Operational Method for Displacement Analysis
First Report
Slope-Deflection Method for Rigid Frames

Norikazu YOSHIZAWA* and Bennosuke TANIMOTO**

(Received March 28, 1967)

1. INTRODUCTION

The slope-deflection method was first proposed by W. Wilson in 1915, and has been widely used for the analysis of rigid frames. It usually ignores the member elongation, and only the bending behavior is transmitted through nodal points by assuming the rigid connection. This is sufficiently correct for simple structures such as portal frames, but complex structures with many bays or stories should call for further scrutiny.

The prevailing procedure of the method is first to exhibit a system of equilibrium moment, and shear equations, the order of which increases with the number of bays or stories. F. Takabeya gave a systematic tabulation of the simultaneous equations which is widely adopted for use in Japan. In order to solve it, however, the iterative procedure has been usually recommended, so that time and labor rapidly increases with complexity in structures.

The present paper shows an operational procedure for topological configuration of structures by introducing the concept of sections or units, wherein a recurrence formula can be derived and no simultaneous equations are necessary.[1)] This can be attained merely by a due rearrangement of rows and columns of the large size coefficient matrix of simultaneous equations, which results in a tridiagonal matrix. However, no exhibition of the tridiagonal matrix is necessary, and hence it will be not given herein. The approach proposed then will result in saving time and labor for treating rigid frames consisting of many bays or stories. It is to be added here that the orthodox

* Assistant of Civil Engineering, Faculty of Engineering, Shinshu University, Nagano, Japan.
** Professor of Civil Engineering, Faculty of Engineering, Shinshu University, Nagano, Japan.

operational procedure, allowing the member elongation, as well as its flexure, can be formulated directly for rigid frames by beginning with general solutions for governing differential equations.

2. BASIC CONCEPTS

A part of a plane rigid frame is shown in Fig. 1, wherein several symbols of physical quantities to be adopted for use in the subsequent discussions and the forces at member ends and their positive directions are illustrated. By the usual assumption of inextensibility for members, the prevailing slope-deflection equations[2] are as follows:

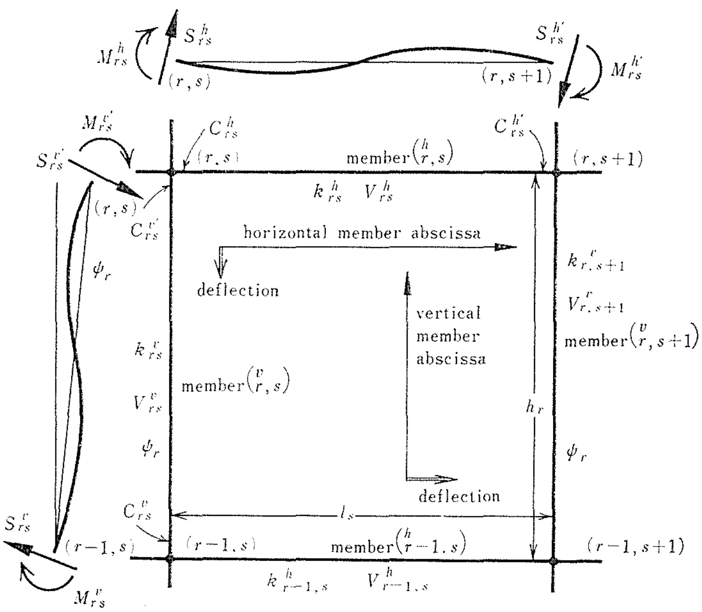

Fig. 1. A Part of Plane Rigid Frames.

For the horizontal member $\binom{h}{r,s}$:

$$\begin{bmatrix} M \\ M' \end{bmatrix}_{rs}^{h} = k_{rs}^{h} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \varphi_s \\ \varphi_{s+1} \end{bmatrix}_r + \begin{bmatrix} C \\ C' \end{bmatrix}_{rs}^{h}, \tag{1}$$

$$\begin{bmatrix} S \\ S' \end{bmatrix}_{rs}^{h} = \begin{bmatrix} V \\ V' \end{bmatrix}_{rs}^{h} - \frac{3k_{rs}^{h}}{l_s} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi_s \\ \varphi_{s+1} \end{bmatrix}_r - \frac{1}{l_s} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ C' \end{bmatrix}_{rs}^{h}. \tag{2}$$

For the vertical member $\binom{v}{r,s}$):

$$\begin{bmatrix} M \\ M' \end{bmatrix}_{rs}^{v} = k_{rs}^{v} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \varphi_{r-1,s} \\ \varphi_{rs} \end{bmatrix} + k_{rs}^{v} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \psi_r + \begin{bmatrix} C \\ C' \end{bmatrix}_{rs}^{v}, \tag{3}$$

$$\begin{bmatrix} S \\ S' \end{bmatrix}_{rs}^{v} = \begin{bmatrix} V \\ V' \end{bmatrix}_{rs}^{v} - \frac{3k_{rs}^{v}}{h_r} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{r-1,s} \\ \varphi_{rs} \end{bmatrix} - \frac{2k_{rs}^{v}}{h_r} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \psi_r - \frac{1}{h_r} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ C' \end{bmatrix}_{rs}^{v}. \tag{4}$$

In these equations, the following symbols are used:

$$k_{rs}^{h} = \frac{K_{rs}^{h}}{K_0}, \qquad k_{rs}^{v} = \frac{K_{rs}^{v}}{K_0}, \tag{5}$$

$$K_{rs}^{h} = \frac{I_{rs}^{h}}{l_s}, \qquad K_{rs}^{v} = \frac{I_{rs}^{v}}{h_r}, \qquad K_0 = \frac{I_0}{l_0}, \tag{6}$$

$$\varphi_{r-1,s} = 2EK_0\theta_{r-1,s}, \quad \varphi_{rs} = 2EK_0\theta_{rs}, \quad \varphi_{r,s+1} = 2EK_0\theta_{r,s+1}, \tag{7}$$

$$\psi_r = -6EK_0R_r. \tag{8}$$

Here k_{rs}^{h}, k_{rs}^{v} = the rigidity ratios, K_{rs}^{h}, K_{rs}^{v} = the member rigidities, K_0 = the standard rigidity, I_{rs}^{h}, I_{rs}^{v} = the moments of inertia, l_s, h_r = the member lengths, E = Young's modulus, $\theta_{r-1,s}$, θ_{rs}, $\theta_{r,s+1}$ = the slope angles, R_r = the member rotation angle, M_{rs}^{h}, $M_{rs}^{h'}$, M_{rs}^{v}, $M_{rs}^{v'}$ = the end moments, S_{rs}^{h}, $S_{rs}^{h'}$, S_{rs}^{v}, $S_{rs}^{v'}$ = the end shears, C_{rs}^{h}, $C_{rs}^{h'}$, C_{rs}^{v}, $C_{rs}^{v'}$ = the load terms for end moments, and V_{rs}^{h}, $V_{rs}^{h'}$, V_{rs}^{v}, $V_{rs}^{v'}$ = the load terms for shearing forces, provided the primed quantities represent end $x = l_s$ or h_r, and the unprimed quantities end $x = 0$.

3. EQUILIBRIUM CONDITIONS

A three-span multistory rigid frame will for example be treated. In Fig. 2 is shown a part of such a system. The concept of the unit will be introduced in the subsequent discussions, which is composed of nodal points on the horizontal line, intervening horizontal members between these points, and vertical members connecting their lower ends with these nodal points. The moment equations at nodal points of the (r–1)-th unit, and the shear equation for the upper portion of the structure cut out by the r-th plane can be put into the following matrix equation:

$$\begin{bmatrix} k_0 & 0 & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & 0 \\ 0 & 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{r-1}^{v} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \psi \end{bmatrix}_{r-2} + \begin{bmatrix} j_{r-1,0} & k^h_{r-1,0} & 0 & 0 & k^v_{r-1,0} \\ k^h_{r-1,0} & j_{r-1,1} & k^h_{r-1,1} & 0 & k^v_{r-1,1} \\ 0 & k^h_{r-1,1} & j_{r-1,2} & k^h_{r-1,2} & k^v_{r-1,2} \\ 0 & 0 & k^h_{r-1,2} & j_{r-1,3} & k^v_{r-1,3} \\ k^v_{r0} & k^v_{r1} & k^v_{r2} & k^v_{r3} & 0 \end{bmatrix} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \psi \end{bmatrix}_{r-1}$$

$$+ \begin{bmatrix} k_0 & 0 & 0 & 0 & k_0 \\ 0 & k_1 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 & k_2 \\ 0 & 0 & 0 & k_3 & k_3 \\ k_0 & k_1 & k_2 & k_3 & f \end{bmatrix}_{r}^{v} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \psi \end{bmatrix}_{r} + \boldsymbol{K}_{r-1} = 0, \tag{9}$$

or

$$\boldsymbol{L}'_{r-2}\boldsymbol{X}_{r-2} + \boldsymbol{M}'_{r-1}\boldsymbol{X}_{r-1} + \boldsymbol{N}_r\boldsymbol{X}_r + \boldsymbol{K}_{r-1} = 0. \tag{10}$$

Here, the following constants and matrices are to be defined:

Nodal constant:

$$j_r = 2\sum(\text{rigidity ratios of members connected at the } r\text{-th node}). \tag{11}$$

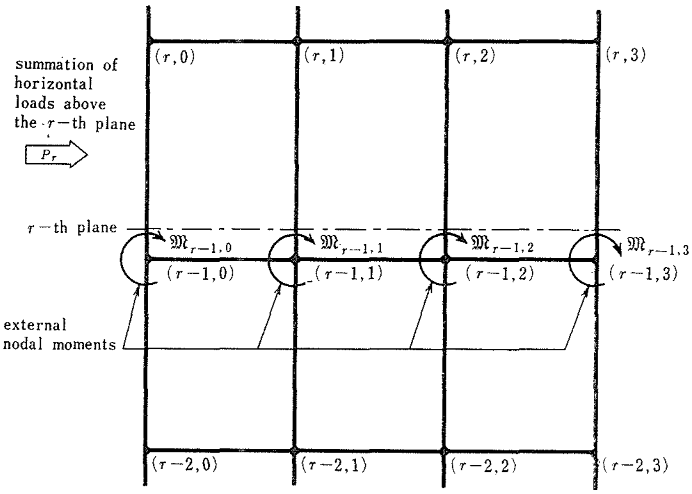

Fig. 2. Part of the Three Span Multi-Story Rigid Frame.

Story constant:

$$f_r^v = \frac{2}{3}\sum(\text{rigidity ratios of vertical members of the } r\text{-th unit}). \tag{12}$$

Load-matrix:

$$\boldsymbol{K}_{r-1} = \begin{bmatrix} C_{r-1,0}^{v'} + C_{r-1,0}^{h} + C_{r0}^{v} - \mathfrak{M}_{r-1,0} \\ C_{r-1,0}^{h'} + C_{r-1,1}^{v'} + C_{r-1,1}^{h} + C_{r1}^{v} - \mathfrak{M}_{r-1,1} \\ C_{r-1,1}^{h'} + C_{r-1,2}^{v'} + C_{r-1,2}^{h} + C_{r2}^{v} - \mathfrak{M}_{r-1,2} \\ C_{r-1,2}^{h'} + C_{r-1,3}^{v'} + C_{r3}^{v} - \mathfrak{M}_{r-1,3} \\ \frac{1}{3}\Big[(C_0 + C'_0 + C_1 + C'_1 + C_2 + C'_2 + C_3 + C'_3)_r^v \\ \qquad - h_r(V_0 + V_1 + V_2 + V_3)_r^v + h_r P_r\Big] \end{bmatrix}. \tag{13}$$

Eigenmatrix:

$$\boldsymbol{X}_r = \{\varphi_0 \quad \varphi_1 \quad \varphi_2 \quad \varphi_3 \quad \psi\}_r. \tag{14}$$

4. SHIFT FORMULAS

In virtue of Eq. 9 or 10, the eigenmatrix of the r-th unit can be represented by the eigenmatrices of the two lower units as follows:

$$\boldsymbol{X}_r = \boldsymbol{L}_{r-2}\boldsymbol{X}_{r-2} + \boldsymbol{M}_{r-1}\boldsymbol{X}_{r-1} + \boldsymbol{F}_r\boldsymbol{K}_{r-1}, \tag{15}$$

in which $\boldsymbol{L}_{r-2}$ and $\boldsymbol{M}_{r-1}$ are designated as the "shift operators," and $\boldsymbol{F}_r$ as the feed operator to the eigenmatrix $\boldsymbol{X}_r$, since the eigenmatrices $\boldsymbol{X}_{r-2}$ and $\boldsymbol{X}_{r-1}$ are to be shifted to $\boldsymbol{X}_r$ by the shift operators, and the load-matrix $\boldsymbol{K}_{r-1}$ is to be fed to $\boldsymbol{X}_r$ by the feed operator, respectively. They are given as follows:

$$\boldsymbol{L}_{r-2} = -\boldsymbol{N}_r^{-1}\boldsymbol{L}'_{r-2}, \tag{16}$$

$$\boldsymbol{M}_{r-1} = -\boldsymbol{N}_r^{-1}\boldsymbol{M}'_{r-1}, \tag{17}$$

$$\boldsymbol{F}_r = -\boldsymbol{N}_r^{-1}. \tag{18}$$

Here the matrix $\boldsymbol{N}_r^{-1}$ is square and nonsingular, and its inverse is given by

$$\boldsymbol{N}_r^{-1} = -\frac{2}{f_r^v}\begin{bmatrix} 1-\dfrac{f}{2k_0} & 1 & 1 & 1 & -1 \\ 1 & 1-\dfrac{f}{2k_1} & 1 & 1 & -1 \\ 1 & 1 & 1-\dfrac{f}{2k_2} & 1 & -1 \\ 1 & 1 & 1 & 1-\dfrac{f}{2k_3} & -1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}_r^v. \tag{19}$$

In this way, the above operators can be evaluated.

5. LOWER BOUNDARY CONDITION.

The lower boundary condition is given by the shear equation at the first unit. Assuming that all the lower ends of vertical members of the first unit are built in a rigid foundation, the shear equation of this unit becomes

$$\lfloor k_0 \quad k_1 \quad k_2 \quad k_3 \quad f \rfloor_1^v \{\varphi_0 \quad \varphi_1 \quad \varphi_2 \quad \varphi_3 \quad \psi\}_1 + \lfloor 0 \quad 0 \quad 0 \quad 0 \quad 1 \rfloor \boldsymbol{K}_0 = 0, \tag{20}$$

provided the load-matrix $\boldsymbol{K}_0$ is in this case given by

$$\boldsymbol{K}_0 = \left\{0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{3}\left[(C_0 + C'_0 + C_1 + C'_1 + C_2 + C'_2 + C_3 + C'_3)_1^v - h_1(V_0 + V_1 + V_2 + V_3)_1^v + h_1 P_1\right]\right\}. \tag{21}$$

Then the order of the eigenmatrix $\boldsymbol{X}_1$ can be degraded to 4-by-1 as follows:

$$\boldsymbol{X}_1 = \boldsymbol{U}_1 \boldsymbol{\Omega} + \boldsymbol{F}_1 \boldsymbol{K}_0, \tag{22}$$

in which

$$\boldsymbol{U}_1 = \frac{1}{f_1^v}\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & f \\ -k_0 & -k_1 & -k_2 & -k_3 \end{bmatrix}_1^v, \quad \boldsymbol{\Omega} = \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}_1, \quad \boldsymbol{F}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{1}{f} \end{bmatrix}_1^v. \tag{23}$$

Here $\boldsymbol{\Omega}$ is designated as the "current-matrix," which will become current to all the units of the structure considered.

6. SHIFT OPERATIONS

Eq. 15 holds for three arbitrary consecutive units in a structure, and is called the "recurrence formula," or the "shift formula." By the recurrent use of this equation, all the eigenmatrices of the structure can be completely represented by the current-matrix $\boldsymbol{\Omega}$ as follows:

$$\boldsymbol{X}_1 = \boldsymbol{U}_1\boldsymbol{\Omega} + \boldsymbol{F}_1\boldsymbol{K}_0, \tag{24}$$

$$\boldsymbol{X}_2 = \boldsymbol{M}_1\boldsymbol{U}_1\boldsymbol{\Omega} + \boldsymbol{M}_1\boldsymbol{F}_1\boldsymbol{K}_0 + \boldsymbol{F}_2\boldsymbol{K}_1 = \boldsymbol{U}_2\boldsymbol{\Omega} + \lfloor \boldsymbol{V}_0 \quad \boldsymbol{V}_1 \rfloor_2 \{\boldsymbol{K}_0 \quad \boldsymbol{K}_1\}, \tag{25}$$

$$\cdots\cdots,$$

$$\begin{aligned}
\boldsymbol{X}_r &= \left[\boldsymbol{L}_{r-2}\boldsymbol{U}_{r-2} + \boldsymbol{M}_{r-1}\boldsymbol{U}_{r-1}\right]\boldsymbol{\Omega} \\
&\quad + \left[\boldsymbol{L}_{r-2}\boldsymbol{V}_{r-2,0} + \boldsymbol{M}_{r-1}\boldsymbol{V}_{r-1,0}\right]\boldsymbol{K}_0 + \left[\boldsymbol{L}_{r-2}\boldsymbol{V}_{r-2,1} + \boldsymbol{M}_{r-1}\boldsymbol{V}_{r-1,1}\right]\boldsymbol{K}_1 + \cdots \\
&\quad + \left[\boldsymbol{L}_{r-2}\boldsymbol{V}_{r-2,r-3} + \boldsymbol{M}_{r-1}\boldsymbol{V}_{r-1,r-3}\right]\boldsymbol{K}_{r-3} + \boldsymbol{M}_{r-1}\boldsymbol{V}_{r-1,r-2}\boldsymbol{K}_{r-2} + \boldsymbol{F}_r\boldsymbol{K}_{r-1} \\
&= \boldsymbol{U}_r\boldsymbol{\Omega} + \lfloor \boldsymbol{V}_0 \quad \boldsymbol{V}_1 \cdots \boldsymbol{V}_{r-1} \rfloor_r \{\boldsymbol{K}_0 \quad \boldsymbol{K}_1 \cdots \boldsymbol{K}_{r-1}\},
\end{aligned} \tag{26}$$

$$\cdots\cdots,$$

$$\boldsymbol{X}_{m-1} = \boldsymbol{U}_{m-1}\boldsymbol{\Omega} + \lfloor \boldsymbol{V}_0 \quad \boldsymbol{V}_1 \cdots \boldsymbol{V}_{m-2} \rfloor_{m-1} \{\boldsymbol{K}_0 \quad \boldsymbol{K}_1 \cdots \boldsymbol{K}_{m-2}\}, \tag{27}$$

$$\boldsymbol{X}_m = \boldsymbol{U}_m\boldsymbol{\Omega} + \lfloor \boldsymbol{V}_0 \quad \boldsymbol{V}_1 \cdots \boldsymbol{V}_{m-2} \quad \boldsymbol{V}_{m-1} \rfloor_m \{\boldsymbol{K}_0 \quad \boldsymbol{K}_1 \cdots \boldsymbol{K}_{m-2} \quad \boldsymbol{K}_{m-1}\}. \tag{28}$$

7. UPPER BOUNDARY CONDITIONS

The upper boundary conditions of the present system are given by moment equations at the top nodal points, $(m, 0)$, $(m, 1)$, $(m, 2)$, and $(m, 3)$, which can be put into one matrix equation

$$\begin{bmatrix} k_0 & 0 & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & 0 \\ 0 & 0 & 0 & k_3 & 0 \end{bmatrix}_m^v \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \psi \end{bmatrix}_{m-1} + \begin{bmatrix} j_{m0} & k^h_{m0} & 0 & 0 & k^v_{m0} \\ k^h_{m0} & j_{m1} & k^h_{m1} & 0 & k^v_{m1} \\ 0 & k^h_{m1} & j_{m2} & k^h_{m2} & k^v_{m2} \\ 0 & 0 & k^h_{m2} & j_{m3} & k^v_{m3} \end{bmatrix} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \psi \end{bmatrix}_m$$

$$+\begin{bmatrix} C^{v'}_{m0} + C^{h}_{m0} - \mathfrak{M}_{m0} \\ C^{h'}_{m0} + C^{v'}_{m1} + C^{h}_{m1} - \mathfrak{M}_{m1} \\ C^{h'}_{m1} + C^{v'}_{m2} + C^{h}_{m2} - \mathfrak{M}_{m2} \\ C^{h'}_{m2} + C^{v'}_{m3} - \mathfrak{M}_{m3} \end{bmatrix} = 0, \tag{29}$$

or

$$\boldsymbol{L}'_{m-1}\boldsymbol{X}_{m-1} + \boldsymbol{M}'_{m}\boldsymbol{X}_{m} + \boldsymbol{K}_{m} = 0. \tag{30}$$

Substituting from Eqs. 27 and 28 into Eq. 30, the current-matrix $\boldsymbol{\Omega}$ can be determined as follows:

$$\begin{aligned}\boldsymbol{\Omega} = &-\left[\boldsymbol{L}'_{m-1}\boldsymbol{U}_{m-1} + \boldsymbol{M}'_{m}\boldsymbol{U}_{m}\right]^{-1} \times \left[\boldsymbol{L}'_{m-1}\lfloor \boldsymbol{V}_0 \quad \boldsymbol{V}_1 \cdots \boldsymbol{V}_{m-2} \quad 0 \quad 0 \rfloor_{m-1}\right. \\ &\left. + \boldsymbol{M}'_{m}\lfloor \boldsymbol{V}_0 \quad \boldsymbol{V}_1 \cdots \boldsymbol{V}_{m-1} \quad 0 \rfloor_{m} + \lfloor 0 \quad 0 \cdots \boldsymbol{E} \rfloor\right] \\ &\times \{\boldsymbol{K}_0 \quad \boldsymbol{K}_1 \cdots \boldsymbol{K}_{m-2} \quad \boldsymbol{K}_{m-1} \quad \boldsymbol{K}_{m}\},\end{aligned} \tag{31}$$

or

$$\begin{aligned}\boldsymbol{\Omega} = &\lfloor \boldsymbol{G}_0 \quad \boldsymbol{G}_1 \cdots \boldsymbol{G}_{m-2} \quad \boldsymbol{G}_{m-1} \quad \boldsymbol{G}_{m} \rfloor \\ &\times \{\boldsymbol{K}_0 \quad \boldsymbol{K}_1 \cdots \boldsymbol{K}_{m-2} \quad \boldsymbol{K}_{m-1} \quad \boldsymbol{K}_{m}\},\end{aligned} \tag{32}$$

or

$$\boldsymbol{\Omega} = \lfloor \boldsymbol{G} \rfloor \{\boldsymbol{K}\}. \tag{33}$$

The first factor on the right side of Eq. 33 is called the "geometry matrix," which can be evaluated from the geometrical configuration and material properties of the structure. Thus, the solution can be obtained in the desirable form in which the load-matrix is separated from the geometry matrix.

For the practical application of this method, it will be recommended to consolidate the solution of the system considered in the form of Eq. 33, since the evaluation of physical quantities can at once be obtained from the geometry matrix.

8. MODIFIED SHEAR EQUATION

Taking the portion near the nodal points of the $(r-1)$-th unit as shown in Fig. 3, the shear equation is given by the form

$$\sum_{s=0}^{3}[S_s]^{v}_{r} - \sum_{s=0}^{3}[S_s]^{v'}_{r-1} + p_{r-1} = 0, \tag{34}$$

in which p_{r-1} = the summation of the horizontal loads applied at the $(r-1)$-th nodal points.

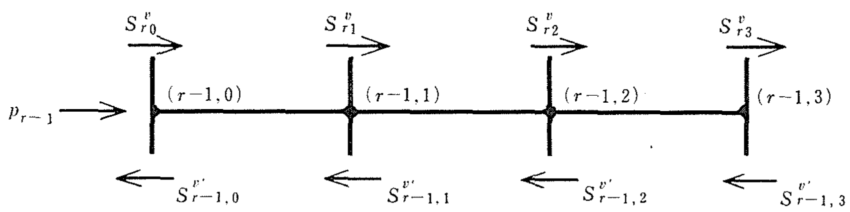

Fig. 3. Portion Near the $(r-1)$-th Nodal Points.

The equilibrium conditions of the $(r-1)$-th unit can also be consolidated in one matrix equation which is similar to Eq. 9, from which the shift formula will be derived. The inverse which is necessary in this case is given below:

$$\begin{bmatrix} k_0 & 0 & 0 & 0 & k_0 \\ 0 & k_1 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 & k_2 \\ 0 & 0 & 0 & k_3 & k_3 \\ \frac{k_0}{h} & \frac{k_1}{h} & \frac{k_2}{h} & \frac{k_3}{h} & \frac{f}{h} \end{bmatrix}_r^{v\,-1} = -\frac{2}{f_r^v}\begin{bmatrix} 1-\frac{f}{2k_0} & 1 & 1 & 1 & -h \\ 1 & 1-\frac{f}{2k_1} & 1 & 1 & -h \\ 1 & 1 & 1-\frac{f}{2k_2} & 1 & -h \\ 1 & 1 & 1 & 1-\frac{f}{2k_3} & -h \\ -1 & -1 & -1 & -1 & h \end{bmatrix}_r^{v}. \tag{35}$$

9. APPLICATION

As a simple application of the above discussions, the analysis of a rigid frame consisting of three spans and four stories will be treated. The configuration and the loading conditions are illustrated in Fig. 4. Referring to this figure, it can be seen that the entire deformation of the frame becomes antisymmetric, and hence sixteen unknown slope angles at the nodal points can be reduced to eight. Consequently, the eigenmatrix of each unit is reduced to an angle of member rotation and two slope angles. Therefore, the upward shift operation is performed by a 3-by-3 shift operator, and the final treatment of the problem requires only a 2-by-2 inverse.

In Table 1 are shown the results obtained from both the operational method and the prevailing iterative method.[3] In the latter procedure, computation was carried out six times repeatedly. From Table 2, it can be observed that the operational method gives accurate results with much less time and labor.

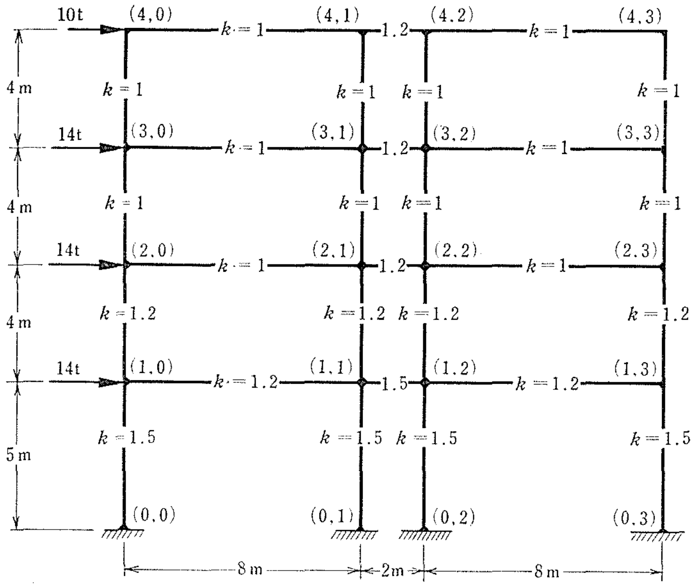

Fig. 4. Three Span Four Story Rigid Frame.

Table 1. Results Obtained.

Unknowns	(A) By Iterative Method	(B) By Operational Method (Two Decimals)	(C) By Operational Method (Three Decimals)
$\varphi_{10} =$	10.20	10.27	10.274
$\varphi_{11} =$	6.35	6.38	6.385
$\phi_1 =$	−34.05	−34.16	−34.160
$\varphi_{20} =$	8.10	8.11	8.107
$\varphi_{21} =$	5.53	5.56	5.559
$\phi_2 =$	−38.40	−38.57	−38.575
$\varphi_{30} =$	4.71	4.70	4.700
$\varphi_{31} =$	3.12	3.12	3.117
$\phi_3 =$	−28.20	−28.18	−28.183
$\varphi_{40} =$	1.83	1.82	1.822
$\varphi_{41} =$	1.07	1.07	1.068
$\phi_4 =$	−13.10	−13.07	−13.065

Table 2. Errors in Check Calculation.

Equations	(A) By Iterative Method	(B) By Operational Method (Two Decimals)	(C) By Operational Method (Three Decimals)
Shear (1st):	0.06	−0.02	−0.002
Moment (1, 0):	−0.26	−0.03	−0.002
Moment (1, 1):	−0.17	−0.05	−0.005
Shear (2nd):	0.11	0.00	0.000
Moment (2, 0):	0.04	0.02	0.000
Moment (2, 1):	−0.04	0.02	0.003
Shear (3rd):	−0.05	0.01	0.000
Moment (3, 0):	0.01	0.00	−0.002
Moment (3, 1):	−0.04	0.03	0.002
Shear (4th):	−0.02	0.00	0.001
Moment (4, 0):	0.00	−0.02	−0.009
Moment (4, 1):	−0.02	0.00	−0.009

10. CONCLUSIONS

In the proposed operational procedure for the slope-deflection equations, the following notes are to be given:

1. For the rigid frame analysis, the concept of the constituent unit is introduced. In virtue of the equilibrium conditions in a unit, the shift formula for three consecutive units is obtained.
2. By the recurrent use of the shift formula, all the unknowns of slope and member rotation of the system can be represented by the current-matrix which consists of the elements of the first unit eigenmatrix degraded by the lower boundary condition.
3. The lower boundary condition is given by the shear equation of the first unit, which results in the formation of the current-matrix from the first unit.
4. The upper boundary conditions are given by the moment equations of the top nodal points of the system. The current-matrix is determined by these conditions. Therefore, the order of inverse necessary for the final step of the operation is given by the number of the top nodal conditions.
5. The number of unknowns in the rigid frame analysis can be extremely decreased by this method, and in addition, accurate results are obtained with less time and labor.

REFERENCES

1) N. YOSHIZAWA and B. TANIMOTO, "Operational Method for Clapeyron's Theorem," Journal of the Faculty of Engineering, Shinshu University, Vol. 18, Dec., 1964, pp. 1–39.

2) I. KONISHI, Y. YOKOO, and M. NARUOKA, "Theory of Structures (in Japanese)," Part II, MARUZEN Co., Inc., 1963, pp. 123–168.

3) S. TANI and H. SUGIYAMA, "Exercise of Structures (in Japanese)," Part II, OHM-Sha Co., Inc., 1966, pp. 365–374.

APPENDIX.—NOTATION

The following symbols have been adopted for use in this paper:

C = load term for slope-deflection method, see Eqs. 1 through 4;
E = Young's modulus;
$\mathbf{F}$ = feed operator, see Eq. 18;
f = story constant, see Eq. 12;
$\mathbf{G}$ = geometry matrix, see Eq. 32;
h = length of vertical-like member;
I = moment of inertia;
j = nodal constant, see Eq. 11;
K = member rigidity, see Eq. 6;
$\mathbf{K}$ = load-matrix, see Eq. 13;
k = rigidity ratio, see Eqs. 5;
$\mathbf{L}$ = shift operator, see Eq. 16;
l = length of horizontal member;
M = end moment of member, see Eqs. 1 and 3;
$\mathbf{M}$ = shift operator, see Eq. 17;
$\mathfrak{M}$ = external nodal moment, see Fig. 2;
m = order number of the top nodal points of frame;
$\mathbf{N}$ = operational matrix, see Eqs. 9 and 10;
P_r = summation of horizontal loads above the r-th plane, see Fig. 2;
p_{r-1} = summation of horizontal loads applied at the $(r-1)$-th nodes;
R = angle of rotation of member, see Eq. 8;
r = symbol representing the story order;
S = end shearing force of member, see Eqs. 2 and 4;
s = symbol representing the column or span order;

$\boldsymbol{U}$ = consolidated operator for current-matrix, see Eqs. 23 through 28;
V = load term for the end shear, see Eqs. 2 and 4;
$\boldsymbol{V}$ = consolidated operator for load-matrix, see Eqs. 25 through 28;
$\boldsymbol{X}_r$ = eigenmatrix of the r-th unit, see Eq. 14;
θ = slope angle;
φ = modified slope angle, see Eqs. 7;
ψ = modified angle of rotation of member, see Eq. 8;
$\boldsymbol{\Omega}$ = current-matrix, see Eq. 23;
$\lfloor \quad \rfloor$ = row vector; and
$\{ \quad \}$ = column vector.