Direct Analysis of Continuous Beam-Columns

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1. INTRODUCTION

The purpose of the present paper is to give the operational procedure of the analysis of continuous beam-columns, in which the intermediate supports may be rigidly or elastically supported. No specific devices or techniques are herein necessary, but the straightforward treatment will lead to the desired solution.

The governing differential equation for rectilinear beam-columns, together with its general solution, has been well known for many years. Basically, the operational method begins only with the general solution of differential equations, and hence it has no key equations. If anything, the general solution must be the key equation, provided that it takes the form of pure and complete classification of data, that is, the form of the product of particular solution assemblage and that of integration constants by making use of the row-column rule for matrix multiplication. In this regard, it seems that if as old as in the early part of the eighteenth century Bernoulli or Euler were eventually conscious of the cited classified form, the structural analysis might have been developed differently. In fact, the operational method is the revival of the most classical approach that begins with the differential equation.

A subsidiary purpose of this paper is to exhibit a typical and rearranged description of the proposed operational method for rectilinear structural systems, as the previous publications[5),6)] by one of the writers are not well arranged.

2. BASIC EQUATIONS

The governing differential equation for the beam-column under no lateral load is

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5) Tanimoto, B., "Operational Method for Continuous Beams," *Journal of the Structural Division*, ASCE, Vol. 90, No. ST6, Proc. Paper 4179, December, 1964.

6) Tanimoto, B., "Some Improvements on Proposed Eigen-Matrix Method," *Journal of the Structural Division*, ASCE, Vol. 92, No. ST1, Proc. Paper 4667, February, 1966.

$$\frac{d^4w}{dx^4} + \frac{P}{EI}\frac{d^2w}{dx^2} = 0, \tag{1}$$

in which $w =$ the deflection (Fig. 1). The general solution for Eq. 1 is

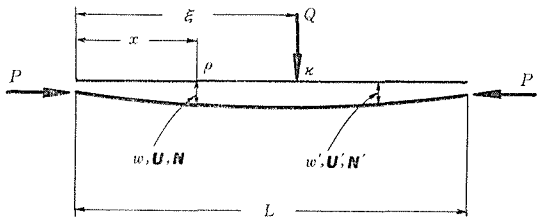

Fig. 1. Beam with a Concentrated Load.

$$w = \frac{L^3}{\alpha^3 EI}\lfloor 1 \quad \alpha\rho \quad \cos\alpha\rho \quad \sin\alpha\rho \rfloor \boldsymbol{N}, \tag{2}$$

in which $\rho = x/L$, and $\boldsymbol{N} = \{A \quad B \quad C \quad D\}$ which is the assemblage of integration constants and is called the eigenmatrix. The state vector then becomes

$$\begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix} = \begin{bmatrix} \frac{L^3}{\alpha^3 EI} & 0 & 0 & 0 \\ 0 & \frac{L^2}{\alpha^2 EI} & 0 & 0 \\ 0 & 0 & \frac{L}{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha\rho & \cos\alpha\rho & \sin\alpha\rho \\ 0 & 1 & -\sin\alpha\rho & \cos\alpha\rho \\ 0 & 0 & \cos\alpha\rho & \sin\alpha\rho \\ 0 & 0 & -\sin\alpha\rho & \cos\alpha\rho \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}, \tag{3}$$

or

$$\boldsymbol{U}(\alpha\rho) = \boldsymbol{DP}(\alpha\rho)\boldsymbol{N}. \tag{4}$$

It should be noticed herein that the right side of Eq. 3 or 4 exhibits the complete classification of data, and then attention can be focused only at attacking the eigenmatrix $\boldsymbol{N}$. It is added also that the prevailing transfer matrix can, if desired, be readily obtained from Eq. 4. The transfer matrix method, however, does not keep the complete classification of data which is of fundamental importance in adopting matrix algebra.

When the beam is subjected to a lateral concentrated load, Q, applied at point $x = \xi$ or $\rho = \kappa$ (Fig. 1), the continuity conditions at the loaded point are

$$\boldsymbol{U}'(\alpha\kappa) = \boldsymbol{U}(\alpha\kappa) - \{0 \quad 0 \quad 0 \quad Q\}. \tag{5}$$

Eq. 5 then gives in view of Eq. 4 the desired equation

$$\boldsymbol{N}' = \boldsymbol{N} + \boldsymbol{K}_\kappa, \tag{6}$$

in which

$$\boldsymbol{K}_\kappa = -\boldsymbol{P}^{-1}(\alpha\kappa)\boldsymbol{D}^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ Q \end{bmatrix} = Q\begin{bmatrix} -\alpha\kappa \\ 1 \\ \sin\alpha\kappa \\ -\cos\alpha\kappa \end{bmatrix}. \tag{7}$$

The $\boldsymbol{K}_\kappa$ matrix is the continuity matrix, or load-matrix, for the concentrated load. Thus,

$$\boldsymbol{U}'(\alpha\rho) = \boldsymbol{DP}(\alpha\rho)[\boldsymbol{N} + \boldsymbol{K}_\kappa], \tag{8}$$

which holds for the conjugate domain $\kappa < \rho < 1$.

For the uniformly distributed load of constant intensity q, Eqs. 8 and 4 may be integrated over the entire span by putting $Q = q\,d\xi$. Then the resulting equation becomes

$$\boldsymbol{U}(\alpha\rho) = \boldsymbol{DP}(\alpha\rho)[\boldsymbol{N} + \boldsymbol{K}(\alpha\rho)], \tag{9}$$

in which

$$\boldsymbol{K}(\alpha\rho) = \int_0^\rho d\boldsymbol{K}_\kappa = \frac{qL}{\alpha}\begin{bmatrix} -\frac{1}{2}(\alpha\rho)^2 \\ \alpha\rho \\ 1 - \cos\alpha\rho \\ -\sin\alpha\rho \end{bmatrix}. \tag{10}$$

The $\boldsymbol{K}(\alpha\rho)$ matrix is the load-matrix for the uniform load, and it takes at ends $\rho = 0$ and $\rho = 1$ the values

$$\boldsymbol{K}(0) = 0, \quad \text{and} \quad \boldsymbol{K}(\alpha) = \frac{qL}{\alpha}\begin{bmatrix} -\frac{\alpha^2}{2} \\ \alpha \\ 1 - \cos\alpha \\ -\sin\alpha \end{bmatrix}. \tag{11}$$

Thus, it can be concluded that the state vector at the ends is always expressed in the forms

$$\boldsymbol{U}(0) = \boldsymbol{DP}(0)\boldsymbol{N}, \quad \text{and} \quad \boldsymbol{U}'(\alpha) = \boldsymbol{DP}(\alpha)[\boldsymbol{N} + \boldsymbol{K}], \tag{12}$$

in which the load-matrix, $\boldsymbol{K}$, represents the assemblage of Eqs. 7, 11b, etc. Note that any kind of lateral loads, including the system of concentrated loads, the partially distributed load, the non-uniform load, the concentrated external moment, etc., can be expressed in the corresponding load-matrix.

3. BEAM-COLUMNS WITH RIGID SUPPORTS

The procedure of arriving at the solution will then consist of the three following steps :

(1) Rigid support conditions that every span has no lateral deflection at both its ends, which are expressed by

$$w_{\rho=0} = 0, \quad \text{and} \quad w'_{\rho=1} = 0. \tag{13}$$

(2) Connection conditions at the common end of any two adjacent spans $r-1$ and r, which are expressed by

$$\begin{bmatrix} \theta \\ M \end{bmatrix}'_{r-1,\,\rho=1} = \begin{bmatrix} \theta \\ M \end{bmatrix}_{r,\,\rho=0}. \tag{14}$$

(3) Boundary conditions at both extreme ends of the continuous structure, which are expressed by

$$\theta_{1,\,\rho=0} = 0, \quad \text{or} \quad M_{1,\,\rho=0} = 0, \tag{15}$$

and

$$\theta'_{n,\,\rho=1} = 0, \quad \text{or} \quad M'_{n,\,\rho=1} = 0, \tag{16}$$

according to the fixed or simple support; n being the number of spans. The three kinds of conditions above, (1), (2), and (3), will be treated successively. Note that Eqs. 13 and 14 may also be treated simultaneously, which will be referred to in the Additional Notes.

First, Eqs. 13 yield

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & \alpha & \cos\alpha & \sin\alpha \end{bmatrix}\boldsymbol{N} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & \alpha & \cos\alpha & \sin\alpha \end{bmatrix}\boldsymbol{K} = 0, \tag{17}$$

which suggests that for example A and B depend on C and D. Then Eq. 17 gives

$$\boldsymbol{N} = \begin{bmatrix} -1 & 0 \\ \dfrac{1-\cos\alpha}{\alpha} & -\dfrac{\sin\alpha}{\alpha} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\boldsymbol{A} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ \dfrac{1}{\alpha} & 1 & \dfrac{\cos\alpha}{\alpha} & \dfrac{\sin\alpha}{\alpha} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\boldsymbol{K}, \tag{18a}$$

and accordingly

$$\boldsymbol{N}' = \begin{bmatrix} -1 & 0 \\ \dfrac{1-\cos\alpha}{\alpha} & -\dfrac{\sin\alpha}{\alpha} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{A} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\dfrac{1}{\alpha} & 0 & -\dfrac{\cos\alpha}{\alpha} & -\dfrac{\sin\alpha}{\alpha} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{K}, \tag{18b}$$

providing $\boldsymbol{A} = \{C \quad D\}$. Eqs. 18 indicate that the fourth-order eigenmatrices, $\boldsymbol{N}$ and $\boldsymbol{N}'$, have been degraded to the second-order semi-eigenmatrix, $\boldsymbol{A}$, by virtue of Eqs. 13.

Secondly, Eq. 14 gives the connection equation

$$\left[\begin{bmatrix} \dfrac{L^2}{\alpha^2 EI} & 0 \\ 0 & \dfrac{L}{\alpha} \end{bmatrix}_{r-1} \begin{bmatrix} 0 & 1 & -\sin\alpha & \cos\alpha \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix}_{r-1}, \right.$$
$$\left. -\begin{bmatrix} \dfrac{L^2}{\alpha^2 EI} & 0 \\ 0 & \dfrac{L}{\alpha} \end{bmatrix}_r \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right] \begin{bmatrix} \boldsymbol{N}'_{r-1} \\ \boldsymbol{N}_r \end{bmatrix} = 0, \tag{19}$$

or

$$\boldsymbol{C}\{\boldsymbol{N}'_{r-1} \quad \boldsymbol{N}_r\} = 0. \tag{20}$$

Here the operational matrix $\boldsymbol{C}$ consists of two 2-by-4 submatrices, and is the connector between the two spans $r-1$ and r.

Substituting from Eqs. 18 into Eq. 19 or 20 yields

$$\left[\begin{bmatrix} \dfrac{L^2}{\alpha^2 EI} & 0 \\ 0 & \dfrac{L}{\alpha} \end{bmatrix}_{r-1} \begin{bmatrix} -\sin\alpha + \dfrac{1-\cos\alpha}{\alpha}, & \cos\alpha - \dfrac{\sin\alpha}{\alpha} \\ \cos\alpha, & \sin\alpha \end{bmatrix}_{-1}, \right.$$
$$\left. -\begin{bmatrix} \dfrac{L^2}{\alpha^2 EI} & 0 \\ 0 & \dfrac{L}{\alpha} \end{bmatrix}_r \begin{bmatrix} \dfrac{1-\cos\alpha}{\alpha}, & 1 - \dfrac{\sin\alpha}{\alpha} \\ 1, & 0 \end{bmatrix}_r \right] \begin{bmatrix} \boldsymbol{A}_{r-1} \\ \boldsymbol{A}_r \end{bmatrix}$$
$$= \left[\begin{bmatrix} \dfrac{L^2}{\alpha^2 EI} & 0 \\ 0 & \dfrac{L}{\alpha} \end{bmatrix}_{r-1} \begin{bmatrix} \dfrac{1}{\alpha}, & 0, & \sin\alpha + \dfrac{\cos\alpha}{\alpha}, & -\cos\alpha + \dfrac{\sin\alpha}{\alpha} \\ 0, & 0, & -\cos\alpha, & -\sin\alpha \end{bmatrix}_{r-1}, \right.$$

$$-\left(\frac{L^2}{EI}\right)_r \begin{bmatrix} \frac{1}{\alpha^3} & \frac{1}{\alpha^2} & \frac{\cos\alpha}{\alpha^3} & \frac{\sin\alpha}{\alpha^3} \\ 0 & 0 & 0 & 0 \end{bmatrix}_r \begin{bmatrix} \boldsymbol{K}_{r-1} \\ \boldsymbol{K}_r \end{bmatrix}, \tag{21}$$

or

$$\boldsymbol{C}'\{\boldsymbol{A}_{r-1} \quad \boldsymbol{A}_r\} = \boldsymbol{T}\{\boldsymbol{K}_{r-1} \quad \boldsymbol{K}_r\}. \tag{22}$$

Here the $\boldsymbol{C}'$ matrix is the reduced connector, consisting of two 2-by-2 square submatrices, each of which is nonsingular.

Eq. 22 or 21 permits extraction of the $\boldsymbol{A}_r$ matrix by premultiplying by the inverse of the second submatrix in the $\boldsymbol{C}'$ matrix. In this way,

$$\boldsymbol{A}_r = \boldsymbol{L}_r\boldsymbol{A}_{r-1} + \lfloor \boldsymbol{V} \quad \boldsymbol{W} \rfloor_r\{\boldsymbol{K}_{r-1} \quad \boldsymbol{K}_r\}, \tag{23}$$

in which

$$\boldsymbol{L}_r = \left(\frac{\alpha}{\alpha - \sin\alpha}\right)_r \begin{bmatrix} 0, & \alpha - \sin\alpha \\ \alpha^2, & -1 + \cos\alpha \end{bmatrix}_r \begin{bmatrix} \frac{EI}{L^2} & 0 \\ 0 & \frac{1}{L} \end{bmatrix}_r \begin{bmatrix} \frac{L^2}{EI} & 0 \\ 0 & L \end{bmatrix}_{r-1}$$

$$\times \begin{bmatrix} -\frac{\sin\alpha}{\alpha^2} + \frac{1-\cos\alpha}{\alpha^3}, & \frac{\cos\alpha}{\alpha^2} - \frac{\sin\alpha}{\alpha^3} \\ \frac{\cos\alpha}{\alpha}, & \frac{\sin\alpha}{\alpha} \end{bmatrix}_{r-1}, \tag{24a}$$

$$\boldsymbol{V}_r = \left(\frac{\alpha}{\alpha - \sin\alpha}\right)_r \begin{bmatrix} 0, & \alpha - \sin\alpha \\ \alpha^2, & -1 + \cos\alpha \end{bmatrix}_r \begin{bmatrix} \frac{EI}{L^2} & 0 \\ 0 & \frac{1}{L} \end{bmatrix}_r \begin{bmatrix} \frac{L^2}{EI} & 0 \\ 0 & L \end{bmatrix}_{r-1}$$

$$\times \begin{bmatrix} -\frac{1}{\alpha^3}, & 0, & -\frac{\sin\alpha}{\alpha^2} - \frac{\cos\alpha}{\alpha^3}, & \frac{\cos\alpha}{\alpha^2} - \frac{\sin\alpha}{\alpha^3} \\ 0, & 0, & \frac{\cos\alpha}{\alpha}, & \frac{\sin\alpha}{\alpha} \end{bmatrix}_{r-1}, \tag{24b}$$

$$\boldsymbol{W}_r = \left(\frac{1}{\alpha - \sin\alpha}\right)_r \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & \alpha & \cos\alpha & \sin\alpha \end{bmatrix}_r. \tag{24c}$$

Here the $\boldsymbol{L}_r$ mtarix is the shift operator or briefly the shiftor, with which the $\boldsymbol{A}_{r-1}$ matrix can be shifted from span $r-1$ to the adjacent span r, and the $\boldsymbol{V}_r$ and $\boldsymbol{W}_r$ matrices are the feed operators or briefly the feeders. Eq. 23 is the desired recurrence formula, with which all the semi-eigenmatrices, $\boldsymbol{A}_r$'s ($r = 2$, 3, 4, $\cdots$, n), can be expressed in terms of the first semi-eigenmatrix, $\boldsymbol{A}_1$. The recurrent application of Eq. 23 then gives

$$\boldsymbol{A}_r = \boldsymbol{Q}_r\boldsymbol{A}_1 + \lfloor \boldsymbol{R} \rfloor_r\{\boldsymbol{K}\}_r, \tag{25}$$

in which

$$\boldsymbol{Q}_r = \boldsymbol{L}_r\, \boldsymbol{Q}_{r-1}, \tag{26a}$$

$$\lfloor \boldsymbol{R} \rfloor_r = \lfloor \boldsymbol{R}_1 \quad \boldsymbol{R}_2 \quad \cdots \quad \boldsymbol{R}_r \rfloor_r = \lfloor \boldsymbol{L}_r \lfloor \boldsymbol{R}_1 \quad \boldsymbol{R}_2 \quad \cdots \quad \boldsymbol{R}_{r-2} \rfloor_{r-1}, \ \ \boldsymbol{L}_r\boldsymbol{W}_{r-1} + \boldsymbol{V}_r, \ \ \boldsymbol{W}_r \rfloor, \tag{26b}$$

with the starting equations

$$\boldsymbol{Q}_2 = \boldsymbol{L}_2, \qquad \lfloor \boldsymbol{R} \rfloor_2 = \lfloor \boldsymbol{V} \quad \boldsymbol{W} \rfloor_2, \tag{27}$$

and $\{\boldsymbol{K}\}_r$ represents the load-matrix assemblage

$$\{\boldsymbol{K}\}_r = \{\boldsymbol{K}_1 \quad \boldsymbol{K}_2 \quad \cdots \quad \boldsymbol{K}_r\}, \tag{28}$$

which is a $4r$-by-1 column matrix. It can be concluded that Eq. 25 has resulted from all the connection conditions of the type of Eq. 14 at intermediate supports, and that the first semi-eigenmatrix, $\boldsymbol{A}_1$, has become current to all the spans. The last step to the solution is then only to attack the $\boldsymbol{A}_1$ matrix.

Thirdly, Eqs. 15 and 16 will be treated. They yield

$$\lfloor 0 \quad 1 \quad 0 \quad 1 \rfloor \boldsymbol{N}_1 = 0, \quad \text{or} \quad \lfloor 0 \quad 0 \quad 1 \quad 0 \rfloor \boldsymbol{N}_1 = 0, \tag{29}$$

and $$\lfloor 0 \quad 1 \quad -\sin\alpha \quad \cos\alpha \rfloor_n \boldsymbol{N}'_n = 0, \quad \text{or} \quad \lfloor 0 \quad 0 \quad \cos\alpha \quad \sin\alpha \rfloor_n \boldsymbol{N}'_n = 0, \tag{30}$$

which, with Eqs. 18, become

$$\lfloor 1 - \cos\alpha, \ \ \alpha - \sin\alpha \rfloor_1 \boldsymbol{A}_1 - \lfloor 1 \quad \alpha \quad \cos\alpha \quad \sin\alpha \rfloor_1 \boldsymbol{K}_1 = 0, \tag{31a}$$

or $$\lfloor 1 \quad 0 \rfloor \boldsymbol{A}_1 = 0, \tag{31b}$$

and $$\lfloor 1 - \cos\alpha - \alpha\sin\alpha, \ \ -\sin\alpha + \alpha\cos\alpha \rfloor_n \boldsymbol{A}_n$$
$$+ \lfloor -1, \ \ 0, \ \ -\cos\alpha - \alpha\sin\alpha, \ \ -\sin\alpha + \alpha\cos\alpha \rfloor_n \boldsymbol{K}_n = 0, \tag{32a}$$

$$\lfloor \cos\alpha \quad \sin\alpha \rfloor_n \boldsymbol{A}_n + \lfloor 0 \quad 0 \quad \cos\alpha \quad \sin\alpha \rfloor_n \boldsymbol{K}_n = 0, \tag{32b}$$

which take the forms

$$\boldsymbol{B}\boldsymbol{A}_1 + \boldsymbol{F}\boldsymbol{K}_1 = 0, \tag{33}$$

and $$\boldsymbol{B}'\boldsymbol{A}_n + \boldsymbol{F}'\boldsymbol{K}_n = 0. \tag{34}$$

Eqs. 33 and 34 are the desired boundary equations obtained from Eqs. 15 and 16, respectively. They are put into one equation, providing Eq. 34 is substituted from Eq. 25. In this way, for the beam problem,

$$\mathbf{A}_1 = -\begin{bmatrix} \mathbf{B} \\ \mathbf{B}'\mathbf{Q}_n \end{bmatrix}^{-1} \left[\begin{bmatrix} 0 \\ \mathbf{B}' \lfloor \mathbf{R} \rfloor_n \end{bmatrix} + \begin{bmatrix} \mathbf{F} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \mathbf{F}' \end{bmatrix} \right] \{\mathbf{K}\}_n, \tag{35}$$

which requires a second-order inverse. Eq. 35 is of the form

$$\mathbf{A}_1 = \lfloor \mathbf{G} \rfloor_n \{\mathbf{K}\}_n. \tag{36}$$

Here $\lfloor \mathbf{G} \rfloor_n$ is a 2-by-$4n$ rectangular matrix and $\{\mathbf{K}\}_n$ is a $4n$-by-1 column matrix. The former, $\lfloor \mathbf{G} \rfloor_n$, depends on only the geometry and material properties of the beam, so that it may be called the geometry matrix, while the latter, $\{\mathbf{K}\}_n$, is the assemblage of all the load-matrices which can correspond to any kinds of load conditions.

For the buckling problem, the desired eigenvalue equation is

$$\begin{vmatrix} \mathbf{B} \\ \mathbf{B}'\mathbf{Q}_n \end{vmatrix} = 0, \tag{37}$$

the left side of which is a 2-by-2 determinant.

It is to be noted here that Eqs. 35 and 37 may be adopted for use when a digital computer is available, but that in case of manual handling it is preferable to treat Eqs. 15 and 16, or Eqs. 33 and 34, separately, in consequence of which the necessary inverse or determinant reduces to the size 1-by-1. Eq. 37 can be readily solved by means of inverse interpolation techniques, as will be given later.

4. BEAM-COLUMNS WITH ELASTIC SUPPORTS

When the beam-column is resting on elastic supports, the connection conditions between any two adjacent spans at their common end are

$$\begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}_r = \begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}'_{r-1} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \lambda_r w'_{r-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \lambda_r & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}'_{r-1}, \tag{38}$$

or

$$\mathbf{U}_r = \mathbf{S}_r \mathbf{U}'_{r-1}. \tag{39}$$

Here the reaction, R_r, at support r is proportional to the end deflection w'_{r-1} at the right end $\rho = 1$ of span $r-1$, so that, with the elastic constant λ_r, $B_r = \lambda_r w'_{r-1}$, which has been considered in Eq. 38.

Eq. 39 at once yields in view of Eqs. 12 the desired recurrence formula

$$\mathbf{N}_r = \mathbf{L}_r \mathbf{N}'_{r-1} = \mathbf{L}_r (\mathbf{N} + \mathbf{K})_{r-1}, \tag{40}$$

in which the shiftor, $\boldsymbol{L}_r$, is

$$\boldsymbol{L}_r = \boldsymbol{P}_r^{-1}(0)\boldsymbol{D}_r^{-1}\boldsymbol{S}_r\cdot\boldsymbol{D}_{r-1}\boldsymbol{P}_{r-1}(\alpha), \tag{41}$$

which may be evaluated as

$$\boldsymbol{L}_r = \begin{bmatrix} \dfrac{\alpha^3 EI}{L^3} & 0 & -\dfrac{\alpha}{L} & 0 \\ -\lambda & \dfrac{\alpha^2 EI}{L^2} & 0 & -1 \\ 0 & 0 & \dfrac{\alpha}{L} & 0 \\ \lambda & 0 & 0 & 1 \end{bmatrix}_r \begin{bmatrix} \dfrac{L^3}{\alpha^3 EI} & 0 & 0 & 0 \\ 0 & \dfrac{L^2}{\alpha^2 EI} & 0 & 0 \\ 0 & 0 & \dfrac{L}{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{r-1} \begin{bmatrix} 1 & \alpha & \cos\alpha & \sin\alpha \\ 0 & 1 & -\sin\alpha & \cos\alpha \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix}_{r-1}. \tag{42}$$

The recurrent use of Eq. 40 permits all the eigenmatrices, $\boldsymbol{N}_r$' s ($r = 2, 3, 4, \cdots, n$), to be expressed in terms of the first eigenmatrix $\boldsymbol{N}_1$. Thus,

$$\boldsymbol{N}_r = \boldsymbol{Q}_r\boldsymbol{N}_1 + \lfloor \boldsymbol{R} \rfloor_{r-1}\{\boldsymbol{K}\}_{r-1}, \tag{43}$$

in which

$$\boldsymbol{Q}_r = \boldsymbol{L}_r\boldsymbol{Q}_{r-1}, \quad \lfloor \boldsymbol{R} \rfloor_{r-1} = \lfloor \boldsymbol{L}_r \lfloor \boldsymbol{R} \rfloor_{r-2} \quad \boldsymbol{L}_r \rfloor, \tag{44}$$

with the starting equations

$$\boldsymbol{Q}_2 = \boldsymbol{L}_2, \qquad \lfloor \boldsymbol{R} \rfloor_1 = \boldsymbol{L}_2. \tag{45}$$

Boundary conditions at both extreme ends are now to be treated. They yield the boundary equations

$$\boldsymbol{B}\boldsymbol{N}_1 = 0, \quad \text{and} \quad \boldsymbol{B}'\boldsymbol{N}'_n = 0, \tag{46}$$

in which, assuming both ends to be simply supported with elastic deflections, the boundary matrices, $\boldsymbol{B}$ and $\boldsymbol{B}'$, take the values

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 1 & -\dfrac{\alpha^3 EI}{\lambda L^3} \\ 0 & 0 & 1 & 0 \end{bmatrix}_1, \tag{47a}$$

$$\boldsymbol{B}' = \begin{bmatrix} 0 & 0 & -\sin\alpha & \cos\alpha \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix}_n + \lambda_{n+1}\left(\frac{L^3}{\alpha^3 EI}\right)_n \begin{bmatrix} 1 & \alpha & \cos\alpha & \sin\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}_n, \tag{47b}$$

providing λ_{n+1} denotes the elastic constant attached to the extreme right end.

Eqs. 46 are put into one equation, providing the latter equation, Eq. 46b, is substituted from Eq. 43 ($r = n$). Thus, the current eigenmatrix $\boldsymbol{N}_1$ for the beam problem can be found to be

$$\boldsymbol{N}_1 = -\begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{B}'\boldsymbol{Q}_n \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \boldsymbol{B}'\lfloor \boldsymbol{R} \rfloor_{n-1}, \ \boldsymbol{B}' \end{bmatrix} \{\boldsymbol{K}\}_n, \tag{48}$$

which requires a fourth-order inverse. Eq. 48 takes the same form as the right side of Eq. 36, and the geometry matrix $\lfloor \boldsymbol{G} \rfloor_n$ in the present case is a 4-by-$4n$ rectangular matrix. For the buckling problem, the desired eigenvalue equation is

$$\begin{vmatrix} \boldsymbol{B} \\ \boldsymbol{B}'\boldsymbol{Q}_n \end{vmatrix} = 0, \tag{49}$$

the left side being a fourth-order determinant. Eqs. 46 may also be treated separately, in which case the 4-by-4 inverse or determinant can be avoided and only a 2-by-2 inverse or determinant is necessary.

It may be seen from the preceding analyses that the beamcolumn with elastic supports is much simpler in philosophy and computation than that with rigid supports, and that the latter is a special case of the former when the elastic constants λ_r' s become very large.

5. ADDITIONAL NOTES

Regarding the beam-column with rigid supports, the following note is to be given. When computer is available, it seems preferable to treat Eqs. 13 and 14 simultaneously. In this case, the connection equation between the two spans $r-1$ and r takes the form

$$\boldsymbol{C}\{\boldsymbol{N}_{r-1} \quad \boldsymbol{N}_r\} = \boldsymbol{T}\{\boldsymbol{K}_{r-1} \quad \boldsymbol{K}_r\}, \tag{50}$$

for which

$$\boldsymbol{C} = \left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{L^2}{\alpha^2 EI} & 0 \\ 0 & \dfrac{L}{\alpha} \end{bmatrix}_{r-1} \begin{bmatrix} 0 & 1 & -\sin\alpha & \cos\alpha \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix}_{r-1}, \quad -\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & \alpha & \cos\alpha & \sin\alpha \\ 0 & \dfrac{L^2}{\alpha^2 EI} & 0 & \dfrac{L^2}{\alpha^2 EI} \\ 0 & 0 & \dfrac{L}{\alpha} & 0 \end{bmatrix}_r \right], \tag{51a}$$

$$\boldsymbol{T} = \left[-\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{L^2}{\alpha^2 EI} & 0 \\ 0 & \dfrac{L}{\alpha} \end{bmatrix}_{r-1} \begin{bmatrix} 0 & 1 & -\sin\alpha & \cos\alpha \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix}_{r-1}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & \alpha & \cos\alpha & \sin\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_r \right]. \tag{51b}$$

Note that the second submatrix on the right side of Eq. 51a is square and non-singular, and hence $\boldsymbol{N}_r$ can be extracted. In this way, Eq. 50 yields the desired recurrence formula

$$\boldsymbol{N}_r = \boldsymbol{L}_r\boldsymbol{N}_{r-1} + \lfloor \boldsymbol{V} \quad \boldsymbol{W} \rfloor_r \{\boldsymbol{K}_{r-1} \quad \boldsymbol{K}_r\}. \tag{52}$$

(The explicit forms of the shiftor and feeders are not recorded herein for saving space.)

Recurrent use of Eq. 52 permits all the eigenmatrices $\boldsymbol{N}_r$' s to be expressed in terms of the first eigenmatrix $\boldsymbol{N}_1$. The four boundary conditions to attack $\boldsymbol{N}_1$ then are

$$w_{1,\rho=0} = 0, \quad w'_{1,\rho=1} = 0, \quad M_{1,\rho=0} = 0, \quad \text{and} \quad M'_{n,\rho=1} = 0, \tag{53}$$

provided that for example both ends of the continuous beam-column are assumed to be simply supported.

When the continuous beam is subjected to the tensile axial force, as well as to lateral loads, it is only necessary to begin with, instead of Eq. 2,

$$w = \frac{L^3}{\alpha^3 EI} \lfloor 1 \quad \alpha\rho \quad \cosh\alpha\rho \quad \sinh\alpha\rho \rfloor \boldsymbol{N}, \tag{54}$$

in which $\alpha = \sqrt{PL^2/EI}$; P denoting the tensile axial force of the span.

The lateral free vibration of the continuous beam can be treated similarly. In this case, the deflection w during vibration is given by

$$w = \frac{L^3}{\alpha^3 EI} \lfloor \cos\alpha\rho \quad \sin\alpha\rho \quad \cosh\alpha\rho \quad \sinh\alpha\rho \rfloor \boldsymbol{N} e^{i\omega t}, \tag{55}$$

in which $\alpha = \sqrt{\gamma A\omega^2 L^2/EIg}$, in which A = the cross-sectional area, g = the acceleration due to gravity, γ = the mass per unit of volume, and ω = the circular frequency.

In conclusion, it is noted that complete analyses regarding Eqs. 54 and 55 have been worked out, though unpublished. The free vibration of some typical rigid frames, together with their successful numerical examples, has also been treated by the operaitonal method, and it will be published some other day.

6. NUMERICAL EXAMPLES

The first numerical example treated was the buckling problem of the continuous column with rigid supports shown in Fig. 2. The critical load, $\bar{P}$, by means of Eq. 37 amounts to

$$\bar{P} = 5.0517 \frac{EI}{L^2}. \tag{56}$$

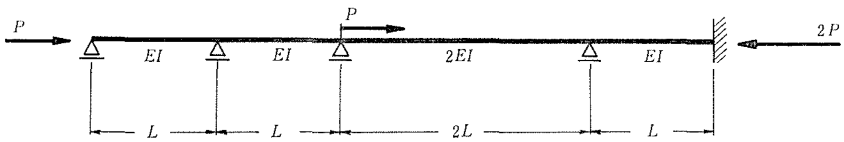

Fig. 2. Four Span Continuous Beam-Column.

Table 1. Critical Loads

λ_r	$\bar{P}/\frac{EI}{L^2}$
10	3.3707
100	4.8088
1 000	5.0274
10 000	5.0493
100 000	5.0515
1 000 000	5.0517

The second numerical example was again the same buckling problem, provided that the continuous column is supported elastically with the spring constant λ_r. The critical loads with increase in λ_r are given in Table 1. This table indicates that the column with rigid supports can be well included in that with elastic supports as a special case in which the spring constant λ_r becomes some larger value, say $\lambda_r = 100\,000$.

7. CONCLUSIONS

The continuous beam-column with rigid or elastic supports is treated by the operational method.[5]

The known key equations to several prevailing methods, including the slope-deflection approach, have been derived from the general solutions of the corresponding differential equations by one of the writers.[7] It has been found there that most of them are not always based on the perfect classification of data, so that the subsequent analyses become more or less tedious, even if matrix form is adopted for use. These facts would throw a doubt on the specific preference of these key equation approaches.

Modern analysis has made it possible to treat the rectilinear structural systems, including the present one, as one of the simplest problems. They can be readily treated through a known straightforward procedure without any specific devices, and no question or difficulty can be raised when the operational method is adopted for use.

Extension or improvement on classical theories would frequently be of little significance. It is, therefore, desirable to pay attention to modern structural

7) Yoshizawa, N., "Comments on Prevailing Methods of Structural Analysis," *Journal of the Shinshu University*, Vol. 22, 1967.

analysis. Emphasis should always be placed on the fact that structural analysis in general reduces easiest when matrix algebra, with the cited classification of data, is adopted for use. This must be the only way of modern structural analysis and also to that in future.

In conclusion, it is noted that the present paper was set about by being stimulated by a paper by Dundurs, Lee, and Hampe, which is due to the slope-deflection approach.[8] An abridged discussion of the cited paper by the present writers has been submitted to and approved by the American Society of Civil Engineers, and it will be published in a forthcoming issue of the proceedings of the Society.

APPENDIX. —NOTATION

The following symbols are used in this paper :

A, B, C, D = elements in eigenmatrix $\boldsymbol{N}$; Eq. 2;
$\boldsymbol{A}$ = semi-eigenmatrix; Eqs. 18;
$\boldsymbol{B}, \boldsymbol{B}'$ = boundary matrices at extreme left and right ends, respectively; Eqs. 33 and 34 for rigid support, and Eqs. 46 for elastic support;
$\boldsymbol{C}$ = connector; Eq. 20;
$\boldsymbol{C}'$ = reduced connector; Eq. 22;
$\boldsymbol{D}$ = diagonal matrix expressing physical properties; Eq. 4;
EI = flexural rigidity; Eq. 2;
$\lfloor \boldsymbol{G} \rfloor_n$ = geometry matrix; Eq. 36;
i = order number of spans and supports;
$\boldsymbol{K}$ = load-matrix consisting of Eqs. 6, 9, etc. ;
$\{\boldsymbol{K}\}_r$ = load-matrix assemblage; Eq. 28;
L = span length; Fig. 1;
$\boldsymbol{L}$ = shiftor; Eq. 48a for rigid support, and Eq. 42 for elastic support;
M = bending moment; Eq. 3;
$\boldsymbol{N}, \boldsymbol{N}'$ = fourth-order eigenmatrix; Eqs. 2 and 6, respectively;
n = number of spans;
P = axial force; Fig. 1;
$\boldsymbol{P}(\alpha\rho)$ = abscissa matrix; Eq. 4;
$\boldsymbol{Q}$ = integrated shiftor; Eq. 26a for rigid support, and Eq. 44a for elastic support;
Q = lateral concentrated load; Fig. 1;

8) John Dundurs, Seng-Lip Lee, and Peter A. Hampe, "Direct Analysis of Continuous Beam-Columns," Journal of the Structural Division, ASCE, Vol. 93, No. ST3, Proc. Paper 5254, June, 1967, pp. 1-10.

q = intensity of distributed load; Eq. 10;

$\lfloor \boldsymbol{R} \rfloor$ = integrated feeder; Eq. 26b for rigid support, and Eq. 44b for elastic support;

$\boldsymbol{S}$ = elastic support matrix; Eq. 39;

$\boldsymbol{T}$ = Eqs. 22 and 50;

$\boldsymbol{U}(\alpha\rho), \boldsymbol{U}'(\alpha\rho)$ = state vector for normal and conjugate domains, respectively; Eqs. 4 and 8;

$\boldsymbol{V}$, $\boldsymbol{W}$ = first and second feeders; Eqs. 24b and 24c;

w, w' = deflection for normal and conjugate domains, respectively; Fig. 1;

x = current abscissa; Fig. 1;

$\alpha = \sqrt{PL^2/EI}$, Eq. 2;

θ = flexural slope; Eq. 3;

$\kappa = \xi/L$, dimensionless load abscissa; Fig. 1;

ξ = load abscissa; Fig. 1;

$\rho = x/L$, dimensionless current abscissa; Fig. 1;

$\lfloor \quad \rfloor$ = row matrix; and

$\{ \quad \}$ = column matrix.