous equations is eliminated in the present procedure, and hence the necessary analytical computation can be facilitated.

INTRODUCTION.

Notation.—The symbols adopted for use in this paper are defined where they first appear and are arranged alphabetically in the Appendix.

The fundamental procedures for solving hyperstatic structures have been proposed by the junior author[1)2)3)] and the present paper is a further development for the analysis of continuous beams with rigid support involving some pin joints, such as Gerber-girder bridges and more complicated systems, provided there takes place no deflection at the rigid support or no bending moment at the pin joint.

The physical quantities of constituent span can be represented by the eigenmatrix consisting of 4-by-1 elements which should be determined by

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given conditions.

Introducing boundary conditions at each end of a constituent span, the above elements can be reduced to the semi-eigenmatrix consisting of 2-by-1 column elements.

There exist two connection conditions between consecutive constituent spans of the continuous beam. In virtue of these conditions, the right or leftward shift formula for various combinations of constituent spans is obtained. In the pin joint coupling, a peculiar property which rejects the ordinary shift operation appears.

For the continuous plate-girder bridge, a preliminary treatment must be made at points of cross-section with abrupt change before the use of connection equatons. A practical example will be given as an illustration.

Using the shift formula, all constituent spans can be expressed by the current semi-eigenmatrix of any span, so that we can compose various shifting procedures for the analysis of continuous beam dispensing with simultaneous equations. Check calculations for results obtained can be provided by the similar manner.

BASIC EQUATIONS.

Definition.—The span orders are denoted by subscripts. The terminology "domain" refers to portions separated by loaded points of a constituent span. The domain orders are counted from the extreme left domain of the span, and represented by superscripts. The first domain is taken as the normal domain where the superscript is omitted, while the last domain is taken as the conjugate domain where the prime (′) is used instead of superscript.

When a beam is bent by a lateral concentrated load, the flexural deflection is given by the equation

$$\left|\begin{matrix} w \\ w' \end{matrix}\right| = \frac{l^3}{6EI}\lfloor 1 \quad \rho \quad \rho^2 \quad \rho^3 \rfloor \left|\begin{matrix} \boldsymbol{N} \\ \boldsymbol{N}' \end{matrix}\right|. \qquad (1)^{2)}$$

The parallel lines denote the correspondence of w to $\boldsymbol{N}$, and w' to $\boldsymbol{N}'$, respectively. In the above equation, w is the deflection, l the span length, EI the flexural rigidity, and ρ the non-dimensional current abscissa defined by

$$\rho = \frac{x}{l}. \qquad (2)$$

$\boldsymbol{N}$ and $\boldsymbol{N}'$ are the eigenmatrices given by

$$\left.\begin{aligned}\boldsymbol{N} &= \{A \quad B \quad C \quad D\},\\ \boldsymbol{N}' &= \{A \quad B \quad C \quad D\}'.\end{aligned}\right\} \tag{3}$$

The dimension of the elements of the above eigenmatrices are the same as that of the concentrated load.

LOAD-MATRICES.

Concentrated Load.

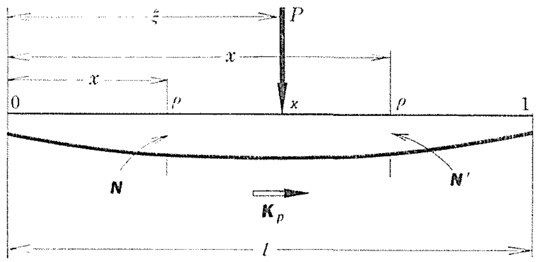


Fig. 1. Concentrated Load.

At the loaded point of the external lateral load P in Fig. 1, the continuity conditions can be written

$$\left\{1 \quad \frac{d}{d\rho} \quad \frac{d^2}{d\rho^2} \quad \frac{d^3}{d\rho^3}\right\}(w-w')_{\rho=\kappa} = \left\{0 \quad 0 \quad 0 \quad -\frac{Pl^3}{EI}\right\}, \tag{4}$$

in which P is the concentrated load, and κ is the non-dimensional load abscissa defined by

$$\kappa = \frac{\xi}{l}. \tag{5}$$

In virtue of Eq. 1, Eq. 4 yields after some rearrangements the following continuity equation:

$$\boldsymbol{N}' = \boldsymbol{N} + \boldsymbol{K}_p, \tag{6}$$

in which $\boldsymbol{K}_p$ is designated as the "load-matrix" of the concentrated load and takes the form

$$\boldsymbol{K}_p = P\{-\kappa^3 \quad 3\kappa^2 \quad -3\kappa \quad 1\}. \tag{7}$$

Distributed Load.

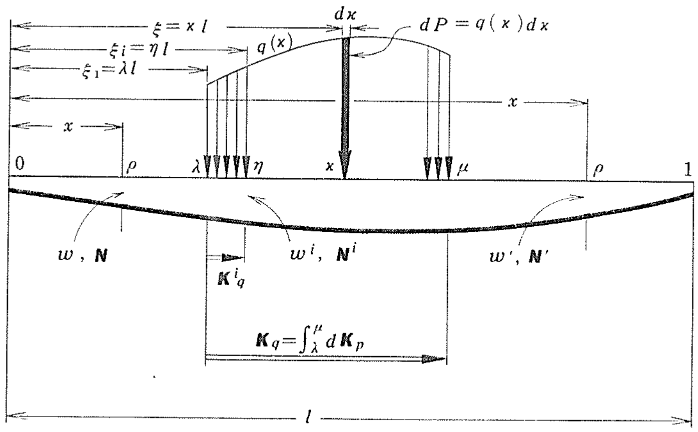


Fig. 2. Partially Distributed Load.

Referring to Fig. 2, the load-matrix for the partially distributed load can be derived by integration of the elementary load dp, in which case the eigenmatrices of normal, intermediate, and conjugate domains are given by the equations

$$\left.\begin{aligned} \boldsymbol{N} &= \{A \quad B \quad C \quad D\}, \\ \boldsymbol{N}^i &= \boldsymbol{N} + \boldsymbol{K}^i{}_q, \\ \boldsymbol{N}' &= \boldsymbol{N} + \boldsymbol{K}_q, \end{aligned}\right\} \tag{8}$$

respectively. The eigenmatrix for the domain with distributed load will take another form. If the non-dimensional abscissa $\eta = \dfrac{\xi_i}{l}$ is introduced to represent this domain, then the load-matrices of an arbitrary point and the conjugate domain become

$$\left.\begin{aligned} \boldsymbol{K}^i{}_q &= \int_\lambda^\eta lq(\kappa)\{-\kappa^3 \quad 3\kappa^2 \quad -3\kappa \quad 1\}d\kappa, \\ \boldsymbol{K}_q &= \int_\lambda^\mu lq(\kappa)\{-\kappa^3 \quad 3\kappa^2 \quad -3\kappa \quad 1\}d\kappa, \end{aligned}\right\} \tag{9}$$

in which $q(\kappa)$ is the intensity of the distributied load.

External Concentrated Moment.

In this case, the continuity conditions at the loaded point become

$$\left\{1 \quad \frac{d}{d\rho} \quad \frac{d^2}{d\rho^2} \quad \frac{d^3}{d\rho^3}\right\}(w - w')_{\rho=\kappa} = \left\{0 \quad 0 \quad -\frac{\mathfrak{M}l^2}{EI} \quad 0\right\}. \tag{10}$$

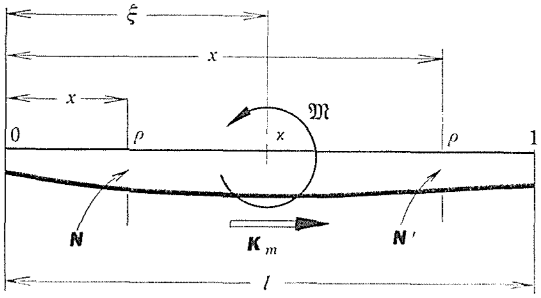


Fig. 3. External Concentrated Moment.

Substituting Eq. 1 into Eq. 10, the following equation and load-matrix are obtained:

$$\left.\begin{aligned} \boldsymbol{N}' &= \boldsymbol{N} + \boldsymbol{K}_m, \\ \boldsymbol{K}_m &= \frac{3}{l}\mathfrak{M}\{\kappa^2 \quad -2\kappa \quad 1 \quad 0\}. \end{aligned}\right\} \tag{11}$$

Generalized Formula.

For the combined action of various external loads, the deflection can be obtained by superposition of the effects caused by respective loading conditions. Taking the general case shown in Fig. 4, the deflection of each domain of the r-th span is given as follows:

$$\begin{vmatrix} w \\ w^i \\ w' \end{vmatrix}_r = \frac{l_r^3}{6EI_r}\lfloor 1 \quad \rho \quad \rho^2 \quad \rho^3 \rfloor \begin{vmatrix} \boldsymbol{N} \\ \boldsymbol{N} + \sum_{\kappa=0}^{\rho} \left(\boldsymbol{K}_p + \int_\lambda^\mu d\boldsymbol{K}_p + \boldsymbol{K}_m\right) \\ \boldsymbol{N} + \boldsymbol{K} \end{vmatrix}_r . \tag{12}$$

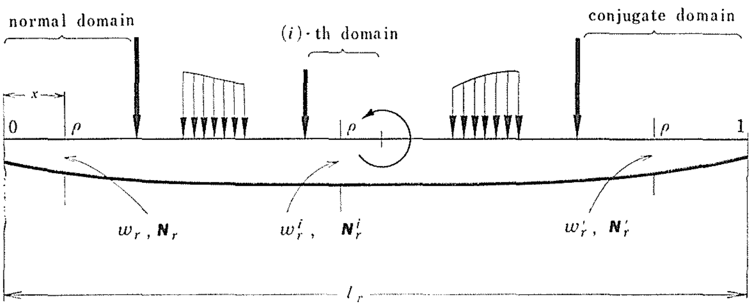


Fig. 4. Combined Action of Various External Loads on the r-th Constituent Span.

The parallel lines denote the correspondence of respective rows on both sides. The second and the third rows on the right side of Eq. 12 represent the eigenmatrices of the i-th intermediate and the conjugate domains respectively. They are given by superposing the successive load-matrices at the loaded points of the left portion of the span.

For later convenience, the "load term" of the i-th span is defined as

$$\boldsymbol{K}_r = \sum_{\kappa=0}^{1} (\boldsymbol{K}_p + \int_{\lambda}^{\mu} d\boldsymbol{K}_p + \boldsymbol{K}_m)_r. \tag{13}$$

Then the eigenmatrices of normal and conjugate domains of the r-th span are given as follows:

$$\left.\begin{aligned} \boldsymbol{N}_r &= \{A \quad B \quad C \quad D\}_r, \\ \boldsymbol{N}'_r &= \boldsymbol{N}_r + \boldsymbol{K}_r. \end{aligned}\right\} \tag{14}$$

BOUNDARY CONDITIONS.

Intermediate Support.

The intermediate rigid support or pin joint in a continuous beam has an independent boundary condition each, that is to say,

$$\left.\begin{aligned} &w_r = 0 \quad \text{for the rigid support, or} \\ &\frac{d^2 w_r}{d\rho^2} = 0 \quad \text{for the pin joint.} \end{aligned}\right\} \tag{15}$$

In virtue of Eq. 1, the above equations can be written in the forms

$$\left.\begin{aligned} &\boldsymbol{B}_I \boldsymbol{N}_r = 0 \qquad \text{for the left end of span, and} \\ &\boldsymbol{B}'_I \boldsymbol{N}'_r = 0 \qquad \text{for the right end of span,} \end{aligned}\right\} \tag{16}$$

in which $\boldsymbol{B}_I$ and $\boldsymbol{B}'_I$ are the "independent boundary-matrices" and are given for respective cases as shown in Table I.

Both Extremities.

After the independent boundary condition has been treated, the boundary conditions at both extremities must be dealt with. They take the forms

$$\left.\begin{aligned} \boldsymbol{B}\boldsymbol{N}_1 = 0 \quad & \text{for the left extremity, and} \\ \boldsymbol{B}'\boldsymbol{N}'_n = 0 \quad & \text{for the right extremity,} \end{aligned}\right\} \tag{17}$$

providing that the continuous beam consists of n spans.

The "extreme boundary-matrices" $\boldsymbol{B}$ and $\boldsymbol{B}'$ are shown in Table I, together with the corresponding independent boundary-matrices.

Table I. Boundary-Matrices.

Left extremity		Intermediate support		Right extremity	
Condition	$\boldsymbol{B}$	Left end $\boldsymbol{B}_l$	Right end $\boldsymbol{B}'_l$	$\boldsymbol{B}'$	Condition
Fix	$\lfloor 0\ 1\ 0\ 0 \rfloor$			$\lfloor 0\ 1\ 2\ 3 \rfloor$	Fix
Simple	$\lfloor 0\ 0\ 1\ 0 \rfloor$	Rigid support: $\lfloor 1\ 0\ 0\ 0 \rfloor$	$\lfloor 1\ 1\ 1\ 1 \rfloor$	$\lfloor 0\ 0\ 1\ 3 \rfloor$	Simple
	$\lfloor 1\ 0\ 0\ 0 \rfloor$	Pin joint: $\lfloor 0\ 0\ 1\ 0 \rfloor$	$\lfloor 0\ 0\ 1\ 3 \rfloor$	$\lfloor 1\ 1\ 1\ 1 \rfloor$	
Free	$\lfloor 0\ 0\ 0\ 1 \rfloor$			$\lfloor 0\ 0\ 0\ 1 \rfloor$	Free

SEMI-EIGENMATRICES.

Initially, the eigenmatrix at each constituent span of a continuous beam has four unknown elements as shown in Eq. 3a. These can be reduced to a 2-by-1 semi-eigenmatrix by independent boundary conditions attached to both ends of the span. For example, a constituent span between pin joint and rigid support is briefly referred to as the "P-R span" shown in Fig. 5, and then, referring to Table I and Eqs. 16, the independent boundary conditions of the span become

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\boldsymbol{N}_r + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\boldsymbol{K}_r = 0, \tag{18}$$

from which

$$A_r = -B_r - D_r - \lfloor 1 \quad 1 \quad 1 \quad 1 \rfloor \boldsymbol{K}_r. \tag{19}$$

Then the eigenmatrix for normal or conjugate domain can be represented by the following reduced form:

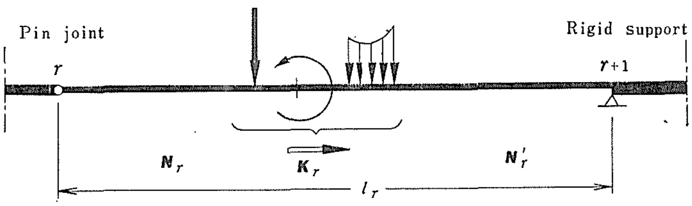


Fig. 5. Constituent Span of Continuous Beam (P-R span).

$$\boldsymbol{N}_r = \boldsymbol{L}\boldsymbol{A}_r + \boldsymbol{P}\boldsymbol{K}_r \qquad \text{or} \qquad \boldsymbol{N}'_r = \boldsymbol{L}\boldsymbol{A}_r + \boldsymbol{Q}\boldsymbol{K}_r, \tag{20}$$

in which

$$\boldsymbol{L} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{21}$$

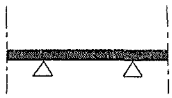
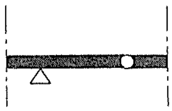
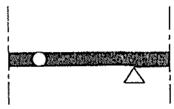
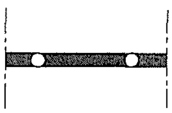
$$\boldsymbol{A}_r = \begin{bmatrix} B \\ D \end{bmatrix}_r, \tag{22}$$

and

$$\boldsymbol{P} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{Q} = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

Eq. 22 is the reduced semi-eigenmatrix of the present span. Values of factors in Eqs. 20 are given in Table II for all possible configurations of the constituent span.

Table II. Semi-Eigenmatrices and Related Matrices.

Configuration	L	A	P	Q
 R-R span	$\begin{bmatrix}0 & 0\\ -1 & -1\\ 1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}C\\ D\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0 & 0\\ -1 & -1 & -1 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0 & 0\\ -1 & 0 & -1 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$
 R-P span	$\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & -3\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}B\\ D\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1 & -3\\ 0 & 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -3\\ 0 & 0 & 0 & 1\end{bmatrix}$
 P-R span	$\begin{bmatrix}-1 & -1\\ 1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}B\\ D\end{bmatrix}$	$\begin{bmatrix}-1 & -1 & -1 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & -1 & -1 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$
 P-P span	$\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\end{bmatrix}$	$\begin{bmatrix}A\\ B\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -\frac{1}{3} & -1\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -\frac{1}{3} & 0\end{bmatrix}$

CONNECTION CONDITIONS.

The continuous beam is composed of the connection of constituent spans. For connection conditions between spans, the rigid support and the pin joint will be considered in the subsequent discussion. Taking the r-th intermediate connection point, the physical conditions are written for the rigid support:

$$\begin{bmatrix}\frac{1}{l} & 0\\ 0 & \frac{EI}{l^2}\end{bmatrix}_{r-1}\begin{bmatrix}\frac{d}{d\rho}\\ \frac{d^2}{d\rho^2}\end{bmatrix}_{\rho=1} w'_{r-1} = \begin{bmatrix}\frac{1}{l} & 0\\ 0 & \frac{EI}{l^2}\end{bmatrix}_{r}\begin{bmatrix}\frac{d}{d\rho}\\ \frac{d^2}{d\rho^2}\end{bmatrix}_{\rho=0} w_r, \qquad (24)$$

and for the pin joint:

$$\begin{bmatrix} 1 & 0 \\ 0 & \dfrac{EI}{l^3} \end{bmatrix}_{r-1} \begin{bmatrix} 1 \\ \dfrac{d^3}{d\rho^3} \end{bmatrix} w'_{r-1} \Bigg|_{\rho=1} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{EI}{l^3} \end{bmatrix}_{r} \begin{bmatrix} 1 \\ \dfrac{d^3}{d\rho^3} \end{bmatrix} w_{r}. \Bigg|_{\rho=0} \tag{25}$$

In virtue of Eq. 12, the above conditions can be written after some rearrangements in the following form:

$$\mathbf{C}_r \{\mathbf{N}'_{r-1} \quad \mathbf{N}_r\} = 0, \tag{26}$$

in which $\mathbf{C}_r$ is designated as the "connection-matrix" at the r-th support and is given by

$$\mathbf{C}_r = \left[\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \ - \begin{bmatrix} 0 & \dfrac{\alpha^2}{\beta} & 0 & 0 \\ 0 & 0 & \alpha & 0 \end{bmatrix}_r \right] \tag{27}$$

for the rigid support, and

$$\mathbf{C}_r = \left[\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ - \begin{bmatrix} \dfrac{\alpha^3}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_r \right] \tag{28}$$

for the pin joint, provided that for shortness

$$\alpha_r = \frac{l_r}{l_{r-1}}, \tag{29}$$

$$\beta_r = \frac{EI_r}{EI_{r-1}}. \tag{30}$$

OPERATORS (SHIFTORS AND FEEDERS).

Referring to Eqs. 20 and Table II, Eq. 26 yields the following shift formulas:

The rightward shift formula:

$$\mathbf{A}_r = \mathbf{S}_r \mathbf{A}_{r-1} + \mathbf{F}_r \mathbf{K}_{r-1} + \mathbf{T}_r \mathbf{K}_r. \tag{31}$$

The leftward shift formula:

$$\mathbf{A}_{r-1} = \mathbf{S}'_r \mathbf{A}_r + \mathbf{F}'_r \mathbf{K}_r + \mathbf{T}'_r \mathbf{K}_{r-1}. \tag{32}$$

In these equations $\mathbf{S}_r$ or $\mathbf{S}'_r$ is the right or leftward shift operator, or briefly the "shiftor," since the semi-eigenmatrix of an arbitrary span can be shifted to the adjacent span.

Similarly, $\boldsymbol{F}_r$ and $\boldsymbol{T}_r$ or $\boldsymbol{F}'_r$ and $\boldsymbol{T}'_r$ are the right or leftward feed operators or briefly the "feeders," since with these all the load terms concerned are fed into the shift formula. The load term of its own span is fed by $\boldsymbol{T}_r$ or $\boldsymbol{T}'_r$, while that of the other span by $\boldsymbol{F}_r$ or $\boldsymbol{F}'_r$. The former $\boldsymbol{T}_r$ or $\boldsymbol{T}'_r$ will be referred to as the feeder of its own span or the "self-feeder," while the latter $\boldsymbol{F}_r$ or $\boldsymbol{F}'_r$ the "shifting feeder," respectively.

Combining the constituent spans shown in Table II, the above operators can be obtained for possible cases as shown in Tables III and IV, in which the sets are referred to, for instance, as follows:

$$\text{Combination of (R-R span) and (R-P span)} = \text{Set RRP}. \tag{33}$$

In such combinations, Set PPP is an unstable or impossible system so that it is omitted from the tables. The singularity will occur in the first step of derivation of operators for this set.

It should be noted here that the rightward operation for Set RPP and the leftward operation for Set PPR are impossible because of the singularity as noted in the case of Set PPP. That is to say, pin joint couplings in a continuous beam must be treated as a particular case in shift operations.

Referring to Fig. 6, the load P_{r-1} causes the deflection on the $(r-1)$-th and the r-th spans, but can not arrive at the $(r+1)$-th span by the reason of physical properties of the pin joint. Similarly, the load P_{r+1} can not bring about any deflection on the $(r-1)$-th span. Contrarily, the load on the r-th P-P span causes the deflection on both consecutive spans.

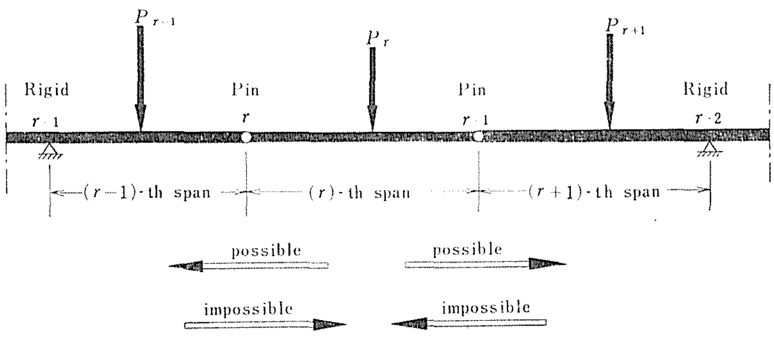


Fig. 6. Physical Property of Pin Joint Coupling.

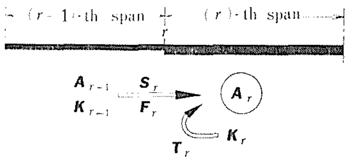
Thus, it can be concluded that the outward operation from the P-P span is possible, but the inward operation for the P-P span is prohibited. Therefore, the operation must be started from the P-P span of a continuous beam.

If there are no loads on a span considered, then the load term vanishes.

Table III. Rightward Operators.

Rightward Shift Formula

$$\boldsymbol{A}_r = \boldsymbol{S}_r \boldsymbol{A}_{r-1} + \boldsymbol{F}_r \boldsymbol{K}_{r-1} + \boldsymbol{T}_r \boldsymbol{K}_r$$

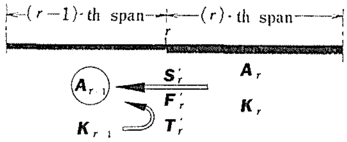


Set	$\boldsymbol{A}_{r-1}$	$\boldsymbol{A}_r$	Shiftor $\boldsymbol{S}$	Shifting Feeder $\boldsymbol{F}$	Self-Feeder $\boldsymbol{T}$
RRR	$\begin{bmatrix} C \\ D \end{bmatrix}$	$\begin{bmatrix} C \\ D \end{bmatrix}$	$\frac{1}{\alpha}\begin{bmatrix} 1 & 3 \\ -\frac{\beta}{\alpha}-1 & -2\frac{\beta}{\alpha}-3 \end{bmatrix}$	$\frac{1}{\alpha}\begin{bmatrix} 0 & 0 & 1 & 3 \\ \frac{\beta}{\alpha} & 0 & -\frac{\beta}{\alpha}-1 & -2\frac{\beta}{\alpha}-3 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix}$
RRP	$\begin{bmatrix} C \\ D \end{bmatrix}$	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\frac{\beta}{3\alpha^2}\begin{bmatrix} 3 & 6 \\ -\frac{\alpha}{\beta} & -3\frac{\alpha}{\beta} \end{bmatrix}$	$\frac{\beta}{3\alpha^2}\begin{bmatrix} -3 & 0 & 3 & 6 \\ 0 & 0 & -\frac{\alpha}{\beta} & -3\frac{\alpha}{\beta} \end{bmatrix}$	$\frac{1}{3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 \end{bmatrix}$
PRR	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\begin{bmatrix} C \\ D \end{bmatrix}$	$\frac{1}{\alpha}\begin{bmatrix} 0 & 3 \\ -\frac{\beta}{\alpha} & -3\frac{\beta}{\alpha}-3 \end{bmatrix}$	$\frac{1}{\alpha}\begin{bmatrix} 0 & 0 & 1 & 3 \\ 0 & -\frac{\beta}{\alpha} & -2\frac{\beta}{\alpha}-1 & -3\frac{\beta}{\alpha}-3 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix}$
PRP	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\frac{1}{\alpha}\begin{bmatrix} \frac{\beta}{\alpha} & 3\frac{\beta}{\alpha} \\ 0 & -1 \end{bmatrix}$	$\frac{1}{3\alpha}\begin{bmatrix} 0 & 3\frac{\beta}{\alpha} & 6\frac{\beta}{\alpha} & 9\frac{\beta}{\alpha} \\ 0 & 0 & -1 & -3 \end{bmatrix}$	$\frac{1}{3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 \end{bmatrix}$
RPR	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\begin{bmatrix} -\frac{\beta}{\alpha^3} & 2\frac{\beta}{\alpha^3}-1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -\frac{\beta}{\alpha^3} & -\frac{\beta}{\alpha^3} & 0 & 2\frac{\beta}{\alpha^3}-1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
RPP	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\begin{bmatrix} A \\ B \end{bmatrix}$	impossible	impossible	impossible
PPR	$\begin{bmatrix} A \\ B \end{bmatrix}$	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\frac{\beta}{\alpha^3}\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$	$\frac{\beta}{3\alpha^3}\begin{bmatrix} -3 & -3 & \frac{\alpha^3}{\beta}-2 & 0 \\ 0 & 0 & -\frac{\alpha^3}{\beta} & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Table IV. Leftward Operators.

Leftward Shift Formula

$$\boldsymbol{A}_{r-1} = \boldsymbol{S}'_r\boldsymbol{A}_r + \boldsymbol{F}'_r\boldsymbol{K}_r + \boldsymbol{T}'_r\boldsymbol{K}_{r-1}$$

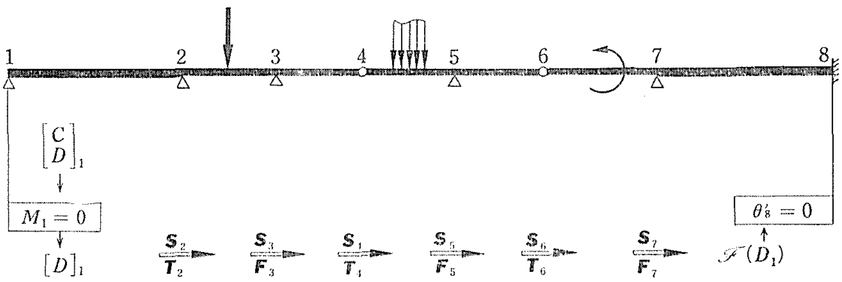


Set	$\boldsymbol{A}_{r-1}$	$\boldsymbol{A}_r$	Shiftor $\boldsymbol{S}'$	Shifting Feeder $\boldsymbol{F}'$	Self-Feeder $\boldsymbol{T}'$
RRR	$\begin{bmatrix} C \\ D \end{bmatrix}$	$\begin{bmatrix} C \\ D \end{bmatrix}$	$\alpha\begin{bmatrix} -3\frac{\alpha}{\beta}-2 & -3\frac{\alpha}{\beta} \\ \frac{\alpha}{\beta}+1 & \frac{\alpha}{\beta} \end{bmatrix}$	$\frac{\alpha^2}{\beta}\begin{bmatrix} -3 & -3 & -3 & -3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$
RRP	$\begin{bmatrix} C \\ D \end{bmatrix}$	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\alpha\begin{bmatrix} 3\frac{\alpha}{\beta} & 6 \\ -\frac{\alpha}{\beta} & -3 \end{bmatrix}$	$\alpha\begin{bmatrix} 0 & 0 & 2 & 6 \\ 0 & 0 & -1 & -3 \end{bmatrix}$	$\begin{bmatrix} 3 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$
PRR	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\begin{bmatrix} C \\ D \end{bmatrix}$	$\frac{\alpha^2}{3\beta}\begin{bmatrix} -3\frac{\beta}{\alpha}-3 & -3 \\ \frac{\beta}{\alpha} & 0 \end{bmatrix}$	$\frac{\alpha^2}{\beta}\begin{bmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\frac{1}{3}\begin{bmatrix} 0 & -3 & -3 & 0 \\ 0 & 0 & -1 & -3 \end{bmatrix}$
PRP	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\begin{bmatrix} \frac{\alpha^2}{\beta} & 3\alpha \\ 0 & -\alpha \end{bmatrix}$	$\frac{\alpha}{3}\begin{bmatrix} 0 & 0 & 3 & 9 \\ 0 & 0 & -1 & -3 \end{bmatrix}$	$\frac{1}{3}\begin{bmatrix} 0 & -3 & -3 & 0 \\ 0 & 0 & -1 & -3 \end{bmatrix}$
RPR	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\begin{bmatrix} -\frac{\alpha^3}{\beta} & -\frac{\alpha^3}{\beta}+2 \\ 0 & 1 \end{bmatrix}$	$\frac{\alpha^3}{\beta}\begin{bmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
RPP	$\begin{bmatrix} B \\ D \end{bmatrix}$	$\begin{bmatrix} A \\ B \end{bmatrix}$	$\frac{\alpha^3}{\beta}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\frac{1}{3}\begin{bmatrix} 0 & 0 & -2 & -6 \\ 0 & 0 & -1 & -3 \end{bmatrix}$	$\begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
PPR	$\begin{bmatrix} A \\ B \end{bmatrix}$	$\begin{bmatrix} B \\ D \end{bmatrix}$	impossible	impossible	impossible

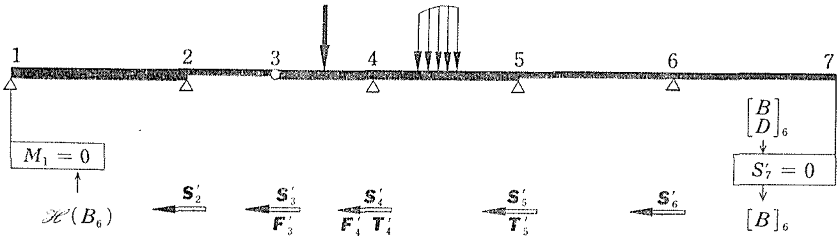
SHIFTING PROCEDURES.

Ordinary Procedure.

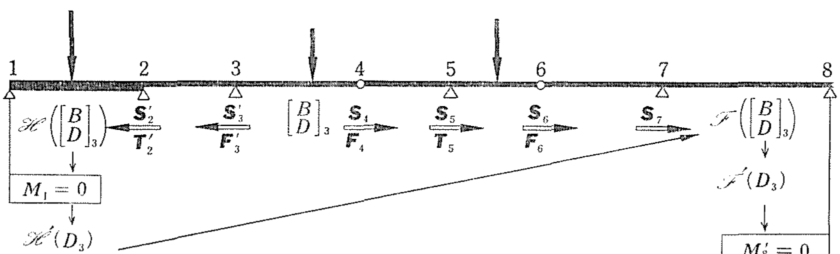
Usually, the operation can be carried out in either direction unless the pin joint coupling is involved in a continuous beam. The typical procedures are shown in Figs. 7a, 7b, and 7c.



(a) Rightward Operation



(b) Leftward Operation



(c) Combined Operation

Fig. 7. Shifting Procedures of Continuous Beam (Ordinary Case).

In Fig. 7a is shown the rightward operation. First, the unkown elements of the first span are reduced to D_1 by means of the extreme left boundary conditions. Secondly, using the rightward operators $\mathbf{S}_r$, $\mathbf{F}_r$, and $\mathbf{T}_r$, the eigenmatrices of respective spans can be represented as functions of D_1, and the eigenmatrix of the extreme right span will be represented by $\mathscr{F}(D_1)$.

In this case, the shiftors $\mathbf{S}_r$ are used at all connection points to shift the current-elements of each span, while the feeders $\mathbf{F}_r$ and $\mathbf{T}_r$ are referred to corresponding to the loading conditions at both adjacent spans of a connection point.

Finally, D_1 can be determined by the extreme right boundary condition. Therefore, the present system has been solved.

In Fig. 7b is shown the leftward operation. The procedure is quite similar to the above case except the direction of operation.

In Fig. 7c, the operation is started from an arbitrary span to both opposite directions. For instance, taking the third span as standard, the unknown elements of this span can be shifted up to both extreme spans. The corresponding operators are also shown with arrows in the figure. Then, at both extremities, the corresponding eigenmatrices are represented by $\mathscr{H}(\{B,\ D\}_3)$ and $\mathscr{F}(\{B,\ D\}_3)$ as shown in the figure.

In virtue of the extreme left boundary condition, the above eigenmatrices will be transformed to the reduced forms $\mathscr{H}'(D_3)$ and $\mathscr{F}'(D_3)$.

Then the residual unknown element D_3 can be determined by the extreme right boundary condition.

In a similar manner, we can compose various operational procedures. Note that there appear no simultaneous equations in the analysis of continuous beams. Besides, the check calculation for the result obtained can be provided by means of other procedures.

Pin Joint Coupling.

A constituent span with pin joint coupling at its ends rejects the inward operation as shown in Fig. 6, so that the operation must be started from the P-P span. For example, taking the case of Fig. 8, the direction of operation should be taken as shown by arrows. The starting shiftors of the intermediate P-P spans are given by Tables III and IV as follows:

For the 4-th span:

$$\mathbf{S}'_4 = \phi_4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{S}_5 = \varphi_5 \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}. \tag{34}$$

For the 8-th span:

$$\mathbf{S}'_8 = \phi_8 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{S}_9 = \varphi_9 \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}. \tag{35}$$

Here, for simplicity,

$$\psi_r = \frac{\alpha_r^3}{\beta_r}, \qquad \varphi_r = \frac{\beta_r}{\alpha_r^3}. \tag{36}$$

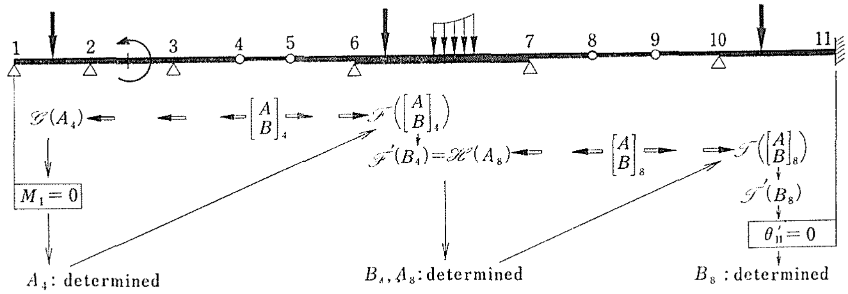


Fig. 8. Operational Procedure for Continuous Beam Involving Several Pin Joint Couplings.

Because of the property of the leftward shiftor given by Eq. 34 or 35, the element which can be shifted leftwards from the P-P span is only the A_r element ($r = 4$ or 8). Then the semi-eigenmatrix in each leftward span is represented by this element A_r. Proceeding operations on, the semi-eigenmatrices of the first and sixth spans become

$$\boldsymbol{A}_1 = \mathscr{G}(A_4) = \boldsymbol{S}'_2\boldsymbol{S}'_3\begin{bmatrix} \psi_4 \\ 0 \end{bmatrix}\boldsymbol{A}_4 + \boldsymbol{S}'_2\big[\boldsymbol{S}'_3(\boldsymbol{F}'_4\boldsymbol{K}_4 + \boldsymbol{T}'_4\boldsymbol{K}_3)$$
$$+ \boldsymbol{F}'_3\boldsymbol{K}_3 + \boldsymbol{T}'_3\boldsymbol{K}_2\big] + \boldsymbol{F}'_2\boldsymbol{K}_2 + \boldsymbol{T}'_2\boldsymbol{K}_1, \tag{37}$$

and

$$\boldsymbol{A}_6 = \mathscr{H}(A_8) = \boldsymbol{S}'_7\begin{bmatrix} \psi_8 \\ 0 \end{bmatrix}\boldsymbol{A}_8 + \boldsymbol{S}'_7(\boldsymbol{F}'_8\boldsymbol{K}_8 + \boldsymbol{T}'_8\boldsymbol{K}_7) + \boldsymbol{F}'_7\boldsymbol{K}_7 + \boldsymbol{T}'_7\boldsymbol{K}_6. \tag{38}$$

In the rightward operation, all the elements of the P-P span can be shifted, so that the semi-eigenmatrices $\boldsymbol{A}_6$ and $\boldsymbol{A}_{10}$ become

$$\boldsymbol{A}_6 = \mathscr{F}\left(\begin{bmatrix} A \\ B \end{bmatrix}_4\right) = -\boldsymbol{S}_6\varphi_5\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}_4$$
$$+ \boldsymbol{S}_6(\boldsymbol{F}_5\boldsymbol{K}_4 + \boldsymbol{T}_5\boldsymbol{K}_5) + \boldsymbol{F}_6\boldsymbol{K}_5 + \boldsymbol{T}_6\boldsymbol{K}_6, \tag{39}$$

and

$$\boldsymbol{A}_{10} = \mathscr{F}\left(\begin{bmatrix} A \\ B \end{bmatrix}_8\right) = -\boldsymbol{S}_{10}\varphi_9\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}_8 + \boldsymbol{S}_{10}(\boldsymbol{F}_9\boldsymbol{K}_8 + \boldsymbol{T}_9\boldsymbol{K}_9) + \boldsymbol{F}_{10}\boldsymbol{K}_9 + \boldsymbol{T}_{10}\boldsymbol{K}_{10}. \tag{40}$$

Next, in virtue of the extreme left boundary condition, the shifted element A_4 in Eq. 37 is determined, so that the eigenmatrices of the first, second, and third spans are readily obtained. The solution for A_4 becomes as follows:

$$A_4 = -\left[\lfloor 1 \quad 0 \rfloor \boldsymbol{S}'_2 \boldsymbol{S}'_3 \begin{bmatrix} \psi_4 \\ 0 \end{bmatrix}\right]^{-1} \lfloor 1 \quad 0 \rfloor \left[\boldsymbol{S}'_2 \left[\boldsymbol{S}'_3 (\boldsymbol{F}'_4 \boldsymbol{K}_4 + \boldsymbol{T}'_4 \boldsymbol{K}_3) \right.\right.$$

$$\left.\left. + \boldsymbol{F}'_3 \boldsymbol{K}_3 + \boldsymbol{T}'_3 \boldsymbol{K}_2 \right] + \boldsymbol{F}'_2 \boldsymbol{K}_2 + \boldsymbol{T}'_2 \boldsymbol{K}_1 \right]. \qquad (41)$$

At the sixth intermediate span, $\boldsymbol{A}_6$ is expressed by Eq. 38 or 39. Since A_4 has been determined, the remaining unknowns B_4 and A_8 are evaluated by equating both equations. The result is as follows:

$$\begin{bmatrix} B_4 \\ A_8 \end{bmatrix} = \left[\boldsymbol{S}_6 \begin{bmatrix} \varphi_5 \\ 0 \end{bmatrix}, \quad \boldsymbol{S}'_7 \begin{bmatrix} \psi_8 \\ 0 \end{bmatrix}\right]^{-1} \left[-\boldsymbol{S}_6 \begin{bmatrix} \varphi_5 \\ 0 \end{bmatrix} A_4 + \boldsymbol{S}_6 (\boldsymbol{F}_5 \boldsymbol{K}_4 + \boldsymbol{T}_5 \boldsymbol{K}_5) \right.$$

$$\left. + \boldsymbol{F}_6 \boldsymbol{K}_5 + \boldsymbol{T}_6 \boldsymbol{K}_6 - \boldsymbol{S}'_7 (\boldsymbol{F}'_8 \boldsymbol{K}_8 + \boldsymbol{T}'_8 \boldsymbol{K}_7) - \boldsymbol{F}'_7 \boldsymbol{K}_7 - \boldsymbol{T}'_7 \boldsymbol{K}_6 \right]. \qquad (42)$$

Then the unknown elements of the last span are reduced to B_8, which is determined by the extreme right boundary condition as follows:

$$B_8 = \left[\lfloor 1 \quad 2 \rfloor \boldsymbol{S}_{10} \begin{bmatrix} \varphi_9 \\ 0 \end{bmatrix}\right]^{-1} \left[\lfloor 1 \quad 2 \rfloor \left[\boldsymbol{S}_{10} \begin{bmatrix} -\varphi_9 \\ 0 \end{bmatrix} A_8 + \boldsymbol{S}_{10} \boldsymbol{F}_9 \boldsymbol{K}_8 + \boldsymbol{S}_{10} \boldsymbol{T}_9 \boldsymbol{K}_9 \right.\right.$$

$$\left.\left. + \boldsymbol{F}_{10} \boldsymbol{K}_9 + \boldsymbol{T}_{10} \boldsymbol{K}_{10} \right] + \lfloor -1 \quad 0 \quad 1 \quad 2 \rfloor \boldsymbol{K}_{10} \right]. \qquad (43)$$

Thus the present system can be solved providing the inverses of Eqs. 41, 42, and 43 exist. In our practical cases this condition is always satisfied.

In this case, the passing-through operation is impossible at the pin joint coupling. However, the other procedures are quite similar to the ordinary procedure, so that we can compose an arbitrary shifting procedure in a group of constituent spans bounded by P-P spans.

Therefore, the check for the result obtained can also be performed by any other procedure.

Continuous Plate-Girder Systems.

After several preliminary treatments are carried out, the continuous plate-girder systems can be analyzed by a manner similar to the preceding articles. In this case, the constituent span is taken as shown in Fig. 9.

The connection conditions at a cross-section with abrupt change become

$$\mathbf{C}_{V_r}\{\boldsymbol{N}'_{r-1} \quad \boldsymbol{N}_r\} = 0, \tag{44}$$

in which $\boldsymbol{N}'_{r-1}$ and $\boldsymbol{N}_r$ are given by Eqs. 14 and $\mathbf{C}_{V_r}$ is the connection-matrix to be derived from the continuity condition of physical quantities at this point. Eq. 44 yields the shift formulas

$$\left.\begin{aligned} \boldsymbol{N}_r &= \boldsymbol{V}_r\boldsymbol{N}'_{r-1}, \\ \boldsymbol{N}'_{r-1} &= \boldsymbol{V}'_r\boldsymbol{N}_r, \end{aligned}\right\} \tag{45}$$

in which $\boldsymbol{V}_r$ or $\boldsymbol{V}'_r$ is the rightward or leftward shiftor given by

$$\boldsymbol{V}_r = \begin{bmatrix} \dfrac{\beta}{\alpha^3} & \dfrac{\beta}{\alpha^3} & \dfrac{\beta}{\alpha^3} & \dfrac{\beta}{\alpha^3} \\ 0 & \dfrac{\beta}{\alpha^2} & 2\dfrac{\beta}{\alpha^2} & 3\dfrac{\beta}{\alpha^2} \\ 0 & 0 & \dfrac{1}{\alpha} & \dfrac{3}{\alpha} \\ 0 & 0 & 0 & 1 \end{bmatrix}_r, \tag{46}$$

$$\boldsymbol{V}'_r = \begin{bmatrix} \dfrac{\alpha^3}{\beta} & -\dfrac{\alpha^2}{\beta} & \alpha & -1 \\ 0 & \dfrac{\alpha^2}{\beta} & -2\alpha & 3 \\ 0 & 0 & \alpha & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}_r. \tag{47}$$

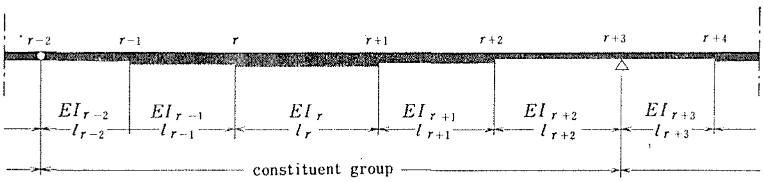


Fig. 9. Constituent Group of Continuous Plate-Girder Systems.

At the point of abrupt change in cross-section, all the elements of eigenmatrix in a constituent span are shifted to the adjacent one by the above shiftors. However, at the rigid support or pin joint, the connection conditions are insufficient to shift all the above elements. Therefore the preliminary treatement must be introduced. Let us designate a set of constituent spans between rigid support or pin joint as the "constituent group." In the preliminary operation for a constituent group, the number of unknown elements can be reduced to two. For example, the group shown in Fig. 9 is treated preliminarily as follows.

In virtue of the boundary condition at the left-hand pin joint, the eigenmatrix in the $(r-2)$-th span becomes

$$\boldsymbol{N}_{r-2}=\{A \quad B \quad 0 \quad D\}_{r-2}. \tag{48}$$

Shifting $\boldsymbol{N}_{r-2}$ rightwards, the conjugate $\boldsymbol{N}'_{r+2}$ of the $(r+2)$-th span becomes

$$\begin{aligned}\boldsymbol{N}'_{r+2}&=\boldsymbol{V}_{r+2}\boldsymbol{V}_{r+1}\boldsymbol{V}_{r}\boldsymbol{V}_{r-1}\{A \quad B \quad 0 \quad D\}_{r-2}\\ &\quad+\boldsymbol{V}_{r+2}\left[\boldsymbol{V}_{r+1}\left[\boldsymbol{V}_{r}(\boldsymbol{V}_{r-1}\boldsymbol{K}_{r-2}+\boldsymbol{K}_{r-1})+\boldsymbol{K}_{r}\right]+\boldsymbol{K}_{r+1}\right]+\boldsymbol{K}_{r+2}\\ &=\boldsymbol{V}_{A}\{A \quad B \quad 0 \quad D\}_{r-2}+\boldsymbol{K}_{A},\end{aligned} \tag{49}$$

in which $\boldsymbol{V}_{A}$ is a 4-by-4 aggregate shiftor, and $\boldsymbol{K}_{A}$ is a 4-by-1 aggregate feeder of this group.

Introducing the boundary condition at the right rigid support, we obtain

$$\begin{aligned}\boldsymbol{A}_{r-2}&=-\left[\lfloor 1 \quad 1 \quad 1 \quad 1\rfloor \boldsymbol{V}_{A}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right]^{-1}\left[\lfloor 1 \quad 1 \quad 1 \quad 1\rfloor\ \boldsymbol{V}_{A}\begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix}\begin{bmatrix}B\\D\end{bmatrix}_{r-2}+\boldsymbol{K}_{A}\right]\\ &=-\lfloor u \quad v\rfloor\begin{bmatrix}B\\D\end{bmatrix}_{r-2}-k\lfloor 1 \quad 1 \quad 1 \quad 1\rfloor\boldsymbol{K}_{A},\end{aligned} \tag{50}$$

in which u, v, and k are mere numbers.

Then, the eigenmatrices $\boldsymbol{N}_{r-2}$ and $\boldsymbol{N}'_{r+2}$ are reduced to the forms

$$\boldsymbol{N}_{r-2}=\begin{bmatrix}-u&-v\\1&0\\0&0\\0&1\end{bmatrix}\begin{bmatrix}B\\D\end{bmatrix}_{r-2}+\begin{bmatrix}-k&-k&-k&-k\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\boldsymbol{K}_{A}, \tag{51}$$

$$\boldsymbol{N}'_{r+2} = \boldsymbol{V}_A \begin{bmatrix} -u & -v \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}_{r-2} + \begin{bmatrix} 1-k & -k & -k & -k \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{K}_A. \tag{52}$$

In virtue of the above preliminary operation, the eigenmatrix of the constituent group can be reduced to the semi-eigenmatrix form. Eqs. 51 and 52 are of quite similar form to those previously given by Eqs. 20 and Table II.

In the second step, we can derive the shift formula between the reduced semi-eigenmatrices of two adjacent groups using the connection equation given by Eq. 24 or 25.

Thus, the present system can be solved. A practical application is given in the following.

APPLICATION.

As a simple application of the preceding discussion, let us take the five-span continuous plate-girder bridge shown in Fig. 10. The constituent spans and groups are taken as shown in the figure, and the operational procedures are shown by arrows. The preliminary operations are performed for the respective groups 1 and 2. Then the reduced semi-eigenmatrix can be shifted to the rightward spans by the connection conditions at the rigid supports 4 and 7. These procedures are as follows.

Load Terms.

The load terms of each constituent spans can be obtained by Eq. 13 and are summarized in Table V.

Table V. Load Terms (*ton*). ($\boldsymbol{K}_3 = \boldsymbol{K}_5 = \boldsymbol{K}_8 = 0$)

$\boldsymbol{K}_1$	$\boldsymbol{K}_2$	$\boldsymbol{K}_4$	$\boldsymbol{K}_6$	$\boldsymbol{K}_7$	$\boldsymbol{K}_9$
$\begin{bmatrix} -5.76 \\ 24 \\ -36 \\ 20 \end{bmatrix}$	$\begin{bmatrix} -0.5 \\ 2 \\ -3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 3.75 \\ -5 \\ 2.5 \end{bmatrix}$	$\begin{bmatrix} 3.75 \\ -15 \\ 15 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1.875 \\ 11.25 \\ -22.5 \\ 15 \end{bmatrix}$	$\begin{bmatrix} -5.76 \\ 24 \\ -36 \\ 20 \end{bmatrix}$

Operators.

The necessary operators for the present system are obtained from Table

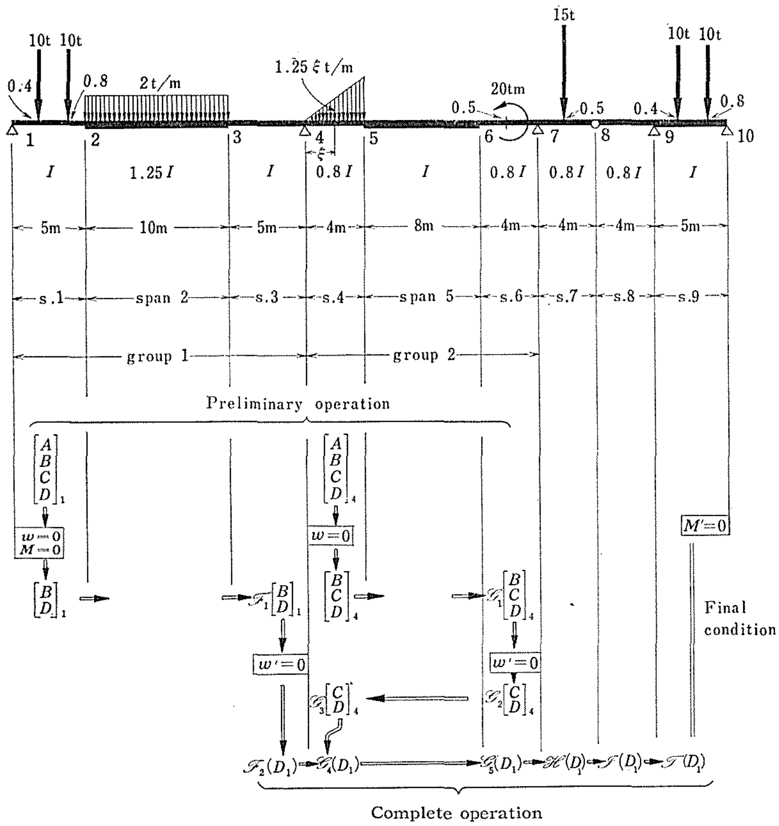


Fig. 10. Continuous Plate-Girder Bridge and Its Operation Chart.

III and Eq. 46. The ratios α_r and β_r at each connection point are summarized in Table VI.

Table VI. Span Ratios and Flexural Rigidity Ratios.

r	2	3	4	5	6	7	8	9
α_r	2	0.5	0.8	2	0.5	1	1	1.25
β_r	1.25	0.8	0.8	1.25	0.8	1	1	1.25

Referring to the operation chart (Fig. 10), the following operators are evaluated.

The rightward shiftors at the cross-sections with abrupt change (Eq. 46):

$$\boldsymbol{V}_2 = \boldsymbol{V}_5 = \begin{bmatrix} 0.15625 & 0.15625 & 0.15625 & 0.15625 \\ 0 & 0.3125 & 0.625 & 0.9375 \\ 0 & 0 & 0.5 & 1.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{53}$$

$$\boldsymbol{V}_3 = \boldsymbol{V}_6 = \begin{bmatrix} 6.4 & 6.4 & 6.4 & 6.4 \\ 0 & 3.2 & 6.4 & 9.6 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{54}$$

The rightward operators at the pin joint 8 and the rigid support 9 (Table III):

$$\left.\begin{aligned} \boldsymbol{S}_8 &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, & \boldsymbol{F}_8 &= \begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ \boldsymbol{S}_9 &= \begin{bmatrix} 0 & 2.4 \\ -0.8 & -4.8 \end{bmatrix}, & \boldsymbol{T}_9 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix}. \end{aligned}\right\} \tag{55}$$

The connection-matrices at the rigid support 4 and 7 (Eq. 27):

$$\boldsymbol{C}_4 = \left[\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \; -\begin{bmatrix} 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix}\right], \tag{56}$$

$$\boldsymbol{C}_7 = \left[\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \; -\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\right]. \tag{57}$$

Preliminary Operations.

——— *First Group.* ———

The extreme left boundary conditions are given by the equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \{A \quad B \quad C \quad D\}_1 = 0, \tag{58}$$

from which $\boldsymbol{N}_1$ becomes

$$\boldsymbol{N}_1 = \{0 \quad B \quad 0 \quad D\}_1. \tag{59}$$

Shifting the above elements rightwards, the conjugate eigenmatrix of the third span is expressed by

$$\boldsymbol{N}'_3 = \boldsymbol{V}_3\boldsymbol{V}_2(\boldsymbol{N}_1 + \boldsymbol{K}_1) + \boldsymbol{V}_3\boldsymbol{K}_2 = \begin{bmatrix} 3 & 23 \\ 1 & 22.2 \\ 0 & 9 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}_1 + \begin{bmatrix} 234.24 \\ 287.2 \\ 150 \\ 22 \end{bmatrix}_{(ton)} = \mathscr{F}_1 \begin{bmatrix} B \\ D \end{bmatrix}_1. \tag{60}$$

The boundary condition at the right end of the third span is given by

$$\lfloor 1 \quad 1 \quad 1 \quad 1 \rfloor \boldsymbol{N}'_3 = 0, \tag{61}$$

from which

$$B_1 = -13.8 D_1 - 173.36\,(ton). \tag{62}$$

Then Eq. 60 becomes

$$\boldsymbol{N}'_3 = \begin{bmatrix} -18.4 \\ 8.4 \\ 9 \\ 1 \end{bmatrix} D_1 + \begin{bmatrix} -285.84 \\ 113.84 \\ 150 \\ 22 \end{bmatrix}_{(ton)} = \mathscr{F}_2(D_1). \tag{63}$$

This is the reduced semi-eigenmatrix of the first group before connection.

——— *Second Group.* ———

At the left end of the fourth span, no deflection takes place, so that $\boldsymbol{N}_4$ becomes

$$\boldsymbol{N}_4 = \{0 \quad B \quad C \quad D\}_4. \tag{64}$$

Shifting rightwards, we obtain

$$\boldsymbol{N}'_6 = \boldsymbol{V}_6 \boldsymbol{V}_5 (\boldsymbol{N}_4 + \boldsymbol{K}_4) + \boldsymbol{K}_6 = \begin{bmatrix} 3 & 8.2 & 23 \\ 1 & 5.2 & 22.2 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B \\ C \\ D \end{bmatrix}_4 + \begin{bmatrix} 30.5 \\ 18.25 \\ 32.5 \\ 2.5 \end{bmatrix}_{(ton)} = \mathscr{G}_1 \begin{bmatrix} B \\ C \\ D \end{bmatrix}_4. \tag{65}$$

At the right end of the sixth span, the deflection vanishes, and then we obtain

$$B_4 = -3.6 C_4 - 13.8 D_4 - 20.9375\,(ton). \tag{66}$$

Therefore, the reduced forms of $\boldsymbol{N}_4$ and $\boldsymbol{N}'_6$ become

$$\boldsymbol{N}_4 = \begin{bmatrix} 0 & 0 \\ -3.6 & -13.8 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}_4 + \begin{bmatrix} 0 \\ -20.9375 \\ 0 \\ 0 \end{bmatrix}_{(ton)} = \mathscr{G}_3 \begin{bmatrix} C \\ D \end{bmatrix}_4, \tag{67}$$

$$\boldsymbol{N}'_6 = \begin{bmatrix} -2.6 & -18.4 \\ 1.6 & 8.4 \\ 1 & 9 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}_4 + \begin{bmatrix} -32.3125 \\ -2.6875 \\ 32.5 \\ 2.5 \end{bmatrix}_{(ton)} = \mathscr{G}_2 \begin{bmatrix} C \\ D \end{bmatrix}_4. \tag{68}$$

Complete Operations

——— *Support* 4. ———

By the connection conditions at the rigid support 4, the unknown elements in the first group can be shifted to the second group. Substituting from Eqs. 56, 63, and 67 into Eq. 26, we obtain after rearrangement the following shift formula:

$$\begin{bmatrix} C \\ D \end{bmatrix}_4 = \begin{bmatrix} 15 \\ -6.576\,087 \end{bmatrix} D_1 + \begin{bmatrix} 270 \\ -115.415\,761 \end{bmatrix}_{(ton)} = \mathscr{G}_4(D_1). \tag{69}$$

Then Eq. 68 becomes

$$\boldsymbol{N}'_6 = \begin{bmatrix} 82.000\,001 \\ -31.239\,131 \\ -44.184\,783 \\ -6.576\,087 \end{bmatrix} D_1 + \begin{bmatrix} -2\,857.962\,502 \\ 1\,398.804\,892 \\ 1\,341.241\,849 \\ 1\,17.915\,761 \end{bmatrix}_{(ton)} = \mathscr{G}_5(D_1). \tag{70}$$

——— *Support* 7. ———

First, using Eq. 20a and Table II, the semi-eigenmatrix of the seventh span becomes

$$\boldsymbol{N}_7 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -3 \\ 0 & 1 \end{bmatrix} \boldsymbol{A}_7 + \begin{bmatrix} 0 \\ 0 \\ -22.5 \\ 0 \end{bmatrix}_{(ton)}. \tag{71}$$

Secondly, substituting from Eqs. 57, 70, and 71 into Eq. 26, it follows that

$$\boldsymbol{A}_7 = \begin{bmatrix} -139.336\,958 \\ 21.304\,348 \end{bmatrix} D_1 + \begin{bmatrix} 4\,435.035\,873 \\ -572.496\,377 \end{bmatrix}_{(ton)} = \mathscr{H}(D_1). \tag{72}$$

——— *Rightward Operations.* ———

From the pin joint 8, we can use the rightward shift formula given by Eq. 31. Using Eqs. 55 and Table V, we obtain

$$\boldsymbol{A}_8 = \boldsymbol{S}_8\boldsymbol{A}_7 + \boldsymbol{F}_8\boldsymbol{K}_7 = \begin{bmatrix} 160.641\,306 \\ 21.304\,348 \end{bmatrix} D_1 + \begin{bmatrix} -5\,001.907\,250 \\ -557.496\,377 \end{bmatrix}_{(ton)} = \mathscr{I}(D_1), \tag{73}$$

$$\boldsymbol{A}_9 = \boldsymbol{S}_9\boldsymbol{A}_8 + \boldsymbol{T}_9\boldsymbol{K}_9 = \begin{bmatrix} 51.130\,435 \\ -230.773\,915 \end{bmatrix} D_1 + \begin{bmatrix} -1\,337.991\,305 \\ 6\,675.268\,410 \end{bmatrix}_{(ton)}. \tag{74}$$

The conjugate of the last span $\boldsymbol{N}'_9$ becomes by Eq. 20b and Table II as follows:

$$\boldsymbol{N}'_9 = \begin{bmatrix} 0 \\ 179.643\,480 \\ 51.130\,435 \\ -230.773\,915 \end{bmatrix} D_1 + \begin{bmatrix} -5.76 \\ -5\,315.517\,105 \\ -1\,373.991\,305 \\ 6\,695.268\,410 \end{bmatrix}_{(ton)} = \mathscr{T}(D_1). \tag{75}$$

The extreme right boundary condition is given by

$$\lfloor 0 \quad 0 \quad 1 \quad 3 \rfloor \boldsymbol{N}'_9 = 0, \tag{76}$$

from which the final solution is obtained as follows:

$$D_1 = 29.182\,888\,(ton). \tag{77}$$

Using this result, we can evaluate the eigenmatrix and the semi-eigenmatrix of each constituent span as shown in Table VII.

Table VII. Eigenmatrix of Each Constituent Span *(ton)*.

$\boldsymbol{N}_r$	$\boldsymbol{N}_1$	$\boldsymbol{N}_2$	$\boldsymbol{N}_3$
$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}_r$	$\begin{bmatrix} 0 \\ -576.083\,854 \\ 0 \\ 29.182\,888 \end{bmatrix}$	$\begin{bmatrix} -85.103\,276 \\ -148.917\,247 \\ 55.774\,332 \\ 49.182\,888 \end{bmatrix}$	$\begin{bmatrix} -822.805\,139 \\ 358.976\,259 \\ 412.645\,992 \\ 51.182\,888 \end{bmatrix}$
$\boldsymbol{N}_r$	$\boldsymbol{N}_4$	$\boldsymbol{N}_5$	$\boldsymbol{N}_6$
$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}_r$	$\begin{bmatrix} 0 \\ 1\,672.271\,686 \\ 707.743\,320 \\ -307.325\,010 \end{bmatrix}$	$\begin{bmatrix} 323.896\,874 \\ 677.197\,905 \\ -105.865\,855 \\ -304.825\,010 \end{bmatrix}$	$\begin{bmatrix} 3\,778.585\,050 \\ -1\,436.828\,270 \\ -2\,040.681\,770 \\ -304.825\,010 \end{bmatrix}$
$\boldsymbol{A}_r$	$\boldsymbol{A}_7$	$\boldsymbol{A}_8$	$\boldsymbol{A}_9$
$\begin{bmatrix} B \\ D \end{bmatrix}_r, \begin{bmatrix} C \\ D \end{bmatrix}_r$	$\begin{bmatrix} 368.781\,034 \\ 49.226\,025 \end{bmatrix}$	$\begin{bmatrix} -313.930\,009 \\ 64.226\,025 \end{bmatrix}$	$\begin{bmatrix} 154.142\,453 \\ -59.380\,905 \end{bmatrix}$
$\boldsymbol{N}_r$	$\boldsymbol{N}_7$	$\boldsymbol{N}_8$	$\boldsymbol{N}_9$
$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}_r$	$\begin{bmatrix} 0 \\ 368.781\,034 \\ -170.178\,075 \\ 49.226\,025 \end{bmatrix}$	$\begin{bmatrix} 249.703\,984 \\ -313.930\,009 \\ 0 \\ 64.226\,025 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -97.001\,548 \\ 154.142\,453 \\ -59.380\,905 \end{bmatrix}$

Check Calculation.

The results above obtained can be checked by other operation. For instance, using the leftward operation from the extreme right span, $\boldsymbol{A}_7$ and $\boldsymbol{A}_8$ are checked as follows:

$$\left.\begin{aligned}
\boldsymbol{A}_8 &= \boldsymbol{S}'_9\boldsymbol{A}_9 + \boldsymbol{F}'_9\boldsymbol{K}_9 = \frac{1.25}{3}\begin{bmatrix} -6 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 154.142\,453 \\ -59.380\,905 \end{bmatrix} \\
&\quad - 1.25\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\{-5.76 \quad 24 \quad -36 \quad 20\} = \begin{bmatrix} -313.930\,001 \\ 64.226\,022 \end{bmatrix}, \\
\boldsymbol{A}_7 &= \boldsymbol{S}'_8\begin{bmatrix} -313.930\,001 \\ 64.226\,021 \end{bmatrix} + \boldsymbol{T}'_8\boldsymbol{K}_7 = \begin{bmatrix} 368.781\,022 \\ 49.226\,022 \end{bmatrix}. \quad \text{O. K.}
\end{aligned}\right\} \tag{78}$$

CONCLUSIONS.

In conclusion, the following notes are given:

1. All physical properties of each constituent span of a continuous beam are expressed by the eigenmatrix.
2. The connection conditions, the support conditions, and the boundary conditions are expressed by the corresponding matrices.
3. The eigenmatrices of two adjacent spans are interconnected with the shiftors and the feeders.
4. The eigenmatrices of respective spans can be expressed by the current-matrix of an arbitrary span.
5. The pin joint coupling shows a peculiar property which necessitates a particular treatment.
6. The analysis of continuous plate-girder bridge can be performed by this procedure after a certain preliminary operation.
7. A typical application is added, which shows the readiness in obtaining solution.

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APPENDIX. —NOTATION.

The following symbols have been adopted for use in this paper:

$\mathbf{A}$ = semi-eigenmatrix consisting of 2-by-1 elements, see Eq. 22;

A, B, C, D = elements of the eigenmatrix, see Eqs. 3a, 8a, and 11a;

$\mathbf{B}, \mathbf{B}'$ = extreme left or right boundary-matrix, see Eqs. 17 and Table I;

$\mathbf{B}_I, \mathbf{B}'_I$ = independent boundary-matrix at the left or right support of a constituent span, see Eqs. 16 and Table I;

$\mathbf{C}_r$ = connection-matrix at the r-th support, see Eqs. 27 and 28;

$\mathbf{C}_{V_r}$ = connection-matrix at the r-th point of abrupt change in cross-section of plate-girder bridge, see Eq. 44;

E = modulus of elasticity;

$\mathbf{F}_r, \mathbf{F}'_r$ = rightward or leftward shifting feed operator, or briefly the shifting feeder at the r-th support, see Eq. 31 or 32, and Table III or IV;

$\mathscr{F}, \mathscr{G}, \mathscr{H}, \mathscr{I}, \mathscr{J}$ = function of the eigenmatrix of a span expressed by the current-matrix;

i = integer representing the domain number;

I = moment of inertia;

$\mathbf{K}_A$ = 4-by-1 aggregate feeder, see Eq. 49;

$\mathbf{K}_m$ = load-matrix at loaded point of external concentrated moment, see Eqs. 11;

$\mathbf{K}_p$ = load-matrix at loaded point of concentrated load, see Eq. 7;

$\mathbf{K}_q$ = load-matrix at loaded point of partially distributed load see Eqs. 9;

$\mathbf{K}_r$ = load term of the r-th span, see Eq. 13;

k = mere constant, see Eqs. 50, 51, and 52;

$\mathbf{L}$ = 4-by-2 related matrix, see Eq. 21 and Table II;

l = span length;

M = bending moment;

$\mathfrak{M}$ = external concentrated moment;
$\boldsymbol{N}_r$ = eigenmatrix of normal domain of the r-th span;
$\boldsymbol{N}^i{}_r$ = eigenmatrix of the i-th domain of the r-th span;
$\boldsymbol{N}'{}_r$ = eigenmatrix of conjugate domain of the r-th span;
$\boldsymbol{P}$ = 4-by-4 related matrix, see Eq. 23a and Table II;
P = external concentrated load;
P = symbol representing the pin joint;
$\boldsymbol{Q}$ = 4-by-4 related matrix, see Eq. 23b and Table II;
q = intensity of distributed load;
r = order of constituent span or connection point;
R = symbol representing the rigid support;
$\boldsymbol{S}_r$, $\boldsymbol{S}'{}_r$ = right or leftward shift operator, or briefly the shiftor at the r-th support;
S = shearing force;
$\boldsymbol{T}_r$, $\boldsymbol{T}'{}_r$ = right or leftward self-feed operator, or briefly the self-feeder, see Eq. 31 or 32 and Table III or IV;
u, v = mere constants, see Eqs. 50, 51, and 52;
$\boldsymbol{V}_A$ = 4-by-4 aggregate shiftor, see Eq. 49;
$\boldsymbol{V}_r$, $\boldsymbol{V}'{}_r$ = right or leftward shift operator at a cross-section with abrupt change, see Eq. 46 or 47;
w_r = beam deflection of normal domain of the r-th span;
$w^i{}_r$ = beam deflection of the i-th domain of the r-th span;
$w'{}_r$ = beam deflection of conjugate domain of the r-th span;
x = current abscissa;
α = span ratio, see Eq. 29;
β = flexural rigidity ratio, see Eq. 30;
η = non-dimensional abscissa of a point under partially distributed load, see Fig. 2;
θ = slope angle;
κ = non-dimensional load abscissa, see Eq. 5;
λ, μ = non-dimensional abscissa of lower or upper boundary of partially distributed load, see Fig. 2;
ξ = load abscissa;
ρ = non-dimensional current abscissa, see Eq. 2;
$\sum$ = summation;
φ = ratio, see Eq. 36b;
ψ = ratio, see Eq. 36a;

$\int$ = integration;

$\lfloor \quad \rfloor$ = row vector; and

$\{ \quad \}$ = column vector.