Operational Method for Displacement Analysis Second Report Vierendeel Trusses

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1. INTRODUCTION

An approach to the analysis of Vierendeel trusses[1] is presented in this paper. The operational procedures[2),3] have been applied to the displacement analysis of structures.[4] In order to describe behaviors of a constituent member, the prevailing slope-deflection formula[5] and Hooke's law will be adopted for use, which may be put into one matrix equation. This equation shows the relationship between two kinds of physical matrices of size 3-by-1: the member force-matrix, consisting of the axial force, the shearing force and the bending moment of a member, and the member displacement-matrix, consisting of the longitudinal and lateral displacements and the slope angle at the member end.

For the displacement analysis of structures, the nodal displacement-matrix of size 3-by-1 is to be defined, whose elements are the horizontal and vertical displacements and the slope angle at this point. Treating beforehand the compatibility conditions between the displacement-matrices of member ends and those of nodal points, the member force-matrices can be represented by two corresponding nodal displacement-matrices. After these treatments, all the member force-matrices of a structure can be reduced to the nodal displacement-matrices. Then the unknowns in the structure are ruduced to the nodal displacements only. As shown in Table I, the nodal equilibrium conditions are equal to the above unknowns in number, and therefore, the system can be solved completely by simple treatment of the nodal equilibrium conditions.

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In Vierendeel trusses, the unit is selected as a couple of upper and lower nodes, and the eigenmatrix is defined by the assemblage of the displacement-matrices at both nodes. Treating the nodal equilibrium conditions at both nodes in a unit, the shift formula between three eigenmatrices of the system can at once be obtained. By the recurrent use of the shift formula, the first eigenmatrix of the first unit, which is of size 6-by-1, will become current to all the other units. Putting the equilibrium conditions at two consecutive units into one matrix equation, the complete shift formula for a group, consisting of adjacent eigenmatrices, can be obtained. Such a procedure will be shown in the subsequent discussions.

The boundary conditions will, in a broad sense, consist of two parts: the given support conditions, and the shifting of eigenmatrix to the boundary unit from the adjacent regular unit. In this treatment, an inverse calculation of the size ranging from 1-by-1 to 6-by-6 appears corresponding to the boundary configuration of the structure.

The current-matrix, the first eigenmatrix, is determined by nodal equilibrium conditions of the last regular unit in shift operation. In this case, an inverse calculation of size 6-by-6 appears. Thus the system can be solved. The sizes of operational matrices requisite for the present analysis are shown inclusively in Conclusions.

Introducing several modifications of the basic equation of vertical members, this method can be extended to the analysis of rib arches, Lohse trusses, and other structural systems.

2. BASIC CONCEPTS

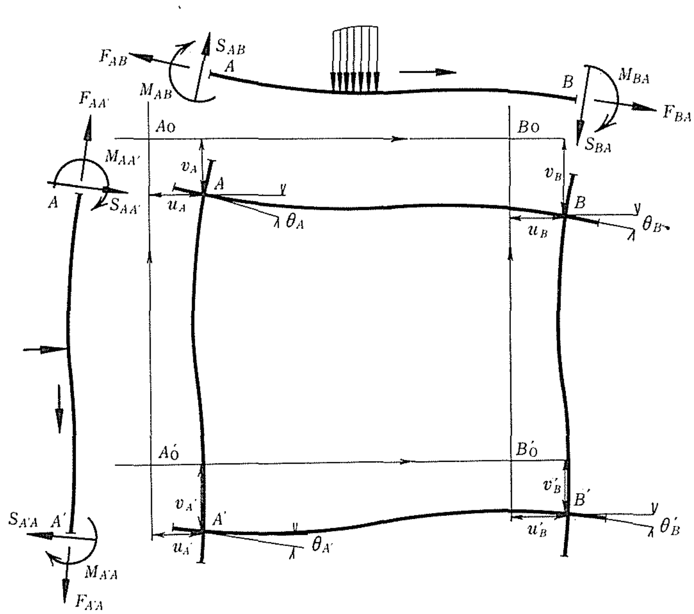
Vierendeel trusses, or frames, will be treated herein, provided the elastic deformation of a member is composed of both flexural deflection and axial elongation in a plane. A part of the network systems is shown in Fig. 1 wherein the positive directions of member coordinates, nodal displacements, forces at member ends, and external loads are illustrated.

For the subsequent discussions, the "force-matrix" at member end, and the "displacement-matrix" at nodal point are to be defined. Referring to Fig. 1, these matrices are given as follows:

Force-matrices:

Member AB:

$$\mathbf{N}_{AB} = \{F \ \ S \ \ M\}_{AB} \ \text{(at } A\text{)}, \quad \mathbf{N}_{BA} = \{F \ \ S \ \ M\}_{BA} \ \text{(at } B\text{)}. \tag{1}$$



u = the horizontal displacement of nodal point,
v = the vertical displacement of nodal point,
θ = the slope angle,
F = the axial force at member end,
S = the shearing force at member end,
M = the bending moment at member end.

Fig. 1. A Part of Plane Network Systems.

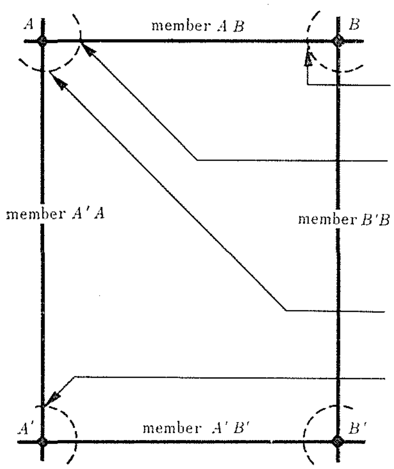
Member $A'A$:

$$\boldsymbol{N}_{A'A}=\{F \quad S \quad M\}_{A'A}\,(\text{at } A'),\quad \boldsymbol{N}_{AA'}=\{F \quad S \quad M\}_{AA'}\,(\text{at } A). \tag{2}$$

Displacement-matrices:

$$\boldsymbol{X}_A=\{u \quad v \quad \theta\}_A,\quad \boldsymbol{X}_B=\{u \quad v \quad \theta\}_B,\quad \boldsymbol{X}_{A'}=\{u \quad v \quad \theta\}_{A'}. \tag{3}$$

Using the slope-deflection equation for the member flexibility and the Hooke's law for the member extensibility, the relations between the above force-matrices and displacement-matrices can be consolidated as shown in Fig. 2 and Eqs. 4 through 7, wherein only the case of the orthogonal configuration of members is to be considered.



Consolidated Formulas

$$\mathbf{N}_{BA} = \lfloor \mathbf{L} \quad \mathbf{H} \rfloor_{BA} \{\mathbf{X}_A \quad \mathbf{X}_B\} + \mathbf{K}_{BA}. \tag{4}$$

$$\mathbf{N}_{AB} = \lfloor \mathbf{H} \quad \mathbf{L} \rfloor_{AB} \{\mathbf{X}_A \quad \mathbf{X}_B\} + \mathbf{K}_{AB}. \tag{5}$$

$$\mathbf{N}_{AA'} = \lfloor \mathbf{L}' \quad \mathbf{H}' \rfloor_{AA'} \{\mathbf{X}_{A'} \quad \mathbf{X}_A\} + \mathbf{K}_{AA'}. \tag{6}$$

$$\underbrace{\mathbf{N}_{A'A}}_{[\text{F. M.}]} = \underbrace{\lfloor \mathbf{H}' \quad \mathbf{L}' \rfloor_{A'A}}_{[\text{O. M.}]} \times \underbrace{\{\mathbf{X}_{A'} \quad \mathbf{X}_A\}}_{[\text{D. M.}]} + \underbrace{\mathbf{K}_{A'A}}_{[\text{L. M.}]}. \tag{7}$$

[F. M.] = Force-matrix, [O. M.] = Operational matrix,
[D. M.] = Displacement-matrix, [L. M.] = Load-matrix.

Fig. 2. Consolidation.

Here the "operational matrix" and the "load-matrix" are to be defined. They are given as follows:

Operational matrices:

$$\mathbf{H}_{AB} = \begin{bmatrix} -f & 0 & 0 \\ 0 & \dfrac{2j}{l} & j \\ 0 & -j & 2k \end{bmatrix}_{AB}, \qquad \mathbf{H}_{BA} = \mathbf{H}_{AB}\mathbf{R}', \tag{8}$$

$$\mathbf{L}_{AB} = \begin{bmatrix} f & 0 & 0 \\ 0 & -\dfrac{2j}{l} & j \\ 0 & j & k \end{bmatrix}_{AB}, \qquad \mathbf{L}_{BA} = \mathbf{L}_{AB}\mathbf{R}', \tag{9}$$

$$\mathbf{H}'_{A'A} = \begin{bmatrix} 0 & f & 0 \\ \dfrac{2j}{l} & 0 & j \\ -j & 0 & 2k \end{bmatrix}_{A'A}, \qquad \mathbf{H}'_{AA'} = \mathbf{H}'_{A'A}\mathbf{R}', \tag{10}$$

$$\mathbf{L}'_{A'A} = \begin{bmatrix} 0 & -f & 0 \\ -\dfrac{2j}{l} & 0 & j \\ j & 0 & k \end{bmatrix}_{A'A}, \qquad \mathbf{L}'_{AA'} = \mathbf{L}'_{A'A}\mathbf{R}', \tag{11}$$

in which

$$f=\frac{EA}{l}, \quad k=\frac{2EI}{l}, \quad j=-\frac{6EI}{l^2}, \tag{12}$$

$$\boldsymbol{R}'=\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{13}$$

$E=$ Young's modulus, $A=$ the cross-sectional area, $I=$ the moment of inertia, $l=$ the member length, and $\boldsymbol{R}=$ the rearrangement-matrix.

Load-matrices:

$$\boldsymbol{K}_{AB}=\begin{bmatrix} 0 \\ V_{AB}-\dfrac{1}{l_{AB}}(C_{AB}+C_{BA}) \\ C_{AB} \end{bmatrix}, \qquad \boldsymbol{K}_{BA}=\begin{bmatrix} -\mathfrak{F}_{AB} \\ V_{BA}-\dfrac{1}{l_{AB}}(C_{AB}+C_{BA}) \\ C_{BA} \end{bmatrix}. \tag{14}$$

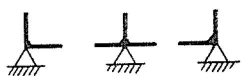
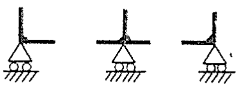
$$\boldsymbol{K}_{A'A}=\begin{bmatrix} 0 \\ V_{A'A}-\dfrac{1}{l_{A'A}}(C_{A'A}+C_{AA'}) \\ C_{A'A} \end{bmatrix}, \quad \boldsymbol{K}_{AA'}=\begin{bmatrix} \mathfrak{F}_{A'A} \\ V_{AA'}-\dfrac{1}{l_{A'A}}(C_{A'A}+C_{AA'}) \\ C_{AA'} \end{bmatrix}, \tag{15}$$

in which $V=$ the load term for shearing force in slope-deflection method, $C=$ the load term for end moment in slope-deflection method, and $\mathfrak{F}=$ the summation of axial loads acting on a member.

3. NODAL CONDITIONS

All the possible states of nodal points in a Vierendeel truss system are

Table I. States of Nodal Points.

Kinds of nodal points	Unknown displacements	Known displacements	Equilibrium conditions
Point I. 	$[\theta]$	$\begin{bmatrix} u \\ v \end{bmatrix}=0$	$\sum[M]=0$
Point II. 	$\begin{bmatrix} u \\ \theta \end{bmatrix}$	$[v]=0$	$\sum\begin{bmatrix} H \\ M \end{bmatrix}=0$
Point III. 	$\begin{bmatrix} u \\ v \\ \theta \end{bmatrix}$	none	$\sum\begin{bmatrix} H \\ V \\ M \end{bmatrix}=0$

classified as shown in Table I, wherein the perfect correspondence between the number of unknown displacements and that of equilibrium conditions is to be noticed.

In this system, the total number of nodal unknown displacements and that of nodal equilibrium conditions are always equal, and therefore, the system can be solved completely by due treatment of the equilibrium conditions at respective nodal points. Referring to Fig. 3, the equilibrium conditions at the nodal points $(r, 1)$ and $(r, 2)$ are given by the following forms:

For upper node $(r, 1)$:

$$\begin{bmatrix} -\cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}_{r-1,1} \begin{bmatrix} F \\ S \\ M \end{bmatrix}'_{r-1,1} + \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}_{r1} \begin{bmatrix} F \\ S \\ M \end{bmatrix}_{r1} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F \\ S \\ M \end{bmatrix}'_{r3} + \begin{bmatrix} P \\ Q \\ R \end{bmatrix}_{r1} = 0, \tag{16}$$

or

$$\boldsymbol{P}'_{r-1,1}\boldsymbol{N}'_{r-1,1} + \boldsymbol{P}_{r1}\boldsymbol{N}_{r1} + \boldsymbol{R}\boldsymbol{N}'_{r3} + \boldsymbol{Q}_{r1} = 0. \tag{17}$$

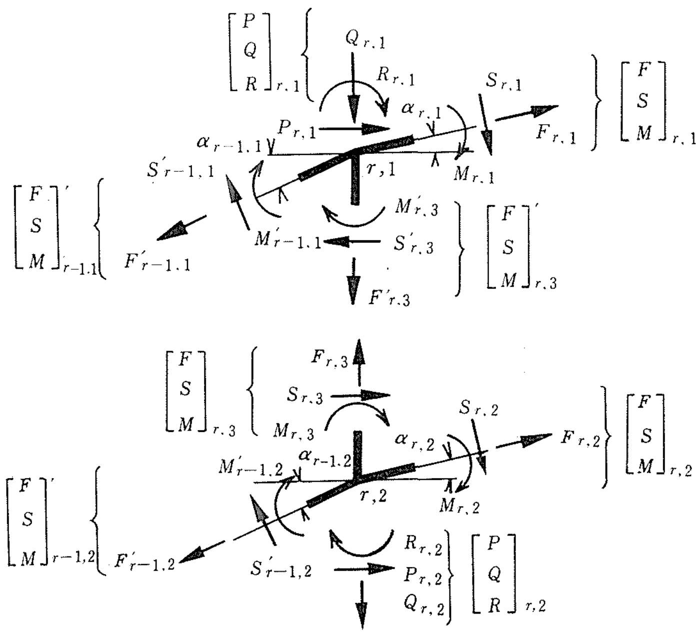


Fig. 3. Force States at Nodal Points.

For lower node $(r, 2)$:

$$\begin{bmatrix} -\cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}_{r-1,2} \begin{bmatrix} F \\ S \\ M \end{bmatrix}'_{r-1,2} + \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}_{r2} \begin{bmatrix} F \\ S \\ M \end{bmatrix}_{r2}$$

$$+ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F \\ S \\ M \end{bmatrix}_{r3} + \begin{bmatrix} P \\ Q \\ R \end{bmatrix}_{r2} = 0, \tag{18}$$

or

$$\mathbf{P}'_{r-1,2}\mathbf{N}'_{r-1,2} + \mathbf{P}_{r2}\mathbf{N}_{r2} + \bar{\mathbf{R}}\mathbf{N}_{r3} + \mathbf{Q}_{r2} = 0. \tag{19}$$

In the above equations, the matrices $\mathbf{P}$ and $\mathbf{P}'$ are designated as the "projection matrices," and $\mathbf{Q}$'s as the "nodal load-matrices."

4. SHIFT OPERATORS

A part of Vierendeel truss is shown in Fig. 4, wherein the letter symbols of nodal points, constituent members, and physical matrices, and the positive directions of member abscissa, member inclination, and deflection are illustrated. The matrices at member end $x = l$ are primed, while those at member end $x = 0$ are unprimed. The nodal displacement-matrix is in this case composed of horizontal and vertical displacements, and slope angle at this point. For the subsequent discussions, it may be recommended to rewrite the key equations to the present system in the following forms:

For upper and lower chord members $(i = 1, 2)$:

$$\mathbf{N}_{ri} = \lfloor \mathbf{H} \quad \mathbf{L} \rfloor_{ri} \mathrm{diag}[\bar{\mathbf{P}} \quad \bar{\mathbf{P}}]_{ri} \{\mathbf{X}_{ri} \quad \mathbf{X}_{r+1,i}\} + \mathbf{K}_{ri}, \tag{20}$$

$$\mathbf{N}'_{ri} = \lfloor \mathbf{L} \quad \mathbf{H} \rfloor_{ri} \mathrm{diag}[\mathbf{R}'\bar{\mathbf{P}} \quad \mathbf{R}'\bar{\mathbf{P}}]_{ri} \{\mathbf{X}_{ri} \quad \mathbf{X}_{r+1,i}\} + \mathbf{K}'_{ri}, \tag{21}$$

$$\mathbf{H}_{ri} = \begin{bmatrix} -f & 0 & 0 \\ 0 & \dfrac{2j}{l} & j \\ 0 & -j & 2k \end{bmatrix}_{ri}, \quad \mathbf{L}_{ri} = \begin{bmatrix} f & 0 & 0 \\ 0 & -\dfrac{2j}{l} & j \\ 0 & j & k \end{bmatrix}_{ri}, \tag{22}$$

$$\mathbf{K}_{ri} = \begin{bmatrix} 0 \\ V - \dfrac{1}{l}(C + C') \\ C \end{bmatrix}_{ri}, \quad \mathbf{K}'_{ri} = \begin{bmatrix} -\mathfrak{F} \\ V' - \dfrac{1}{l}(C + C') \\ C' \end{bmatrix}_{ri}. \tag{23}$$

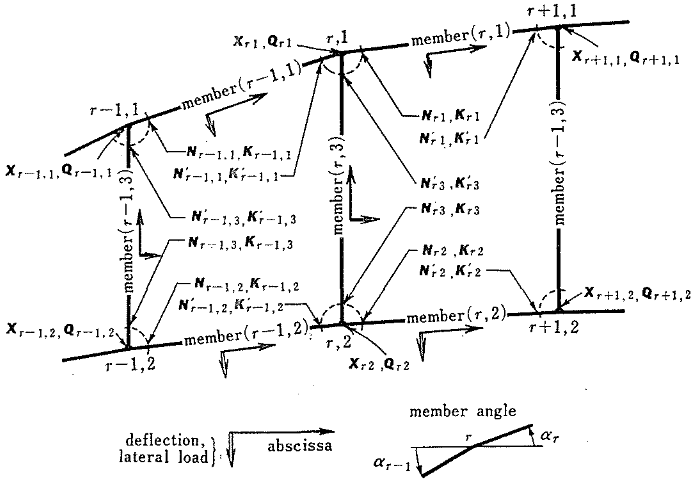


Fig. 4. Vierendeel Truss with Inclined Chords.

For vertical members:

$$\mathbf{N}_{r3} = \lfloor \mathbf{H} \quad \mathbf{L} \rfloor_{r3} \mathrm{diag}[\mathbf{R} \quad \mathbf{R}]\{\mathbf{X}_2 \quad \mathbf{X}_1\}_r + \mathbf{K}_{r3}, \tag{24}$$

$$\mathbf{N}'_{r3} = \lfloor \mathbf{L} \quad \mathbf{H} \rfloor_{r3} \mathrm{diag}[\overline{\mathbf{R}} \quad \overline{\mathbf{R}}]\{\mathbf{X}_2 \quad \mathbf{X}_1\}_r + \mathbf{K}'_{r3}, \tag{25}$$

in which $\mathbf{H}_{r3}$ and $\mathbf{L}_{r3}$ are given by Eqs. 22 provided $i=3$, and $\mathbf{K}_{r3}$ and $\mathbf{K}'_{r3}$ are as follows:

$$\mathbf{K}_{r3} = \begin{bmatrix} 0 \\ V - \dfrac{1}{l}(C+C') \\ C \end{bmatrix}_{r3}, \quad \mathbf{K}'_{r3} = \begin{bmatrix} \mathfrak{F} \\ V' - \dfrac{1}{l}(C+C') \\ C' \end{bmatrix}_{r3}. \tag{26}$$

Here the following matrices are to be defined:

Rearrangement matrices:

$$\mathbf{R} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \overline{\mathbf{R}} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{R}' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{27}$$

Projection matrices:

$$\overline{\mathbf{P}}_{ri} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}_{ri} = \mathbf{P}_{ri}^{-1}, \tag{28}$$

in which α_{ri} = the inclination of member $(r,\ i)$.

By substitution from these equations to Eqs. 17 and 19, the equilibrium conditions at nodal points $(r, 1)$ and $(r, 2)$ are consolidated into one matrix equation as follows:

$$\boldsymbol{A}'_{r-1}\boldsymbol{Z}_{r-1} + \boldsymbol{B}'_r\boldsymbol{Z}_r + \boldsymbol{G}'_{r+1}\boldsymbol{Z}_{r+1} + \boldsymbol{C}'_{r-1}\boldsymbol{F}_{r-1} + \boldsymbol{D}'_r\boldsymbol{F}_r = 0. \tag{29}$$

Here,

$$\boldsymbol{Z}_{r-1} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}_{r-1}, \quad \boldsymbol{Z}_r = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}_r, \quad \boldsymbol{Z}_{r+1} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}_{r+1}, \tag{30}$$

which are designated as the "eigenmatrices" of the $(r-1)$-, (r)-, and $(r+1)$-th units, and the other symbols are given by

$$\boldsymbol{F}_r = \{\boldsymbol{K}_1 \quad \boldsymbol{K}'_1 \quad \boldsymbol{K}_2 \quad \boldsymbol{K}'_2 \quad \boldsymbol{K}_3 \quad \boldsymbol{K}'_3 \quad \boldsymbol{Q}_1 \quad \boldsymbol{Q}_2\}_r, \tag{31}$$

$$\boldsymbol{A}'_{r-1} = \begin{bmatrix} (\boldsymbol{P}'\boldsymbol{L}\boldsymbol{R}'\overline{\boldsymbol{P}})_1 & 0 \\ 0 & (\boldsymbol{P}'\boldsymbol{L}\boldsymbol{R}'\overline{\boldsymbol{P}})_2 \end{bmatrix}_{r-1}, \tag{32}$$

$$\boldsymbol{B}'_r = \begin{bmatrix} (\boldsymbol{P}'\boldsymbol{H}\boldsymbol{R}'\overline{\boldsymbol{P}})_{r-1,1} + (\boldsymbol{P}\boldsymbol{H}\overline{\boldsymbol{P}})_{r1} + \boldsymbol{R}\boldsymbol{H}_{r3}\overline{\boldsymbol{R}}, & \boldsymbol{R}\boldsymbol{L}_{r3}\overline{\boldsymbol{R}} \\ \overline{\boldsymbol{R}}_r\boldsymbol{L}_3\boldsymbol{R}, & (\boldsymbol{P}'\boldsymbol{H}\boldsymbol{R}'\overline{\boldsymbol{P}})_{r-1,2} + (\boldsymbol{P}\boldsymbol{H}\overline{\boldsymbol{P}})_{r2} + \overline{\boldsymbol{R}}\boldsymbol{H}_{r3}\boldsymbol{R} \end{bmatrix}, \tag{33}$$

$$\boldsymbol{G}'_{r+1} = \begin{bmatrix} (\boldsymbol{P}\boldsymbol{L}\overline{\boldsymbol{P}})_1 & 0 \\ 0 & (\boldsymbol{P}\boldsymbol{L}\overline{\boldsymbol{P}})_2 \end{bmatrix}_r, \tag{34}$$

$$\boldsymbol{C}'_{r-1} = \begin{bmatrix} 0 & \boldsymbol{P}'_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{P}'_2 & 0 & 0 & 0 & 0 \end{bmatrix}_{r-1}, \tag{35}$$

$$\boldsymbol{D}'_r = \begin{bmatrix} \boldsymbol{P}_1 & 0 & 0 & 0 & 0 & \boldsymbol{R} & \boldsymbol{E} & 0 \\ 0 & 0 & \boldsymbol{P}_2 & 0 & \overline{\boldsymbol{R}} & 0 & 0 & \boldsymbol{E} \end{bmatrix}_r. \tag{36}$$

Eq. 29 yields the recurrence or shift formula

$$\boldsymbol{Z}_{r+1} = \boldsymbol{A}_{r-1}\boldsymbol{Z}_{r-1} + \boldsymbol{B}_r\boldsymbol{Z}_r + \boldsymbol{C}_{r-1}\boldsymbol{F}_{r-1} + \boldsymbol{D}_r\boldsymbol{F}_r. \tag{37}$$

Here $\boldsymbol{A}_{r-1}$ and $\boldsymbol{B}_r$ are the "shift-operators" or briefly "shiftors," and $\boldsymbol{C}_{r-1}$ and $\boldsymbol{D}_r$ are the "feed-operators" or "feeders." They are given as follows:

$$\begin{bmatrix} \boldsymbol{A}_{r-1} \\ \boldsymbol{B}_r \\ \boldsymbol{C}_{r-1} \\ \boldsymbol{D}_r \end{bmatrix} = -\boldsymbol{G}'^{-1}_{r+1} \begin{bmatrix} \boldsymbol{A}_{r-1} \\ \boldsymbol{B}_r \\ \boldsymbol{C}_{r-1} \\ \boldsymbol{D}_r \end{bmatrix}'. \tag{38}$$

The matrix $\boldsymbol{G}'_{r+1}$ is square and nonsingular, and therefore, the above operators can be evaluated.

Eq. 37 is the shift formula between three eigenmatrices $\mathbf{Z}_{r-1}$, $\mathbf{Z}_r$, and $\mathbf{Z}_{r+1}$. In virtue of the recurrent use of the above shift formula, all the displacements of each unit of the system can be represented by the displacements of an arbitrary unit, called the "current-matrix." Treating boundary conditions of a given system, the current-matrix is determined, and therefore, the sytem can be solved completely.

5. TABULAR TREATMENT

The size of the above shiftors is 6-by-6 square, and that of the feeders 6-by-24 rectangular. These may be treated with manual operation because of their bearable sizes. In this case, it may be recommended to carry out the operation by tabular form. The following notes are given.

Table II. Nodal Equilibrium Conditions.

Formula: $\lfloor \mathbf{A}_{r-1}\ \ \mathbf{B}_r\ \ \mathbf{G}_{r+1}\rfloor'\{\mathbf{Z}_{r-1}\ \ \mathbf{Z}_r\ \ \mathbf{Z}_{r+1}\} + \lfloor \mathbf{C}_{r-1}\ \ \mathbf{D}_r\rfloor'\{\mathbf{F}_{r-1}\ \ \mathbf{F}_r\} = 0.$

Nodal unit	Displacement-matrices							Load-matrices						R.S.
	$\mathbf{Z}_{r-3}$	$\mathbf{Z}_{r-2}$	$\mathbf{Z}_{r-1}$	$\mathbf{Z}_r$	$\mathbf{Z}_{r+1}$	$\mathbf{Z}_{r+2}$	$\mathbf{Z}_{r+3}$	$\mathbf{F}_{r-3}$	$\mathbf{F}_{r-2}$	$\mathbf{F}_{r-1}$	$\mathbf{F}_r$	$\mathbf{F}_{r+1}$	$\mathbf{F}_{r+2}$	
$r-2$	$\mathbf{A}'_{r-3}$	$\mathbf{B}'_{r-2}$	$\mathbf{G}'_{r-1}$					$\mathbf{C}'_{r-3}$	$\mathbf{D}'_{r-2}$					$=0$
$r-1$		$\mathbf{A}'_{r-2}$	$\mathbf{B}'_{r-1}$	$\mathbf{G}'_r$					$\mathbf{C}'_{r-2}$	$\mathbf{D}'_{r-1}$				$=0$
r			$\mathbf{A}'_{r-1}$	$\mathbf{B}'_r$	$\mathbf{G}'_{r+1}$					$\mathbf{C}'_{r-1}$	$\mathbf{D}'_r$			$=0$
$r+1$				$\mathbf{A}'_r$	$\mathbf{B}'_{r+1}$	$\mathbf{G}'_{r+2}$					$\mathbf{C}'_r$	$\mathbf{D}'_{r+1}$		$=0$
$r+2$					$\mathbf{A}'_{r+1}$	$\mathbf{B}'_{r+2}$	$\mathbf{G}'_{r+3}$					$\mathbf{C}'_{r+1}$	$\mathbf{D}'_{r+2}$	$=0$

Table III. Shift Formulas.

Formula: $\mathbf{Z}_{r+1} = \lfloor \mathbf{A}_{r-1}\ \ \mathbf{B}_r\rfloor\{\mathbf{Z}_{r-1}\ \ \mathbf{Z}_r\} + \lfloor \mathbf{C}_{r-1}\ \ \mathbf{D}_r\rfloor\{\mathbf{F}_{r-1}\ \ \mathbf{F}_r\}.$

Nodal unit	Displacement-matrices							Load-matrices					
	$\mathbf{Z}_{r-3}$	$\mathbf{Z}_{r-2}$	$\mathbf{Z}_{r-1}$	$\mathbf{Z}_r$	$\mathbf{Z}_{r+1}$	$\mathbf{Z}_{r+2}$	$\mathbf{Z}_{r+3}$	$\mathbf{F}_{r-3}$	$\mathbf{F}_{r-2}$	$\mathbf{F}_{r-1}$	$\mathbf{F}_r$	$\mathbf{F}_{r+1}$	$\mathbf{F}_{r+2}$
$r-2$	$\mathbf{A}_{r-3}$	$\mathbf{B}_{r-2}$	$\mathbf{E}$					$\mathbf{C}_{r-3}$	$\mathbf{D}_{r-2}$				
$r-1$		$\mathbf{A}_{r-2}$	$\mathbf{B}_{r-1}$	$\mathbf{E}$					$\mathbf{C}_{r-2}$	$\mathbf{D}_{r-1}$			
r			$\mathbf{A}_{r-1}$	$\mathbf{B}_r$	$\mathbf{E}$					$\mathbf{C}_{r-1}$	$\mathbf{D}_r$		
$r+1$				$\mathbf{A}_r$	$\mathbf{B}_{r+1}$	$\mathbf{E}$					$\mathbf{C}_r$	$\mathbf{D}_{r+1}$	
$r+2$					$\mathbf{A}_{r+1}$	$\mathbf{B}_{r+2}$	$\mathbf{E}$					$\mathbf{C}_{r+1}$	$\mathbf{D}_{r+2}$

1. The nodal equilibrium conditions are written up by recurrent use of Eq. 29 as shown in Table II.
2. The premultiplication by $-[\boldsymbol{G}'_{ri}]^{-1}$ yields the shift formula as shown in Table III.

6. COMPLETE SHIFT FORMULA

It is roundabout to carry out the analysis of Vierendeel trusses with many panels by recurrent use of Eq. 37, because the eigenmatrices of three units always appear in this equation. In such a case, the following procedure may be preferable. Taking two rows from Table III out, the equilibrium conditions at nodal points of the r-th and $(r+1)$-th units are written as

$$\begin{bmatrix} \boldsymbol{A}_{r-1} & \boldsymbol{B}_r \\ 0 & \boldsymbol{A}_r \end{bmatrix} \begin{bmatrix} \boldsymbol{Z}_{r-1} \\ \boldsymbol{Z}_r \end{bmatrix} + \begin{bmatrix} -\boldsymbol{E} & 0 \\ \boldsymbol{B}_{r+1} & -\boldsymbol{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{Z}_{r+1} \\ \boldsymbol{Z}_{r+2} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_{r-1} & \boldsymbol{D}_r \\ 0 & \boldsymbol{C}_r \end{bmatrix} \begin{bmatrix} \boldsymbol{F}_{r-1} \\ \boldsymbol{F}_r \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \boldsymbol{D}_{r+1} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{F}_{r+1} \\ \boldsymbol{F}_{r+2} \end{bmatrix} = 0. \tag{39}$$

Then we obtain

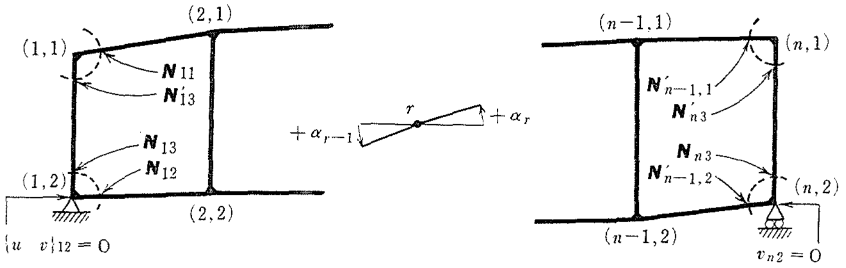
$$\begin{bmatrix} \boldsymbol{Z}_{r+1} \\ \boldsymbol{Z}_{r+2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{r-1} & \boldsymbol{B}_r \\ \boldsymbol{B}_{r+1}\boldsymbol{A}_{r-1} & \boldsymbol{B}_{r+1}\boldsymbol{B}_r + \boldsymbol{A}_r \end{bmatrix} \begin{bmatrix} \boldsymbol{Z}_{r-1} \\ \boldsymbol{Z}_r \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_{r-1} & \boldsymbol{D}_r \\ \boldsymbol{B}_{r+1}\boldsymbol{C}_{r-1} & \boldsymbol{B}_{r+1}\boldsymbol{D}_r + \boldsymbol{C}_r \end{bmatrix} \begin{bmatrix} \boldsymbol{F}_{r-1} \\ \boldsymbol{F}_r \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \boldsymbol{D}_{r+1} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{F}_{r+1} \\ \boldsymbol{F}_{r+2} \end{bmatrix}. \tag{40}$$

This is the complete shift formula. Eigenmatrices of the $(r-1)$-th and r-th units are at once shifted to adjacent $(r+1)$-th and $(r+2)$-th units. Although no additional treatments are herein necessary, the size of the shiftor is 12-by-12 and that of feeders is 12-by-48. This formula would be recommended when computers are available.

7. BOUNDARY CONDITIONS

In general, at the supports of a structural system, the nodal displacements in the direction of support are externally restrained, and then these displacements become known quantities or are represented by the given support conditions. Contrary to this, equilibrium conditions of forces in the direction of support can not be used for determination of the unknown nodal displacements in the structure. They can only contribute to the determination of corresponding support reactions.

From the viewpoint of shift operation between three units of regular form presented in the previous articles, boundary conditions are considered as one of branched treatments. As an illustration, the boundary conditions of Vierendeel truss in question are taken as shown in Fig. 5. The left end of the system is supported with hinge, and the right end with roller, respectively. They can be treated as follows.



(a) Left Boundary. (b) Right Boundary.

Fig. 5. Boundary Conditions.

7.1. Left Boundary Conditions.

The boundary conditions at the extreme left end consist of the following (Fig. 5a):

(i) One equilibrium condition of bending moment at the lower node (1, 2).

(ii) Two support conditions at the lower node (1, 2), which are expressed by the equation

$$u_{12} = v_{12} = 0. \tag{41}$$

(iii) Three force equilibrium conditions of horizontal forces, vertical forces, and bending moments at the upper node (1, 1).

Taking the support conditions given by Eq. 41 into account, the force equilibrium conditions, (i) and (iii), can be put into one matrix equation as follows:

$$\begin{bmatrix} j_3 & 0 & k_3 & 2(k_2+k_3) \\ & & & -j_3 \\ \multicolumn{3}{c}{(\boldsymbol{P}\boldsymbol{H}\overline{\boldsymbol{P}})_1 + (\boldsymbol{R}\boldsymbol{H}\overline{\boldsymbol{R}})_3} & 0 \\ & & & k_3 \end{bmatrix}_1 \begin{bmatrix} \mathbf{X}_1 \\ \theta_2 \end{bmatrix}_1$$

$$= -\begin{bmatrix} 0 & 0 & 0 & 0 & j_2 & k_2 \\ & (\boldsymbol{P}\ \boldsymbol{L}\ \overline{\boldsymbol{P}})_1 & & 0 & 0 & 0 \end{bmatrix}_1 \boldsymbol{Z}_2 - \begin{bmatrix} C_2 + C_3 + R_2 \\ \boldsymbol{P}_1\boldsymbol{K}_1 + \boldsymbol{R}\,\boldsymbol{K}'_3 + \boldsymbol{Q}_1 \end{bmatrix}_1. \tag{42}$$

The 4-by-4 square matrix on the left side of this equation is nonsingular, and then, the extreme left nodal displacements $\boldsymbol{Z}'_1$ can be reduced to the second eigenmatrix $\boldsymbol{Z}_2$ as follows:

$$\boldsymbol{Z}'_1 = \begin{bmatrix} \boldsymbol{X}_1 \\ \theta_2 \end{bmatrix}_1 = \boldsymbol{A}''_2\boldsymbol{Z}_2 + \boldsymbol{C}''_1\boldsymbol{F}_1. \tag{43}$$

7.2. Right Boundary Conditions.

At the extreme right nodal points $(n, 1)$ and $(n, 2)$, shown in Fig. 5b, the following boundary conditions are to be given:

(i) One support condition at the lower node $(n, 2)$, which is expressed by the equation

$$v_{n2} = 0. \tag{44}$$

(ii) Two force equilibrium conditions of horizontal forces and bending moments at the lower node $(n, 2)$.

(iii) Three force equilibrium conditions of horizontal forces, vertical forces, and bending moments at the upper node $(n, 1)$.

Considering the support condition given by Eq. 44, the above force equilibrium conditions, (ii) and (iii), can be expressed by the following matrix equation:

$$-\begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{L}_{n3}\boldsymbol{R}, & \begin{bmatrix} -f\cos^2\alpha + \dfrac{2j}{l}\sin^2\alpha & -j\sin\alpha \\ j\sin\alpha & 2k \end{bmatrix}_{n-1,2} + \begin{bmatrix} \dfrac{2j}{l} & j \\ -j & 2k \end{bmatrix}_{n3} \\ [(\boldsymbol{P}'\boldsymbol{H}\boldsymbol{R}'\overline{\boldsymbol{P}})_{n-1,1} + (\boldsymbol{R}\,\boldsymbol{H}_{n3}\overline{\boldsymbol{R}})], & \begin{bmatrix} -\dfrac{2j}{l} & -j \\ 0 & 0 \\ -j & k \end{bmatrix}_{n3} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_1 \\ u_2 \\ \theta_2 \end{bmatrix}_n$$

$$= \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ & \boldsymbol{P}'\,\boldsymbol{L}\,\boldsymbol{R}'\overline{\boldsymbol{P}} & \end{bmatrix}_1, & \begin{bmatrix} -\cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_2 (\boldsymbol{L}\,\boldsymbol{R}'\overline{\boldsymbol{P}})_2 \end{bmatrix} \end{bmatrix}_{n-1} \boldsymbol{Z}_{n-1}$$

$$+\begin{bmatrix} 0 & 0 & 0 & -\cos\alpha_2 & -\sin\alpha_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ & & & 0 & 0 & 0 \\ & \boldsymbol{P}'_1 & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{bmatrix}_{n-1} \begin{bmatrix} \boldsymbol{K}_1 \\ \boldsymbol{K}_2 \end{bmatrix}'_{n-1}$$

$$+\left[\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{K}_3 \\ \boldsymbol{K}'_3 \\ \boldsymbol{Q}_1 \\ \boldsymbol{Q}_2 \end{bmatrix}_n\right]. \tag{45}$$

The 5-by-5 square matrix on the left side of Eq. 45 is nonsingular, and then, the extreme right nodal displacements $\boldsymbol{Z}'_n$ can be represented as a function of the adjacent eigenmatrix $\boldsymbol{Z}_{n-1}$ of the form

$$\boldsymbol{Z}'_n = \begin{bmatrix} \boldsymbol{X}_1 \\ u_2 \\ \theta_2 \end{bmatrix}_n = \boldsymbol{B}''_{n-1}\boldsymbol{Z}_{n-1} + \boldsymbol{C}''_{n-1}\boldsymbol{F}_{n-1} + \boldsymbol{D}''_n\boldsymbol{F}_n. \tag{46}$$

8. SHIFT OPERATIONS

Referring to Eq. 43 and Table III, the shift formulas between the eigen-matrices of units at the left end of the structure are summarized in Table IV.

Table IV. Shift Formulas at Left End.

Nodal unit	Displacement-matrices				Load-matrices		
	$\boldsymbol{Z}'_1$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_3$	$\boldsymbol{Z}_4$	$\boldsymbol{F}_1$	$\boldsymbol{F}_2$	$\boldsymbol{F}_3$
1	$\boldsymbol{E}$	$\boldsymbol{A}''_2$			$\boldsymbol{C}''_1$		
2	$\boldsymbol{A}_1$	$\boldsymbol{B}_2$	$\boldsymbol{E}$		$\boldsymbol{C}_1$	$\boldsymbol{D}_2$	
3		$\boldsymbol{A}_2$	$\boldsymbol{B}_3$	$\boldsymbol{E}$		$\boldsymbol{C}_2$	$\boldsymbol{D}_3$

From this table, the rightward shift operation starting from the second eigenmatrix $\boldsymbol{Z}_2$ can be written down as shown in the equations below, in which the currency of the $\boldsymbol{Z}_2$ matrix to the succeeding eigenmatrices is observed. Then the $\boldsymbol{Z}_2$ matrix is called as the "current-matrix" in the present shift operation, and will be represented by $\boldsymbol{\Omega}$.

$$\boldsymbol{Z}_3 = \boldsymbol{U}_3\boldsymbol{Z}_2 + \lfloor \boldsymbol{V}_1 \quad \boldsymbol{V}_2 \rfloor_2\{\boldsymbol{F}_1 \quad \boldsymbol{F}_2\} = \boldsymbol{U}_3\boldsymbol{\Omega} + \lfloor \boldsymbol{V} \rfloor_2\{\boldsymbol{F}\}_2, \tag{47}$$

$$\boldsymbol{Z}_4 = \boldsymbol{U}_4\boldsymbol{Z}_2 + \lfloor \boldsymbol{V}_1 \quad \boldsymbol{V}_2 \quad \boldsymbol{V}_3 \rfloor_3\{\boldsymbol{F}_1 \quad \boldsymbol{F}_2 \quad \boldsymbol{F}_3\} = \boldsymbol{U}_4\boldsymbol{\Omega} + \lfloor \boldsymbol{V} \rfloor_3\{\boldsymbol{F}\}_3. \tag{48}$$

Here the following symbols have been used:

$$\boldsymbol{U}_3 = \boldsymbol{A}_1\boldsymbol{A}''_2 + \boldsymbol{B}_2, \quad \boldsymbol{U}_4 = \boldsymbol{A}_2 + \boldsymbol{B}_3\boldsymbol{U}_3, \tag{49}$$

$$\boldsymbol{V}_{21} = \boldsymbol{A}_1\boldsymbol{C}''_1 + \boldsymbol{C}_1, \quad \boldsymbol{V}_{22} = \boldsymbol{D}_2, \tag{50}$$

$$\lfloor \boldsymbol{V}_1 \quad \boldsymbol{V}_2 \quad \boldsymbol{V}_3 \rfloor_3 = \lfloor \boldsymbol{B}_3\boldsymbol{V}_{21} \quad \boldsymbol{B}_3\boldsymbol{V}_{22} + \boldsymbol{C}_2 \quad \boldsymbol{D}_3 \rfloor. \tag{51}$$

Thus the generalized form of shift operation of the current-matrix $\boldsymbol{\Omega}$ to the r-th unit is given as follows:

$$\begin{aligned}\boldsymbol{Z}_r &= \boldsymbol{U}_r\boldsymbol{\Omega} + \lfloor \boldsymbol{V}_1 \quad \boldsymbol{V}_2 \quad \cdots \quad \boldsymbol{V}_{r-1} \rfloor_{r-1}\{\boldsymbol{F}_1 \quad \boldsymbol{F}_2 \quad \cdots \quad \boldsymbol{F}_{r-1}\} \\ &= \boldsymbol{U}_r\boldsymbol{\Omega} + \lfloor \boldsymbol{V} \rfloor_{r-1}\{\boldsymbol{F}\}_{r-1}.\end{aligned} \tag{52}$$

This is the desired recurrence formula for the present structure.

By the recurrent use of this equation, the eigenmatrix $\boldsymbol{Z}_{n-1}$, adjacent to the extreme right unit of the structure, is given as follows:

$$\boldsymbol{Z}_{n-1} = \boldsymbol{U}_{n-1}\boldsymbol{\Omega} + \lfloor \boldsymbol{V}_1 \quad \boldsymbol{V}_2 \quad \cdots \quad \boldsymbol{V}_{n-2} \rfloor_{n-2}\{\boldsymbol{F}_1 \quad \boldsymbol{F}_2 \quad \cdots \quad \boldsymbol{F}_{n-2}\}. \tag{53}$$

The equilibrium conditions at nodal points at the right end of the structure is summarized in Table V.

Table V. Nodal Equilibrium Conditions at Right End.

Nodal unit	Displacement-matrices			Load-matrices			R.S.
	$\boldsymbol{Z}_{n-2}$	$\boldsymbol{Z}_{n-1}$	$\boldsymbol{Z}'_n$	$\boldsymbol{F}_{n-2}$	$\boldsymbol{F}_{n-1}$	$\boldsymbol{F}_n$	
$n-1$	$\boldsymbol{A}'_{n-2}$	$\boldsymbol{B}'_{n-1}$	$\boldsymbol{G}'_n$	$\boldsymbol{C}'_{n-2}$	$\boldsymbol{D}'_{n-1}$		$=0$
n		$\boldsymbol{B}''_{n-1}$	$-\boldsymbol{E}$		$\boldsymbol{C}''_{n-1}$	$\boldsymbol{D}''_n$	$=0$

Referring to this table and Eq. 53, the extreme left nodal displacements $\boldsymbol{Z}'_n$ are given by the form

$$\begin{aligned}\boldsymbol{Z}'_n = \boldsymbol{B}''_{n-1}\big[\boldsymbol{U}_{n-1}\boldsymbol{\Omega} + \lfloor \boldsymbol{V}_1 \quad \cdots \quad \boldsymbol{V}_{n-2} \rfloor_{n-2}\{\boldsymbol{F}_1 \quad \cdots \quad \boldsymbol{F}_{n-2}\}\big] \\ + \lfloor \boldsymbol{C}''_{n-1} \quad \boldsymbol{D}''_n \rfloor\{\boldsymbol{F}_{n-1} \quad \boldsymbol{F}_n\},\end{aligned} \tag{54}$$

or

$$\boldsymbol{Z}'_n = \boldsymbol{U}'_n\boldsymbol{\Omega} + \lfloor \boldsymbol{V}_1 \quad \boldsymbol{V}_2 \quad \cdots \quad \boldsymbol{V}_n \rfloor'_n \{\boldsymbol{F}_1 \quad \boldsymbol{F}_2 \quad \cdots \quad \boldsymbol{F}_n\}. \tag{55}$$

Then the equilibrium conditions at the $(n-1)$-th nodal points can be written

$$\begin{aligned}&\big[\boldsymbol{A}'_{n-2}\boldsymbol{U}_{n-2} + \boldsymbol{B}'_{n-1}\boldsymbol{U}_{n-1} + \boldsymbol{G}'_n\boldsymbol{U}'_n\big]\boldsymbol{\Omega}\\ &\quad + \big[\boldsymbol{A}'_{n-2}\lfloor \boldsymbol{V}_1 \quad \boldsymbol{V}_2 \quad \cdots \quad \boldsymbol{V}_{n-3} \quad 0 \quad 0 \quad 0 \rfloor_{n-3}\\ &\qquad + \boldsymbol{B}'_{n-1}\lfloor \boldsymbol{V}_1 \quad \boldsymbol{V}_2 \quad \cdots \quad \boldsymbol{V}_{n-2} \quad 0 \quad 0 \rfloor_{n-2} + \boldsymbol{G}'_n\lfloor \boldsymbol{V}_1 \quad \boldsymbol{V}_2 \quad \cdots \quad \boldsymbol{V}_n \rfloor'_n\\ &\qquad\quad + \lfloor 0 \quad 0 \quad \cdots \quad 0 \quad \boldsymbol{C}'_{n-2} \quad \boldsymbol{D}'_{n-1} \quad 0 \rfloor\big]\{\boldsymbol{F}_1 \quad \boldsymbol{F}_2 \quad \cdots \quad \boldsymbol{F}_n\} = 0,\end{aligned} \tag{56}$$

or

$$\boldsymbol{U}_n\boldsymbol{\Omega} + \lfloor \boldsymbol{V}_1 \quad \boldsymbol{V}_2 \quad \cdots \quad \boldsymbol{V}_n \rfloor_n \{\boldsymbol{F}_1 \quad \boldsymbol{F}_2 \quad \cdots \quad \boldsymbol{F}_n\} = 0. \tag{57}$$

From this equation, the current-matrix $\boldsymbol{\Omega}$ is determined as follows:

$$\boldsymbol{\Omega} = -\boldsymbol{U}_n^{-1}\lfloor \boldsymbol{V}_1 \quad \boldsymbol{V}_2 \quad \cdots \quad \boldsymbol{V}_n \rfloor_n \{\boldsymbol{F}_1 \quad \boldsymbol{F}_2 \quad \cdots \quad \boldsymbol{F}_n\}, \tag{58}$$

and hence the present system can be solved.

9. EXTENDED APPLICATION

The preceding discussions can also be extended to the analysis of the rib arches or Lohse trusses, whose upper and lower chords are subjected to both the axial and bending stresses, while the vertical members can experience only the axial force. A part of such a system is shown in Fig. 6, wherein the physical matrices to be treated in the subsequent discussions are illustrated. In this case, the physical matrices of upper and lower chord members are also given by Eqs. 20 through 23. On the other hand, the force-matrices of vertical member $(r, 3)$ are given as follows:

$$\boldsymbol{n}_{r3} = \{F \quad 0 \quad 0\}_{r3}, \qquad \boldsymbol{n}'_{r3} = \{F \quad 0 \quad 0\}'_{r3} \tag{59}$$

Referring to Eqs. 24 and 25, Eqs. 59 become

$$\begin{aligned}\boldsymbol{n}_{r3} &= \boldsymbol{S}\lfloor \boldsymbol{H} \quad \boldsymbol{L} \rfloor_{r3}\mathrm{diag}[\boldsymbol{R} \quad \boldsymbol{R}]\{\boldsymbol{X}_2 \quad \boldsymbol{X}_1\}_r + \boldsymbol{S}\boldsymbol{K}_{r3}\\ &= \begin{bmatrix}0 & f & 0 & 0 & -f & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}_{r3}\begin{bmatrix}\boldsymbol{X}_2\\ \boldsymbol{X}_1\end{bmatrix}_r,\end{aligned} \tag{60}$$

$$\boldsymbol{n}'_{r3} = \boldsymbol{S}\lfloor \boldsymbol{L} \quad \boldsymbol{H} \rfloor_{r3}\mathrm{diag}[\bar{\boldsymbol{R}} \quad \bar{\boldsymbol{R}}]\{\boldsymbol{X}_2 \quad \boldsymbol{X}_1\}_r + \boldsymbol{S}\boldsymbol{K}'_{r3}$$

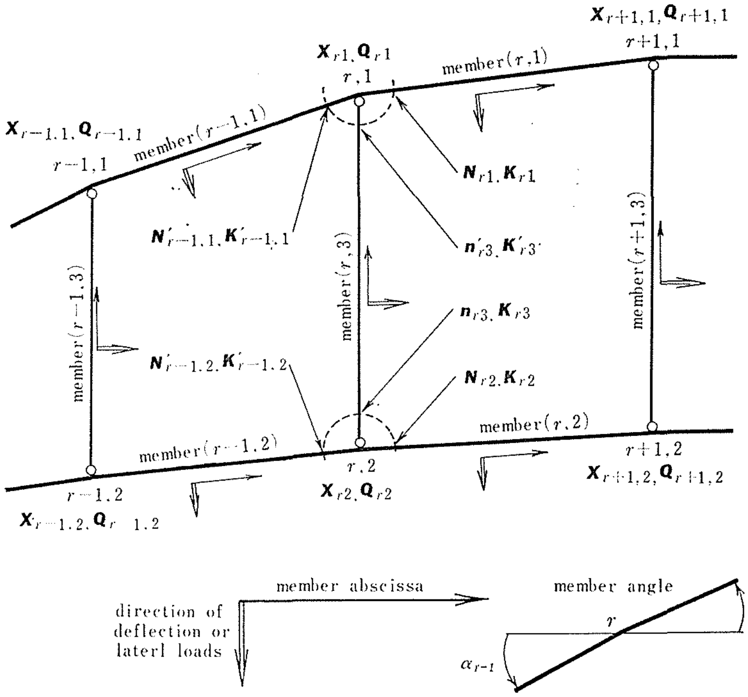


Fig. 6. Part of Rib Arch and Lohse Truss.

$$= \begin{bmatrix} 0 & f & 0 & 0 & -f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{r3} \begin{bmatrix} \mathbf{X}_2 \\ \mathbf{X}_1 \end{bmatrix}_r + \begin{bmatrix} \mathfrak{F} \\ 0 \\ 0 \end{bmatrix}_{r3}. \tag{61}$$

Here, $\mathbf{S}$ is designated as the selection operator or briefly the "selector," which takes the form

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{62}$$

Then the previous nodal equilibrium conditions, Eqs. 17 and 19, must be rewritten in the forms

$$\mathbf{P}'_{r-1,1}\mathbf{N}'_{r-1,1} + \mathbf{P}_{r1}\mathbf{N}_{r1} + \mathbf{RSN}'_{r3} + \mathbf{Q}_{r1} = 0, \tag{63}$$

$$\mathbf{P}'_{r-1,2}\mathbf{N}'_{r-1,2} + \mathbf{P}_{r2}\mathbf{N}_{r2} + \bar{\mathbf{R}}\mathbf{SN}_{r3} + \mathbf{Q}_{r2}] = 0. \tag{64}$$

In virtue of these equilibrium conditions, the shift formula between the eigenmatrices at the $(r-1)$-, (r)-, and $(r+1)$-th units can be obtained in the following form:

$$\boldsymbol{Z}_{r+1} = \boldsymbol{A}_{r-1}\boldsymbol{Z}_{r-1} + \boldsymbol{b}_r\boldsymbol{Z}_r + \boldsymbol{C}_{r-1}\boldsymbol{F}_{r-1} + \boldsymbol{d}_r\boldsymbol{F}_r. \tag{65}$$

Here, the operators $\boldsymbol{A}_{r-1}$ and $\boldsymbol{C}_{r-1}$ are the same as those of Eq. 38, while the operators $\boldsymbol{b}_r$ and $\boldsymbol{d}_r$ take the form

$$\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{d} \end{bmatrix}_r = -\boldsymbol{G}'^{-1}_{r+1}\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{d} \end{bmatrix}'_r, \tag{66}$$

in which

$$\boldsymbol{b}'_r = \begin{bmatrix} (\boldsymbol{P}'\boldsymbol{H}\boldsymbol{R}'\overline{\boldsymbol{P}})_{r-1,1} + (\boldsymbol{P}\boldsymbol{H}\overline{\boldsymbol{P}})_{r1} + \boldsymbol{R}\boldsymbol{S}\boldsymbol{H}_{r3}\overline{\boldsymbol{R}}, & \boldsymbol{R}\boldsymbol{S}\boldsymbol{L}_{r3}\overline{\boldsymbol{R}} \\ \overline{\boldsymbol{R}}\boldsymbol{S}\boldsymbol{L}_{r3}\boldsymbol{R}, & (\boldsymbol{P}'\boldsymbol{H}\boldsymbol{R}'\overline{\boldsymbol{P}})_{r-1,2} + (\boldsymbol{P}\boldsymbol{H}\overline{\boldsymbol{P}})_{r2} + \overline{\boldsymbol{R}}\boldsymbol{S}\boldsymbol{H}_{r3}\boldsymbol{R} \end{bmatrix}, \tag{67}$$

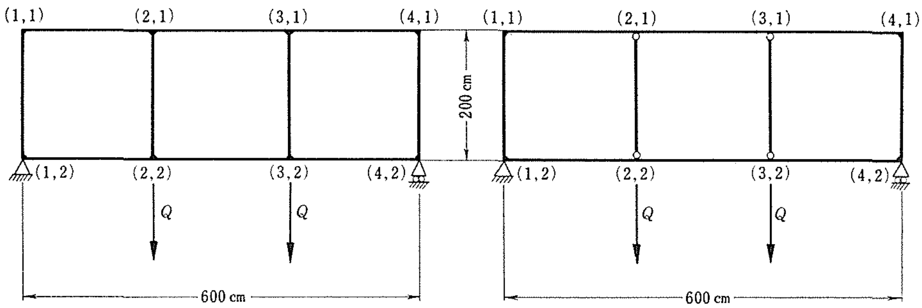
$$\boldsymbol{d}'_r = \begin{bmatrix} \boldsymbol{P}_1 & 0 & 0 & 0 & 0 & \boldsymbol{R}\boldsymbol{S} & \boldsymbol{E} & 0 \\ 0 & 0 & \boldsymbol{P}_2 & 0 & \overline{\boldsymbol{R}}\boldsymbol{S} & 0 & 0 & \boldsymbol{E} \end{bmatrix}_r. \tag{68}$$

Therefore, the shift operation of the present system can also be carried out by the recurrent use of Eq. 65.

The boundary conditions and the determination of the current-matrix will be treated in a similar manner.

10. ILLUSTRATIVE EXAMPLE

As an illustration of the preceding analyses, the three panel orthogonal trusses of Vierendeel type and of Lohse type will be referred to, as shown in Fig. 7. In this case, both trusses have the same geometry, every member of which has the dimension 10cm × 10cm × 200cm. For simplicity, the vertical loads Q's are applied symmetrically at the nodal points (2, 2) and (3, 2). The solutions of these trusses are summarized in Table VI, wherein the symmetry of nodal displacements in each truss is to be observed, and the magnitudes of corresponding nodal displacements in both trusses are to be compared with.



(a) Vierendeel truss (b) Lohse truss

Fig. 7. Three Panel Orthogonal Rigid Trusses.

Table VI. Nodal Displacements of Vierendeel and Lohse Trusses
($\times Q$(nodal load)$/E$(Young's modulus)).

Vierendeel truss			Node	Lohse truss		
u	v	θ		u	v	θ
3.674612	0.999205	1.948241	11	3.597302	0.999375	2.412440
2.707634	750.8389	1.553336	21	2.398201	1122.158	4.801648
0.966978	750.8389	−1.553336	31	1.199101	1122.158	−4.801648
0.000000	0.999205	−1.948241	41	0.000000	0.999375	−2.412440
0.000000	0.000000	1.956416	12	0.000000	0.000000	2.419936
0.966978	751.8381	1.558786	22	1.199101	1123.158	4.809144
2.707634	751.8381	−1.558786	32	2.398201	1123.158	−4.809144
3.674612	0.000000	−1.956416	42	3.597302	0.000000	−2.419936

11. CONCLUSIONS

In the present approach to the analysis of Vierendeel truss systems, the following notes are to be given:

1. The flexural and axial behavior of a constituent member are given by the prevailing slope-deflection formula and Hooke's law, respectively.

2. The physical quantities of a member are represented by two kinds of 3-by-1 matrices: the force-matrix and the displacement-matrix. The force-matrix of a member shows the perfect dependency on the displacement-matrices at member ends.

3. The single shift formula between the displacement-matrices for three

consecutive units can be obtained from nodal equilibrium conditions at the middle unit. The complete shift formula for group consisting of two adjacent units can also be derived from assembling two single shift formulas.

4. In practical systems of Vierendeel truss type, the number of nodal displacement components is always equal to that of nodal equilibrium conditions, and therefore, the system can be analyzed completely by recurrent use of the shift formula and treatment of given boundary conditions.

5. The sizes of operational matrices are as follows:
The current-matrix: 6-by-1. The shiftors: 6-by-6 for single shifting, and 12-by-12 for compelte shifting. The feeders: 6-by-24 for single shifting, and 12-by-48 for complete shifting. Inverse matrix for the boundary treatment: from 1-by-1 to 6-by-6. Inverse matrix for determination of the current-matrix : 6-by-6.

6. In virtue of a little modification, the procedures for the Vierendeel truss analysis can at once be extended to the analysis of the rib arch, Lohse truss, and other similar structures.

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APPENDIX. —NOTATION

The following symbols have been used in this paper;

A = cross-sectional area ;
$\boldsymbol{A}$ = shift operator in single shifting ;
$\boldsymbol{B}$ = shift operator in single shifting ;
$\boldsymbol{b}$ = shift operator of rib arch systems ;
C = load term of the slope-deflection method, see Eqs. 14 and 15 ;
$\boldsymbol{C}$ = feed operator in single shifting ;

$\mathbf{D}$ = feed operator in single shifting;
$\mathbf{d}$ = feed operator of rib arch systems;
E = Young's Modulus;
$\mathbf{E}$ = unit matrix;
F = axial force;
$\mathfrak{F}$ = external axial load;
$\mathbf{F}_r$ = load-matrix of the r-th unit;
f = member stiffness, see Eq. 12;
$\mathbf{H}$ = operational matrix, see Eqs. 8 and 10;
I = moment of inertia;
j = member stiffness, see Eq. 12;
$\mathbf{K}$ = member load-matrix, see Eqs. 14 and 15;
k = member stiffness, see Eq. 12;
$\mathbf{L}$ = operational matrix, see Eqs. 9 and 11;
l = member length;
M = bending moment;
$\mathbf{N}$ = force-matrix, see Eqs. 1 and 2;
$\mathbf{n}$ = force-matrix of vertical member of rib arch systems;
P = horizontal nodal load, see Fig. 3;
$\mathbf{P}, \mathbf{P}', \overline{\mathbf{P}}$ = projection matrices, see Eqs. 17, 19 and 28;
Q = vertical nodal load, see Fig. 3;
$\mathbf{Q}$ = nodal load-matrix;
R = external nodal moment, see Fig. 3;
$\mathbf{R}, \mathbf{R}', \overline{\mathbf{R}}$ = rearrangement-matrices, see Eqs. 27;
r = integer denoting the order of node, member or unit;
S = shearing force;
$\mathbf{S}$ = selection matrix, see Eq. 62;
$\mathbf{U}$ = operational matrix, see Eqs. 49;
u = horizontal displacement of nodal point;
V = load-term of the slope-deflection method, see Eqs. 14 and 15;
$\mathbf{V}$ = operational matrix, see Eqs. 50,
v = vertical displacement of nodal point;
$\mathbf{X}$ = displacement-matrix, see Eqs. 3;
$\mathbf{Z}_r$ = eigenmatrix of the r-th unit;
α = angle of member inclination;
θ = slope angle at member end;
$\boldsymbol{\Omega}$ = current-matrix;
$\lfloor \quad \rfloor$ = row vector; and
$\{ \quad \}$ = column vector.